GAUSSIAN FLUCTUATIONS OF THE 2D KPZ EQUATION

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Abstract. We prove the 2D KPZ equation with a logarithmically tuned nonlinearity and a small coupling constant, scales to the Edwards-Wilkinson equation with an effective variance.

MSC 2010: 35R60, 60H07, 60H15.

Keywords: KPZ equation, Edwards-Wilkinson equation, Feynman-Kac formula.

1. Introduction

1.1. **Main result.** We are interested in the 2D KPZ equation driven by a mollified spacetime white noise and starting from flat initial data:

$$\partial_t h_\varepsilon = \tfrac{1}{2}\Delta h_\varepsilon + \tfrac{\beta}{2\sqrt{|\log\varepsilon|}}|\nabla h_\varepsilon|^2 + \dot{W}_\varepsilon(t,x), \quad h_\varepsilon(0,x)\equiv 0, \quad x\in\mathbb{R}^2, \tag{1.1}$$

where

$$\dot{W}_\varepsilon(t,x) = \tfrac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\varphi(\tfrac{x-y}{\varepsilon})\dot{W}(t,y)dy,$$

with $\dot{W}$ a spacetime white noise built on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $0\le\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^2)$. The covariance function of $\dot{W}_\varepsilon$ is formally written as

$$\mathbb{E}[\dot{W}_\varepsilon(t,x)\dot{W}_\varepsilon(s,y)] = \delta(t-s)\tfrac{1}{\varepsilon^2}R(\tfrac{x-y}{\varepsilon}), \quad \text{with} \quad R(x)=\int_{\mathbb{R}^2}\varphi(x+y)\varphi(y)dy. \tag{1.2}$$

Without loss of generality, assume $\varphi(x)=0$ for $|x|\ge\frac{1}{2}$ and $\int_{\mathbb{R}^2}\varphi(x)dx=1$. The following is our main result:

Theorem 1.1. *There exists β_0 depending on φ such that if $\beta<\beta_0\le\sqrt{2\pi}$, then for any $t>0$ and test function $g\in\mathcal{C}_c^\infty(\mathbb{R}^2)$, we have*

$$\int_{\mathbb{R}^2}(h_\varepsilon(t,x)-\mathbb{E}[h_\varepsilon(t,x)])\,g(x)dx \Rightarrow \int_{\mathbb{R}^2}\mathscr{H}(t,x)g(x)dx \tag{1.3}$$

in distribution as $\varepsilon\to0$, where $\mathscr{H}$ solves the Edwards-Wilkinson equation

$$\partial_t\mathscr{H} = \tfrac{1}{2}\Delta\mathscr{H} + \nu_{\text{eff}}\dot{W}(t,x), \quad \mathscr{H}(0,x)\equiv0,$$

with the effective variance

$$\nu_{\text{eff}}^2 = \tfrac{2\pi}{2\pi-\beta^2}. \tag{1.4}$$

There is a lot of activities on the study of the 1D KPZ equation over the past decade. We refer to the reviews [7, 17] for a summary of some recent developments. Progresses in $d\ge3$ can be found in [8, 13], where results similar to Theorem 1.1 were proved. In two dimensions, the tightness of $\{h_\varepsilon\}_{\varepsilon\in(0,1)}$, as a sequence of random distributions, was proved in the recent work of Chatterjee-Dunlap [5]. To prove Theorem 1.1, we implement the same strategy laid out in [8].

The convergence in (1.3) is expected to hold for all $\beta\in(0,\sqrt{2\pi})$, and our proof seems to only work for β small enough. After the completion of this paper, we learnt the very recent work of Caravenna-Sun-Zygouras [4] which proved Theorem 1.1 for all $\beta \in (0,\sqrt{2\pi})$, using a different method. While their result is more general and covers the entire "subcritical" regime, the proof presented here seems to be simpler and offers another perspective. We compare the two approaches in Section 2.4.

At the critical $\beta = \sqrt{2\pi}$, the early work of Bertini-Cancrini [1] identified the limiting covariance function of the corresponding stochastic heat equation. While the limiting distribution remains an open question, we refer to the work of [3, 9] in this direction.

1.2. **Connection to the stochastic heat equation and heuristics.** Through a Hopf-Cole transformation, the h_ε defined in (1.1) is related to the solution of the heat equation with a weak random potential

$$\partial_t u = \tfrac{1}{2}\Delta u + \beta_\varepsilon V(t,x)u, \quad u(0,x) \equiv 1, \quad x \in \mathbb{R}^2, \tag{1.5}$$

with

$$\beta_\varepsilon = \frac{\beta}{\sqrt{|\log \varepsilon|}}, \tag{1.6}$$

and

$$V(t,x) = \int_{\mathbb{R}^2} \varphi(x-y)\dot{W}(t,y)dy.$$

Here, the product $V(t,x)u$ in (1.5) is interpreted in the Itô sense. Consider $u_\varepsilon(t,x) = u(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})$, which solves

$$\partial_t u_\varepsilon = \tfrac{1}{2}\Delta u_\varepsilon + \frac{\beta_\varepsilon}{\varepsilon^2}V(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon})u_\varepsilon.$$

By the scaling property of the spacetime white noise and the fact that $d = 2$, we have

$$\frac{1}{\varepsilon^2}V(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}) \overset{\text{law}}{=} \dot{W}_\varepsilon(t,x). \tag{1.7}$$

Applying Itô's formula yields

$$\beta_\varepsilon^{-1}(\log u_\varepsilon - \mathbb{E}[\log u_\varepsilon]) \overset{\text{law}}{=} h_\varepsilon - \mathbb{E}[h_\varepsilon].$$

From now on, we focus on $\log u_\varepsilon$ rather than h_ε.

Our proof of Theorem 1.1 implies a similar result of u_ε: for $\beta < \beta_0$,

$$\beta_\varepsilon^{-1}\int_{\mathbb{R}^2}(u_\varepsilon(t,x)-1)g(x)dx \Rightarrow \int_{\mathbb{R}^2}\mathscr{H}(t,x)g(x)dx \tag{1.8}$$

in distribution. This was previously proved in [2, Theorem 2.17] for all $\beta \in (0,\sqrt{2\pi})$. Let us explain the mechanism behind the convergence of (1.8) for the stochastic heat equation (SHE) and how it relates to the convergence for KPZ in (1.3).

First, the variance of the l.h.s. of (1.8) is

(1.9)

$$\mathrm{Var}[\beta_\varepsilon^{-1}\int_{\mathbb{R}^2}u_\varepsilon(t,x)g(x)dx] = \beta^{-2}|\log\varepsilon|\int_{\mathbb{R}^4}\mathrm{Cov}[u(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}),u(\tfrac{t}{\varepsilon^2},\tfrac{y}{\varepsilon})]g(x)g(y)dxdy.$$

The covariance is written explicitly by the Feynman-Kac formula:

(1.10)

$$u(t,x) = \mathbb{E}_B[e^{\beta_\varepsilon\int_0^t V(t-s,x+B_s)ds - \frac{1}{2}\beta_\varepsilon^2 R(0)t}],$$

$$\mathrm{Cov}[u(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}),u(\tfrac{t}{\varepsilon^2},\tfrac{y}{\varepsilon})] = \mathbb{E}_B\big[e^{\beta^2|\log\varepsilon|^{-1}\int_0^{t/\varepsilon^2}R(\frac{x-y}{\varepsilon}+B_s^1-B_s^2)ds}\big] - 1 = F(\tfrac{t}{\varepsilon^2},\tfrac{x-y}{\varepsilon}) - 1,$$

where B^1, B^2 are independent Brownian motions starting from the origin and $\mathbb{E}_B$ denotes the expectation with respect to the Brownian motions. Here F solves the deterministic PDE

$$\partial_t F = \Delta F + \beta^2|\log\varepsilon|^{-1}R(x)F, \quad F(0,x) \equiv 1.$$

Similar to u, we have omitted the dependence of F on ε. The above equation can be written in the mild formulation as

$$F(\tfrac{t}{\varepsilon^2},\tfrac{x-y}{\varepsilon}) = 1+\beta^2|\log\varepsilon|^{-1}\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^2} G_{2(t/\varepsilon^2-\ell)}(\tfrac{x-y}{\varepsilon}-w)R(w)F(\ell,w)dwd\ell,$$

where we denote $G_t(x) = (2\pi t)^{-1}\exp(-|x|^2/2t)$ as the standard heat kernel. After a change of variable $\ell \mapsto \ell/\varepsilon^2$, we have

$$\beta^{-2}|\log\varepsilon|\,[F(\tfrac{t}{\varepsilon^2},\tfrac{x-y}{\varepsilon})-1] = \int_0^t\int_{\mathbb{R}^2} G_{2(t-\ell)}(x-y-\varepsilon w)R(w)F(\tfrac{\ell}{\varepsilon^2},w)dwd\ell. \tag{1.11}$$

By the Feynman-Kac representation of F in (1.10), we know that $F(\ell/\varepsilon^2,w)$ measures the intersection time of the two Brownian motions during $[0,\ell/\varepsilon^2]$. By a classical result of Kallianpur-Robbins [12, Theorem 1], for $\beta>0,\ell>0$ and $w\in\mathbb{R}^2$, the following convergence in distribution holds:

$$|\log\varepsilon|^{-1}\int_0^{\ell/\varepsilon^2} R(w+B_s^1-B_s^2)ds \overset{\text{law}}{=} (2|\log\varepsilon|)^{-1}\int_0^{2\ell/\varepsilon^2} R(w+B_s)ds \Rightarrow \tfrac{1}{2\pi}\mathrm{Exp}(1), \tag{1.12}$$

where we used the fact that $\int R = 1$. Together with some uniform integrability we will establish later, this implies

$$F(\tfrac{\ell}{\varepsilon^2},w) = \mathbb{E}_B[e^{\beta^2|\log\varepsilon|^{-1}\int_0^{\ell/\varepsilon^2}R(w+B_s^1-B_s^2)ds}] \to \tfrac{2\pi}{2\pi-\beta^2} = \nu_{\text{eff}}^2 \tag{1.13}$$

for small β, as $\varepsilon\to 0$. Combining (1.11) and (1.13), the variance in (1.9) converges:

$$\begin{aligned}\mathrm{Var}[\beta_\varepsilon^{-1}\int_{\mathbb{R}^2} u_\varepsilon(t,x)g(x)dx] &\to \nu_{\text{eff}}^2\int_0^t\int_{\mathbb{R}^6} G_{2(t-\ell)}(x-y)R(w)g(x)g(y)dxdydwd\ell\\ &= \mathrm{Var}[\int_{\mathbb{R}^2}\mathscr{H}(t,x)g(x)dx].\end{aligned} \tag{1.14}$$

The above calculation and the convergence in (1.13) interprets the effective variance ν_{eff}^2 in terms of the intersection of two Brownian paths.

Now we explain the origin of the Gaussianity. It is important to note that the main contribution to the integral in (1.12) comes from $s\in[0,K_\varepsilon]$ provided that $|\log(\varepsilon^2K_\varepsilon)| \ll |\log\varepsilon|$. Actually, the heat kernel in $d=2$ satisfies that $G_t(x)\sim t^{-1}$ for those x around the origin, so we have

$$\begin{aligned}&|\log\varepsilon|^{-1}\left(\int_0^{\ell/\varepsilon^2}-\int_0^{K_\varepsilon}\right)\mathbb{E}_B[R(w+B_s^1-B_s^2)]ds\\ &\qquad= |\log\varepsilon|^{-1}\int_{K_\varepsilon}^{\ell/\varepsilon^2}\mathbb{E}_B[R(w+B_s^1-B_s^2)]ds\\ &\qquad\sim |\log\varepsilon|^{-1}|\log(\varepsilon^2K_\varepsilon)|\to 0.\end{aligned} \tag{1.15}$$

Thus, we can e.g. pick $K_\varepsilon = \frac{1}{\varepsilon^2|\log\varepsilon|} = o(\varepsilon^{-2})$ and replace $F(\ell/\varepsilon^2,w)$ in (1.11) by $F(K_\varepsilon,w)$ without changing the asymptotic covariance:

$$\beta^{-2}|\log\varepsilon|\,[F(\tfrac{t}{\varepsilon^2},\tfrac{x-y}{\varepsilon})-1] \approx \int_0^t\int_{\mathbb{R}^2} G_{2(t-\ell)}(x-y-\varepsilon w)R(w)F(K_\varepsilon,w)dwd\ell. \tag{1.16}$$

Recall that

$$\mathrm{Var}[\beta_\varepsilon^{-1}\int_{\mathbb{R}^2} u_\varepsilon(t,x)g(x)dx] = \int_{\mathbb{R}^4}\beta^{-2}|\log\varepsilon|\,[F(\tfrac{t}{\varepsilon^2},\tfrac{x-y}{\varepsilon})-1]g(x)g(y)dxdy.$$

The r.h.s. of (1.16) shows the main contribution to the variance of our interested quantity, from the perspective of Brownian paths intersections. In microscopic variables, we have two Brownian paths, starting from $\frac{x}{\varepsilon}$ and $\frac{y}{\varepsilon}$ respectively and running backwards in time. After first meeting each other at the time $(t-\ell)/\varepsilon^2$ for some $\ell\in(0,t)$, the two paths spend $K_\varepsilon = o(\varepsilon^{-2})$ amount of time "intersecting" before splitting again. As a result, the random environment involved in this "intersection" only consists of $\dot{W}(s,\cdot)$ with $s \in [\ell/\varepsilon^2 - o(\varepsilon^{-2}), \ell/\varepsilon^2]$, which induces a temporal decorrelation for different $\ell_1 \neq \ell_2 \in (0,t)$ and creates the Gaussianity. Together with the variance convergence in (1.14), this implies the Edwards-Wilkinson limit in (1.8). The results in [10] for $d \geq 3$ is based on the above heuristics.

