Leading-Order Nonlocal Kinetic Energy in Peridynamics for Consistent Energetics and Wave Dispersion

Kaushik Dayal*

Center for Nonlinear Analysis, Carnegie Mellon University

Department of Civil and Environmental Engineering, Carnegie Mellon University

Department of Materials Science and Engineering, Carnegie Mellon University

May 8, 2017

Abstract

This paper first considers the approximation of peridynamics by strain-gradient models in the linear, one-dimensional setting. Strain-gradient expansions that approximate the peridynamic dispersion relation using Taylor series are compared to strain-gradient models that approximate the peridynamic elastic energy. The dynamic and energetic expansions differ from each other, and neither captures an important feature of peridynamics that mimics atomic-scale dynamics, namely that the frequency of short waves is bounded and non-zero.

The paper next examines peridynamics as the limit model along a sequence of strain-gradient models that consistently approximate both the energetics and the dispersion properties of peridynamics. Formally examining the limit suggests that the inertial term in the dynamical equation of peridynamics – or equivalently, the peridynamic kinetic energy – is necessarily nonlocal in space to balance the spatial nonlocality in the elastic energy. The nonlocality in the kinetic energy is of leading-order in the following sense: classical elasticity is the zeroth-order theory in both the kinetically nonlocal peridynamics and the classical peridynamics, but once nonlocality in the elastic energy is introduced, it must be balanced by nonlocality in the kinetic energy at the same order. In that sense, the kinetic nonlocality is not a higher-order correction; rather, the kinetic nonlocality is essential for consistent energetics and dynamics even in the simplest setting. The paper then examines the implications of kinetically nonlocal peridynamics in the context of stationary and propagating discontinuities of the kinematic fields.

1 Introduction

Peridynamics is a nonlocal continuum model that is formulated in terms of integral operators, and was introduced in the foundational work by Silling [Sil00]. The use of integral operators allows for the possibility of discontinuous solutions, providing important advantages for the modeling of fracture [Sil00, Lip14, Lip15, LSL16].

*Email: Kaushik.Dayal@cmu.edu

To aid understanding and analysis, particularly of wave propagation, recent works such as [WA05, Sil16] have found it useful to examine linearized peridynamics by approximating the integral operator, when the nonlocality is small, as an expansion in terms of strain gradients. In those works, the starting point is the peridynamic equation of momentum balance, and the expansion is conducted so that the gradient models approximate the dispersion relation $\omega = \omega(k)$ at $k \to 0$. An important feature of the peridynamic dispersion relation is that, under mild and reasonable assumptions on the model, $\lim_{k\to\infty} \omega(k)$ goes to a finite non-zero value ω_∞ ([Sil00, WA05] and Appendix A). Therefore, short waves have zero phase and group velocity, and this is an important feature in peridynamics that mimics the dispersive behavior of lattice waves [BH98]. However, the approximation by strain-gradient models does not have this feature. In the approximation by strain-gradients models of higher and higher order, the dispersion relation for the approximate model has the form of a polynomial with even powers of k and coefficients of alternating sign. When truncated after a finite number of terms, the approximate polynomial is dominated at large k by the highest power. This leads to important qualitative failings: first, there is the possibility of instability depending on the order of truncation; second, the velocity of short-waves is unbounded even if there is stability.

Another approach to approximating peridynamics is to start from the peridynamic elastic energy and use an expansion starting from the small nonlocality limit. In the static setting, it has been shown that peridynamics can be considered as the minimization of a physically-motivated energy. If the original energy is positive-definite, the approximate gradient models retain this stability. In this approach, the dispersion relation for the approximate models has the form of a polynomial with even powers of k. The velocity of short waves is again unbounded, though the models have stable energies.

Importantly, the static equilibrium equations of the gradient models obtained by the energy expansion do not match the corresponding equations obtained by the dispersion expansion. As noted above, both approaches fail to capture the bounded velocity of short waves.

The tension between energetic approximation and dispersion approximation is closely related to the "sign paradox" in gradient elasticity [MS07]. Both characteristics discussed above, namely approximating the energy and approximating the dispersion, are important features to capture in a gradient model. The former is important for stability and to capture the static solutions correctly, while the latter is important for approximating the dynamics. In this paper, we examine the use of rational function approximations for the approximation of the dispersion relation. Our approach is motivated by the work in strain-gradient models that introduce inertial terms that involve not just the standard second derivative in time of the displacement, but also the second derivative in time of displacement gradients of various orders [Min64] (so-called "micro-inertia"). These models can approximate both the dispersion and energy consistently, i.e., the energy is stable and the dispersion has a finite short wavelength limit .

We now take the perspective that peridynamics arises as a limit of a sequence of strain-gradient models, and further require that gradient models in this sequence are energetically and dynamically consistent as in gradient models with micro-inertia. A formal calculation then shows that the limit operator of the inertial terms is nonlocal. This sequence *defines* what we mean here by an energetically and dynamically consistent peridynamic model. We note that this sequence of gradient models is different from those obtained by expanding either the peridynamic energy or the peridynamic dispersion, because these models either do not approximate short waves well, or have instabilities, or both.

We emphasize that the kinetic nonlocality is a leading-order term and not a small higher-order correction. That is, the zeroth-order approximation of peridynamics is classical linear elastodynamics, but once there is any level of nonlocality in the peridynamic elastic energy, there must be a similar level of nonlocality in the peridynamic kinetic energy for dynamic and energetic consistency.

1.1 Organization

- Section 2 outlines the peridynamic model and the dispersion-based and energy-based gradient expansions.
- Section 3 briefly outlines the relevant aspects of the notion of micro-inertia from the strain-gradient literature.
- Section 4 examines the limit of energetically and dynamically consistent strain-gradient models to arrive at kinetically nonlocal peridynamics.
- Section 5 examines the consistency of kinetically nonlocal peridynamics with the fundamental balances of continuum mechanics in a 3D finite deformation setting, and uses the consistency to constrain the new terms that have been introduced.
- Section 6 examines the behavior of discontinuities in classical kinetically-local peridynamics and the proposed kinetically nonlocal peridynamics, in particular, (1) the jump condition, (2) dissipation, and (3) construction of solutions for a specific problem.

1.2 Notation and Definitions

To enable the construction of simple arguments, we assume that micromodulus and nonlocal inertia are both non-negative for all values of their argument. The former is a sufficient though not necessary condition for positive-definiteness of the energy.

$\hat{(\cdot)} \equiv \mathcal{F}\left[(\cdot)\right]$ represents the Fourier transform, defined as $\hat{f}(k) = \mathcal{F}[f](k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{\imath kx}\ dx$. The inverse Fourier transform is defined as $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(k) e^{-\imath kx}\ dk$. In this convention, convolution in real space translates to a product in real space as $\mathcal{F}[f * g] = \sqrt{2\pi}\mathcal{F}[f]\mathcal{F}[g]$, and $\mathcal{F}[1] = \sqrt{2\pi}\delta(k)$. Only the 1D versions are used in this paper.

$\dot{(\cdot)}$ represents the time derivative of $(\cdot)$.

For brevity, we use *classical* or *kinetically-local peridynamics* to refer to the universally-used formulation of peridynamics with the local kinetic energy, and *kinetically nonlocal peridynamics* to refer to the model proposed in this paper.

ℓ: intrinsic model/material lengthscale
k: wavenumber
ρ: mass density
E: classical linear elastic modulus
$u(x,t)$ and $y(x,t)$: displacement and deformation respectively as a function of position x and time t. Boldface versions $\boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{y}(\boldsymbol{x},t)$ are used for the corresponding vector quantities.
$\delta(x)$: Dirac mass at $x = 0$
$H(x)$: Heaviside function, defined as $H(x \leq 0) = 0, H(x > 0) = 1$.

C_ℓ: kernel for the peridynamic elastic energy nonlocal operator, *micromodulus* for short
P_ℓ: kernel for the peridynamic kinetic energy nonlocal operator, *nonlocal inertia* for short
$\tilde{C}(\cdot/\ell) = \dfrac{\ell^3}{E} C_\ell(\cdot)$: the ℓ-independent part of the micromodulus
$\tilde{P}(\cdot/\ell) = \dfrac{\ell^3}{\rho} P_\ell(\cdot)$: the ℓ-independent part of the nonlocal inertia
$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}', t) - \boldsymbol{y}(\boldsymbol{x}, t))$: nonlinear (geometrically and materially) peridynamic response function

T: the total kinetic energy in classical peridynamics
T_{kn}: the total kinetic energy in kinetically nonlocal peridynamics
U: the total peridynamic elastic energy

L^2: the space of square-integrable functions
H^1: the space of functions that are square-integrable and with square-integrable weak derivatives.

2 Formulation and Gradient Expansion of Classical Peridynamics

For a linear, homogeneous, one-dimensional, infinite body, the peridynamic equation of motion is [Sil00]:

$$\rho \ddot{u}(x,t) = \int_{x' \in \mathbb{R}} C_\ell\left(x' - x\right) \cdot \left[u(x',t) - u(x,t)\right] \, dx' \tag{2.1}$$

The micromodulus C_ℓ decays with increasing separation; typical models often assume that it has compact support, but that is not necessary. Here, we assume that the various integrals involving C_ℓ are finite and meaningful, see also Appendix A. Also, ℓ is the internal lengthscale associated with the material and sets the range of the interactions. A useful symmetry of C_ℓ, derived from linear momentum balance, is that $C_\ell(a) = C_\ell(-a)$. In the general inhomogeneous setting, $C_\ell = C_\ell(x', x)$, and the symmetry condition is then $C_\ell(x', x) = C_\ell(x, x')$.

Following [WA05], we use the scaling $C_\ell(x' - x) = \dfrac{E}{\ell^3} \tilde{C}\left(\dfrac{x' - x}{\ell}\right)$, where $\tilde{C}$ is independent of ℓ and satisfies $\dfrac{1}{2}\displaystyle\int_{\xi \in \mathbb{R}} \xi^2 \tilde{C}(\xi) \, d\xi = 1$. This scaling will ensure later that the classical elastic modulus derived from the zeroth-order theory (i.e., $\ell \to 0$) is finite, non-zero, and equal to E.

The peridynamic dispersion relation is obtained by substituting $u(x,t) = \exp(\imath(kx - \omega t))$ in (2.1):

$$\rho \omega^2 = \underbrace{\int_{\xi \in \mathbb{R}} (1 - \cos k\xi) C_\ell(\xi) \, d\xi}_{M(k), \text{ the acoustic tensor}} \tag{2.2}$$

We have used the property $C_\ell(a) = C_\ell(-a)$ above. The peridynamic acoustic tensor $M(k)$ is closely related to the Fourier transform of the micromodulus C_ℓ.

The peridynamic elastic energy can be written [Sil00]:

$$U = \int_{x \in \mathbb{R}} \underbrace{\int_{x' \in \mathbb{R}} \frac{1}{4} C_\ell \cdot \left[u(x',t) - u(x,t)\right]^2 \, dx' \, dx}_{\text{energy density at } x} \tag{2.3}$$

The physical picture is that of a linear spring connecting x and x' with energy $\frac{1}{2}C_\ell(x'-x)\left[u(x',t)-u(x,t)\right]^2$. Partitioning the energy equally between x and x' gives another factor of $\frac{1}{2}$.

It suffices for positive-definiteness of U that $C \geq 0$ for all values of the arguments, and we assume that hereon.

The classical peridynamic kinetic energy is simply $T = \int_{x\in\mathbb{R}} \frac{1}{2}\rho\dot{u}(x,t)^2 \, dx$ as in classical elasticity. The evolution equation (2.1) can be derived by requiring the time derivative of the total energy $T+U$ to vanish, as in classical elasticity.

2.1 Dispersion Expansion

Following [WA05], we expand $M(k)$ as a Taylor series about $\ell k = 0$ to obtain the "dispersion expansion" strain-gradient model. Substituting for $\tilde{C}$ in (2.2) and using the non-dimensional variable $z = \xi/\ell$, the dispersion relation can be written:

$$\rho\omega^2 = k^2 \int_{z\in\mathbb{R}} (1-\cos k\ell z)\frac{E}{(kl)^2}\tilde{C}(z)\, dz \tag{2.4}$$

Using the Taylor expansion of $\cos k\ell z$ around $k\ell = 0$ gives:

$$\rho\omega^2 = Ek^2 \sum_{n=1} (-1)^{n+1}(k\ell)^{2n-2} \int_{z\in\mathbb{R}} \frac{z^{2n}\tilde{C}(z)}{(2n)!}\, dz \tag{2.5}$$

As observed by [WA05], the zeroth-order term (i.e., using only $n = 1$ above) gives classical linear elasticity.

For short, we write the expansion above as $\rho\omega^2 = \sum_{n=1}(-1)^{n+1}\tilde{A}_n k^{2n}$. We define $\tilde{A}_n := E\ell^{2n-2}\int_{z\in\mathbb{R}} \frac{z^{2n}\tilde{C}(z)}{(2n)!}\, dz$ by comparing to (2.5). Note that all $\tilde{A}_n$ are non-negative because $\tilde{C}$ has been assumed non-negative for all values of its argument. We note from the normalization condition on $\tilde{C}$ that:

$$\tilde{A}_1 = E\int_{z\in\mathbb{R}} \frac{z^2\tilde{C}(z)}{2}\, dz = E \tag{2.6}$$

The approximate dispersion relation can be inverted to find the dispersion expansion dynamical equation:

$$\rho\ddot{u}(x,t) = \sum_{n=1} \tilde{A}_n \frac{\partial^{2n}u}{\partial x^{2n}} \tag{2.7}$$

by noting that integer powers of k correspond to standard derivatives in real-space.

