

# REGULARITY OF MILNE PROBLEM WITH GEOMETRIC CORRECTION IN 3D

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**ABSTRACT.** Consider the Milne problem with geometric correction in a 3D convex domain. Via bootstrapping arguments, we establish  $W^{1,\infty}$  regularity for its solutions. Combined with a uniform  $L^6$  estimate, such regularity leads to the validity of diffusive expansion for the neutron transport equation with diffusive boundary conditions.

**Keywords:** Geometric correction; Bootstrapping;  $W^{1,\infty}$  estimates.

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## 1. INTRODUCTION

**1.1. Motivation and Formulation.** Milne problem is the main tool to study boundary layer effect in kinetic equations. Here to motivate 3D  $\epsilon$ -Milne problem with geometric correction, we consider the steady neutron transport equation in a three-dimensional convex domain with diffusive boundary. In the space domain  $\vec{x} = (x_1, x_2, x_3) \in \Omega$  where  $\partial\Omega \in C^3$  and the velocity domain  $\vec{w} = (w_1, w_2, w_3) \in \mathcal{S}^2$ , the neutron density  $u^\epsilon(\vec{x}, \vec{w})$  satisfies

$$\begin{cases} \epsilon \vec{w} \cdot \nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon &= 0 \text{ in } \Omega, \\ u^\epsilon(\vec{x}_0, \vec{w}) &= \mathcal{P}[u^\epsilon](\vec{x}_0) + \epsilon g(\vec{x}_0, \vec{w}) \text{ for } \vec{w} \cdot \vec{\nu} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases} \quad (1.1)$$

where

$$\bar{u}^\epsilon(\vec{x}) = \frac{1}{4\pi} \int_{\mathcal{S}^2} u^\epsilon(\vec{x}, \vec{w}) d\vec{w}, \quad (1.2)$$

$$\mathcal{P}[u^\epsilon](\vec{x}_0) = \frac{1}{4\pi} \int_{\vec{w} \cdot \vec{\nu} > 0} u^\epsilon(\vec{x}_0, \vec{w}) (\vec{w} \cdot \vec{\nu}) d\vec{w}, \quad (1.3)$$

$\vec{\nu}$  is the outward unit normal vector, with the Knudsen number  $0 < \epsilon \ll 1$ . Also,  $u^\epsilon$  satisfies the normalization condition

$$\int_{\Omega \times \mathcal{S}^2} u^\epsilon(\vec{x}, \vec{w}) d\vec{w} d\vec{x} = 0, \quad (1.4)$$

and  $g$  satisfies the compatibility condition

$$\int_{\partial\Omega} \int_{\vec{w} \cdot \vec{\nu} < 0} g(\vec{x}_0, \vec{w}) (\vec{w} \cdot \vec{\nu}) d\vec{w} d\vec{x}_0 = 0. \quad (1.5)$$

A classical problem is to study the diffusive limit of (1.1) as  $\epsilon \rightarrow 0$ . Generally speaking, the solution  $u^\epsilon$  varies smoothly and slowly in the interior of  $\Omega$ , and behaves like  $u^\epsilon - \bar{u}^\epsilon = 0$  which ignores  $\epsilon \vec{w} \cdot \nabla_x u^\epsilon$ . However, its value changes severely when approaching boundary  $\partial\Omega$  and  $\epsilon \vec{w} \cdot \nabla_x u^\epsilon$  becomes non-negligible. The smaller  $\epsilon$  is, the more severe  $u^\epsilon$  changes. This indicates that  $u^\epsilon$  actually contains two separate parts with different scalings, i.e. interior solution and boundary layer. In particular, the boundary layer is a function with scaled variable defined in a thin layer of thickness  $O(\epsilon)$  close to boundary in  $\Omega$ .

In the region of boundary layer, assume  $\mu$  denotes the distance to the boundary in the inward normal direction and  $(\tau_1, \tau_2)$  denote a local orthogonal curvilinear coordinate system for  $\partial\Omega$ . Then we have

$$\epsilon \vec{w} \cdot \nabla_x = -\epsilon (\vec{w} \cdot \vec{\nu}) \frac{\partial}{\partial \mu} + \frac{\epsilon}{R_1 - \mu} (\vec{w} \cdot \vec{t}_1) \frac{\partial}{\partial \tau_1} + \frac{\epsilon}{R_2 - \mu} (\vec{w} \cdot \vec{t}_2) \frac{\partial}{\partial \tau_2}, \quad (1.6)$$

where  $(\vec{t}_1, \vec{t}_2)$  are orthogonal tangential vectors associated with  $(\tau_1, \tau_2)$ , and  $(R_1(\tau_1, \tau_2), R_2(\tau_1, \tau_2))$  denote two radius of principle curvature. Since  $\vec{w} \in \mathcal{S}^2$ , we define the spherical velocity substitution as

$$\begin{cases} -\vec{w} \cdot \vec{\nu} &= \sin \phi, \\ \vec{w} \cdot \vec{t}_1 &= \cos \phi \sin \psi, \\ \vec{w} \cdot \vec{t}_2 &= \cos \phi \cos \psi. \end{cases} \quad (1.7)$$

With the rescaled distance  $\eta = \frac{\mu}{\epsilon}$ , we may represent

$$\epsilon \vec{w} \cdot \nabla_x = \sin \phi \frac{\partial}{\partial \eta} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right) \cos \phi \frac{\partial}{\partial \phi} + \text{higher-order terms}. \quad (1.8)$$

Therefore, in order to construct boundary layer, it suffices to study the 3D  $\epsilon$ -Milne problem with geometric correction for  $f^\epsilon(\eta, \phi, \psi)$  in  $(\eta, \phi, \psi) \in [0, L] \times [-\pi/2, \pi/2] \times [-\pi, \pi]$  as

$$\begin{cases} \sin \phi \frac{\partial f^\epsilon}{\partial \eta} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right) \cos \phi \frac{\partial f^\epsilon}{\partial \phi} + f^\epsilon - \bar{f}^\epsilon &= S^\epsilon(\eta, \phi, \psi), \\ f^\epsilon(0, \phi, \psi) &= h^\epsilon(\phi, \psi) \text{ for } \sin \phi > 0, \\ f^\epsilon(L, \phi, \psi) &= f^\epsilon(L, -\phi, \psi), \end{cases} \quad (1.9)$$

where  $h^\epsilon$  and  $S^\epsilon$  are two functions given a priori and

$$\bar{f}^\epsilon(\eta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} f^\epsilon(\eta, \phi, \psi) \cos \phi d\phi d\psi \quad (1.10)$$

in which  $\cos \phi$  shows up as the Jacobian of spherical coordinates in integration, and  $L = \epsilon^{-n}$  for some  $n$  to be specified later. Note that actually  $f^\epsilon$ ,  $S^\epsilon$ ,  $h^\epsilon$  and  $R_1$ ,  $R_2$  are all related to  $(\tau_1, \tau_2)$ . Since boundary layer is defined locally on  $\partial\Omega$  and our analysis focuses on the case for fixed  $(\tau_1, \tau_2)$ , we do not need to specify such dependence explicitly unless necessary.

**1.2. Main Result.** Define norms

$$\|f\|_{L^\infty L^\infty} = \sup_{(\eta, \phi, \psi)} |f(\eta, \phi, \psi)|, \quad (1.11)$$

$$\|f\|_{L^\infty}(\eta) = \sup_{(\phi, \psi) \text{ with } \sin \phi > 0} |f(\eta, \phi, \psi)|, \quad (1.12)$$

$$\|f\|_{L^2 L^2} = \left( \int_0^L \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} |f(\eta, \phi, \psi)|^2 \cos \phi d\phi d\psi d\eta \right)^{1/2}, \quad (1.13)$$

$$\|f\|_{L^2}(\eta) = \left( \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} |f(\eta, \phi, \psi)|^2 \cos \phi d\phi d\psi \right)^{1/2}, \quad (1.14)$$

and the inner product as

$$\langle f, g \rangle(\eta) = \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} f(\eta, \phi, \psi) g(\eta, \phi, \psi) \cos \phi d\phi d\psi. \quad (1.15)$$

**Theorem 1.1.** (*Well-Posedness and Decay*) Assume  $0 < n < \frac{2}{5}$  and there exist some constants  $M, K > 0$  uniform in  $\epsilon$ , such that

$$\|h^\epsilon\|_{L^\infty} \leq M, \quad (1.16)$$

and

$$\|e^{K\eta} S^\epsilon\|_{L^\infty L^\infty} \leq M.$$

Then for  $K_0 > 0$  sufficiently small, there exists a constant  $f_L^\epsilon$  and the solution  $f^\epsilon(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (1.9) satisfies

$$\|e^{K_0\eta} (f^\epsilon - f_L^\epsilon)\|_{L^\infty L^\infty} \leq C. \quad (1.17)$$

Here  $C \geq 0$  denotes a universal constant independent of  $\epsilon$ .

**Theorem 1.2.** (*Weighted Regularity*) Assume  $0 < n < \frac{2}{5}$  and there exist some constants  $M, K > 0$  uniform in  $\epsilon$ , such that

$$\|h^\epsilon\|_{L^\infty} + \left\| \frac{\partial h^\epsilon}{\partial \phi} \right\|_{L^\infty} \leq M, \quad (1.18)$$

and

$$\|e^{K\eta} S^\epsilon\|_{L^\infty L^\infty} + \left\| e^{K\eta} \frac{\partial S^\epsilon}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| e^{K\eta} \frac{\partial S^\epsilon}{\partial \phi} \right\|_{L^\infty L^\infty} \leq M.$$

Then for  $K_0 > 0$  sufficiently small, we have

$$\left\| e^{K_0\eta} \zeta \frac{\partial (f^\epsilon - f_L^\epsilon)}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| e^{K_0\eta} \zeta \frac{\partial (f^\epsilon - f_L^\epsilon)}{\partial \phi} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8, \quad (1.19)$$

where the weight function

$$\zeta(\eta, \phi, \psi) = \left( 1 - \left( \frac{R_1 - \epsilon\eta}{R_1} \right)^{2 \sin^2 \psi} \left( \frac{R_2 - \epsilon\eta}{R_2} \right)^{2 \cos^2 \psi} \cos^2 \phi \right)^{1/2}. \quad (1.20)$$

If further for  $i = 1, 2$ ,

$$\left\| \frac{\partial h^\epsilon}{\partial \psi} \right\|_{L^\infty} + \left\| \frac{\partial h^\epsilon}{\partial \tau_i} \right\|_{L^\infty} + \left\| e^{K_0 \eta} \frac{\partial S^\epsilon}{\partial \psi} \right\|_{L^\infty L^\infty} + \left\| e^{K_0 \eta} \frac{\partial S^\epsilon}{\partial \tau_i} \right\|_{L^\infty L^\infty} \leq M, \quad (1.21)$$

we have

$$\left\| e^{K_0 \eta} \frac{\partial (f^\epsilon - f_L^\epsilon)}{\partial \psi} \right\|_{L^\infty L^\infty} + \left\| e^{K_0 \eta} \frac{\partial (f^\epsilon - f_L^\epsilon)}{\partial \tau_i} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8. \quad (1.22)$$

**Remark 1.3.** It is easy to see  $\zeta \geq |\sin \phi|$ . Then Theorem 1.2 naturally implies

$$\left\| e^{K_0 \eta} \sin \phi \frac{\partial (f^\epsilon - f_L^\epsilon)}{\partial \eta} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8, \quad (1.23)$$

which is bounded away from the grazing set. More importantly, due to the half-line geometry of the Milne problem, the tangential derivatives are bounded in (1.22) up to the grazing set. This is a sharp contrast to [17] in a bounded domain.

As an application, thanks to the uniform bounds for tangential derivatives (1.22), we finally establish the diffusive limit of neutron transport equation.

**Corollary 1.4.** Assume  $g(\vec{x}_0, \vec{w}) \in C^2(\Gamma^-)$  satisfying (1.5). Then for the steady neutron transport equation (1.1), there exists a unique solution  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  satisfying (1.4). Moreover, for any  $0 < \delta \ll 1$ , the solution obeys the estimate

$$\|u^\epsilon(\vec{x}, \vec{w}) - U_0^\epsilon(\vec{x})\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C(\delta, \Omega) \epsilon^{\frac{1}{3} - \delta}, \quad (1.24)$$

where  $U_0^\epsilon(\vec{x})$  satisfies

$$\left\{ \begin{array}{l} \Delta_x U_0^\epsilon = 0 \text{ in } \Omega, \\ \frac{\partial U_0^\epsilon}{\partial \vec{\nu}} = \frac{1}{\pi^2} \int_{\vec{w} \cdot \vec{\nu} < 0} g(\vec{x}, \vec{w}) |\vec{w} \cdot \vec{\nu}| d\vec{w} \text{ on } \partial\Omega, \\ \int_\Omega U_0^\epsilon(\vec{x}) d\vec{x} = 0, \end{array} \right. \quad (1.25)$$

in which  $C(\delta, \Omega) > 0$  denotes a constant that depends on  $\delta$  and  $\Omega$ .

**1.3. Background and Methods.** At the core of boundary layer analysis, the study of Milne problem is consistent with the development of asymptotic analysis of kinetic equations in bounded domains. Since 1960s, people have discovered several methods to study the well-posedness of Milne problem, and apply them to asymptotic expansion. We refer to the references [15], [3], [4], [29], [20], [25], [13], [12], [1], [8], [11], [16], [19], [26], [5], [2], [10], [21], [22], [23], and [24] for more details. In 1979, diffusive limit of steady neutron transport equation was systematically investigated in [9] (see also [6] and [7]).

The key idea of [9], [6] and [7] is to study the classical Milne problem as

$$\left\{ \begin{array}{l} \sin \phi \frac{\partial f^\epsilon}{\partial \eta} + f^\epsilon - \bar{f}^\epsilon = S^\epsilon(\eta, \phi, \psi), \\ f^\epsilon(0, \phi, \psi) = h^\epsilon(\phi, \psi) \text{ for } \sin \phi > 0, \\ \lim_{\eta \rightarrow \infty} f^\epsilon(\eta, \phi, \psi) = f_\infty^\epsilon. \end{array} \right. \quad (1.26)$$

In [9], the authors proved that  $f^\epsilon$  is well-posed and decays exponentially fast to some constant  $f_\infty^\epsilon$  in  $L^\infty$ .

Unfortunately, as discovered recently in [27], the lack of regularity of such classical Milne problem (1.26) has been overlooked for non-flat bounded domains. The solutions of (1.26) are singular in the normal direction, which leads to singularity in the tangential directions, resulting in break-down of diffusive expansion with classical Milne boundary layers.

The regularity of the Milne problem is the central issue. In [27] and [28], a new approach with geometric correction from the next-order diffusive expansion has been introduced to ensure regularity in the cases of

2D plate and annulus, i.e. to solve for  $f^\epsilon(\eta, \phi)$  satisfying

$$\begin{cases} \sin \phi \frac{\partial f^\epsilon}{\partial \eta} - \frac{\epsilon}{R - \epsilon \eta} \cos \phi \frac{\partial f^\epsilon}{\partial \phi} + f^\epsilon - \bar{f}^\epsilon = S^\epsilon(\eta, \phi), \\ f^\epsilon(0, \phi) = h^\epsilon(\phi) \text{ for } \sin \phi > 0, \\ f^\epsilon(L, \phi) = f^\epsilon(L, -\phi), \end{cases} \quad (1.27)$$

where  $R$  denotes the radius of curvature. Also, in [18], weighted  $W^{1, \infty}$  estimates was proved to treat more general 2D convex domains.

There are three main ingredients to generalize our previous results to 3D convex domains.

The first difficulty is the lack of conserved energy. Consider the simplest case that  $S = 0$  and we omit  $\epsilon$  temporarily. Assume

$$F(\eta, \psi) = -\epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right). \quad (1.28)$$

Taking inner product with  $f$  on both sides of (1.9), we obtain

$$\frac{1}{4} \frac{\partial}{\partial \eta} \langle f, f \sin(2\phi) \rangle + \frac{1}{2} \left\langle F(\eta, \psi), \frac{\partial(f^2)}{\partial \phi} \cos^2 \phi \right\rangle + \|f - \bar{f}\|_{L^2}^2 = 0. \quad (1.29)$$

We may integrate by parts to get

$$\frac{1}{2} \left\langle F(\eta, \psi), \frac{\partial(f^2)}{\partial \phi} \cos^2 \phi \right\rangle = \frac{1}{2} \langle F(\eta, \psi) f, f \sin(2\phi) \rangle. \quad (1.30)$$

Since  $F(\eta, \psi)$  depends on  $\psi$  when  $R_1 \neq R_2$  in 3D, we cannot further pull  $F$  out of the integral, i.e.

$$\frac{1}{2} \langle F(\eta, \psi) f, f \sin(2\phi) \rangle \stackrel{?}{=} \frac{1}{2} F(\eta, \psi) \langle f, f \sin(2\phi) \rangle. \quad (1.31)$$

This important equality is true in (1.27) for 2D domains and yields an ordinary differential equation for  $\langle f, f \sin(2\phi) \rangle$ , leading to the closure of the  $L^2$  estimate of the microscopic part  $f - \bar{f}$ . Similarly, taking inner product with 1 on both sides of (1.27) and integrating by parts, we easily get the orthogonality relation

$$\langle f, \sin \phi \rangle = 0, \quad (1.32)$$

which plays a crucial role in estimating hydrodynamical part  $\bar{f}$ . Unfortunately, both (1.31) and (1.32) break down in 3D domains.