For the KPZ equation, the Gaussianity comes from a similar temporal decorrelation discussed above (we will prove it by a different method though). The convergence of the variance

$$\begin{aligned}\mathrm{Var}[\int_{\mathbb{R}^2} h_\varepsilon(t,x)g(x)dx] &= \mathrm{Var}[\beta_\varepsilon^{-1}\int_{\mathbb{R}^2} \log u_\varepsilon(t,x)g(x)dx]\\ &\to \mathrm{Var}[\int_{\mathbb{R}^2} \mathscr{H}(t,x)g(x)dx]\end{aligned}$$

is however more involved. While we do not have a Feynman-Kac representation for $\mathrm{Cov}[\log u_\varepsilon(t,x), \log u_\varepsilon(t,y)]$ as (1.10), an application of the Clark-Ocone formula will help us express the covariance in terms of an integral of

$$\mathbb{E}[D\log u_\varepsilon(t,z)|\mathcal{F}_r] = \mathbb{E}[u_\varepsilon^{-1}(t,z)Du_\varepsilon(t,z)|\mathcal{F}_r], \quad z = x,y, \quad r \leq \tfrac{t}{\varepsilon^2}. \tag{1.17}$$

Here D is the Malliavin derivative with respect to the random noise and $\mathcal{F}_r$ is the filtration generated by $\{\dot{W}(s,\cdot), s \leq r\}$. The key difficulty in analyzing (1.17) is to deal with the factor u_ε^{-1} and to evaluate the conditional expectation given $\mathcal{F}_r$. By the same discussion for (1.15), the random variable $u_\varepsilon(t,\cdot)$ mainly depends on the noise $\dot{W}(s,\cdot)$ for $s \in [t/\varepsilon^2 - o(\varepsilon^{-2}), t/\varepsilon^2]$, so we could replace the factor $u_\varepsilon^{-1}(t,\cdot)$ in (1.17) with a small error by something that is independent of $\mathcal{F}_r$ for those $r < \frac{t}{\varepsilon^2} - o(\varepsilon^{-2})$. The rest of the discussion is similar to the SHE case.

1.3. **Notation.** We use the following notation and conventions.

(i) We use $a \lesssim b$ for $a \leq Cb$ for some constant C independent of ε.

(ii) We use (p,q) to denote the Hölder exponents $\frac{1}{p} + \frac{1}{q} = 1$, and always choose $p \gg 1$.

(iii) $G_t(x) = (2\pi t)^{-1}\exp(-|x|^2/2t)$ denotes the standard heat kernel.

(iv) We let H denote the Hilbert space $L^2(\mathbb{R}^{2+1})$, with norm $\|\cdot\|_H$ and inner product $\langle\cdot,\cdot\rangle_H$.

(v) We use $\|\cdot\|_p$ to denote the L^p norm of the probability space $(\Omega,\mathcal{F},\mathbb{P})$ for $p \geq 1$.

(vi) $\{B_t^j : t \geq 0, j = 1,\ldots\}$ is a family of standard independent 2−dimensional Brownian motions built on another probability space $(\Sigma,\mathcal{A},\mathbb{P}_B)$. We will use $\mathbb{E}_B, \mathbb{P}_B$ when taking the expectation and the probability with respect to B.

(vii) We use $d_{\mathrm{TV}}(\cdot,\cdot)$ to denote the total variation distance between two distributions.

(viii) We let $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm.

(viiii) We use $[0,t]^n_<$ to denote the n−dimensional simplex $\{0 \leq t_1 < \ldots < t_n \leq t\}$.

Acknowledgments. We would like to thank Li-Cheng Tsai for his initial involvement in this project and multiple inspiring discussions. The research is supported by NSF grant DMS-1613301/1807748 and the Center for Nonlinear Analysis of CMU.

2. Sketch of the proof

The main result (1.3) is equivalent with the convergence in distribution of

$$\beta_\varepsilon^{-1}\int_{\mathbb{R}^2}(\log u_\varepsilon(t,x)-\mathbb{E}[\log u_\varepsilon(t,x)])g(x)dx \Rightarrow \int_{\mathbb{R}^2}\mathscr{H}(t,x)g(x)dx. \tag{2.1}$$

We rely on the Feynman-Kac representation of the solution to (1.5):

$$u(t,x)=\mathbb{E}_B\left[e^{\beta_\varepsilon\int_0^t V(t-s,x+B_s)ds-\frac12\beta_\varepsilon^2R(0)t}\right], \tag{2.2}$$

which has the same distribution, if viewed as a random field in x with t fixed, as

$$Z(t,x)=\mathbb{E}_B[M(t,x)],\quad \text{with}\quad M(t,x)=\exp\left(\beta_\varepsilon\int_0^t V(s,x+B_s)ds-\tfrac12\beta_\varepsilon^2R(0)t\right).$$

We keep in mind that M,Z depend on ε through the small factor β_ε defined in (1.6) but omit its dependence to simplify the notation. For fixed B and x, $M(\cdot,x)$ is a martingale. Defining

$$Z_\varepsilon(t,x)=Z(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}),\quad M_\varepsilon(t,x)=M(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon}),$$

and

$$X_\varepsilon(t)=\int_{\mathbb{R}^2}\log Z_\varepsilon(t,x)g(x)dx.$$

The convergence in (2.1) is equivalent to

$$\beta_\varepsilon^{-1}(X_\varepsilon(t)-\mathbb{E}[X_\varepsilon(t)])\Rightarrow\int_{\mathbb{R}^2}\mathscr{H}(t,x)g(x)dx.$$

Throughout the paper, we fix the variable $t>0$ and sometimes omit its dependence. Define

$$\sigma_t^2=\mathrm{Var}[\int\mathscr{H}(t,\cdot)g(\cdot)]=\nu_{\mathrm{eff}}^2\int_0^t\int_{\mathbb{R}^4}g(x_1)g(x_2)G_{2s}(x_1-x_2)dx_1dx_2ds, \tag{2.3}$$

where we recall that $G_t(x)$ is the standard heat kernel. The proof of Theorem 1.1 consists of two steps:

Proposition 2.1. *As $\varepsilon\to0$, $\beta_\varepsilon^{-2}\mathrm{Var}[X_\varepsilon(t)]\to\sigma_t^2$.*

Proposition 2.2. *As $\varepsilon\to0$,*

$$\frac{X_\varepsilon(t)-\mathbb{E}[X_\varepsilon(t)]}{\sqrt{\mathrm{Var}[X_\varepsilon(t)]}}\Rightarrow N(0,1).$$

2.1. **Negative moments.**
Throughout the paper, we rely on the existence of negative moments of $Z_\varepsilon(t,x)$ for small β.

Proposition 2.3. *There exits $\beta_0>0$ such that if $\beta<\beta_0$,*

$$\sup_{t\in[0,T]}\sup_{\varepsilon\in(0,1)}\mathbb{E}[Z_\varepsilon(t,x)^{-n}]\le C_{\beta,n,T}.$$

The proof is presented in Appendix B.

2.2. **The Clark-Ocone representation.**
For each realization of the Brownian motion B, we can write

$$\begin{aligned}\int_0^{t/\varepsilon^2}V(s,\tfrac{x}{\varepsilon}+B_s)ds&=\int_0^{t/\varepsilon^2}\left(\int_{\mathbb{R}^2}\varphi(\tfrac{x}{\varepsilon}+B_s-y)\dot W(s,y)dy\right)ds\\&=\int_{\mathbb{R}^3}\Phi^\varepsilon_{t,x,B}(s,y)dW(s,y),\end{aligned}$$

with

$$\Phi^\varepsilon_{t,x,B}(s,y)=\mathbb{1}_{[0,t/\varepsilon^2]}(s)\varphi(\tfrac{x}{\varepsilon}+B_s-y).$$

Therefore,

$$D_{s,y}Z_\varepsilon(t,x)=D_{s,y}\mathbb{E}_B[M_\varepsilon(t,x)]=\beta_\varepsilon\mathbb{E}_B\left[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)\right],$$

where $D_{s,y}$ denotes the Malliavin derivative operator with respect to $\dot{W}$. By [8, Lemma A.1], we have

$$D_{s,y}\log Z_\varepsilon(t,x)=\frac{D_{s,y}Z_\varepsilon(t,x)}{Z_\varepsilon(t,x)},$$

and the Clark-Ocone formula says

$$\begin{aligned}X_\varepsilon-\mathbb{E}[X_\varepsilon] &= \int_{\mathbb{R}^3}\mathbb{E}[D_{s,y}X_\varepsilon|\mathcal{F}_s]dW(s,y)\\ &= \int_{\mathbb{R}^3}\mathbb{E}\Big[\int_{\mathbb{R}^2}\frac{D_{s,y}Z_\varepsilon(t,x)}{Z_\varepsilon(t,x)}g(x)dx\Big|\mathcal{F}_s\Big]dW(s,y) \qquad (2.4)\\ &= \beta_\varepsilon\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}g(x)\mathbb{E}\Big[\frac{\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)]}{Z_\varepsilon(t,x)}\Big|\mathcal{F}_s\Big]dx\right)dW(s,y).\end{aligned}$$

Here $\mathcal{F}_s$ is the filtration generated by $\dot{W}(\ell,\cdot)$ up to $\ell\le s$.

For

$$K_\varepsilon=\tfrac{1}{\varepsilon^2|\log\varepsilon|^\alpha} \qquad (2.5)$$

with some $\alpha>0$ to be determined, we decompose the stochastic integral in (2.4) into three parts:

$$\beta_\varepsilon^{-1}(X_\varepsilon-\mathbb{E}[X_\varepsilon])=I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon},$$

with

$$I_{1,\varepsilon}=\int_0^{K_\varepsilon}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}g(x)\mathbb{E}\Big[\frac{\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)]}{Z_\varepsilon(t,x)}\Big|\mathcal{F}_s\Big]dx\right)dW(s,y), \qquad (2.6)$$

(2.7)

$$I_{2,\varepsilon}=\int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}g(x)\mathbb{E}\Big[\frac{\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)]}{Z(K_\varepsilon,x/\varepsilon)}\left(\frac{Z(K_\varepsilon,x/\varepsilon)}{Z(t/\varepsilon^2,x/\varepsilon)}-1\right)\Big|\mathcal{F}_s\Big]dx\right)dW(s,y),$$

and

$$I_{3,\varepsilon}=\int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}g(x)\mathbb{E}\Big[\frac{\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)]}{Z(K_\varepsilon,x/\varepsilon)}\Big|\mathcal{F}_s\Big]dx\right)dW(s,y). \qquad (2.8)$$

Since $1\ll K_\varepsilon\ll\varepsilon^{-2}$ and we expect that $Z(K_\varepsilon,x/\varepsilon)$ is close to $Z(t/\varepsilon^2,x/\varepsilon)$, the contribution from $I_{1,\varepsilon},I_{2,\varepsilon}$ is small compared to that from $I_{3,\varepsilon}$. For $I_{3,\varepsilon}$, the integration is in $s\ge K_\varepsilon$, so the random variable $Z(K_\varepsilon,x/\varepsilon)$ is $\mathcal{F}_s$–measurable, and
(2.9)

$$I_{3,\varepsilon}=\int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{g(x)}{Z(K_\varepsilon,x/\varepsilon)}\mathbb{E}\big[\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)]\big|\mathcal{F}_s\big]dx\right)dW(s,y).$$

Note that the procedure we took here is slightly different from the heuristics provided in Section 1.2 due to the time reversal and the fact that we considered $Z(t,x)$ rather than $u(t,x)$. Mathematically they are equivalent.

2.3. **The second order Poincaré inequality.** To simplify the notation, we define

$$Y_\varepsilon=\tfrac{X_\varepsilon-\mathbb{E}[X_\varepsilon]}{\sqrt{\mathrm{Var}[X_\varepsilon]}}.$$

To show that $Y_\varepsilon\Rightarrow N(0,1)$, we apply the second order Poincaré inequality, which was originally proved in the discrete setting in [6] and generalized to the continuous setting in [15]. Since $\mathbb{E}[Y_\varepsilon]=0$ and $\mathrm{Var}[Y_\varepsilon]=1$, with ζ a standard centered Gaussian random variable, by [15, Theorem 1.1], we have

$$d_{\mathrm{TV}}(Y_\varepsilon,\zeta)\lesssim\mathbb{E}[\|DY_\varepsilon\|_H^4]^{1/4}\mathbb{E}[\|D^2Y_\varepsilon\|_{\mathrm{op}}^4]^{1/4}, \qquad (2.10)$$

where $\|D^2Y_\varepsilon\|_{\mathrm{op}}$ denotes the operator norm of the mapping $H\otimes H \ni D^2Y_\varepsilon : H \mapsto H$ defined as $D^2Y_\varepsilon h := \langle h, D^2Y_\varepsilon\rangle_H$.