We can further deduce the dispersion expansion elastic energy by multiplying the dynamical equation by $\dot{u}$ and integrating over the domain. Using integration-by-parts gives the time derivative of the elastic energy, which has the form:

$$\int_{x\in\mathbb{R}} \sum_{n=1}(-1)^{n+1}\frac{\tilde{A}_n}{2}\left(\frac{\partial^n u}{\partial x^n}\right)^2\, dx \tag{2.8}$$

The expansion above has undesirable features for large $k\ell$ if truncated after a finite number of terms. Notice that the dispersion relation (2.5) is dominated by the largest power of $k\ell$ for short wavelengths. If the expansion above is truncated with an even number of terms, there is instability (ω imaginary); if truncated after an odd number of terms, $\omega \to \infty$ as $k \to \infty$. An indication of possible instability can also be seen in the energy (2.8): the sign flips between terms and recall that $\tilde{A}_n$ are all non-negative. This undesirable feature of the dispersion expansion strain-gradient model was also noticed by [WA05, SPGL09].

2.2 Energy Expansion

We now consider the expansion of U in terms of ℓk to obtain the "energy expansion" strain-gradient model. Notice that the classical kinetic energy T is independent of ℓ. The zeroth-order term, i.e. the convergence of peridynamics to classical elasticity, has been studied in the nonlinear setting by [SL08].

Consider the quantity $u(x',t) - u(x,t) = u\left(x + k\ell\frac{z'-z}{k}, t\right) - u(x,t)$ from (2.3), where we use the non-dimensional variable $z = x/\ell$. The Taylor expansion of this quantity around $k\ell = 0$ is given by:

$$u(x',t) - u(x,t) = u\left(x + k\ell\frac{z'-z}{k}, t\right) - u(x,t) = \sum_{n=1} \ell^n \frac{\partial^n u}{\partial x^n}\frac{(z'-z)^n}{n!} \tag{2.9}$$

Substituting this expansion in (2.3) gives the expression for the energy. We do not write it down here because it involves a large number of terms due to the square of the series; the general formula can be found using the multinomial theorem. However, the key aspects can be noted without the explicit expressions but rather by noticing the following points.

1. Since the energy involves the square of a sum and the micromodulus is non-negative, it is immediately clear that the elastic energy of the strain-gradient model is positive-definite.
2. Terms involving odd powers of $(z'-z)$ vanish when we integrate over $\mathbb{R}$ because of the symmetry $\tilde{C}(a) = \tilde{C}(-a)$.
3. The energy contains terms of the form $\left(\frac{\partial^m u}{\partial x^m}\frac{\partial^{2n-m} u}{\partial x^{2n-m}}\right)$. Using integration-by-parts to move derivatives, we can write all such terms in the form $\left(\frac{\partial^n u}{\partial x^n}\right)^2$, and the boundary terms at $x \to \pm\infty$ are assumed to vanish. Therefore, the energy can be written in the form $\int_{x\in\mathbb{R}} \sum_{n=1} \frac{A_n}{2}\left(\frac{\partial^n u}{\partial x^n}\right)^2$. It follows that the dispersion relation is a polynomial with terms $k^2, k^4, \dots$ and positive coefficients, thereby being stable at all k and becoming unbounded as $k \to \infty$.
4. The zeroth-order term (use $n = 1$ only in the series above) gives classical elasticity and agrees also with the zeroth-order dispersion expansion model, i.e. $A_1 = \tilde{A}_1 = E$. The energy expansion and the dispersion expansion models differ from the next order onwards, i.e. $A_n \neq \tilde{A}_n, n > 1$, and there are the disagreements in the sign of these quantities as noted previously.

5. For illustration, we write out the first 2 terms of the series above explicitly. The leading-order term, using the normalization of $\tilde{C}$, gives us the classical linear elasticity expression:

$$\int_{x\ in\mathbb{R}} \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \int_{z'\in\mathbb{R}} E\frac{(z'-z)^2\tilde{C}(z'-z)}{2}\ dz'\ dx = \int_{x\in\mathbb{R}} \frac{1}{2}E\left(\frac{\partial u}{\partial x}\right)^2\ dx \tag{2.10}$$

The next order term has the form:

$$\int_{x\in\mathbb{R}} \frac{1}{8}El^2\left[\int_{\xi\in\mathbb{R}} \tilde{C}(\xi)\xi^4\ d\xi\right]\left(\frac{\partial^2 u}{\partial x^2}\right)^2\ dx \tag{2.11}$$

The cross-term vanishes in this case because $\tilde{C}$ is an even function. However, keeping more terms will lead to algebraically formidable models because of terms of the form, e.g., $\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial^3 u}{\partial x^3}\right)$, which require balancing of the derivatives.

3 Spatio-Temporal Derivatives in Strain Gradient Models

Gradients of the inertial term were introduced by Mindlin [Min64], and explored further in the strain-gradient literature to better approximate the dispersion behavior of real materials while also having a stable elastic energy. Recent terminology denotes these terms as "micro-inertia" [AA06, AMPB08, AA11] Consider a simple one-dimensional model below[1].

$$\rho\ddot{u} - \rho\gamma_1\frac{\partial^2\ddot{u}}{\partial x^2} + \rho\gamma_2\frac{\partial^4\ddot{u}}{\partial x^4} = E\frac{\partial^2 u}{\partial x^2} - E\kappa\frac{\partial^4 u}{\partial x^4} \tag{3.1}$$

The non-standard terms involving both strain gradients and inertia on the left side correspond to "micro-inertia". It was introduced for very similar reasons as the motivation in this paper, namely to better match the observed dispersion behavior of short waves. Writing the dispersion relation provides the connection to rational functions[2]:

$$\rho\omega^2 = E\frac{k^2+\kappa k^4}{1+\gamma_1 k^2+\gamma_2 k^4} \tag{3.2}$$

If all the constants are positive: we recover classical linear elastodynamics at long wavelengths; at short wavelengths, the dispersion relation has a non-zero finite limit; and the right side is positive for all values of $k > 0$ implying that there are no instabilities. Further, static solutions are governed by $E\frac{\partial^2 u}{\partial x^2} - E\kappa\frac{\partial^4 u}{\partial x^4} = 0$ which corresponds to minimizing an energy of the form $\int_{\mathbb{R}} E\left(\frac{\partial u}{\partial x}\right)^2 + E\kappa\left(\frac{\partial^2 u}{\partial x^2}\right)^2$ which is positive definite. Figure 1 illustrates different approximations to the dispersion relation.

The essential physical idea that we draw from this model is that derivatives of the displacement field in the elastic energy must be balanced by derivatives of the inertial term.

[1] While this particular model does not seem to have been considered in the body of work on micro-inertia, the broader observations therein motivate our approach.

[2] It does not appear to have been posed as a rational function approximation to the dispersion relation in the strain gradient literature.

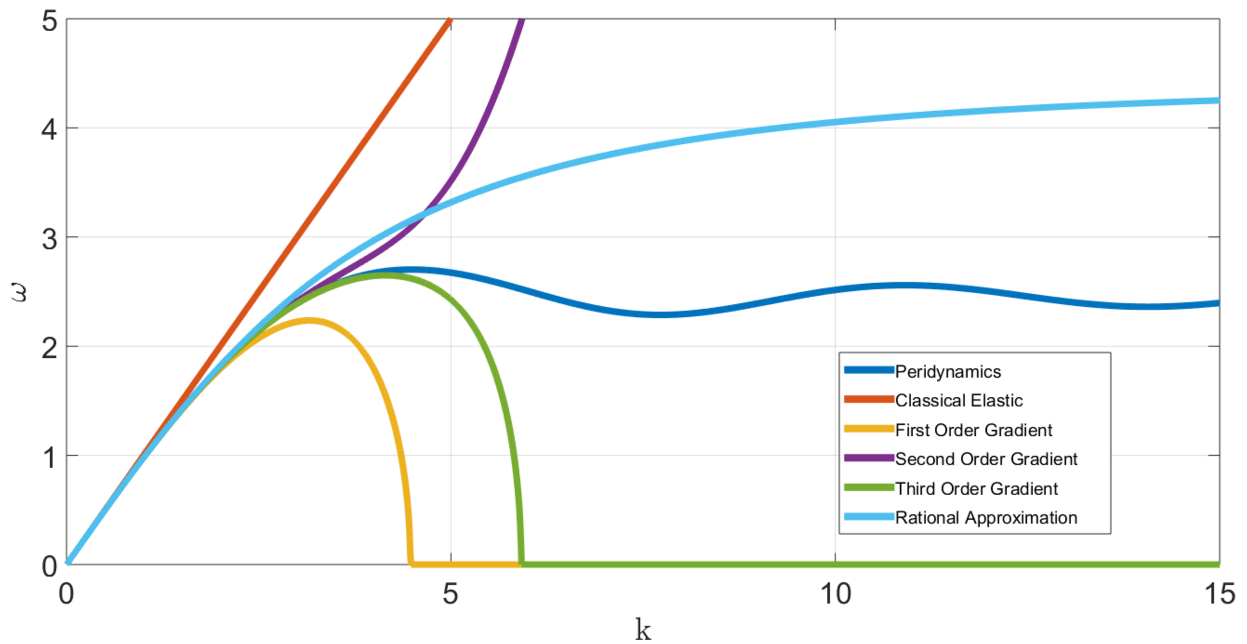

Figure 1: A comparison of the peridynamic dispersion relation with the rational function approximation and various orders of polynomial approximation for the material with micromodulus $\tilde{C}(a) = \begin{cases} 3 \text{ if } |a| < 1 \\ 0 \text{ if } |a| \geq 1 \end{cases}$.

4 Nonlocal Kinetic Energy in Peridynamics for Consistent Energetics and Dispersion

We now examine peridynamics as a limit model of the gradient expansions. Loosely speaking, we take the limit along the sequence of strain-gradient models that approximate both the energetics and the dispersion properties of peridynamics. This sequence of strain-gradient models is different from those described in Sections 2.1 and 2.2 which either approximate the peridynamic energy or the peridynamic dispersion but not both – and of course neither is a good approximation for dispersion at $k\ell \to \infty$. Instead, the sequence of strain-gradient models is motivated by the micro-inertia models using rational function approximations. This sequence defines what we mean here by energetically and dynamically consistent. In a sense, we are going backwards: rather than expanding peridynamics, we ask what peridynamics should look like to have energetically and dynamically consistent strain-gradient models as approximations.

We write the peridynamic dispersion using a rational function approximation as:

$$\rho\omega^2 = \frac{\sum_{n=1}^{M} N_n (k\ell)^{2n}}{1 + \sum_{n=1}^{M} D_n (k\ell)^{2n}} = \frac{\hat{N}_\ell(k)}{1 + \hat{D}_\ell(k)} \tag{4.1}$$

By using the same order M of polynomial in the numerator and denominator, we ensure that the limit value of ω is finite as $k\ell \to \infty$. Further, at $k = 0$, we recover the classical linear elastodynamic limit if we set $N_1 = E$. Furthermore, we use only even powers of k to reflect the symmetry of the dispersion relation and the stability of the energy. Finally, we notice that the equilibrium solution is determined by $N(k)$, so it should be chosen to match the energy expansion gradient model in Section 2.2.

From the dispersion relation immediately above, we write the Fourier space dynamical equation:

$$\rho\hat{\ddot{u}}(k,t)(1+\hat{D}_\ell(k)) = -\hat{N}_\ell(k)\hat{u}(k) \tag{4.2}$$

We will use the inverse Fourier transform to recover the real-space equation.

Define $\hat{C}_\ell(k) := \dfrac{\bar{C}_\ell - \hat{N}_\ell(k)}{\sqrt{2\pi}}$. Note that this definition implies $\hat{C}_\ell(0) = \dfrac{\bar{C}_\ell}{\sqrt{2\pi}}$ because $\hat{N}_\ell(k)$ is of order k^2 and higher. Using the Fourier convolution / product duality, we can write:

$$\int_{x'\in\mathbb{R}} C_\ell(x'-x)\,dx' = \int_{x'\in\mathbb{R}} 1\times C_\ell(x'-x)\,dx' = \sqrt{2\pi}\mathcal{F}^{-1}\left[\hat{C}_\ell(k)\sqrt{2\pi}\delta(k)\right](x) = \bar{C}_\ell\mathcal{F}^{-1}\left[\sqrt{2\pi}\delta(k)\right](x) = \bar{C}_\ell \tag{4.3}$$

Using the Fourier convolution / product duality and applying the inverse Fourier transform to the right side of (4.2) gives:

$$\mathcal{F}^{-1}\left[-\bar{C}_\ell\hat{u} + \underbrace{\left(\bar{C}_\ell - \hat{N}_\ell\right)}_{\sqrt{2\pi}\hat{C}_\ell}\hat{u}\right](x) = -\bar{C}_\ell u(x) + \mathcal{F}^{-1}\left[\sqrt{2\pi}\hat{C}_\ell\hat{u}\right](x) = -\bar{C}_\ell u(x) + \int_{x'\in\mathbb{R}} C_\ell(x'-x)u(x')\,dx' \tag{4.4}$$

We can use these two results to arrive at the classical peridynamic nonlocal interaction operator:

$$\begin{aligned} -\mathcal{F}^{-1}\left[\hat{N}_\ell(k)\hat{u}_\ell(k)\right] &= \int_{x'\in\mathbb{R}} C_\ell(x'-x)\left[u(x',t)-u(x,t)\right]\,dx' \\ &\text{with } \sqrt{2\pi}\hat{C}_\ell(k) := \bar{C}_\ell - \hat{N}_\ell(k), \int_{x'\in\mathbb{R}} C_\ell(x'-x)\,dx' = \bar{C}_\ell \end{aligned} \tag{4.5}$$

Essentially, the calculation above shows that if a function in Fourier space $\hat{G}(k)$ can be represented through a Taylor expansion of the form $\sum_{n=1}\hat{G}_n k^{2n}$, then the real-space function $\mathcal{F}^{-1}\left[\hat{G}(k)\hat{u}(k)\right](x)$ has the form of a peridynamic operator. That is, it acts to leading-order as a second derivative because the leading-order term in Fourier space is k^2.

