A MODEL FOR VORTEX NUCLEATION IN THE GINZBURG-LANDAU EQUATIONS.

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Abstract. This paper studies questions related to the dynamic transition between local and global minimizers in the Ginzburg-Landau theory of superconductivity. We derive a heuristic equation governing the dynamics of vortices that are close to the boundary, and of dipoles with small inter vortex separation. We consider a small random perturbation of this equation, and study the asymptotic regime under which vortices nucleate.

1. Introduction.

This paper studies questions related to the dynamic transition between local and global minimizers in the Ginzburg-Landau theory of superconductivity. The Ginzburg-Landau theory provides a mezoscopic description of the state of a superconductor through the *order parameter* – a specific function $\mathbb{C}$-valued function $u \in H^1(\Omega)$ for which the local density of superconducting Cooper pairs is given by $|u(x)|$. Here $\Omega \subseteq \mathbb{R}^2$ is the region occupied by the superconductor. A fundamental feature of superconductors are the presence of localized regions called *vortices*, where the superconductor drops into a normal state. In these regions the degree of u is nontrivial about each vortex, and the induced magnetic field pierces through the superconductor.

The mechanism by which vortices become energetically favorable was proved by Serfaty using a careful energy decomposition. Recall, the Ginzburg-Landau energy is defined by

$$G_\varepsilon(u, A) \stackrel{\text{def}}{=} \int_\Omega \Big(\frac{1}{2} |\nabla_A u|^2 + \frac{1}{2} |\nabla \times A - h_{ex}|^2 + \frac{1}{4\varepsilon^2} \left(1 - |u|^2\right)^2 \Big) \, dx \,, \tag{1.1}$$

where A is the magnetic potential, $h_{ex} = h_{ex}(\varepsilon)$ is the strength of the external magnetic field, and $\nabla_A \stackrel{\text{def}}{=} \nabla - iA$. Physically, ε is the non-dimensional ratio of the super-conductors coherence length to the London penetration depth.

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To understand the energy decomposition, define the *Meissner* potential ξ_m to be the solution of

$$-\Delta\xi_m + \xi_m + 1 = 0 \quad \text{in } \Omega \qquad \xi_m = 0 \quad \text{on } \partial\Omega\,. \tag{1.2}$$

When h_{ex} is small enough (explicitly, when $h_{ex} < h_{c_1}$, defined below), the purely super-conducting state with no vortices gives a *global* minimizer of the Ginzburg-Landau energy G_ε, see [14]. This state corresponds to $u \equiv 1$ and $A = h_{ex}\nabla^\perp\xi_m$, and the minimizing energy (called the *Meissner energy*) is given by

$$G_m(h_{ex}) \overset{\text{def}}{=} G_\varepsilon(1, h_{ex}\nabla^\perp\xi_m) = h_{ex}^2 \int_\Omega \frac{1}{2}\left|\nabla\xi_m\right|^2 + \frac{1}{2}\left|\Delta\xi_m - 1\right|^2 \, dx\,. \tag{1.3}$$

If there are a finite number of vortices at points a_j, with degrees $d_j \in \{\pm 1\}$ respectively which are reasonably separated and away from the boundary, then Serfaty [16, 17] shows that Ginzburg-Landau energy can be decomposed as

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_m + \sum_{j=1}^{n} \left(\pi\left|\ln\varepsilon\right| + 2\pi d_j h_{ex}\xi_m(a_j)\right) + o_\varepsilon(\left|\ln\varepsilon\right|)\,. \tag{1.4}$$

and the order parameter u_ε takes the form,

$$u_\varepsilon(x) \approx \prod_{j=1}^{n} \rho_\varepsilon(|x-a_j|)\left(\frac{x-a_j}{|x-a_j|}\right)^{d_j} e^{i\psi_\varepsilon^*}\,, \tag{1.5}$$

where $\rho_\varepsilon(s)$ is the equivariant vortex profile with $\rho_\varepsilon(0) = 0$, $\rho_\varepsilon(s) \to 1$ for $s \gg \varepsilon$ (see Appendix II of [2]), $d_j \in \pm 1$, and ψ_ε^* is a harmonic function that ensures $\partial_\nu u_\varepsilon = 0$ on $\partial\Omega$.

Note that $\xi_m(a_i)$ is always negative, since the maximum principle implies $-1 < \xi_m < 0$ in Ω. Thus, examining (1.4) one sees that vortices with negative degrees are never energetically favorable. Furthermore, if the applied magnetic field h_{ex} is very large, then a positively oriented vortex can be energetically favorable. The critical threshold at which such vortices become energetically favorable is explicitly given by

$$h_{c_1} \overset{\text{def}}{=} \frac{\left|\ln\varepsilon\right|}{2\max\left|\xi_m\right|}\,, \tag{1.6}$$

and is known as the *first critical field*. In this case, the optimal location for a single positively oriented, energetically favorable vortex is at the point where ξ_m achieves its minimum and is located in the interior of Ω.

Our main interest in this paper is to study the dynamic transition between the Meissner state and the energetically favorable state with an interior vortex. We recall that the dynamics of a type-II superconductor are governed by the Gor'kov-Éliashberg system [9], a coupled system of equations describing the evolution of the order parameter u_ε and the electromagnetic field potentials $\Phi_\varepsilon \in H^1(\mathbb{R}^2, \mathbb{R}^1)$, $A_\varepsilon \in H^1(\mathbb{R}^2, \mathbb{R}^2)$. Explicitly, these equations are

$$\partial_{\Phi_\varepsilon} u_\varepsilon = \nabla_{A_\varepsilon}^2 u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon\left(1 - |u_\varepsilon|^2\right) \quad \text{in } \Omega\,, \tag{1.7}$$

$$E_\varepsilon = \nabla^\perp h_\varepsilon + j_{A_\varepsilon}(u_\varepsilon) \quad \text{in } \Omega\,, \tag{1.8}$$

$$\nu \cdot \nabla_{A_\varepsilon} u_\varepsilon = \nu \cdot E_\varepsilon = 0 \quad \text{on } \partial\Omega\,, \tag{1.9}$$

$$h_\varepsilon = h_{ex} \quad \text{on } \partial\Omega\,, \tag{1.10}$$

where

$$\partial_{\Phi_\varepsilon} \stackrel{\text{def}}{=} \partial_t - i\Phi_\varepsilon\,, \quad E_\varepsilon \stackrel{\text{def}}{=} \partial_t A_\varepsilon - \nabla\Phi_\varepsilon\,, \quad h_\varepsilon \stackrel{\text{def}}{=} \nabla\times A_\varepsilon\,, \quad j_{A_\varepsilon}(u_\varepsilon) \stackrel{\text{def}}{=} (iu_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon)\;,$$

and $(a,b) \stackrel{\text{def}}{=} \frac{1}{2}(a\bar{b} + \bar{a}b)$ is the real part of the complex inner product.

Now consider the Gor'kov-Éliashberg system with initial data $(u^0_\varepsilon, A^0_\varepsilon)$ corresponding to the Meissner state $(1, h_{ex}\nabla^\perp\xi_m)$. Since energy satisfies the diffusive identity

$$G_\varepsilon(u_\varepsilon(t), A_\varepsilon(t)) + \int_0^t \int_\Omega \Big(|\partial_{\Phi_\varepsilon} u_\varepsilon|^2 + |E_\varepsilon|^2\Big)\, dx\, ds = G_\varepsilon(u^0_\varepsilon, A^0_\varepsilon)\,, \tag{1.11}$$

we will assume that h_{ex} is large enough so that

$$G_\varepsilon(u^0_\varepsilon, A^0_\varepsilon) = G_m(h_{ex}) > G_\varepsilon(u^\varepsilon_{min}, A^\varepsilon_{min})\,.$$

Here $(u^\varepsilon_{min}, A^\varepsilon_{min})$ denotes the global minimizer of the energy G_ε with applied magnetic field h_{ex}. In this case the energy minimizing configuration has lower energy than the Meissner state, and the dynamic transition to the global minimizer will involve nucleating vortices.

The process by which vortices are nucleated is not yet well understood. It was shown in [1] and [3] that the Meissner state is linearly stable until the applied magnetic field h_{ex} crosses the second critical field

$$h_{c_2} \stackrel{\text{def}}{=} \frac{C}{\varepsilon},$$

where C is a constant depending on the domain Ω. Since h_{c_2} is much larger than h_{c_1}, this points at a very significant hysteresis phenomena and the process by which this dynamically generates vortices is highly nontrivial, see [11].