Since

$$DY_\varepsilon = \frac{DX_\varepsilon}{\sqrt{\mathrm{Var}[X_\varepsilon]}}, \quad D^2Y_\varepsilon = \frac{D^2X_\varepsilon}{\sqrt{\mathrm{Var}[X_\varepsilon]}},$$

and $\mathrm{Var}[X_\varepsilon] \sim |\log\varepsilon|^{-1}$ from Proposition 2.1, to show $d_{\mathrm{TV}}(Y_\varepsilon,\zeta)\to 0$ using (2.10), we only need to prove

$$\mathbb{E}[\|DX_\varepsilon\|_H^4]^{1/4}\mathbb{E}[\|D^2X_\varepsilon\|_{\mathrm{op}}^4]^{1/4} = o(|\log\varepsilon|^{-1}), \text{ as } \varepsilon\to 0. \tag{2.11}$$

Another possible way to prove the Gaussianity is to utilize the fast temporal mixing, as explained heuristically in Section 1.2 and implemented in $d\ge 3$ in [10].

2.4. **A comparison with** [4]**.** The basic ideas behind both approaches are similar, and the key is to modify the partition function so that $\log Z(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})$ can be "linearized" in some sense. As we will prove later, $I_{3,\varepsilon}$ is the main contribution to the random fluctuations, which essentially corresponds to the partition function of a directed polymer $\{B_s\}_{s\ge 0}$ that interacts only with the random environment $\dot W(s,\cdot)$ in $s\ge K_\varepsilon$. The initial layer in $s<K_\varepsilon$ only determines the starting point B_{K_ε} for this interaction. By our choice of $K_\varepsilon = o(\varepsilon^{-2})$, it is easy to show that in the weak disorder regime (β small), $\varepsilon B_{K_\varepsilon}\to 0$ as $\varepsilon\to 0$ under the polymer measure, indicating that the initial layer plays no role in the limit. Given this heuristics, if we ignore the factor $Z(K_\varepsilon, x/\varepsilon)^{-1}$ in (2.9), then $I_{3,\varepsilon}$ becomes

$$\begin{aligned} I_{3,\varepsilon} \mapsto & \int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} g(x)\mathbb{E}\big[\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}(s,y)]\big|\mathcal{F}_s\big]dx\right)dW(s,y)\\ &= \int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} g(x)\mathbb{E}_B\big[M(s,\tfrac{x}{\varepsilon})\Phi^\varepsilon_{t,x,B}(s,y)\big]dx\right)dW(s,y)\\ &= \int_{\mathbb{R}^2} g(x)\left(\int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\mathbb{E}[D_{s,y}Z(\tfrac{t}{\varepsilon^2},\tfrac{x}{\varepsilon})|\mathcal{F}_s]dW(s,y)\right)dx. \end{aligned}$$

The last expression precisely describes the fluctuation of the partition function that only involves the environment in $s\ge K_\varepsilon$. A similar heuristics was given in [4, Section 2.1], and the $Z^{B\ge}_{N,\beta_N}(x)-1$ defined in [4, Equation (2.11)] corresponds to the above expression. The Clark-Ocone formula seems to be particularly handy for this "linearization". It is worth mentioning that a naive Taylor expansion does not necessarily work for $\mathfrak{f}(Z(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}))$ with arbitrary smooth $\mathfrak{f}$, as shown in [8, Theorem 1.2].

Another difference between the two approaches is the proof of the Gaussianity. After the "linearization" in [4, Proposition 2.3], the convergence to the Edwards-Wilkinson limit follows from the convergence of SHE proved in [2, Theorem 2.17], which was based on a polynomial chaos expansion and the fourth moment theorem [14, 16]. In our case, we directly employ the second order Poincaré inequality to the KPZ equation, which simplifies some analysis. On the other hand, the fourth moment theorem covers more general distributions of the random environment and the convergence of a discrete directed polymer model to the Edwards-Wilkinson limit was proved in [4, Theorem 1.6], while we only deal with the continuous Gaussian environment in our setting.

3. Variance convergence

To simplify the notation, we define

$$M_{\varepsilon,j}(t,x) := \exp\left(\beta_\varepsilon \int_0^{t/\varepsilon^2} V(s, \tfrac{x}{\varepsilon} + B_s^j)ds - \tfrac{\beta_\varepsilon^2 R(0)t}{2\varepsilon^2}\right),$$

where $\{B^j\}_j$ are independent Brownian motions. For any set $I \subset \mathbb{R}_+, x \in \mathbb{R}^2$ and Brownian motions B^i, B^j, we define

$$\mathcal{R}(I, x, B^i, B^j) = \int_I R(x + B_s^i - B_s^j)ds \tag{3.1}$$

as the intersection time of B^i, B^j during the interval I, and x is the initial distance. For $I = [0,T]$, we write $\mathcal{R}(T, x, B^i, B^j) = \mathcal{R}([0,T], x, B^i, B^j)$.

The following lemma will be used repeatedly and is taken from [8, Lemma 3.1].

Lemma 3.1. *For any $n \in \mathbb{Z}_+$ and $q > 1$, there exists $\beta(n,q) > 0$ such that if $\beta < \beta(n,q)$, then for any random variable $F(B^1, \ldots, B^n) \geq 0$, $t > 0$ and $\{x_j \in \mathbb{R}^2\}_{j=1,\ldots,n}$, we have*

$$\mathbb{E}\left[\frac{\mathbb{E}_B[\prod_{j=1}^n M_{\varepsilon,j}(t,x_j) F(B^1, \ldots, B^n)]}{\prod_{j=1}^n Z_\varepsilon(t,x_j)}\right] \lesssim \mathbb{E}_B[F(B^1, \ldots, B^n)^q]^{1/q}. \tag{3.2}$$

Proof. By the Cauchy-Schwarz inequality and Proposition 2.3, the square of the l.h.s. of (3.2) is bounded by

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}_B[\prod_{j=1}^n M_{\varepsilon,j}(t,x_j)F(B^1,\ldots,B^n)]\right|^2\right] \\
&= \mathbb{E}_B\mathbb{E}\left[\prod_{j=1}^{2n} M_{\varepsilon,j}(t,x_j)F(B^1,\ldots,B^n)F(B^{n+1},\ldots,B^{2n})\right],
\end{aligned}$$

where $x_{j+n} = x_j$ for $j = 1, \ldots, n$. Evaluating the expectation with respect to $\dot{W}$, we obtain

$$\mathbb{E}\left[\prod_{j=1}^{2n} M_{\varepsilon,j}(t,x_j)\right] = \exp\left(\frac{\beta_\varepsilon^2}{2}\sum_{j,k=1}^{2n} \mathbb{1}_{j\neq k}\mathcal{R}(\tfrac{t}{\varepsilon^2}, \tfrac{x_j - x_k}{\varepsilon}, B^j, B^k)\right).$$

With $p = \frac{q}{q-1}$, Lemma A.1 shows that the r.h.s. of the above expression has an L^p norm that is bounded uniformly in ε and x_j, provided that β is chosen small. We apply Hölder inequality to complete the proof. □

3.1. The analysis of $I_{1,\varepsilon}$.

Recall that $I_{1,\varepsilon}$ is defined in (2.6).

Lemma 3.2. *For $K_\varepsilon = \frac{1}{\varepsilon^2|\log\varepsilon|^\alpha}$ with $\alpha > 1$, we have $\mathbb{E}[I_{1,\varepsilon}^2] \to 0$ as $\varepsilon \to 0$.*

Proof. Writing $I_{1,\varepsilon} = \int_0^{K_\varepsilon}\int_{\mathbb{R}^2} \mathbb{E}[\mathbf{Y}_{s,y}|\mathcal{F}_s]dW(s,y)$ for the appropriate $\mathbf{Y}_{s,y}$, we have by Itô's isometry that

$$\mathbb{E}[I_{1,\varepsilon}^2] = \int_0^{K_\varepsilon}\int_{\mathbb{R}^2} \mathbb{E}[|\mathbb{E}[\mathbf{Y}_{s,y}|\mathcal{F}_s]|^2]dyds \leq \int_0^K \int_{\mathbb{R}^2} \mathbb{E}[\mathbf{Y}_{s,y}^2]dyds,$$

and

$$\mathbb{E}[\mathbf{Y}_{s,y}^2] = \int_{\mathbb{R}^4} g(x_1)g(x_2)\mathbb{E}\left[\frac{\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x)\Phi^\varepsilon_{t,x_j,B^j}(s,y)]}{Z_\varepsilon(t,x_1)Z_\varepsilon(t,x_2)}\right]dx_1dx_2.$$

Using the fact that

$$\int_0^{K_\varepsilon}\int_{\mathbb{R}^2}\prod_{j=1}^2 \Phi^\varepsilon_{t,x_j,B^j}(s,y)dyds = \int_0^{K_\varepsilon} R(\tfrac{x_1-x_2}{\varepsilon} + B_s^1 - B_s^2)ds = \mathcal{R}(K_\varepsilon, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2),$$

where we recall that $R(x) = \int \varphi(x+y)\varphi(y)dy$, we have

$$\mathbb{E}[I_{1,\varepsilon}^2] \lesssim \int_{\mathbb{R}^4} |g(x_1)g(x_2)|\mathbb{E}\left[\frac{\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x_j)\mathcal{R}(K_\varepsilon, \frac{x_1-x_2}{\varepsilon}, B^1, B^2)]}{Z_\varepsilon(t,x_1)Z_\varepsilon(t,x_2)}\right]dx_1dx_2.$$

By Lemma 3.1, we have

$$\begin{aligned}\mathbb{E}[I_{1,\varepsilon}^2] &\lesssim \int_{\mathbb{R}^4} |g(x_1)g(x_2)|\sqrt{\mathbb{E}_B[\mathcal{R}^2(K_\varepsilon, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]}dx_1dx_2 \\ &\lesssim \sqrt{\int_{\mathbb{R}^4} |g(x_1)g(x_2)|\mathbb{E}_B[\mathcal{R}^2(K_\varepsilon, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]dx_1dx_2}.\end{aligned} \tag{3.3}$$

We apply Lemma A.2 to deduce

$$\mathbb{E}[I_{1,\varepsilon}^2] \lesssim \sqrt{(1+\log K_\varepsilon)\varepsilon^2 K_\varepsilon} \lesssim \tfrac{1}{|\log\varepsilon|^{(\alpha-1)/2}} \to 0.$$

The proof is complete. □

3.2. **The analysis of $I_{2,\varepsilon}$.** Recall that $I_{2,\varepsilon}$ is defined in (2.7).

Lemma 3.3. *For $K_\varepsilon = \frac{1}{\varepsilon^2|\log\varepsilon|^\alpha}$ with $\alpha > 0$, we have $\mathbb{E}[I_{2,\varepsilon}^2] \to 0$ as $\varepsilon \to 0$.*

Proof. By the same calculation as in the proof of Lemma 3.2, we have

$$\mathbb{E}[I_{2,\varepsilon}^2] \lesssim \int_{\mathbb{R}^4} g(x_1)g(x_2)A_\varepsilon(x_1,x_2)dx_1dx_2,$$

with

$$\begin{aligned}&A_\varepsilon(x_1,x_2)\\ &= \mathbb{E}\left[\frac{\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x_j)\mathcal{R}([K_\varepsilon, t/\varepsilon^2], \frac{x_1-x_2}{\varepsilon}, B^1, B^2)]}{Z(K_\varepsilon, x_1/\varepsilon)Z(K_\varepsilon, x_2/\varepsilon)}\prod_{j=1}^2\left(\frac{Z(K_\varepsilon, x_j/\varepsilon)}{Z(t/\varepsilon^2, x_j/\varepsilon)} - 1\right)\right].\end{aligned}$$

Applying Proposition 2.3, Hölder inequality, and the fact that $Z(t,x)$ is stationary in x, we have

$$|A_\varepsilon(x_1,x_2)| \lesssim \|a\|_8\|b\|_2\|b\|_4,$$

where we simply denoted

$$a = \mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x_j)\mathcal{R}([K_\varepsilon, t/\varepsilon^2], \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)], \quad b = Z(K_\varepsilon, 0) - Z(t/\varepsilon^2, 0).$$

First, we write $b^4 = b^{2+\delta}b^{2-\delta}$ and apply Hölder inequality to bound $\|b\|_4 \lesssim \|b\|_2^{\frac{2-\delta}{4}}$. Further applying Lemma 3.4 below yields

$$\|b\|_2\|b\|_4 \lesssim \|b\|_2^{\frac{6-\delta}{4}} \lesssim \frac{1}{|\log\varepsilon|^{\frac{3}{4}-\delta'}}$$

for some $\delta' > 0$ that is sufficiently close to zero. Now we apply Lemma 3.1 to derive

$$\begin{aligned}\|a\|_8^8 &= \mathbb{E}[|\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x_j)\mathcal{R}([K_\varepsilon, t/\varepsilon^2], \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]|^8] \\ &\lesssim \|\mathcal{R}([K_\varepsilon, t/\varepsilon^2], \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)\|_2^8.\end{aligned}$$

This implies

$$\begin{aligned}&\int_{\mathbb{R}^4} |g(x_1)g(x_2)| \times \|a\|_8\, dx_1dx_2 \\ &\lesssim \sqrt{\int_{\mathbb{R}^4} |g(x_1)g(x_2)| \times \|\mathcal{R}([K_\varepsilon, t/\varepsilon^2], \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)\|_2^2 dx_1dx_2} \lesssim \sqrt{|\log\varepsilon|},\end{aligned}$$

where the second $\lesssim$ comes from Lemma A.2. The proof is completed by choosing $\delta' < \frac{1}{4}$. □

Lemma 3.4. *For any $\delta > 0$, there exists $\beta(\delta) > 0$ such that if $\beta < \beta(\delta)$, we have*

$$\mathbb{E}[|Z(\tfrac{t}{\varepsilon^2},0) - Z(K_\varepsilon,0)|^2] \lesssim \frac{1}{|\log\varepsilon|^{1-\delta}}.$$