Using precisely the same ideas, we can invert the left side of (4.2):

$$\begin{aligned} \rho\mathcal{F}^{-1}\left[\hat{\ddot{u}}(k,t)(1+\hat{D}_\ell(k))\right] =& \rho\ddot{u}(x,t) + \int_{x'\in\mathbb{R}} P_\ell(x'-x)\left[\ddot{u}(x,t)-\ddot{u}(x',t)\right]\,dx' \\ &\text{with } \sqrt{2\pi}\hat{P}_\ell(k) := \bar{P}_\ell + \rho\hat{D}_\ell(k), \int_{x'\in\mathbb{R}} P_\ell(x'-x)\,dx' = \bar{P}_\ell \end{aligned} \tag{4.6}$$

Putting these together, we write the kinetically nonlocal peridynamic dynamical equation as:

$$\begin{aligned} \rho\ddot{u}(x,t) + \int_{x'\in\mathbb{R}} P_\ell(x'-x)\left[\ddot{u}(x,t)-\ddot{u}(x',t)\right]\,dx' &= \int_{x'\in\mathbb{R}} C_\ell(x'-x)\left[u(x',t)-u(x,t)\right]\,dx' \\ \Leftrightarrow \left(\rho+\bar{P}_\ell\right)\ddot{u}(x,t) - \int_{x'\in\mathbb{R}} P_\ell(x'-x)\ddot{u}(x',t)\,dx' &= \int_{x'\in\mathbb{R}} C_\ell(x'-x)\left[u(x',t)-u(x,t)\right]\,dx' \end{aligned} \tag{4.7}$$

Using this, we can then define the nonlocal kinetic energy:

$$T_{kn} = \int_{x \in \mathbb{R}} \frac{1}{2} \rho \dot{u}^2(x,t)\, dx + \int_{x \in \mathbb{R}} \int_{x' \in \mathbb{R}} \frac{1}{4} P_\ell(x'-x) \left[\dot{u}(x',t) - \dot{u}(x,t)\right]^2 \, dx' dx \tag{4.8}$$

The nonlocal kinetic energy and kinetically nonlocal dynamical equation can be related by using $\frac{d}{dt}\left(T_{kn}+U\right) = 0$. We notice that a sufficient, but possibly not necessary, condition for positive-definite T_{kn} is to require that $P_\ell \geq 0$ for all values of the arguments.

Note that $\bar{C}_\ell$ and $\bar{P}_\ell$ are arbitrary at this point. We will return to this question below in Section 4.2.

4.1 Scaling and Dispersion Expansion of Kinetically Nonlocal Peridynamics

We consider the dispersion expansion of kinetically nonlocal peridynamics. First, to find the dispersion relation, we use $u(x,t) = \exp(\imath(kx - \omega t))$ in (4.7) to get:

$$\rho\omega^2 + \omega^2 \int_{\xi \in \mathbb{R}} P_\ell(\xi) \left(1 - \cos k\xi\right) \, d\xi = \int_{\xi \in \mathbb{R}} C_\ell(\xi) \left(1 - \cos k\xi\right) \, d\xi \tag{4.9}$$

We follow closely the calculations in Section 2.1 for both the nonlocal terms above.

Using the scaling $P_\ell(x'-x) = \frac{\rho}{\ell^3} \tilde{P}\left(\frac{x'-x}{\ell}\right)$, we obtain:

$$\rho\omega^2 + \omega^2 k^2 \int_{z \in \mathbb{R}} (1 - \cos k\ell z) \frac{\rho}{(kl)^2} \tilde{P}(z) \, dz = k^2 \int_{z \in \mathbb{R}} (1 - \cos k\ell z) \frac{E}{(kl)^2} \tilde{C}(z) \, dz \tag{4.10}$$

where $z = \xi/\ell$.

Using the Taylor expansion of $\cos k\ell z$ around $k\ell = 0$ gives:

$$\rho\omega^2 + \rho\omega^2 k^2 \sum_{n=1} (-1)^{n+1} (k\ell)^{2n-2} \int_{z \in \mathbb{R}} \frac{z^{2n} \tilde{P}(z)}{(2n)!} \, dz = E k^2 \sum_{n=1} (-1)^{n+1} (k\ell)^{2n-2} \int_{z \in \mathbb{R}} \frac{z^{2n} \tilde{C}(z)}{(2n)!} \, dz \tag{4.11}$$

This provides the polynomial expansion of $\hat{N}_\ell(k)$ and $\hat{D}_\ell(k)$ by comparing with (4.1).

An essential requirement of the rational function approximation is that the numerator and denominator have the same largest exponent of k. A physical interpretation of this requirement is that the kinetic energy nonlocality and the elastic energy nonlocality must both be of the same order for energetic and dynamic consistency. This is a significant constraint on peridynamic models, and we discuss this further in Section 7.

We notice that the energy expansion is unchanged by kinetic nonlocality.

4.2 Determining the Mean Values of the Peridynamic Kernels

We recall that the quantities $\bar{C}_\ell, \bar{P}_\ell$ have not yet been determined by the arguments in Section 4. Positive-definiteness of the energy (4.8) has as a sufficient condition that these quantities be positive, but does not provide a specific value. We examine this question further here.

The classical linear elastic momentum operator in 1D is $\frac{d^2}{dx^2}u(x)$. At leading order, peridynamics aims to mimic this behavior; hence, a possible Fourier space representation is $\hat{B}(k)\hat{u}(k)$, with $\hat{B}(k) \sim k^2$ to leading order. Inverting to find the real-space momentum operator gives $\int_{x'\in\mathbb{R}} B(x-x')u(x')\ dx'$, and the corresponding elastic energy is $\frac{1}{2}\int_{x'\in\mathbb{R}}\int_{x\in\mathbb{R}} B(x-x')u(x)u(x')\ dx\ dx'$. However, this elastic energy is *not* generally stable positive-definite, even for seemingly-reasonable choices for $B(x-x')$. For instance, one can notice that a discrete version of $\frac{1}{2}\int_{x'\in\mathbb{R}}\int_{x\in\mathbb{R}} B(x-x')u(x)u(x')\ dx\ dx'$ corresponds to the matrix multiplication operation $u.Bu$, and it is difficult to know if B is positive definite without evaluating the spectrum. For instance, simply choosing $B(x-x') \geq 0$ for all values of the argument does *not* guarantee positive-definiteness – as in the discrete case (see also Section 3 in [MPD17]).

The approach in peridynamics is not as described in the previous paragraph. Instead, as described in Section 4, the peridynamic momentum operator has the form $\int_{x'\in\mathbb{R}} C(x-x')u(x')\ dx' - u(x)\int_{x'\in\mathbb{R}} C(x-x')\ dx'$ and the corresponding elastic energy is $\int_{x'\in\mathbb{R}}\int_{x\in\mathbb{R}} \frac{1}{2}C(x-x')\left(u(x)-u(x')\right)^2\ dx\ dx'$. Using $C(x-x') \geq 0$ for all values of the argument is a sufficient – though possibly not necessary – condition for positive-definiteness of the elastic energy, as can be readily seen. Further, as can be observed from the calculations in Section 4, $C(x-x')$ does not have to be chosen to have leading order in k^2; the additional term $-u(x)\int_{x'\in\mathbb{R}} C(x-x')\ dx'$ precisely eliminates the Fourier mode at $k=0$. However, in doing so, the additional quantity $\bar{C}_\ell := \int_{x'\in\mathbb{R}} C(x-x')\ dx'$ is introduced. In this regard, note from Sections 2.1 and 2.2 that only the higher even moments of C appear in the approximating PDE models but $\bar{C}_\ell$ does not.

Since $\bar{C}_\ell$ is directly related to the value of the Fourier transform of $C(x-x')$ at $k=0$, we might expect that $\bar{C}_\ell$ can be deduced from the behavior of long-wavelength waves. Remarkably, it is instead short-wavelength objects that determine $\bar{C}_\ell$: namely, the limit of the dispersion relation as $k\to\infty$, as noted in (A.2). This also highlights the fact that the expansions discussed in previous sections are formal.

We will see (Section 6) that in peridynamics – both the classical and kinetically nonlocal models – discontinuities in the displacement field must be stationary, whereas a continuous approximation to a discontinuity need not generally be stationary. Therefore, approximation in a norm such as L^2 is not sufficient to completely capture the behavior of solutions; rather, approximation in a norm such as H^1 that carries information about the derivatives is important. The specific examples of L^2 and H^1 are chosen because there are standard theorems in Fourier analysis regarding the convergence of Fourier approximations in these norms, and the dispersion relation is essentially the Fourier transform of the peridynamic micromodulus kernel. In particular, we expect slow convergence in H^1 of the Fourier representation of discontinuous functions such as $H(x)$, e.g. as in the Gibbs phenomenon [Fol92].

We notice that using the theory of distributions [Zie12], we can write $u(x) = \int_{x'\in\mathbb{R}} \delta(x-x')u(x')\ dx'$ if $u(x')$ is sufficiently smooth. Therefore, the peridynamic nonlocal operator can be written as a convolution: $\int_{x'\in\mathbb{R}} \left[C(x-x') - \bar{C}_\ell\delta(x-x')\right]u(x')\ dx'$. As discussed above, this operator goes as k^2 to leading order, and $\bar{C}_\ell$ cancels out. Since the Fourier representation does not represent derivatives well for discontinuous functions, we can expect that the cancellation fails in those cases and allows one to isolate $\bar{C}_\ell$. Therefore, we examine the case where $u(x) = H(x)$ at $x = 0$.

Consider $H_\epsilon(x)$ that is a smooth and symmetric approximation of $H(x)$ that recovers $H(x)$ as $\epsilon \to 0$. Using $u(x) = H_\epsilon(x)$ and evaluating the convolution form of the peridynamic operator at $x = 0$ gives:

$$\int_{x'\in\mathbb{R}} \left[C(-x') - \bar{C}_\ell\delta(-x')\right] H_\epsilon(x')\, dx' \to \int_{x'\in\mathbb{R}^+} C(-x')\, dx' - \frac{1}{2}\bar{C}_\ell = \frac{1}{2}\bar{C}_\ell \tag{4.12}$$

On the other hand, we can evaluate the standard form of the peridynamic operator $\int_{x'\in\mathbb{R}} C(x-x')\left(u(x') - u(x)\right)\, dx'$ for $u(x) = H(x)$ at $x = 0$, which does not require smoothness of $u(x)$:

$$\int_{x'\in\mathbb{R}} C(-x')H(x')\, dx' - H(0)\int_{x'\in\mathbb{R}} C(-x')\, dx' = \int_{x'\in\mathbb{R}^+} C(-x')\, dx' - H(0)\bar{C}_\ell = \bar{C}_\ell \tag{4.13}$$

The calculation above is an elementary implication of the theory of distributions. Essentially, treating the discontinuous field as a limit of smooth fields leads to a different answer than directly treating the discontinuous field. Since the term involving the Dirac mass has $\bar{C}_\ell$ as the only information from the kernel $C(x - x')$, the discrepancy is necessarily related to $\bar{C}_\ell$ alone. We can think of the first approach – treating the discontinuity as a limit – as related to evaluation in terms of Fourier representations. As noted above, in the Fourier representation, convergence of derivatives is slow for discontinuous functions; loosely, this smooths out the function, and we need higher wavelengths ($k \to \infty$). On the other hand, the real-space evaluation does not cause smoothing. Therefore, the dispersion relation and the real-space evaluation differ from each other precisely by a quantity proportional to $\bar{C}_\ell$.

For the kinetically nonlocal case, closely-analogous arguments can be used for the nonlocal inertia term, noting that the dispersion relation leads to additional factors of ω.

5 Extension to Three Dimensions and Consistency with Balance Laws

We extend the formulation to three dimensions in the general geometrically and materially nonlinear setting, and examine the consistency of the model with the balances of linear momentum, angular momentum and energy for finite bodies. The balances provide important constraints on the nonlocal inertia. In summary, if the conditions (5.6) and (5.21) are satisfied, we obtain consistency with the balance laws of continuum mechanics.

Consider a three-dimensional finite body Ω. Following [SL10], we begin by writing the balance of linear momentum in the reference for any sub-body V:

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\boldsymbol{x}\in V} \rho(\boldsymbol{x})\dot{\boldsymbol{y}}(\boldsymbol{x},t)\, dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in V}\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x},\boldsymbol{x}')\left(\dot{\boldsymbol{y}}(\boldsymbol{x},t) - \dot{\boldsymbol{y}}(\boldsymbol{x}',t)\right)\, dV_{\boldsymbol{x}}\, dV_{\boldsymbol{x}'}\right) \\ &= \int_{\boldsymbol{x}\in V}\int_{\boldsymbol{x}'\in\Omega\setminus V} \boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x},t) - \boldsymbol{y}(\boldsymbol{x}',t))\, dV_{\boldsymbol{x}}\, dV_{\boldsymbol{x}'} + \int_{\boldsymbol{x}\in V} \boldsymbol{b}(\boldsymbol{x},t)\, dV_{\boldsymbol{x}} \end{aligned} \tag{5.1}$$

The terms within the time derivative are the linear momentum content in V, the term containing $\boldsymbol{f}$ represents the force on V due to the remainder of the body $\Omega \backslash V$. The external body force is $\boldsymbol{b}$.