Along these lines, vortices can also dynamically nucleate as a way of tunneling to lower energy states. Due to topological considerations, vortices should either nucleate at the boundary or nucleate as a dipole in the interior of the domain. In the first case, let $a_\varepsilon = a_\varepsilon(t)$ be the distance of the center of a vortex from the boundary of Ω. We know from [15] that when $a_\varepsilon(0) > \exp(-|\ln\varepsilon|^{1/2})$ the evolution of a_ε is governed by the ODE

$$\pi\dot{a}_\varepsilon = -d_j\lambda\nabla\xi_m(a_\varepsilon). \tag{1.12}$$

Hence, any positive vortices move towards the interior to a lower energy state and any negative vortices move to the boundary and become excised (see [15]). However, the energy of a vortex a distance of order $\exp(-|\ln\varepsilon|^{1/2})$ away from $\partial\Omega$ is

$$G_m(h_{ex}) + \pi\,|\ln\varepsilon| + o_\varepsilon(|\ln\varepsilon|) \gg G_m(h_{ex})\,.$$

This is an extremely large barrier to overcome, and is even more dramatic when a dipole is nucleated in the interior. In particular once a dipole has separation of at least $\exp(-|\ln\varepsilon|^{1/2})$, the associated energy is

$$G_m(h_{ex}) + 2\pi\,|\ln\varepsilon| + o_\varepsilon(|\ln\varepsilon|) \gg G_m(h_{ex})\,.$$

Thus the energy barrier to nucleate a vortex at the boundary or via a dipole is much larger than the energy gap between the Meissner state and the configuration with the vortex at the minimizer.

The main purpose of this paper is to better understand how this energy barrier can be overcome through the study of vortices close to the boundary and dipoles with small inter-vortex separation. In particular, for every $\alpha \in (0,1)$ we study the dynamics of vortices a distance of ε^α away from the boundary. The energy barrier to nucleate such vortices is $\pi(1-\alpha)\,|\ln\varepsilon|$, which is a much smaller energy barrier to overcome. We show the following results:

1. We obtain a heuristic ODE governing the motion of these vortices (equation (2.1), below).
2. We rigorously estimate the annihilation times of vortices $O(\varepsilon^\alpha)$ away from the boundary, and show that this agrees with the annihilation times of (2.1) (Theorem 2.1 and Proposition 2.2, below).
3. We consider a stochastically perturbed version of the heuristic ODE governing vortex motion, and estimate the chance that the vortex nucleates (and thus achieving a lower energy state) before annihilating. This is Theorem 2.3, below.

The same analysis can be made for vortex dipoles with inter-vortex separation ε^α.

A more physically relevant problem is the direct study of a stochastically perturbed version of (1.7)–(1.10), without relying on the simplified heuristics. This is a much harder question requiring a deep understanding of the long time dynamics of the underlying nonlinear stochastic PDE. The problem is described briefly at the end of Section 2, below, but its resolution is beyond the scope of the current investigation.

Plan of this paper. In Section 2 we state the main results of this paper. In Section 3 we formally derive the heuristic ODE (2.1) by matching terms of leading order. In Section 4 prove Theorem 2.1, rigorously estimate the annihilation times of vortices a distance $O(\varepsilon^\alpha)$ away from the boundary. Confirming that these annihilation times agrees with that of (2.1) is relegated to Appendix A. Finally, in Section 5 we prove Theorem 2.3, estimating the chance of vortex nucleation.

2. Main Results.

2.1. Boundary Vortex Dynamics and Annihilation Times.

We begin with a heuristic ODE governing the motion of a vortex close to the boundary of the domain Ω. Since the scales we are interested in are very small, we locally flatten the boundary of $\partial\Omega$ and state the governing equation on the half plane.

Heuristic ODE. *Let $(0, a_\varepsilon(t))$ be the position of a vortex at time t in the domain $\mathbb{R}^2_+ \stackrel{\text{def}}{=} \mathbb{R}\times\mathbb{R}_+$. If $\alpha\in(0,1)$ and $a(0)=\varepsilon^\alpha$ then, to leading order the motion of the vortex is governed by the ODE*

$$\dot a_\varepsilon = \frac{\lambda h_{ex}}{|\ln\varepsilon|} - \frac{1}{|\ln\varepsilon|\,a_\varepsilon}\,, \tag{2.1}$$

where $\lambda = -2\partial_y\xi_m(0,0) > 0$, which corresponds an order parameter of the form (1.5).

We provide a formal derivation of (2.1) in Section 3 by matching terms of leading order. To obtain a rigorous result supporting (2.1) as a model for boundary vortex dynamics, we show that annihilation time of vortices a distance of ε^α from the boundary is $O(\varepsilon^{2\alpha})$ in the full Gor'kov-Éliashberg system (1.7)-(1.10) (Theorem 2.1, below). This this agrees with the annihilation time predicted by the ODE (2.1) (Proposition 2.2, below).

In order to state Theorem 2.1 we need to introduce some notation. Define the Jacobian J by

$$J(w) \stackrel{\text{def}}{=} \det \nabla w = \frac{1}{2}\nabla \times j(w)\,,$$

where $j(w) \stackrel{\text{def}}{=} (iw, \nabla w)$. Recall that if

$$w = \Big(\prod_{j=1}^{n}\Big(\frac{x-a_j}{|x-a_j|}\Big)^{d_j}\Big)$$

is the order parameter associated with n point vortices located at $a_1, \ldots, a_n$ with degrees $d_1, \ldots, d_n \in \{\pm 1\}$ respectively, then a direct calculation shows

$$J\Big(\prod_{j=1}^{n}\Big(\frac{x-a_j}{|x-a_j|}\Big)^{d_j}\Big) = \pi \sum_{j=1}^{n} d_j \delta_{a_j}\,.$$

Consequently, J can be used to describe the location of vortices.

More precisely, the measure of vortex separation used throughout this paper is the $(C_0^{0,\gamma}(\Omega))^*$ norm of the differences in the Jacobian of the order parameters. Here $0 < \gamma \leqslant 1$ if a fixed parameter. It is convenient to note that if $a, b \in \Omega$ are such that $\min\{d(a, \partial\Omega), d(b, \partial\Omega)\} \geqslant |a-b|$, then

$$\|\delta_a - \delta_b\|_{(C_0^{0,\gamma}(\Omega))^*} = |a-b|^\gamma.$$

We now state our first result.

Theorem 2.1. *Let $(u_\varepsilon(t), A_\varepsilon(t), \Phi_\varepsilon(t))$ be a solution to the system (1.7)–(1.10) under the Coulomb gauge such that $h_{ex} \leqslant e^{\sqrt{|\ln \varepsilon|}}$ with initial data $(u_\varepsilon^0, A_\varepsilon^0, \Phi_\varepsilon^0)$ that satisfies*

$$\big|G_\varepsilon(u_\varepsilon^0, A_\varepsilon^0) - G_m(h_{ex})\big| \leqslant \pi(1-\alpha)\,|\ln \varepsilon| + C$$

for a constant C and $0 < \alpha < 1$. Moreover, for some $\gamma \in (0, 1]$ suppose either

$$\Big\| J(u_\varepsilon^0) - \pi\left(\delta_{a_\varepsilon^{0,+}} - \delta_{a_\varepsilon^{0,-}}\right)\Big\|_{(C_0^{0,\gamma})^*} = o_\varepsilon\Big(\frac{|\ln \varepsilon|}{h_{ex}}\Big) \text{ and } \big|a_\varepsilon^{0,+} - a_\varepsilon^{0,-}\big| = \varepsilon^\alpha\,, \tag{2.2}$$

or

$$\big\| J(u_\varepsilon^0) - \pi\delta_{a_\varepsilon^0}\big\|_{(C_0^{0,\gamma})^*} = o_\varepsilon\Big(\frac{|\ln \varepsilon|}{h_{ex}}\Big) \text{ and } d(a_\varepsilon^0, \partial\Omega) = \varepsilon^\alpha\,. \tag{2.3}$$

Then there exists time $t_\varepsilon \leqslant \varepsilon^{2\alpha}$ such that

$$\big\| 1 - |u_\varepsilon(t_\varepsilon)| \big\|_{L^\infty(\Omega)} = o_\varepsilon(1). \tag{2.4}$$

In particular, there are no vortices in the domain at time t_ε.

Remark. Initial data satisfying (2.2) can be constructed as follows. Let ρ_ε be the profile of the equivariant vortex as in (1.5), and define

$$u_\varepsilon^0 = \prod_{j=1}^{2} \rho_\varepsilon(|x-a_j^\varepsilon|)\Big(\frac{x-a_j^\varepsilon}{|x-a_j^\varepsilon|}\Big)^{d_j} e^{i\psi_\varepsilon^*}\,,$$

with $d_1 = 1, d_2 = -1$, $a_1^\varepsilon = a_\varepsilon^{0,+}$, $a_2^\varepsilon = a_\varepsilon^{0,-}$. We set $A_\varepsilon^0 = h_{ex}\nabla^\perp \xi_m$ (where ξ_m is given by (1.2)) and $\Phi_\varepsilon^0 \equiv 0$. A short calculation shows that $G_\varepsilon(u_\varepsilon^0, A_\varepsilon^0) = G_m + \pi(1-\alpha)\,|\ln \varepsilon| + o_\varepsilon(|\ln \varepsilon|)$.

For completeness, we also estimate the annihilation time of the ODE (2.1), and show that it agrees with the time scales obtained in Theorem 2.1.

Proposition 2.2. *Let $\alpha \in (0,1]$ and suppose a satisfies the ODE (2.1) with initial data $a_\varepsilon(0) = \varepsilon^\alpha$. Let t_ε be the vortex annihilation time (i.e. a time such that $a_\varepsilon(t_\varepsilon) = 0$). Then*

$$\lim_{\varepsilon \to 0} \frac{t_\varepsilon}{\varepsilon^{2\alpha} |\ln \varepsilon|} = \frac{1}{2} \tag{2.5}$$

for any $\lambda \geqslant 0$.