Proof. By the second moment calculation, we have

$$\mathbb{E}[|Z(\tfrac{t}{\varepsilon^2},0) - Z(K_\varepsilon,0)|^2] = \mathbb{E}[Z(\tfrac{t}{\varepsilon^2},0)^2] - \mathbb{E}[Z(K_\varepsilon,0)^2]$$
$$= \mathbb{E}_B\left[e^{\beta_\varepsilon^2\int_0^{t/\varepsilon^2} R(B_{2s})ds} - e^{\beta_\varepsilon^2\int_0^{K_\varepsilon} R(B_{2s})ds}\right].$$

Applying the simple inequality $|e^x - e^y| \le (e^x + e^y)|x-y|$, Hölder inequality and Lemma A.1, we have

$$\mathbb{E}_B\left[e^{\beta_\varepsilon^2\int_0^{t/\varepsilon^2} R(B_{2s})ds} - e^{\beta_\varepsilon^2\int_0^{K_\varepsilon} R(B_{2s})ds}\right] \lesssim \frac{1}{|\log\varepsilon|}\left\|\int_{K_\varepsilon}^{t/\varepsilon^2} R(B_{2s})ds\right\|_q \tag{3.4}$$

for any $q > 1$ (provided that $\beta < \beta(q)$). To estimate the above L^q norm, we note that

$$\left\|\int_{K_\varepsilon}^{t/\varepsilon^2} R(B_{2s})ds\right\|_1 \lesssim \log|\log\varepsilon|, \qquad \left\|\int_{K_\varepsilon}^{t/\varepsilon^2} R(B_{2s})ds\right\|_2 \lesssim \sqrt{|\log\varepsilon|(\log|\log\varepsilon|)},$$

with both estimates coming from the same proof of Lemma A.2. For $\theta \in (0,1)$ and $q = \frac{2}{2-\theta}$, by the L^p–interpolation inequality we have

$$\left\|\int_{K_\varepsilon}^{t/\varepsilon^2} R(B_{2s})ds\right\|_q \lesssim \left\|\int_{K_\varepsilon}^{t/\varepsilon^2} R(B_{2s})ds\right\|_1^{1-\theta}\left\|\int_{K_\varepsilon}^{t/\varepsilon^2} R(B_{2s})ds\right\|_2^{\theta} \lesssim |\log\varepsilon|^{\delta}, \tag{3.5}$$

provided that θ is chosen sufficiently close to zero (i.e., q is sufficiently close to 1). The proof is complete. □

3.3. **The analysis of $I_{3,\varepsilon}$.** Recall that $I_{3,\varepsilon}$ is defined in (2.8). Using the fact that $\mathbb{E}[M(t/\varepsilon^2, x/\varepsilon)|\mathcal{F}_s] = M(s,x/\varepsilon)$, we have that

$$I_{3,\varepsilon} = \int_{K_\varepsilon}^{t/\varepsilon^2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\frac{g(x)}{Z(K_\varepsilon,x/\varepsilon)}\mathbb{E}_B[M(s,x/\varepsilon)\Phi^\varepsilon_{t,x,B}(s,y)]dx\right)dW(s,y).$$

For any $T > 0, x_1, x_2 \in \mathbb{R}^2$ and a standard 2-dimensional Brownian motion $\bar{B}$, we define the deterministic function

$$\mathcal{H}_\varepsilon(T,x_1,x_2) = \mathbb{E}_{\bar{B}}\left[e^{\beta_\varepsilon^2\int_0^T R(x_1+\bar{B}_{2s})ds}\big|\bar{B}_{2T} = x_2\right].$$

We introduce the following notation: for any $x,y \in \mathbb{R}^2$, the expectation $\hat{\mathbb{E}}_{x,y}$ is defined as

$$\hat{\mathbb{E}}_{x,y}[F] = \mathbb{E}\left[\frac{\mathbb{E}_B[M_1(K_\varepsilon,x)M_2(K_\varepsilon,y)F]}{Z(K_\varepsilon,x)Z(K_\varepsilon,y)}\right]$$

for any random variable F. In particular, we will consider functional of

$$\mathbf{X}_{K_\varepsilon} = B^1_{K_\varepsilon} - B^2_{K_\varepsilon},$$

so

$$\hat{\mathbb{E}}_{x,y}[F(\mathbf{X}_{K_\varepsilon})] = \mathbb{E}\left[\frac{\mathbb{E}_B[M_1(K_\varepsilon,x)M_2(K_\varepsilon,y)F(B^1_{K_\varepsilon} - B^2_{K_\varepsilon})]}{Z(K_\varepsilon,x)Z(K_\varepsilon,y)}\right].$$

Note that we have omitted the dependence of the expectation $\hat{\mathbb{E}}_{x,y}$ on ε to simplify the notation.

The following three lemmas combine to show the convergence of

$$\mathbb{E}[I^2_{3,\varepsilon}] \to \sigma_t^2 \tag{3.6}$$

with σ_t^2 given in (2.3).

Lemma 3.5. $\mathbb{E}[I_{3,\varepsilon}^2] = \int_0^{t-\varepsilon^2 K_\varepsilon} \mathcal{G}_\varepsilon(s)ds$, *with*

$$\begin{aligned}\mathcal{G}_\varepsilon(s) = \int_{\mathbb{R}^6} & g(x-w)g(x)R(y) \\ & \times \hat{\mathbb{E}}_{-w/\varepsilon,0}\left[G_{2s}(w+\varepsilon y-\varepsilon \mathbf{X}_{K_\varepsilon})\mathcal{H}_\varepsilon(\tfrac{s}{\varepsilon^2}, y, \mathbf{X}_{K_\varepsilon} - \tfrac{w}{\varepsilon} - y)\right] dxdydw.\end{aligned} \tag{3.7}$$

Lemma 3.6. *There exists $\beta_0 > 0$ so that there exists $\gamma \in (0,1)$ such that, for all $\beta < \beta_0$, $\mathcal{G}_\varepsilon(s) \lesssim s^{-\gamma}$ for $s \in (0,t)$.*

Lemma 3.7. *For any $s \in (0,t)$, $\mathcal{G}_\varepsilon(s) \to \nu_{\mathrm{eff}}^2 \int_{\mathbb{R}^4} g(x-w)g(x)G_{2s}(w)dwdx$, as $\varepsilon \to 0$.*

The proof of Lemmas 3.5 and 3.6 is the same as [8, Lemma 3.5, 3.6].

Proof of Lemma 3.7. Recall that

$$\begin{aligned}\mathcal{G}_\varepsilon(s) = \int_{\mathbb{R}^6} & g(x-w)g(x)R(y) \\ & \times \hat{\mathbb{E}}_{-w/\varepsilon,0}\left[G_{2s}(w+\varepsilon y-\varepsilon \mathbf{X}_{K_\varepsilon})\mathcal{H}_\varepsilon(\tfrac{s}{\varepsilon^2}, y, \mathbf{X}_{K_\varepsilon} - \tfrac{w}{\varepsilon} - y)\right] dxdydw.\end{aligned}$$

Since $s > 0$ is fixed and the term $\mathcal{H}_\varepsilon$ is uniformly bounded by Lemma A.1, the expectation in the above expression is bounded uniformly in x, y, w, ε, so we only need to pass to the limit of the expectation for fixed $x, y, w \in \mathbb{R}^2$ and $w \neq 0$. The proof is divided into three steps.

(i) We show that $\hat{\mathbb{E}}_{-w/\varepsilon,0}[|G_{2s}(w+\varepsilon y-\varepsilon \mathbf{X}_{K_\varepsilon}) - G_{2s}(w)|] \to 0$ as $\varepsilon \to 0$. Using the fact that

$$|G_{2s}(w+\varepsilon y-\varepsilon \mathbf{X}_{K_\varepsilon}) - G_{2s}(w)| \lesssim \varepsilon|y| + \varepsilon|\mathbf{X}_{K_\varepsilon}|,$$

it suffices to show $\hat{\mathbb{E}}_{-w/\varepsilon,0}[|\varepsilon \mathbf{X}_{K_\varepsilon}|] \to 0$. We apply Lemma 3.1 to derive

$$\hat{\mathbb{E}}_{-w/\varepsilon,0}[|\varepsilon \mathbf{X}_{K_\varepsilon}|] \lesssim \sqrt{\mathbb{E}_B[|\varepsilon \mathbf{X}_{K_\varepsilon}|^2]} = \sqrt{2\varepsilon^2 K_\varepsilon} \to 0. \tag{3.8}$$

(ii) Define $s_\varepsilon = \frac{s}{\varepsilon^2|\log\varepsilon|}$ and

$$\tilde{\mathcal{H}}_\varepsilon = \mathbb{E}_{\bar{B}}\left[e^{\beta_\varepsilon^2 \int_0^{s_\varepsilon} R(y+\bar{B}_{2r})dr} \,\middle|\, \bar{B}_{2s/\varepsilon^2} = \mathbf{X}_{K_\varepsilon} - \tfrac{w}{\varepsilon} - y\right],$$

we show that

$$\hat{\mathbb{E}}_{-w/\varepsilon,0}[\tilde{\mathcal{H}}_\varepsilon] \to \tfrac{2\pi}{2\pi-\beta^2}, \qquad \text{as } \varepsilon \to 0. \tag{3.9}$$

We first note that $\tilde{\mathcal{H}}_\varepsilon$ can be written more explicitly by conditioning on $\bar{B}_{2s_\varepsilon}$:

$$\begin{aligned}\tilde{\mathcal{H}}_\varepsilon =& \mathbb{E}_{\bar{B}}\left[e^{\beta_\varepsilon^2 \int_0^{s_\varepsilon} R(y+\bar{B}_{2r})dr} \times \frac{G_{2s(1-|\log\varepsilon|^{-1})}(\varepsilon \mathbf{X}_{K_\varepsilon} - w - \varepsilon y - \varepsilon \bar{B}_{2s_\varepsilon})}{G_{2s}(\varepsilon \mathbf{X}_{K_\varepsilon} - w - \varepsilon y)}\right] \\ =& \frac{1}{(1-|\log\varepsilon|^{-1})}\mathbb{E}_{\bar{B}}\left[e^{\beta_\varepsilon^2 \int_0^{s_\varepsilon} R(y+\bar{B}_{2r})dr} e^{-\frac{(\varepsilon \mathbf{X}_{K_\varepsilon} - w - \varepsilon y - \varepsilon \bar{B}_{2s_\varepsilon})^2}{4s(1-|\log\varepsilon|^{-1})}} e^{\frac{(\varepsilon \mathbf{X}_{K_\varepsilon} - w - \varepsilon y)^2}{4s}}\right].\end{aligned}$$

There are three factors inside the above expectation. By an application of Lemma 3.1 again and the fact that $\varepsilon^2 K_\varepsilon \to 0$ as $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon\to 0} \hat{\mathbb{E}}_{-w/\varepsilon,0}[e^{\lambda|\varepsilon \mathbf{X}_{K_\varepsilon}|^2}] \lesssim 1,$$

for any $\lambda > 0$. Thus by the same proof as for (i), we can replace the second factor by $e^{-w^2/4s}$ with a negligible error. Similarly, we use the inequality $|e^x - e^y| \le (e^x + e^y)|x-y|$ to replace the third factor by $e^{w^2/4s}$ with a negligible error. In the end, we note that by [12, Theorem 1],

$$\beta_\varepsilon^2 \int_0^{s_\varepsilon} R(y+\bar{B}_{2r})dr = \frac{\beta^2 \log 2s_\varepsilon}{2|\log\varepsilon|} \frac{1}{\log 2s_\varepsilon} \int_0^{2s_\varepsilon} R(y+\bar{B}_r)dr \Rightarrow \lambda_\beta \mathrm{Exp}(1), \quad \lambda_\beta = \frac{\beta^2}{2\pi}.$$

Lemma A.1 ensures the uniform integrability, and we pass to the limit to obtain (3.9).