In classical kinetically local peridynamics, $\boldsymbol{f}$ is assumed to satisfy two conditions [Sil00]. First, $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}', t) - \boldsymbol{y}(\boldsymbol{x}, t)) = -\boldsymbol{f}(\boldsymbol{x}', \boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t))$ for consistency with linear momentum balance. Second, $\boldsymbol{f}$ must be parallel to $\boldsymbol{y}(\boldsymbol{x}', t) - \boldsymbol{y}(\boldsymbol{x}, t) := \boldsymbol{x}' + \boldsymbol{u}(\boldsymbol{x}', t) - \boldsymbol{x} - \boldsymbol{u}(\boldsymbol{x}, t)$ for consistency with angular momentum balance. We retain these conditions on $\boldsymbol{f}$. Further, classical kinetically local peridynamics has no way to apply traction boundary conditions. Instead, tractions can be approximately applied by using appropriate body forces over a boundary volume. This feature is also unchanged in kinetically nonlocal peridynamics.

We note the following useful result, where $\boldsymbol{A}$ can be a scalar, vector or tensor function:

$$\begin{aligned} &\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}', t) - \boldsymbol{y}(\boldsymbol{x}, t)) = -\boldsymbol{A}(\boldsymbol{x}', \boldsymbol{x}, \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t)) \\ &\Rightarrow \int_{\boldsymbol{x}\in V} \int_{\boldsymbol{x}'\in V} \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t))\, dV_{\boldsymbol{x}}\, dV_{\boldsymbol{x}'} = 0 \text{ for any } V \end{aligned} \tag{5.2}$$

We use this below a number of times, and refer to it as the "anti-symmetry property" for short.

We notice that $\boldsymbol{f}$ satisfies the anti-symmetry property. Physically, this corresponds to the fact that V does not exert a net force on itself. Adding $\int_{\boldsymbol{x}\in V} \int_{\boldsymbol{x}'\in V} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t))\, dV_{\boldsymbol{x}}\, dV_{\boldsymbol{x}'} = 0$ to the right side of (5.1), and combining it with the term representing the force on V due to the remainder of the body $\Omega \backslash V$, lets us write (5.1) as:

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\boldsymbol{x}\in V} \rho(\boldsymbol{x})\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in V}\int_{\boldsymbol{x}'\in \Omega} P_\ell(\boldsymbol{x}, \boldsymbol{x}')\left(\dot{\boldsymbol{y}}(\boldsymbol{x}, t) - \dot{\boldsymbol{y}}(\boldsymbol{x}', t)\right)\, dV_{\boldsymbol{x}}\, dV_{\boldsymbol{x}'}\right) \\ &= \int_{\boldsymbol{x}\in V}\int_{\boldsymbol{x}'\in \Omega} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t))\, dV_{\boldsymbol{x}}\, dV_{\boldsymbol{x}'} + \int_{\boldsymbol{x}\in V} \boldsymbol{b}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} \end{aligned} \tag{5.3}$$

Assuming that motions are sufficiently smooth that we can move the time derivative within the integral, and using that (5.3) holds for all sub-bodies V, we can write the local form of the linear momentum balance as follows:

$$\rho(\boldsymbol{x})\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) + \int_{\boldsymbol{x}'\in \Omega} P_\ell(\boldsymbol{x}, \boldsymbol{x}')\left(\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) - \ddot{\boldsymbol{y}}(\boldsymbol{x}', t)\right)\, dV_{\boldsymbol{x}'} = \int_{\boldsymbol{x}'\in \Omega} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t))\, dV_{\boldsymbol{x}'} + \boldsymbol{b}(\boldsymbol{x}, t) \tag{5.4}$$

5.1 Balance of Linear Momentum

To examine the consistency of kinetically nonlocal peridynamics with linear momentum balance, we integrate (5.4) over Ω:

$$\int_{\boldsymbol{x}\in \Omega} \rho(\boldsymbol{x})\ddot{\boldsymbol{y}}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in \Omega}\int_{\boldsymbol{x}'\in \Omega} P_\ell(\boldsymbol{x}, \boldsymbol{x}')\left(\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) - \ddot{\boldsymbol{y}}(\boldsymbol{x}', t)\right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} = \int_{\boldsymbol{x}\in \Omega} \boldsymbol{b}(\boldsymbol{x}, t) \tag{5.5}$$

The contribution due to $\boldsymbol{f}$ vanishes due to the anti-symmetry property.

If we set:

$$P_\ell(\boldsymbol{x}, \boldsymbol{x}') = P_\ell(\boldsymbol{x}', \boldsymbol{x}) \tag{5.6}$$

then the integral containing P_ℓ satisfies the anti-symmetry property and it vanishes in (5.5).

If we further define:

$$\text{Net force: } \boldsymbol{F} = \int_{\boldsymbol{x}\in\Omega} \boldsymbol{b}(\boldsymbol{x}, t) \tag{5.7a}$$

$$\text{Mass: } M = \int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\, dV_{\boldsymbol{x}} \tag{5.7b}$$

$$\text{Center of Mass: } \boldsymbol{y}_c(t) = \frac{1}{M}\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\boldsymbol{y}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} \tag{5.7c}$$

then we can write (5.5) in the classical form:

$$\boldsymbol{F} = M\ddot{\boldsymbol{y}}_c(t) \tag{5.8}$$

We recall that peridynamics provides no mechanism to apply tractions, though a close analogy to traction can be achieved by applying the appropriate body forces.

5.2 Balance of Energy

To examine the consistency of kinetically nonlocal peridynamics with energy balance, we take the inner product of (5.4) with $\dot{\boldsymbol{y}}(\boldsymbol{x}, t)$ and integrate over Ω:

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x}, \boldsymbol{x}')\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\left(\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) - \ddot{\boldsymbol{y}}(\boldsymbol{x}', t)\right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} \\
&= \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}(\boldsymbol{x}, t) - \boldsymbol{y}(\boldsymbol{x}', t))\cdot\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in\Omega} \dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\boldsymbol{b}(\boldsymbol{x}, t)
\end{aligned} \tag{5.9}$$

We can rewrite the first term in (5.9) as $\dfrac{d}{dt}\displaystyle\int_{\boldsymbol{x}\in\Omega} \frac{1}{2}\rho(\boldsymbol{x})|\dot{\boldsymbol{y}}(\boldsymbol{x}, t)|^2\, dV_{\boldsymbol{x}}$.

Using (5.6), we can rewrite the second term in (5.9) as follows:

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x}, \boldsymbol{x}')\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\left(\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) - \ddot{\boldsymbol{y}}(\boldsymbol{x}', t)\right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} \\
&= \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} \frac{P_\ell(\boldsymbol{x}, \boldsymbol{x}')}{4}\left(4\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) - 4\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}', t)\right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} \\
&= \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} \frac{P_\ell(\boldsymbol{x}, \boldsymbol{x}')}{4}\left(2\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}, t) + 2\dot{\boldsymbol{y}}(\boldsymbol{x}', t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}', t) - 2\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}', t) - 2\dot{\boldsymbol{y}}(\boldsymbol{x}', t)\cdot\ddot{\boldsymbol{y}}(\boldsymbol{x}, t)\right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} \\
&= \frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} \frac{P_\ell(\boldsymbol{x}, \boldsymbol{x}')}{4}\left(|\dot{\boldsymbol{y}}(\boldsymbol{x}, t)|^2 + |\dot{\boldsymbol{y}}(\boldsymbol{x}', t)|^2 - \dot{\boldsymbol{y}}(\boldsymbol{x}, t)\cdot\dot{\boldsymbol{y}}(\boldsymbol{x}', t) - \dot{\boldsymbol{y}}(\boldsymbol{x}', t)\cdot\dot{\boldsymbol{y}}(\boldsymbol{x}, t)\right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} \\
&= \frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} \frac{P_\ell(\boldsymbol{x}, \boldsymbol{x}')}{4}\left|\dot{\boldsymbol{y}}(\boldsymbol{x}, t) - \dot{\boldsymbol{y}}(\boldsymbol{x}', t)\right|^2\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}}
\end{aligned} \tag{5.10}$$

We consider next the third term in (5.9). Following [Sil00] in defining the peridynamic analog of an elastic material, assume that $\boldsymbol{f}$ is related to a stored energy w such that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{\eta}) = \dfrac{\partial}{\partial\boldsymbol{\eta}} w(\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{\eta})$.

Using that $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t)) = -\boldsymbol{f}(\boldsymbol{x}',\boldsymbol{x},\boldsymbol{y}(\boldsymbol{x},t)-\boldsymbol{y}(\boldsymbol{x}',t))$, we can rewrite this term as follows:

$$\begin{aligned} &-\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\frac{1}{2}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\cdot(\dot{\boldsymbol{y}}(\boldsymbol{x}',t)-\dot{\boldsymbol{y}}(\boldsymbol{x},t))\ dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \\ &= -\frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\frac{1}{2}w(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \end{aligned} \tag{5.11}$$

The fourth term in (5.9) is the external working due to $\boldsymbol{b}$. Putting (5.10) and (5.11) in (5.9), we obtain the analog to the classical balance of energy:

$$\begin{aligned} &\frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\frac{\rho(\boldsymbol{x})}{2}|\dot{\boldsymbol{y}}(\boldsymbol{x},t)|^2\,dV_{\boldsymbol{x}} + \frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\frac{P_\ell(\boldsymbol{x},\boldsymbol{x}')}{4}\left|\dot{\boldsymbol{y}}(\boldsymbol{x},t)-\dot{\boldsymbol{y}}(\boldsymbol{x}',t)\right|^2\ dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \\ &+\frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\frac{1}{2}w(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} = \int_{\boldsymbol{x}\in\Omega}\dot{\boldsymbol{y}}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t) \end{aligned} \tag{5.12}$$

5.3 Balance of Angular Momentum

To consider the consistency of kinetically nonlocal peridynamics with angular momentum balance, we take the cross product of (5.4) with $\dot{\boldsymbol{y}}(\boldsymbol{x},t)$ and integrate over Ω:

$$\begin{aligned} &\int_{\boldsymbol{x}\in\Omega}\rho(\boldsymbol{x})\dot{\boldsymbol{y}}(\boldsymbol{x},t)\times\ddot{\boldsymbol{y}}(\boldsymbol{x},t)\,dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}P_\ell(\boldsymbol{x},\boldsymbol{x}')\dot{\boldsymbol{y}}(\boldsymbol{x},t)\times(\ddot{\boldsymbol{y}}(\boldsymbol{x},t)-\ddot{\boldsymbol{y}}(\boldsymbol{x}',t))\ dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \\ &= \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\dot{\boldsymbol{y}}(\boldsymbol{x},t)\times\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x},t)-\boldsymbol{y}(\boldsymbol{x}',t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} + \int_{\boldsymbol{x}\in\Omega}\dot{\boldsymbol{y}}(\boldsymbol{x},t)\times\boldsymbol{b}(\boldsymbol{x},t) \end{aligned} \tag{5.13}$$

We rewrite the first term in (5.13) as $\frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\rho(\boldsymbol{x})\boldsymbol{y}(\boldsymbol{x},t)\times\dot{\boldsymbol{y}}(\boldsymbol{x},t)\,dV_{\boldsymbol{x}}$.

We rewrite the second term in (5.13) as follows:

$$\begin{aligned} &\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}P_\ell(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{y}(\boldsymbol{x},t)\times(\ddot{\boldsymbol{y}}(\boldsymbol{x},t)-\ddot{\boldsymbol{y}}(\boldsymbol{x}',t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \\ &= \frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}P_\ell(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{y}(\boldsymbol{x},t)\times(\dot{\boldsymbol{y}}(\boldsymbol{x},t)-\dot{\boldsymbol{y}}(\boldsymbol{x}',t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \end{aligned} \tag{5.14}$$

where the term $\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}P_\ell(\boldsymbol{x},\boldsymbol{x}')\dot{\boldsymbol{y}}(\boldsymbol{x},t)\times\dot{\boldsymbol{y}}(\boldsymbol{x}',t)\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}}$ vanishes due to the anti-symmetry property, recalling (5.6).

We show that the third term in (5.13) vanishes. Recall that $\boldsymbol{f}$ must be parallel to $\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t)$. Therefore, we write $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t)) = (\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\,F(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))$ where F is a scalar function that is invariant under $\boldsymbol{x}\leftrightarrow\boldsymbol{x}'$. We can then write this term as

$$\begin{aligned} &\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\boldsymbol{y}(\boldsymbol{x},t)\times(\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\,F(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \\ &= \int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega}\boldsymbol{y}(\boldsymbol{x},t)\times\boldsymbol{y}(\boldsymbol{x}',t)F(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{y}(\boldsymbol{x}',t)-\boldsymbol{y}(\boldsymbol{x},t))\,dV_{\boldsymbol{x}'}\,dV_{\boldsymbol{x}} \end{aligned} \tag{5.15}$$

which then vanishes from the anti-symmetry property.

We rewrite the fourth term in (5.13) as:

$$\boldsymbol{M_0}(t) := \int_{\boldsymbol{x}\in\Omega} \boldsymbol{y}(\boldsymbol{x},t) \times \boldsymbol{b}(\boldsymbol{x},t)\ dV_{\boldsymbol{x}} \tag{5.16}$$

where $\boldsymbol{M_0}(t)$ is the total moment about $\boldsymbol{0}$ acting on Ω.