The proof of Theorem 2.1 (presented in Section 4) is similar to arguments found in [19] in the gauge-free situation. The proof of Proposition 2.2 follows quickly from well known properties of the Lambert W function, and is relegated to Appendix A.

2.2. **Stochastic Models for Driving Dipoles.** In light of Theorem 2.1, one requires an applied field *larger* than $\exp(|\ln \varepsilon|^{1/2})$ for vortices to be pulled away from the boundary and nucleate. Since this is extremely large, we introduce a random perturbation into (2.1) and study the probability of nucleation. Explicitly, the equation we consider is

$$da_\varepsilon = -b_\varepsilon(a_\varepsilon)\, dt + \sqrt{2\beta_\varepsilon}\, dW_t, \tag{2.6}$$

where

$$b_\varepsilon(a) \stackrel{\text{def}}{=} -\frac{\lambda h_{ex}}{|\ln \varepsilon|} + \frac{1}{|\ln \varepsilon| a} \tag{2.7}$$

is the right hand side of (2.1), and W is a standard Brownian motion and β_ε is a scaling parameter depending on ε. We remark that (2.6) is similar to that of a Bessel process of dimension $1 + 1/(\beta_\varepsilon |\ln \varepsilon|)$.

The question of how stochastic forcing in the Gor'kov-Éliashberg equations can induce stochastic ODE's for the vortex position (2.6)-(2.7) is not well understood. Some initial forays in this direction in the gauge-less case can be found in [4].

Once vortices reach a distance of $1/|\ln \varepsilon|$ away from the boundary, we know (see [15], briefly described in Section 1), we know they are driven into the interior and move into a stable, lower energy state. Thus, in order to investigate nucleation, we study the chance that solutions to (2.6) starting a distance of ε^α away from the boundary, reach a distance $O(1/|\ln \varepsilon|)$ *before* annihilating (i.e. before $a_\varepsilon = 0$, corresponding to the vortex with center a_ε reaching the boundary of the domain).

Precisely, define the stopping time τ_ε by

$$\tau_\varepsilon = \inf\{t \mid a_\varepsilon(t) \notin (0, \hat{A}_\varepsilon)\}, \tag{2.8}$$

where $\hat{A}_\varepsilon > 0$ is a parameter that will later be chosen to be of order $1/|\ln \varepsilon|$. Since the vortices we consider have radius ε^α, the closest to the boundary they can start is a distance of ε^α away. In this case the chance of nucleating one such vortex is

$$\boldsymbol{P}^{\varepsilon^\alpha}\big(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon\big) \stackrel{\text{def}}{=} \boldsymbol{P}\big(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon \bigm| a_\varepsilon(0) = \varepsilon^\alpha\big).$$

On a bounded domain Ω, we locally flatten the boundary and interpret $a_\varepsilon(t)$ as the distance of the vortex from the boundary. In this case it is natural to consider the motion of many vortices simultaneously, near different points on the boundary. Since the size of these vortices is $O(\varepsilon^\alpha)$, we can fit $O(\varepsilon^{-\alpha})$ such vortices simultaneously on $\partial\Omega$. Assuming the motion of each of these vortices is independent, and governed by (2.6), then the chance that *at least* one of these vortices nucleates is given by

$$N_\varepsilon \stackrel{\text{def}}{=} 1 - (1 - \boldsymbol{P}^{\varepsilon^\alpha}(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon))^{\varepsilon^{-\alpha}}. \tag{2.9}$$

Under the physically relevant assumption $\hat{A}_\varepsilon = O(1/|\ln \varepsilon|)$ we show that the nucleation probability N_ε transitions from 0 to 1 at the threshold

$$\beta_\varepsilon \approx \frac{\alpha}{\ln|\ln \varepsilon|}\,.$$

This, along with more precise asymptotics, is our next result.

Theorem 2.3. *Suppose a_ε solves the SDE (2.6)–(2.7),*

$$c_0 > 0\,, \quad h_{ex} = \frac{c_0 \lambda}{|\ln \varepsilon|}\,, \quad \hat{A} = \frac{1}{c_0 |\ln \varepsilon|}\,, \tag{2.10}$$

and τ_ε, defined by (2.8), is the first exit time of a from the interval $(0, \hat{A}_\varepsilon)$.

(1) If

$$\limsup_{\varepsilon \to 0} \beta_\varepsilon \ln|\ln \varepsilon| < \alpha,$$

then the nucleation probability $N_\varepsilon \to 0$ as $\varepsilon \to 0$.

(2) On the other hand, if

$$\lim_{\varepsilon \to 0} \beta_\varepsilon = 0 \quad \textit{and} \quad \liminf_{\varepsilon \to 0} \beta_\varepsilon \ln|\ln \varepsilon| > \alpha,$$

then the nucleation probability $N_\varepsilon \to 1$ as $\varepsilon \to 0$.

We remark that the dependence of h_{ex} and $\hat{A}_\varepsilon$ on c_0 in (2.10) is chosen so that $\hat{A}_\varepsilon$ is an equilibrium of the ODE (2.1). The proof of Theorem 2.3 is in Section 5, and also provides asymptotics in the transition regime when

$$\beta_\varepsilon \ln|\ln \varepsilon| \to \alpha\,,$$

In this case limiting value of N_ε depends on the rate of convergence and is described in Remark 5.1, below.

We conclude this section with the description of an open question that is more physically realistic. Consider a stochastically forced version of the full Gor'kov-Éliashberg equations (1.7)–(1.10), instead of the heuristic ODE (2.1). The noise should spontaneously generate vortices, and due to topological constraints these vortices will appear either near the boundary, or as dipoles with small inter vortex separation. Using the heuristic ODE (2.1) and Theorem 2.3 we expect that when the noise is strong enough, these vortices (or dipoles) will nucleate providing a mechanism by which the system "tunnels" to a lower energy state. This leads us to make the following conjecture.

Conjecture 2.4. *Consider a stochastically forced version of (1.7)–(1.10). If the forcing is strong enough, the system admits a unique invariant measure for all $\varepsilon > 0$. In this case, the invariant measure converges weakly as $\varepsilon \to 0$ to a measure supported on the set of all functions that are limits of global minimizers of the Ginzburg-Landau energy functional. Depending on the relation between h_{ex} and h_{c_1} such functions correspond to the purely superconducting state, or a nucleated state with finitely many vortices.*

In light of Theorem 2.3 one would guess that the above conjecture holds when the variance of the noise is at least $O(1/\ln|\ln \varepsilon|)$. However, this would require the stochastic forcing to spontaneously nucleate enough dipoles (or vortices near the boundary). Moreover, truly nonlinear effects may change the threshold significantly.

Proving existence (and possibly uniqueness) of the invariant measure is likely to be amenable to currently available techniques. Understanding the limiting behaviour of the invariant measure, however, is more delicate. In finite dimensions, the small noise limit of the invariant measure of a randomly perturbed potential flow is a sum of delta masses located at the global minima of the potential, with the relative mass at each minima depending on its basin of attraction. In infinite dimensions the situation is more complicated as there may be a continuum of global minima, and understanding the limiting behaviour is a much more involved.

The remainder of this paper is devoted to proving the results stated in this section.

3. **Formal Derivation of Boundary Vortex Dynamics.**

The purpose of this section is to provide a short, heuristic derivation of (2.1). Recall that a standard calculation (see for example [8]) shows that

$$\partial_t g_\varepsilon(u_\varepsilon, A_\varepsilon) + |\partial_{\Phi_\varepsilon} u_\varepsilon|^2 + |E_\varepsilon|^2 = \nabla\cdot(\partial_{\Phi_\varepsilon} u_\varepsilon, \nabla_{A_\varepsilon} u_\varepsilon) + \nabla\times(E_\varepsilon\,(h_\varepsilon - h_{ex}))\,, \tag{3.1}$$

where $g_\varepsilon(u,A)$, defined by

$$g_\varepsilon(u,A) \stackrel{\text{def}}{=} \frac{1}{2}|\nabla_A u|^2 + \frac{1}{2}|h-h_{ex}|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2\,,$$

is the energy density associated to $G_\varepsilon(u,A)$. We will split this energy density into simpler terms. Following Bethuel et. al. [2], let

$$\mathcal{E}_\varepsilon(u) \stackrel{\text{def}}{=} \int_\Omega e_\varepsilon(u)\,dx\,, \quad \text{where} \quad e_\varepsilon(u) \stackrel{\text{def}}{=} \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2\,.$$

as introduced and studied by Bethuel et. al. [2].