(iii) We show that

$$\hat{\mathbb{E}}_{-w/\varepsilon,0}[|\mathcal{H}_\varepsilon(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|]\to 0 \tag{3.10}$$

as $\varepsilon\to 0$. For the fixed $w\neq 0$, define the event $A_w:=\{|\varepsilon\mathbf{X}_{K_\varepsilon}|>w/2\}$. First, we have

$$\begin{aligned}
&|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|\\
&=|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|\mathbb{1}_{A_w}+|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|\mathbb{1}_{A_w^c}\\
&\lesssim \mathbb{1}_{A_w}+|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|\mathbb{1}_{A_w^c},
\end{aligned}$$

where we applied Lemma A.1. Thus

$$\begin{aligned}
&\hat{\mathbb{E}}_{-w/\varepsilon,0}[|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|]\\
&\lesssim \hat{\mathbb{E}}_{-w/\varepsilon,0}[\mathbb{1}_{A_w}]+\hat{\mathbb{E}}_{-w/\varepsilon,0}[|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|\mathbb{1}_{A_w^c}].
\end{aligned}$$

The first term on the r.h.s. goes to zero as $\varepsilon\to 0$ by (3.8). For the second term, we have

$$\begin{aligned}
&\hat{\mathbb{E}}_{-w/\varepsilon,0}[|\mathcal{H}(\tfrac{s}{\varepsilon^2},y,\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y)-\tilde{\mathcal{H}}_\varepsilon|\mathbb{1}_{A_w^c}]\\
&\leq \hat{\mathbb{E}}_{-w/\varepsilon,0}\Big[\mathbb{E}_B\Big[e^{\beta_\varepsilon^2\int_0^{s/\varepsilon^2}R(y+\bar{B}_{2r})dr}\beta_\varepsilon^2\int_{s_\varepsilon}^{s/\varepsilon^2}R(y+\bar{B}_{2r})dr\Big|\bar{B}_{2s/\varepsilon^2}=\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y\Big]\mathbb{1}_{A_w^c}\Big].
\end{aligned}$$

The conditional expectation can be bounded by

$$\begin{aligned}
&\mathbb{E}_B\Big[e^{\beta_\varepsilon^2\int_0^{s/\varepsilon^2}R(y+\bar{B}_{2r})dr}\beta_\varepsilon^2\int_{s_\varepsilon}^{s/\varepsilon^2}R(y+\bar{B}_{2r})dr\Big|\bar{B}_{2s/\varepsilon^2}=\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y\Big]\\
&\lesssim\frac{1}{|\log\varepsilon|}\sqrt{\mathbb{E}_B\Big[|\int_{s_\varepsilon}^{s/\varepsilon^2}R(y+\bar{B}_{2r})dr|^2\Big|\bar{B}_{2s/\varepsilon^2}=\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y\Big]}.
\end{aligned} \tag{3.11}$$

In the event A_w^c, we have $|\varepsilon\mathbf{X}_{K_\varepsilon}|\leq w/2$, thus $c_1w\leq|\varepsilon\mathbf{X}_{K_\varepsilon}-w-\varepsilon y|\leq c_2w$ for some $c_1,c_2>0$ (note that y is fixed). Recall that $s_\varepsilon=\frac{s}{\varepsilon^2|\log\varepsilon|}$, we apply Lemma A.3 to derive

$$\mathbb{E}_B\Big[|\int_{s_\varepsilon}^{s/\varepsilon^2}R(y+\bar{B}_{2r})dr|^2\Big|\bar{B}_{2s/\varepsilon^2}=\mathbf{X}_{K_\varepsilon}-\tfrac{w}{\varepsilon}-y\Big]\lesssim|\log\varepsilon|(\log|\log\varepsilon|),$$

uniformly in $|\varepsilon\mathbf{X}_{K_\varepsilon}|\leq w/2$, so we pass to the limit in (3.11), then obtain (3.10).

To summarize, we have

$$\begin{aligned}
\mathcal{G}_\varepsilon(s)\to&\tfrac{2\pi}{2\pi-\beta^2}\int_{\mathbb{R}^{3d}}g(x-w)g(x)R(y)G_{2s}(w)dxdydw\\
&=\nu_{\text{eff}}^2\int_{\mathbb{R}^4}g(x-w)g(x)G_{2s}(w)dxdw,
\end{aligned}$$

which completes the proof. □

3.4. **Proof of Proposition 2.1.**

Recall that $X_\varepsilon-\mathbb{E}[X_\varepsilon]=\beta_\varepsilon(I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon})$. We combine Lemmas 3.2, 3.3 and (3.6) to derive

$$\beta_\varepsilon^{-2}\text{Var}[X_\varepsilon]=\mathbb{E}[|I_{1,\varepsilon}+I_{2,\varepsilon}+I_{3,\varepsilon}|^2]\to\sigma_t^2.$$

4. Gaussianity

Recall the goal is to show

$$\mathbb{E}[\|DX_\varepsilon\|_H^4]^{1/4}\mathbb{E}[\|D^2X_\varepsilon\|_{\mathrm{op}}^4]^{1/4} = o(|\log\varepsilon|^{-1}), \text{ as } \varepsilon\to 0,$$

where $X_\varepsilon = \int_{\mathbb{R}^2}\log Z_\varepsilon(t,x)g(x)dx$. Since

$$DX_\varepsilon = \int_{\mathbb{R}^2}\frac{DZ_\varepsilon(t,x)}{Z_\varepsilon(t,x)}g(x)dx,$$

we have

$$\begin{aligned}D^2X_\varepsilon &= D\int_{\mathbb{R}^2}\frac{DZ_\varepsilon(t,x)}{Z_\varepsilon(t,x)}g(x)dx\\ &= \int_{\mathbb{R}^2}\frac{Z_\varepsilon(t,x)D^2Z_\varepsilon(t,x) - DZ_\varepsilon(t,x)\otimes DZ_\varepsilon(t,x)}{Z_\varepsilon^2(t,x)}g(x)dx.\end{aligned}$$

Using the Feynman-Kac representation (2.2),

$$D^2Z_\varepsilon(t,x) = \beta_\varepsilon^2\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B}\otimes\Phi^\varepsilon_{t,x,B}],$$

so

$$Z_\varepsilon(t,x)D^2Z_\varepsilon(t,x) = \beta_\varepsilon^2\mathbb{E}_B\left[\prod_{j=1}^2 M_{\varepsilon,j}(t,x)\Phi^\varepsilon_{t,x,B^2}\otimes\Phi^\varepsilon_{t,x,B^2}\right],$$

and

$$DZ_\varepsilon(t,x)\otimes DZ_\varepsilon(t,x) = \beta_\varepsilon^2\mathbb{E}_B\left[\prod_{j=1}^2 M_{\varepsilon,j}(t,x)\Phi^\varepsilon_{t,x,B^1}\otimes\Phi^\varepsilon_{t,x,B^2}\right].$$

Thus we can write

$$D^2X_\varepsilon = \beta_\varepsilon^2\int_{\mathbb{R}^2}\frac{\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x)(\Phi^\varepsilon_{t,x,B^2}-\Phi^\varepsilon_{t,x,B^1})\otimes\Phi^\varepsilon_{t,x,B^2}]}{Z_\varepsilon^2(t,x)}g(x)dx = \mathcal{P}_2-\mathcal{P}_1,$$

where

$$H\otimes H\ni\mathcal{P}_k = \beta_\varepsilon^2\int_{\mathbb{R}^2}\frac{\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x)\Phi^\varepsilon_{t,x,B^k}\otimes\Phi^\varepsilon_{t,x,B^2}]}{Z_\varepsilon^2(t,x)}g(x)dx.$$

Thus,

$$\|D^2X_\varepsilon\|_{\mathrm{op}}^4 \lesssim \|\mathcal{P}_1\|_{\mathrm{op}}^4 + \|\mathcal{P}_2\|_{\mathrm{op}}^4,$$

and we only need to estimate $\mathbb{E}[\|\mathcal{P}_k\|_{\mathrm{op}}^4]$.

4.1. **The first derivative.**

Lemma 4.1. *For any $\delta>0$, there exists $\beta(\delta)>0$ such that if $\beta<\beta(\delta)$,*

$$\mathbb{E}[\|DX_\varepsilon\|_H^4]^{1/4}\lesssim|\log\varepsilon|^{-\frac12+\delta}.$$

Proof. A direct calculation gives

$$\|DX_\varepsilon\|_H^4 = \beta_\varepsilon^4\int_{\mathbb{R}^8}\prod_{j=1}^4\frac{g(x_j)}{Z_\varepsilon(t,x_j)}\mathbb{E}_B\left[\prod_{j=1}^4 M_{\varepsilon,j}(t,x_j)\mathcal{R}(\tfrac{t}{\varepsilon^2},\tfrac{x_1-x_2}{\varepsilon},B^1,B^2)\mathcal{R}(\tfrac{t}{\varepsilon^2},\tfrac{x_3-x_4}{\varepsilon},B^3,B^4)\right]dx,$$

with $\mathcal{R}$ defined in (3.1). Taking the expectation and applying Lemma 3.1, we have

$$\begin{aligned}\beta_\varepsilon^{-4}\mathbb{E}[\|DX_\varepsilon\|_H^4] &\lesssim \int_{\mathbb{R}^8}\prod_{j=1}^4|g(x_j)|\mathbb{E}_B\left[\mathcal{R}^q(\tfrac{t}{\varepsilon^2},\tfrac{x_1-x_2}{\varepsilon},B^1,B^2)\mathcal{R}^q(\tfrac{t}{\varepsilon^2},\tfrac{x_3-x_4}{\varepsilon},B^3,B^4)\right]^{1/q}dx\\ &\lesssim\left(\int_{\mathbb{R}^4}|g(x_1)g(x_2)|\mathbb{E}_B[\mathcal{R}^q(\tfrac{t}{\varepsilon^2},\tfrac{x_1-x_2}{\varepsilon},B^1,B^2)]dx\right)^{2/q}.\end{aligned}$$

We can view the factor $|g(x_1)g(x_2)|$ as a weight (without loss of generality assume $\int |g| = 1$), so the integral

$$\int_{\mathbb{R}^4} |g(x_1)g(x_2)| \mathbb{E}_B[\mathcal{R}^q(\tfrac{t}{\varepsilon^2}, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]dx$$

can be viewed as an expectation of $\mathcal{R}^q(\frac{t}{\varepsilon^2}, \frac{x_1-x_2}{\varepsilon}, B^1, B^2)$ with x_1, x_2 independently sampled from the density $|g|$. Therefore, by the L^p–interpolation inequality and arguing similarly as (3.5), we have

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^4} |g(x_1)g(x_2)| \mathbb{E}_B[\mathcal{R}^q(\tfrac{t}{\varepsilon^2}, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]dx\Big)^{1/q} \\
&\qquad \lesssim \Big(\int_{\mathbb{R}^4} |g(x_1)g(x_2)| \mathbb{E}_B[\mathcal{R}(\tfrac{t}{\varepsilon^2}, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]dx\Big)^{1-\theta} \\
&\qquad \times \Big(\int_{\mathbb{R}^4} |g(x_1)g(x_2)| \mathbb{E}_B[\mathcal{R}^2(\tfrac{t}{\varepsilon^2}, \tfrac{x_1-x_2}{\varepsilon}, B^1, B^2)]dx\Big)^{\theta/2}
\end{aligned} \tag{4.1}$$

for $\theta = 2 - \frac{2}{q}$. Applying Lemma A.2, we know that the first factor on the r.h.s. is uniformly bounded and the second factor is bounded by $|\log \varepsilon|^{\theta/2}$. Thus,

$$\mathbb{E}[\|DX_\varepsilon\|_H^4] \lesssim \frac{|\log \varepsilon|^\theta}{|\log \varepsilon|^2}.$$

By choosing q sufficiently close to 1, we can make θ arbitrarily small, which completes the proof. □

4.2. **The second derivative.** To estimate $\|\mathcal{P}_k\|_{\text{op}}$, we use the contraction inequality [15, Proposition 4.1], which says that

$$\|\mathcal{P}_k\|_{\text{op}}^4 \le \|\mathcal{P}_k \otimes_1 \mathcal{P}_k\|_{H\otimes H}^2.$$

4.2.1. *The case $k = 1$.* A direct calculation gives

$$\begin{aligned}
&\mathcal{P}_1 \otimes_1 \mathcal{P}_1 \\
&= \beta_\varepsilon^4 \int_{\mathbb{R}^4} \frac{\mathbb{E}_B[\prod_{j=1}^4 M_{\varepsilon,j}(t, x_j)\mathcal{R}(\frac{t}{\varepsilon^2}, \frac{x-y}{\varepsilon}, B^1, B^3)\Phi^\varepsilon_{t,x,B^2} \otimes \Phi^\varepsilon_{t,y,B^4}]}{Z_\varepsilon^2(t,x)Z_\varepsilon^2(t,y)} g(x)g(y)dxdy,
\end{aligned}$$

where we write $x_1 = x_2 = x, x_3 = x_4 = y$ to simplify the notations. Thus,

$$\begin{aligned}
\|\mathcal{P}_1 \otimes_1 \mathcal{P}_1\|_{H\otimes H}^2 = \beta_\varepsilon^8 &\int_{\mathbb{R}^8} g(x)g(y)g(z)g(w) \left(\prod_{j=1}^8 Z_\varepsilon(t, x_j) \right)^{-1} \\
&\times \mathbb{E}_B \left[\prod_{j=1}^8 M_{\varepsilon,j}(t, x_j) \prod_{(i,k)\in\mathcal{O}} \mathcal{R}(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k) \right] dxdydzdw,
\end{aligned}$$

where $x_5 = x_6 = z, x_7 = x_8 = w$, and the set $\mathcal{O}$ is $\mathcal{O} = \{(1,3), (5,7), (2,6), (4,8)\}$.