Putting together (5.14), (5.15), (5.16), we can write the analog of the balance of angular momentum:

$$\frac{d}{dt}\left(\int_{\boldsymbol{x}\in\Omega} \boldsymbol{y}(\boldsymbol{x},t) \times \left(\rho(\boldsymbol{x})\dot{\boldsymbol{y}}(\boldsymbol{x},t) + \int_{\boldsymbol{x}'\in\Omega} \left(P_\ell(\boldsymbol{x},\boldsymbol{x}')(\dot{\boldsymbol{y}}(\boldsymbol{x},t) - \dot{\boldsymbol{y}}(\boldsymbol{x}',t))\right)\ dV_{\boldsymbol{x}'}\right)\ dV_{\boldsymbol{x}}\right) = \boldsymbol{M_0}(t) \tag{5.17}$$

5.3.1 Rigid Deformations and the Euler Equation

We notice that the balance of angular momentum has not provided any further constraints on P_ℓ up to this point. However, we now go further and require that kinetically nonlocal peridynamics recover the Euler equation for rigid body rotation.

Following the calculations in [JR03] in the context of classical continuum mechanics, we begin with the assumption[3] that the deformation is rigid:

$$\boldsymbol{y}(\boldsymbol{x},t) = \boldsymbol{R}(t)\,(\boldsymbol{x} - \boldsymbol{x}_c) + \boldsymbol{y}_c(t) \tag{5.18}$$

where $\boldsymbol{R}(t)$ is any rotation, and $\boldsymbol{y}_c(t)$ and $\boldsymbol{x}_c$ is the center of mass in the deformed and reference configurations respectively. Since $\boldsymbol{R}(t)$ is a rotation, we can define the skew tensor $\boldsymbol{W}(t)$ through $\dot{\boldsymbol{R}}(t) = \boldsymbol{R}(t)\boldsymbol{W}(t)$, and also the axial vector $\boldsymbol{\omega}(t)$ through $\boldsymbol{W}\boldsymbol{a} = \boldsymbol{\omega} \times \boldsymbol{a}$ for any vector $\boldsymbol{a}$. This lets us write $\dot{\boldsymbol{y}}(\boldsymbol{x},t) = \boldsymbol{R}(t)\,(\boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{x}_c)) + \dot{\boldsymbol{y}}_c(t)$.

Notice that $\boldsymbol{y}(\boldsymbol{x}_c,t) = \boldsymbol{y}_c(t)$. Recalling that $\boldsymbol{y}_c(t) = \dfrac{1}{M}\displaystyle\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\boldsymbol{y}(\boldsymbol{x},t)\ dV_{\boldsymbol{x}}$, we substitute the rigid ansatz on the right side of this definition to get $\boldsymbol{x}_c = \dfrac{1}{M}\displaystyle\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\boldsymbol{x}\ dV_{\boldsymbol{x}}$.

The identities are used below: $\boldsymbol{R}(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{R}\boldsymbol{a} \times \boldsymbol{R}\boldsymbol{b}$ and $\boldsymbol{a} \times (\boldsymbol{\omega} \times \boldsymbol{b}) = [(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{I} - \boldsymbol{b}\otimes\boldsymbol{a}]\,\boldsymbol{\omega}$ for any rotation $\boldsymbol{R}$ and any vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}$.

We now substitute the rigid ansatz in the left side of (5.17) and examine the terms individually.

Consider the term $\dfrac{d}{dt}\displaystyle\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\boldsymbol{y}(\boldsymbol{x},t) \times \dot{\boldsymbol{y}}(\boldsymbol{x},t)\ dV_{\boldsymbol{x}}$ in (5.17). Substituting the rigid ansatz gives 4

[3] It appears to be an open problem to show the consistency of Euler's equations with continuum mechanics without this assumption even in classical continuum models.

distinct terms:

$$\frac{d}{dt}\boldsymbol{y}_c(t) \times \dot{\boldsymbol{y}}_c(t) \int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\, dV_{\boldsymbol{x}} = \frac{d}{dt} M \boldsymbol{y}_c(t) \times \dot{\boldsymbol{y}}_c(t) \tag{5.19a}$$

$$\frac{d}{dt}\boldsymbol{y}_c(t) \times \boldsymbol{R}(t) \left(\boldsymbol{\omega}(t) \times \int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{x}_c)\, dV_{\boldsymbol{x}} \right) = 0 \tag{5.19b}$$

$$\frac{d}{dt}\left(\boldsymbol{R}(t) \left(\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})(\boldsymbol{x}-\boldsymbol{x}_c)\, dV_{\boldsymbol{x}} \right) \right) \times \dot{\boldsymbol{y}}_c(t) = 0 \tag{5.19c}$$

$$\frac{d}{dt}\boldsymbol{R}(t) \underbrace{\left(\int_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x}) \left((\boldsymbol{x}-\boldsymbol{x}_c)\cdot(\boldsymbol{x}-\boldsymbol{x}_c)\boldsymbol{I} - (\boldsymbol{x}-\boldsymbol{x}_c)\otimes(\boldsymbol{x}-\boldsymbol{x}_c) \right)\, dV_{\boldsymbol{x}} \right)}_{\text{moment of inertia tensor } \mathcal{I}} \boldsymbol{\omega}(t) \tag{5.19d}$$

The terms that go to 0 do so directly from the definition of $\boldsymbol{x}_c$. The last term takes the classical form $\frac{d}{dt}\boldsymbol{R}(t)\mathcal{I}\boldsymbol{\omega}(t)$, e.g. [JR03, O'R17].

Consider the term $\frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{y}(\boldsymbol{x},t) \times (\dot{\boldsymbol{y}}(\boldsymbol{x},t) - \dot{\boldsymbol{y}}(\boldsymbol{x}',t))\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}}$ in (5.17). Substituting the rigid motion ansatz gives:

$$\frac{d}{dt}\int_{\boldsymbol{x}\in\Omega}\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x},\boldsymbol{x}') \left(\boldsymbol{R}(t)(\boldsymbol{x}-\boldsymbol{x}_c) + \boldsymbol{y}_c(t) \right) \times \boldsymbol{R}(t) \left(\boldsymbol{\omega}(t) \times (\boldsymbol{x}-\boldsymbol{x}') \right)\, dV_{\boldsymbol{x}'}\, dV_{\boldsymbol{x}} \tag{5.20}$$

which goes to $\mathbf{0}$ if P_ℓ satisfies:

$$\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x},\boldsymbol{x}')(\boldsymbol{x}'-\boldsymbol{x})\, dV_{\boldsymbol{x}'} = 0 \Leftrightarrow \int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x},\boldsymbol{x}')\boldsymbol{x}'\, dV_{\boldsymbol{x}'} = \boldsymbol{x}\int_{\boldsymbol{x}'\in\Omega} P_\ell(\boldsymbol{x},\boldsymbol{x}')\, dV_{\boldsymbol{x}'} \tag{5.21}$$

Substituting from (5.19) in (5.17) gives us the Euler equation for rigid body dynamics:

$$\frac{d}{dt}\left(M\boldsymbol{y}_c(t) \times \dot{\boldsymbol{y}}_c(t) + \boldsymbol{R}(t)\mathcal{I}\boldsymbol{\omega}(t) \right) = \boldsymbol{M}_{\mathbf{0}}(t) \tag{5.22}$$

6 Displacement and Velocity Discontinuities

We examine the behavior of displacement and velocity discontinuities in peridynamics. For simplicity, we confine the calculations to 1D, though it appears straightforward to extend them to 3D. Key elements of the approach here parallel [WA05]. In particular, we find that velocity discontinuities are stationary in the reference configuration in kinetically nonlocal peridynamics, as is also the case in classical peridynamics [WA05]. However, we further examine the dissipation associated with displacement discontinuities that can propagate, and find that there can be dissipation even in nominally elastic peridynamics – both the classical and the kinetically nonlocal versions. Unexpectedly, we find that the dissipation does not vanish in the linear setting.

Consider a finite body in one dimension[4] that we denote by Ω in the reference. Consider a discontinuity in displacement and velocity located at $x = s(t)$, and V an arbitrary sub-body of Ω that contains the discontinuity.

[4]Due to the difficulty in peridynamics of defining the elastic stored energy of sub-bodies (Section 2.4 in [SL10]), we do not begin with an infinite body and then consider a finite part of it as we did in previous sections. This is related to the presence of long-range interactions, and analogous issues arise in atomistic models and continuum models with electromagnetic interactions [MD14, Liu13].

We begin by finding the jump conditions at a discontinuity. Consider balance of linear momentum over V:

$$\frac{d}{dt}\int_{x\in V}\left(\rho(x)\dot{u}(x,t)+\int_{x'\in\Omega}P_\ell(x,x')\left(\dot{u}(x,t)-\dot{u}(x',t)\right)\,dx'\right)\,dx=\int_{x\in V}\int_{x'\in\Omega\backslash V}f(x,x',u(x',t)-u(x,t))\,dx'\,dx \tag{6.1}$$

where the left side is the rate of change of linear momentum and the right side is the force exerted on V by the remainder of Ω.

The antisymmetry of f with respect to $x\leftrightarrow x'$ implies that $\int_{x\in V}\int_{x'\in V}f(x,x',u(x',t)-u(x,t))\,dx'\,dx=0$. Adding this expression to the right side of the equation above, we can write:

$$\frac{d}{dt}\int_{x\in V}\left(\rho(x)\dot{u}(x,t)+\int_{x'\in\Omega}P_\ell(x,x')\left(\dot{u}(x,t)-\dot{u}(x',t)\right)\,dx'\right)\,dx=\int_{x\in V}\int_{x'\in\Omega}f(x,x',u(x',t)-u(x,t))\,dx'\,dx \tag{6.2}$$

Using the Leibniz rule, we can write:

$$\begin{aligned}\int_{x\in V}&\left(\rho(x)\ddot{u}(x,t)+\dot{s}[\![P_\ell(x,s)\dot{u}(s)]\!]+\int_{x'\in\Omega}\left(P_\ell(x,x')\left(\ddot{u}(x,t)-\ddot{u}(x',t)\right)-f(x,x',u(x',t)-u(x,t))\right)\,dx'\right)\,dx\\&=\dot{s}\left[\!\!\left[\dot{u}\left(\rho+\int_{x'\in\Omega}P_\ell(x',s)\,dx'\right)\right]\!\!\right]\end{aligned} \tag{6.3}$$

Using the arbitrariness of V, we can localize to write the field equation away from the discontinuity:

$$\rho(x)\ddot{u}(x,t)+\dot{s}[\![P_\ell(x,s)\dot{u}(s)]\!]+\int_{x'\in\Omega}\left(P_\ell(x,x')\left(\ddot{u}(x,t)-\ddot{u}(x',t)\right)-f(x,x',u(x',t)-u(x,t))\right)\,dx'=0 \tag{6.4}$$

and the jump condition on the discontinuity:

$$\dot{s}\left[\!\!\left[\dot{u}\left(\rho(s)+\int_{x'\in\Omega}P_\ell(x',s)\,dx'\right)\right]\!\!\right]=0 \tag{6.5}$$

Assuming that $\rho(x)$ and $\int_{x'\in\Omega}P_\ell(x',x)\,dx'$ are continuous in x, we conclude that either $\dot{s}=0$ or $[\![\dot{u}]\!]=0$ from the jump condition. This further implies that the field equation reduces to:

$$\rho(x)\ddot{u}(x,t)+\int_{x'\in\Omega}P_\ell(x,x')\left(\ddot{u}(x,t)-\ddot{u}(x',t)\right)\,dx'=\int_{x'\in\Omega}f(x,x',u(x',t)-u(x,t))\,dx' \tag{6.6}$$

The result that either $\dot{s}=0$ or $[\![\dot{u}]\!]=0$ in kinetically nonlocal peridynamics matches with the findings of [WA05] in classical peridynamics. However, it appears that the setting $\dot{s}\neq 0$ and $[\![\dot{y}]\!]=0$, but with $[\![y]\!]\neq 0$, is allowable in both classical and kinetically nonlocal peridynamics. That is, we can have a discontinuity in displacement that moves if it is *not* associated with a genuine velocity discontinuity. We examine below stationary discontinuities briefly and propagating discontinuities in more detail.

6.1 Evolution of Stationary Discontinuities in Linear Kinetically Nonlocal Peridynamics

[WA05] performed an insightful calculation into the time evolution of a stationary displacement discontinuity ($\dot{s} = 0$) in classical peridynamics. We perform a very similar calculation here in kinetically nonlocal peridynamics.

We begin by writing the field equation in the form:

$$\left(\rho(x) + \bar{P}_\ell\right) \ddot{u}(x,t) + \bar{C}_\ell u(x,t) = \int_{x' \in \Omega} \left(P_\ell(x,x')\ddot{u}(x',t) + C_\ell(x,x')u(x',t)\right) \, dx' \tag{6.7}$$

Evaluate this equation at $x = s + \epsilon$ and $x = s - \epsilon$ and subtract these equations, and then take the limit $\epsilon \to 0$:

$$\left(\rho(s) + \bar{P}_\ell\right) [\![\ddot{u}(s,t)]\!] + \bar{C}_\ell [\![u(s,t)]\!] = 0 \tag{6.8}$$

where the integral terms vanish if $P_\ell(x,x')$ and $C_\ell(x,x')$ are bounded.

This provides a linear ordinary differential equation for the evolution of $[\![u(s,t)]\!]$. [WA05] examine this for various cases in classical peridynamics, and we refer the reader to that work for more details. In our context, we can directly note from (6.8) that a sufficient condition for the stability of the evolution of $[\![u(s,t)]\!]$ is that $\bar{C}_\ell > 0$ and $\bar{P}_\ell > 0$.