Suppose now that our domain Ω is the half ball $\Omega = B_1(0)\cap\mathbb{R}^2_+$, and consider a vortex located at $(0, a_\varepsilon(t))$ at time t, where $a_\varepsilon(0) = a^0_\varepsilon = \varepsilon^\alpha$. From [16] we know

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_m(h_{ex}) + \mathcal{E}_\varepsilon(u_\varepsilon) + 2\pi h_{ex}\int_\Omega \xi_m J(u_\varepsilon)dx + \text{ lower order terms}\,,$$

where $G_m(h_{ex})$ is the Meissner energy associated to applied field h_{ex}. We can approximate the energy $\mathcal{E}_\varepsilon(u_\varepsilon)$ by

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \pi\ln\frac{a_\varepsilon(t)}{\varepsilon} + \text{ lower order terms}\,.$$

Combining the previous two equations, and using the fact that on small scales $J(u_\varepsilon)$ is concentrated at the site of the vortex (see [10]), yields

$$G_\varepsilon(u_\varepsilon, A_\varepsilon) = \pi\ln\frac{a_\varepsilon(t)}{\varepsilon} + 2\pi h_{ex}\xi_m(0, a_\varepsilon(t)) + G_m(h_{ex}) + \text{ lower order terms}\,.$$

On the other hand one can formally show that

$$\int_0^t\int_\Omega\left(|\partial_{\Phi_\varepsilon} u_\varepsilon|^2 + |E_\varepsilon|^2\right)dx\,ds = \int_0^t \pi\ln\frac{a_\varepsilon(s)}{\varepsilon}\,|\dot{a}_\varepsilon(s)|^2\,ds + \text{ lower order terms.}$$

Combined with (1.11) this yields

$$\begin{aligned}\left[\pi\log\frac{a_\varepsilon(t)}{\varepsilon} + 2\pi h_{ex}\xi_m(0,a_\varepsilon(t))\right] + \int_0^t \pi\ln\frac{a_\varepsilon(s)}{\varepsilon}\,|\dot{a}_\varepsilon(s)|^2\,ds \\ = \left[\pi\log\frac{a^0_\varepsilon}{\varepsilon} + 2\pi h_{ex}\xi_m(a^0_\varepsilon)\right] + \text{ lower order terms.}\end{aligned} \tag{3.2}$$

Differentiating (3.2) in time and neglecting the lower order terms yields the ODE

$$\dot{a}_\varepsilon \ln \frac{a_\varepsilon}{\varepsilon} = -2h_{ex}\partial_y \xi_m(0, a_\varepsilon) - \frac{1}{a_\varepsilon}.$$

Using (1.2) and the Hopf lemma, we know that the outward normal derivative of ξ_m is strictly positive on the boundary. Thus, to leading order, we obtain (2.1).

4. **Dipole Annihilation Times.**

The main goal in this section is to prove Theorem 2.1. We do this through the following η-compactness result.

Proposition 4.1 (η-compactness)**.** *Fix $C_1, C_2 > 0$ and suppose that*

$$C_1 |\ln \varepsilon| \leqslant h_{ex} \leqslant C_2 \exp\big(|\ln \varepsilon|^{1/2}\big) .$$

Let $(u_\varepsilon(t), A_\varepsilon(t), \Phi_\varepsilon(t))$ be a solution to (1.7)-(1.10) under a Coulomb gauge that satisfies

$$\big|G_\varepsilon(u_\varepsilon^0, A_\varepsilon^0) - G_m\big| \leqslant \eta\, |\ln \varepsilon| + C$$

for some constants $C > 0$ and $\eta \in (0, \pi)$. If further

$$\big\|J(u_\varepsilon^0)\big\|_{(C_0^{0,\gamma})^*} = o_\varepsilon\Big(\frac{|\ln \varepsilon|}{h_{ex}}\Big) \tag{4.1}$$

for some $0 < \gamma \leqslant 1$, then for any $\delta < 2 - \frac{2\eta}{\pi}$, there exists a time $t_\varepsilon \leqslant \varepsilon^\delta$ such that (2.4) holds.

Momentarily postponing the proof of Proposition 4.1, we prove Theorem 2.1.

Proof of Theorem 2.1. We first consider the case where the initial data is a dipole with separation ε^α (i.e. satisfies (2.2)). In this case,

$$\begin{aligned}\|J(u_\varepsilon^0)\|_{(C_0^{0,\gamma})^*} &\leqslant \Big\|J(u_\varepsilon^0) - \pi\big(\delta_{a_\varepsilon^{0,+}} - \delta_{a_\varepsilon^{0,-}}\big)\Big\|_{(C_0^{0,\gamma})^*} + \pi\Big\|\delta_{a_\varepsilon^{0,+}} - \delta_{a_\varepsilon^{0,-}}\Big\|_{(C_0^{0,\gamma})^*} \\ &= o_\varepsilon\Big(\frac{|\ln \varepsilon|}{h_{ex}}\Big) + \pi\big|a_\varepsilon^{0,+} - a_\varepsilon^{0,-}\big|^\gamma = o_\varepsilon\Big(\frac{|\ln \varepsilon|}{h_{ex}}\Big) .\end{aligned}$$

Thus, Proposition 4.1 guarantees that for any $\delta \in [0, 2 - 2\eta/\pi)$, there exists a $t_\varepsilon < \varepsilon^\delta$ such that equation (2.4) holds. Since $\eta = \pi(1-\alpha)$, the restriction $\delta < 2 - 2\eta/\pi$ is precisely $\delta < 2\alpha$. Taking the infimum of the times t_ε as $\delta \to 2\alpha$ will guarantee the existence of a time in the interval $[0, \varepsilon^{2\alpha}]$ for which (2.4) holds. This proves Theorem 2.1 in the case that (2.2) holds.

The proof when the initial data is a vortex located a distance of ε^α away from the boundary (i.e. when (2.3) is satisfied) is similar. Indeed, let $\varphi \in C_0^{0,\gamma}$ and $y \in \partial\Omega$ be the point that is closest to the vortex center a_ε^0, and observe

$$\Big|\int_\Omega \varphi \delta_{a_\varepsilon^0}\, dx\Big| = \big|\varphi(a_\varepsilon^0) - \varphi(y)\big| \leqslant d(a_\varepsilon^0, \partial\Omega)^\gamma \|\varphi\|_{C_0^{0,\gamma}} .$$

Thus $\|\delta_{a_\varepsilon^0}\|_{(C_0^{0,\gamma})^*} = O(\varepsilon^{\alpha\gamma})$, and hence

$$\|J(u_\varepsilon^0)\|_{(C_0^{0,\gamma})^*} \leqslant \|J(u_\varepsilon^0) - \delta_{a_\varepsilon^0}\|_{(C_0^{0,\gamma})^*} + \|\delta_{a_\varepsilon^0}\|_{(C_0^{0,\gamma})^*} = o_\varepsilon\Big(\frac{|\ln \varepsilon|}{h_{ex}}\Big) .$$

Now Proposition 4.1, and the same argument as in the previous case, finishes the proof. □

The remainder of this section is devoted to the proof of Proposition 4.1. We begin by recalling a regularity result from [7, 20].

Lemma 4.2 (Lemma 3.7 in [7] or Propositions 2.7–2.8 in [20])**.** *Let* $(u_\varepsilon, A_\varepsilon, \Phi_\varepsilon)$ *be a solution to* (1.7)–(1.10) *under the Coulomb gauge with* $\|u_\varepsilon^0\|_{L^\infty} \leqslant 1$, $\|\nabla u_\varepsilon^0\|_{L^\infty} \leqslant \frac{C}{\varepsilon}$ *and* $G_\varepsilon(u_\varepsilon^0, A_\varepsilon^0) \leqslant C h_{ex}^2$*. Then we have*

$$\|u_\varepsilon(t)\|_{L^\infty} \leqslant 1\,, \tag{4.2}$$

$$\|\nabla u_\varepsilon(t)\|_{L^\infty} \leqslant \frac{C}{\varepsilon}\,, \tag{4.3}$$

for all $t \geqslant 0$.

Remark. The hypotheses in [7] and [20] for (4.2) and (4.3) respectively, are for smaller energies (when $G_\varepsilon(0) = O(|\ln\varepsilon|)$) and under the parabolic gauge. The proofs, however, can be easily adjusted to the higher energy level $O(h_{ex}^2)$ and the Coulomb gauge as stated in Lemma 4.2 above.

The main step in the proof of Proposition 4.1 is an energy-splitting argument, which we now describe. Define the free energy $G_{F,\varepsilon}(u,A)$ by

$$G_{F,\varepsilon}(u,A) \stackrel{\text{def}}{=} \int_\Omega \Big(\frac{1}{2}\left|\nabla_A u\right|^2 + \frac{1}{2}\left|\nabla\times A\right|^2 + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2\Big)\,dx\,. \tag{4.4}$$

Even though $G_\varepsilon(u_\varepsilon, A_\varepsilon)$ is of order $O(h_{ex}^2)$, we claim that the $\mathcal{E}_\varepsilon(u_\varepsilon)$ is of order $O(|\ln\varepsilon|)$ for short time. This is our next result.

Proposition 4.3. *Suppose that* $(u_\varepsilon(t), A_\varepsilon(t), \Phi_\varepsilon(t))$ *is a solution to* (1.7)-(1.10) *in the Coulomb gauge with* $|G_\varepsilon(u_\varepsilon^0, A_\varepsilon^0) - G_m| \leqslant \eta\,|\ln\varepsilon|$*. If* $\|J(u_\varepsilon^0)\|_{(C_0^{0,\gamma})^*} = o_\varepsilon(\frac{|\ln\varepsilon|}{h_{ex}})$ *for some* $0 < \gamma \leqslant 1$*, then for all* $0 \leqslant t \ll \big(\frac{|\ln\varepsilon|}{h_{ex}^3}\big)^{2/\gamma}$ *we have*

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) \leqslant \eta\,|\ln\varepsilon| + o_\varepsilon(|\ln\varepsilon|). \tag{4.5}$$

Remark. By the assumptions on h_{ex}, we have

$$\varepsilon^\beta \ll \Big(\frac{|\ln\varepsilon|}{h_{ex}^3}\Big)^{2/\gamma}$$

for any $0 \leqslant \beta \leqslant 1$ and all $\varepsilon \leqslant \varepsilon_0$, independent of β, γ.