Lemma 4.2. *For any $\delta > 0$, there exists $\beta(\delta)$ such that if $\beta < \beta(\delta)$,*

$$\mathbb{E}[\|\mathcal{P}_1 \otimes_1 \mathcal{P}_1\|_{H\otimes H}^2] \lesssim |\log \varepsilon|^{-4+\delta}.$$

Proof. Applying Lemma 3.1, we have

$$\begin{aligned}
&\beta_\varepsilon^{-8} \mathbb{E}[\|\mathcal{P}_1 \otimes_1 \mathcal{P}_1\|_{H\otimes H}^2] \\
&\lesssim \int_{\mathbb{R}^8} |g(x)g(y)g(z)g(w)| \mathbb{E}_B\Big[\prod_{(i,k)\in\mathcal{O}} \mathcal{R}^q(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k) \Big]^{1/q} dxdydzdw \\
&\lesssim \left(\int_{\mathbb{R}^8} |g(x)g(y)g(z)g(w)| \mathbb{E}_B\Big[\prod_{(i,k)\in\mathcal{O}} \mathcal{R}^q(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k) \Big] dxdydzdw \right)^{1/q} =: a_q
\end{aligned}$$

for some $q > 1$ and we used the simplified notation a_q. Again we apply the L^p–interpolation inequality as in (4.1), with $\theta = 2 - \frac{2}{q}$, we have

$$a_q \le a_1^{1-\theta} a_2^{\theta}.$$

Note that the process (B^i, B^k) are independent for different pairs of $(i,k) \in \mathcal{O}$. Applying Lemma A.2 yields

$$\mathbb{E}_B\big[\prod_{(i,k)\in\mathcal{O}} \mathcal{R}(\tfrac{t}{\varepsilon^2}, \tfrac{x_i - x_k}{\varepsilon}, B^i, B^k)\big] \lesssim \prod_{(i,k)\in\mathcal{O}} (1 + |\log|x_i - x_k||),$$
$$\mathbb{E}_B\big[\prod_{(i,k)\in\mathcal{O}} \mathcal{R}^2(\tfrac{t}{\varepsilon^2}, \tfrac{x_i - x_k}{\varepsilon}, B^i, B^k)\big] \lesssim |\log\varepsilon|^4 \prod_{(i,k)\in\mathcal{O}} (1 + |\log|x_i - x_k||).$$

Thus we have $a_1 \lesssim 1$ and $a_2 \lesssim |\log\varepsilon|^2$, which implies $a_q \lesssim |\log\varepsilon|^{2\theta}$. By choosing q sufficiently close to 1 so that θ is sufficiently close to 0, we complete the proof. □

4.2.2. *The case $k = 2$.* In this case,

$$\mathcal{P}_2 = \beta_\varepsilon^2 \int_{\mathbb{R}^2} \frac{\mathbb{E}_B[M_\varepsilon(t,x)\Phi^\varepsilon_{t,x,B} \otimes \Phi^\varepsilon_{t,x,B}]}{Z_\varepsilon(t,x)} g(x)dx,$$

so

$$\mathcal{P}_2 \otimes_1 \mathcal{P}_2$$
$$= \beta_\varepsilon^4 \int_{\mathbb{R}^4} \frac{\mathbb{E}_B[\prod_{j=1}^2 M_{\varepsilon,j}(t,x_j)\mathcal{R}(\frac{t}{\varepsilon^2}, \frac{x_1-x_2}{\varepsilon}, B^1, B^2)\Phi^\varepsilon_{t,x_1,B^1} \otimes \Phi^\varepsilon_{t,x_2,B^2}]}{Z_\varepsilon(t,x_1)Z_\varepsilon(t,x_2)} g(x_1)g(x_2)dx_1dx_2,$$

and

$$\|\mathcal{P}_2 \otimes_1 \mathcal{P}_2\|^2_{H\otimes H} = \beta_\varepsilon^8 \int_{\mathbb{R}^8} \prod_{j=1}^4 \frac{g(x_j)}{Z_\varepsilon(t,x_j)} \mathbb{E}_B\big[\prod_{j=1}^4 M_{\varepsilon,j}(t,x_j) \prod_{(i,k)\in\tilde{\mathcal{O}}} \mathcal{R}(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k)\big]dx,$$

with the set $\tilde{\mathcal{O}} = \{(1,2),(3,4),(1,3),(2,4)\}$.

Lemma 4.3. *For any $\delta > 0$, there exists $\beta(\delta)$ such that if $\beta < \beta(\delta)$,*

$$\mathbb{E}[\|\mathcal{P}_2 \otimes_1 \mathcal{P}_2\|^2_{H\otimes H}] \lesssim |\log\varepsilon|^{-3+\delta}.$$

Proof. By Lemma 3.1 and the fact that g is compactly supported, we have

$$\beta_\varepsilon^{-8}\mathbb{E}[\|\mathcal{P}_2 \otimes_1 \mathcal{P}_2\|^2_{H\otimes H}] \lesssim \int_{\mathbb{R}^8} \prod_{j=1}^4 |g(x_j)| \mathbb{E}_B\big[\prod_{(i,k)\in\tilde{\mathcal{O}}} \mathcal{R}^q(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k)\big]^{1/q} dx$$
$$\lesssim \left(\int_{\mathbb{R}^8} \prod_{j=1}^4 |g(x_j)| \mathbb{E}_B\big[\prod_{(i,k)\in\tilde{\mathcal{O}}} \mathcal{R}^q(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k)\big]dx\right)^{1/q} =: a_q.$$

Arguing in the same way as in the proof of Lemma 4.2, we have $a_q \le a_1^{1-\theta}a_2^{\theta}$ with $\theta = 2 - \frac{2}{q}$. By Lemma 4.4, we know that $a_1 \le |\log\varepsilon|$. For a_2, to simplify the notation we write

$$\prod_{(i,k)\in\tilde{\mathcal{O}}} \mathcal{R}^2(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k) = \mathcal{R}_1^2\mathcal{R}_2^2\mathcal{R}_3^2\mathcal{R}_4^2,$$

with $\mathcal{R}_j$ denoting $\mathcal{R}(\frac{t}{\varepsilon^2}, \frac{x_i-x_k}{\varepsilon}, B^i, B^k)$ for different $(i,k) \in \tilde{\mathcal{O}}$. Applying Hölder inequality and Lemma A.2, we derive

$$\mathbb{E}_B\big[\prod_{(i,k)\in\tilde{\mathcal{O}}} \mathcal{R}^2(\tfrac{t}{\varepsilon^2}, \tfrac{x_i-x_k}{\varepsilon}, B^i, B^k)\big] \le \|\mathcal{R}_1\|_4^2\|\mathcal{R}_2\|_8^2\|\mathcal{R}_3\|_{16}^2\|\mathcal{R}_4\|_{16}^2$$
$$\lesssim |\log\varepsilon|^{2(\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\frac{15}{16})} \prod_{(i,k)\in\tilde{\mathcal{O}}} (1 + |\log|x_i - x_k||)^{\alpha_{i,k}}$$

for some $\alpha_{i,k} > 0$. After integration in x_j, we have $a_2 \lesssim |\log\varepsilon|^{\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\frac{15}{16}}$. Thus, by choosing q sufficiently close to 1, the proof is complete. □

Lemma 4.4. *Assume* $0 \le f, h \in \mathcal{C}_c^\infty(\mathbb{R}^2)$*, then*

$$\int_{\mathbb{R}^8} \mathbb{E}_B \left[\prod_{j=1}^4 f(x_j) \prod_{(i,k)\in\tilde{\mathcal{O}}} \int_0^{t/\varepsilon^2} h(\tfrac{x_i - x_k}{\varepsilon} + B_s^i - B_s^k) ds \right] dx \lesssim |\log\varepsilon|.$$

Proof. Without loss of generality, assume h is even. We write

$$\prod_{(i,k)\in\tilde{\mathcal{O}}} \int_0^{t/\varepsilon^2} h(\tfrac{x_i - x_k}{\varepsilon} + B_s^i - B_s^k) ds = \int_{[0,t/\varepsilon^2]^4} \prod_{j=1}^4 h(\tfrac{x_j - x_{j-1}}{\varepsilon} + B_{s_j}^j - B_{s_j}^{j-1}) ds,$$

where $x_0 = x_4, B^0 = B^4$. Denoting $\hat{f}(\xi) = \int f(x) e^{-i\xi\cdot x} dx$ as the Fourier transform of f, we have

$$\begin{aligned}
&\int_{\mathbb{R}^8} \prod_{j=1}^4 f(x_j) h(\tfrac{x_j - x_{j-1}}{\varepsilon} + B_{s_j}^j - B_{s_j}^{j-1}) dx \\
&= \frac{1}{(2\pi)^8} \int_{\mathbb{R}^{16}} \prod_{j=1}^4 f(x_j) \hat{h}(\eta_j) e^{i\eta_j\cdot(x_j - x_{j-1})/\varepsilon} e^{i\eta_j\cdot(B_{s_j}^j - B_{s_j}^{j-1})} d\eta dx \\
&= \frac{1}{(2\pi)^8} \int_{\mathbb{R}^8} \prod_{j=1}^4 \hat{f}(\tfrac{\eta_j - \eta_{j-1}}{\varepsilon}) \hat{h}(\eta_j) e^{i(\eta_j\cdot B_{s_j}^j - \eta_{j+1}\cdot B_{s_{j+1}}^j)} d\eta,
\end{aligned}$$

with $\eta_0 = \eta_4, \eta_5 = \eta_1, s_5 = s_1$. Thus, it suffices to estimate

$$\begin{aligned}
&\int_{[0,t/\varepsilon^2]^4} \int_{\mathbb{R}^8} \prod_{j=1}^4 \hat{f}(\tfrac{\eta_j - \eta_{j-1}}{\varepsilon}) \hat{h}(\eta_j) \mathbb{E}_B[e^{i(\eta_j\cdot B_{s_j} - \eta_{j+1}\cdot B_{s_{j+1}})}] d\eta ds \\
&= \int_{[0,t]^4} \int_{\mathbb{R}^8} \prod_{j=1}^4 \hat{f}(\eta_j - \eta_{j-1}) \hat{h}(\varepsilon\eta_j) \mathbb{E}_B[e^{i(\eta_j\cdot B_{s_j} - \eta_{j+1}\cdot B_{s_{j+1}})}] d\eta ds,
\end{aligned}$$

where we changed variables $s_j \mapsto s_j/\varepsilon^2, \eta_j \mapsto \varepsilon\eta_j$ and used the scaling property of the Brownian motion. Without loss of generality, consider the set $A_1 = \{(s_1, \ldots, s_4) \in [0,t]^4 : s_1 \ge s_j, j \ne 1\}$, it is clear that in A_1 we have

$$\prod_{j=1}^4 \mathbb{E}_B[e^{i(\eta_j\cdot B_{s_j} - \eta_{j+1}\cdot B_{s_{j+1}})}] \le e^{-\frac{1}{2}|\eta_1|^2(s_1 - s_2)},$$

which implies

$$\begin{aligned}
&\int_{A_1} \int_{\mathbb{R}^8} \prod_{j=1}^4 |\hat{f}(\eta_j - \eta_{j-1}) \hat{h}(\varepsilon\eta_j)| \mathbb{E}_B[e^{i(\eta_j\cdot B_{s_j} - \eta_{j+1}\cdot B_{s_{j+1}})}] d\eta ds \\
&\le \int_{A_1} \int_{\mathbb{R}^8} e^{-\frac{1}{2}|\eta_1|^2(s_1 - s_2)} \prod_{j=1}^4 |\hat{f}(\eta_j - \eta_{j-1}) \hat{h}(\varepsilon\eta_j)| d\eta ds \\
&\lesssim \int_{A_1} \int_{\mathbb{R}^8} e^{-\frac{1}{2}|\eta_1|^2(s_1 - s_2)} |\hat{h}(\varepsilon\eta_1) \hat{f}(\tilde{\eta}_2) \hat{f}(\tilde{\eta}_3) \hat{f}(\tilde{\eta}_4)| d\eta_1 d\tilde{\eta} ds.
\end{aligned}$$

In the last "$\lesssim$" we bounded $|\hat{f}(\eta_1 - \eta_4)| \lesssim 1$ and changed variables $\eta_j - \eta_{j-1} \mapsto \tilde{\eta}_j, j = 2, 3, 4$. The last integral can be computed explicitly, and we use the fact that

$$\int_0^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}|\eta_1|^2 s} |\hat{h}(\varepsilon\eta_1)| d\eta_1 ds = \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^2} e^{-\frac{1}{2}|\eta_1|^2 s} |\hat{h}(\eta_1)| d\eta_1 ds \lesssim |\log\varepsilon|$$

to complete the proof. □

4.3. **Proof of Proposition 2.2.** Recall that $Y_\varepsilon = \frac{X_\varepsilon - \mathbb{E}[X_\varepsilon]}{\sqrt{\mathrm{Var}[X_\varepsilon]}}$. Since

$d_{\mathrm{TV}}(Y_\varepsilon, \zeta) \lesssim \mathbb{E}[\|DY_\varepsilon\|_H^4]^{1/4}\mathbb{E}[\|D^2 Y_\varepsilon\|_{\mathrm{op}}^4]^{1/4} = \frac{1}{\mathrm{Var}[X_\varepsilon]}\mathbb{E}[\|DX_\varepsilon\|_H^4]^{1/4}\mathbb{E}[\|D^2 X_\varepsilon\|_{\mathrm{op}}^4]^{1/4}$,

using the fact that $\mathrm{Var}[X_\varepsilon] \sim |\log\varepsilon|^{-1}$ and applying Lemmas 4.1, 4.2 and 4.3, we have

$$d_{\mathrm{TV}}(Y_\varepsilon, \zeta) \lesssim |\log\varepsilon| \times |\log\varepsilon|^{-\frac{1}{2}+\delta} \times \left(|\log\varepsilon|^{-4+\delta} + |\log\varepsilon|^{-3+\delta}\right)^{\frac{1}{4}}.$$

By choosing δ small, the r.h.s. goes to zero as $\varepsilon \to 0$.

Appendix A. Auxiliary lemmas

Recall that $\beta_\varepsilon = \frac{\beta}{\sqrt{|\log\varepsilon|}}$ and $G_t(x)$ is the standard heat kernel.