To circumvent these two major difficulties, as Lemma 3.1 reveals, we decompose

$$F(\eta, \psi) \cos \phi \frac{\partial f}{\partial \phi} = \tilde{F}(\eta) \cos \phi \frac{\partial f}{\partial \phi} + G(\eta) \cos^2 \psi \cos \phi \frac{\partial f}{\partial \phi}, \quad (1.33)$$

where

$$\tilde{F}(\eta) = -\frac{\epsilon}{R_1 - \epsilon \eta}, \quad (1.34)$$

$$G(\eta) = -\frac{\epsilon(R_1 - R_2)}{(R_1 - \epsilon \eta)(R_2 - \epsilon \eta)}, \quad (1.35)$$

in which  $\tilde{F}$  behaves like 2D force (independent of  $\psi$ ) and  $G$  can be regarded as a source term. Roughly speaking, taking inner product with  $f$  in (1.9), we obtain

$$\begin{aligned} \|f - \bar{f}\|_{L^2 L^2}^2 &\lesssim C \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right| + \text{lower-order terms} \\ &\lesssim C + C \|G\|_{L^\infty L^\infty} \|f\|_{L^2 L^2}^2, \end{aligned} \quad (1.36)$$

which means we cannot close the estimate for  $f - \bar{f}$  alone without invoking  $\|\bar{f}\|_{L^2 L^2}$ . On the other hand, taking inner product with  $\sin \phi$  in (1.9) indicates

$$\begin{aligned} \|\bar{f}\|_{L^2 L^2}^2 &\lesssim C \int_0^L \left| \int_s^L \int_z^L G(y) \langle \cos^2 \psi, (f - \bar{f}) \sin \phi \rangle (y) dy dz \right|^2 ds + \text{lower-order terms} \\ &\lesssim C + CL^3 \|G\|_{L^2 L^2} \|f - \bar{f}\|_{L^2 L^2}. \end{aligned} \quad (1.37)$$

(1.36) and (1.37) form a coupled system for  $f - \bar{f}$  and  $\bar{f}$  and require careful analysis of the interplay between microscopic and hydrodynamic parts. Also, we have to delicately choose  $L = \epsilon^{-n}$  with  $0 < n < \frac{2}{5}$  to create a small constant such that an intricate bootstrapping argument can finally close the  $L^2$  estimates.

The second key ingredient in our analysis is to establish the regularity estimate of  $\epsilon$ -Milne problem. Proving diffusive limit in transport equations requires boundary layer expansion higher than leading-order term, which means we need  $L^\infty$  estimate of the tangential derivatives

$$\frac{\partial f}{\partial \tau_1}, \quad \frac{\partial f}{\partial \tau_2}, \quad \text{and} \quad \frac{\partial f}{\partial \psi}. \quad (1.38)$$

In the case when  $R_1 = R_2$  are constant independent of  $\tau_1$  and  $\tau_2$ , as in a perfect ball  $\{|\vec{x}| = 1\}$ ,  $\frac{\partial f}{\partial \tau_i}$  for  $i = 1, 2$  is smooth, since the tangential derivative commutes with the equation. On the other hand, when  $R_1$  and  $R_2$  are functions of  $\tau_i$ , then  $\frac{\partial f}{\partial \tau_i}$  relates to the normal derivative  $\frac{\partial f}{\partial \eta}$ , and  $\frac{\partial f}{\partial \psi}$  relates to the velocity derivative  $\frac{\partial f}{\partial \phi}$ .

Our main contribution is to show  $\frac{\partial f}{\partial \tau_i}$  and  $\frac{\partial f}{\partial \psi}$  are bounded even if  $R_1$  and  $R_2$  are not identical constant for a general convex domain (see Theorem 4.12). Our proof is intricate and delicate, which relies on the weighted  $L^\infty$  estimates for the normal derivative with detailed analysis along the characteristic curves in the presence of non-local operator  $\bar{f}$ . The convexity and invariant kinetic distance  $\zeta$  defined in (1.20) play the key role.

The third ingredient is a new  $L^6 - L^\infty$  framework developed to improve remainder estimates in

$$\begin{cases} \epsilon \vec{w} \cdot \nabla_x u + u - \bar{u} = S(\vec{x}, \vec{w}) & \text{in } \Omega, \\ u(\vec{x}_0, \vec{w}) = \mathcal{P}[u](\vec{x}_0) + h(\vec{x}_0, \vec{w}) & \text{for } \vec{w} \cdot \vec{\nu} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases} \quad (1.39)$$

The main idea is to introduce special test functions in the weak formulation to treat kernel and non-kernel parts separately. In principle, we get  $L^6$  estimate

$$\begin{aligned} & \frac{1}{\epsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)} + \|\bar{u}\|_{L^6(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} \\ & \leq C \left( o(1)\epsilon^{\frac{1}{2}} \|u\|_{L^\infty(\Gamma^+)} + \frac{1}{\epsilon} \|S\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^2} \|S\|_{L^{\frac{6}{5}}(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|h\|_{L^2(\Gamma^-)} + \|h\|_{L^4(\Gamma^-)} \right), \end{aligned} \quad (1.40)$$

where  $o(1)$  denotes a sufficiently small constant (see Theorem A.3). The proof relies on a careful analysis using sharp interpolation and Young's inequality. Finally, the utilization of modified double Duhamel's principle and a bootstrapping argument yield the  $L^\infty$  estimate as

$$\|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C \left( \frac{1}{\epsilon^{\frac{1}{2}}} \|\bar{u}\|_{L^6(\Omega \times \mathcal{S}^2)} + \|S\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|g\|_{L^\infty(\Gamma^-)} \right). \quad (1.41)$$

Our methods are currently being applied to the study of hydrodynamic limit of Boltzmann equation in the bounded domains with boundary layer corrections.

**1.4. Notation and Structure.** Throughout this paper, unless specified,  $C > 0$  denotes a universal constant which does not depend on the data and can change from one inequality to another. When we write  $C(z)$ , it means a positive constant depending on the quantity  $z$ .

Our paper is organized as follows: in Section 2, we present the asymptotic analysis of the equation (1.1); in Section 3, we prove the well-posedness and decay of  $\epsilon$ -Milne problem, i.e. Theorem 1.1; in Section 4, we prove the weighted  $W^{1,\infty}$  estimates in  $\epsilon$ -Milne problem, i.e. Theorem 1.2; finally, in appendix, we prove the improved  $L^\infty$  estimate of remainder equation and the diffusive limit, i.e. Corollary 1.4.

## 2. ASYMPTOTIC ANALYSIS

**2.1. Interior Expansion.** We first try to approximate the solution of neutron transport equation (1.1). We define the interior expansion as follows:

$$U^\epsilon(\vec{x}, \vec{w}) \sim U_0^\epsilon(\vec{x}, \vec{w}) + \epsilon U_1^\epsilon(\vec{x}, \vec{w}) + \epsilon^2 U_2^\epsilon(\vec{x}, \vec{w}), \quad (2.1)$$

where  $U_k^\epsilon$  can be determined by comparing the order of  $\epsilon$  by plugging (2.1) into the equation (1.1). Thus we have

$$U_0^\epsilon - \bar{U}_0^\epsilon = 0, \quad (2.2)$$

$$U_1^\epsilon - \bar{U}_1^\epsilon = -\vec{w} \cdot \nabla_x U_0^\epsilon, \quad (2.3)$$

$$U_2^\epsilon - \bar{U}_2^\epsilon = -\vec{w} \cdot \nabla_x U_1^\epsilon. \quad (2.4)$$

Plugging (2.2) into (2.3), we obtain

$$U_1^\epsilon = \bar{U}_1^\epsilon - \vec{w} \cdot \nabla_x \bar{U}_0^\epsilon. \quad (2.5)$$

Plugging (2.5) into (2.4), we get

$$U_2^\epsilon - \bar{U}_2^\epsilon = -\vec{w} \cdot \nabla_x (\bar{U}_1^\epsilon - \vec{w} \cdot \nabla_x \bar{U}_0^\epsilon) = -\vec{w} \cdot \nabla_x \bar{U}_1^\epsilon + w_1^2 \partial_{x_1 x_1} \bar{U}_0^\epsilon + w_2^2 \partial_{x_2 x_2} \bar{U}_0^\epsilon + 2w_1 w_2 \partial_{x_1 x_2} \bar{U}_0^\epsilon. \quad (2.6)$$

Integrating (2.6) over  $\vec{w} \in \mathcal{S}^1$ , we achieve the final form

$$\Delta_x \bar{U}_0^\epsilon = 0, \quad (2.7)$$

which further implies  $U_0^\epsilon(\vec{x}, \vec{w})$  satisfies the equation

$$\begin{cases} U_0^\epsilon &= \bar{U}_0^\epsilon, \\ \Delta_x \bar{U}_0^\epsilon &= 0. \end{cases} \quad (2.8)$$

In a similar fashion, for  $k = 1, 2$ ,  $U_k^\epsilon$  satisfies

$$\begin{cases} U_k^\epsilon &= \bar{U}_k^\epsilon - \vec{w} \cdot \nabla_x U_{k-1}^\epsilon, \\ \Delta_x \bar{U}_k^\epsilon &= -\int_{\mathcal{S}^2} \vec{w} \cdot \nabla_x U_{k-1}^\epsilon d\vec{w}. \end{cases} \quad (2.9)$$

It is easy to see  $\bar{U}_k^\epsilon$  satisfies an elliptic equation. However, the boundary condition of  $\bar{U}_k^\epsilon$  is unknown at this stage, since generally  $U_k^\epsilon$  does not necessarily satisfy the diffusive boundary condition of (1.1). Therefore, we have to resort to boundary layer.

**2.2. Local Coordinate System.** Basically, we use two types of coordinate systems: Cartesian coordinate system for interior solution, which is stated above, and a local coordinate system in a neighborhood of the boundary for boundary layer. We need several substitution to describe solution near boundary.

Substitution 1: spacial substitution:

We consider the three-dimensional transport operator  $\vec{w} \cdot \nabla_x$ . In the boundary surface, locally we can always define an orthogonal curvilinear coordinates system  $(\tau_1, \tau_2)$  and the surface is described as  $\vec{r}(\tau_1, \tau_2)$ . From the differential geometry, we know  $\partial_1 \vec{r}$  and  $\partial_2 \vec{r}$  denote two orthogonal tangential vectors. Then assume the outward unit normal vector is

$$\vec{\nu} = \frac{\partial_1 \vec{r} \times \partial_2 \vec{r}}{|\partial_1 \vec{r} \times \partial_2 \vec{r}|}. \quad (2.10)$$

Here  $|\cdot|$  denotes the length and  $\partial_i$  denotes derivative with respect to  $\tau_i$ . Let

$$P = |\partial_1 \vec{r} \times \partial_2 \vec{r}| = |\partial_1 \vec{r}| |\partial_2 \vec{r}| = P_1 P_2, \quad (2.11)$$

with the unit tangential vectors are

$$\vec{t}_1 = \frac{\partial_1 \vec{r}}{P_1}, \quad \vec{t}_2 = \frac{\partial_2 \vec{r}}{P_2}. \quad (2.12)$$

Then in the new coordinates  $(\mu, \tau_1, \tau_2)$  where  $\mu$  denotes the normal distance to boundary surface, we have

$$\vec{x} = \vec{r} - \mu \vec{\nu}. \quad (2.13)$$

which further implies the operator becomes

$$\begin{aligned} \vec{w} \cdot \nabla_x = & - \frac{\left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \right) \cdot \vec{w}}{\left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \right) \cdot \vec{v}} \frac{\partial f}{\partial \mu} \\ & + \frac{\left( (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \times \vec{v} \right) \cdot \vec{w}}{\left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \right) \cdot \vec{v}} \frac{\partial f}{\partial \tau_1} - \frac{\left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times \vec{v} \right) \cdot \vec{w}}{\left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \right) \cdot \vec{v}} \frac{\partial f}{\partial \tau_2}. \end{aligned} \quad (2.14)$$

Based on differential geometry, we define the first fundamental form as  $(E, F, G)$  and second fundamental form as  $(L, M, N)$ , then we have  $F = M = 0$  and the principal curvatures are given by

$$\kappa_1 = \frac{L}{E}, \quad \kappa_2 = \frac{N}{G}, \quad (2.15)$$

and also

$$\partial_1 \vec{v} = \kappa_1 \partial_1 \vec{r}, \quad \partial_2 \vec{v} = \kappa_2 \partial_2 \vec{r}. \quad (2.16)$$

Hence, we know

$$\begin{aligned} \left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \right) \cdot \vec{v} &= \left( \kappa_1 \kappa_2 \mu^2 - (\kappa_1 + \kappa_2) \mu + 1 \right) P \\ &= \left( (\kappa_1 \mu - 1)(\kappa_2 \mu - 1) \right) P. \end{aligned} \quad (2.17)$$

$$\begin{aligned} \left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \right) \cdot \vec{w} &= \left( \kappa_1 \kappa_2 \mu^2 - (\kappa_1 + \kappa_2) \mu + 1 \right) P(\vec{v} \cdot \vec{w}) \\ &= \left( (\kappa_1 \mu - 1)(\kappa_2 \mu - 1) \right) P(\vec{w} \cdot \vec{v}). \end{aligned} \quad (2.18)$$

$$\left( (\partial_2 \vec{r} - \mu \partial_2 \vec{v}) \times \vec{v} \right) \cdot \vec{w} = (1 - \kappa_2 \mu) \frac{P_2}{P_1} (\vec{w} \cdot \partial_1 \vec{r}). \quad (2.19)$$

$$\left( (\partial_1 \vec{r} - \mu \partial_1 \vec{v}) \times \vec{v} \right) \cdot \vec{w} = -(1 - \kappa_1 \mu) \frac{P_1}{P_2} (\vec{w} \cdot \partial_2 \vec{r}). \quad (2.20)$$

Hence, we have the transport operator as

$$\vec{w} \cdot \nabla_x = -(\vec{w} \cdot \vec{v}) \frac{\partial}{\partial \mu} - \frac{\vec{w} \cdot \vec{t}_1}{P_1(\kappa_1 \mu - 1)} \frac{\partial}{\partial \tau_1} - \frac{\vec{w} \cdot \vec{t}_2}{P_2(\kappa_2 \mu - 1)} \frac{\partial}{\partial \tau_2}. \quad (2.21)$$

Therefore, under substitution  $(x_1, x_2, x_3) \rightarrow (\mu, \tau_1, \tau_2)$ , the equation (1.1) is transformed into

$$\begin{cases} \epsilon \left( -(\vec{w} \cdot \vec{v}) \frac{\partial u^\epsilon}{\partial \mu} - \frac{\vec{w} \cdot \vec{t}_1}{P_1(\kappa_1 \mu - 1)} \frac{\partial u^\epsilon}{\partial \tau_1} - \frac{\vec{w} \cdot \vec{t}_2}{P_2(\kappa_2 \mu - 1)} \frac{\partial u^\epsilon}{\partial \tau_2} \right) + u^\epsilon - \bar{u}^\epsilon = 0 & \text{in } \Omega, \\ u^\epsilon(0, \tau_1, \tau_2, \vec{w}) = \mathcal{P}[u^\epsilon](0, \tau_1, \tau_2) + \epsilon g(\tau_1, \tau_2, \vec{w}) & \text{for } \vec{w} \cdot \vec{v} < 0, \end{cases} \quad (2.22)$$

where

$$\mathcal{P}[u^\epsilon](0, \tau_1, \tau_2) = \frac{1}{2\pi} \int_{\vec{w} \cdot \vec{v} > 0} u^\epsilon(0, \tau_1, \tau_2, \vec{w}) (\vec{w} \cdot \vec{v}) d\vec{w}, \quad (2.23)$$

Substitution 2: velocity substitution.

Define the orthogonal velocity substitution

$$\begin{cases} -\vec{w} \cdot \vec{v} &= \sin \phi, \\ \vec{w} \cdot \vec{t}_1 &= \cos \phi \sin \psi, \\ \vec{w} \cdot \vec{t}_2 &= \cos \phi \cos \psi. \end{cases} \quad (2.24)$$



Then we have

$$\frac{\partial}{\partial \tau_1} \rightarrow \frac{\partial}{\partial \tau_1} - \kappa_1 P_1 \sin \psi \frac{\partial}{\partial \phi} + \left( \frac{(\partial_2 \vec{r} \times (\partial_{21} \vec{r} \times \partial_2 \vec{r})) \cdot \vec{t}_2}{P_2^3} - \kappa_1 P_1 \tan \phi \cos \psi \right) \frac{\partial}{\partial \psi}, \quad (2.25)$$

$$\frac{\partial}{\partial \tau_2} \rightarrow \frac{\partial}{\partial \tau_2} - \kappa_2 P_2 \cos \psi \frac{\partial}{\partial \phi} + \left( \frac{(\partial_1 \vec{r} \times (\partial_{12} \vec{r} \times \partial_1 \vec{r})) \cdot \vec{t}_1}{P_1^3} + \kappa_2 P_2 \tan \phi \sin \psi \right) \frac{\partial}{\partial \psi}. \quad (2.26)$$

Then the transport operator is as

$$\begin{aligned} \vec{w} \cdot \nabla_x &= \sin \phi \frac{\partial}{\partial \mu} - \left( \frac{\sin^2 \psi}{R_1 - \mu} + \frac{\cos^2 \psi}{R_2 - \mu} \right) \cos \phi \frac{\partial}{\partial \phi} \\ &+ \left( \frac{\cos \phi \sin \psi}{P_1(1 - \kappa_1 \mu)} \frac{\partial}{\partial \tau_1} + \frac{\cos \phi \cos \psi}{P_2(1 - \kappa_2 \mu)} \frac{\partial}{\partial \tau_2} \right) \\ &+ \left( \frac{\sin \psi}{1 - \kappa_1 \mu} (\vec{t}_2 \times (\partial_{21} \vec{r} \times \vec{t}_2)) \cdot \vec{t}_2 + \frac{\cos \psi}{1 - \kappa_2 \mu} (\vec{t}_1 \times (\partial_{12} \vec{r} \times \vec{t}_1)) \cdot \vec{t}_1 \right) \frac{\cos \phi}{P_1 P_2} \frac{\partial}{\partial \psi}, \end{aligned} \quad (2.27)$$

where  $R_1 = \frac{1}{\kappa_1}$  and  $R_2 = \frac{1}{\kappa_2}$ . Hence, under substitution  $(w_1, w_2, w_3) \rightarrow (\phi, \psi)$ , the equation (1.1) is transformed into

$$\left\{ \begin{aligned} &\epsilon \sin \phi \frac{\partial u^\epsilon}{\partial \mu} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \mu} + \frac{\cos^2 \psi}{R_2 - \mu} \right) \cos \phi \frac{\partial u^\epsilon}{\partial \phi} \\ &+ \epsilon \left( \frac{\cos \phi \sin \psi}{P_1(1 - \kappa_1 \mu)} \frac{\partial u^\epsilon}{\partial \tau_1} + \frac{\cos \phi \cos \psi}{P_2(1 - \kappa_2 \mu)} \frac{\partial u^\epsilon}{\partial \tau_2} \right) \\ &+ \epsilon \left( \frac{\sin \psi}{1 - \kappa_1 \mu} (\vec{t}_2 \times (\partial_{21} \vec{r} \times \vec{t}_2)) \cdot \vec{t}_2 + \frac{\cos \psi}{1 - \kappa_2 \mu} (\vec{t}_1 \times (\partial_{12} \vec{r} \times \vec{t}_1)) \cdot \vec{t}_1 \right) \frac{\cos \phi}{P_1 P_2} \frac{\partial u^\epsilon}{\partial \psi} + u^\epsilon - \bar{u}^\epsilon = 0, \\ &u^\epsilon(0, \tau_1, \tau_2, \phi, \psi) = \mathcal{P}[u^\epsilon](0, \tau_1, \tau_2) + \epsilon g(\tau_1, \tau_2, \phi, \psi) \text{ for } \sin \phi > 0, \end{aligned} \right. \quad (2.28)$$

where

$$\mathcal{P}[u^\epsilon](0, \tau_1, \tau_2) = \frac{1}{4\pi} \iint_{\sin \phi > 0} u^\epsilon(0, \tau_1, \tau_2, \phi, \psi) \sin \phi \cos \phi d\phi d\psi, \quad (2.29)$$

due to Jacobian  $J = \cos \phi$ , in a neighborhood of the boundary.