Remark. A dipole separated by ε^α or a vortex at a distance ε^α from the boundary satisfy the hypotheses.

Proof of Proposition 4.3. 1. We first establish some regularity results on solutions of the equation. We will fix a Coulomb Gauge that ensures

$$\nabla\cdot A_\varepsilon = 0 \text{ in } \Omega \qquad A_\varepsilon\cdot\nu = 0 \text{ on } \partial\Omega\,.$$

In this gauge, a solution satisfies

$$\begin{aligned}\partial_t u_\varepsilon - i\Phi_\varepsilon u_\varepsilon &= \nabla_{A_\varepsilon}^2 u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon\left(1-|u_\varepsilon|^2\right)\,,\\ \partial_t A_\varepsilon - \nabla\Phi_\varepsilon &= \Delta A_\varepsilon + j_{A_\varepsilon}(u_\varepsilon)\,,\end{aligned}$$

in Ω with boundary conditions

$$\begin{aligned}\partial_\nu u_\varepsilon &= \nu\cdot A_\varepsilon = \partial_\nu\Phi_\varepsilon = 0\,,\\ h_\varepsilon &= h_{ex}\,,\end{aligned}$$

on $\partial\Omega$. Using the boundary conditions (1.9)–(1.10) and (3.1) we have the energy bound

$$G_\varepsilon(u_\varepsilon(t),A_\varepsilon(t))+\int_0^t\int_\Omega |\partial_{\Phi_\varepsilon}u_\varepsilon|^2+|E_\varepsilon|^2\,dx\,ds=G_\varepsilon(u_\varepsilon^0,A_\varepsilon^0), \tag{4.6}$$

and by assumptions on h_{ex}, $G_\varepsilon(t)\leqslant Ch_{ex}^2$. (We assume here, and subsequently, that C is a constant independent of ε that may increase from line to line.) Therefore,

$$\|\nabla_{A_\varepsilon}u_\varepsilon\|_{L^2}\leqslant Ch_{ex},\quad \|E_\varepsilon\|_{L^2([0,t];L^2(\Omega))}\leqslant Ch_{ex},\quad\text{and}\quad \|A_\varepsilon\|_{H^1}\leqslant Ch_{ex},$$

where the last bound on A_ε follows from the Coulomb gauge and a standard Hodge argument.

2. Next we claim that for all $0\leqslant t\leqslant 1$ and any $0\leqslant\gamma\leqslant 1$,

$$\left\|J(u_\varepsilon(t))-J(u_\varepsilon^0)\right\|_{(C_0^{0,\gamma})^*}\leqslant Ch_{ex}^2\left(\max\{\varepsilon,t^{\frac12}\}\right)^\gamma, \tag{4.7}$$

and if $0\leqslant t\ll\left(\frac{|\ln\varepsilon|}{h_{ex}^3}\right)^{2/\gamma}$, then

$$\left\|J(u_\varepsilon(t))-J(u_\varepsilon^0)\right\|_{(C_0^{0,\gamma})^*}=o_\varepsilon\Big(\frac{|\ln\varepsilon|}{h_{ex}}\Big). \tag{4.8}$$

This will enable us to split the full Ginzburg-Landau energy sufficiently well.

We first establish an estimate on the continuity of the Jacobian in certain weak topologies. Recalling $J_A(u)=\frac12\nabla\times(j_A(u)+A)$, a direct calculation shows

$$\partial_t J_{A_\varepsilon}(u_\varepsilon)=\nabla\times(i\partial_{\Phi_\varepsilon}u_\varepsilon,\nabla_{A_\varepsilon}u_\varepsilon)+\nabla\times\left(E_\varepsilon\left(\frac{1-|u_\varepsilon|^2}{2}\right)\right)\ . \tag{4.9}$$

Now for any $\varphi\in C_0^{0,\gamma}$ we have

$$\begin{aligned}
\left|\int_\Omega\big(J_{A_\varepsilon(t)}(u_\varepsilon(t))-J_{A_\varepsilon^0}(u_\varepsilon^0)\big)\,\varphi\,dx\right| &= \left|\int_0^t\int_\Omega\varphi(x)\frac{d}{ds}J_{A_\varepsilon(s)}(u_\varepsilon(s))\,dx\,ds\right|\\
&=2\left|\int_0^t\int_\Omega\nabla^\perp\varphi\cdot(i\partial_{\Phi_\varepsilon}u_\varepsilon,\nabla_{A_\varepsilon}u_\varepsilon)\ dx\,ds\right|\\
&\quad+\left|\int_0^t\int_\Omega\nabla^\perp\varphi\cdot E_\varepsilon\left(1-|u_\varepsilon|^2\right)\ dx\,ds\right|\\
&\leqslant 2\|\nabla\varphi\|_{L^\infty}\|\partial_{\Phi_\varepsilon}u_\varepsilon\|_{L^2([0,t]\times\Omega)}\|\nabla_{A_\varepsilon}u_\varepsilon\|_{L^2([0,t]\times\Omega)}\\
&\quad+\varepsilon\|\nabla\varphi\|_{L^\infty}\|E_\varepsilon\|_{L^2([0,t]\times\Omega)}\left\|\frac{1-|u_\varepsilon|^2}{\varepsilon}\right\|_{L^2([0,t]\times\Omega)}\\
&\leqslant 2\sqrt{t}\,\|\nabla\varphi\|_{L^\infty}\|\partial_{\Phi_\varepsilon}u_\varepsilon\|_{L^2([0,t]\times\Omega)}\|\nabla_{A_\varepsilon}u_\varepsilon\|_{L^\infty([0,t];L^2(\Omega))}\\
&\quad+\varepsilon\sqrt{t}\,\|\nabla\varphi\|_{L^\infty}\|E_\varepsilon\|_{L^2([0,t]\times\Omega)}\left\|\frac{1-|u_\varepsilon|^2}{\varepsilon}\right\|_{L^\infty([0,t];L^2(\Omega))}\ ,
\end{aligned}$$

and so

$$\left|\int_\Omega\big(J_{A_\varepsilon(t)}(u_\varepsilon(t))-J_{A_\varepsilon^0}(u_\varepsilon^0)\big)\,\varphi\,dx\right|\leqslant C\,\|\nabla\varphi\|_{L^\infty}\,h_{ex}^2\sqrt{t}\,.$$

This implies

$$\left\|J_{A_\varepsilon(t)}(u_\varepsilon(t))-J_{A_\varepsilon^0}(u_\varepsilon^0)\right\|_{\dot W^{-1,1}(\Omega)}\leqslant Ch_{ex}^2\sqrt{t}\,. \tag{4.10}$$

However, we note that for any $\varphi \in W_0^{1,\infty}$,

$$\begin{aligned}\left|\int_\Omega \varphi\,(J_{A_\varepsilon}(u_\varepsilon) - J(u_\varepsilon))\,dx\right| &= \left|\int_\Omega \nabla^\perp\varphi\cdot(j_{A_\varepsilon}(u_\varepsilon) + A_\varepsilon - j(u_\varepsilon))\right|\\ &= \left|\int_\Omega \nabla^\perp\varphi\cdot\left(A_\varepsilon(1-|u_\varepsilon|^2)\right)\right|\\ &\leqslant \varepsilon\,\|\nabla\varphi\|_{L^\infty}\,\|A_\varepsilon\|_{L^2}\left\|\frac{1-|u_\varepsilon|^2}{\varepsilon}\right\|_{L^2}\\ &\leqslant C\varepsilon\,\|\nabla\varphi\|_{L^\infty}\,h_{ex}^2\,,\end{aligned}$$

where we recall $j(u) \stackrel{\text{def}}{=} (iu, \nabla u)$. Consequently

$$\left\|J_{A_\varepsilon(t)}(u_\varepsilon(t)) - J(u_\varepsilon(t))\right\|_{\dot W^{-1,1}(\Omega)} \leqslant C\varepsilon h_{ex}^2\,, \tag{4.11}$$

for all $t \leqslant \varepsilon^{-\frac12}$. In particular (4.10) and (4.11) imply

$$\left\|J(u_\varepsilon(t)) - J(u_\varepsilon^0)\right\|_{(C^{0,1}(\Omega))^*} \leqslant \left\|J(u_\varepsilon(t)) - J(u_\varepsilon^0)\right\|_{\dot W^{-1,1}(\Omega)} \leqslant C\max\{\varepsilon,\sqrt{t}\}h_{ex}^2. \tag{4.12}$$

A similar calculation shows that

$$\left\|J(u_\varepsilon(t)) - J(u_\varepsilon^0)\right\|_{(C_0^0(\Omega))^*} \leqslant Ch_{ex}^2. \tag{4.13}$$

Using (4.12) and (4.13), along with interpolation of dual Hölder spaces, (see Jerrard-Soner [10]), yields (4.7).

