DISSIPATION ENHANCEMENT BY MIXING

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Abstract. We quantitatively study the interaction between diffusion and mixing in both the continuous, and discrete time setting. In discrete time, we consider a mixing dynamical system interposed with diffusion. In continuous time, we consider the advection diffusion equation where the advecting vector field is assumed to be sufficiently mixing. The main results of this paper estimate the dissipation time and energy decay based based on an assumption quantifying the mixing rate.

1. **Introduction.**

Diffusion and mixing are two fundamental phenomena that arise in a wide variety of applications ranging from micro-fluids to meteorology, and even cosmology. In incompressible fluids, stirring induces mixing by filamentation and facilitates the formation of small scales. Diffusion, on the other hand, efficiently damps small scales and the balance between these two phenomena is the main subject of our investigation. Specifically, our aim in this paper is to quantify the interaction between diffusion and mixing in a manner that often arises in the context of fluids [DT06, CKRZ08, LTD11, Thi12].

In the absence of diffusion, the mixing of tracer particles passively advected by an incompressible flow has been extensively studied. Several authors [MMP05, LTD11, Thi12] measured mixing using *multi-scale* norms and studied how efficiently incompressible flows can mix (see for instance [Bre03, LLN+12, IKX14, ACM16, YZ17] and references therein). In this scenario, however, there is no apriori limit to the resolution attainable via mixing.

In contrast, in the presence of diffusion, the effects of mixing may be enhanced, balanced, or even counteracted by diffusion (see for instance [FP94, TC03, FNW04, CKRZ08, INRZ10, KX15, MDTY18, MD18]). In this paper we quantify this interaction by studying the energy dissipation rate. Roughly speaking, our main results can be stated as follows:

(1) In the continuous time setting we show (Theorem 2.9) that if the flow is strongly mixing with a *square integrable* rate function, then the dissipation time is bounded by $C/(\nu|\ln \nu|^{\delta})$ for some explicit $\delta > 0$. Here ν is the diffusivity, and we recall that the dissipation time (see Definition 2.1, below) is the time required for the system to dissipate a constant fraction of its initial energy. We also obtain a similar result if the flow is weakly mixing instead (Theorem 2.10).

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(2) Under similar assumptions in the discrete time setting we obtain stronger results (Theorems 2.3 and 2.4), and show that the dissipation time is now at most C/ν^{δ} for some explicit $\delta \in (0,1)$.
(3) In the discrete time setting we show (Theorem 2.7) that the energy can not decay faster than double exponentially in time. Moreover, using elementary Galois theory, we obtain a family of examples where the energy indeed decays double exponentially in time. (In the continuous time setting the double exponential lower bound is known [Poo96], however, to the best of our knowledge there are no smooth flows which are known to attain this lower bound.)
(4) In bounded domains, Berestycki et. al. [BHN05] studied asymptotics of the principal eigenvalue of the operator $-\nu\Delta + u \cdot \nabla$ as $\nu \to 0$. We show (Proposition 2.13) that one can use the dissipation time to obtain quantitative bounds on the rate at which the principal eigenvalue approaches 0.

We remark that our results on strongly mixing flows only require the mixing rate function to be square integrable. If the mixing rate function has exponentially decaying tails, then we expect that stronger bounds on the dissipation time can be obtained. Our techniques, however, do not yield stronger bounds for exponentially mixing maps.

Plan of this paper. We begin by defining mixing rates, and state our main results in Section 2. Next, in Section 2.1, we prove the dissipation time bounds in the discrete time setting (Theorems 2.3 and 2.4). In Section 4 we study toral automorphisms, and use them to prove our result on energy decay (Theorem 2.7). These proofs require certain facts on algebraic number fields, and may be skipped by readers who are not familiar with this material. In Section 2.2 we prove the dissipation time bounds in the continuous time setting. The proofs are similar to the discrete case, with a few key differences that we highlight. Finally we conclude this paper with two appendices. The first (Appendix A) provides a brief introduction to mixing and the notion of mixing rates we use to formulate our results. The second (Appendix B) shows that the characterization of relaxation enhancing flows in [CKRZ08, KSZ08] still applies in the context of pulsed diffusions.

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2. Main Results.

We devote this section to stating our main results. In the discrete time setting we consider *pulsed diffusions* (mixing maps interposed with diffusion), and our results concerning these are stated in Section 2.1, below. In the continuous time setting we consider the advection diffusion equation, and our results in this setting are stated in Section 2.2, below.

2.1. Pulsed Diffusions.

In our setup we will consider a mixing map on a closed Riemannian manifold. While the primary manifold we are interested in is the torus, there are, to the best of our knowledge, no known examples of smooth exponentially mixing maps on the torus that can be realized as the time one map of the flow of a smooth incompressible vector field. There are, however, several examples of closed

Riemannian manifolds that admit such maps (see [Dol98, BW16] and references therein). Since working on closed Riemannian manifolds does not increase the complexity by much, we state our results in this context instead of restricting our attention to the torus.

Let M be a closed d-dimensional Riemannian manifold, and $\varphi\colon M \to M$ be a smooth volume preserving diffeomorphism. For simplicity we will subsequently assume that the volume form on M is normalized so that the total volume, $|M|$, is 1. Let $\nu > 0$ be the strength of the diffusion, Δ denote the Laplace-Beltrami operator on M, and $L^2_0 = L^2_0(M)$ denote the space of all mean zero square integrable functions on M. Given $\theta_0 \in L^2_0$, we consider the *pulsed diffusion* defined by

$$\theta_{n+1} = e^{\nu\Delta} U \theta_n \,. \tag{2.1}$$

Here $U\colon L^2(M) \to L^2(M)$ is the Koopman operator associated with φ, and is defined by $Uf = f \circ \varphi$. Our aim is to understand the asymptotic behaviour of the energy $\|\theta_n\|_{L^2_0}$ in the long time, small diffusivity limit. For notational convenience, we will use $\|\cdot\|$ to denote the L^2_0 norm, and $\langle \cdot, \cdot \rangle$ to denote the L^2_0 inner-product.

Since φ is volume preserving, the operator U is unitary and hence if $\nu = 0$ the system (2.1) conserves energy. If $\nu > 0$ and φ is mixing, then Koopman operator U produces fine scales which are rapidly damped by the diffusion. We quantify this using the notion of *dissipation time* in [FW03] (see also [FNW04, FNW06]).

Definition 2.1 (Dissipation time)**.** We define the *dissipation time* of the operator U by

$$\begin{aligned} \tau_d &\stackrel{\text{def}}{=} \inf\Big\{ n \in \mathbb{N} \;\Big|\; \|(e^{\nu\Delta}U)^n\|_{L^2_0 \to L^2_0} < \frac{1}{e} \Big\} \\ &= \inf\Big\{ n \in \mathbb{N} \;\Big|\; \|\theta_n\| < \frac{\|\theta_0\|}{e} \quad \text{for all } \theta_0 \in L^2_0 \Big\} \,. \end{aligned}$$

Since U is unitary we clearly have $\|\theta_n\| \leqslant e^{-\nu\lambda_1} \|\theta_{n-1}\|$, where $\lambda_1 > 0$ is the smallest non-zero eigenvalue of $-\Delta$ on M. Consequently, we always have

$$\tau_d \leqslant \frac{1}{\nu\lambda_1} \,, \tag{2.2}$$

Our aim is to investigate how (2.2) can be improved given an assumption on the "mixing rate" of φ. Recall, (strongly) mixing maps are those for which the correlations $\langle U^n f, g \rangle$ decay to 0 as $n \to \infty$ for all $f, g \in L^2_0$. Weakly mixing maps are those for which the Cesàro averages of $|\langle U^n f, g \rangle|^2$ decay to 0 (see Appendix A for a brief introduction and [EFHN15, KH95, SOW06] for a comprehensive treatment). We quantify the mixing rate of φ by imposing a rate at which these convergences occur.

Definition 2.2. Let $h\colon \mathbb{N} \to (0, \infty)$ be a decreasing function that vanishes at infinity.

(1) Given $\alpha, \beta > 0$, we say that φ is *strongly α, β mixing with rate function h* if for all $f \in \dot{H}^\alpha$, $g \in \dot{H}^\beta$ and $n \in \mathbb{N}$ the associated Koopman operator U satisfies

$$\big|\langle U^n f, g \rangle\big| \leqslant h(n) \|f\|_\alpha \|g\|_\beta \,. \tag{2.3}$$

(2) Given $\alpha, \beta \geqslant 0$, we say that φ is *weakly α, β mixing with rate function h* if for all $f \in \dot{H}^\alpha$, $g \in \dot{H}^\beta$ and $n \in \mathbb{N}$ the associated Koopman operator U

satisfies

$$\Big(\frac{1}{n}\sum_{k=0}^{n-1}\big|\langle U^k f, g\rangle\big|^2\Big)^{1/2} \leqslant h(n)\|f\|_\alpha \|g\|_\beta\,. \tag{2.4}$$

Here $\dot H^\alpha = \dot H^\alpha(M)$ is the homogeneous Sobolev space of order α, and $\|\cdot\|_\alpha$ denotes the norm in $\dot H^\alpha$. In the dynamical systems literature it is common to use Hölder spaces instead of Sobolev spaces, and study strongly mixing maps that are *exponentially mixing* (i.e. $h(t) = c_1 e^{-c_2 t}$ for some $c_1 < \infty$ and $c_2 > 0$). Using Sobolev spaces and asymmetric norms on f and g, however, is more convenient for our purposes. In order not to detract from our main results, we briefly motivate and study the above notions of mixing in Appendix A. Our main results on the dissipation time are as follows:

Theorem 2.3. *Suppose φ is strongly α, β mixing with rate function h, where $\alpha, \beta > 0$ and $\sum_0^\infty h(n)^2 < \infty$. Then there exists a constant $C > 0$ such that*

$$\tau_d \leqslant C\Big(\sum_0^\infty h(n)^2\Big)^{\frac{2}{1+2\alpha+2\beta}}\,\nu^{-\frac{2\alpha+2\beta}{1+2\alpha+2\beta}}\,,$$

for all ν sufficiently small.

Theorem 2.4. *Suppose φ is weakly α, β mixing with rate function h, where $\alpha, \beta > 0$. Then the dissipation time τ_d is bounded by*

$$\tau_d \leqslant \frac{17}{\nu\lambda_N}\,. \tag{2.5}$$

Here $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ are the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity, and $N = N(\nu)$ is the largest integer which satisfies

$$\lambda_N\Big(h^{-1}\Big(\big(2N\lambda_N^{\alpha+\beta}\big)^{-1/2}\Big) + 1\Big)^2 \leqslant \frac{1}{\nu}\,.$$

where h^{-1}, defined by

$$h^{-1}(x) \stackrel{\text{def}}{=} \inf\big\{n \in \mathbb{N} \bigm| h(n) \leqslant x\big\}\,.$$

is the upper inverse function of h.

If further the rate function h is the power law

$$h(n) = \frac{c_1}{n^s}\,, \tag{2.6}$$

for some $s \in (0, 1/2]$, then the dissipation time is bounded by

$$\tau_d \leqslant C\nu^{-\frac{d+2\beta+2\alpha}{d+2s+2\beta+2\alpha}}\,. \tag{2.7}$$

for some explicit constant $C = C(c_1, s)$.

Note that in both the bounds provided Theorem 2.3 and 2.4, we have $\nu\tau_d \to 0$. Thus the results of both Theorems 2.3 and 2.4 are stronger than the elementary bound (2.2).

Remark 2.5. Notice that if φ is strongly α, β mixing with rate function h, then it is also weakly α, β mixing with rate function

$$h_w(n) \stackrel{\text{def}}{=} \Big(\frac{1}{n}\sum_{k=0}^{n-1} h(k)^2\Big)^{1/2}$$

If further $H \stackrel{\text{def}}{=} \sum h(n)^2 < \infty$, then $h_w(n) \leqslant \sqrt{H/n}$. In this case, however, one immediately sees that the bound provided by Theorem 2.4 is weaker than that provided by Theorem 2.3.

Before proceeding further, we note that Fannjiang et. al. [FNW04] (see also [FW03, FNW06]) also obtain bounds on the dissipation time τ_d assuming the time decay of the correlations of the *diffusive operator* $e^{\nu\Delta}U$ for sufficiently small ν. Explicitly they assume sufficient decay of $\langle (e^{\nu\Delta}U)^n f, g\rangle$ as $n \to \infty$, and then show that the dissipation time τ_d is at most $C/|\ln \nu|$. In contrast, our results only assume decay of the correlations of the operator U (without diffusion) as in Definition 2.2.

In continuous time, Constantin et. al. [CKRZ08] (see also [KSZ08]) characterized flows for which the dissipation time is $o(1/\nu)$. Their result can directly be adapted to pulsed diffusions as follows.

Proposition 2.6. *The Koopman operator U has no eigenfunctions in $\dot{H}^1$ if and only if*

$$\lim_{\nu \to 0} \nu\tau_d = 0\,.$$

Since the proof is a direct adaptation of [CKRZ08, KSZ08], we relegate it to Appendix B. We remark, however, that without a quantitative assumption on the mixing rate of φ, it does not seem possible to obtain more information regarding the rate at which $\nu\tau_d \to 0$.