Substitution 3: scaling substitution.

We define the scaled variable  $\eta = \frac{\mu}{\epsilon}$ , which implies  $\frac{\partial}{\partial \mu} = \frac{1}{\epsilon} \frac{\partial}{\partial \eta}$ . Then, under the substitution  $\mu \rightarrow \eta$ , the equation (1.1) is transformed into

$$\left\{ \begin{aligned} &\sin \phi \frac{\partial u^\epsilon}{\partial \eta} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right) \cos \phi \frac{\partial u^\epsilon}{\partial \phi} \\ &+ \epsilon \left( \frac{\cos \phi \sin \psi}{P_1(1 - \epsilon \kappa_1 \eta)} \frac{\partial u^\epsilon}{\partial \tau_1} + \frac{\cos \phi \cos \psi}{P_2(1 - \epsilon \kappa_2 \eta)} \frac{\partial u^\epsilon}{\partial \tau_2} \right) \\ &+ \epsilon \left( \frac{\sin \psi}{1 - \epsilon \kappa_1 \eta} (\vec{t}_2 \times (\partial_{21} \vec{r} \times \vec{t}_2)) \cdot \vec{t}_2 + \frac{\cos \psi}{1 - \epsilon \kappa_2 \eta} (\vec{t}_1 \times (\partial_{12} \vec{r} \times \vec{t}_1)) \cdot \vec{t}_1 \right) \frac{\cos \phi}{P_1 P_2} \frac{\partial u^\epsilon}{\partial \psi} + u^\epsilon - \bar{u}^\epsilon = 0, \\ &u^\epsilon(0, \tau_1, \tau_2, \phi, \psi) = \mathcal{P}[u^\epsilon](0, \tau_1, \tau_2) + \epsilon g(\tau_1, \tau_2, \phi, \psi) \text{ for } \sin \phi > 0, \end{aligned} \right. \quad (2.30)$$

where

$$\mathcal{P}[u^\epsilon](0, \tau_1, \tau_2) = \frac{1}{4\pi} \iint_{\sin \phi > 0} u^\epsilon(0, \tau_1, \tau_2, \phi, \psi) \sin \phi \cos \phi d\phi d\psi, \quad (2.31)$$

in a neighborhood of the boundary.

**2.3. Boundary Layer Expansion with Geometric Correction.** We define the boundary layer expansion as follows:

$$\mathcal{U}^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) \sim \mathcal{U}_0^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) + \epsilon \mathcal{U}_1^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi), \quad (2.32)$$

where  $\mathcal{U}_k^\epsilon$  can be defined by comparing the order of  $\epsilon$  via plugging (2.32) into the equation (2.30). Thus, in a neighborhood of the boundary, we have

$$\sin \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \eta} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right) \cos \phi \frac{\partial \mathcal{U}_0^\epsilon}{\partial \phi} + \mathcal{U}_0^\epsilon - \bar{\mathcal{U}}_0^\epsilon = 0, \quad (2.33)$$

$$\sin \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \eta} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right) \cos \phi \frac{\partial \mathcal{U}_1^\epsilon}{\partial \phi} + \mathcal{U}_1^\epsilon - \bar{\mathcal{U}}_1^\epsilon = -G_0, \quad (2.34)$$

where

$$\begin{aligned} G_0 = & \left( \frac{\cos \phi \sin \psi}{P_1(1 - \epsilon \kappa_1 \eta)} \frac{\partial \mathcal{U}_0^\epsilon}{\partial \tau_1} + \frac{\cos \phi \cos \psi}{P_2(1 - \epsilon \kappa_2 \eta)} \frac{\partial \mathcal{U}_0^\epsilon}{\partial \tau_2} \right) \\ & + \left( \frac{\sin \psi}{1 - \epsilon \kappa_1 \eta} (\vec{t}_2 \times (\partial_{21} \vec{r} \times \vec{t}_2)) \cdot \vec{t}_2 + \frac{\cos \psi}{1 - \epsilon \kappa_2 \eta} (\vec{t}_1 \times (\partial_{12} \vec{r} \times \vec{t}_1)) \cdot \vec{t}_1 \right) \frac{\cos \phi}{P_1 P_2} \frac{\partial \mathcal{U}_0^\epsilon}{\partial \psi}, \end{aligned} \quad (2.35)$$

and

$$\bar{\mathcal{U}}_k^\epsilon(\eta, \tau_1, \tau_2) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \mathcal{U}_k^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) \cos \phi d\phi d\psi. \quad (2.36)$$

Note that this formulation is always valid locally on the boundary. By open covering theorem, we can find finite open domains to cover the whole surface. For convenience, we will not change our notation in each open domain.

**2.4. Matching of Interior Solution and Boundary Layer.** The bridge between the interior solution and the boundary layer solution is the boundary condition of (1.1), so we first consider the boundary condition expansion:

$$(U_0^\epsilon + \mathcal{U}_0^\epsilon) = \mathcal{P}[U_0^\epsilon + \mathcal{U}_0^\epsilon], \quad (2.37)$$

$$(U_1^\epsilon + \mathcal{U}_1^\epsilon) = \mathcal{P}[U_1^\epsilon + \mathcal{U}_1^\epsilon] + g. \quad (2.38)$$

Note the fact that  $\bar{U}_k^\epsilon = \mathcal{P}[\bar{U}_k^\epsilon]$ , we can simplify above conditions as follows:

$$\mathcal{U}_0^\epsilon = \mathcal{P}[\mathcal{U}_0^\epsilon], \quad (2.39)$$

$$\mathcal{U}_1^\epsilon = \mathcal{P}[\mathcal{U}_1^\epsilon] + (\vec{w} \cdot U_0^\epsilon - \mathcal{P}[\vec{w} \cdot U_0^\epsilon]) + g. \quad (2.40)$$

The construction of  $U_k^\epsilon$  and  $\mathcal{U}_k^\epsilon$  are as follows:

Step 0: Preliminaries.

Assume the cut-off function  $\Upsilon_0 \in C^\infty[0, \infty)$  are defined as

$$\Upsilon_0(\mu) = \begin{cases} 1 & 0 \leq \mu \leq \frac{1}{4} R_{\min}, \\ 0 & \frac{1}{2} R_{\min} \leq \mu \leq \infty. \end{cases} \quad (2.41)$$

where

$$R_{\min} = \min_{\tau_1, \tau_2} \{R_1(\tau_1, \tau_2), R_2(\tau_1, \tau_2)\}. \quad (2.42)$$

Define the length of boundary layer  $L = \epsilon^{-n}$  for  $0 < n < \frac{2}{5}$  and the force as

$$F(\epsilon; \eta, \psi) = -\epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right). \quad (2.43)$$

Also, denote  $\mathcal{R}\phi = -\phi$ .

Step 1: Construction of  $\mathcal{W}_0^\epsilon$ .

Define the zeroth-order boundary layer as

$$\left\{ \begin{array}{l} \mathcal{W}_0^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) = \Upsilon_0(\epsilon^n \eta) \left( f_0^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) - f_{0,L}^\epsilon(\tau_1, \tau_2) \right), \\ \sin \phi \frac{\partial f_0^\epsilon}{\partial \eta} + F(\epsilon; \eta, \psi) \cos \phi \frac{\partial f_0^\epsilon}{\partial \phi} + f_0^\epsilon - \bar{f}_0^\epsilon = 0, \\ f_0^\epsilon(0, \tau_1, \tau_2, \phi, \psi) = \mathcal{P}[f_0^\epsilon](0, \tau_1, \tau_2) \text{ for } \sin \phi > 0, \\ f_0^\epsilon(L, \tau_1, \tau_2, \phi, \psi) = f_0^\epsilon(L, \tau_1, \tau_2, \mathcal{R}\phi, \psi), \end{array} \right. \quad (2.44)$$

with

$$\mathcal{P}[f_0^\epsilon](0, \tau_1, \tau_2) = 0. \quad (2.45)$$

By Theorem 3.11,  $\mathcal{W}_0^\epsilon$  is well-defined. It is obvious to see  $f_0^\epsilon = f_{0,L}^\epsilon = 0$  is the only solution.

Step 2: Construction of  $\mathcal{W}_1^\epsilon$  and  $U_0^\epsilon$ .

Define the first-order boundary layer as

$$\left\{ \begin{array}{l} \mathcal{W}_1^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) = \Upsilon_0(\epsilon^n \eta) \left( f_1^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi) - f_{1,L}^\epsilon(\tau_1, \tau_2) \right), \\ \sin \phi \frac{\partial f_1^\epsilon}{\partial \eta} + F(\epsilon; \eta, \psi) \cos \phi \frac{\partial f_1^\epsilon}{\partial \phi} + f_1^\epsilon - \bar{f}_1^\epsilon = -G_0, \\ f_1^\epsilon(0, \tau_1, \tau_2, \phi, \psi) = \mathcal{P}[f_1^\epsilon](0, \tau_1, \tau_2) + g_1(\tau_1, \tau_2, \phi, \psi) \text{ for } \sin \phi > 0, \\ f_1^\epsilon(L, \tau_1, \tau_2, \phi, \psi) = f_1^\epsilon(L, \tau_1, \tau_2, \mathcal{R}\phi, \psi), \end{array} \right. \quad (2.46)$$

with

$$\mathcal{P}[f_1^\epsilon](0, \tau_1, \tau_2) = 0, \quad (2.47)$$

where

$$g_1 = (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0) - \mathcal{P}[\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)]) + g, \quad (2.48)$$

with  $\vec{x}_0$  is the same boundary point as  $(0, \tau_1, \tau_2)$ . To solve (2.46), we require the compatibility condition (3.102) for the boundary data

$$\iint_{\sin \phi > 0} \left( g + \vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0) - \mathcal{P}[\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)] \right) \sin \phi \cos \phi d\phi d\psi - \int_0^L \int_{-\pi}^{\pi} \int_0^{\pi/2} e^{-V(s)} G_0 \cos \phi d\phi d\psi ds = 0, \quad (2.49)$$

where  $V(0) = 0$  and  $\frac{\partial V}{\partial \eta} = -F$ . Note the fact  $\vec{w} = (\sin \phi) \vec{v} + (\cos \phi \sin \psi) \vec{t}_1 + (\cos \phi \cos \psi) \vec{t}_2$  and

$$\begin{aligned} & \iint_{\sin \phi > 0} \left( \vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0) - \mathcal{P}[\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)] \right) \sin \phi \cos \phi d\phi d\psi \\ &= \iint_{\sin \phi > 0} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi \cos \phi d\phi d\psi - 2\pi \mathcal{P}[\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)] \\ &= \iint_{\sin \phi > 0} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi \cos \phi d\phi d\psi + \iint_{\sin \phi < 0} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi \cos \phi d\phi d\psi \\ &= \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} (\vec{w} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi \cos \phi d\phi d\psi \\ &= \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} (\vec{v} \cdot \nabla_x U_0^\epsilon(\vec{x}_0)) \sin \phi \cos \phi d\phi d\psi \\ &= -\pi^2 \nabla_x \bar{U}_0^\epsilon(\vec{x}_0) \cdot \vec{v} = -\pi^2 \frac{\partial \bar{U}_0^\epsilon(\vec{x}_0)}{\partial \vec{v}}. \end{aligned} \quad (2.50)$$

We can simplify the compatibility condition as follows:

$$\iint_{\sin \phi > 0} g(\tau_1, \tau_2, \phi, \psi) \sin \phi \cos \phi d\phi d\psi - \pi^2 \frac{\partial \bar{U}_0^\epsilon(\vec{x}_0)}{\partial \bar{v}} - \int_0^L \int_{-\pi}^\pi \int_{-\pi/2}^{\pi/2} e^{-V(s)} G_0 \cos \phi d\phi d\psi ds = 0. \quad (2.51)$$

Then we have

$$\begin{aligned} \frac{\partial \bar{U}_0^\epsilon(\vec{x}_0)}{\partial \bar{v}} &= \frac{1}{\pi^2} \iint_{\sin \phi > 0} g(\tau_1, \tau_2, \phi, \psi) \sin \phi \cos \phi d\phi d\psi + \frac{1}{\pi^2} \int_0^L \int_{-\pi}^\pi \int_{-\pi/2}^{\pi/2} e^{-V(s)} G_0 \cos \phi d\phi d\psi ds \\ &= \frac{1}{\pi^2} \iint_{\sin \phi > 0} g(\tau_1, \tau_2, \phi, \psi) \sin \phi \cos \phi d\phi d\psi. \end{aligned} \quad (2.52)$$

Hence, we define the zeroth order interior solution  $U_0^\epsilon(\vec{x}, \vec{w})$  as

$$\left\{ \begin{array}{l} U_0^\epsilon = \bar{U}_0^\epsilon, \\ \Delta_x \bar{U}_0^\epsilon = 0 \text{ in } \Omega, \\ \frac{\partial \bar{U}_0^\epsilon}{\partial \bar{v}} = \frac{1}{\pi^2} \iint_{\sin \phi > 0} g(\tau_1, \tau_2, \phi, \psi) \sin \phi \cos \phi d\phi d\psi \text{ on } \partial\Omega, \\ \int_\Omega \bar{U}_0^\epsilon(\vec{x}_0) d\vec{x}_0 = 0. \end{array} \right. \quad (2.53)$$

Step 3: Construction of  $U_1^\epsilon$ .

We do not expand the boundary layer to  $\mathcal{W}_2^\epsilon$  and just terminate at  $\mathcal{W}_1^\epsilon$ . Then we define the first order interior solution  $U_1^\epsilon(\vec{x})$  as

$$\left\{ \begin{array}{l} U_1^\epsilon = \bar{U}_1^\epsilon - \vec{w} \cdot \nabla_x U_0^\epsilon, \\ \Delta_x \bar{U}_1^\epsilon = - \int_{S^2} (\vec{w} \cdot \nabla_x U_0^\epsilon) d\vec{w} \text{ in } \Omega, \\ \frac{\partial \bar{U}_1^\epsilon}{\partial \bar{v}} = - \int_\Omega \int_{S^2} (\vec{w} \cdot \nabla_x U_0^\epsilon) d\vec{w} d\vec{x} \text{ on } \partial\Omega, \\ \int_\Omega \bar{U}_1^\epsilon(\vec{x}) d\vec{x} = \int_\Omega \int_{S^2} (\vec{w} \cdot \nabla_x U_0^\epsilon - \mathcal{W}_1^\epsilon) d\vec{w} d\vec{x}. \end{array} \right. \quad (2.54)$$

Note that here we only require the trivial boundary condition since we cannot resort to the compatibility condition in  $\epsilon$ -Milne problem with geometric correction. Based on [27], this might lead to  $O(\epsilon^2)$  error to the boundary approximation. Thanks to the improved remainder estimate, this error is acceptable.

Step 4: Construction of  $U_2^\epsilon$ .

By a similar fashion, we define the second order interior solution as

$$\left\{ \begin{array}{l} U_2^\epsilon = \bar{U}_2^\epsilon - \vec{w} \cdot \nabla_x U_1^\epsilon, \\ \Delta_x \bar{U}_2^\epsilon = - \int_{S^2} (\vec{w} \cdot \nabla_x U_1^\epsilon) d\vec{w} \text{ in } \Omega, \\ \frac{\partial \bar{U}_2^\epsilon}{\partial \bar{v}} = - \int_\Omega \int_{S^2} (\vec{w} \cdot \nabla_x U_1^\epsilon) d\vec{w} d\vec{x} \text{ on } \partial\Omega, \\ \int_\Omega \bar{U}_2^\epsilon(\vec{x}) d\vec{x} = \int_\Omega \int_{S^2} \vec{w} \cdot \nabla_x U_1^\epsilon d\vec{w} d\vec{x}. \end{array} \right. \quad (2.55)$$

As the case of  $U_1^\epsilon$ , we might have  $O(\epsilon^3)$  error in this step due to the trivial boundary data. However, it will not affect the diffusive limit.