6.2 Dissipation due to Propagating Discontinuities in Classical and Kinetically Nonlocal Peridynamics

We next consider the possibility of dissipation due to the motion of the displacement discontinuity. We assume that there is no discontinuity in the velocity.

The dissipation rate D is defined as the difference between the rates of external work done on Ω by the external body force b, and the stored kinetic and elastic energy:

$$\begin{aligned} D = &\int_{x \in \Omega} b(x,t)\dot{u}(x,t) \, dx - \frac{d}{dt}\int_{x \in \Omega} \frac{1}{2}\rho(x)\dot{u}^2(x,t) \, dx - \frac{d}{dt}\int_{x \in \Omega}\int_{x' \in \Omega} \frac{1}{4}P_\ell(x,x')\left(\dot{u}(x,t) - \dot{u}(x',t)\right)^2 \, dx' \, dx \\ &- \frac{d}{dt}\int_{x \in \Omega}\int_{x' \in \Omega} \frac{1}{2} w(x,x',u(x',t) - u(x,t)) \, dx \, dx' \end{aligned} \tag{6.9}$$

The factor of $\frac{1}{2}$ in front of w is due to the double-counting of each bond.

The kinetic energy term can be written as

$$\int_{x \in \Omega} \rho(x)\ddot{u}(x,t)\dot{u}(x,t) \, dx + \int_{x \in \Omega}\int_{x' \in \Omega} \frac{1}{4}P_\ell(x,x')\left(\ddot{u}(x,t) - \ddot{u}(x',t)\right)\dot{u}(x,t) \, dx' \, dx \tag{6.10}$$

because there is no velocity discontinuity, and using the symmetry of P_ℓ in its arguments.

The stored energy term can be written as:

$$-\int_{x \in \Omega}\int_{x' \in \Omega} \frac{1}{2} f(x,x',u(x',t) - u(x,t)) \cdot \left(\dot{u}(x',t) - \dot{u}(x,t)\right) \, dx \, dx' + \dot{s}\int_{x \in \Omega} [\![w(x,s,u(s,t) - u(x,t))]\!] \tag{6.11}$$

The first term comes from applying the chain rule. The second term comes from applying the Leibniz rule to the inner and outer integrals in turn and then combining the result, noting that integration and the jump operation commute.

We can use the antisymmetry of f to rewrite

$$\begin{aligned}&\int_{x\in\Omega}\int_{x'\in\Omega}\frac{1}{2}f(x,x',u(x',t)-u(x,t))\cdot(\dot{u}(x',t)-\dot{u}(x,t))\ dx\,dx'\\&=-\int_{x\in\Omega}\int_{x'\in\Omega}f(x,x',u(x',t)-u(x,t))\dot{u}(x,t)\,dx\,dx'\end{aligned}\tag{6.12}$$

Using all of these in the expression for D cancels out almost everything by noting the balance of linear momentum, leaving only the term:

$$D=\dot{s}\int_{x\in\Omega}[\![w(x,s,u(s,t)-u(x,t))]\!]\,dx\tag{6.13}$$

Silling [Sil00] points out that the expression $\int_{x'\in\Omega}\frac{1}{2}w(x,x',u(x')-u(x))$ can be considered as the energy density W at x: it splits equally between x and x' the energy of the bond connecting x and x', and then integrates over x'. This lets us write the dissipation as $D=2\dot{s}[\![W]\!]$, noting that integration and the jump operation commute.

Notice that the expression for dissipation in classical nonlinear elasticity has additional terms that are missing here, e.g. [AK06]. When specialized to classical linear elasticity, the terms in the dissipation balance exactly and give the result that there is no dissipation for any motion even if there are discontinuities present.

An unexpected consequence of those missing terms is that linear peridynamics has dissipation, or at least it is not clear that the dissipation will vanish for generic motions. To see this, we note that w has the form $\frac{1}{2}C(x,x')\cdot(u(x')-u(x))^2$ in the linear peridynamic setting, and assume for simplicity that C is continuous in both arguments[5]. Then the dissipation is

$$D=\dot{s}\int_{x\in\Omega}\frac{C(x,s)}{4}\left((u(s^+)-u(x))^2-(u(s^-)-u(x))^2\right)\ dx\tag{6.14}$$

While superposition is valid in linear peridynamics as in classical linear elasticity, the latter model allows one to examine shocks as a limit of smooth solutions while the former model does not, as this calculation illustrates. In particular, smooth solutions in linear peridynamics do not have dissipation, while discontinuities that move can have dissipation. In contrast, nonsmooth solutions in classical linear elasticity do not dissipate. This is closely related to the discussion of convergence in different norms in Section 4.2.

One possible way for nonsmooth deformation fields to develop is through the application of suitable external body forces. Note that the working due to b is already accounted for in (6.9), and there are no smoothness requirements on b.

Finally, all of our conclusions, including the expression $D=\dot{s}\int_{x\in\Omega}[\![w(x,s,u(s,t)-u(x,t))]\!]\,dx$, carry over unchanged to classical peridynamics. We can simply set $P_\ell\equiv 0$ in the calculations above to examine this setting.

[5] It is also required to be symmetric in the arguments to achieve anti-symmetry of f.

6.3 Numerical Construction of Steadily Propagating Discontinuities using Traveling Waves

In this section, we construct explicit numerical solutions for problems with propagating discontinuities for a specific linear homogeneous peridynamic material. We use the framework of traveling waves to examine steadily propagating discontinuities. Traveling wave problems generically lead to eigenvalue problems. The approach that we use below is to use a traveling wave form with an assumed discontinuous part, and solve for the correction to the discontinuous part such that it satisfies the traveling wave equation. Thus we convert from a eigenvalue problem to a forced linear problem to ensure that the solutions that we find contain a discontinuous part.

We make the following choices for C_ℓ and P_ℓ, motivated by the reason that these allow easy closed-form expressions and have a lot of smoothness:

$$C_\ell(x,x') = \bar{C}_\ell \frac{e^{-(x'-x)^2/2\ell^2}}{\sqrt{2\pi}\ell}, \quad P_\ell(x,x') = \bar{P}_\ell \frac{e^{-(x'-x)^2/2\ell^2}}{\sqrt{2\pi}\ell} \tag{6.15}$$

We work on an infinite body to be consistent with the traveling wave assumption, and this drives the choice of normalization above. The dispersion relation is:

$$\rho\omega^2(k) = \frac{\bar{C}_\ell \left(1 - e^{-\frac{1}{2}k^2 l^2}\right)}{1 + \frac{\bar{P}_\ell}{\rho}\left(1 - e^{-\frac{1}{2}k^2 l^2}\right)} \tag{6.16}$$

We notice that the phase velocity $c(k) := \omega(k)/k$ has the long wavelength limit $c_0 = \lim_{k\to 0} c(k) = \sqrt{\frac{\ell^2 \bar{C}_\ell}{2\rho}} = \sqrt{\frac{E}{\rho}}$.

Consider now a traveling wave solution:

$$u(x,t) = U(x - V_0 t) \Rightarrow \ddot{u}(x,t) = V_0^2 U''(x - V_0 t) \tag{6.17}$$

We work in the traveling frame and define $w := x - V_0 t$. The governing equation for U is obtained by substituting into the balance of momentum:

$$\frac{M^2\ell^2}{2}U''(w) + \frac{M^2\ell^2}{2}\left(\frac{\bar{P}_\ell}{\rho}\right)\int_{w'\in\mathbb{R}} \frac{e^{-(w'-w)^2/2\ell^2}}{\sqrt{2\pi}\ell}\left(U''(w) - U''(w')\right)\,dw' = \int_{w'\in\mathbb{R}} \frac{e^{-(w'-w)^2/2\ell^2}}{\sqrt{2\pi}\ell}\left(U(w') - U(w)\right)\,dw' \tag{6.18}$$

We have nondimensionalized and defined $M := V_0/c_0$. This is a generalized eigenvalue problem with M corresponding to the eigenvalue.

We decompose $U(w) = U_1(w) + U_2(w)$, where $U_1(w)$ is to be solved for, and $U_2(w) = \begin{cases} A^- w, w < 0 \\ A^+ w + B, w \geq 0 \end{cases}$
This forces U to have a discontinuity – and a discontinuity in slope – at $w = 0$. Substituting this decomposition into the governing equation gives:

$$\begin{aligned} &\frac{M^2\ell^2}{2}U_1''(w) + \frac{M^2\ell^2}{2}\left(\frac{\bar{P}_\ell}{\rho}\right)\int_{w'\in\mathbb{R}} \frac{e^{-(w'-w)^2/2\ell^2}}{\sqrt{2\pi}\ell}\left(U_1''(w) - U_1''(w')\right)\,dw' - \int_{w'\in\mathbb{R}} \frac{e^{-(w'-w)^2/2\ell^2}}{\sqrt{2\pi}\ell}\left(U_1(w') - U_1(w)\right)\,dw' \\ &= \frac{(A^+ - A^-)\ell e^{-w^2/2\ell^2}}{\sqrt{2\pi}} + \frac{1}{2}\left(B + (A^+ - A^-)w\right)\left(\operatorname{erf}\left(w/\sqrt{2}\ell\right) - \operatorname{sign}(w)\right) \end{aligned} \tag{6.19}$$

This equation is no longer an eigenvalue problem, but instead simply a linear equation for U_1 given the forcing on the right side.

We assume that the singularities that develop due to the discontinuity of U_2 are balanced by singularities in the applied body force b. Discontinuity in time of the velocity roughly implies infinite accelerations that requires a singular force. Similarly, discontinuity in time of the displacement without discontinuity of the velocity implies infinite acceleration followed instantly by infinite deceleration, and requires a singular force dipole. Physically, we are then considering a problem with an imposed moving body force monopole and dipole, and examining the displacement field that it sets up. While we do not consider further the origin of this external body force system, potentially shock waves in electromechanical materials can lead to charge double layers that travel with the shock causing body force distributions similar to that assumed here [AB17].

We briefly examine (6.19) in Fourier space:

$$-\left(\frac{M^2\ell^2k^2}{2}+\frac{M^2\ell^2k^2}{2}\frac{\bar{P}_\ell}{\rho}\left(1-e^{-\frac{1}{2}k^2l^2}\right)-\left(1-e^{-\frac{1}{2}k^2l^2}\right)\right)\hat{U}_1(k)=\frac{\left(1-e^{-\frac{1}{2}k^2l^2}\right)(A^+-A^--\imath Bk)}{\sqrt{2\pi}k^2} \tag{6.20}$$

As expected, the left side is precisely the dispersion relation (6.16).

We recall from Appendix A some important properties of dispersion relations in peridynamics. For this particular model, we have from the dispersion relation that the fastest waves are at the long wave limit ($k \to 0$), and their velocity is c_0. Further, the phase velocity is a smooth function of k and goes to 0 as $k \to \infty$. Therefore, for every V_0 such that $0 \leq V_0 \leq c_0$ – or equivalently, for every M such that $0 \leq M \leq 1$ – there is a corresponding value of k for which the left side above goes to 0. However, the right side of (6.20) is nonzero for all k if $A^+ \neq A^-$, and nonzero for all k except $k \to 0$ if $A^+ = A^-$ Therefore, to balance the term going to 0 on the left side, we must have that $\hat{U}_1(k)$ is singular at the value of k that corresponds to M through the dispersion relation. This implies that there is a distinguished frequency that is strongly represented, and physically it corresponds to the radiation of waves due to defect motion, e.g. [DB06] in peridynamics and [Sha16, LD17] in lattice models. We note further that supersonic defects, i.e. $M > 1$, do not radiate waves because the left side does not go to 0 for any value of k. This feature is readily observed in Figures 2 and 3.

We next turn to the solution of (6.19). Unfortunately, closed-form inversion of (6.20) appears intractable. Further, the singularities in Fourier space make accurate numerical inversion challenging. Therefore, we solve the (6.19) directly in real space by adapting techniques from [DB06, AD15] as we describe below. As a preliminary step, we nondimensionalize lengths by ℓ, and also use that the equations are linear to examine the cases of $B = 1$ and $A^+ - A^- = 1$ independently. We then discretize using finite differences and approximate the integrations using Riemann sums and derivatives in the standard way. A key element is the treatment of boundaries since the numerical solution is on a finite domain. Further, peridynamics requires the specification of boundary conditions on a set of finite measure. In 1D, that corresponds to specifying the boundary condition over a finite length rather than on discrete points as in local models. We define a finite domain on which we solve for U_1, and a subregion in the interior on which we impose (6.19); the outer boundary layer (defined as the complement of the interior subregion with the domain of U_1) is taken to be large enough that the interior subregion interacts negligibly with the portion of the body outside the domain of U_1. This leads to an underdetermined problem that we solve using a least-squares approach. Since our problem is linear, it corresponds to using the Moore-Penrose generalized inverse, i.e. $\min_x \|A.x - b\|^2$ where A has has more columns than rows and x is longer than b. We refer to [DB06, AD15] for more discussion of this approach.

The results of our calculations are shown in Figures 2 and 3. Figure 2 shows the result for a unit displacement discontinuity ($B = 1, A^+ - A^- = 0$) for a subsonic defect and a supersonic defect, for various values of $\bar{P}_\ell/\rho$. Figure 3 shows the result for a unit slope discontinuity ($B = 0, A^+ - A^- = 1$) for a subsonic defect and a supersonic defect, for various values of $\bar{P}_\ell/\rho$. We notice that the subsonic defect fields in both types of defects are far more sensitive to the level of nonlocal inertia: an increase in nonlocal inertia increases the wavelength of the radiation. This can readily rationalized by noting that the nonlocal inertia behaves like a regularized strain gradient term in the traveling wave setting, and therefore penalizes gradients. In the supersonic case, there is a decrease in the magnitude of the gradient, but much less pronounced.