Lemma A.1. *Fix $t > 0$, there exists $\beta_0 > 0$ such that if $\beta < \beta_0$, we have in $d = 2$ that*

$$\sup_{x\in\mathbb{R}^2, \varepsilon\in(0,1)} \mathbb{E}_B\left[e^{\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} R(x+B_s)ds}\right] < \infty, \tag{A.1}$$

and

$$\sup_{x,y\in\mathbb{R}^2, \varepsilon\in(0,1)} \mathbb{E}_B\left[e^{\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} R(x+B_s)ds}\middle| B_{t/\varepsilon^2} = y\right] < \infty. \tag{A.2}$$

Proof. It suffices to prove (A.2) since (A.1) follows from an integration in y. We claim (A.2) is implied by

$$\sup_{x,y\in\mathbb{R}^2, \varepsilon\in(0,1)} |\log\varepsilon|^{-1} \int_0^{t/\varepsilon^2} \mathbb{E}_B[R(x+B_s)|B_{t/\varepsilon^2} = y]ds \le C. \tag{A.3}$$

The proof essentially follows Portenko's lemma and we sketch it here for the convenience of readers. For any $n \ge 1$, we can write

$$\mathbb{E}_B\left[\left(\int_0^{t/\varepsilon^2} R(x+B_s)ds\right)^n \middle| B_{t/\varepsilon^2} = y\right] = n! \int_{[0,t/\varepsilon^2]_<^n} \mathbb{E}_B\left[\prod_{j=1}^n R(x+B_{s_j})\middle| B_{t/\varepsilon^2} = y\right]ds.$$

By conditioning on $B_{s_{n-1}}$ and applying (A.3) to the integral in s_n, we have

$$\int_{[0,t/\varepsilon^2]_<^n} \mathbb{E}_B\left[\prod_{j=1}^n R(x+B_{s_j})\middle| B_{t/\varepsilon^2} = y\right]ds \le C|\log\varepsilon| \int_{[0,t/\varepsilon^2]_<^{n-1}} \mathbb{E}_B\left[\prod_{j=1}^{n-1} R(x+B_{s_j})\right]ds.$$

Iterating this procedure yields

$$\mathbb{E}_B\left[\left(\int_0^{t/\varepsilon^2} R(x+B_s)ds\right)^n \middle| B_{t/\varepsilon^2} = y\right] \le n!(C|\log\varepsilon|)^n,$$

which completes the proof of (A.2).

It remains to show (A.3). For the Brownian bridge, we only need to show

$$\int_0^{t/2\varepsilon^2} \mathbb{E}_B[R(x+B_s)|B_{t/\varepsilon^2} = y]ds \le C|\log\varepsilon|$$

for some constant C independent of x, y, ε. Note that B_s has the Gaussian distribution with mean $\frac{ys}{t/\varepsilon^2}$ and variance $\frac{s(t/\varepsilon^2 - s)}{t/\varepsilon^2}$, so its density is bounded from above by $\frac{1}{s}$ when $s \le \frac{t}{2\varepsilon^2}$. Thus, we have

$$\int_0^{t/2\varepsilon^2} \mathbb{E}_B[R(x+B_s)|B_{t/\varepsilon^2} = y]ds \lesssim 1 + \int_1^{t/2\varepsilon^2}\int_{\mathbb{R}^2} R(x+w)\tfrac{1}{s}dwds \lesssim 1 + \log\tfrac{t}{2\varepsilon^2}.$$

The proof is complete. □

Lemma A.2. *For any $0 \le g \in \mathcal{C}_c(\mathbb{R}^2)$ and $t > 1$, we have*

$$\begin{aligned}&\int_{\mathbb{R}^4} g(x_1)g(x_2)\mathbb{E}_B\Big[\big|\int_0^t R(\tfrac{x_1-x_2}{\varepsilon}+B_s)ds\big|^2\Big]dx_1dx_2 \le C(1+\log t)\varepsilon^2 t,\\ &\int_{\mathbb{R}^4} g(x_1)g(x_2)\mathbb{E}_B\Big[\int_0^t R(\tfrac{x_1-x_2}{\varepsilon}+B_s)ds\Big]dx_1dx_2 \le C\varepsilon^2 t,\end{aligned} \tag{A.4}$$

with some constant C independent of t,ε. Fix $t>0, n\in\mathbb{Z}_+$, we also have

$$\mathbb{E}_B\Big[\big|\int_0^{t/\varepsilon^2} R(\tfrac{x}{\varepsilon}+B_s)ds\big|^n\Big] \le C|\log\varepsilon|^{n-1}(1+|\log|x||) \tag{A.5}$$

with some constant C independent of x,ε.

Proof. To prove (A.4), we write the expectation explicitly:

(A.6)

$$\begin{aligned}\mathbb{E}_B\Big[\big|\int_0^t R(\tfrac{x}{\varepsilon}+B_s)ds\big|^2\Big] =&2\int_0^t ds\int_0^s du\,\mathbb{E}_B[R(\tfrac{x}{\varepsilon}+B_s)R(\tfrac{x}{\varepsilon}+B_u)]\\ =&2\int_0^t ds\int_0^s du\int_{\mathbb{R}^4} R(y)R(z)G_u(z-\tfrac{x}{\varepsilon})G_{s-u}(y-z)dzdy.\end{aligned}$$

Integrating in s and y yields

$$\int_u^t\int_{\mathbb{R}^2} R(y)G_{s-u}(y-z)dyds \lesssim 1+\int_{u+1}^t\int_{\mathbb{R}^2} R(y)\tfrac{1}{s-u}dyds \lesssim 1+\log t,$$

which implies

$$\mathbb{E}_B\Big[\big|\int_0^t R(\tfrac{x}{\varepsilon}+B_s)ds\big|^2\Big] \lesssim (1+\log t)\int_0^t\int_{\mathbb{R}^2} R(z)G_u(z-\tfrac{x}{\varepsilon})dzdu. \tag{A.7}$$

Since the integral $\int_0^t\int_{\mathbb{R}^2} R(z)G_u(z-\frac{x}{\varepsilon})dzdu = \mathbb{E}_B[\int_0^t R(\frac{x}{\varepsilon}+B_s)ds]$, to prove (A.4), we only need to note that

$$\begin{aligned}&\int_{\mathbb{R}^4} g(x_1)g(x_2)\int_0^t du\int_{\mathbb{R}^2} R(z)G_u(z-\tfrac{x_1-x_2}{\varepsilon})dz\\ =&\tfrac{\varepsilon^2}{(2\pi)^2}\int_0^t\int_{\mathbb{R}^2}|\hat{g}(\xi)|^2\hat{R}(\varepsilon\xi)e^{-\frac12|\varepsilon\xi|^2u}d\xi du \lesssim \varepsilon^2 t.\end{aligned}$$

To prove (A.5), by the same argument above, we have

$$\mathbb{E}_B\Big[\big|\int_0^{t/\varepsilon^2} R(\tfrac{x}{\varepsilon}+B_s)ds\big|^n\Big] \lesssim |\log\varepsilon|^{n-1}\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^2} R(z)G_u(z-\tfrac{x}{\varepsilon})dzdu.$$

We estimate the integral in u by

$$\int_0^t G_u(z-\tfrac{x}{\varepsilon})du \lesssim \int_0^t u^{-1}e^{-\frac{|z-x/\varepsilon|^2}{2u}}du \lesssim \int_{\frac{|z-x/\varepsilon|^2}{2t}}^\infty \lambda^{-1}e^{-\lambda}d\lambda \lesssim 1+|\log|\varepsilon z-x||+|\log\varepsilon^2 t|.$$

For the integral in z, recall that $R(z)=0$ for $|z|\ge 1$, we have

$$\int_{\mathbb{R}^2} R(z)|\log|\varepsilon z-x||dz \lesssim \int_{|z|\le 1}|\log|\varepsilon z-x||dz \lesssim 1+|\log|x||+\mathbb{1}_{|x|\le 3\varepsilon}|\log\varepsilon| \lesssim 1+|\log|x||.$$

The proof is complete. □

Lemma A.3. *Fix $t>0$ and $w\neq 0$, we have*

$$\mathbb{E}\Big[\Big(\int_{\frac{t}{\varepsilon^2|\log\varepsilon|}}^{\frac{t}{\varepsilon^2}} R(B_s)ds\Big)^2\Big|B_{t/\varepsilon^2}=\frac{w}{\varepsilon}\Big] \lesssim |\log\varepsilon|(\log|\log\varepsilon|).$$

Proof. To simplify the notation, denote $t_1 = \frac{t}{\varepsilon^2|\log\varepsilon|}$ and $t_2 = \frac{t}{\varepsilon^2}$, and write

$$\Big(\int_{t_1}^{t_2} R(B_s)ds\Big)^2 = 2\int_{[t_1,t_2]^2_<} R(B_s)R(B_u)\mathbb{1}_{s>u}duds.$$

Now we compute the conditional expectation

$$\begin{aligned}\mathbb{E}[R(B_s)R(B_u)|B_{t_2}=w/\varepsilon] &= \int_{\mathbb{R}^4} R(x)R(y)\frac{G_u(y)G_{s-u}(x-y)G_{t_2-s}(\frac{w}{\varepsilon}-x)}{G_{t_2}(\frac{w}{\varepsilon})}dxdy\\ &= \int_{\mathbb{R}^4} R(x)R(y)\frac{G_u(y)G_{s-u}(x-y)G_{t-\varepsilon^2 s}(w-\varepsilon x)}{G_t(w)}dxdy.\end{aligned}$$

Since $w\neq 0$ and $t>0$ is fixed, and R is compactly supported (so the $x-$variable in the above integral is bounded), we have

$$\frac{G_{t-\varepsilon^2 s}(w-\varepsilon x)}{G_t(w)} \lesssim 1,$$

uniformly in $s\leq t/\varepsilon^2$ and $|x|\leq 1$, so the conditional expectation is bounded by

$$\mathbb{E}[R(B_s)R(B_u)|B_{t_2}=w/\varepsilon] \lesssim \int_{\mathbb{R}^4} R(x)R(y)G_u(y)G_{s-u}(x-y)dxdy,$$

which implies

$$\begin{aligned}&\mathbb{E}\left[\Big(\int_{\frac{t}{\varepsilon^2|\log\varepsilon|}}^{\frac{t}{\varepsilon^2}} R(B_s)ds\Big)^2\Big| B_{t/\varepsilon^2}=w/\varepsilon\right]\\ &\lesssim \int_{[t_1,t_2]^2_<}\int_{\mathbb{R}^4} R(x)R(y)G_u(y)G_{s-u}(x-y)dxdyduds.\end{aligned}$$

By the same proof as for Lemma 3.4, the above integral bounded by

$$\log(t_2-t_1)\log\tfrac{t_2}{t_1} \lesssim |\log\varepsilon|(\log|\log\varepsilon|),$$

which completes the proof. □

Appendix B. Negative moments of $Z_\varepsilon(t,x)$

The goal is to show there exists $\beta_0>0$ such that if $\beta<\beta_0$ and $n\in\mathbb{Z}_+$, we have

$$\sup_{t\in[0,T]}\sup_{\varepsilon\in(0,1)} \mathbb{E}[Z_\varepsilon(t,x)^{-n}] \leq C_{\beta,n,T} \tag{B.1}$$

for some constant $C_{\beta,n,T}>0$. The result is essentially implied by [11, Theorem 4.6], and we only present the details here for the convenience of the readers. Since $Z_\varepsilon(t,x)$ has the same distribution as $u(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon})$ and is stationary in the $x-$variable, it suffices to estimate the small ball probability $\mathbb{P}[u(\frac{t}{\varepsilon^2},x)\leq r]$ for $r\ll 1$. From now on, we will fix $\varepsilon>0$ and derive an estimate that is uniform in $\varepsilon>0$ and $t\in[0,T]$. We fix $t>0, x\in\mathbb{R}^2$.

We first define an approximation of the spacetime white noise

$$\dot W_\delta(t,x) = e^{-\delta(t^2+|x|^2)}\int_{\mathbb{R}^3}\phi_\delta(t-s,x-y)dW(s,y),$$

where $\phi_\delta(t,x)=\varepsilon^{-4}\phi(\frac{t}{\delta^2},\frac{x}{\delta})$ with $\phi\in\mathcal{C}_c^\infty(\mathbb{R}^3)$ such that ϕ is even and $\int\phi=1$. Thus, we have almost surely that $\dot W_\delta\in L^2(\mathbb{R}^3)\cap\mathcal{C}^\infty(\mathbb{R}^3)$. Define

$$V_\delta(t,x)=\int_{\mathbb{R}^2}\varphi(x-y)\dot W_\delta(t,y)dy,\quad \mathscr{R}_\delta(t,s,x,y)=\mathbb{E}[V_\delta(t,x)V_\delta(s,y)],$$

and $\mathscr{U}_{\varepsilon,\delta}(t,x)=\mathbb{E}_B\left[e^{\mathcal{V}_{\varepsilon,\delta}(t,B)}\right]$, with

$$\mathcal{V}_{\varepsilon,\delta}(t,B)=\beta_\varepsilon\int_0^{t/\varepsilon^2} V_\delta(\tfrac{t}{\varepsilon^2}-s,x+B_s)ds-\tfrac12\beta_\varepsilon^2\mathcal{Q}_\delta(\tfrac{t}{\varepsilon^2},x,x,B,B),$$

where

$$\mathcal{Q}_\delta(t,x,y,B^1,B^2) = \int_{[0,t]^2} \mathscr{R}_\delta(t-s,t-\ell,x+B^1_s,y+B^2_\ell)dsd\ell.$$

By [11, Proposition 4.2], for each fixed $\varepsilon>0$, $\mathscr{U}_{\varepsilon,\delta}(t,x)\to u(\frac{t}{\varepsilon^2},x)$ in probability as $\delta\to 0$, so we only need to estimate $\mathbb{P}[\mathscr{U}_{\varepsilon,\delta}(t,x)\le r]$ for $r\ll 1$, uniformly in $\varepsilon,\delta>0$ and $t\in[0,T]$.