3. Next, we claim that

$$G_\varepsilon(u_\varepsilon(t), A_\varepsilon(t)) = G_m + G_{F,\varepsilon}(u_\varepsilon(t), A'_\varepsilon(t)) + o_\varepsilon(|\ln\varepsilon|)\,, \tag{4.14}$$

where $A'_\varepsilon(t) = A_\varepsilon(t) - h_{ex}\nabla^\perp\xi_m$ and $G_{F,\varepsilon}$ is defined in (4.4). To see this, we decompose

$$\begin{aligned}G_\varepsilon(u_\varepsilon, A_\varepsilon) = G_{F,\varepsilon}(u_\varepsilon, A'_\varepsilon) &+ h_{ex}^2\int_\Omega |\nabla\xi_m|^2\,|u_\varepsilon|^2 + |h_m - 1|^2\,dx\\ &- h_{ex}\int_\Omega \nabla^\perp\xi_m\cdot j_{A'_\varepsilon}(u_\varepsilon) + h'_\varepsilon(h_m-1)\,dx\,,\end{aligned}$$

where $h'_\varepsilon = \nabla\times A'_\varepsilon$.Note $h_m - 1 = \xi_m$, where $h_m \stackrel{\text{def}}{=} \Delta\xi_m$ is the Meissner magnetic field. Therefore,

$$\begin{aligned}&- h_{ex}\int_\Omega \nabla^\perp\xi_m\cdot j_{A'_\varepsilon}(u_\varepsilon) + h'_\varepsilon(h_m-1)\,dx\\ &\qquad= -h_{ex}\int_\Omega \nabla^\perp\xi_m\cdot j(u_\varepsilon) - \nabla^\perp\xi_m\cdot A'_\varepsilon|u_\varepsilon|^2 + h'_\varepsilon\xi_m\,dx\\ &\qquad= 2h_{ex}\int_\Omega \xi_m J(u_\varepsilon)\,dx + h_{ex}\int_\Omega \nabla^\perp\xi_m\cdot A'_\varepsilon(|u_\varepsilon|^2-1)\,dx\,,\end{aligned}$$

and so

$$\begin{aligned}G_\varepsilon(u_\varepsilon, A_\varepsilon) &= G_m + G_{F,\varepsilon}(u_\varepsilon, A'_\varepsilon) + 2h_{ex}\int_\Omega \xi_m J(u_\varepsilon)\,dx\\ &\qquad + h_{ex}\int_\Omega \nabla^\perp\xi_m\cdot A'_\varepsilon(|u_\varepsilon|^2-1)\,dx + h_{ex}^2\int_\Omega |\nabla\xi_m|^2\,(|u_\varepsilon|^2-1)\,dx\\ &= G_m + G_{F,\varepsilon}(u_\varepsilon, A'_\varepsilon) + 2h_{ex}\int_\Omega \xi_m J(u_\varepsilon)\,dx + o_\varepsilon(|\ln\varepsilon|)\,.\end{aligned}$$

Since ξ_m is a smooth function (and hence in any Hölder space), then

$$h_{ex}\int_\Omega \xi_m J(u_\varepsilon^0)\,dx = o_\varepsilon(|\ln\varepsilon|)$$

holds by assumption and (4.14) follows from (4.7).

4. Finally, we prove that

$$\mathcal{E}_\varepsilon(u_\varepsilon(t)) \leqslant \eta\,|\ln\varepsilon| + o_\varepsilon(|\ln\varepsilon|) \tag{4.15}$$

for all $t \leqslant \left(\frac{|\ln\varepsilon|}{h_{ex}^3}\right)^{2/\gamma}$.

By (1.11) and our assumptions

(4.16)

$$G_\varepsilon(u_\varepsilon(t),A_\varepsilon(t)) + \int_0^t\int_\Omega |\partial_{\Phi_\varepsilon}u_\varepsilon|^2 + |\nabla\Phi_\varepsilon|^2 + |\partial_t A_\varepsilon|^2\,dx\,ds \leqslant G_m + \eta\,|\ln\varepsilon|\,.$$

By steps 2 and 3, we have

$$G_{F,\varepsilon}(u_\varepsilon(t),A'_\varepsilon(t)) \leqslant \eta\,|\ln\varepsilon| + o_\varepsilon(|\ln\varepsilon|). \tag{4.17}$$

as long as $t \ll \left(\frac{|\ln\varepsilon|}{h_{ex}^3}\right)^{2/\gamma}$. Thus, we can continue our decomposition and use

$$\begin{aligned}\int_\Omega \left|\nabla_{A'_\varepsilon}u_\varepsilon\right|^2\,dx &= \int_\Omega |\nabla u_\varepsilon|^2 - 2\nabla^\perp\xi'_\varepsilon\cdot j(u_\varepsilon) + |A'_\varepsilon|^2|u_\varepsilon|^2\,dx\\ &= \int_\Omega |\nabla u_\varepsilon|^2 - 2\xi'_\varepsilon J(u_\varepsilon) + |A'_\varepsilon|^2|u_\varepsilon|^2\,dx\,.\end{aligned}$$

By (4.17) we have $\|A'_\varepsilon\|_{H^1} \leqslant Ch_{ex}$ then $\|\xi'_\varepsilon(t)\|_{C^{0,\gamma}} \leqslant Ch_{ex}$ for all $t \geqslant 0$. Using $\|J(u_\varepsilon^0)\|_{(C_0^{0,\gamma})^*} = o_\varepsilon(\frac{|\ln\varepsilon|}{h_{ex}})$ and (4.7), along with the bound on ξ'_ε yields (4.15). □

We now state the η-compactness result for the gauge-less energy, $\mathcal{E}_\varepsilon$.

Proposition 4.4 (Proposition 2.2 in [18])**.** *Suppose u_ε satisfies the* static *equation*

$$\begin{aligned}\Delta u_\varepsilon + \frac{1}{\varepsilon^2}u_\varepsilon\left(1-|u_\varepsilon|^2\right) &= f_\varepsilon \text{ in } \Omega\,,\\ \partial_\nu u_\varepsilon &= 0 \text{ on } \partial\Omega\,.\end{aligned}$$

Further, assume $|u_\varepsilon| \leqslant 1$, $\mathcal{E}_\varepsilon(u_\varepsilon) \leqslant M\,|\ln\varepsilon|$, $|\nabla u_\varepsilon| \leqslant \frac{C}{\varepsilon}$, and $\|f_\varepsilon\|_{L^2} \leqslant \frac{1}{\varepsilon^\beta}$ for some $\beta < 2$. Then, after extraction of a subsequence $\varepsilon \to 0$, we can find $R_\varepsilon \to +\infty$ with $R_\varepsilon \leqslant C\,|\ln\varepsilon|$ and a family of balls $\cup_{i=1}^n B_i = \cup_{i=1}^n B(a_i, R_\varepsilon\varepsilon)$, with a_i depending on ε and n bounded independently of ε, such that the following hold.

(1) As $\varepsilon \to 0$ we have

$$\|1-|u_\varepsilon|\|_{L^\infty(\Omega\setminus\cup_i B(a_i,R_\varepsilon\varepsilon))} \to 0\,. \tag{4.18}$$

(2) For every $\beta < 1$ and every subset I of $[1,n]$, we have

$$\beta\pi\sum_{i\in I} d_i^2 \leqslant \int_{\cup_{i\in I}B(a_i,R_\varepsilon\varepsilon^{1-\beta})}\frac{e_\varepsilon(u_\varepsilon)}{|\ln\varepsilon|}dx + C\,|\ln\varepsilon|^{\frac{7}{2}}\,\varepsilon^{1-\beta}\,\|f_\varepsilon\|_{L^2} + o_\varepsilon(1) \tag{4.19}$$

We now prove Proposition 4.1.

Proof of Proposition 4.1. Using the above bounds on the gauged problem above, we follow the template laid out in [19].

1. We first claim that for any $0 < \delta < 1$, there exists a time $t_\varepsilon \in (\frac{\varepsilon^\delta}{2}, \varepsilon^\delta)$ such that

$$\|\partial_{\Phi_\varepsilon} u_\varepsilon(t_\varepsilon)\|_{L^2}^2 + \|E_\varepsilon(t_\varepsilon)\|_{L^2}^2 \leqslant C\varepsilon^{-\delta} h_{ex}^2. \tag{4.20}$$

(We will later choose δ to ensure there are no vortices at time t_ε.) To prove (4.20), we use the parabolic bound

$$\int_0^t \left(\|\partial_{\Phi_\varepsilon} u_\varepsilon(s)\|_{L^2}^2 + \|E_\varepsilon(s)\|_{L^2}^2 \right) ds \leqslant G_\varepsilon(u_\varepsilon^0, A_\varepsilon^0) \leqslant C h_{ex}^2.$$

Since we also know

$$Ch_{ex}^2 \geqslant \int_{\frac{t}{2}}^t \|\partial_{\Phi_\varepsilon} u_\varepsilon(s)\|_{L^2}^2 + \|E_\varepsilon(s)\|_{L^2}^2 \, ds \geqslant \frac{t}{2} \inf_{s \in [\frac{t}{2}, t]} \left(\|\partial_{\Phi_\varepsilon} u_\varepsilon(s)\|_{L^2}^2 + \|E_\varepsilon(s)\|_{L^2}^2 \right)$$

then there exists a $t_\varepsilon \in (\frac{\varepsilon^\delta}{2}, \varepsilon^\delta)$ such that (4.20) holds.