We now turn to studying the energy decay as $n \to \infty$. Clearly

$$\|\theta_n\| \leqslant \Big\| \big((e^{\nu\Delta}U)^{\tau_d}\big)^{\lfloor n/\tau_d \rfloor}\theta_0 \Big\| \leqslant \big\|(e^{\nu\Delta}U)^{\tau_d}\big\|^{\lfloor n/\tau_d \rfloor}\|\theta_0\| \leqslant e^{-n/\tau_d}\|\theta_0\|\,,$$

and thus the energy $\|\theta_n\|$ decays at least exponentially with rate $1/\tau_d$ as $n \to \infty$. We believe, however, that that the above bound is not from optimal. In fact, we conjecture that for *any* exponentially mixing map the energy decays *double exponentially* in n. The reason we expect a double exponential decay rate is twofold: First, for the continuous time advection diffusion a double exponential *lower bound* was proved in [Poo96]. Adapting this to the discrete time setting we can prove the same lower bound. Second, if φ is the Arnold cat map, it is well known [TC03] that the energy decays double exponentially. We show that this remains true for a large class of toral automorphisms. The following theorem states our results concerning energy decay.

Theorem 2.7 (Energy decay)**.** *For any $\theta_0 \in \dot{H}^1$, there exist finite constants $C = C(\varphi) > 0$ and $\gamma = \gamma(\varphi) > 1$ for which the double exponential lower bound*

$$\|\theta_n\|^2 \geqslant \|\theta_0\|^2 \exp\Big(-\frac{C\nu\|\theta_0\|_1^2}{\|\theta_0\|^2}\gamma^n\Big)\,, \tag{2.8}$$

holds. Moreover, there exists a map φ for which the above bound is achieved. Explicitly, if φ is an toral automorphisms which has no invariant proper rational subspaces, and no eigenvalues that are roots of unity, then there exists finite constants C and $\gamma > 1$ such that

$$\|\theta_n\|^2 \leqslant \|\theta_0\|^2 \exp\Big(-\frac{\nu\gamma^n}{C}\Big)\,, \tag{2.9}$$

for all $\theta_0 \in L^2_0$.

We prove Theorem 2.7 in Section 4. Recall *toral automorphisms* are diffeomorphisms of the torus onto itself that can be lifted to a linear transformation on the covering space $\mathbb{R}^d$, and Section 4 also contains a brief introduction to such maps.

Note that for maps φ that achieve the upper bound (2.9), the dissipation time satisfies

$$\tau_d \leqslant C|\ln \nu|\,. \tag{2.10}$$

We will show (Proposition 4.1) that the toral automorphisms considered in Theorem 2.7 are all exponentially mixing. This gives a large class of exponentially mixing diffeomorphisms for which the dissipation time is of order $|\ln \nu|$. For general exponentially mixing diffeomorphisms, however, it is not known whether the dissipation time is still of order $|\ln \nu|$.

Observe that the bound on the dissipation time provided by Theorem 2.3 is only algebraic, and hence weaker than (2.10). This is because Theorem 2.3 only requires the mixing rate function h to have a square integrable tail. In principle, the faster the tail of h decays, the smaller the dissipation time should be. Our proof of Theorem 2.3, however, does not utilize the decay rate of the tail of h, and provides the same upper bound (up to constants) for all square integrable rate functions.

2.2. **Advection Diffusion Equation.** We now turn to the continuous time setting. Let M be a (smooth) closed Riemannian manifold, and u be a smooth, time dependent, divergence free vector field on M. Let θ be a solution to the advection-diffusion equation

$$\begin{cases} \partial_t \theta_s + (u(t)\cdot\nabla)\theta_s - \nu\Delta\theta_s = 0 & \text{in } M, \text{ for } t > s,\\ \theta_s(t) = \theta_{s,0} & \text{for } t = s. \end{cases} \tag{2.11}$$

for $t > s$, with initial data $\theta_s(0) = \theta_{s,0} \in L^2_0(M)$. Since u is divergence free we have

$$\frac{1}{2}\partial_t \|\theta_s(t)\|^2 + \nu \|\theta_s(t)\|_1^2 = 0\,, \tag{2.12}$$

and hence

$$\|\theta_s(t)\| \leqslant e^{-\nu\lambda_1(t-s)}\|\theta_{s,0}\|\,. \tag{2.13}$$

Our interest, again, is to to investigate how this decay rate can be quantifiably improved when the flow of u is mixing. Similar to our treatment of pulsed diffusions, we define the *dissipation time* of u by

$$\begin{aligned}\tau_d &\stackrel{\text{def}}{=} \sup_{s\in\mathbb{R}} \inf\Big\{ t-s \;\Big|\; t \geqslant s, \text{ and } \|\theta_s(t)\| \leqslant \frac{\|\theta_{s,0}\|}{e} \quad \text{for all } \theta_{s,0} \in L^2_0 \Big\}\\ &= \sup_{s\in\mathbb{R}} \inf\Big\{ t-s \;\Big|\; t \geqslant s, \text{ and } \|\mathcal{S}_{s,t}\|_{L^2_0 \to L^2_0} \leqslant \frac{1}{e} \Big\}\,,\end{aligned}$$

where $\mathcal{S}_{s,t}$ is the solution operator to (2.11).

From (2.13) we immediately see that for any smooth divergence free advecting field u we again have

$$\tau_d \leqslant \frac{1}{\nu\lambda_1}\,,$$

where λ_1 is the smallest non-zero eigenvalue of $-\Delta$ on M. If the flow of u is mixing, then we expect that τ_d to be much smaller than than $1/(\lambda_1\nu)$. It turns out that all stationary vector fields for which $\nu\tau_d \to 0$ can be elegantly characterized in terms of the spectrum of the operator $u\cdot\nabla$. Indeed, seminal work of Constantin et. al. [CKRZ08] shows[1] that for time independent incompressible vector fields u, $\nu\tau_d \to 0$ if and only if $\nu\tau_d \to 0$. Consequently, it follows that if the flow generated by u is weakly mixing, we must have $\nu\tau_d \to 0$ as $\nu \to 0$.

Our aim is to obtain bounds on the rate at which $\nu\tau_d \to 0$, under an assumption on the rate at which the flow of u mixes. The analogue of Definition 2.2 in continuous time is as follows.

Definition 2.8. Let $h\colon [0,\infty) \to [0,\infty)$ be a continuous, decreasing function that vanishes at ∞, and $\alpha, \beta \geqslant 0$. Let $\varphi_{s,t}\colon M \to M$ be the flow map of u defined by

$$\partial_t \varphi_{s,t} = u(\varphi_{s,t}, t) \qquad \text{and} \qquad \varphi_{s,s} = \mathrm{Id}\,.$$

(1) We say that the vector field u is *strongly α, β mixing with rate function h* if for all $f \in \dot H^\alpha$, $g \in \dot H^\beta$ we have

$$\big|\langle f \circ \varphi_{s,t}, g\rangle\big| \leqslant h(t-s) \|f\|_\alpha \|g\|_\beta\,. \tag{2.14}$$

(2) We say that φ is *weakly α, β mixing with rate function h* if for all $f \in \dot H^\alpha$, $g \in \dot H^\beta$ we have

$$\Big(\frac{1}{t-s}\int_s^t \big|\langle f \circ \varphi_{s,r}, g\rangle\big|^2\,dr\Big)^{1/2} \leqslant h(t-s) \|f\|_\alpha \|g\|_\beta\,. \tag{2.15}$$

Our first result bounds the dissipation time of vector fields u that are strongly α, β mixing with a square integrable rate function.

Theorem 2.9. *Suppose u is strongly α, β mixing with rate function h, where $\alpha, \beta > 0$, and $H \stackrel{\text{def}}{=} \int_0^\infty h^2 < \infty$. Let $N = N(\nu)$ to be the largest integer such that*

$$\lambda_N^{1-\alpha-\beta} \exp\Big(8H\|\nabla u\|_{L^\infty} \lambda_N^{\alpha+\beta}\Big) \leqslant \frac{2H\|\nabla u\|_{L^\infty}^2}{\nu}\,, \tag{2.16}$$

where $\|\nabla u\|_{L^\infty}$ denotes the space time L^∞ norm of ∇u. Then

$$\tau_d \leqslant \frac{5}{\nu\lambda_N}\,, \tag{2.17}$$

and consequently

$$\tau_d \leqslant \frac{C}{\nu|\ln \nu|^{1/(\alpha+\beta)}}$$

for some finite constant $C = C(\alpha, \beta, H, \|\nabla u\|_{L^\infty})$.

Theorem 2.10. *Suppose u is weakly α, β mixing with rate function h, where alpha, $\beta > 0$. Let $N = N(\nu)$ to be the largest integer such that*

$$\frac{\lambda_N \exp\Big(2\|\nabla u\|_{L^\infty} h^{-1}\big(\frac{1}{2}\lambda_N^{-\frac{\alpha+\beta}{2}}\big)\Big)}{h^{-1}\big(\frac{1}{2}\lambda_N^{-\frac{\alpha+\beta}{2}}\big)} \leqslant \frac{\|\nabla u\|_{L^\infty}^2}{2\nu}\,, \tag{2.18}$$

where h^{-1} denotes the inverse function of h. Then

$$\tau_d \leqslant \frac{5}{\nu\lambda_N}\,. \tag{2.19}$$

[1] More precisely, in [CKRZ08] the authors show that an incompressible, time independent, vector field u is *relaxation enhancing* if and only if $(u \cdot \nabla)$ has no eigenfunctions in $\dot H^1$. It is, however, easy to see that a vector field is relaxation enhancing if and only if $\nu\tau_d \to 0$.

If further the rate function h is power law

$$h(t) = \frac{c_1}{t^s}, \tag{2.20}$$

for some $s \in (0, \frac{1}{2}]$, consequently we have

$$\tau_d \leqslant \frac{C}{\nu |\ln \nu|^{\frac{2s}{\alpha+\beta}}}, \tag{2.21}$$

for some finite constant $C = C(\alpha, \beta, \|\nabla u\|_{L^\infty})$.

Remark 2.11 (Comparison with pulsed diffusions). In continuous time, the estimate on the dissipation time (2.17) is *weaker* than that of a pulsed diffusion, with the same mixing rate function. In particular, if h decays algebraically, then $\nu\tau_d$ decays algebraically for pulsed diffusions (as in Theorem 2.3) but only logarithmically (as in Theorem 2.9) for the advection diffusion equation. The reason the bound in Theorem 2.9 is weaker than that in Theorem 2.3 is because pulsed diffusions are better approximated by the underlying dynamical system than solutions to (2.11) are. Thus when studying pulsed diffusions one is able to better use the mixing properties of the underlying dynamical system.

Remark 2.12 (Shear Flows). In the particular case of shear flows a stronger estimate on the dissipation time can be obtained using Theorem 1.1 in [BCZ17]. Namely let $u = u(y)$ be a smooth shear flow on the 2-dimensional torus with non-degenerate critical points, and let L^2_0 denote the space of all functions whose horizontal average is 0. Now Theorem 1.1 in [BCZ17] guarantees that the dissipation time is bounded by

$$\tau_d \leqslant C \frac{|\ln \nu|^2}{\nu^{1/2}}, \tag{2.22}$$

for some constant $C > 0$.

To place this in the context of our results, we restrict our attention to L^2_0 functions on $\mathbb{T}^2$ whose horizontal averages are all 0. On this space, the method of stationary phase one can show that the flow generated by u is strongly 1, 1 mixing with rate function $h(t) = Ct^{-1/2}$ (see equation (1.8) in [BCZ17]). Since the rate function is not square integrable in time, Theorem 2.9 will *not* apply. However, after time averaging, we see that the flow of u is weakly 1, 1 mixing with rate function $(\ln t/t)^{1/2}$. In this case Theorem 2.10 will apply, however, the dissipation time bound provided by Theorem 2.10 is weaker than (2.22).

In terms of optimality, we recall that Poon [Poo96] showed the double exponential *lower bound*

$$\|\theta_s(t)\| \geqslant \exp\big(-C\nu\gamma^{t-s}\big) \|\theta_{s,0}\|, \tag{2.23}$$

for some constants $C > 0$ and $\gamma > 1$. To the best of our knowledge, there are no incompressible smooth divergence free vector fields for which the lower bound (2.23) is attained. Similar to the case of pulsed diffusions, we conjecture that (2.23) is attained whenever the flow generated by u is exponentially mixing. However, recent work of Miles and Doering [MD18] suggests that the Batchelor length scale may limit the long term effectiveness of mixing leading to slower energy decay.

Notice that if (2.23) is indeed attained for exponentially mixing flows, then the dissipation time would be bounded by

$$\tau_d \leqslant C |\ln \nu|. \tag{2.24}$$

This is a stronger than the bound provided by Theorem 2.9. As in the case of pulsed diffusions Theorem 2.9 only requires the tail of the mixing rate function h to be square integrable. We expect that one should be able to obtain better bounds on τ_d using the precise decay rate of h, however, we are presently unable to do so using our methods.

Finally, we turn our attention to studying the principal eigenvalue of the operator $-\nu\Delta + u\cdot\nabla$ in a bounded domain Ω with Dirichlet boundary conditions. In this case, in addition to u being smooth and divergence free, we also assume u is time independent and tangential on the boundary (i.e. $u \cdot \hat{n} = 0$ on $\partial\Omega$, where $\hat{n}$ denotes the outward pointing unit normal). Let $\mu_0(\nu, u)$ denote the principal eigenvalue of $-\nu\Delta + u\cdot\nabla$ with Dirichlet boundary conditions.

By Rayleigh's principle we note

$$\mu_0(\nu, u) \geqslant \mu_0(\nu, 0) = \nu\mu_0(1, 0)$$

where $\mu_0(1, 0)$ is the principal eigenvalue of the Laplacian. Our interest is in understanding the behaviour of $\mu_0(\nu, u)/\nu$ as $\nu \to 0$. Berestycki et. al. [BHN05] showed that $\mu_0(\nu, u)/\nu \to \infty$ if and only if $u\cdot\nabla$ has no first integrals in H^1_0. That is, $\mu_0(\nu, u)/\nu \to \infty$ if and only if there does not exist $w \in H^1_0(\Omega)$ such that $u \cdot \nabla w = 0$.