3. WELL-POSEDNESS AND DECAY OF  $\epsilon$ -MILNE PROBLEM

We consider the  $\epsilon$ -Milne problem for  $f^\epsilon(\eta, \phi, \psi)$  in the domain  $(\eta, \phi, \psi) \in [0, L] \times [-\pi/2, \pi/2] \times [-\pi, \pi]$  as

$$\begin{cases} \sin \phi \frac{\partial f^\epsilon}{\partial \eta} + F(\epsilon; \eta, \psi) \cos \phi \frac{\partial f^\epsilon}{\partial \phi} + f^\epsilon - \bar{f}^\epsilon &= S^\epsilon(\eta, \phi, \psi), \\ f^\epsilon(0, \phi, \psi) &= h^\epsilon(\phi, \psi) \text{ for } \sin \phi > 0, \\ f^\epsilon(L, \phi, \psi) &= f^\epsilon(L, \mathcal{R}\phi, \psi), \end{cases} \quad (3.1)$$

where

$$\bar{f}^\epsilon(\eta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} f^\epsilon(\eta, \phi, \psi) \cos \phi d\phi d\psi \quad (3.2)$$

in which  $\cos \phi$  shows up as the Jacobian of spherical coordinates in integration. Note that for  $\phi \in [-\pi/2, \pi/2]$ , we always have  $\cos \phi \geq 0$ , which means this will not destroy the positivity of integral. Also, we have

$$F(\epsilon; \eta, \psi) = -\epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right). \quad (3.3)$$

for  $R_1$  and  $R_2$  radius of two principle curvature, and  $L = \epsilon^{-n}$  for some  $n > 0$  which will be specified later.

In this section, for convenience, we temporarily ignore the superscript  $\epsilon$ . Note that all the estimates we get will be uniform in  $\epsilon$ . We define the norms in the space  $(\eta, \phi, \psi) \in [0, L] \times [-\pi/2, \pi/2] \times [-\pi, \pi]$  as follows:

$$\|f\|_{L^2 L^2} = \left( \int_0^L \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} |f(\eta, \phi, \psi)|^2 \cos \phi d\phi d\psi d\eta \right)^{1/2}, \quad (3.4)$$

$$\|f\|_{L^\infty L^\infty} = \sup_{(\eta, \phi, \psi)} |f(\eta, \phi, \psi)|. \quad (3.5)$$

Also, we define the inner product as

$$\langle f, g \rangle(\eta) = \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} f(\eta, \phi, \psi) g(\eta, \tau_1, \tau_2, \phi, \psi) \cos \phi d\phi d\psi. \quad (3.6)$$

Similarly, we can define the norm at in-flow boundary as

$$\|f\|_{L^2}(\eta) = \left( \iint_{\sin \phi > 0} |f(\eta, \phi, \psi)|^2 \cos \phi d\phi d\psi \right)^{1/2}, \quad (3.7)$$

$$\|f\|_{L^\infty}(\eta) = \sup_{(\phi, \psi) \text{ with } \sin \phi > 0} |f(\eta, \phi, \psi)|, \quad (3.8)$$

We further assume

$$\|h\|_{L^\infty} \leq M, \quad (3.9)$$

and

$$\|e^{K\eta} S\|_{L^\infty L^\infty} \leq M,$$

for  $M > 0$  and  $K > 0$  uniform in  $\epsilon$ .

3.1.  $L^2$  Estimates.

3.1.1.  $L^2$  Estimates when  $\bar{S} = 0$ . Consider the equation

$$\begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} &= S(\eta, \phi, \psi), \\ f(0, \phi, \psi) &= h(\phi, \psi) \text{ for } \sin \phi > 0, \\ f(L, \phi, \psi) &= f(L, \mathcal{R}\phi, \psi). \end{cases} \quad (3.10)$$

where

$$F(\eta, \psi) \cos \phi \frac{\partial f}{\partial \phi} = \tilde{F}(\eta) \cos \phi \frac{\partial f}{\partial \phi} + G(\eta) \cos^2 \psi \cos \phi \frac{\partial f}{\partial \phi}, \quad (3.11)$$

for

$$\tilde{F}(\eta) = -\frac{\epsilon}{R_1 - \epsilon\eta}, \quad (3.12)$$

$$G(\eta) = -\frac{\epsilon(R_1 - R_2)}{(R_1 - \epsilon\eta)(R_2 - \epsilon\eta)}. \quad (3.13)$$

Also,  $\mathcal{R}\phi = -\phi$ . We may decompose the solution

$$f(\eta, \phi, \psi) = q(\eta) + r(\eta, \phi, \psi), \quad (3.14)$$

where the hydrodynamical part  $q$  is in the null space of the operator  $f - \bar{f}$ , and the microscopic part  $r$  is the orthogonal complement, i.e.

$$q(\eta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} f(\eta, \phi, \psi) \cos \phi d\phi d\psi, \quad r(\eta, \phi, \psi) = f(\eta, \phi, \psi) - q(\eta). \quad (3.15)$$

Furthermore, we define a potential function  $\tilde{V}(\eta)$  satisfying  $\tilde{V}(0) = 0$  and  $\frac{\partial \tilde{V}}{\partial \eta} = -\tilde{F}(\eta)$ . It is easy to compute  $\tilde{V}(\eta) = \ln\left(\frac{R_1}{R_1 - \epsilon\eta}\right)$ .

**Lemma 3.1.** *Assume  $\bar{S} = 0$  satisfying (3.9) and (3.10) and  $0 < n < \frac{2}{5}$ . There exists a solution  $f(\eta, \phi, \psi)$  to the equation (3.10), satisfying for some constant  $|f_L| \leq C$ ,*

$$\|f - f_L\|_{L^2 L^2} \leq C. \quad (3.16)$$

*The solution is unique among functions such that (A.1) holds.*

*Proof.* As in [27, Section 6], the existence can be proved using a standard approximation argument, so we will only focus on the a priori estimates. We divide the proof into several steps:

Step 1:  $r$  Estimates.

Multiplying  $f \cos \phi$  on both sides of (3.10) and integrating over  $(\phi, \psi) \in [-\pi/2, \pi/2] \times [-\pi, \pi]$ , we get the energy estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{d\eta} \langle f, f \sin \phi \rangle (\eta) &= -\|r(\eta)\|_{L^2}^2 - \tilde{F}(\eta) \left\langle \frac{\partial f}{\partial \phi}, f \cos \phi \right\rangle (\eta) \\ &\quad - G(\eta) \left\langle \frac{\partial f}{\partial \phi} \cos^2 \psi, f \cos \phi \right\rangle (\eta) + \langle S, f \rangle (\eta). \end{aligned} \quad (3.17)$$

A further integration by parts in  $\phi$  reveals

$$-\tilde{F}(\eta) \left\langle \frac{\partial f}{\partial \phi}, f \cos \phi \right\rangle (\eta) = -\tilde{F}(\eta) \langle f, f \sin \phi \rangle (\eta), \quad (3.18)$$

$$-G(\eta) \left\langle \frac{\partial f}{\partial \phi} \cos^2 \psi, f \cos \phi \right\rangle (\eta) = -G(\eta) \langle f \cos^2 \psi, f \sin \phi \rangle (\eta). \quad (3.19)$$

Hence, we can simplify (3.17) as

$$\begin{aligned} \frac{1}{2} \frac{d}{d\eta} \langle f, f \sin \phi \rangle (\eta) &= -\|r(\eta)\|_{L^2}^2 - \tilde{F}(\eta) \langle f, f \sin \phi \rangle (\eta) \\ &\quad - G(\eta) \langle f \cos^2 \psi, f \sin \phi \rangle (\eta) + \langle S, f \rangle (\eta). \end{aligned} \quad (3.20)$$

Define

$$\alpha(\eta) = \frac{1}{2} \langle f, f \sin \phi \rangle (\eta). \quad (3.21)$$

Then (3.20) can be rewritten as

$$\frac{d\alpha}{d\eta} = -\|r(\eta)\|_{L^2}^2 - 2\tilde{F}(\eta)\alpha(\eta) - G(\eta) \langle f \cos^2 \psi, f \sin \phi \rangle (\eta) + \langle S, f \rangle (\eta). \quad (3.22)$$

We can solve above in  $[\eta, L]$  and  $[0, \eta]$  respectively to obtain

$$\alpha(\eta) = e^{2\tilde{V}(\eta)-2\tilde{V}(L)}\alpha(L) + \int_{\eta}^L e^{2\tilde{V}(\eta)-2\tilde{V}(y)} \left( \|r(y)\|_{L^2}^2 + G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) - \langle S, f \rangle (y) \right) dy, \quad (3.23)$$

$$\alpha(\eta) = e^{2\tilde{V}(\eta)}\alpha(0) + \int_0^{\eta} e^{2\tilde{V}(\eta)-2\tilde{V}(y)} \left( -\|r(y)\|_{L^2}^2 - G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) + \langle S, f \rangle (y) \right) dy. \quad (3.24)$$

The specular reflexive boundary  $f(L, \phi) = f(L, \mathcal{R}\phi)$  ensures  $\alpha(L) = 0$ . Hence, based on (3.23), we have

$$\alpha(\eta) \geq \int_{\eta}^L e^{2\tilde{V}(\eta)-2\tilde{V}(y)} \left( G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) - \langle S, f \rangle (y) \right) dy. \quad (3.25)$$

Also, (3.24) implies

$$\begin{aligned} \alpha(\eta) &\leq C\alpha(0) + \int_0^{\eta} e^{2\tilde{V}(\eta)-2\tilde{V}(y)} \left( -G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) + \langle S, f \rangle (y) \right) dy \\ &\leq C\|h\|_{L^2}^2 + \int_0^{\eta} e^{2\tilde{V}(\eta)-2\tilde{V}(y)} \left( -G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) + \langle S, f \rangle (y) \right) dy, \end{aligned} \quad (3.26)$$

due to the fact

$$\alpha(0) = \frac{1}{2} \langle f \sin \phi, f \rangle (0) \leq \frac{1}{2} \left( \int_{\sin \phi > 0} h(\phi)^2 \sin \phi \cos \phi d\phi \right) \leq \frac{1}{2} \|h\|_{L^2}^2. \quad (3.27)$$

Then in (3.24) taking  $\eta = L$ , from  $\alpha(L) = 0$ , we have

$$\begin{aligned} &\int_0^L e^{-2\tilde{V}(y)} \|r(y)\|_{L^2}^2 dy \\ &\leq \alpha(0) + \int_0^L e^{-2\tilde{V}(y)} \left( -G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) + \langle S, f \rangle (y) \right) dy \\ &\leq C\|h\|_{L^2}^2 + \int_0^L e^{-2\tilde{V}(y)} \left( -G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) + \langle S, f \rangle (y) \right) dy. \end{aligned} \quad (3.28)$$

On the other hand, we can directly estimate

$$\int_0^L e^{-2\tilde{V}(y)} \|r(y)\|_{L^2}^2 dy \geq C\|r\|_{L^2 L^2}^2. \quad (3.29)$$

Combining above yields

$$\|r\|_{L^2 L^2}^2 \leq C \left( \|h\|_{L^2}^2 + \int_0^L e^{-2\tilde{V}(y)} \left( -G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) + \langle S, f \rangle (y) \right) dy \right). \quad (3.30)$$

Since  $\langle S, f \rangle = \langle S, r \rangle$  due to  $\bar{S} = 0$ , by Cauchy's inequality, we have

$$\left| \int_0^L e^{-2\tilde{V}(y)} \langle S, r \rangle (y) dy \right| \leq C'\|r\|_{L^2 L^2}^2 + C\|S\|_{L^2 L^2}^2, \quad (3.31)$$

for  $C' > 0$  sufficiently small. Therefore, absorbing  $C'\|r\|_{L^2 L^2}^2$  term, we deduce

$$\begin{aligned} \|r\|_{L^2 L^2}^2 &\leq C \left( \|h\|_{L^2}^2 + \|S\|_{L^2 L^2}^2 + \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right| \right) \\ &\leq C \left( 1 + \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right| \right) \\ &\leq C \left( 1 + \|G\|_{L^\infty L^\infty} \|f\|_{L^2 L^2}^2 \right) \\ &\leq C \left( 1 + \epsilon \|f\|_{L^2 L^2}^2 \right). \end{aligned} \quad (3.32)$$

Note that this estimate is not closed since it depends on  $f$ .

Step 2: Quasi-Orthogonality relation.

Multiplying  $\cos \phi$  on both sides of (3.10) and integrating over  $(\phi, \psi) \in [-\pi/2, \pi/2) \times [-\pi, \pi)$  imply

$$\begin{aligned} \frac{d}{d\eta} \langle \sin \phi, f \rangle (\eta) &= -\tilde{F} \left\langle \cos \phi, \frac{df}{d\phi} \right\rangle (\eta) - G \left\langle \cos \phi \cos^2 \psi, \frac{df}{d\phi} \right\rangle (\eta) + \bar{S}(\eta) \\ &= -2\tilde{F} \langle \sin \phi, f \rangle (\eta) - 2G \langle \sin \phi \cos^2 \psi, f \rangle (\eta). \end{aligned} \quad (3.33)$$

The specular reflexive boundary  $f(L, \phi) = f(L, \mathcal{R}\phi)$  implies  $\langle \sin \phi, f \rangle (L) = 0$ . Then we have

$$\langle \sin \phi, f \rangle (\eta) = -2 \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, f \rangle (y) dy. \quad (3.34)$$

It is easy to see

$$\langle \sin \phi, q \rangle (\eta) = 0. \quad (3.35)$$

Hence, we may derive

$$\begin{aligned} \langle \sin \phi, r \rangle (\eta) &= -2 \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, f \rangle (y) dy, \\ &= -2 \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r \rangle (y) dy. \end{aligned} \quad (3.36)$$

Step 3:  $q$  Estimates.

Multiplying  $\sin \phi \cos \phi$  on both sides of (3.10) and integrating over  $(\phi, \psi) \in [-\pi/2, \pi/2) \times [-\pi, \pi)$  lead to

$$\begin{aligned} \frac{d}{d\eta} \langle \sin^2 \phi, f \rangle (\eta) &= -\langle \sin \phi, r \rangle (\eta) - \tilde{F}(\eta) \left\langle \sin \phi \cos \phi, \frac{\partial f}{\partial \phi} \right\rangle (\eta) \\ &\quad - G(\eta) \left\langle \sin \phi \cos \phi \cos^2 \psi, \frac{\partial f}{\partial \phi} \right\rangle (\eta) + \langle \sin \phi, S \rangle (\eta). \end{aligned} \quad (3.37)$$

We can further integrate by parts in  $\phi$  as

$$-\tilde{F}(\eta) \left\langle \sin \phi \cos \phi, \frac{\partial f}{\partial \phi} \right\rangle (\eta) = \tilde{F}(\eta) \langle 1 - 3 \sin^2 \phi, f \rangle (\eta) = \tilde{F}(\eta) \langle 1 - 3 \sin^2 \phi, r \rangle (\eta), \quad (3.38)$$

$$-G(\eta) \left\langle \sin \phi \cos \phi \cos^2 \psi, \frac{\partial f}{\partial \phi} \right\rangle (\eta) = G(\eta) \langle 1 - 3 \sin^2 \phi, f \cos^2 \psi \rangle (\eta) = G(\eta) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (\eta), \quad (3.39)$$

to obtain

$$\begin{aligned} \frac{d}{d\eta} \langle \sin^2 \phi, f \rangle (\eta) &= -\langle \sin \phi, r \rangle (\eta) + \tilde{F}(\eta) \langle 1 - 3 \sin^2 \phi, r \rangle (\eta) \\ &\quad + G(\eta) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (\eta) + \langle \sin \phi, S \rangle (\eta). \end{aligned} \quad (3.40)$$

Define

$$\beta(\eta) = \langle \sin^2 \phi, f \rangle (\eta). \quad (3.41)$$

Then we can simplify (3.37) as

$$\frac{d\beta}{d\eta} = D(\eta), \quad (3.42)$$

where

$$\begin{aligned} D(\eta) &= -\langle \sin \phi, r \rangle (\eta) + \tilde{F}(\eta) \langle 1 - 3 \sin^2 \phi, r \rangle (\eta) \\ &\quad + G(\eta) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (\eta) + \langle \sin \phi, S \rangle (\eta). \end{aligned} \quad (3.43)$$

We can integrate over  $[0, \eta]$  in (3.42) to obtain

$$\beta(\eta) = \beta(0) + \int_0^{\eta} D(y) dy. \quad (3.44)$$



The quasi-orthogonal relation implies

$$D(\eta) = 2 \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, f \rangle (y) dy + \tilde{F}(\eta) \langle 1 - 3 \sin^2 \phi, r \rangle (\eta) \quad (3.45)$$

$$+ G(\eta) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (\eta) + \langle \sin \phi, S \rangle (\eta).$$

Hence, we deduce

$$\beta(\eta) - \beta(0) = 2 \int_0^{\eta} \int_z^L e^{2\tilde{V}(z) - 2\tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r \rangle dy dz + \int_0^{\eta} \tilde{F}(y) \langle 1 - 3 \sin^2 \phi, r \rangle (y) dy \quad (3.46)$$

$$+ \int_0^{\eta} G(y) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (y) dy + \int_0^{\eta} \langle \sin \phi, S \rangle (y) dy.$$

For the boundary data,

$$\beta(0) = \langle \sin^2 \phi, f \rangle (0) \leq \left( \langle f, f |\sin \phi| \rangle (0) \right)^{1/2} \|\sin \phi\|_{L^2}^{3/2} \leq C \left( \langle f, f |\sin \phi| \rangle (0) \right)^{1/2}. \quad (3.47)$$

Obviously, we have

$$\langle f, f |\sin \phi| \rangle (0) = \int_{\sin \phi > 0} h^2(\phi) \sin \phi \cos \phi d\phi - \int_{\sin \phi < 0} \left( f(0, \phi) \right)^2 \sin \phi \cos \phi d\phi. \quad (3.48)$$

However, based on the definition of  $\alpha(\eta)$ , we can obtain

$$\int_{\sin \phi > 0} h^2(\phi) \sin \phi \cos \phi d\phi + \int_{\sin \phi < 0} \left( f(0, \phi) \right)^2 \sin \phi \cos \phi d\phi = 2\alpha(0) \quad (3.49)$$

$$\geq 2 \int_0^L e^{-2\tilde{V}(y)} \left( G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) - \langle S, r \rangle (y) \right) dy.$$