7 Discussion

We have made a number of simplifying assumptions in deriving the kinetically nonlocal model above. Conceptually it is easy to relax many of these assumptions, at the expense of increasing the algebraic complexity. For instance, going to 3D, using more complex material models [TR14, SEW$^+$07], nonlinear nonlocal kinetic energies, and so on. However, our aim in this paper is not to find the most general peridynamic model. On the contrary, we aim to find the simplest possible energetically and dynamically consistent peridynamic model. Our key conclusion is that even in this simplest case, we must have nonlocality in the kinetic energy for the energetics and dynamics to be consistent in the sense of peridynamics being a limiting model. Further, the nonlocality in the kinetic energy is of leading-order, i.e. the nonlocality in the kinetic energy appears at the same order as the nonlocality in the strain energy.

A natural question about the kinetic nonlocality is if it is physically important, or merely a curiosity that has little practical significance as has been argued for strain gradients in some contexts, e.g. [MS07]. A heuristic argument for the former case is as follows. As we have noticed, capturing the physically-observed dispersive behavior of short waves in gradient models requires the gradient expansion of the non-local inertial terms to balance out the strain gradients on the right side of the momentum balance. Given that the gradient expansions of the non-local kinetic energy and the non-local strain energy are of the same order, we can reasonably expect that the original non-local terms for each of these are also of the same order. Hence, it seems appropriate to require that the peridynamic model be augmented to include the non-local kinetic terms, not as a correction but as a leading-order contribution. That is, the zeroth-order approximation of peridynamics is classical linear elasticity when the non-locality vanishes on both sides, but once we go beyond this to require non-local terms, they should appear on both sides of the momentum balance – or equivalently, in both the kinetic energy and the potential energy.

We notice that there are in principle infinitely many possible approximations of the dispersion relation. Polynomial approximations have been widely used, e.g. [WA05], because they enable the immediate construction of the real-space dynamical equation. The disadvantage is that polynomial approximations lead to instability and unphysical behavior in certain regimes. On the other hand, approximating the dispersion relation by more complicated functions may lead to a very good fit, but constructing the real-space dynamical equation can be difficult or even impossible; as a particular example, even models based on fractional powers of k are extremely challenging to solve [DPZP13]. Rational functions provide a middle ground between these two extremes: they are better than polynomials in important regimes, while still being invertible to obtain standard differential operators.

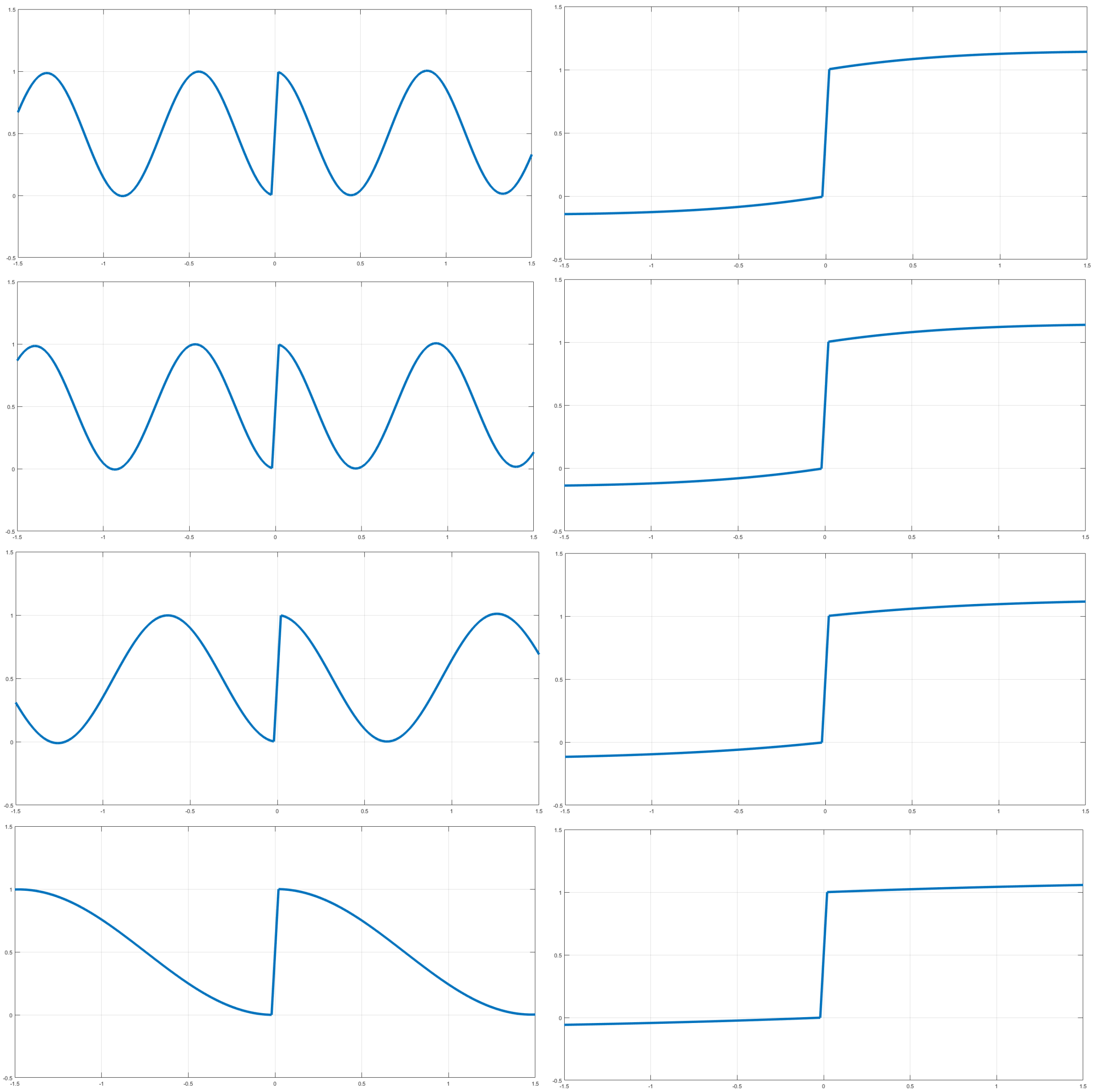

Figure 2: Displacement field around a displacement discontinuity. The left column is subsonic ($M = 0.2$) and the right column is supersonic ($M = 2.0$). The ratio of nonlocal inertia to local inertia $\bar{P}_\ell/\rho$ is $0, 0.1, 1, 10$ in the first, second, third, fourth rows respectively.

This paper has argued for nonlocal kinetic energies in peridynamics so that the model is consistent with energetics and wave propagation; roughly, a top-down approach. It is important to examine this issue – and more broadly, all aspects of peridynamics – from lower-scale models. This is largely an open problem. We briefly note work in static [DW96] and dynamic homogenization [MW07] where nonlocality is observed in the effective behavior both in the inertial term as well as in the elastic energy term. While those models and the resulting nonlocality are significantly different from that proposed here, some elements of those approaches appear promising for the peridynamic setting. Another possibility to understand the origin of kinetic nonlocality may be through the use of microscopic rotational degrees

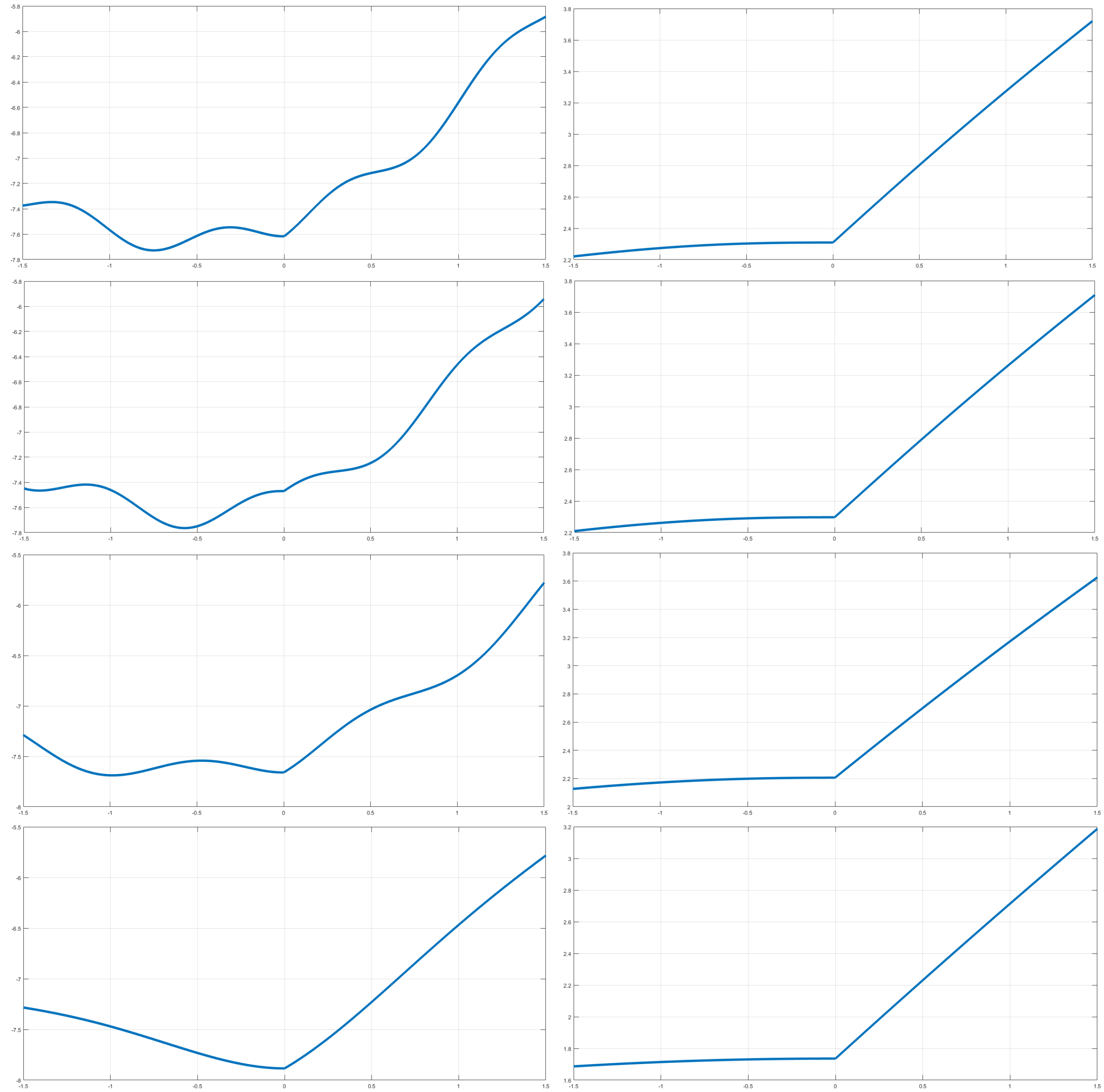

Figure 3: Displacement field around a displacement-derivative discontinuity. The left column is subsonic ($M = 0.2$) and the right column is supersonic ($M = 2.0$). The ratio of nonlocal inertia to local inertia $\bar{P}_\ell/\rho$ is $0, 0.1, 1, 10$ in the first, second, third, fourth rows respectively.

of freedom that couples material points in a neighborhood due to coordinated rotations, e.g. discussed in a toy model in [ZP16] or through Cosserat-like models. Further, all the arguments in this paper are formal, and rigorous analysis, following e.g. [HR12, EP14, MD15, BMC14] for classical peridynamics, is essential.

Rapid spatial variations in the displacement and velocity fields are typical in the vicinity of defects such as interfaces and cracks. In addition, while the nonlocal kinetic energy terms are not dissipative, it has been suggested that the dispersion of short waves may be important for the effective dissipation due to radiation during defect motion in nonlinear peridynamic models [DB06]. Therefore, the classical and kinetically nonlocal peridynamics can potentially have qualitatively different behaviors in defect dynamics, both in the near-field defect structure as well as the far-field energy transport. Careful analyses such as [Lip14, Lip15] are essential to clarify these questions. We notice that the particular defects – discontinuities driven by singular external force dipoles – that we study in Section 6.3 do not show qualitative differences between classical and kinetically nonlocal peridynamics. It can be argued that this is not representative of more relevant and interesting defects such as interfaces and cracks in materials with nonconvex response. In the defects in Section 6.3, the discontinuity is sharp and has vanishing measure. It therefore does not contribute to the nonlocal inertia integral. The discontinuity also does not contribute to the classical inertia, because the singular terms cannot be balanced except by singular external forces, but these singular forces then cancel out the effect of the discontinuity completely. Defects that have large but nonsingular gradients will likely have a significantly larger – and possibly qualitatively different – interaction with the nonlocal inertia. Such defects typically require some nonlinearity and nonconvexity in the material response, and are hence much more challenging to compute compared to the relatively simple cases studied in Section 6.3; however, the study of such defects with nonlocal inertia is the natural next question to examine.