In general it does not appear to be possible to obtain a rate at which $\mu_0(\nu, u)/\nu \to \infty$. If, however, the flow generated by u is sufficiently mixing then we can show $\mu_0(\nu, u)/\nu \to \infty$ logarithmically.

Proposition 2.13. *If u is a smooth, time independent, incompressible vector field which is tangential on $\partial\Omega$, then*

$$\mu_0(\nu, u) \geqslant \frac{1}{\tau_d}\,. \tag{2.25}$$

Consequently, if $\alpha, \beta > 0$ and u is strongly α, β mixing with a square integrable rate function h, then there exists a constant $C = C(\alpha, \beta, h)$ such that

$$\frac{\mu_0(\nu, u)}{\nu} \geqslant \frac{|\ln \nu|^{1/(\alpha+\beta)}}{C}\,. \tag{2.26}$$

We prove Proposition 2.13 in Section 5.3. We remark, however, that in view of (2.24) and (2.25), we expect that if u that generates an exponentially mixing flow, then

$$\frac{\mu_0(\nu, u)}{\nu} \geqslant \frac{1}{C\nu|\ln \nu|}\,.$$

We are, however, presently unable to prove this stronger bound.

The rest of this paper is devoted to the proofs of the main results, and a brief plan can be found at the end of Section 1.

3. Dissipation Enhancement for Pulsed Diffusions.

In this section we prove Theorems 2.3 and 2.4. The main idea behind the proof is to split the analysis into two cases. In the first case, we assume $\|\theta_n\|_1/\|\theta_n\|$ is large, and obtain decay of $\|\theta_n\|$ using the energy inequality. In the second case, $\|\theta_n\|_1/\|\theta_n\|$ is small, and hence the dynamics are well approximated by that of the underlying dynamical system. The mixing assumption now forces the generation of high frequencies, and the rapid dissipation of these gives an enhanced decay of $\|\theta_n\|$.

3.1. **The Strongly Mixing Case.** We begin by stating two lemmas handling each of the cases stated above.

Lemma 3.1. *Given $\theta \in L^2_0$, define $\mathcal{E}_\nu \theta$ by*

$$\mathcal{E}_\nu \theta \stackrel{\text{def}}{=} \frac{1}{\nu} \big\| (1 - e^{2\nu\Delta})^{1/2} U\theta \big\|^2 . \tag{3.1}$$

If for $\theta_0 \in L^2_0$ and $c_0 > 0$ we have

$$\mathcal{E}_\nu \theta_0 \geqslant c_0 \|\theta_0\|^2 , \tag{3.2}$$

then

$$\|\theta_1\|^2 \leqslant e^{-\nu c_0} \|\theta_0\|^2 .$$

Lemma 3.2. *Let $0 < \lambda_1 < \lambda_2 \leqslant \cdots$ be the eigenvalues of the Laplacian, where each eigenvalue is repeated according to its multiplicity. Let $H = \sum_0^\infty h(n)^2$, and define $N = N(\nu)$ to be the largest integer such that*

$$\lambda_N \leqslant \Big(\frac{1}{16 H^2 \nu} \Big)^{1/(1+2\alpha+2\beta)} . \tag{3.3}$$

If

$$\mathcal{E}_\nu \theta_0 < \lambda_N \|\theta_0\|^2 , \tag{3.4}$$

then for $m_0 = \lceil 2H \lambda_N^{\alpha+\beta} \rceil + 1$, we have

$$\|\theta_{m_0}\|^2 \leqslant \exp\Big(-\frac{\nu \lambda_N m_0}{8} \Big) \|\theta_0\|^2 . \tag{3.5}$$

Momentarily postponing the proofs of Lemmas 3.1 and 3.2 we prove Theorem 2.3.

Proof of Theorem 2.3. Choosing $c_0 = \lambda_N$ and repeatedly applying Lemmas 3.1 and 3.2 we obtain an increasing sequence of times n_k such that

$$\|\theta_{n_k}\|^2 \leqslant \exp\Big(-\frac{\nu \lambda_N n_k}{8} \Big) \|\theta_0\|^2 , \qquad \text{and} \qquad n_{k+1} - n_k \leqslant m_0 .$$

This immediately implies

$$\tau_d \leqslant \frac{16}{\nu \lambda_N} + m_0 . \tag{3.6}$$

Note by choice of m_0 and λ_N we have

$$m_0 \leqslant 2H \Big(\frac{1}{16 H^2 \nu} \Big)^{\frac{\alpha+\beta}{2\alpha+2\beta+1}} + 2 ,$$

and

$$\frac{1}{\nu \lambda_N} \geqslant (16 H^2)^{\frac{1}{2\alpha+2\beta+1}} \nu^{-\frac{2\alpha+2\beta}{2\alpha+2\beta+1}} .$$

Thus when ν is sufficiently small, we can ensure $m_0 < 1/(\nu\lambda_N)$. Using this in (3.6) gives

$$\tau_d \leqslant \frac{17}{\nu \lambda_N} . \tag{3.7}$$

To finish the proof, we only need the asymptotic growth of the eigenvalues of the Laplacian. Recall by Weyl's lemma (see for instance [MP49]) we know

$$\lambda_j \approx \frac{4\pi\, \Gamma(\frac{d}{2} + 1)^{2/d}}{\operatorname{vol}(M)^{2/d}} j^{2/d} , \tag{3.8}$$

asymptotically as $j \to \infty$. This implies $\lambda_{j+1} - \lambda_j = o(\lambda_j)$, and hence the eigenvalue λ_N chosen in (3.3) will satisfy

$$\frac{1}{2}\Big(\frac{1}{16H^2\nu}\Big)^{1/(1+2\alpha+2\beta)} \leqslant \lambda_N \leqslant \Big(\frac{1}{16H^2\nu}\Big)^{1/(1+2\alpha+2\beta)},$$

when ν is sufficiently small. Substituting this in (3.7) finishes the proof. $\square$

It remains to prove Lemmas 3.1 and 3.2.

Proof of Lemma 3.1. Note first that (2.1) and (3.1) imply the energy equality

$$\begin{aligned}\|\theta_1\|^2 &= \sum_{i=1}^{\infty} e^{-2\nu\lambda_i} |\langle U\theta_0, e_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle U\theta_0, e_i\rangle|^2 - \nu\mathcal{E}_\nu\theta_0 \\ &= \|\theta_0\|^2 - \nu\mathcal{E}_\nu\theta_0 \,. \end{aligned} \tag{3.9}$$

Now using (3.2) immediately implies

$$\|\theta_1\|^2 \leqslant (1 - c_0\nu)\|\theta_0\|^2 \leqslant e^{-c_0\nu}\|\theta_0\|^2 \,. \tag{3.10}$$

$\square$

In order to prove Lemma 3.2, we first need to estimate the difference between the pulsed diffusion and the underlying dynamical system. We do this as follows.

Lemma 3.3. *Let ϕ_n, defined by*

$$\phi_n = U^n\theta_0 \,,$$

be the evolution of θ_0 under the dynamical system generated by φ. Then for all $n \geqslant 0$ we have

$$\|\theta_n - \phi_n\| \leqslant \sum_{k=0}^{n-1} \sqrt{\nu\mathcal{E}_\nu\theta_k} \,. \tag{3.11}$$

Proof. Since $\phi_n = U\phi_{n-1}$, we have

$$\begin{aligned}\|\theta_n - \phi_n\| &\leqslant \|(e^{\nu\Delta} - 1)U\theta_{n-1}\| + \|U(\theta_{n-1} - \phi_{n-1})\| \\ &= \Big(\sum_{i=1}^{\infty}(e^{-\nu\lambda_i} - 1)^2 |\langle U\theta_{n-1}, e_i\rangle|^2\Big)^{1/2} + \|\theta_{n-1} - \phi_{n-1}\| \\ &\leqslant \Big(\sum_{i=1}^{\infty}(1 - e^{-2\nu\lambda_i}) |\langle U\theta_{n-1}, e_i\rangle|^2\Big)^{1/2} + \|\theta_{n-1} - \phi_{n-1}\| \\ &\leqslant \sqrt{\nu\mathcal{E}_\nu\theta_{n-1}} + \|\theta_{n-1} - \phi_{n-1}\| \,,\end{aligned}$$

and hence (3.11) follows by induction. $\square$

We now prove Lemma 3.2.

Proof of Lemma 3.2. By (3.9), we have

$$\|\theta_{m_0}\|^2 = \|\theta_1\|^2 - \nu \sum_{m=1}^{m_0-1} \mathcal{E}_\nu\theta_m \,. \tag{3.12}$$

Thus the decay of $\|\theta_{m_0}\|$ is governed by the growth of $\sum_{m=1}^{m_0-1} \mathcal{E}_\nu\theta_m$. In order to estimate $\mathcal{E}_\nu\theta_m$ we claim

$$2\|\theta_{m+1}\|_1^2 \leqslant \mathcal{E}_\nu\theta_m \leqslant 2\|U\theta_m\|_1^2 \,, \quad \text{for all } m \in \mathbb{N} \,. \tag{3.13}$$

Indeed, by definition of $\mathcal{E}_\nu$ (equation (3.1)) we have

$$\nu \mathcal{E}_\nu \theta_m = \sum_{k=1}^{\infty} \bigl(1 - e^{-2\nu\lambda_k}\bigr) |(U\theta_m)^\wedge(k)|^2 \,,$$

where $(U\theta_m)^\wedge(k) \stackrel{\text{def}}{=} \langle U\theta_m, e_k\rangle$ is the k-th Fourier coefficient of $U\theta_m$, and e_k is the eigenfunction of the Laplacian corresponding to the eigenvalue λ_k. Now (3.13) follows from the inequalities

$$2\nu\lambda_k e^{-2\nu\lambda_k} \leqslant 1 - e^{-2\nu\lambda_k} \leqslant 2\nu\lambda_k \,.$$

We next claim that for all sufficiently small ν we have

$$\|\theta_1\|_1^2 < \lambda_N \|\theta_1\|^2 \,. \tag{3.14}$$

To see this, note that (3.4) and (3.13) imply

$$\|\theta_1\|_1^2 \leqslant \frac{1}{2} \mathcal{E}_\nu \theta_0 < \frac{\lambda_N}{2} \|\theta_0\|^2 \,. \tag{3.15}$$

Moreover, our choice of λ_N (in equation (3.3)) guarantees $\lambda_N \leqslant 1/(2\nu)$ for all ν sufficiently small. Thus

$$\|\theta_1\|^2 = \|\theta_0\|^2 - \nu \mathcal{E}_\nu \theta_0 \geqslant (1 - \nu\lambda_N)\|\theta_0\|^2 \geqslant \frac{1}{2}\|\theta_0\|^2 \,,$$

and substituting this in equation (3.15) gives (3.14) as claimed.

We now claim that for N and m_0 as in the statement of Lemma 3.2 we have

$$\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant \frac{\lambda_N (m_0 - 1)}{4} \|\theta_1\|^2 \,. \tag{3.16}$$

Note equation (3.16) immediately implies (3.5). Indeed, by (3.12), we have

$$\begin{aligned}
\|\theta_{m_0}\|^2 &\leqslant \Bigl(1 - \frac{\nu\lambda_N(m_0-1)}{4}\Bigr)\|\theta_1\|^2 \\
&\leqslant \exp\Bigl(-\frac{\nu\lambda_N(m_0-1)}{4}\Bigr)\|\theta_1\|^2 \leqslant \exp\Bigl(-\frac{\nu\lambda_N m_0}{8}\Bigr)\|\theta_0\|^2 \,,
\end{aligned}$$

since $m_0/2 \leqslant m_0 - 1$ and $\|\theta_1\| \leqslant \|\theta_0\|$.

Thus it only remains to prove equation (3.16). For this we let ϕ_m, defined by

$$\phi_m = U^{m-1}\theta_1 \,,$$

be the evolution of θ_1 under the dynamical system generated by φ. Let $P_N \colon L_0^2 \to L_0^2$ defined by

$$P_N f = \sum_{k=1}^{N} \hat{f}(k) e_k \,,$$

be the projection operator onto $\operatorname{span}\{e_1, \dots, e_N\}$. Using (3.13) we have

$$\begin{aligned}
\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m &\geqslant 2 \sum_{m=1}^{m_0-1} \|\theta_{m+1}\|_1^2 \\
&\geqslant 2\lambda_N \sum_{m=1}^{m_0-1} \|(I - P_N)\theta_{m+1}\|^2 \\
&\geqslant 2\lambda_N \Bigl(\frac{1}{2} \sum_{m=1}^{m_0-1} \|(I - P_N)\phi_{m+1}\|^2 - \sum_{m=1}^{m_0-1} \|(I - P_N)(\theta_{m+1} - \phi_{m+1})\|^2\Bigr)
\end{aligned}$$

$$\geqslant \lambda_N \Big((m_0 - 1)\|\phi_1\|^2 - \sum_{m=1}^{m_0-1} \|P_N \phi_{m+1}\|^2 - 2\sum_{m=1}^{m_0-1} \|\theta_{m+1} - \phi_{m+1}\|^2 \Big) . \tag{3.17}$$

Now using Lemma 3.3 we estimate the last term on the right of (3.17) by

$$\begin{aligned}\sum_{m=1}^{m_0-1} \|\theta_{m+1} - \phi_{m+1}\|^2 &\leqslant \sum_{m=1}^{m_0-1} \Big(\sum_{l=1}^{m} \sqrt{\nu \mathcal{E}_\nu \theta_l} \Big)^2 \leqslant \sum_{m=1}^{m_0-1} m\nu \sum_{l=1}^{m} \mathcal{E}_\nu \theta_l \\ &\leqslant \frac{m_0(m_0-1)\nu}{2} \sum_{l=1}^{m_0-1} \mathcal{E}_\nu \theta_l \leqslant \frac{m_0^2 \nu}{2} \sum_{l=1}^{m_0-1} \mathcal{E}_\nu \theta_l \,. \end{aligned} \tag{3.18}$$

For the second term on the right of (3.17) we note that since U is strongly α, β mixing with rate function h, we have

$$\|U^m f\|_{-\beta} \leqslant h(m) \|f\|_\alpha \,,$$

for every $f \in \dot{H}^\alpha$ (see also (A.5) in Appendix A). This implies

$$\begin{aligned}\sum_{m=1}^{m_0-1} \|P_N \phi_{m+1}\|^2 &\leqslant \sum_{m=1}^{m_0-1} \lambda_N^\beta \|\phi_{m+1}\|_{-\beta}^2 \leqslant \sum_{m=1}^{m_0-1} \lambda_N^\beta h(m)^2 \|\phi_1\|_\alpha^2 \\ &\leqslant H \lambda_N^\beta \|\phi_1\|_\alpha^2 \leqslant H \lambda_N^\beta \|\theta_1\|^{2-2\alpha} \|\theta_1\|_1^{2\alpha} \leqslant H \lambda_N^{\alpha+\beta} \|\theta_1\|^2 \,, \end{aligned} \tag{3.19}$$

where the last inequality followed from (3.14).