Hence, we can deduce

$$- \int_{\sin \phi < 0} \left( f(0, \phi) \right)^2 \sin \phi \cos \phi d\phi \quad (3.50)$$

$$\leq \int_{\sin \phi > 0} h^2(\phi) \sin \phi \cos \phi d\phi - 2 \int_0^L e^{-2\tilde{V}(y)} \left( G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) - \langle S, r \rangle (y) \right) dy$$

$$\leq \|h\|_{L^2}^2 + C \left| \int_0^L \left( G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) - \langle S, r \rangle (y) \right) dy \right|.$$

Hence, we obtain

$$\left( \beta(0) \right)^2 \leq \langle f, f |\sin \phi| \rangle (0) \leq C \|h\|_{L^2}^2 + C \left| \int_0^L \left( G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) - \langle S, r \rangle (y) \right) dy \right|. \quad (3.51)$$

Note that  $\|G\|_{L^\infty L^\infty} \leq C\epsilon$ . Since  $\tilde{F} \in L^1[0, L] \cap L^2[0, L]$ ,  $r \in L^2([0, L] \times [-\pi, \pi])$ , and  $S$  exponentially decays, using Cauchy's inequality, we have

$$\begin{aligned}
|\beta(L)| &\leq C \|h\|_{L^2} + C \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right|^{1/2} + C \left| \int_0^L \langle S, r \rangle (y) dy \right|^{1/2} \\
&\quad + \left| \int_0^L \tilde{F}(y) \langle 1 - 3 \sin^2 \phi, r \rangle (y) dy \right| + \left| \int_0^L \int_z^L e^{2\tilde{V}(z) - 2\tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r \rangle dy dz \right| \\
&\quad \left| \int_0^L G(y) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (y) dy \right| + \left| \int_0^L \langle \sin \phi, S \rangle (y) dy \right| \\
&\leq C \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right|^{1/2} + C \|h\|_{L^2} + C \|r\|_{L^2 L^2} \\
&\quad + C \left\| \tilde{F} \right\|_{L^2 L^2} \|r\|_{L^2 L^2} + \|G\|_{L^2 L^2} \|r\|_{L^2 L^2} + L \|G\|_{L^2 L^2} \|r\|_{L^2 L^2} + \|S\|_{L^2 L^2} \\
&\leq C + C(1 + \epsilon^{1 - \frac{3n}{2}}) \|r\|_{L^2 L^2} + \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right|^{1/2} \\
&\leq C(1 + \epsilon^{1 - \frac{3n}{2}}) \left( 1 + \left| \int_0^L G(y) \langle f \cos^2 \psi, f \sin \phi \rangle (y) dy \right|^{1/2} \right) \\
&\leq C(1 + \epsilon^{1 - \frac{3n}{2}}) \left( 1 + \epsilon^{\frac{1}{2}} \|f\|_{L^2 L^2} \right) \\
&\leq C(1 + \epsilon^{1 - \frac{3n}{2}}) \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} + \epsilon^{\frac{1}{2}} \|f_L\|_{L^2 L^2} \right) \\
&\leq C(1 + \epsilon^{1 - \frac{3n}{2}}) \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} + \epsilon^{\frac{1}{2} - \frac{n}{2}} |f_L| \right),
\end{aligned} \tag{3.52}$$

where we define

$$f_L = q_L = \frac{\beta(L)}{\|\sin \phi\|_{L^2}^2}. \tag{3.53}$$

Therefore, for  $0 < n < \frac{2}{3}$  and  $\epsilon$  sufficiently small, absorbing  $|f_L|$ , we have

$$|f_L| \leq C + C\epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2}. \tag{3.54}$$

Thus, we naturally have

$$\begin{aligned}
\|r\|_{L^2 L^2} &\leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f\|_{L^2 L^2} \right) \\
&\leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} + \epsilon^{\frac{1}{2}} \|f_L\|_{L^2 L^2} \right) \\
&\leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} \right),
\end{aligned} \tag{3.55}$$

Furthermore, we have

$$\begin{aligned}
\beta(L) - \beta(\eta) &= \int_\eta^L D(y) dy \\
&= \left| \int_\eta^L \tilde{F}(y) \langle 1 - 3 \sin^2 \phi, r \rangle (y) dy \right| + \left| \int_\eta^L \int_z^L e^{2\tilde{V}(z) - 2\tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r \rangle dy dz \right| \\
&\quad \left| \int_\eta^L G(y) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (y) dy \right| + \left| \int_\eta^L \langle \sin \phi, S \rangle (y) dy \right|
\end{aligned} \tag{3.56}$$

Note

$$\beta(\eta) = \langle \sin^2 \phi, f \rangle (\eta) = \langle \sin^2 \phi, q \rangle (\eta) + \langle \sin^2 \phi, r \rangle (\eta) = q(\eta) \|\sin \phi\|_{L^2}^2 + \langle \sin^2 \phi, r \rangle (\eta). \tag{3.57}$$

Thus considering  $S$  decays exponentially, we can estimate

$$\begin{aligned}
\|q - q_L\|_{L^2 L^2}^2 &\leq C\|r\|_{L^2 L^2}^2 + C\|\beta(\eta) - \beta(L)\|_{L^2 L^2}^2 \\
&\leq C\|r\|_{L^2 L^2}^2 + \int_0^L \left| \int_\eta^L \tilde{F}(y) \langle 1 - 3 \sin^2 \phi, r \rangle (y) dy \right|^2 d\eta \\
&\quad + \int_0^L \left| \int_\eta^L \int_z^L e^{2\tilde{V}(z) - 2\tilde{V}y} G(y) \langle \sin \phi \cos^2 \psi, r \rangle dy dz \right|^2 d\eta \\
&\quad + \int_0^L \left| \int_\eta^L G(y) \langle 1 - 3 \sin^2 \phi, r \cos^2 \psi \rangle (y) dy \right|^2 d\eta + \int_0^L \left| \int_\eta^L \langle \sin \phi, S \rangle (y) dy \right|^2 d\eta \\
&\leq C\|r\|_{L^2 L^2}^2 + \|r\|_{L^2 L^2}^2 \int_0^L \int_\eta^L |\tilde{F}(y)|^2 dy d\eta + L^3 \|G\|_{L^2 L^2}^2 \|r\|_{L^2 L^2}^2 + L \|G\|_{L^2 L^2}^2 \|r\|_{L^2 L^2}^2 \\
&\quad + \int_0^L \left| \int_\eta^L S(y) dy \right|^2 d\eta \\
&\leq C(1 + \epsilon^{2-5n} \|r\|_{L^2 L^2}^2) \\
&\leq C(1 + \epsilon^{2-5n}) \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} \right).
\end{aligned} \tag{3.58}$$

Therefore, for  $0 < n < \frac{2}{5}$ , we have

$$\|q - q_L\|_{L^2 L^2} \leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} \right). \tag{3.59}$$

Step 4: Synthesis.

For  $0 < n < \frac{2}{5}$ , we have

$$\|r\|_{L^2 L^2} \leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} \right), \tag{3.60}$$

$$\|q - q_L\|_{L^2 L^2} \leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} \right), \tag{3.61}$$

which further implies

$$\|f - f_L\|_{L^2 L^2} \leq \|r\|_{L^2 L^2} + \|q - q_L\|_{L^2 L^2} \leq C \left( 1 + \epsilon^{\frac{1}{2}} \|f - f_L\|_{L^2 L^2} \right). \tag{3.62}$$

Hence, for  $\epsilon$  sufficiently small, we have  $|f_L| \leq C$  and

$$\|f - f_L\|_{L^2 L^2} \leq C. \tag{3.63}$$

In order to show the uniqueness of the solution, we assume there are two solutions  $f_1$  and  $f_2$  to the equation (3.10) satisfying above estimates. Then  $f' = f_1 - f_2$  satisfies the equation

$$\begin{cases} \sin \phi \frac{\partial f'}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f'}{\partial \phi} + f' - \bar{f}' = 0, \\ f'(0, \phi, \psi) = 0 \text{ for } \sin \phi > 0, \\ f'(L, \phi, \psi) = f'(L, \mathcal{R}\phi, \psi). \end{cases} \tag{3.64}$$

Assume  $|f'_L| \leq C$  and

$$\|f' - f'_L\|_{L^2 L^2} \leq C. \tag{3.65}$$

Then we can repeat the proof procedure and obtain

$$\|f' - f'_L\|_{L^2 L^2} \leq C + C\epsilon^{\frac{1}{2}} \|f' - f'_L\|_{L^2 L^2}. \tag{3.66}$$

Note that in this proof,  $O(1)$  term  $C$  purely comes from the boundary data and source term. Since all data are zero in  $f'$  equation, we have

$$\|f' - f'_L\|_{L^2 L^2} \leq C\epsilon^{\frac{1}{2}} \|f' - f'_L\|_{L^2 L^2}, \tag{3.67}$$

which implies  $f' = f'_L$  is a constant. Then based on zero boundary data, we must have  $f' = 0$ .  $\square$

3.1.2.  $\bar{S} \neq 0$  Case. Consider the  $\epsilon$ -Milne problem for  $f(\eta, \phi)$  in  $(\eta, \phi, \psi) \in [0, L] \times [-\pi/2, \pi/2] \times [-\pi, \pi]$  with a general source term

$$\begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} = S(\eta, \phi, \psi), \\ f(0, \phi, \psi) = h(\phi, \psi) \text{ for } \sin \phi > 0, \\ f(L, \phi, \psi) = f(L, \mathcal{R}\phi, \psi), \end{cases} \quad (3.68)$$

where  $F = \bar{F} + G$ .

**Lemma 3.2.** *Assume  $S = 0$  satisfying (3.9) and (3.10) and  $0 < n < \frac{2}{5}$ . There exists a solution  $f(\eta, \phi, \psi)$  of the problem (3.10), satisfying for some constant  $|f_L| \leq C$ ,*

$$\|f - f_L\|_{L^2 L^2} \leq C. \quad (3.69)$$

The solution is unique among functions such that (3.69) holds

*Proof.* We can utilize superposition property for this linear problem, i.e. write  $S = \bar{S} + (S - \bar{S}) = S_Q + S_R$ . Then we solve the problem by the following steps.

Step 1: Construction of auxiliary function  $f^1$ .

We first solve  $f^1$  as the solution to

$$\begin{cases} \sin \phi \frac{\partial f^1}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^1}{\partial \phi} + f^1 - \bar{f}^1 = S_R(\eta, \phi, \psi), \\ f^1(0, \phi, \psi) = h(\phi, \psi) \text{ for } \sin \phi > 0, \\ f^1(L, \phi, \psi) = f^1(L, \mathcal{R}\phi, \psi). \end{cases} \quad (3.70)$$

Since  $\bar{S}_R = 0$ , by Lemma 3.1, we know there exists a unique solution  $f^1$  satisfying the  $L^2$  estimate.

Step 2: Construction of auxiliary function  $f^2$ .

We seek a function  $f^2$  satisfying

$$-\frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \left( \sin \phi \frac{\partial f^2}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} \right) \cos \phi d\phi d\psi + S_Q = 0. \quad (3.71)$$

The following analysis shows this type of function can always be found. An integration by parts transforms the equation (3.71) into

$$-\int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{\partial f^2}{\partial \eta} \sin \phi \cos \phi d\phi d\psi - \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} F(\eta) f^2 \sin \phi \cos \phi d\phi d\psi + 4\pi S_Q = 0. \quad (3.72)$$

Setting

$$f^2(\phi, \eta) = a(\eta) \sin \phi. \quad (3.73)$$

and plugging this ansatz into (3.72), we have

$$-\frac{da}{d\eta} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \sin^2 \phi \cos \phi d\phi d\psi - a(\eta) \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} F(\eta) \sin^2 \phi \cos \phi d\phi d\psi + 4\pi S_Q = 0. \quad (3.74)$$

Hence, we have

$$-\frac{da}{d\eta} - \bar{F}(\eta) a(\eta) + 2S_Q = 0, \quad (3.75)$$

where

$$\bar{F}(\eta) = \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} F(\eta) \sin^2 \phi \cos \phi d\phi d\psi \sim \left( \frac{\epsilon}{R_1 - \epsilon\eta} + \frac{\epsilon}{R_2 - \epsilon\eta} \right) \quad (3.76)$$

This is a first order linear ordinary differential equation, which possesses infinite solutions. We can directly solve it to obtain

$$a(\eta) = e^{-\int_0^\eta \bar{F}(y)dy} \left( a(0) + \int_0^\eta e^{\int_0^y \bar{F}(z)dz} 2S_Q(y)dy \right). \quad (3.77)$$

We may take

$$a(0) = - \int_0^L e^{\int_0^y \bar{F}(z)dz} 2S_Q(y)dy. \quad (3.78)$$

Based on the exponential decay of  $S_Q$ , we can directly verify  $a(\eta)$  decays exponentially to zero as  $\eta \rightarrow L$  and  $f^2$  satisfies the  $L^2$  estimate.

Step 3: Construction of auxiliary function  $f^3$ .

Based on above construction, we can directly verify

$$\int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \left( -\sin \phi \frac{\partial f^2}{\partial \eta} - F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} - f^2 + \bar{f}^2 + S_Q \right) \cos \phi d\phi d\psi = 0. \quad (3.79)$$

Then we can solve  $f^3$  as the solution to

$$\begin{cases} \sin \phi \frac{\partial f^3}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^3}{\partial \phi} + f^3 - \bar{f}^3 &= -\sin \phi \frac{\partial f^2}{\partial \eta} - F(\eta) \cos \phi \frac{\partial f^2}{\partial \phi} - f^2 + \bar{f}^2 + S_Q, \\ f^3(0, \phi, \psi) &= -a(0) \sin \phi \text{ for } \sin \phi > 0, \\ f^3(L, \phi, \psi) &= f^3(L, \mathcal{R}\phi, \psi). \end{cases} \quad (3.80)$$

By (3.79), we can apply Lemma 3.1 to obtain a unique solution  $f^3$  satisfying the  $L^2$  estimate.

Step 4: Construction of auxiliary function  $f^4$ .

We now define  $f^4 = f^2 + f^3$  and an explicit verification shows

$$\begin{cases} \sin \phi \frac{\partial f^4}{\partial \eta} + F(\eta) \cos \phi \frac{\partial f^4}{\partial \phi} + f^4 - \bar{f}^4 &= S_Q(\eta, \phi, \psi), \\ f^4(0, \phi, \psi) &= 0 \text{ for } \sin \phi > 0, \\ f^4(L, \phi, \psi) &= f^4(L, \mathcal{R}\phi, \psi), \end{cases} \quad (3.81)$$

and  $f^4$  satisfies the  $L^2$  estimate.

In summary, we deduce that  $f^1 + f^4$  is the solution of (3.68) and satisfies the  $L^2$  estimate.  $\square$

Combining all above, we have the following theorem.

**Theorem 3.3.** *For the  $\epsilon$ -Milne problem (3.1), there exists a unique solution  $f(\eta, \phi, \psi)$  satisfying the estimates*

$$\|f - f_L\|_{L^2 L^2} \leq C \quad (3.82)$$

for some constant  $f_L$  satisfying

$$|f_L| \leq C. \quad (3.83)$$

**3.2.  $L^\infty$  Estimates.** This section is similar to Section 3 of [18] with obvious modifications, so we omit the proof here and only present the main results.

**Theorem 3.4.** *The solution  $f(\eta, \phi, \psi)$  to the Milne problem (3.1) satisfies*

$$\|f - f_L\|_{L^\infty L^\infty} \leq C \left( 1 + \|f - f_L\|_{L^2 L^2} \right). \quad (3.84)$$

**Theorem 3.5.** *There exists a unique solution  $f(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (3.1) satisfying*

$$\|f - f_L\|_{L^\infty L^\infty} \leq C. \quad (3.85)$$

**3.3. Exponential Decay.** In this section, we prove the spatial decay of the solution to the Milne problem.

**Theorem 3.6.** *Assume (3.9) and (3.10) hold and  $0 < n < \frac{2}{5}$ . For  $K_0 > 0$  sufficiently small, the solution  $f(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (3.1) satisfies*

$$\|e^{K_0\eta}(f - f_L)\|_{L^\infty L^\infty} \leq C. \quad (3.86)$$

*Proof.* Let  $\mathcal{V} = f - f_L$ . Then  $\mathcal{V}$  satisfies

$$\begin{cases} \sin \phi \frac{\partial \mathcal{V}}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \mathcal{V}}{\partial \phi} + \mathcal{V} - \bar{\mathcal{V}} = S, \\ \mathcal{V}(0, \phi, \psi) = p(\phi, \psi) = h(\phi, \psi) - f_L \text{ for } \sin \phi > 0, \\ \mathcal{V}(L, \phi, \psi) = \mathcal{V}(L, \mathcal{R}\phi, \psi). \end{cases} \quad (3.87)$$

We divide the analysis into several steps:

Step 1:  $L^2$  Estimates.