2. We claim that this implies that

$$\|A_\varepsilon(t_\varepsilon)\|_{L^\infty} \leqslant C\varepsilon^{-\delta/2} h_{ex}. \tag{4.21}$$

To see this we note that from (4.20),

$$\begin{aligned} \|\nabla h_\varepsilon(t_\varepsilon)\|_{L^2} &\leqslant \|j_{A_\varepsilon}(u_\varepsilon(t_\varepsilon))\|_{L^2} + \|E_\varepsilon(t_\varepsilon)\|_{L^2} \\ &\leqslant Ch_{ex} + C\varepsilon^{-\delta/2} h_{ex}. \end{aligned}$$

In particular, $\|h_\varepsilon(t_\varepsilon)\|_{H^1} \leqslant C\varepsilon^{-\delta/2} h_{ex}$. Using standard elliptic theory and a Hodge decomposition of A_ε we find that

$$\|A_\varepsilon(t_\varepsilon)\|_{L^\infty} \leqslant C \|A_\varepsilon(t_\varepsilon)\|_{H^2} \leqslant C \left\| \nabla \times (-\Delta_0^{-1}) h_\varepsilon(t_\varepsilon) \right\|_{H^2} \leqslant C \|h_\varepsilon(t_\varepsilon)\|_{H^1},$$

and (4.21) follows by embedding.

3. We now show that the first two steps and the hypotheses allow us to use Proposition 4.3. From the evolution equation for u_ε we can write

$$\Delta u_\varepsilon + \frac{1}{\varepsilon^2} u_\varepsilon \left(1 - |u_\varepsilon|^2\right) = f_\varepsilon,$$

where

$$f_\varepsilon \equiv 2iA_\varepsilon \cdot \nabla_{A_\varepsilon} u_\varepsilon + |A_\varepsilon|^2 u_\varepsilon - \partial_{\Phi_\varepsilon} u_\varepsilon. \tag{4.22}$$

Each of these terms can be estimated in L^2 for some time $t \in (\frac{1}{2}\varepsilon^\delta, \varepsilon^\delta)$. From (4.21) and (1.11)

$$\|A_\varepsilon \cdot \nabla_{A_\varepsilon} u_\varepsilon\|_{L^2} \leqslant \|A_\varepsilon\|_{L^\infty} \|\nabla_{A_\varepsilon} u_\varepsilon\|_{L^2} \leqslant C\varepsilon^{-\frac{\delta}{2}} h_{ex} \tag{4.23}$$

and

$$\left\| |A_\varepsilon|^2 u_\varepsilon \right\|_{L^2} \leqslant \|A_\varepsilon\|_{H^1}^2 \leqslant Ch_{ex}^2. \tag{4.24}$$

$$\|\partial_{\Phi_\varepsilon} u_\varepsilon(t_\varepsilon)\|_{L^2} \leqslant C\varepsilon^{-\frac{\delta}{2}} h_{ex}. \tag{4.25}$$

Then by (4.23)-(4.25) for the $t_\varepsilon \in (\frac{\varepsilon^\delta}{2}, \varepsilon^\delta)$ we have

$$\|f_\varepsilon(t_\varepsilon)\|_{L^2} \leqslant C\varepsilon^{-\frac{\delta}{2}} h_{ex}. \tag{4.26}$$

4. We can now follow Proposition 2.1 in [19]; by the structure result Proposition 4.4 at t_ε defined above, there are vortices $\{a_j\}$ of degree $\{d_j\}$ such that for any $\beta < 1$,

$$\beta\pi \sum_j d_j^2 \leqslant \eta + C\,|\ln\varepsilon|^{\frac{7}{2}}\,\varepsilon^{1-\beta-\frac{\delta}{2}} h_{ex} + o_\varepsilon(1). \tag{4.27}$$

We can choose $\beta > \frac{\eta}{2\pi}$ and $\delta < 2-2\beta$, then by (4.19) we have that for all $\varepsilon \leqslant \varepsilon_0$, $\sum_j d_j^2 < 2$; consequently, $\sum d_j^2 \in \{0,1\}$. However, since the d_j's are nontrivial then either $\|1-u_\varepsilon(t_\varepsilon)\|_{L^\infty} = o_\varepsilon(1)$ or there exists one vortex a with degree ± 1.

Suppose now that there exists a single vortex a, we again use the argument from [19] to get that $E_\varepsilon(u_\varepsilon(t_\varepsilon)) \geqslant \pi \ln\frac{\ell}{\varepsilon} + O(1)$, where $\ell = d(a,\partial\Omega)$. But upper bound (4.5) implies that $E_\varepsilon(u_\varepsilon(t_\varepsilon)) \leqslant \eta\,|\ln\varepsilon| + o_\varepsilon(|\ln\varepsilon|)$ which implies that the vortex must satisfy $\ell \leqslant \varepsilon^\mu$ for some

$$\mu \geqslant 1 - \frac{\eta}{\pi}. \tag{4.28}$$

By Theorem 2 of [18] we have that

$$\|f_\varepsilon(t_\varepsilon)\|_{L^2} \geqslant C\frac{\varepsilon^{-\mu}}{|\ln\varepsilon|}, \tag{4.29}$$

for some constant C depending on β and Ω. Given (4.26) and (4.29) we see that

$$2\mu < \delta. \tag{4.30}$$

On the other hand we can further restrict $\delta < 2 - 2\frac{\eta}{\pi}$, and so with (4.28) we get

$$\delta < 2\mu, \tag{4.31}$$

and a contradiction between (4.30) and (4.31). Therefore, there are no vortices at time t_ε, which implies $\|1-|u_\varepsilon(t_\varepsilon)|\|_{L^\infty} = o_\varepsilon(1)$. □

5. **Stochastic Models for Driving Dipoles.**

In this section we prove Theorem 2.3.

Proof of Theorem 2.3. When $\beta_\varepsilon = 0$ the SDE (2.6) reduces to (2.1) which has a strongly attractive stable equilibrium at $a_\varepsilon = 0$. Standard large deviation results can now be used to estimate the chance that a escapes from 0. These results, however, don't directly apply here since the initial position, strength of the noise, and the interval length all depend on ε. While these obstructions can likely be overcome abstractly, the problem at hand admits an explicit solution and we handle it directly instead.

We know (see for instance [12, §9]) that the function φ_ε defined by

$$\varphi_\varepsilon(x) \stackrel{\text{def}}{=} \boldsymbol{P}^x(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon)\,.$$

satisfies the equation

$$\beta_\varepsilon \partial_x^2 \varphi_\varepsilon - b_\varepsilon \partial_x \varphi_\varepsilon = 0\,, \tag{5.1}$$

with boundary conditions

$$\varphi_\varepsilon(0) = 0\,, \quad \text{and} \quad \varphi_\varepsilon(\hat{A}) = 1\,.$$

Let $B_\varepsilon = \int b_\varepsilon$ be a primitive of b_ε. The solution to (5.1) is given by

$$\varphi_\varepsilon(z) = \frac{\int_0^z e^{B_\varepsilon/\beta_\varepsilon}}{\int_0^{\hat{A}_\varepsilon} e^{B_\varepsilon/\beta_\varepsilon}} = \frac{\int_0^z x^{\frac{1}{\beta_\varepsilon |\ln \varepsilon|}} \exp\Big(\frac{-\lambda h_{ex} x}{\beta_\varepsilon |\ln \varepsilon|}\Big)\, dx}{h_{ex} \int_0^{\hat{A}_\varepsilon} x^{\frac{1}{\beta_\varepsilon |\ln \varepsilon|}} \exp\Big(\frac{-\lambda h_{ex} x}{\beta_\varepsilon |\ln \varepsilon|}\Big)\, dx}\,.$$

Using (2.10) and making the substitution $y = c_0 x/\beta_\varepsilon$ yields

$$\boldsymbol{P}^z(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon) = \varphi_\varepsilon(z) = \frac{\int_0^{c_0 z/\beta_\varepsilon} y^{\frac{1}{\beta_\varepsilon |\ln \varepsilon|}} e^{-y}\, dy}{\int_0^{1/(\beta_\varepsilon |\ln \varepsilon|)} y^{\frac{1}{\beta_\varepsilon |\ln \varepsilon|}} e^{-y}\, dy}\,.$$

Thus, Theorem 2.3 now reduces to understanding the asymptotic behaviour of the right hand side as $\varepsilon \to 0$.

To this end, define

$$m_\varepsilon = \frac{c_0 \varepsilon^\alpha}{\beta_\varepsilon} \quad \text{and} \quad n_\varepsilon = \frac{1}{\beta_\varepsilon |\ln \varepsilon|}\,,$$

and observe

$$\boldsymbol{P}^{\varepsilon^\alpha}(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon) = \varphi_\varepsilon(\varepsilon^\alpha) = \frac{\gamma(n_\varepsilon + 1, m_\varepsilon)}{\gamma(n_\varepsilon + 1, n_\varepsilon)}\,, \tag{5.2}$$

where

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t}\, dt$$

is the incomplete lower gamma function. We now split the analysis into cases.