Substituting (3.18) and (3.19) in (3.17) we obtain

$$\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant \frac{(m_0-1)\lambda_N}{1 + \lambda_N \nu m_0^2} \Big(1 - \frac{H \lambda_N^{\alpha+\beta}}{m_0 - 1} \Big) \|\theta_1\|^2 \,.$$

By choice of N and m_0, equation (3.16) follows. This finishes the proof of Lemma 3.2. $\square$

3.2. **The Weakly Mixing Case.**

We now turn our attention to Theorem 2.4. The proof is very similar to the proof of Theorem 2.3, the only difference is that the analogue of Lemma 3.4 is not as explicit.

Lemma 3.4. *Define $N = N(\nu)$ to be the largest integer such that*

$$\lambda_N \Big(h^{-1}\Big(\big(2N \lambda_N^{\alpha+\beta} \big)^{-1/2} \Big) + 1 \Big)^2 \leqslant \frac{1}{\nu} \,. \tag{3.20}$$

If

$$\mathcal{E}_\nu \theta_0 < \lambda_N \|\theta_0\|^2 \,,$$

then for

$$m_0 = h^{-1}\Big(\big(2N \lambda_N^{\alpha+\beta} \big)^{-1/2} \Big) + 1 \,,$$

we have

$$\|\theta_{m_0}\|^2 \leqslant \exp\Big(-\frac{\nu \lambda_N m_0}{8} \Big) \|\theta_0\|^2 \,.$$

Given Lemma 3.4, the proof of Theorem 2.4 is essentially the same as the proof of Theorem 2.3.

Proof of Theorem 2.4. Choosing $c_0 = \lambda_N$ and repeatedly applying Lemmas 3.1 and 3.4 we obtain an increasing sequence of times n_k such that

$$\|\theta_{n_k}\|^2 \leqslant \exp\Big(-\frac{\nu\lambda_N n_k}{8}\Big)\|\theta_0\|^2\,, \qquad \text{and} \qquad n_{k+1}-n_k \leqslant m_0\,.$$

This immediately implies

$$\tau_d \leqslant \frac{16}{\nu\lambda_N} + m_0\,. \tag{3.21}$$

By the choice of m_0 and N, we notice that

$$m_0 \leqslant \frac{1}{\sqrt{\nu\lambda_N}} + 1 \leqslant \frac{1}{\nu\lambda_N}\,.$$

The last inequality follows because (3.20) implies $\nu\lambda_N \leqslant 1/4$. This proves (2.5).

In the case when h is given by (2.6), observe

$$h^{-1}(x) = \Big\lceil \Big(\frac{c_1}{x}\Big)^{1/s}\Big\rceil\,.$$

Substituting this in (3.20) gives

$$\lambda_N\Big(c_1^{\frac{1}{s}}(2N)^{\frac{1}{2s}}\lambda_N^{\frac{\alpha+\beta}{2s}} + 2\Big)^2 \leqslant \frac{1}{\nu}\,.$$

Using Weyl's law (3.8), this gives

$$N \approx \nu^{-\frac{ds}{d+2s+2\beta+2\alpha}}\,, \qquad \lambda_N \approx \nu^{-\frac{2s}{d+2s+2\beta+2\alpha}}\,.$$

This proves (2.7) as desired. □

It remains to prove Lemma 3.4.

Proof of Lemma 3.4. We first claim that (3.16) still holds if λ_N, m_0 chosen as in the statement of Lemma 3.4. Once (3.16) is established, then the remainder of the proof is identical to that of Lemma 3.2.

To prove (3.16), we observe that the lower bound (3.17) (from the proof of Lemma 3.2) still holds in this case. For last term on the right of (3.17), we use the bound (3.18). The only difference here is to estimate the second term using the weak mixing assumption (2.4) instead. Observe

$$\frac{1}{m_0-1}\sum_{m=1}^{m_0-1}\|P_N\phi_{m+1}\|^2 = \sum_{l=1}^{N}\frac{1}{m_0-1}\sum_{m=1}^{m_0-1}|\langle e_l, U^m\theta_1\rangle|^2\,.$$

Since φ is weak α,β-mixing with rate function h, the assumption (2.4) yields

$$\begin{aligned}\frac{1}{m_0-1}\sum_{m=1}^{m_0-1}|\langle e_l, U^m\theta_1\rangle|^2 &\leqslant h(m_0-1)^2\|U\theta_1\|_\alpha^2\lambda_l^\beta \leqslant h(m_0-1)^2\lambda_N^\beta\|\theta_1\|_\alpha^2\\ &\leqslant h(m_0-1)^2\lambda_N^\beta\|\theta_1\|^{2-2\alpha}\|\theta_1\|_1^{2\alpha} \leqslant h(m_0-1)^2\lambda_N^{\beta+\alpha}\|\theta_1\|^2\,.\end{aligned}$$

Together with (3.14) this gives

$$\frac{1}{m_0-1}\sum_{m=1}^{m_0-1}\|P_N\phi_{m+1}\|^2 \leqslant h(m_0-1)^2 N\lambda_N^{\beta+\alpha}\|\theta_1\|^2\,.$$

Substituting this and (3.18) in (3.17) gives

$$\sum_{m=1}^{m_0-1} \mathcal{E}_\nu \theta_m \geqslant \frac{\lambda_N (m_0-1)}{1+m_0^2 \nu \lambda_N} \Big(1 - h(m_0-1)^2 N \lambda_N^{\beta+\alpha} \Big) \|\theta_1\|^2 \,.$$

By the choice of N and m_0, this yields (3.15) as claimed. □

4. Toral Automorphisms and the Energy Decay of Pulsed Diffusions.

In this section we study pulsed diffusions where the underlying map φ is a toral automorphism, and prove Theorem 2.7. Recall a *toral automorphism* is a map of the form

$$\varphi(x) = Ax \pmod{\mathbb{Z}^d}\,, \tag{4.1}$$

where $A \in SL_d(\mathbb{Z})$ is an integer valued $d \times d$ matrix with determinant 1. Maps of this form are known as *"cat maps"*, and one particular example is when $d = 2$ and

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

The reason for the somewhat unusual name is that originally "CAT" was an abbreviation for Continuous Automorphism of the Torus. However, it has now become tradition to demonstrate the mixing effects of this map using the image of a cat [SOW06].

4.1. Mixing Rates of Toral Automorphisms.

It is well known that no eigenvalue of A is a root of unity, if and only if φ is ergodic, if and only if φ is strongly mixing (see [Kat71], Page 160, problem 4.2.11 in [KH95], or Proposition 3 in [FW03]). Our interest is in understanding the mixing rates in the sense of Definition 2.2.

Proposition 4.1. *Let $A \in SL_d(\mathbb{Z})$ be such that:*

(C1) No eigenvalue of A is a root of unity,

(C2) and the characteristic polynomial of A is irreducible over $\mathbb{Q}$.

If $\alpha, \beta > 0$ then the toral automorphism $\varphi \colon \mathbb{T}^d \to \mathbb{T}^d$ defined by (4.1) is strongly α, β mixing with rate function

$$h(n) = C_{\alpha,\beta} \exp\Big(-\frac{n}{C_0} \Big(\alpha \wedge \frac{\beta}{d-1} \Big) \Big)\,, \tag{4.2}$$

for some finite non-zero constants $C_{\alpha,\beta} = C_{\alpha,\beta}(A, \alpha, \beta)$ and $C_0 = C_0(A)$.

For completeness, we also mention that if A satisfies Condition (C1) above, then A is also weakly α, β if either $\alpha = 0$ or $\beta = 0$ (but not both).

Proposition 4.2. *Let $A \in SL_d(\mathbb{Z})$ satisfy the condition (C1) in Proposition 4.1.*

(1) If either $\alpha > 0$ and $\beta = 0$, or $\alpha = 0$ and $\beta > 0$, then there exists a finite constant $C_{\alpha,\beta} = C(\alpha, \beta)$ such that φ is weakly α, β mixing with rate function

$$h(n) = \begin{cases} \dfrac{C_{\alpha,\beta}}{\sqrt{n}}\,, & \alpha \vee \beta > \dfrac{d}{2}\,, \\[2ex] C_{\alpha,\beta} \Big(\dfrac{\ln n}{n} \Big)^{1/2}, & \alpha \vee \beta = \dfrac{d}{2}\,, \\[2ex] \dfrac{C_{\alpha,\beta}}{n^{(\alpha\vee\beta)/d}}\,, & \alpha \vee \beta < \dfrac{d}{2}\,. \end{cases} \tag{4.3}$$

(2) *If further A satisfies condition (C2) in Proposition 4.2, and both $\alpha > 0$ and $\beta > 0$, then there exists a finite constant $C_{\alpha,\beta} = C(A, \alpha, \beta)$ such that φ is weakly α, β mixing with rate function*

$$h(n) = \frac{C_{\alpha,\beta}}{\sqrt{n}}\,. \tag{4.4}$$

Remark 4.3. Condition (C2) above is equivalent to assuming that A has no proper invariant subspaces in $\mathbb{Q}^d$.

When $d = 2$, Proposition 4.1 is well known and can be proved elementarily. In higher dimensions, Proposition 4.1 can be deduced from the results on the algebraic structure of toral automorphisms developed in [FW03]. These arguments, however, rely on three sophisticated results from number theory: the Schmidt subspace theorem [Sch80], Minkowski's theorem on linear forms [New72, Chapter VI] and van der Waerdern's theorem on arithmetic progressions [vdW27, Luk48]. We will avoid using these results, and instead instead prove Proposition 4.1 directly using the following two algebraic lemmas.

Lemma 4.4. *Suppose $A \in SL_d(\mathbb{Z})$ satisfies the assumptions (C1) and (C2) in Proposition 4.1. There exists a basis $\{v_1, \dots, v_d\}$ of $\mathbb{C}^d$ such that the following hold:*

(1) *Each v_i is an eigenvector of A.*
(2) *If $k \in \mathbb{Z}^d - 0$, and $a_i = a_i(k) \in \mathbb{C}$ are such that*

$$k = \sum_1^d a_i(k) v_i = \sum_1^d a_i v_i\,,$$

then we must have

$$\prod_{i=1}^d |a_i(k)| \geqslant 1\,. \tag{4.5}$$

Lemma 4.5 (Kronecker [Kro57])**.** *Let p be a monic polynomial with integer coefficients that is irreducible over $\mathbb{Q}$. If all the roots of p are contained in the unit disk, they must be roots of unity.*

The proofs of Lemma 4.4 and 4.5 use elementary facts about algebraic number fields, and to avoid breaking continuity, we defer the proofs to Section 4.3. These lemmas will also be used to prove sharpness of the double exponential bound (2.8) in Theorem 2.7. The reason these lemmas arise here is as follows. Lemma 4.5 will guarantee that guarantee $(A^T)^{-1}$ has at least one eigenvalue, λ_1, strictly outside the unit disk. Lemma 4.4 now guarantees that all non-zero Fourier frequencies have a certain minimum component in the eigenspace of λ_1. This will of course dominate the long time behaviour, leading to exponential mixing of φ and rapid energy dissipation of the associated pulsed diffusion.

Proof of Proposition 4.1. Let $B = (A^T)^{-1}$, and $f \in L^2_0$. Observe

$$(Uf)^\wedge(k) = \int_{\mathbb{T}^d} e^{-2\pi i k\cdot x} f(Ax)\,dx = \int_{\mathbb{T}^d} e^{-2\pi i (Bk)\cdot x} f(x)\,dx = \hat{f}(Bk)\,,$$

and hence

$$(U^n f)^\wedge(k) = \hat{f}(B^n k)\,, \tag{4.6}$$

for all $n \geqslant 0$. Now to prove that φ is exponentially mixing, let $f \in \dot{H}^\alpha$, and $g \in \dot{H}^\beta$. Using (4.6) we have

$$\langle U^n f, g\rangle = \sum_{k \in \mathbb{Z}^d - 0} \hat{f}(B^n k)\overline{\hat{g}(k)} = \sum_{k \in \mathbb{Z}^d - 0} \frac{1}{|B^n k|^\alpha |k|^\beta} |B^n k|^\alpha \hat{f}(B^n k) |k|^\beta \overline{\hat{g}(k)}$$

Consequently

$$|U^n f, g| \leqslant \Big(\sup_{k \in \mathbb{Z}^d - 0} \frac{1}{|B^n k|^\alpha |k|^\beta} \Big) \|f\|_\alpha \|g\|_\beta \tag{4.7}$$

We now estimate the pre-factor on the right of (4.7) using Lemmas 4.4 and 4.5. First note that $B \in SL_d(\mathbb{Z})$ also satisfies the assumptions (C1) and (C2). Let v_1, …, v_d be the basis given by Lemma 4.4, and λ_1, …, λ_d be the corresponding eigenvalues. Since the characteristic polynomial of B satisfies the conditions of Lemma 4.5, we see that B has at least one eigenvalue outside the unit disk. Without loss of generality we suppose $|\lambda_1| > 1$.