Assume  $\bar{S} = 0$ . We continue using the notation  $F = \tilde{F} + G$  and the decomposition  $\mathcal{V} = r_{\mathcal{V}} + q_{\mathcal{V}}$ . Now we naturally have  $(q_{\mathcal{V}})_L = 0$ . The quasi-orthogonal property reveals

$$\begin{aligned} \langle \mathcal{V}, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) &= \langle r_{\mathcal{V}}, r_{\mathcal{V}} \sin \phi \rangle_{\phi}(\eta) + 2 \langle r_{\mathcal{V}}, q_{\mathcal{V}} \sin \phi \rangle_{\phi}(\eta) + \langle q_{\mathcal{V}}, q_{\mathcal{V}} \sin \phi \rangle_{\phi}(\eta) \\ &= \langle r_{\mathcal{V}}, r_{\mathcal{V}} \sin \phi \rangle_{\phi}(\eta) - 4q_{\mathcal{V}}(\eta) \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r_{\mathcal{V}} \rangle(y) dy. \end{aligned} \quad (3.88)$$

Multiplying  $e^{2K_0\eta} \mathcal{V} \cos \phi$  on both sides of equation (3.87) and integrating over  $(\phi, \psi) \in [-\pi/2, \pi/2] \times [-\pi, \pi)$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\eta} \left( e^{2K_0\eta} \langle \mathcal{V}, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) \right) + F(\eta) \left( e^{2K_0\eta} \langle \mathcal{V}, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) \right) \\ &= e^{2K_0\eta} K_0 \langle \mathcal{V}, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) - \langle r_{\mathcal{V}}, r_{\mathcal{V}} \rangle_{\phi}(\eta) \\ & \quad - G(\eta) \left( e^{2K_0\eta} \langle \mathcal{V} \cos^2 \psi, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) \right) + e^{2K_0\eta} \langle S, r_{\mathcal{V}} \rangle_{\phi}(\eta) \\ &= e^{2K_0\eta} \left( K_0 \langle r_{\mathcal{V}}, r_{\mathcal{V}} \sin \phi \rangle_{\phi}(\eta) - \langle r_{\mathcal{V}}, r_{\mathcal{V}} \rangle_{\phi}(\eta) \right) \\ & \quad + 4e^{2K_0\eta} K_0 q_{\mathcal{V}}(\eta) \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r_{\mathcal{V}} \rangle(y) dy \\ & \quad - G(\eta) \left( e^{2K_0\eta} \langle \mathcal{V} \cos^2 \psi, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) \right) + e^{2K_0\eta} \langle S, r_{\mathcal{V}} \rangle_{\phi}(\eta). \end{aligned} \quad (3.89)$$

For  $K_0 < \min\{1/2, K\}$ , we have

$$\frac{3}{2} \|r_{\mathcal{V}}(\eta)\|_{L^2}^2 \geq -K_0 \langle r_{\mathcal{V}}, r_{\mathcal{V}} \sin \phi \rangle_{\phi}(\eta) + \langle r_{\mathcal{V}}, r_{\mathcal{V}} \rangle_{\phi}(\eta) \geq \frac{1}{2} \|r_{\mathcal{V}}(\eta)\|_{L^2}^2. \quad (3.90)$$

Similar to the proof of Lemma 3.1, formula as (3.89) and (3.90) imply

$$\begin{aligned} \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 &\leq \left| \int_0^L 4e^{2K_0\eta} K_0 q_{\mathcal{V}}(\eta) \int_{\eta}^L e^{2\tilde{V}(\eta) - \tilde{V}(y)} G(y) \langle \sin \phi \cos^2 \psi, r_{\mathcal{V}} \rangle(y) dy d\eta \right| \\ & \quad + \left| \int_0^L G(\eta) \left( e^{2K_0\eta} \langle \mathcal{V} \cos^2 \psi, \mathcal{V} \sin \phi \rangle_{\phi}(\eta) \right) d\eta \right| + \left| \int_0^L e^{2K_0\eta} \langle S, r_{\mathcal{V}} \rangle_{\phi}(\eta) d\eta \right| \\ &\leq CL \|G\|_{L^2 L^2} \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2} \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2} + C\epsilon \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2 \\ & \quad + C \|e^{K_0\eta} S\|_{L^2 L^2} \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2} \\ &\leq C\epsilon \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 + C\epsilon^{1-\frac{3}{2}n} \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 + \epsilon \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2 + C \|S\|_{L^2 L^2}^2 \\ &\leq C + C \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 + C\epsilon^{1-\frac{3}{2}n} \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 + \epsilon \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2. \end{aligned} \quad (3.91)$$

Hence, for  $\epsilon$  sufficiently small, we know

$$\|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 \leq C + C\epsilon \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 + C\epsilon \|\mathcal{V}\|_{L^2 L^2}^2. \quad (3.92)$$

Then similar to the proof of Lemma 3.1, we deduce

$$\begin{aligned} & \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 \\ & \leq \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 + \int_0^L e^{2K_0\eta} \left| \int_{\eta}^L \tilde{F}(y) \langle 1 - 3 \sin^2 \phi, r_{\mathcal{V}} \rangle (y) dy \right|^2 d\eta \\ & \quad + \int_0^L e^{2K_0\eta} \left| \int_{\eta}^L \int_z^L e^{2\tilde{V}(z)-2\tilde{V}y} G(y) \langle \sin \phi \cos^2 \psi, r_{\mathcal{V}} \rangle dy dz \right|^2 d\eta \\ & \quad + \int_0^L e^{2K_0\eta} \left| \int_{\eta}^L G(y) \langle 1 - 3 \sin^2 \phi, r_{\mathcal{V}} \cos^2 \psi \rangle (y) dy \right|^2 d\eta + \int_0^L e^{2K_0\eta} \left| \int_{\eta}^L \langle \sin \phi, S \rangle (y) dy \right|^2 d\eta \\ & \leq C + C \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 + C \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2} \left( \int_0^L \int_{\eta}^L e^{2K_0(\eta-y)} F^2(y) dy d\eta \right) \\ & \quad + L^3 \|G\|_{L^2 L^2}^2 \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 + L \|G\|_{L^2 L^2}^2 \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 + \int_0^L e^{2K_0\eta} \left( \int_{\eta}^L \|S(y)\|_{L^\infty} dy \right)^2 d\eta \\ & \leq C + C(1 + \epsilon^{2-5n}) \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 \\ & \leq C + C \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 \\ & \leq C + C\epsilon \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 + C\epsilon \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2, \end{aligned} \quad (3.94)$$

which implies

$$\|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 \leq C + C\epsilon \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2. \quad (3.95)$$

In summary, we have

$$\|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2 \leq \|e^{K_0\eta} q_{\mathcal{V}}\|_{L^2 L^2}^2 + \|e^{K_0\eta} r_{\mathcal{V}}\|_{L^2 L^2}^2 \leq C + C\epsilon \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}^2, \quad (3.96)$$

which yields

$$\|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2} \leq C. \quad (3.97)$$

This is the desired result when  $\bar{S} = 0$ . By the method introduced in Lemma 3.2, we can extend above  $L^2$  estimates to the general  $S$  case. Note all the auxiliary functions constructed in Lemma 3.2 satisfy the desired estimates.

Step 2:  $L^\infty$  Estimates.

This is similar to the proof of exponential decay in [18], so we omit the details here. We have

$$\|e^{K_0\eta} \mathcal{V}\|_{L^\infty L^\infty} \leq C + C \|e^{K_0\eta} \mathcal{V}\|_{L^2 L^2}. \quad (3.98)$$

Combining (3.97) and (3.98), we deduce the desired result

$$\|e^{K_0\eta} (f - f_L)\|_{L^\infty L^\infty} \leq C. \quad (3.99)$$

□

**3.4. Diffusive Boundary.** In this subsection, we consider the  $\epsilon$ -Milne problem with diffusive boundary as

$$\begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} = S(\eta, \phi, \psi), \\ f(0, \phi, \psi) = h(\phi, \psi) + \mathcal{P}[f](0) \text{ for } \sin \phi > 0, \\ f(L, \phi, \psi) = f(L, \mathcal{R}\phi, \psi), \end{cases} \quad (3.100)$$

where

$$\mathcal{P}[f](0) = -\frac{1}{4\pi} \iint_{\sin \phi < 0} f(0, \phi, \psi) \sin \phi \cos \phi d\phi d\psi, \quad (3.101)$$

Similar to [27, Section 6], we can easily prove that

**Lemma 3.7.** *In order for the equation (3.100) to have a solution  $f(\eta, \phi, \psi) \in L^\infty([0, L] \times [-\pi, \pi] \times [-\pi, \pi] \times [-\pi/2, \pi/2])$ , the boundary data  $h$  and the source term  $S$  must satisfy the compatibility condition*

$$\iint_{\sin \phi > 0} h(\phi, \psi) \sin \phi \cos \phi d\phi d\psi + \int_0^L \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} e^{-V(s)} S(s, \phi, \psi) \cos \phi d\phi d\psi ds = 0. \quad (3.102)$$

In particular, if  $S = 0$ , then the compatibility condition reduces to

$$\iint_{\sin \phi > 0} h(\phi, \psi) \sin \phi \cos \phi d\phi d\psi = 0. \quad (3.103)$$

It is easy to see if  $f$  is a solution to (3.100), then  $f + C$  is also a solution for any constant  $C$ . Hence, in order to obtain a unique solution, we need a normalization condition

$$\mathcal{P}[f](0) = 0. \quad (3.104)$$

The following lemma in [27, Section 6] tells us the problem (3.100) can be reduced to the  $\epsilon$ -Milne problem with in-flow boundary (3.1).

**Lemma 3.8.** *If the boundary data  $h$  and  $S$  satisfy the compatibility condition (3.102), then the solution  $f$  to the  $\epsilon$ -Milne problem (3.1) with in-flow boundary as  $f = h$  on  $\sin \phi > 0$  is also a solution to the  $\epsilon$ -Milne problem (3.100) with diffusive boundary, which satisfies the normalization condition (3.104). Furthermore, this is the unique solution to (3.100) among the functions satisfying (3.104) and  $\|f(\eta, \phi, \psi) - f_L\|_{L^2 L^2} \leq C$ .*

In summary, based on above analysis, we can utilize the known result for  $\epsilon$ -Milne problem (3.1) to obtain the desired results of the solution to the  $\epsilon$ -Milne problem (3.100).

**Theorem 3.9.** *There exists a unique solution  $f(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (3.100) with the normalization condition (3.104) satisfying for some constant  $|f_L| < C$ ,*

$$\|f(\eta, \phi, \psi) - f_L\|_{L^2 L^2} \leq C. \quad (3.105)$$

**Theorem 3.10.** *The unique solution  $f(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (3.100) with the normalization condition (3.104) satisfying for some constant  $|f_L| < C$ ,*

$$\|f(\eta, \phi, \psi) - f_L\|_{L^\infty L^\infty} \leq C. \quad (3.106)$$

**Theorem 3.11.** *There exists  $K_0 > 0$  such that the solution  $f(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (3.100) with the normalization condition (3.104) satisfies*

$$\left\| e^{K_0 \eta} \left( f(\eta, \phi, \psi) - f_L \right) \right\|_{L^\infty L^\infty} \leq C. \quad (3.107)$$



4. REGULARITY OF  $\epsilon$ -MILNE PROBLEM

We continue studying the  $\epsilon$ -Milne problem with in-flow boundary as

$$\begin{cases} \sin \phi \frac{\partial f}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial f}{\partial \phi} + f - \bar{f} = S(\eta, \phi, \psi), \\ f(0, \phi, \psi) = h(\phi, \psi) \text{ for } \sin \phi > 0, \\ f(L, \phi, \psi) = f(L, \mathcal{R}\phi, \psi). \end{cases} \quad (4.1)$$

Here we already omit the superscript  $\epsilon$  and dependence on  $(\tau_1, \tau_2)$ . Besides (3.9) and (3.10), we further assume

$$\left\| \frac{\partial h}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial h}{\partial \psi} \right\|_{L^\infty} + \left\| \frac{\partial h}{\partial \tau_1} \right\|_{L^\infty} + \left\| \frac{\partial h}{\partial \tau_2} \right\|_{L^\infty} \leq M, \quad (4.2)$$

and

$$\left\| e^{K\eta} \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| e^{K\eta} \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| e^{K\eta} \frac{\partial S}{\partial \psi} \right\|_{L^\infty L^\infty} + \left\| e^{K\eta} \frac{\partial S}{\partial \tau_1} \right\|_{L^\infty L^\infty} + \left\| e^{K\eta} \frac{\partial S}{\partial \tau_2} \right\|_{L^\infty L^\infty} \leq M, \quad (4.3)$$

for some  $M, K > 0$ . Define a potential function  $V(\eta, \psi)$  satisfying  $V(0, \psi) = 0$  and  $\frac{\partial V}{\partial \eta} = -F(\eta, \psi)$ . Also, we know  $L = \epsilon^{-n}$  for  $0 < n < \frac{2}{5}$ .

**Lemma 4.1.** *We have  $e^{-V(0, \psi)} = 1$  and*

$$e^{-V(L, \psi)} = \left( 1 - \frac{\epsilon^{1-n}}{R_1} \right)^{\sin^2 \psi} \left( 1 - \frac{\epsilon^{1-n}}{R_2} \right)^{\cos^2 \psi}. \quad (4.4)$$

Also, for  $R = \max\{R_1, R_2\}$  and  $R' = \min\{R_1, R_2\}$  which are the maximum and minimum of  $R_1$  and  $R_2$ , we have

$$\frac{R' - \epsilon\eta}{R'} \leq e^{-V(\eta, \psi)} \leq \frac{R - \epsilon\eta}{R}. \quad (4.5)$$

*Proof.* We directly compute

$$V(\eta, \psi) = \sin^2 \psi \ln \left( \frac{R_1}{R_1 - \epsilon\eta} \right) + \cos^2 \psi \ln \left( \frac{R_2}{R_2 - \epsilon\eta} \right), \quad (4.6)$$

and

$$e^{-V(\eta, \psi)} = \left( \frac{R_1 - \epsilon\eta}{R_1} \right)^{\sin^2 \psi} \left( \frac{R_2 - \epsilon\eta}{R_2} \right)^{\cos^2 \psi}. \quad (4.7)$$

Hence, our result naturally follows.  $\square$

**4.1. Preliminaries.** It is easy to see  $\mathcal{V}(\eta, \phi, \psi) = f(\eta, \phi, \psi) - f_L$  satisfies the equation

$$\begin{cases} \sin \phi \frac{\partial \mathcal{V}}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \mathcal{V}}{\partial \phi} + \mathcal{V} - \bar{\mathcal{V}} = S(\eta, \phi, \psi), \\ \mathcal{V}(0, \phi, \psi) = p(\phi, \psi) \text{ for } \sin \phi > 0, \\ \mathcal{V}(L, \phi, \psi) = \mathcal{V}(L, \mathcal{R}\phi, \psi). \end{cases} \quad (4.8)$$

where

$$p(\phi, \psi) = h(\phi, \psi) - f_L. \quad (4.9)$$

We intend to estimate the normal, tangential and velocity derivative. This idea is motivated by [17] and [18]. Define a distance function  $\zeta(\eta, \phi, \psi)$  as

$$\zeta(\eta, \phi, \psi) = \left( 1 - \left( e^{-V(\eta, \psi)} \cos \phi \right)^2 \right)^{1/2}. \quad (4.10)$$

Note that the closer  $(\eta, \phi, \psi)$  is to the grazing set which satisfies  $\eta = 0$  and  $\sin \phi = 0$ , the smaller  $\zeta$  is. In particular, at grazing set,  $\zeta = 0$ . Also, we have  $0 \leq \zeta \leq 1$ .

**Lemma 4.2.** *We have*

$$\sin \phi \frac{\partial \zeta}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \zeta}{\partial \phi} = 0. \quad (4.11)$$

*Proof.* We may directly compute

$$\frac{\partial \zeta}{\partial \eta} = \frac{1}{2} \left( 1 - \left( e^{-V(\eta, \psi)} \cos \phi \right)^2 \right)^{-1/2} \left( -2e^{-2V(\eta, \psi)} \cos^2 \phi \right) F(\eta, \psi) = -\frac{e^{-2V(\eta, \psi)} F(\eta, \psi) \cos^2 \phi}{\zeta} \quad (4.12)$$

$$\frac{\partial \zeta}{\partial \phi} = \frac{1}{2} \left( 1 - \left( e^{-V(\eta, \psi)} \cos \phi \right)^2 \right)^{-1/2} \left( -2e^{-2V(\eta, \psi)} \cos \phi \right) (-\sin \phi) = \frac{e^{-2V(\eta, \psi)} \cos \phi \sin \phi}{\zeta}. \quad (4.13)$$

Hence, we know

$$\sin \phi \frac{\partial \zeta}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \zeta}{\partial \phi} = \frac{-\sin \phi \left( e^{-2V(\eta, \psi)} F(\eta, \psi) \cos^2 \phi \right) + F(\eta, \psi) \cos \phi \left( e^{-2V(\eta, \psi)} \cos \phi \sin \phi \right)}{\zeta} = 0. \quad (4.14)$$

□

As a matter of fact, we are able to prove some preliminary estimates that are based on the characteristics of  $\mathcal{V}$  itself instead of the derivative. In the following, let  $0 < \delta_0 \ll 1$  be a small quantity.

**Lemma 4.3.** *Assume (3.9), (3.10), (4.2) and (4.3). For  $\sin \phi > \delta_0$ , we have*

$$\left| \sin \phi \frac{\partial \mathcal{V}}{\partial \eta}(\eta, \phi) \right| \leq C \left( 1 + \frac{1}{\delta_0^3} \right). \quad (4.15)$$

**Lemma 4.4.** *Assume (3.9), (3.10), (4.2) and (4.3). For  $\sin \phi < 0$  with  $|E(\eta, \phi)| \leq e^{-V(L)}$ , if it satisfies  $\min_{\phi'} \sin \phi' \geq \delta_0$  where  $(\eta', \phi')$  are on the same characteristics as  $(\eta, \phi)$  with  $\sin \phi' \geq 0$ , then we have*

$$\left| \sin \phi \frac{\partial \mathcal{V}}{\partial \eta}(\eta, \phi) \right| \leq C \left( 1 + \frac{1}{\delta_0^3} \right). \quad (4.16)$$

**Lemma 4.5.** *Assume (3.9), (3.10), (4.2), (4.3) and  $\left| \frac{\partial \bar{\mathcal{V}}}{\partial \eta} \right| \leq C(1 + |\ln(\epsilon)| + |\ln(\eta)|)$ . For  $\sin \phi \leq 0$  and  $|E(\eta, \phi)| \geq e^{-V(L)}$ , we have*

$$\left| \sin \phi \frac{\partial \mathcal{V}}{\partial \eta}(\eta, \phi) \right| \leq C(1 + |\ln(\epsilon)|). \quad (4.17)$$

The proofs of Lemma 4.3, Lemma 4.4 and Lemma 4.5 are similar to those in [18] with obvious modifications, so we omit the details here.