The strain-gradient expansion of peridynamics has provided important insights into peridynamics for analysis, e.g. [Sil16]. In addition, numerical discretizations of peridynamics can induce similar gradient-like effects, e.g. [WG14, SDP16]. However, it is important to emphasize that strain-gradient models impose more and more continuity, whereas the attraction of peridynamics is the possibility to model fracture with reduced continuity. The primary contribution of this work is therefore not in the development of consistent strain-gradient models from peridynamics, but rather in the suggestion that the kinetic energy in peridynamics must have an essentially nonlocal character. While motivated by simple settings, the conclusion that there must be kinetic nonlocality is potentially most useful and relevant in the setting of nonlinear dynamic fracture and defect dynamics. That is, if the kinetic nonlocality is present even in the simple case considered here, that provides a minimum level of kinetic nonlocality for the more complex settings such as dynamic fracture.

We notice here, following [WA05], that discontinuities must be stationary in both classical and kinetically nonlocal peridynamics. While relatively simple compared to the classical jump conditions, the condition that the discontinuity must be stationary is nonetheless a jump condition for peridynamics. Though much less formidable than tracking a dynamic defect, stationary discontinuities must be resolved. Smoothing out the discontinuity can relax this constraint and allow them to move, potentially leading to qualitatively-incorrect answers. Careful numerical methods that respect this peridynamic jump condition must be formulated, e.g. following [JL17].

We note that it may not be numerically overly expensive to account for the nonlocal inertia in peridynamics due to the linearity of the term. In a discretized setting, computing the nonlocal inertia will require a matrix-vector multiplication operation. The matrix does not change with time, even in the large deformation setting because it is based in the reference, and therefore can be pre-computed and stored, e.g. adapting strategies from [WT12].

In principle, classical peridynamics can be calibrated entirely from static experiments. If such a calibrated model is then compared to dynamic experiments and the matching is unsatisfactory, it is difficult to correct the dynamic behavior without affecting the static calibration. In kinetically nonlocal peridynamics, dynamics and statics are incorporated independently in the model and this situation can be easily handled by calibrating the nonlocal inertia kernel to dynamics and the elastic interaction kernel to statics.

Acknowledgments

I thank Pradeep Sharma for valuable discussions and pointing me to the use of rational function approximations in gradient models; Richard James, Mehrdad Massoudi, Michael Parks, Oliver O'Reilly for suggestions about Section 5; and Tony Rollett, Rohan Abeyaratne, and the anonymous reviewers for many useful remarks. I thank (1) ARO Numerical Analysis (YI W911NF-12-1-0156 and W911NF-17-1-0084), ONR Applied and Computational Analysis (N00014-14-1-0715), NSF DMREF program (1628994) for support; (2) the Department of Mathematics at the University of Bath for support through the Parkin Visiting Professorship; (3) support in part by an appointment to the National Energy Technology Laboratory Faculty Research Participation Program sponsored by the U.S. Department of Energy and administered by the Oak Ridge Institute for Science and Education; and (4) support in part by the National Science Foundation through TeraGrid resources provided by Pittsburgh Supercomputing Center.

A Limit of the Dispersion Relation at Large Wavenumber

As noted in [WA05, Sil00], the dispersion relation $\omega(k)$ for specific peridynamic kernels appears to have a finite non-zero limit ω_∞ as $k \to \infty$. We examine a characterization of when this limit exists here.

The dispersion relation can be written:

$$\rho\omega^2 = M(k) := \int_{\mathbb{R}} (1 - \cos k\xi)\, C(\xi)\, d\xi \tag{A.1}$$

where $M(k)$ is the peridynamic analog of the acoustic tensor, and we use $C(a) = C(-a)$. It is immediate that the integral above is bounded for all k if C is square-integrable (for instance), but there is the possibility that the limit with k does not exist, e.g. if the dispersion relation is oscillating even as $k \to \infty$. We examine a sufficient characterization of C for this not to occur, because we do not expect this on physical grounds, and also because we cannot approximate such behavior with the rational functions as proposed here.

For a well-defined limit at $k \to \infty$, it is sufficient to look at the limit of $\lim_{k\to\infty} \int_{\mathbb{R}} \cos k\xi \; C(\xi)\; d\xi$. Using integration by parts, we can write this as:

$$\int_{\mathbb{R}} \cos k\xi \; C(\xi)\; d\xi = \left. C(\xi)\frac{\sin k\xi}{k} \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{\sin k\xi}{k} C'(\xi)\; d\xi \tag{A.2}$$

If $C(\xi)$ is chosen that the integrals are finite, formally the limit $k \to \infty$ suggests that the short wavelength limit is $\rho\omega_\infty^2 = \int_{\mathbb{R}} C(\xi)\; d\xi$.

We examine the condition for $\hat{C}$ be uniformly continuous[6]; in combination with $\hat{C} \in L_p, p < \infty$, the limit of $\hat{C}$ can readily be shown to be 0. The sufficient condition for $\hat{C}$ uniformly continuous can be shown immediately by noting the following from Holders inequality:

$$\left|\hat{C}(k+h) - \hat{C}(k)\right| = \left| \int_{\mathbb{R}} C(\xi) \left(e^{\imath h\xi} - 1\right) e^{\imath k\xi}\; d\xi \right| \leq \|C(\xi)\|_1 \cdot \left\| e^{\imath k\xi} \left(e^{\imath h\xi} - 1\right) \right\|_\infty \tag{A.3}$$

[6]We note that simply continuity is not sufficient, e.g. Example 2 in Chapter 2 of [Gie00].

and then take the limit $h \to 0$ and use Theorems 2.3.1 and 2.3.3 (#2) from [Ber09]. Alternately, one can also see this using Dominated Convergence in L_1 since $|e^{\imath \cdots}| \leq 1$. Therefore, it is sufficient that $C \in L_1$. We recall that it is sufficient that $C \in L_2$ for the Fourier transform to exist, and in that case it follows that $\hat{C} \in L_2$.

The most common choice in peridynamics of compactly-supported kernels satisfies this condition. Other choices such as e^{-x^2} in [DB06] and others in [WA05] which are not compact also satisfy this condition, and the closed-form dispersion curves in those papers can be seen to have a limit. Interestingly, C can be singular and $C \in L_\infty$ is not necessary.

References

[AA06] Harm Askes and Elias C Aifantis. Gradient elasticity theories in statics and dynamics-a unification of approaches. *International journal of fracture*, 139(2):297–304, 2006.

[AA11] Harm Askes and Elias C Aifantis. Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results. *International Journal of Solids and Structures*, 48(13):1962–1990, 2011.

[AB17] Vinamra Agrawal and Kaushik Bhattacharya. Impact induced depolarization and electro-thermomechanical coupling of ferroelectric materials. *submitted*, 2017.

[AD15] Vaibhav Agrawal and Kaushik Dayal. A dynamic phase-field model for structural transformations and twinning: Regularized interfaces with transparent prescription of complex kinetics and nucleation. part i: Formulation and one-dimensional characterization. *Journal of the Mechanics and Physics of Solids*, 85:270–290, 2015.

[AK06] Rohan Abeyaratne and James K Knowles. *Evolution of phase transitions: a continuum theory*. Cambridge University Press, 2006.

[AMPB08] Harm Askes, Andrei V Metrikine, Aleksey V Pichugin, and Terry Bennett. Four simplified gradient elasticity models for the simulation of dispersive wave propagation. *Philosophical magazine*, 88(28-29):3415–3443, 2008.

[Ber09] Christian Berg. Fourier analysis (unpublished class notes). 2009.

[BH98] Max Born and Kun Huang. *Dynamical theory of crystal lattices*. Oxford university press, 1998.

[BMC14] José C Bellido and Carlos Mora-Corral. Existence for nonlocal variational problems in peridynamics. *SIAM Journal on Mathematical Analysis*, 46(1):890–916, 2014.

[DB06] Kaushik Dayal and Kaushik Bhattacharya. Kinetics of phase transformations in the peridynamic formulation of continuum mechanics. *Journal of the Mechanics and Physics of Solids*, 54(9):1811–1842, 2006.

[DPZP13] Luca Deseri, Mario Di Paola, Massimiliano Zingales, and Pietro Pollaci. Power-law hereditariness of hierarchical fractal bones. *International journal for numerical methods in biomedical engineering*, 29(12):1338–1360, 2013.

[DW96] WJ Drugan and JR Willis. A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites. *Journal of the Mechanics and Physics of Solids*, 44(4):497–524, 1996.

[EP14] Etienne Emmrich and Dimitri Puhst. Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics. *Nonlinearity*, 28(1):285, 2014.

[Fol92] Gerald B Folland. *Fourier analysis and its applications*, volume 4. American Mathematical Soc., 1992.

[Gie00] François Gieres. Mathematical surprises and dirac's formalism in quantum mechanics. *Reports on Progress in Physics*, 63(12):1893, 2000.

[HR12] Brittney Hinds and Petronela Radu. DirichletâĂŹs principle and wellposedness of solutions for a nonlocal p-laplacian system. *Applied Mathematics and Computation*, 219(4):1411–1419, 2012.

[JL17] Prashant K Jha and Robert Lipton. Numerical analysis of peridynamic models in h\" older space. *arXiv preprint arXiv:1701.02818*, 2017.

[JR03] RD James and Raffaella Rizzoni. Piecewise rigid body mechanics. *Journal of Nonlinear Science*, 13(1):65–114, 2003.

[LD17] Chang-Tsan Lu and Kaushik Dayal. Temperature and energy transport in the kinetics of phase transformations. *in preparation*, 2017.

[Lip14] Robert Lipton. Dynamic brittle fracture as a small horizon limit of peridynamics. *Journal of Elasticity*, 117(1):21–50, 2014.

[Lip15] Robert Lipton. Cohesive dynamics and brittle fracture. *Journal of Elasticity*, pages 1–49, 2015.

[Liu13] Liping Liu. On energy formulations of electrostatics for continuum media. *Journal of the Mechanics and Physics of Solids*, 61(4):968–990, 2013.

[LSL16] Robert Lipton, Stewart Silling, and Richard Lehoucq. Complex fracture nucleation and evolution with nonlocal elastodynamics. *arXiv preprint arXiv:1602.00247*, 2016.

[MD14] Jason Marshall and Kaushik Dayal. Atomistic-to-continuum multiscale modeling with long-range electrostatic interactions in ionic solids. *Journal of the Mechanics and Physics of Solids*, 62:137–162, 2014.

[MD15] Tadele Mengesha and Qiang Du. On the variational limit of a class of nonlocal functionals related to peridynamics. *Nonlinearity*, 28(11):3999, 2015.

[Min64] Raymond David Mindlin. Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis*, 16(1):51–78, 1964.

[MPD17] Robert Macedo, Hossein Pourmatin, and Kaushik Dayal. Disclinations without gradients: A nonlocal model for topological defects in liquid crystals. *submitted*, 2017.

[MS07] R Maranganti and P Sharma. A novel atomistic approach to determine strain-gradient elasticity constants: tabulation and comparison for various metals, semiconductors, silica, polymers and the (ir) relevance for nanotechnologies. *Journal of the Mechanics and Physics of Solids*, 55(9):1823–1852, 2007.

[MW07] Graeme W Milton and John R Willis. On modifications of newton's second law and linear continuum elastodynamics. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 463(2079):855–880, 2007.

[O'R17] Oliver M O'Reilly. *Modeling Nonlinear Problems in the Mechanics of Strings and Rods*. Springer, 2017.

[SDP16] Pablo Seleson, Qiang Du, and Michael L Parks. On the consistency between nearest-neighbor peridynamic discretizations and discretized classical elasticity models. *Computer Methods in Applied Mechanics and Engineering*, 311:698–722, 2016.

[SEW+07] Stewart A Silling, M Epton, O Weckner, J Xu, and E Askari. Peridynamic states and constitutive modeling. *Journal of Elasticity*, 88(2):151–184, 2007.

[Sha16] Basant Lal Sharma. On energy balance and the structure of radiated waves in kinetics of crystalline defects. *Journal of the Mechanics and Physics of Solids*, 96:88–120, 2016.

[Sil00] Stewart A Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48(1):175–209, 2000.

[Sil16] Stewart A Silling. Solitary waves in a peridynamic elastic solid. *to appear in Journal of the Mechanics and Physics of Solids*, 2016.

[SL08] Stewart A Silling and Richard B Lehoucq. Convergence of peridynamics to classical elasticity theory. *Journal of Elasticity*, 93(1):13–37, 2008.

[SL10] SA Silling and RB Lehoucq. Peridynamic theory of solid mechanics. *Advances in applied mechanics*, 44:73–168, 2010.

[SPGL09] Pablo Seleson, Michael L Parks, Max Gunzburger, and Richard B Lehoucq. Peridynamics as an upscaling of molecular dynamics. *Multiscale Modeling & Simulation*, 8(1):204–227, 2009.

[TR14] MR Tupek and R Radovitzky. An extended constitutive correspondence formulation of peridynamics based on nonlinear bond-strain measures. *Journal of the Mechanics and Physics of Solids*, 65:82–92, 2014.

[WA05] Olaf Weckner and Rohan Abeyaratne. The effect of long-range forces on the dynamics of a bar. *Journal of the Mechanics and Physics of Solids*, 53(3):705–728, 2005.

[WG14] Raymond A Wildman and George A Gazonas. A finite difference-augmented peridynamics method for reducing wave dispersion. *International Journal of Fracture*, 190(1-2):39–52, 2014.

[WT12] Hong Wang and Hao Tian. A fast galerkin method with efficient matrix assembly and storage for a peridynamic model. *Journal of Computational Physics*, 231(23):7730–7738, 2012.

[Zie12] William P Ziemer. *Weakly differentiable functions: Sobolev spaces and functions of bounded variation*, volume 120. Springer Science & Business Media, 2012.

[ZP16] Qingze Zhao and Prashant K Purohit. (adiabatic) phase boundaries in a bistable chain with twist and stretch. *Journal of the Mechanics and Physics of Solids*, 92:176–194, 2016.