By equivalence of norms on finite dimensional spaces, we know there exists $c_* > 0$ such that

$$\frac{1}{c_*} |k'| \leqslant \Big(\sum |a_i(k')|^2 \Big)^{1/2} \leqslant c_* |k'| \,, \quad \text{for all } k' \in \mathbb{Z}^d \,. \tag{4.8}$$

Using Lemma 4.4, we note

$$|B^n k| = \Big| \sum a_i \lambda_i^n v_i \Big| \geqslant \frac{|a_1||\lambda_1|^n}{c_*} \geqslant \frac{|\lambda_1|^n}{c_* |a_2| \cdots |a_d|} \geqslant \frac{|\lambda_1|^n}{c_*^d |k|^{d-1}} \,.$$

Thus

$$\sup_{k \in \mathbb{Z}^d - 0} \frac{1}{|B^n k|^\alpha |k|^\beta} \leqslant |\lambda_1|^{-n\alpha} \Big(\sup_{k \in \mathbb{Z}^d - 0} \frac{c_*^{d\alpha}}{|k|^{\beta - (d-1)\alpha}} \Big) \,.$$

If $(d-1)\alpha \leqslant \beta$, (4.7) and the above shows that φ is strongly α, β mixing with rate function $h(n) = C|\lambda_1|^{-n\alpha}$. This proves (4.2) in the case $(d-1)\alpha \leqslant \beta$.

On the other hand, if $(d-1)\alpha > \beta$, we let $\alpha' = \beta/(d-1)$. By the previous argument we know φ is α', β mixing with rate function $h(n) = C|\lambda_1|^{-n\alpha'}$. Since $\alpha > \alpha'$, $\|f\|_{\alpha'} \leqslant \|f\|_\alpha$ and it immediately follows that φ is also α, β mixing with the same rate function. This proves (4.2) when $(d-1)\alpha > \beta$ completing the proof. $\square$

Proof of Proposition 4.2. The second assertion follows immediately from Proposition 4.1. Indeed, when both $\alpha, \beta > 0$, Proposition 4.1 implies φ is strongly α, β mixing with rate function h given by (4.2). Since the rate function decays exponentially, it is square summable and equation (4.4) holds with $C_{\alpha,\beta} = (\sum_{i=1}^\infty h(i)^2)^{1/2}$.

To prove the first assertion, suppose first $\alpha = 0$ and $\beta > 0$. As before set $B = (A^T)^{-1}$, and let $f, g \in L^2_0$ and observe

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^{n-1} |\langle U^i f, g\rangle|^2 &= \frac{1}{n} \sum_{i=0}^{n-1} \Big| \sum_{k \in \mathbb{Z}^d - 0} \hat{f}(B^i k) \overline{\hat{g}(k)} \Big|^2 \\ &\leqslant \frac{\|g\|_\beta^2}{n} \sum_{i=0}^{n-1} \sum_{k \in \mathbb{Z}^d - 0} \frac{|\hat{f}(B^i k)|^2}{|k|^{2\beta}} \,. \end{aligned} \tag{4.9}$$

We now split the analysis into cases. First suppose $\beta > d/2$. By Kronecker's theorem (Lemma 4.5) we see that the matrix B can not have finite order, and hence

k, Bk, B^2k, …, $B^{n-1}k$ are all distinct. Thus (4.9) implies

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \frac{\|g\|_\beta^2}{n}\sum_{k\in\mathbb{Z}^d-0}\sum_{i=0}^{n-1}\frac{|\hat{f}(B^ik)|^2}{|k|^{2\beta}} \leqslant \frac{\|g\|_\beta^2}{n}\sum_{k\in\mathbb{Z}^d-0}\frac{\|f\|^2}{|k|^{2\beta}}\,.$$

Since $\beta > d/2$, the sum on the right is finite, showing φ is 0, β mixing with rate function $C/n^{1/2}$ as desired.

Suppose now $\beta < d/2$. Let $m \in \mathbb{N}$ be a large integer that will be chosen shortly, and split the above sum as

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \frac{\|g\|_\beta^2}{n}\Big(\sum_{0<|k|\leqslant m}\sum_{i=0}^{n-1}\frac{|\hat{f}(B^ik)|^2}{|k|^{2\beta}} + \sum_{i=0}^{n-1}\sum_{|k|>m}\frac{|\hat{f}(B^ik)|^2}{|k|^{2\beta}}\Big) \tag{4.10}$$

$$\leqslant \|f\|^2\|g\|_\beta^2\Big[\Big(\frac{1}{n}\sum_{0<|k|\leqslant m}\frac{1}{|k|^{2\beta}}\Big) + \frac{1}{m^{2\beta}}\Big] \tag{4.11}$$

$$\leqslant \|f\|^2\|g\|_\beta^2\Big(\frac{Cm^{d-2\beta}}{n} + \frac{1}{m^{2\beta}}\Big)\,, \tag{4.12}$$

for some (explicit) constant $C = C(d)$, independent of n. (Note, we again used the fact that k, B^k, …, are all distinct when computing the first sum on the right of (4.10) to obtain (4.11).) We now choose $m = Cn^{1/d}$ in order to minimize the right hand side. This implies

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \frac{C\|f\|^2\|g\|_\beta^2}{n^{2/d}}$$

proving (4.3) when $\beta < d/2$.

Finally, when $\beta = d/2$ we repeat the same argument above to obtain (4.11). When summed (4.11) now yields

$$\frac{1}{n}\sum_{i=0}^{n-1}|\langle U^i f, g\rangle|^2 \leqslant \|f\|^2\|g\|_\beta^2\Big(\frac{C\ln m}{n} + \frac{1}{m^d}\Big)\,, \tag{4.13}$$

and choosing $m = n$ yields (4.3) as desired. This finishes the proof. □

4.2. **Energy Decay, and the proof of Theorem 2.7.** We now turn our attention to studying the energy decay of pulsed diffusions. Our first result shows that if a toral automorphism satisfies conditions (C1) and (C2) in Proposition 4.1, then the energy of the associated pulsed diffusion decays double exponentially. This will prove sharpness of the lower bound (2.8) in Theorem 2.7. Following this we will prove lower bound (2.8) itself using a convexity argument.

Proposition 4.6. *Suppose $A \in SL_d(\mathbb{Z})$ satisfies the assumptions (C1) and (C2) in Proposition 4.1. Let φ be the associated toral automorphism defined in (4.1), and θ_n be the pulsed diffusion defined by (2.1). Then there exists a constants $c > 0$ and $\gamma > 1$ such that*

$$\|\theta_n\| \leqslant \exp\Big(-\frac{\nu\gamma^n}{c}\Big) \tag{4.14}$$

Proof. Using (4.6) we see

$$\hat{\theta}_{n+1}(k) = e^{-\nu|k|^2}\hat{\theta}_n(Bk)\,.$$

Setting $A_* = A^T$, iterating the above, squaring and summing in k gives

$$\|\theta_n\|^2 = \sum_{k \in \mathbb{Z}^d - 0} \exp\Big(-2\nu \sum_{j=1}^{n} |A_*^j k|^2\Big) |\hat{\theta}_0(k)|^2 \,. \tag{4.15}$$

Observe that the matrix A_* also satisfies the conditions (C1) and (C2) in Proposition 4.1. Let $v_1, \ldots, v_d$ be the basis of $\mathbb{C}^d$ given by Lemma 4.4, and $\lambda_1, \ldots, \lambda_d$ be the corresponding eigenvalues. Now (4.15) implies

$$\begin{aligned}
\|\theta_n\|^2 &\leqslant \sum_{k \in \mathbb{Z}^d - 0} \exp\Big(-\frac{2\nu}{c_*} \sum_{j=1}^{n} \sum_{i=1}^{d} |a_i|^2 |\lambda_i|^{2j}\Big) |\hat{\theta}_0(k)|^2 \\
&= \sum_{k \in \mathbb{Z}^d - 0} \exp\Big(-\frac{2\nu}{c_*} \sum_{i=1}^{d} |a_i|^2 \Big(\frac{|\lambda_i|^{2(n+1)} - |\lambda_i|^2}{|\lambda_i|^2 - 1}\Big)\Big) |\hat{\theta}_0(k)|^2 \\
&\leqslant \|\theta_0\|^2 \sup_{k \in \mathbb{Z}^d - 0} \exp\Big(-\frac{2\nu}{c_*} \sum_{i=1}^{d} |a_i|^2 \Big(\frac{|\lambda_i|^{2(n+1)} - |\lambda_i|^2}{|\lambda_i|^2 - 1}\Big)\Big) \,. \qquad (4.16)
\end{aligned}$$

where c_* is the constant in (4.8).

We will now show that the last term decays double exponentially in n. Indeed, the inequality of the means implies

$$\begin{aligned}
\sum_{i=1}^{d} |a_i|^2 \Big(\frac{|\lambda_i|^{2(n+1)} - |\lambda_i|^2}{|\lambda_i|^2 - 1}\Big) &\geqslant d\Big(\prod_{i=1}^{d} |a_i|^2 \Big(\frac{|\lambda_i|^{2(n+1)} - |\lambda_i|^2}{|\lambda_i|^2 - 1}\Big)\Big)^{1/d} \\
&= d\Big(\prod_{i=1}^{d} |a_i|^2\Big)^{1/d} \Big(\prod_{i=1}^{d} \Big(\frac{|\lambda_i|^{2(n+1)} - |\lambda_i|^2}{|\lambda_i|^2 - 1}\Big)\Big)^{1/d} \\
&\geqslant d\Big(\prod_{i=1}^{d} \Big(\frac{|\lambda_i|^{2(n+1)} - |\lambda_i|^2}{|\lambda_i|^2 - 1}\Big)\Big)^{1/d} \,, \qquad (4.17)
\end{aligned}$$

where the last inequality followed from Lemma 4.4. As in the proof of Proposition 4.1, Lemma 4.5 guarantees that $\max_i |\lambda_i| > 1$. The right hand side of (4.17) is of order $(\max_i |\lambda_i|)^{2n/d}$ and substituting this in (4.16) gives (4.14) as desired. □

We now prove Theorem 2.7.

Proof. Proposition 4.6 immediately shows that the double exponential upper bound equation (2.9) is achieved for the desired class of toral automorphisms. Thus it only remains to prove the double exponential lower bound (2.8). For this, observe

$$\ln\|\theta_{n+1}\|^2 - \ln\|\theta_n\|^2 = \ln\Big(\frac{\|\theta_{n+1}\|^2}{\|\theta_n\|^2}\Big) = \ln\Big(\frac{\|\theta_{n+1}\|^2}{\|U\theta_n\|^2}\Big) = \ln\Big(\frac{\sum_i e^{-2\nu\lambda_i} |\langle U\theta_n, e_i\rangle|^2}{\sum_i |\langle U\theta_n, e_i\rangle|^2}\Big) \,,$$

where we recall that λ_i are the eigenvalues of the Laplacian, and e_i's are the corresponding eigenfunctions. Using concavity of the logarithm and Jensen's inequality to bound the last term on the right we obtain

$$\begin{aligned}
\ln\|\theta_{n+1}\|^2 - \ln\|\theta_n\|^2 &\geqslant \frac{-2\nu \sum_i \lambda_i |\langle U\theta_n, e_i\rangle|^2}{\sum_i |\langle U\theta_n, e_i\rangle|^2} = -2\nu \frac{\|U\theta_n\|_1^2}{\|U\theta_n\|^2} \\
&\geqslant -2\nu \|\nabla\varphi\|_{L^\infty}^2 \frac{\|\theta_n\|_1^2}{\|\theta_n\|^2} \,. \qquad (4.18)
\end{aligned}$$

We now claim

$$\frac{\|\theta_n\|_1^2}{\|\theta_n\|^2} \leqslant \|\nabla\varphi\|_{L^\infty}^{2n} \frac{\|\theta_0\|_1^2}{\|\theta_0\|^2}. \tag{4.19}$$

Note that substituting (4.19) in (4.18) and summing in n immediately implies (2.8). Thus to finish the proof we only need to prove (4.19).