**Remark 4.6.** *Estimates in Lemma 4.3, Lemma 4.4 and Lemma 4.5 can provide pointwise bounds of derivatives. However, they are not uniform estimates due to presence of  $\delta_0$  and  $\ln(\epsilon)$ . We need weighted  $L^\infty$  estimates of derivatives to close the proof. Also, the estimate  $\left| \frac{\partial \bar{\mathcal{V}}}{\partial \eta} \right| \leq C(1 + |\ln(\epsilon)| + |\ln(\eta)|)$  are not known a priori, so we need an iteration argument.*

**4.2. Mild Formulation of Normal Derivative.** In this and next subsection, we will prove stronger a priori estimates of derivatives. Consider the  $\epsilon$ -transport problem for  $\mathcal{A} = \zeta \frac{\partial \mathcal{V}}{\partial \eta}$  as

$$\begin{cases} \sin \phi \frac{\partial \mathcal{A}}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \mathcal{A}}{\partial \phi} + \mathcal{A} &= \tilde{\mathcal{A}} + S_{\mathcal{A}}, \\ \mathcal{A}(0, \phi, \psi) &= p_{\mathcal{A}}(\phi, \psi) \text{ for } \sin \phi > 0, \\ \mathcal{A}(L, \phi, \psi) &= \mathcal{A}(L, \mathcal{R}\phi, \psi), \end{cases} \quad (4.18)$$

where  $p_{\mathcal{A}}$  and  $S_{\mathcal{A}}$  will be specified later with

$$\tilde{\mathcal{A}}(\eta, \phi, \psi) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{\zeta(\eta, \phi, \psi)}{\zeta(\eta, \phi_*, \psi)} \mathcal{A}(\eta, \phi_*, \psi) \cos \phi_* d\phi_* d\psi. \quad (4.19)$$

**Lemma 4.7.** *We have*

$$\begin{aligned} \|\mathcal{A}\|_{L^\infty L^\infty} &\leq C \left( \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} \right) \\ &+ C |\ln(\epsilon)|^8 \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \end{aligned} \quad (4.20)$$

The rest of this subsection will be devoted to the proof of this lemma. We first introduce some notation. Define the energy as before

$$E(\eta, \phi, \psi) = e^{-V(\eta, \psi)} \cos \phi. \quad (4.21)$$

Along the characteristics, where this energy is conserved and  $\zeta$  is a constant, the equation can be simplified as follows:

$$\sin \phi \frac{d\mathcal{A}}{d\eta} + \mathcal{A} = \tilde{\mathcal{A}} + S_{\mathcal{A}}. \quad (4.22)$$

An implicit function  $\eta^+(\eta, \phi, \psi)$  can be determined through

$$|E(\eta, \phi, \psi)| = e^{-V(\eta^+, \psi)}. \quad (4.23)$$

which means  $(\eta^+, \phi_0, \psi)$  with  $\sin \phi_0 = 0$  is on the same characteristics as  $(\eta, \phi, \psi)$ . Define the quantities for  $0 \leq \eta' \leq \eta^+$  as follows:

$$\phi'(\phi, \eta, \eta', \psi) = \cos^{-1}(e^{V(\eta', \psi) - V(\eta, \psi)} \cos \phi), \quad (4.24)$$

$$\mathcal{R}\phi'(\phi, \eta, \eta', \psi) = -\cos^{-1}(e^{V(\eta', \psi) - V(\eta, \psi)} \cos \phi) = -\phi'(\phi, \eta, \eta', \psi), \quad (4.25)$$

where the inverse trigonometric function can be defined single-valued in the domain  $[0, \pi/2]$  and the quantities are always well-defined due to the monotonicity of  $V$ . Note that  $\sin \phi' \geq 0$ , even if  $\sin \phi < 0$ . Finally we put

$$G_{\eta, \eta', \psi}(\phi) = \int_{\eta'}^{\eta} \frac{1}{\sin(\phi'(\phi, \eta, \xi, \psi))} d\xi. \quad (4.26)$$

Similar to  $\epsilon$ -Milne problem, we can define the solution along the characteristics as follows:

$$\mathcal{A}(\eta, \phi, \psi) = \mathcal{K}[p_{\mathcal{A}}] + \mathcal{T}[\tilde{\mathcal{A}} + S_{\mathcal{A}}], \quad (4.27)$$

where

Region I:

For  $\sin \phi > 0$ ,

$$\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}}(\phi'(0), \psi) \exp(-G_{\eta, 0, \psi}) \quad (4.28)$$

$$\mathcal{T}[\tilde{\mathcal{A}} + S_{\mathcal{A}}] = \int_0^{\eta} \frac{(\tilde{\mathcal{A}} + S_{\mathcal{A}})(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta, \eta', \psi}) d\eta'. \quad (4.29)$$

Region II:

For  $\sin \phi < 0$  and  $|E(\eta, \phi, \psi)| \leq e^{-V(L, \psi)}$ ,

$$\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}}(\phi'(0), \psi) \exp(-G_{L, 0, \psi} - G_{L, \eta, \psi}) \quad (4.30)$$

$$\begin{aligned} \mathcal{T}[\tilde{\mathcal{A}} + S_{\mathcal{A}}] &= \int_0^L \frac{(\tilde{\mathcal{A}} + S)(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{L, \eta', \psi} - G_{L, \eta, \psi}) d\eta' \\ &+ \int_{\eta}^L \frac{(\tilde{\mathcal{A}} + S)(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta', \eta, \psi}) d\eta'. \end{aligned} \quad (4.31)$$

Region III:

For  $\sin \phi < 0$  and  $|E(\eta, \phi, \psi)| \geq e^{-V(L, \psi)}$ ,

$$\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}}(\phi'(\phi, \eta, 0), \psi) \exp(-G_{\eta^+, 0, \psi} - G_{\eta^+, \eta, \psi}) \quad (4.32)$$

$$\begin{aligned} \mathcal{T}[\tilde{\mathcal{A}} + S_{\mathcal{A}}] &= \int_0^{\eta^+} \frac{(\tilde{\mathcal{A}} + S_{\mathcal{A}})(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta^+, \eta', \psi} - G_{\eta^+, \eta, \psi}) d\eta' \\ &\quad + \int_{\eta}^{\eta^+} \frac{(\tilde{\mathcal{A}} + S_{\mathcal{A}})(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta', \eta, \psi}) d\eta'. \end{aligned} \quad (4.33)$$

Then we need to estimate  $\mathcal{K}[p_{\mathcal{A}}]$  and  $\mathcal{T}[\tilde{\mathcal{A}} + S_{\mathcal{A}}]$  in each region. We assume  $0 < \delta \ll 1$  and  $0 < \delta_0 \ll 1$  are small quantities which will be determined later.

4.2.1. *Region I*:  $\sin \phi > 0$ . A direct computation reveals

$$|\mathcal{K}[p_{\mathcal{A}}]| \leq \|p_{\mathcal{A}}\|_{L^\infty}, \quad (4.34)$$

$$|\mathcal{T}[S_{\mathcal{A}}]| \leq \|S_{\mathcal{A}}\|_{L^\infty L^\infty}. \quad (4.35)$$

Hence, we only need to estimate  $I = \mathcal{T}[\tilde{\mathcal{A}}]$ . We divide it into several steps:

Step 0: Preliminaries.

We have

$$E(\eta', \phi') = \left( \frac{R_1 - \epsilon\eta'}{R_1} \right)^{\sin^2 \psi} \left( \frac{R_2 - \epsilon\eta'}{R_2} \right)^{\cos^2 \psi} \cos \phi'. \quad (4.36)$$

We can directly obtain

$$\begin{aligned} \zeta(\eta', \phi', \psi) &\leq \frac{1}{R'} \sqrt{R'^2 - \left( (R' - \epsilon\eta') \cos \phi' \right)^2} = \frac{1}{R'} \sqrt{R'^2 - (R' - \epsilon\eta')^2 + (R' - \epsilon\eta')^2 \sin^2 \phi'}, \\ &\leq \frac{\sqrt{R'^2 - (R' - \epsilon\eta')^2} + \sqrt{(R' - \epsilon\eta')^2 \sin^2 \phi'}}{R'} \leq C \left( \sqrt{\epsilon\eta'} + \sin \phi' \right), \end{aligned} \quad (4.37)$$

and

$$\zeta(\eta', \phi', \psi) \geq \frac{1}{R} \sqrt{R^2 - (R - \epsilon\eta')^2} \geq C \sqrt{\epsilon\eta'}. \quad (4.38)$$

Also, we know for  $0 \leq \eta' \leq \eta$ ,

$$\sin \phi' = \sqrt{1 - \cos^2 \phi'} \leq \sqrt{1 - \left( \frac{R' - \epsilon\eta'}{R' - \epsilon\eta'} \right)^2 \cos^2 \phi} \quad (4.39)$$

$$= \frac{\sqrt{(R' - \epsilon\eta')^2 \sin^2 \phi + (2R' - \epsilon\eta - \epsilon\eta')(\epsilon\eta - \epsilon\eta') \cos^2 \phi}}{R' - \epsilon\eta'}. \quad (4.40)$$

Since

$$0 \leq (2R' - \epsilon\eta - \epsilon\eta')(\epsilon\eta - \epsilon\eta') \cos^2 \phi \leq 2R'\epsilon(\eta - \eta'), \quad (4.41)$$

we have

$$\sin \phi \leq \sin \phi' \leq 2\sqrt{\sin^2 \phi + \epsilon(\eta - \eta')}, \quad (4.42)$$

which means

$$\frac{1}{2\sqrt{\sin^2 \phi + \epsilon(\eta - \eta')}} \leq \frac{1}{\sin \phi'} \leq \frac{1}{\sin \phi}. \quad (4.43)$$

Therefore,

$$\begin{aligned}
-\int_{\eta'}^{\eta} \frac{1}{\sin \phi'(y)} dy &\leq -\int_{\eta'}^{\eta} \frac{1}{2\sqrt{\sin^2 \phi + \epsilon(\eta - y)}} dy \\
&= \frac{1}{\epsilon} \left( \sin \phi - \sqrt{\sin^2 \phi + \epsilon(\eta - \eta')} \right) \\
&= -\frac{\eta - \eta'}{\sin \phi + \sqrt{\sin^2 \phi + \epsilon(\eta - \eta')}} \\
&\leq -\frac{\eta - \eta'}{2\sqrt{\sin^2 \phi + \epsilon(\eta - \eta')}}.
\end{aligned} \tag{4.44}$$

Define a cut-off function  $\chi \in C^\infty[-\pi, \pi]$  satisfying

$$\chi(\phi) = \begin{cases} 1 & \text{for } |\sin \phi| \leq \delta, \\ 0 & \text{for } |\sin \phi| \geq 2\delta, \end{cases} \tag{4.45}$$

In the following, we will divide the estimate of  $I$  into several cases based on the value of  $\sin \phi$ ,  $\sin \phi'$ ,  $\epsilon\eta'$  and  $\epsilon(\eta - \eta')$ . Let  $\mathbf{1}$  denote the indicator function. We write

$$\begin{aligned}
I &= \int_0^\eta \mathbf{1}_{\{\sin \phi \geq \delta_0\}} + \int_0^\eta \mathbf{1}_{\{0 \leq \sin \phi \leq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) < 1\}} + \int_0^\eta \mathbf{1}_{\{0 \leq \sin \phi \leq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \geq \sin \phi'\}} \\
&\quad + \int_0^\eta \mathbf{1}_{\{0 \leq \sin \phi \leq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \leq \sin \phi'\}} \mathbf{1}_{\{\sin^2 \phi \leq \epsilon(\eta - \eta')\}} \\
&\quad + \int_0^\eta \mathbf{1}_{\{0 \leq \sin \phi \leq \delta_0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \leq \sin \phi'\}} \mathbf{1}_{\{\sin^2 \phi \geq \epsilon(\eta - \eta')\}} \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.46}$$

Step 1: Estimate of  $I_1$  for  $\sin \phi \geq \delta_0$ .

Based on Lemma 4.3, we know

$$\left| \sin \phi \frac{\partial \mathcal{V}}{\partial \eta} \right| \leq C \left( 1 + \frac{1}{\delta_0^3} \right) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \tag{4.47}$$

Hence, we have

$$|I_1| \leq C \left| \frac{\partial \mathcal{V}}{\partial \eta} \right| \leq \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \tag{4.48}$$

Step 2: Estimate of  $I_2$  for  $0 \leq \sin \phi \leq \delta_0$  and  $\chi(\phi_*) < 1$ .

We have

$$\begin{aligned}
I_2 &= \frac{1}{4\pi} \int_0^\eta \left( \int_{-\pi}^\pi \int_{-\pi/2}^{\pi/2} \frac{\zeta(\eta', \phi', \psi)}{\zeta(\eta', \phi_*, \psi)} (1 - \chi(\phi_*)) \mathcal{A}(\eta', \phi_*, \psi) \cos \phi_* d\phi_* d\psi \right) \frac{1}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta' \\
&= \frac{1}{4\pi} \int_0^\eta \left( \int_{-\pi}^\pi \int_{-\pi/2}^{\pi/2} \zeta(\eta', \phi', \psi) (1 - \chi(\phi_*)) \frac{\mathcal{V}(\eta', \phi_*, \psi)}{\partial \eta'} \cos \phi_* d\phi_* d\psi \right) \frac{1}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta'.
\end{aligned} \tag{4.49}$$

Based on the  $\epsilon$ -Milne problem of  $\mathcal{V}$  as

$$\sin \phi_* \frac{\partial \mathcal{V}(\eta', \phi_*, \psi)}{\partial \eta'} + F(\eta', \psi) \cos \phi_* \frac{\partial \mathcal{V}(\eta', \phi_*, \psi)}{\partial \phi_*} + \mathcal{V}(\eta', \phi_*, \psi) - \bar{\mathcal{V}}(\eta') = S(\eta', \phi_*, \psi), \tag{4.50}$$

we have

$$\frac{\partial \mathcal{V}(\eta', \phi_*, \psi)}{\partial \eta'} = -\frac{1}{\sin \phi_*} \left( F(\eta', \psi) \cos \phi_* \frac{\partial \mathcal{V}(\eta', \phi_*, \psi)}{\partial \phi_*} + \mathcal{V}(\eta', \phi_*, \psi) - \bar{\mathcal{V}}(\eta') - S(\eta', \phi_*, \psi) \right). \tag{4.51}$$

Hence, we have

$$\begin{aligned}
\tilde{\mathcal{A}} &= \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \zeta(\eta', \phi', \psi)(1 - \chi(\phi_*)) \frac{\partial \mathcal{V}(\eta', \phi_*)}{\partial \eta'} \cos \phi_* d\phi_* d\psi \\
&= - \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \zeta(\eta', \phi', \psi)(1 - \chi(\phi_*)) \frac{1}{\sin \phi_*} \left( \mathcal{V}(\eta', \phi_*, \psi) - \bar{\mathcal{V}}(\eta') - S(\eta', \phi_*, \psi) \right) \cos \phi_* d\phi_* d\psi \\
&\quad - \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \zeta(\eta', \phi', \psi)(1 - \chi(\phi_*)) \frac{1}{\sin \phi_*} F(\eta', \psi) \cos \phi_* \frac{\partial \mathcal{V}(\eta', \phi_*, \psi)}{\partial \phi_*} \cos \phi_* d\phi_* d\psi \\
&= \tilde{\mathcal{A}}_1 + \tilde{\mathcal{A}}_2.
\end{aligned} \tag{4.52}$$

We may directly obtain

$$\begin{aligned}
|\tilde{\mathcal{A}}_1| &\leq \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \zeta(\eta', \phi', \psi)(1 - \chi(\phi_*)) \frac{1}{\sin \phi_*} \left( \mathcal{V}(\eta', \phi_*, \psi) - \bar{\mathcal{V}}(\eta') - S(\eta', \phi_*, \psi) \right) \cos \phi_* d\phi_* d\psi \\
&\leq \frac{R}{\delta} \left| \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \left( \mathcal{V}(\eta', \phi_*, \psi) - \bar{\mathcal{V}}(\eta') - S(\eta', \phi_*, \psi) \right) \cos \phi_* d\phi_* d\psi \right| \\
&\leq C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).
\end{aligned} \tag{4.53}$$

On the other hand, an integration by parts yields

$$\tilde{\mathcal{A}}_2 = \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial \phi_*} \left( \zeta(\eta', \phi', \psi)(1 - \chi(\phi_*)) \frac{1}{\sin \phi_*} F(\eta', \psi) \cos \phi_* \right) \mathcal{V}(\eta', \phi_*, \psi) \cos \phi_* d\phi_* d\psi, \tag{4.54}$$

which further implies

$$|\tilde{\mathcal{A}}_2| \leq \frac{C\epsilon}{\delta^2} \|\mathcal{V}\|_{L^\infty L^\infty} \leq C(\delta) \|\mathcal{V}\|_{L^\infty L^\infty}. \tag{4.55}$$

Since we can use substitution to show

$$\int_0^\eta \frac{1}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta' \leq 1, \tag{4.56}$$

we have

$$\begin{aligned}
|I_2| &\leq C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right) \int_0^\eta \frac{1}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta' \\
&\leq C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).
\end{aligned} \tag{4.57}$$

Step 3: Estimate of  $I_3$  for  $0 \leq \sin \phi \leq \delta_0$ ,  $\chi(\phi_*) = 1$  and  $\sqrt{\epsilon \eta'} \geq \sin \phi'$ .

Based on (4.37), this implies

$$\zeta(\eta', \phi', \psi) \leq C \sqrt{\epsilon \eta'}.$$

Then combining this with (4.38), we can directly obtain

$$\int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{\zeta(\eta', \phi', \psi)}{\zeta(\eta', \phi_*, \psi)} \chi(\phi_*) \mathcal{A}(\eta', \phi_*, \psi) \cos \phi_* d\phi_* d\psi \leq C \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} \mathcal{A}(\eta', \phi_*, \psi) \cos \phi_* d\phi_* d\psi \leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty}. \tag{4.58}$$

Hence, we have

$$|I_3| \leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta' \leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty}. \tag{4.59}$$

Step 4: Estimate of  $I_4$  for  $0 \leq \sin \phi \leq \delta_0$ ,  $\chi(\phi_*) = 1$ ,  $\sqrt{\epsilon \eta'} \leq \sin \phi'$  and  $\sin^2 \phi \leq \epsilon(\eta - \eta')$ .