Case I: $\beta_\varepsilon \ll 1/|\ln \varepsilon|$. In this case $m_\varepsilon, n_\varepsilon \to \infty$, and $m_\varepsilon/n_\varepsilon \to 0$. Clearly

$$\gamma(n_\varepsilon + 1, m_\varepsilon) \leqslant \int_0^{m_\varepsilon} t^{n_\varepsilon}\, dt = \frac{m_\varepsilon^{n_\varepsilon + 1}}{n_\varepsilon + 1}. \tag{5.3}$$

To estimate $\gamma(n_\varepsilon + 1, n_\varepsilon)$, observe first that for any $x > 1$, $\gamma(s, x)$ is decreasing in s when s is sufficiently large. Thus, without we can without loss of generality, assume $n_\varepsilon \in \mathbb{N}$. Repeatedly integrating by parts we obtain the identity

$$\gamma(n_\varepsilon + 1, x) = n_\varepsilon!(1 - e^{-x} e_{n_\varepsilon}(x))\,, \quad \text{where } e_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

is the truncated exponential. Since $e^{-n} e_n(n) \to 1/2$ as $n \to \infty$ we must have

$$\lim_{n \to \infty} \frac{\gamma(n+1, n)}{n!} = \frac{1}{2}\,. \tag{5.4}$$

For the numerator $\gamma(n_\varepsilon + 1, m_\varepsilon)$, clearly

$$\gamma(n_\varepsilon + 1, m_\varepsilon) \leqslant \int_0^{m_\varepsilon} t^{n_\varepsilon}\, dt = \frac{m_\varepsilon^{n_\varepsilon + 1}}{n_\varepsilon + 1}\,,$$

and using (5.2), (5.4) and Sterlings formula we have

$$\lim_{\varepsilon \to 0} \frac{\boldsymbol{P}^{\varepsilon^\alpha}(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon)}{\varepsilon^\alpha} = 0\,.$$

Finally, to estimate N_ε, equation (2.9) shows

$$\frac{\varphi_\varepsilon(\varepsilon^\alpha)}{2\varepsilon^\alpha} \leqslant N_\varepsilon = 1 - (1 - \boldsymbol{P}^{\varepsilon^\alpha}(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon))^{\varepsilon^{-\alpha}} \leqslant \frac{2\varphi_\varepsilon(\varepsilon^\alpha)}{\varepsilon^\alpha}\,, \tag{5.5}$$

and hence $N_\varepsilon \to 0$ as $\varepsilon \to 0$.

Case II: $\beta_\varepsilon \approx 1/|\ln\varepsilon|$. In this case we assume

$$\lim_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|\beta_\varepsilon} = n_0 > 0\,,$$

and so $n_\varepsilon \to n_0$ and $m_\varepsilon \to 0$ as $\varepsilon \to 0$.

Now the denominator $\gamma(n_\varepsilon + 1, n_\varepsilon) \to c_2 > 0$ as $n_\varepsilon \to n_0$, and the numerator $\gamma(n_\varepsilon + 1, m_\varepsilon)$ can again be bounded by (5.3). This shows

$$\boldsymbol{P}^{\varepsilon^\alpha}(a_\varepsilon(\tau_\varepsilon) = \hat{A}_\varepsilon) \leqslant c\varepsilon^{\alpha(1+n_0)}$$

for some constant $c > 0$. Consequently, using (5.5), we have $N_\varepsilon \leqslant c\varepsilon^{\alpha n_0} \to 0$ as $\varepsilon \to 0$.

Case III: $\beta_\varepsilon \gg 1/|\ln\varepsilon|$. In this case both $m_\varepsilon \to 0$, $n_\varepsilon \to 0$ and $m_\varepsilon/n_\varepsilon \to 0$. Using the estimate

$$\frac{e^{-x}x^s}{s} \leqslant \gamma(s,x) \leqslant \frac{x^s}{s},$$

we see

$$e^{-m_\varepsilon}\Big(\frac{m_\varepsilon}{n_\varepsilon}\Big)^{n_\varepsilon+1} \leqslant \varphi_\varepsilon(\varepsilon^\alpha) \leqslant e^{n_\varepsilon}\Big(\frac{m_\varepsilon}{n_\varepsilon}\Big)^{n_\varepsilon+1}. \tag{5.6}$$

To compute the limiting behaviour of N_ε, observe

$$\lim_{\varepsilon\to 0}\frac{-\ln(1-N_\varepsilon)}{\varepsilon^{-\alpha}\varphi_\varepsilon(\varepsilon^\alpha)} = \frac{-\ln(1-\varphi_\varepsilon(\varepsilon^\alpha))}{\varphi_\varepsilon(\varepsilon^\alpha)} = 1\,,$$

where, for simplicity, we assumed the existence of the limit. Using (5.6),

$$\begin{aligned}
\lim_{\varepsilon\to 0}\ln\Big(\frac{\varphi_\varepsilon(\varepsilon^\alpha)}{\varepsilon^\alpha}\Big) &= \lim_{\varepsilon\to 0}\ln\Big(\frac{(m_\varepsilon/n_\varepsilon)^{1+n_\varepsilon}}{\varepsilon^\alpha}\Big)\\
&= \lim_{\varepsilon\to 0}\Big[\Big(1+\frac{1}{\beta_\varepsilon|\ln\varepsilon|}\Big)\ln\Big(c_0|\ln\varepsilon|\Big) - \frac{\alpha}{\beta_\varepsilon}\Big]\\
&= \lim_{\varepsilon\to 0}\Big[\frac{1}{\beta_\varepsilon}\Big(\frac{\ln\big(c_0|\ln\varepsilon|\big)}{|\ln\varepsilon|} - \alpha\Big) + \ln|\ln\varepsilon| + \ln c_0\Big]\,, \qquad (5.7)
\end{aligned}$$

provided the limits exists. From this it follows that $\liminf \beta_\varepsilon \ln|\ln\varepsilon| > \alpha$, then $N_\varepsilon \to 1$ as $\varepsilon \to 0$. On the other hand if $\limsup \beta_\varepsilon \ln|\ln\varepsilon| < \alpha$, then $N_\varepsilon \to 0$ as $\varepsilon \to 0$. □

Remark 5.1. In the transition regime (when $\beta_\varepsilon \ln|\ln\varepsilon| \to \alpha$), we note that N_ε can be estimated from above and below using (5.7). The bounds obtained, however, depend on the rate at which $\beta_\varepsilon \ln|\ln\varepsilon|$ converges to α.

Appendix A. Annihilation Times of the Heuristic ODE.

We devote this appendix to proving Proposition 2.2, estimating the annihilation times of the heuristic equation (2.1).

Proof of Proposition 2.2. A direct calculation shows that when $\lambda > 0$ the solution to (2.1) is given by

$$a_\varepsilon(t) = \frac{1}{\lambda h_{ex}}\Big[1 + W_0(-C\exp\Big(\frac{\lambda^2 h_{ex}^2 t}{|\ln\varepsilon|}\Big)\Big],$$

for some constant C. Here W_0 is the principal branch of the Lambert W function. We recall (see [5], or Section 4.13 in [13]) that W_0 is defined by the functional relation

$$W_0(ze^z) = z$$

when $z \geqslant -1$.

Using the initial data $a_\varepsilon(0) = \varepsilon^\alpha$ we find

$$C = (1 - \lambda h_{ex}\varepsilon^\alpha)e^{-(1-\lambda h_{ex}\varepsilon^\alpha)}.$$

Annihilation occurs when $W_0 = -1$ which is precisely when

$$C \exp\Big(\frac{\lambda^2 h_{ex}^2 t_\varepsilon}{|\ln \varepsilon|}\Big) = \frac{1}{e}.$$

Substituting C above gives

$$\exp\Big(\lambda h_{ex}\varepsilon^\alpha + \frac{\lambda^2 h_{ex}^2 t_\varepsilon}{|\ln \varepsilon|}\Big) = \frac{1}{1 - \lambda h_{ex}\varepsilon^\alpha},$$

and hence

$$t_\varepsilon = \frac{|\ln \varepsilon|}{\lambda^2 h_{ex}^2}\Big(|\ln(1 - \lambda h_{ex}\varepsilon^\alpha)| - \lambda h_{ex}\varepsilon^\alpha\Big) = \frac{\varepsilon^{2\alpha}|\ln \varepsilon|}{2} + \frac{1}{3}\lambda h_{ex}\varepsilon^{3\alpha}|\ln \varepsilon| + \cdots ,$$

from which (2.5) follows.

It remains to prove (2.5) when $\lambda = 0$. In this case, the exact solution to (2.1) is given by

$$a_\varepsilon(t) = \Big(\varepsilon^{2\alpha} - \frac{2t}{|\ln \varepsilon|}\Big)^{1/2},$$

and hence

$$t_\varepsilon = \frac{\varepsilon^{2\alpha}|\ln \varepsilon|}{2}, \tag{A.1}$$

for any $\varepsilon > 0$. This concludes the proof. ☐

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