For this we observe

$$\begin{aligned}
\frac{\|\theta_{n+1}\|_1^2}{\|\theta_{n+1}\|^2} - \frac{\|U\theta_n\|_1^2}{\|U\theta_n\|^2} &= \frac{\|\theta_{n+1}\|_1^2\|U\theta_n\|^2 - \|\theta_{n+1}\|^2\|U\theta_n\|_1^2}{\|\theta_n\|^2\|U\theta_n\|^2} \\
&= \frac{1}{\|\theta_n\|^2\|U\theta_n\|^2}\Big(\sum_{i,j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2\Big) \\
&= \frac{1}{\|\theta_n\|^2\|U\theta_n\|^2}\Big(\sum_{i<j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2 \\
&\qquad + \sum_{i>j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2\Big) \\
&\leqslant \frac{1}{\|\theta_n\|^2\|U\theta_n\|^2}\Big(\sum_{i<j} e^{-2\nu\lambda_i}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2 \\
&\qquad + \sum_{i>j} e^{-2\nu\lambda_j}(\lambda_i-\lambda_j)|\langle U\theta_n, e_i\rangle|^2|\langle U\theta_n, e_j\rangle|^2\Big) \\
&= 0\,.
\end{aligned}$$

Thus

$$\frac{\|\theta_{n+1}\|_1^2}{\|\theta_{n+1}\|^2} \leqslant \frac{\|U\theta_n\|_1^2}{\|U\theta_n\|^2} = \frac{\|U\theta_n\|_1^2}{\|\theta_n\|^2} \leqslant \|\nabla\varphi\|_{L^\infty}^2 \frac{\|\theta_n\|_1^2}{\|\theta_n\|^2},$$

and iterating yields (4.19). This finishes the proof. □

4.3. **Diophantine Approximation and Kronecker's Theorem.** We now prove Lemmas 4.4 and 4.5. The proofs rely on standard facts on algebraic number fields, and we refer the reader to the books [Mar77] and [Rib01] for a comprehensive treatment.

Before beginning the proof, we remark that Lemma 4.4 is a weaker version of the celebrated Schmidt subspace theorem [Sch80]. In this case the Schmidt subspace theorem would guarantee that for any $\varepsilon > 0$ we have

$$\Big|\prod_{i=1}^d a_i(k)\Big| \geqslant \frac{1}{|k|^\varepsilon},$$

at all integer points $k \in \mathbb{Z}^d$, *except* on finitely many proper rational subspaces. To use the Schmidt subspace theorem in our context we would need to handle the exceptional subspaces. The approach taken by Fannjiang et. al. in [FW03] is to use van der Waerdern's theorem on arithmetic progressions [vdW27, Luk48] to construct an equivalent minimization problem whose minimizer is guaranteed to lie outside the exceptional subspaces. Our approach is to avoid the Schmidt subspace theorem entirely by making do with the weaker bound (4.5).

Proof of Lemma 4.4. Let p be the characteristic polynomial of A, and $\lambda_1, \ldots, \lambda_d$ be the roots of p. Let $F = \mathbb{Q}(\lambda_1, \ldots, \lambda_d)$ and $\mathcal{G} = \mathrm{Gal}(F/\mathbb{Q})$ denote the Galois group. Let $G_i \subseteq \mathcal{G}$ be the group of field automorphisms that fix λ_i, and $F_i = \{x \in F \mid \sigma(x) = x \ \forall \sigma \in G_i\}$ be the fixed field of G_i. Since $\det(A - \lambda_i I) = 0$, there must exist

v_i in the F_i vector space F_i^d such that $Av_i = \lambda_i v_i$. Viewing v_i as an element of $\mathbb{C}^d$, we let $V \in GL_d(\mathbb{C})$ be the matrix with columns $v_1, \ldots, v_d$. Dividing each v_i by a large integer if necessary, we may assume that each entry of V^{-1} is an algebraic integer. We claim that $v_1, \ldots, v_d$ is the desired basis.

To see this suppose $k = \sum a_i v_i$. By construction of the basis note that if $\sigma \in \mathcal{G}$ is such that $\sigma(\lambda_i) = \lambda_j$, then $\sigma(v_i) = v_j$. This implies that $\sigma(a_i) = a_j$. Note also that since the groups G_i are conjugate, they all have the same cardinality. Consequently

$$p_* \stackrel{\text{def}}{=} \prod_{\sigma \in \mathcal{G}} \sigma(a_1) = \Big(\prod_{\sigma \in \mathcal{G}} a_i \Big)^m ,$$

where $m = |G_1|$. Thus p_* is in the fixed field of $\mathcal{G}$, and hence must be rational.

Further, since $a_i = (V^{-1}k) \cdot e_i$, each a_i must also be an algebraic integer. This forces p_* to be a rational algebraic integer, and hence an integer. By transitivity of the Galois group we see that if $a_i = 0$ for some i, then we must have $a_j = 0$ for all j. Thus p_* must be a non-zero integer if $k \neq 0$. Hence $|p_*| \geqslant 1$ and (4.5) follows. $\square$

Lemma 4.5 is due to Kronecker [Kro57]. This result was improved by Stewart [Ste78] and Dobrowolski [Dob79]. More generally Lehmer's conjecture [Leh33] asserts that if $\lambda_1, \ldots, \lambda_d$ are the roots of p and the product $\prod(1 \vee |\lambda_i|)$ is smaller than an absolute constant μ (widely believed to be approximately $1.176\ldots$), then each λ_i is a root of unity. For our purposes, however, Kronecker's original result will suffice. Since the proof is short and elementary, we present it below.

Proof of Lemma 4.5. Let $\lambda_1, \ldots, \lambda_d$ be the roots of p. For any $n \in \mathbb{N}$, let p_n be the minimal monic polynomial satisfied by λ_1^n. Since the Galois conjugates of λ_1^n are precisely $\lambda_2^n, \ldots, \lambda_d^n$, the coefficients of p_n are symmetric functions of $\lambda_1^n, \ldots, \lambda_d^n$. By assumption $|\lambda_i| \leqslant 1$, which implies $|\lambda_i^n| \leqslant 1$, which in turn implies that the coefficients of p_n are uniformly bounded as functions of n. There are only finitely many polynomials with degree at most d, and uniformly bounded integer coefficients. Thus for some distinct $m, n \in \mathbb{N}$ we must have $p_m = p_n$. This forces $\lambda_1^m = \lambda_1^n$ showing λ_1 is a root of unity. $\square$

5. **Dissipation Enhancement for the advection diffusion equation.**

We now prove Theorems 2.9 and 2.10, bounding the dissipation time in the continuous time setting. The main idea is similar to the discrete time case. However, in the continuous time setting the approximation of the diffusive system by the underlying dynamical system is not as good as in the discrete time setting. This is the reason why the estimates in Theorems 2.9 and 2.10 are not as strong as those in Theorems 2.3 and 2.4.

5.1. **The Strongly Mixing Case.** As in Section 2.2, let $\theta_{s,0} \in L_0^2(M)$, let $\theta_s(t)$ be the solution of (2.11). By the energy inequality (2.12) we know

$$\|\theta_s(t)\|^2 = \|\theta_s(s)\|^2 \exp\Big(-2\nu \int_s^t \frac{\|\theta_s(r)\|_1^2}{\|\theta_s(r)\|^2} \, dr \Big) .$$

Thus, $\|\theta_s(t)\|$ decays rapidly when the ratio $\|\theta_s(t)\|_1 / \|\theta_s(t)\|$ remains large. Precisely, if for some $c_0 > 0$, we have

$$\|\theta_s(t)\|_1^2 \geqslant c_0 \|\theta_s(t)\|^2 , \qquad \text{for all } s \leqslant t \leqslant t_0 ,$$

then

$$\|\theta_s(t)\|^2 \leqslant e^{-2\nu c_0(t-s)}\|\theta_{s,0}\|^2\,, \qquad \text{for all } s \leqslant t \leqslant t_0\,. \tag{5.1}$$

As in the proof of Theorems 2.3 and 2.4, we will show that if the ratio $\|\theta_{s,0}\|_1/\|\theta_{s,0}\|$ is small, then the mixing properties of u will guarantee that for some later time $t_0 > s$, $\|\theta_s(t_0)\|$ becomes sufficiently small. This is the content of the following lemma.

Lemma 5.1. *If λ_N is chosen according to (2.16) and*

$$\|\theta_{s,0}\|_1^2 < \lambda_N\|\theta_{s,0}\|^2\,, \tag{5.2}$$

then for $t_0 = 4H\lambda_N^{\alpha+\beta} + s$, we have

$$\|\theta_s(t_0)\|^2 \leqslant \exp\Big(-\frac{\nu\lambda_N(t_0-s)}{2}\Big)\|\theta_{s,0}\|^2\,. \tag{5.3}$$

Momentarily postponing the proof of Lemma 5.1, we prove Theorem 2.9.

Proof of Theorem 2.9. Choosing $c_0 = \lambda_N$ and repeatedly applying the inequality (5.1) and Lemma 5.1, we obtain an increasing sequence of times (t'_k), such that

$$\|\theta_s(t'_k)\|^2 \leqslant \exp\Big(-\frac{\nu\lambda_N(t'_k-s)}{2}\Big)\|\theta_{s,0}\|^2\,, \text{ and } t'_{k+1}-t'_k \leqslant t_0\,.$$

This immediately implies

$$\tau_d \leqslant \frac{4}{\nu\lambda_N} + (t_0 - s)\,. \tag{5.4}$$

By choice of λ_N and t_0, we know that $t_0 - s \leqslant 1/(\nu\lambda_N)$ for ν sufficiently small. This implies (2.17) as desired. ◻

It remains to prove Lemma 5.1. For this we will need a standard result estimating the difference between θ and solutions to the inviscid transport equation.

Lemma 5.2. *Let ϕ_s, defined by*

$$\phi_s = \theta_{s,0}\circ\varphi_{s,t}\,,$$

be the evolution of $\theta_{s,0}$ under the dynamical system generated by $\varphi_{s,t}$. If $\theta_{s,0} \in \dot H^1(M)$, then for all $t \geqslant s$, we have

$$\|\theta_s(t)-\phi_s(t)\|^2 \leqslant \frac{\nu}{2\|\nabla u\|_{L^\infty}}\exp\big(2\|\nabla u\|_{L^\infty}(t-s)\big)\,\|\theta_{s,0}\|_1^2\,. \tag{5.5}$$

Proof. Let $w(t) = \theta_s(t) - \phi_s(t)$. Note $w(s) = 0$, and for $t \geqslant s$ we have

$$\partial_t w + u\cdot\nabla w - \nu\Delta w = \nu\Delta\phi_s\,.$$

Multiplying both sides by w and integrating over M gives

$$\frac{1}{2}\partial_t\|w\|^2 + \nu\|w\|_1^2 = \nu\int_M w\Delta\phi_s\,dx \leqslant \frac{\nu}{2}\|w\|_1^2 + \frac{\nu}{2}\|\phi_s\|_1^2\,,$$

and hence

$$\partial_t\|w\|^2 \leqslant \nu\|\phi_s\|_1^2\,. \tag{5.6}$$

Since $\phi_s(t) = \theta_{s,0}\circ\varphi_{s,t}$ we know

$$\|\phi_s(t)\|_1 \leqslant \exp\big(\|\nabla u\|_{L^\infty}(t-s)\big)\|\theta_{s,0}\|_1\,.$$

Substituting this into (5.6) and integrating in time yields (5.5) as claimed. ◻

We can now prove Lemma 5.1.

Proof of Lemma 5.1. Integrating the energy equality (2.12) gives

$$\|\theta_s(t_0)\|^2 = \|\theta_{s,0}\|^2 - 2\nu \int_s^{t_0} \|\theta_s(r)\|_1^2 \, dr \,. \tag{5.7}$$

We claim that our choice of λ_N and t_0 will guarantee

$$\int_s^{t_0} \|\theta_s(r)\|_1^2 \, dr \geqslant \frac{\lambda_N (t_0 - s) \|\theta_{s,0}\|^2}{4} \,. \tag{5.8}$$

This immediately yields (5.3), and so to finish the proof we only have to prove (5.8).

Note first

$$\begin{aligned} \int_s^{t_0} \|\theta_s(r)\|_1^2 \, dr &\geqslant \lambda_N \int_s^{t_0} \|(I - P_N)\theta_s(r)\|^2 \, dr \\ &\geqslant \frac{\lambda_N}{2} \int_s^{t_0} \|(I - P_N)\phi_s(r)\|^2 \, dr \\ &\qquad - \lambda_N \int_s^{t_0} \|(I - P_N)\big(\theta_s(r) - \phi_s(r)\big)\|^2 \, dr \\ &\geqslant \frac{\lambda_N (t_0 - s)}{2} \|\theta_{s,0}\|^2 - \frac{\lambda_N}{2} \int_s^{t_0} \|P_N \phi_s(r)\|^2 \, dr \\ &\qquad - \lambda_N \int_s^{t_0} \|\theta_s(r) - \phi_s(r)\|^2 \, dr \,. \end{aligned} \tag{5.9}$$

We will now bound the last two terms in (5.9). For the second term, note the strong mixing assumption (2.14) gives

$$\begin{aligned} \int_s^{t_0} \|P_N \phi_s(r)\|^2 \, dr &\leqslant \lambda_N^\beta \int_s^{t_0} \|\phi_s(r)\|_{-\beta}^2 \, dr \leqslant \lambda_N^\beta \int_s^{t_0} h(r-s)^2 \|\theta_{s,0}\|_\alpha^2 \, dr \\ &\leqslant H \lambda_N^\beta \|\theta_{s,0}\|_\alpha^2 \leqslant H \lambda_N^\beta \|\theta_{s,0}\|^{2-2\alpha} \|\theta_{s,0}\|_1^{2\alpha} \,. \end{aligned} \tag{5.10}$$

Using the assumption (5.2), we obtain

$$\int_s^{t_0} \|P_N \phi_s(r)\|^2 \, dr \leqslant H \lambda_N^{\beta+\alpha} \|\theta_{s,0}\|^2 \,. \tag{5.11}$$

Now we bound the last term in (5.9). Using Lemma 5.2 we obtain

$$\begin{aligned} \int_s^{t_0} \|\theta_s(r) - \phi_s(r)\|^2 \, dr &\leqslant \frac{\nu}{4\|\nabla u\|_{L^\infty}^2} e^{2\|\nabla u\|_{L^\infty}(t_0 - s)} \|\theta_{s,0}\|_1^2 \\ &\leqslant \frac{\nu \lambda_N}{4\|\nabla u\|_{L^\infty}^2} e^{2\|\nabla u\|_{L^\infty}(t_0 - s)} \|\theta_{s,0}\|^2 \,. \end{aligned} \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.9) gives

$$\int_s^{t_0} \|\theta_s(r)\|_1^2 \, dr \geqslant \lambda_N (t_0 - s) \|\theta_{s,0}\|^2 \Big(\frac{1}{2} - \frac{H \lambda_N^{\alpha+\beta}}{2(t_0 - s)} - \frac{\nu \lambda_N e^{2\|\nabla u\|_{L^\infty}(t_0 - s)}}{4\|\nabla u\|_{L^\infty}^2 (t_0 - s)} \Big)$$

By our choice of λ_N in (2.16) and t_0, equation (5.8) follows. This finishes the proof of Lemma 5.1. $\square$

5.2. **The Weakly Mixing Case.** We now turn our attention to Theorem 2.10. The proof is similar to the proof of Theorem 2.9. The main difference is that the analogue of Lemma 5.1 is weaker.