With any given $\dot W_\delta$, define the expectation

$$\mathbb{E}_B^{\dot W_\delta}[F(B^1,B^2)] = \frac{\mathbb{E}_B[F(B^1,B^2)e^{\mathcal{V}_{\varepsilon,\delta}(t,B^1)+\mathcal{V}_{\varepsilon,\delta}(t,B^2)}]}{\mathbb{E}_B[e^{\mathcal{V}_{\varepsilon,\delta}(t,B^1)+\mathcal{V}_{\varepsilon,\delta}(t,B^2)}]}.$$

To emphasize the dependence of $\mathscr{U}_{\varepsilon,\delta}$ on $\dot W_\delta$, we write $\mathscr{U}_{\varepsilon,\delta}(t,x)=\mathscr{U}_\varepsilon(t,x,\dot W_\delta)$. For any $\lambda>0$, define the set

$$A_\lambda(t,x) = \left\{\dot W_\delta : \mathscr{U}_\varepsilon(t,x,\dot W_\delta)>\tfrac12,\quad \beta_\varepsilon^2\int_0^{t/\varepsilon^2}\mathbb{E}_B^{\dot W_\delta}[R(B^1_s-B^2_s)]ds\le\lambda\right\}.$$

Lemma B.1. *For any $\dot{\mathscr{W}}_\delta\in A_\lambda(t,x)$, we have*

$$\mathscr{U}_\varepsilon(t,x,\dot W_\delta)\ge \tfrac12 e^{-\sqrt{\lambda}\|\dot W_\delta-\dot{\mathscr{W}}_\delta\|_{L^2(\mathbb{R}^3)}}.$$

Proof. We write

$$\begin{aligned}\mathscr{U}_\varepsilon(t,x,\dot W_\delta) = \mathbb{E}_B[e^{\mathcal{V}_{\varepsilon,\delta}(t,B)}] =&\mathbb{E}_B[e^{\mathscr{V}_{\varepsilon,\delta}(t,B)}]\frac{\mathbb{E}_B[e^{\mathcal{V}_{\varepsilon,\delta}(t,B)-\mathscr{V}_{\varepsilon,\delta}(t,B)}e^{\mathscr{V}_{\varepsilon,\delta}(t,B)}]}{\mathbb{E}_B[e^{\mathscr{V}_{\varepsilon,\delta}(t,B)}]}\\ =&\mathscr{U}_\varepsilon(t,x,\dot{\mathscr{W}}_\delta)\mathbb{E}_B^{\dot{\mathscr{W}}_\delta}[e^{\mathcal{V}_{\varepsilon,\delta}(t,B)-\mathscr{V}_{\varepsilon,\delta}(t,B)}],\end{aligned}$$

where $\mathscr{V}_{\varepsilon,\delta}(t,B)$ is obtained by replacing $\dot W_\delta$ by $\dot{\mathscr{W}}_\delta$ in the expression of $\mathcal{V}_{\varepsilon,\delta}(t,B)$. By the fact that $\dot{\mathscr{W}}_\delta\in A_\lambda$ and Jensen's inequality, we have

$$\mathscr{U}_\varepsilon(t,x,\dot W_\delta)\ge\tfrac12\exp(\mathbb{E}_B^{\dot{\mathscr{W}}_\delta}[\mathcal{V}_{\varepsilon,\delta}(t,B)-\mathscr{V}_{\varepsilon,\delta}(t,B)]).$$

It remains to show that

$$|\mathbb{E}_B^{\dot{\mathscr{W}}_\delta}[\mathcal{V}_{\varepsilon,\delta}(t,B)-\mathscr{V}_{\varepsilon,\delta}(t,B)]|\le\sqrt{\lambda}\|W_\varepsilon-\tilde W_\varepsilon\|_{L^2(\mathbb{R}^3)}. \tag{B.2}$$

We write

$$\mathcal{V}_{\varepsilon,\delta}(t,B)-\mathscr{V}_{\varepsilon,\delta}(t,B)=\beta_\varepsilon\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^2}\varphi(x+B_s-y)[\dot W_\delta(\tfrac{t}{\varepsilon^2}-s,y)-\dot{\mathscr{W}}_\delta(\tfrac{t}{\varepsilon^2}-s,y)]dyds,$$

and apply Cauchy-Schwarz to derive

$$\begin{aligned}&|\mathbb{E}_B^{\dot{\mathscr{W}}_\delta}[\mathcal{V}_{\varepsilon,\delta}(t,B)-\mathscr{V}_{\varepsilon,\delta}(t,B)]|\\ &\le\|\dot W_\delta-\dot{\mathscr{W}}_\delta\|_{L^2(\mathbb{R}^3)}\sqrt{\beta_\varepsilon^2\int_0^{t/\varepsilon^2}\int_{\mathbb{R}^2}|\mathbb{E}_B^{\dot{\mathscr{W}}_\delta}[\varphi(x+B_s-y)]|^2dyds}\\ &=\|\dot W_\delta-\dot{\mathscr{W}}_\delta\|_{L^2(\mathbb{R}^3)}\sqrt{\beta_\varepsilon^2\int_0^{t/\varepsilon^2}\mathbb{E}_B^{\dot{\mathscr{W}}_\delta}[R(B^1_s-B^2_s)]ds}\le\sqrt{\lambda}\|\dot W_\delta-\dot{\mathscr{W}}_\delta\|_{L^2(\mathbb{R}^3)},\end{aligned}$$

which completes the proof. □

Lemma B.2. *There exists constants $\lambda,c>0$ independent of $\varepsilon,\delta>0$ and $t\in[0,T]$ such that $\mathbb{P}[A_\lambda(t,x)]\ge c$.*

Proof. We have

$$\mathbb{P}[A_\lambda(t,x)]\ge\mathbb{P}[\mathscr{U}_\varepsilon(t,x,\dot W_\delta)>\tfrac12]-\mathbb{P}[B_\lambda(t,x)],$$

with

$$B_\lambda(t,x)=\left\{\dot W_\delta:\mathscr{U}_\varepsilon(t,x,\dot W_\delta)>\tfrac12,\beta_\varepsilon^2\int_0^{t/\varepsilon^2}\mathbb{E}_B^{\dot W_\delta}[R(B^1_s-B^2_s)]ds>\lambda\right\}.$$

Using the fact that $\mathbb{E}[\mathscr{U}_\varepsilon(t,x,\dot W_\delta)] = 1$ and the Paley-Zygmund's inequality, we have

$$\mathbb{P}[\mathscr{U}_\varepsilon(t,x,\dot W_\delta) > \tfrac{1}{2}] \geq \frac{1}{4\mathbb{E}[\mathscr{U}_\varepsilon(t,x,\dot W_\delta)^2]} = \frac{1}{4\mathbb{E}_B[e^{\beta_\varepsilon^2 \mathcal{Q}_\delta(t/\varepsilon^2,x,x,B^1,B^2)}]}.$$

For $B_\lambda(t,x)$, we have

$$\begin{aligned}\mathbb{P}[B_\lambda(t,x)] \leq & \mathbb{P}\left[\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{E}_B[R(B_s^1 - B_s^2)e^{\mathcal{V}_{\varepsilon,\delta}(t,B^1)+\mathcal{V}_{\varepsilon,\delta}(t,B^2)}]ds > \tfrac{\lambda}{4}\right] \\ \leq & \tfrac{4}{\lambda}\mathbb{E}_B\left[e^{\beta_\varepsilon^2 \mathcal{Q}_\delta(t/\varepsilon^2,x,x,B^1,B^2)}\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} R(B_s^1 - B_s^2)ds\right] \\ \leq & \tfrac{4C}{\lambda}\mathbb{E}_B\left[e^{2\beta_\varepsilon^2 \mathcal{Q}_\delta(t/\varepsilon^2,x,x,B^1,B^2)}\right]^{1/2}\end{aligned}$$

for some constant $C > 0$, where the last "$\leq$" comes from an application of Cauchy-Schwarz inequality and Lemma A.1. By Lemma B.3 and choosing λ large, there exists some constants $c, \lambda > 0$ independent of $\varepsilon, \delta > 0$ such that $\mathbb{P}[A_\lambda(t,x)] \geq c$, which completes the proof. □

Lemma B.3. *There exists $\beta_0 > 0$ such that if $\beta < \beta_0$, we have*

$$1 \leq \sup_{t\in[0,T]} \sup_{\varepsilon,\delta\in(0,1)} \mathbb{E}_B\left[e^{\beta_\varepsilon^2 \mathcal{Q}_\delta(t/\varepsilon^2,x,x,B^1,B^2)}\right] \leq C_{\beta,T}.$$

Proof. Recall that $\mathcal{Q}_\delta(t,x,x,B^1,B^2) = \int_{[0,t]^2} \mathscr{R}_\delta(t-s,t-\ell,x+B_s^1,x+B_\ell^2)dsd\ell$. We write $\mathscr{R}_\delta$ explicitly:

$$\begin{aligned}\mathscr{R}_\delta(t_1,t_2,x_1,x_2) &= \int_{\mathbb{R}^4} \varphi(x_1-y_1)\varphi(x_2-y_2)\mathbb{E}[\dot W_\delta(t_1,y_1)\dot W_\delta(t_2,y_2)]dy_1dy_2 \\ &\leq \int_{\mathbb{R}^4} \varphi(x_1-y_1)\varphi(x_2-y_2)\phi_\delta \star \phi_\delta(t_1-t_2,y_1-y_2)dy_1dy_2,\end{aligned}$$

with "$\star$" denoting the convolution. By the fact that φ, ϕ have compact supports, it is clear that

$$\mathscr{R}_\delta(t_1,t_2,x_1,x_2) \lesssim \delta^{-2}\mathbb{1}_{|x_1-x_2|\leq c,|t_1-t_2|\leq c\delta^2}$$

for some $c > 0$. Thus, we have

$$\begin{aligned}\mathcal{Q}_\delta(t/\varepsilon^2,x,x,B^1,B^2) &\lesssim \int_{[0,t/\varepsilon^2]^2} \delta^{-2}\mathbb{1}_{|s-\ell|\leq c\delta^2}\mathbb{1}_{|B_s^1-B_\ell^2|\leq c}dsd\ell \\ &\lesssim \int_0^c \left(\int_0^{t/\varepsilon^2} \mathbb{1}_{|B^1_{\ell+\delta^2 s}-B_\ell^2|\leq c}d\ell\right)ds.\end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}_B[e^{\beta_\varepsilon^2 \mathcal{Q}_\delta(t/\varepsilon^2,x,x,B^1,B^2)}] \leq & \mathbb{E}_B\left[\exp\left(c'\int_0^c \left(\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{1}_{|B^1_{\ell+\delta^2 s}-B_\ell^2|\leq c}d\ell\right)ds\right)\right] \\ \leq & \tfrac{1}{c}\int_0^c \mathbb{E}_B[\exp\left(cc'\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{1}_{|B^1_{\ell+\delta^2 s}-B_\ell^2|\leq c}d\ell\right)]ds,\end{aligned}$$

for some $c, c' > 0$. Clearly we have

$$\begin{aligned}&\sup_{s\in[0,c]} \mathbb{E}_B[\exp\left(cc'\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{1}_{|B^1_{\ell+\delta^2 s}-B_\ell^2|\leq c}d\ell\right)] \\ &\leq \sup_{x\in\mathbb{R}^2} \mathbb{E}_B[\exp\left(cc'\beta_\varepsilon^2 \int_0^{t/\varepsilon^2} \mathbb{1}_{|x+B_\ell^1-B_\ell^2|\leq c}d\ell\right)] \lesssim 1\end{aligned}$$

for small β, where the last "$\lesssim$" comes from Lemma A.1. The proof is complete. □

Now we can write

$$
\text{(B.3)} \qquad \begin{aligned} \mathbb{P}[\mathscr{U}_{\varepsilon,\delta}(t,x,\dot W_\delta) \le r] \le & \mathbb{P}[\tfrac{1}{2}e^{-\sqrt{\lambda}\mathrm{dist}(\dot W_\delta, A_\lambda(t,x))} \le r] \\ \le & \mathbb{P}\Big[\mathrm{dist}(\dot W_\delta, A_\lambda(t,x)) \ge \tfrac{\log(2r)^{-1}}{\sqrt{\lambda}}\Big], \end{aligned}
$$

where $\mathrm{dist}(\dot W_\delta, A_\lambda(t,x)) = \inf\{\|\dot W_\delta - \dot{\mathscr{W}}_\delta\|_{L^2(\mathbb{R}^3)} : \dot{\mathscr{W}}_\delta \in A_\lambda(t,x)\}$. Now we can apply [11, Lemma 4.5] to derive that

$$
\text{(B.4)} \qquad \mathbb{P}\Big[\mathrm{dist}(\dot W_\delta, A_\lambda(t,x)) \ge \tau + 2\sqrt{\log\tfrac{2}{c}}\Big] \le 2e^{-\tau^2/4}
$$

for all $\tau > 0$, where $\lambda, c > 0$ are chosen as in Lemma B.2 and are independent of $\varepsilon, \delta > 0$ and $t \in [0,T]$. Combining (B.3) and (B.4), we have

$$
\mathbb{P}[\mathscr{U}_\varepsilon(t,x,\dot W_\delta) \le r] \le 2\exp\left(-\frac{1}{4}\left(\frac{\log(2r)}{\sqrt{\lambda}} + 2\sqrt{\log\tfrac{2}{c}}\right)^2\right),
$$

which implies $\mathbb{E}[\mathscr{U}_\varepsilon(t,x,\dot W_\delta)^{-n}] \lesssim 1$ and completes the proof of (B.1).

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