Lemma 5.3. *If λ_N is chosen according to (2.18) and*

$$\|\theta_{s,0}\|_1^2 < \lambda_N \|\theta_{s,0}\|^2 \,, \tag{5.13}$$

then for $t_0 = h^{-1}\big(\frac{1}{2}\lambda_N^{-\frac{\alpha+\beta}{2}}\big)$, we have

$$\|\theta_s(t_0)\|^2 \leqslant \exp\Big(-\frac{\nu\lambda_N(t_0-s)}{2}\Big)\|\theta_{s,0}\|^2 \,. \tag{5.14}$$

Proof of Theorem 2.10. Given Lemma 5.3, the proof of Theorem 2.10 is identical to that of Theorem 2.9. □

Proof of Lemma 5.3. Following the proof of Lemma 5.1, we claim that (5.8) still holds in our case, provided λ_N and t_0 are chosen correctly. Indeed, note that (5.9) and (5.12) still hold, and the only difference here is that we need to bound the second term in (5.9) using the weak mixing assumption. Explicitly, (2.15) gives

$$\begin{aligned}\int_s^{t_0} \|P_N\phi_s(r)\|^2\,dr &\leqslant \lambda_N^\beta \int_s^{t_0} \|\phi_s(r)\|_{-\beta}^2\,dr \leqslant (t_0-s)\lambda_N^\beta h(t_0-s)^2\|\theta_{s,0}\|_\alpha^2 \\ &\leqslant (t_0-s)\lambda_N^\beta h(t_0-s)^2\|\theta_{s,0}\|^{2\alpha-2}\|\theta_{s,0}\|_1^{2\alpha} \\ &\leqslant (t_0-s)\lambda_N^{\beta+\alpha} h(t_0-s)^2\|\theta_{s,0}\|^2 \,. \end{aligned} \tag{5.15}$$

Substituting (5.12) and (5.15) into (5.9), we obtain

$$\int_s^{t_0} \|\theta_s(r)\|_1^2\,dr \geqslant \lambda_N(t_0-s)\|\theta_{s,0}\|^2\Big(\frac{1}{2} - \frac{\lambda_N^{\beta+\alpha}h(t_0-s)^2}{2} - \frac{\nu\lambda_N e^{2\|\nabla u\|_{L^\infty}(t_0-s)}}{4\|\nabla u\|_{L^\infty}^2(t_0-s)}\Big)\,.$$

By our choice of λ_N (in (2.18)) and t_0, equation (5.8) follows. □

5.3. **The Principal Eigenvalue with Dirichlet Boundary Conditions.** We now prove Proposition 2.13 estimating the principal eigenvalue of $-\nu\Delta + (u\cdot\nabla)$ in a bounded domain with Dirichlet boundary conditions.

Proof of Proposition 2.13. For notational convenience we will write μ_0 to denote $\mu_0(\nu,u)$. Let $\phi_0 = \phi_0(\nu,u)$ be the principal eigenfunction of the operator $-\nu\Delta + (u\cdot\nabla)$. Then we know

$$\psi(x,t) \stackrel{\text{def}}{=} \phi_0(x)e^{-\mu_0 t}$$

satisfies the advection diffusion equation

$$\partial_t\psi + u\cdot\nabla\psi - \nu\Delta\psi = 0\,,$$

with initial data ϕ_0. Consequently $\|\psi(t)\| = e^{-\mu_0 t}\|\psi(0)\|$. This forces $\tau_d \geqslant 1/\mu_0$ proving (2.25).

Now we note the proof of Theorem 2.9 only uses the spectral decomposition of the Laplacian, and is unaffected by the presence of boundaries. Thus Theorem 2.9 still applies in this context. The eigenvalue bound now follows immediately from Theorem 2.9 and (2.25). □

Appendix A. Weak and Strong Mixing Rates

In this appendix we provide a brief introduction to mixing and, in particular, analyze the notions of weak and strong weak mixing rates as in Definition 2.2. Recall that M is a d-dimensional Riemannian manifold with volume form normalized so that the total volume of M is 1. A volume preserving diffeomorphism $\varphi\colon M \to M$ is said to be *mixing* (or *strongly mixing*) if for every pair of Borel sets $A, B \subseteq M$, we have

$$\lim_{n\to\infty} \operatorname{vol}(\varphi^{-n}(A) \cap B) = \operatorname{vol}(A) \operatorname{vol}(B)\,. \tag{A.1}$$

Roughly speaking, this says that for every Borel set A, successive iterations of the map φ will stretch and fold it over M so that it eventually the fraction of *every* fixed region $B \subseteq M$ occupied by A will approach $\operatorname{vol}(A)$. For a comprehensive review of mixing we refer the reader to [KH95, SOW06].

Approximating by simple functions we see that (A.1) immediately implies that for any $f, g \in L^2_0$, we have[2]

$$\lim_{n\to\infty} \langle U^n f, g\rangle = 0\,.$$

Thus, one can quantify the *mixing rate* by requiring the correlations $\langle U^n f, g\rangle$ to decay at a particular rate. Since these are linear in f, g, a natural first attempt is to require

$$\bigl|\langle U^n f, g\rangle\bigr| \leqslant h(n) \|f\|\,\|g\|\,, \tag{A.2}$$

for some decreasing sequence $h(n)$ that vanishes at infinity. This, however, is impossible. Indeed using duality, equation (A.2) immediately implies

$$\|U^n f\| \leqslant h(n) \xrightarrow{n\to\infty} 0\,. \tag{A.3}$$

Of course, U is a unitary operator and hence we must also have $\|U^n f\| = \|f\|$, which is in direct contradiction to (A.3).

To circumvent this difficulty, one uses stronger norms of f and g on the right of (A.2) as in the series of papers by Fannjiang et. al. [FW03, FNW04, FNW06]. This is the content of the first part of Definition 2.2, and is repeated here for convenience.

Definition A.1. Let $h\colon \mathbb{N} \to (0, \infty)$ be a decreasing function that vanishes at infinity, and $\alpha, \beta > 0$. We say that φ is *strongly α, β mixing with rate function h* if for all $f \in \dot H^\alpha$, $g \in \dot H^\beta$ the associated Koopman operator U satisfies

$$\bigl|\langle U^n f, g\rangle\bigr| \leqslant h(n) \|f\|_\alpha \|g\|_\beta\,. \tag{A.4}$$

If U is simply a unitary operator, then the rate function h can decay arbitrarily fast. However, when U is the Koopman operator associated with a smooth map φ, the rate function can decay at most exponentially. To see this, note that for $k \in \mathbb{N}$ we have $\|Uf\|_k \leqslant c_k \|f\|_1$ for some finite constant $c_k = c_k(\|\varphi\|_{C^k}) > 1$. Iterating this n times, choosing $k = \lceil \beta \rceil$, and $g = U^n f$ in (A.4) gives

$$\|f\|^2 = \|U^n f\|^2 \leqslant h(n) \|f\|_\alpha \|f\|_k c_k^n\,,$$

forcing

$$h(n) \geqslant \frac{\|f\|^2 c_k^{-n}}{\|f\|_\alpha \|f\|_k}\,.$$

[2] Recall L^2_0 is the set of all mean zero square integrable functions, and $U\colon L^2_0 \to L^2_0$ is the Koopman operator defined by $Uf = f \circ \varphi$.

Notice that by duality equation (A.4) implies that if φ is α, β mixing with rate function h, then

$$\|U^n f\|_{-\beta} \leqslant h(n)\|f\|_{\alpha}\,. \tag{A.5}$$

In particular, this implies $\|U^n f\|_{-\beta} \to 0$ as $n \to \infty$, and this has been used by many authors [MMP05, LTD11, Thi12, IKX14] to quantify (strong) mixing.

We now turn our attention to weak mixing. Recall that the dynamical system generated by φ is said to be *weakly mixing* if for every pair of Borel sets $A, B \subseteq M$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \bigl|\operatorname{vol}(\varphi^{-k}(A) \cap B) - \operatorname{vol}(A)\operatorname{vol}(B)\bigr| = 0\,. \tag{A.6}$$

Clearly strongly mixing implies weakly mixing, but the converse is false (see for instance [AK70]). Approximating by simple functions, and using the fact that U is L^2 bounded, one can show that (A.6) holds if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \bigl|\langle U^n f, g\rangle\bigr|^2 = 0\,, \tag{A.7}$$

for all $f, g \in L^2_0$ (see for instance [EFHN15, Theorem 9.19 (iv)]). We can now quantify the *weak mixing rate* by by imposing a rate of convergence in (A.7). This is the content of the second part of Definition 2.2, and is repeated here for convenience.

Definition A.2. Let $h\colon \mathbb{N} \to (0, \infty)$ be a decreasing function that vanishes at infinity. Given $\alpha, \beta \geqslant 0$, we say that φ is *weakly α, β mixing with rate function h* if for all $f \in \dot H^\alpha$, $g \in \dot H^\beta$ and $n \in \mathbb{N}$ the associated Koopman operator U satisfies

$$\Bigl(\frac{1}{n} \sum_{k=0}^{n-1} \bigl|\langle U^k f, g\rangle\bigr|^2\Bigr)^{1/2} \leqslant h(n)\|f\|_\alpha \|g\|_\beta\,.$$

Unlike Definition A.1, the convergence rate need not involve stronger norms of *both* f and g. Indeed Proposition 4.2 shows that for toral automorphisms, either α or β may be chosen to be 0

By choosing $f = g$ one immediately sees that for weakly α, β mixing maps, we must have $h(n) \geqslant 1/\sqrt{n}$. Moreover, if φ is strongly α, β mixing with rate function h, then φ is also *weakly* α, β mixing. In this case, the weak α, β rate function, denoted by h_w, is given by

$$h_w(n) \stackrel{\text{def}}{=} \Bigl(\frac{1}{n} \sum_{k=0}^{n-1} h(k)^2\Bigr)^{1/2}\,.$$

If additionally h is square summable, then $h_w(n) \leqslant \sqrt{H/n}$, where $H^2 = \sum_0^\infty h(k)^2$.

Appendix B. A characterization of relaxation enhancing maps on the torus

We devote this appendix to proving Proposition 2.6 characterizing maps φ for which $\nu\tau_d \to 0$. Showing that $\lim_{\nu\to 0} \nu\tau_d = 0$ implies there is no eigenvector in $\dot H^1$ is simpler and we present this part first. The proof follows along the lines of [CKRZ08, KSZ08].

Proof of the backward implication in Proposition 2.6. We assume $\nu\tau_d \to 0$, and will show that U has no non-constant eigenfunction. Suppose, for sake of contradiction, that $f \in L^2_0 - 0$ is an eigenfunction, normalized so that $\|f\| = 1$, and let λ be the corresponding eigenvalue. Choosing $\theta_0 = f$, and defining θ_n by (2.1) we observe

$$\begin{aligned}
|\langle \theta_{n+1}, f\rangle - \langle U\theta_n, f\rangle| &= \Big|\sum_k (1-e^{-\nu\lambda_k})(U\theta_n)^\wedge(k)\overline{\hat{f}(k)}\Big| \\
&\leqslant \nu\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|(U\theta_n)^\wedge(k)|^2\Big)^{1/2}\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|\hat{f}(k)|^2\Big)^{1/2} \\
&\leqslant \nu\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|(U\theta_n)^\wedge(k)|^2\Big)^{1/2}\Big(\sum_k \frac{1-e^{-\nu\lambda_k}}{\nu}|\hat{f}(k)|^2\Big)^{1/2} \\
&\leqslant \nu(\mathcal{E}_\nu\theta_n)^{1/2}\|f\|_1 \leqslant \frac{\nu}{2}\mathcal{E}_\nu\theta_n + \frac{\nu}{2}\|f\|_1^2 \,.
\end{aligned}$$

Using equation (3.9), this gives

$$|\langle \theta_{n+1}, f\rangle - \langle U\theta_n, f\rangle| \leqslant \frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n+1}\|^2) + \frac{\nu}{2}\|f\|_1^2 \,,$$

which implies

$$|\langle \theta_{n+1}, f\rangle| - |\langle U\theta_n, f\rangle| \geqslant -\frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n+1}\|^2) - \frac{\nu}{2}\|f\|_1^2 \,.$$

Since $\langle U\theta_n, f\rangle = \langle \theta_n, U^* f\rangle = \lambda\langle \theta_n, f\rangle$, and $|\lambda| = 1$, the above implies

$$|\langle \theta_{n+1}, f\rangle| - |\langle \theta_n, f\rangle| \geqslant -\frac{1}{2}(\|\theta_n\|^2 - \|\theta_{n+1}\|^2) - \frac{\nu}{2}\|f\|_1^2 \,,$$

and iterating this gives

$$|\langle \theta_n, f\rangle| - |\langle f, f\rangle| \geqslant -\frac{1}{2}(\|f\|^2 - \|\theta_n\|^2) - \frac{n\nu}{2}\|f\|_1^2 \,,$$

since $\theta_0 = f$. Thus

$$|\langle \theta_n, f\rangle| \geqslant \frac{1}{2}\|f\|^2 + \frac{1}{2}\|\theta_n\|^2 - \frac{n\nu}{2}\|f\|_1^2 \geqslant \frac{1}{2} - \frac{n\nu}{2}\|f\|_1^2 \,.$$

Now choosing n to be the dissipation time τ_d gives

$$\frac{1}{e} \geqslant |\langle \theta_{\tau_d}, f\rangle| \geqslant \frac{1}{2} - \frac{\tau_d\nu}{2}\|f\|_1^2 \,,$$

and hence

$$\nu\tau_d \geqslant \frac{e-2}{e\|f\|_1^2}$$

This contradicts the assumption $\lim_{\nu\to 0}\nu\tau_d = 0$. □

For the other direction, we need two lemmas. The first is an application of the discrete RAGE theorem.

Lemma B.1. *Let $K \subset S = \{\phi \in L^2_0 \,|\, \|\phi\| = 1\}$ be a compact. Let P_c be the spectral projection on the continuous spectral subspace in the spectral decomposition of U. For any $N, \delta > 0$, there exists $n_c(N, \delta, K)$ such that for all $n \geqslant n_c$ and any $\phi \in K$, we have*

$$\text{(B.1)} \qquad \frac{1}{n-1}\sum_{i=1}^{n-1}\|P_N U^i P_c\phi\|^2 \leqslant \delta \,.$$

Proof. Define

$$f(n,\phi) \stackrel{\text{def}}{=} \frac{1}{n-1}\sum_{i=1}^{n-1}\|P_N U^i P_c \phi\|^2 \,.$$

Recall that by the RAGE theorem [CFKS87] we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\|AU^i P_c\phi\|^2 = 0\,,\quad \text{for any compact operator } A\,,$$

and hence for all ϕ, $f(\phi,n)\to 0$ as $n\to\infty$. Thus, to finish the proof, we only need to show that this convergence is uniform on compact sets.

To prove this, it is enough to show that f is uniformly continuous in ϕ. The uniform continuity of f can be proved as follows. For any $\phi_1,\phi_2\in S$, observe

$$\begin{aligned}
&|f(n,\phi_1)-f(n,\phi_2)|\\
&\quad\leqslant \frac{1}{n-1}\sum_{i=1}^{n-1}\big|\|P_N U^i P_c\phi_1\| - \|P_N U^i P_c\phi_2\|\big|\big(\|P_N U^i P_c\phi_1\| + \|P_N U^i P_c\phi_2\|\big)\\
&\quad\leqslant \frac{1}{n-1}\sum_{i=1}^{n-1}\|\phi_1-\phi_2\|\big(\|\phi_1\|+\|\phi_2\|\big)\\
&\quad\leqslant 2\|\phi_1-\phi_2\|\,.
\end{aligned}$$

Thus proves uniform continuity of f, concluding the proof of Lemma B.2. □

Lemma B.2. *Assume the Koopman operator U has no eigenfunctions in $\dot H^1$. Let P_p be the spectral projection on its point spectral subspace. Let K be a compact subset of S. Define the set $K_1=\{\phi\in K \mid \|P_p\phi\|\geqslant \frac12\}$. Then for any $C>0$, there exist $N_p(C,K)$ and $n_p(C,K)$ such that for any $N\geqslant N_p(C,K)$, any $n\geqslant n_p(C,K)$, and any $\phi\in K_1$,*

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\|P_N U^i P_p\phi\|_1^2 \geqslant C\,. \tag{B.2}$$

The proof of this is the same as Lemma 3.3 in [CKRZ08] and we do not present it here. We can now finish the proof of Proposition 2.6.

Proof of the forward implication in Proposition 2.6. Suppose now U has no eigenfunctions in $\dot H^1$. We will show that for any $\eta>0$,

$$\Big\|\theta\Big(\Big\lceil\frac{\eta}{\nu}\Big\rceil\Big)\Big\| \to 0 \text{ as } \nu\to 0\,, \tag{B.3}$$

which immediately implies $\lim_{\nu\to0}\nu\tau_d = 0$ as desired. To prove (B.3), we need to show for any given η,ε, there exists ν_0, such that for any $\nu\leqslant\nu_0$, we have $\|\theta(\lceil\frac{\eta}{\nu}\rceil)\|^2\leqslant\varepsilon$ for any initial $\theta_0\in H$ with $\|\theta_0\|=1$.

We choose N large enough satisfying $e^{-\lambda_N\eta/80}\leqslant\varepsilon$. Denote $K=\{\phi\in S \mid \|\phi\|^2\leqslant\lambda_N\}$, and $K_1=\{\phi\in K \mid \|P_p\phi\|\geqslant\frac12\}$. Let n_1 be

$$n_1 = \max\Big\{2, n_p(5\lambda_N, K), n_c\Big(N, \frac{1}{20}, K\Big)\Big\}\,.$$

Lastly we choose ν_0 satisfying $n_1\leqslant\eta/(2\nu_0)$, $\nu_0 n_1^2\leqslant\frac{1}{\lambda_N}$ and $\frac{n_1^2\nu\gamma^{n_1+1}}{(n_1-1)(\gamma-1)}\leqslant 1/4$, where $\gamma>1$ satisfying $\|U\theta\|_1^2\leqslant\gamma\|\theta\|_1^2$.

Note that if $\mathcal{E}_\nu\theta_n \geqslant \lambda_N\|\theta_n\|^2$ for all $n \in [0, \lceil \frac{\eta}{\nu} \rceil]$, then we have $\|\theta(\lceil \frac{\eta}{\nu} \rceil)\|^2 \leqslant e^{-\nu\lambda_N \lceil \frac{\eta}{\nu} \rceil} \leqslant e^{-\lambda_N \eta} \leqslant \varepsilon$. Otherwise, we denote $n_0 \in [0, \lceil \frac{\eta}{\nu} \rceil]$ to be the first time satisfying $\mathcal{E}_\nu\theta_{n_0} < \lambda_N\|\theta_{n_0}\|^2$. Similar to (3.14) we have $\|\theta_{n_0+1}\|_1^2 < \lambda_N\|\theta_{n_0+1}\|^2$. We now claim

$$\|\theta_{n_0+n_1}\|^2 \leqslant e^{-\lambda_N \nu n_1/40}\|\theta_{n_0}\|^2 \,. \tag{B.4}$$

Let ϕ_m be defined as $\phi_m = U^{m-1}\theta_{n_0+1}$, then $\phi_1/\|\phi_1\| = \theta_{n_0+1}/\|\theta_{n_0+1}\| \in K$. And we have $P_c\phi_m = U^{m-1}P_c\theta_{n_0+1}$, $P_p\phi_m = U^{m-1}P_p\theta_{n_0+1}$. Next we consider two cases.

Case 1: $\|P_c\theta_{n_0+1}\|^2 \geqslant \frac{3}{4}\|\theta_{n_0+1}\|^2$, i.e., $\|P_p\theta_{n_0+1}\|^2 \leqslant \frac{1}{4}\|\theta_{n_0+1}\|^2$. In this case, we have

$$\begin{aligned}
\sum_{m=1}^{n_1-1} \mathcal{E}_\nu\theta_{n_0+m} &\geqslant 2\sum_{m=1}^{n_1-1} \|\theta_{n_0+1+m}\|_1^2 \\
&\geqslant 2\lambda_N \sum_{m=1}^{n_1-1} \|(I-P_N)\theta_{n_0+1+m}\|^2 \\
&\geqslant \lambda_N \sum_{m=1}^{n_1-1} \|(I-P_N)\phi_{m+1}\|^2 - 2\lambda_N \sum_{m=1}^{n_1-1} \|(I-P_N)(\theta_{n_0+1+m}-\phi_{m+1})\| \,.
\end{aligned} \tag{B.5}$$

By direct calculation, we also have

$$\begin{aligned}
\|(I-P_N)\phi_{m+1}\|^2 &\geqslant \frac{1}{2}\|(I-P_N)P_c\phi_{m+1}\|^2 - \|(I-P_N)P_p\phi_{m+1}\|^2 \\
&\geqslant \frac{1}{2}\|U^m P_c\theta_{n_0+1}\|^2 - \frac{1}{2}\|P_N U^m P_c\theta_{n_0+1}\|^2 - \|U^m P_p\theta_{n_0+1}\|^2 \\
&= \frac{1}{2}\|P_c\theta_{n_0+1}\|^2 - \frac{1}{2}\|P_N U^m P_c\theta_{n_0+1}\|^2 - \|P_p\theta_{n_0+1}\|^2 \,.
\end{aligned}$$

By Lemmas B.1,B.2, and the choice of n_1, we have

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1} \|(I-P_N)\phi_{m+1}\|^2 \geqslant \frac{1}{10}\|\theta_{n_0+1}\|^2 \,. \tag{B.6}$$

Substituting (3.18) and (B.6) in (B.5) gives

$$\sum_{m=1}^{n_1-1} \mathcal{E}_\nu\theta_{n_0+m} \geqslant \frac{\lambda_N(n_1-1)}{20}\|\theta_{n_0+1}\|^2 \,.$$

Since $\|\theta_{n_0+n_1}\|^2 = \|\theta_{n_0+1}\|^2 - \nu\sum_{m=1}^{n_1-1}\mathcal{E}_\nu\theta_{n_0+m}$, we further have

$$\begin{aligned}
\|\theta_{n_0+n_1}\|^2 &\leqslant \Big(1 - \frac{\nu\lambda_N(n_1-1)}{20}\Big)\|\theta_{n_0+1}\|^2 \\
&\leqslant (1 - \frac{\nu\lambda_N n_1}{40})\|\theta_{n_0}\|^2 \leqslant e^{-\frac{\nu\lambda_N n_1}{40}}\|\theta_{n_0}\|^2 \,.
\end{aligned}$$

Case 2: $\|P_p\theta_{n_0+1}\|^2 \geqslant \frac{1}{4}\|\theta_{n_0+1}\|^2$, i.e., $\|P_c\theta_{n_0+1}\|^2 \leqslant \frac{3}{4}\|\theta_{n_0+1}\|^2$.

By Lemma B.2, we have

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1} \|P_N U^m P_p\theta_{n_0+1}\|_1^2 \geqslant 5\lambda_N\|\theta_{n_0+1}\|^2 \,. \tag{B.7}$$

And Lemma B.1 yields that

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N U^m P_c \theta_{n_0+1}\|_1^2 \leqslant \frac{\lambda_N}{20}\|\theta_{n_0+1}\|^2\,. \tag{B.8}$$

Combining (B.7) and (B.7), we get

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N U^m \theta_{n_0+1}\|_1^2 \geqslant 2\lambda_N\|\theta_{n_0+1}\|^2\,. \tag{B.9}$$

By (3.18) and (3.13), we have

$$\begin{aligned}\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|\theta_{n_0+1+m}-\phi_{m+1}\|^2 &\leqslant \frac{n_1^2\nu}{n_1-1}\sum_{m=1}^{n_1-1}\|U\theta_{n_0+1+m}\|^2\\ &\leqslant \frac{n_1^2\nu}{n_1-1}\sum_{m=1}^{n_1-1}\gamma^{m+1}\|\theta_{n_0+1}\|^2\\ &\leqslant \frac{n_1^2\nu\gamma^{n_1+1}}{(n_1-1)(\gamma-1)}\|\theta_{n_0+1}\|^2\\ &\leqslant \frac{1}{4}\|\theta_{n_0+1}\|^2\end{aligned}$$

This implies

$$\frac{1}{n_1-1}\sum_{m=1}^{n_1-1}\|P_N(\theta_{n_0+1+m}-\phi_{m+1})\|_1^2 \leqslant \frac{\lambda_N}{4}\|\theta_{n_0+1}\|^2\,. \tag{B.10}$$

Equation (B.9) together with (B.10) gives

$$\sum_{m=1}^{n_1-1}\|\theta_{n_0+1+m}\|_1^2 \geqslant \sum_{m=1}^{n_1-1}\|P_N\theta_{n_0+1+m}\|_1^2 \geqslant \frac{\lambda_N}{2}(n_1-1)\|\theta_{n_0+1}\|^2\,. \tag{B.11}$$

We now use (3.13) again to get

$$\sum_{m=1}^{n_1-1}\mathcal{E}_\nu\theta_{n_0+m} \geqslant \lambda_N(n_1-1)\|\theta_{n_0+1}\|^2\,.$$

As before, we get

$$\|\theta_{n_0+n_1}\|^2 \leqslant e^{-\frac{\nu\lambda_N n_1}{2}}\|\theta_{n_0}\|^2\,.$$

This proves (B.4) as desired.

Given (B.4), we can find $\tilde n \in [\eta/(2\nu), \eta/\nu]$ such that $\|\theta(\lceil\eta/\nu\rceil)\|^2 \leqslant \|\theta_{\tilde n}\|^2 \leqslant e^{-\lambda_N\nu\tilde n/40} \leqslant e^{-\lambda_N\eta/80} \leqslant \varepsilon$, proving (B.3) as desired. This finishes the proof. $\square$

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