Based on (4.37), this implies

$$\zeta(\eta', \phi', \psi) \leq C \sin \phi'. \tag{4.60}$$

Based on (4.44), we have

$$-G_{\eta, \eta', \psi} = - \int_{\eta'}^{\eta} \frac{1}{\sin \phi'(y)} dy \leq - \frac{\eta - \eta'}{2\sqrt{\epsilon(\eta - \eta')}} \leq -C\sqrt{\frac{\eta - \eta'}{\epsilon}}. \quad (4.61)$$

Hence, we know

$$\begin{aligned} |I_4| &\leq C \int_0^{\eta} \left( \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \frac{\zeta(\eta', \phi', \psi)}{\zeta(\eta', \phi_*, \psi)} \chi(\phi_*) \mathcal{A}(\eta', \phi_*, \psi) \cos \phi_* d\phi_* d\psi \right) \frac{1}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta' \quad (4.62) \\ &\leq C \int_0^{\eta} \left( \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*, \psi)} \mathcal{A}(\eta', \phi_*, \psi) \cos \phi_* d\phi_* d\psi \right) \frac{\zeta(\eta', \phi', \psi)}{\sin \phi'} \exp(-G_{\eta, \eta', \psi}) d\eta' \\ &\leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty} \int_0^{\eta} \frac{1}{\sqrt{\epsilon\eta'}} \exp(-G_{\eta, \eta', \psi}) d\eta' \\ &\leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty} \int_0^{\eta} \frac{1}{\sqrt{\epsilon\eta'}} \exp\left(-C\sqrt{\frac{\eta - \eta'}{\epsilon}}\right) d\eta' \end{aligned}$$

Define  $z = \frac{\eta'}{\epsilon}$ , which implies  $d\eta' = \epsilon dz$ . Substituting this into above integral, we have

$$\begin{aligned} |I_4| &\leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty} \int_0^{\eta/\epsilon} \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z}\right) dz \quad (4.63) \\ &= C\delta \|\mathcal{A}\|_{L^\infty L^\infty} \left( \int_0^1 \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z}\right) dz + \int_1^{\eta/\epsilon} \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z}\right) dz \right). \end{aligned}$$

We can estimate these two terms separately.

$$\int_0^1 \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z}\right) dz \leq \int_0^1 \frac{1}{\sqrt{z}} dz = 2. \quad (4.64)$$

$$\int_1^{\eta/\epsilon} \frac{1}{\sqrt{z}} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z}\right) dz \leq \int_1^{\eta/\epsilon} \exp\left(-C\sqrt{\frac{\eta}{\epsilon} - z}\right) dz \stackrel{t^2 = \frac{\eta}{\epsilon} - z}{\leq} 2 \int_0^{\infty} te^{-Ct} dt < \infty. \quad (4.65)$$

Hence, we know

$$|I_4| \leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty}. \quad (4.66)$$

Step 5: Estimate of  $I_5$  for  $0 \leq \sin \phi \leq \delta_0$ ,  $\chi(\phi_*) = 1$ ,  $\sqrt{\epsilon\eta'} \leq \sin \phi'$  and  $\sin^2 \phi \geq \epsilon(\eta - \eta')$ . Based on (4.37), this implies

$$\zeta(\eta', \phi', \psi) \leq C \sin \phi'.$$

Based on (4.44), we have

$$-G_{\eta, \eta', \psi} = - \int_{\eta'}^{\eta} \frac{1}{\sin \phi'(y)} dy \leq - \frac{C(\eta - \eta')}{\sin \phi}. \quad (4.67)$$

Hence, we have

$$|I_5| \leq C \|\mathcal{A}\|_{L^\infty L^\infty} \int_0^{\eta} \left( \int_{-\pi}^{\pi} \int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*, \psi)} \cos \phi_* d\phi_* d\psi \right) \exp\left(-\frac{C(\eta - \eta')}{\sin \phi}\right) d\eta' \quad (4.68)$$

Here, we use a different way to estimate the inner integral. We use substitution to find

$$\begin{aligned}
\int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*, \psi)} \cos \phi_* d\phi_* &\leq \int_{-\delta}^{\delta} \frac{1}{\left(R^2 - (R - \epsilon\eta')^2 \cos \phi_*^2\right)^{1/2}} d\phi_* \\
&\stackrel{\sin \phi_* \text{ small}}{\leq} C \int_{-\delta}^{\delta} \frac{\cos \phi_*}{\left(R^2 - (R - \epsilon\eta')^2 \cos \phi_*^2\right)^{1/2}} d\phi_* \\
&= C \int_{-\delta}^{\delta} \frac{\cos \phi_*}{\left(R^2 - (R - \epsilon\eta')^2 + (R - \epsilon\eta')^2 \sin \phi_*^2\right)^{1/2}} d\phi_* \\
&\stackrel{y = \sin \phi_*}{=} C \int_{-\delta}^{\delta} \frac{1}{\left(R^2 - (R - \epsilon\eta')^2 + (R - \epsilon\eta')^2 y^2\right)^{1/2}} dy.
\end{aligned} \tag{4.69}$$

Define

$$p = \sqrt{R^2 - (R - \epsilon\eta')^2} = \sqrt{2R\epsilon\eta' - \epsilon^2\eta'^2} \leq C\sqrt{\epsilon\eta'}, \tag{4.70}$$

$$q = R - \epsilon\eta' \geq C, \tag{4.71}$$

$$r = \frac{p}{q} \leq C\sqrt{\epsilon\eta'}. \tag{4.72}$$

Then we have

$$\begin{aligned}
\int_{-\delta}^{\delta} \frac{1}{\zeta(\eta', \phi_*, \psi)} d\phi_* &\leq C \int_{-\delta}^{\delta} \frac{1}{(p^2 + q^2 y^2)^{1/2}} dy \\
&\leq C \int_{-2}^2 \frac{1}{(p^2 + q^2 y^2)^{1/2}} dy \leq C \int_{-2}^2 \frac{1}{(r^2 + y^2)^{1/2}} dy \\
&\leq C \int_0^2 \frac{1}{(r^2 + y^2)^{1/2}} dy = \left( \ln(y + \sqrt{r^2 + y^2}) - \ln(r) \right) \Big|_0^2 \\
&\leq C \left( \ln(2 + \sqrt{r^2 + 4}) - \ln r \right) \leq C \left( 1 + \ln(r) \right) \\
&\leq C \left( 1 + |\ln(\epsilon)| + |\ln(\eta')| \right).
\end{aligned} \tag{4.73}$$

Hence, we know

$$|I_5| \leq C \|\mathcal{A}\|_{L^\infty L^\infty} \int_0^\eta \left( 1 + |\ln(\epsilon)| + |\ln(\eta')| \right) \exp\left(-\frac{C(\eta - \eta')}{\sin \phi}\right) d\eta' \tag{4.74}$$

We may directly compute

$$\left| \int_0^\eta \left( 1 + |\ln(\epsilon)| \right) \exp\left(-\frac{C(\eta - \eta')}{\sin \phi}\right) d\eta' \right| \leq C \sin \phi (1 + |\ln(\epsilon)|). \tag{4.75}$$

Hence, we only need to estimate

$$\left| \int_0^\eta |\ln(\eta')| \exp\left(-\frac{C(\eta - \eta')}{\sin \phi}\right) d\eta' \right|. \tag{4.76}$$

If  $\eta \leq 2$ , using Cauchy's inequality, we have

$$\begin{aligned}
\left| \int_0^\eta |\ln(\eta')| \exp\left(-\frac{C(\eta - \eta')}{\sin \phi}\right) d\eta' \right| &\leq \left( \int_0^\eta \ln^2(\eta') d\eta' \right)^{1/2} \left( \int_0^\eta \exp\left(-\frac{2C(\eta - \eta')}{\sin \phi}\right) d\eta' \right)^{1/2} \\
&\leq \left( \int_0^2 \ln^2(\eta') d\eta' \right)^{1/2} \left( \int_0^\eta \exp\left(-\frac{2C(\eta - \eta')}{\sin \phi}\right) d\eta' \right)^{1/2} \\
&\leq \sqrt{\sin \phi}.
\end{aligned} \tag{4.77}$$



If  $\eta \geq 2$ , we decompose and apply Cauchy's inequality to obtain

$$\begin{aligned}
& \left| \int_0^\eta |\ln(\eta')| \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) d\eta' \right| \\
& \leq \left| \int_0^2 |\ln(\eta')| \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) d\eta' \right| + \left| \int_2^\eta \ln(\eta') \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) d\eta' \right| \\
& \leq \left( \int_0^2 \ln^2(\eta') d\eta' \right)^{1/2} \left( \int_0^2 \exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right) d\eta' \right)^{1/2} + \ln(2) \left| \int_2^\eta \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) d\eta' \right| \\
& \leq C \left( \sqrt{\sin\phi} + \sin\phi \right) \leq C\sqrt{\sin\phi}.
\end{aligned} \tag{4.78}$$

Hence, we have

$$|I_5| \leq C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty}. \tag{4.79}$$

Step 6: Synthesis.

Collecting all the terms in previous steps, we have proved

$$\begin{aligned}
|I| & \leq C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \\
& \quad + \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right) + C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).
\end{aligned} \tag{4.80}$$

Therefore, we know

$$\begin{aligned}
|\mathcal{A}|_I & \leq \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} + C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \\
& \quad + \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right) + C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).
\end{aligned} \tag{4.81}$$

4.2.2. *Region II:  $\sin\phi < 0$  and  $|E(\eta, \phi, \psi)| \leq e^{-V(L)}$ .*

$$\mathcal{K}[p_{\mathcal{A}}] = p_{\mathcal{A}}(\phi'(0), \psi) \exp(-G_{L,0,\psi} - G_{L,\eta,\psi}) \tag{4.82}$$

$$\begin{aligned}
\mathcal{T}[\tilde{\mathcal{A}} + S_{\mathcal{A}}] & = \int_0^L \frac{(\tilde{\mathcal{A}} + S)(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{L,\eta',\psi} - G_{L,\eta,\psi}) d\eta' \\
& \quad + \int_\eta^L \frac{(\tilde{\mathcal{A}} + S)(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta',\eta,\psi}) d\eta'.
\end{aligned} \tag{4.83}$$

A direct computation reveals

$$|\mathcal{K}[p_{\mathcal{A}}]| \leq \|p_{\mathcal{A}}\|_{L^\infty}, \tag{4.84}$$

$$|\mathcal{T}[S_{\mathcal{A}}]| \leq \|S_{\mathcal{A}}\|_{L^\infty L^\infty}. \tag{4.85}$$

Hence, we only need to estimate  $II = \mathcal{T}[\tilde{\mathcal{A}}]$ . In particular, we can decompose

$$\begin{aligned}
\mathcal{T}[\tilde{\mathcal{A}}] & = \int_0^L \frac{\tilde{\mathcal{A}}(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{L,\eta',\psi} - G_{L,\eta,\psi}) d\eta' + \int_\eta^L \frac{\tilde{\mathcal{A}}(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta',\eta,\psi}) d\eta' \\
& = \int_0^\eta \frac{\tilde{\mathcal{A}}(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{L,\eta',\psi} - G_{L,\eta,\psi}) d\eta' \\
& \quad + \int_\eta^L \frac{\tilde{\mathcal{A}}(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{L,\eta',\psi} - G_{L,\eta,\psi}) d\eta' + \int_\eta^L \frac{\tilde{\mathcal{A}}(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta',\eta,\psi}) d\eta'.
\end{aligned} \tag{4.86}$$

The integral  $\int_0^\eta \dots$  can be estimated as in Region I, so we only need to estimate the integral  $\int_\eta^L \dots$ . Also, noting that fact that

$$\exp(-G_{L,\eta',\psi} - G_{L,\eta,\psi}) \leq \exp(-G_{\eta',\eta,\psi}), \tag{4.87}$$

we only need to estimate

$$\int_{\eta}^L \frac{\tilde{\mathcal{A}}(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta', \eta, \psi}) d\eta'. \quad (4.88)$$

Here the proof is almost identical to that in Region I, so we only point out the key differences.

Step 0: Preliminaries.

We need to update one key result. For  $0 \leq \eta \leq \eta'$ ,

$$\sin \phi' = \sqrt{1 - \cos^2 \phi'} \leq \sqrt{1 - \left( \frac{R - \epsilon\eta}{R - \epsilon\eta'} \right)^2 \cos^2 \phi} \quad (4.89)$$

$$\begin{aligned} &= \frac{\sqrt{(R - \epsilon\eta')^2 \sin^2 \phi + (2R - \epsilon\eta - \epsilon\eta')(\epsilon\eta' - \epsilon\eta) \cos^2 \phi}}{R - \epsilon\eta'} \\ &\leq |\sin \phi|. \end{aligned} \quad (4.90)$$

Then we have

$$- \int_{\eta}^{\eta'} \frac{1}{\sin \phi'(y)} dy \leq - \frac{\eta' - \eta}{|\sin \phi|}. \quad (4.91)$$

In the following, we will divide the estimate of  $II$  into several cases based on the value of  $\sin \phi$ ,  $\sin \phi'$  and  $\epsilon\eta'$ . We write

$$\begin{aligned} II &= \int_{\eta}^L \mathbf{1}_{\{\sin \phi \leq -\delta_0\}} + \int_{\eta}^L \mathbf{1}_{\{-\delta_0 \leq \sin \phi \leq 0\}} \mathbf{1}_{\{\chi(\phi_*) < 1\}} \\ &\quad + \int_{\eta}^L \mathbf{1}_{\{-\delta_0 \leq \sin \phi \leq 0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \geq \sin \phi'\}} + \int_{\eta}^L \mathbf{1}_{\{-\delta_0 \leq \sin \phi \leq 0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \leq \sin \phi'\}} \\ &= II_1 + II_2 + II_3 + II_4. \end{aligned} \quad (4.92)$$

Step 1: Estimate of  $II_1$  for  $\sin \phi \leq -\delta_0$ .

We first estimate  $\sin \phi'$ . Along the characteristics, we know

$$e^{-V(\eta', \psi)} \cos \phi' = e^{-V(\eta, \psi)} \cos \phi, \quad (4.93)$$

which implies

$$\cos \phi' = e^{V(\eta', \psi) - V(\eta, \psi)} \cos \phi \leq e^{V(L, \psi) - V(0, \psi)} \cos \phi = e^{V(L, \psi) - V(0, \psi)} \sqrt{1 - \delta_0^2}. \quad (4.94)$$

Based on Lemma 4.1, we can further deduce that

$$\cos \phi' \leq \left( 1 - \frac{\epsilon^{1-n}}{R'} \right)^{-1} \sqrt{1 - \delta_0^2}. \quad (4.95)$$

Then we have

$$\sin \phi' \geq \sqrt{1 - \left( 1 - \frac{\epsilon^{1-n}}{R'} \right)^{-2} (1 - \delta_0^2)} \geq \delta_0 - \epsilon^{\frac{1}{2} - \frac{n}{2}} > \frac{\delta_0}{2}, \quad (4.96)$$

when  $\epsilon$  is sufficiently small. Based on Lemma 4.4, we know

$$\left| \sin \phi \frac{\partial \mathcal{Y}}{\partial \eta} \right| \leq C \left( 1 + \frac{1}{\delta_0^3} \right) \left( \|\mathcal{Y}\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \quad (4.97)$$

Hence, we have

$$|II_1| \leq \frac{1}{|\sin \phi|} \left| \frac{\partial \mathcal{Y}}{\partial \eta} \right| \leq \frac{C}{\delta_0^4} \left( \|\mathcal{Y}\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \quad (4.98)$$

Step 2: Estimate of  $II_2$  for  $-\delta_0 \leq \sin \phi \leq 0$  and  $\chi(\phi_*) < 1$ .

This is similar to the estimate of  $I_2$  based on the integral

$$\int_{\eta}^L \frac{1}{\sin \phi'} \exp(-G_{\eta', \eta, \psi}) d\eta' \leq 1. \quad (4.99)$$

Then we have

$$|II_2| \leq C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right). \quad (4.100)$$

Step 3: Estimate of  $II_3$  for  $-\delta_0 \leq \sin \phi \leq 0$ ,  $\chi(\phi_*) = 1$  and  $\sqrt{\epsilon \eta'} \geq \sin \phi'$ .

This is identical to the estimate of  $I_4$ , we have

$$|II_3| \leq C\delta \|\mathcal{A}\|_{L^\infty L^\infty}. \quad (4.101)$$

Step 4: Estimate of  $II_4$  for  $-\delta_0 \leq \sin \phi \leq 0$ ,  $\chi(\phi_*) = 1$  and  $\sqrt{\epsilon \eta'} \leq \sin \phi'$ .

This step is different. We do not need to further decompose the cases. Based on (4.91), we have,

$$-G_{\eta, \eta'} \leq -\frac{\eta' - \eta}{|\sin \phi|}. \quad (4.102)$$

Then following the same argument in estimating  $I_5$ , we obtain

$$|II_4| \leq C \|\mathcal{A}\|_{L^\infty L^\infty} \int_{\eta}^L \left( 1 + |\ln(\epsilon)| + |\ln(\eta')| \right) \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \quad (4.103)$$

If  $\eta \geq 2$ , we directly obtain

$$\begin{aligned} \left| \int_{\eta}^L |\ln(\eta')| \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right| &\leq \left| \int_2^L \ln(\eta') \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right| \\ &\leq \ln(2) \left| \int_2^L \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right| \\ &\leq C \sqrt{|\sin \phi|}. \end{aligned} \quad (4.104)$$

If  $\eta \leq 2$ , we decompose as

$$\begin{aligned} &\left| \int_{\eta}^L |\ln(\eta')| \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right| \\ &\leq \left| \int_{\eta}^2 |\ln(\eta')| \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right| + \left| \int_2^L |\ln(\eta')| \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right|. \end{aligned} \quad (4.105)$$

The second term is identical to the estimate in  $\eta \geq 2$ . We apply Cauchy's inequality to the first term

$$\begin{aligned} \left| \int_{\eta}^2 |\ln(\eta')| \exp\left(-\frac{\eta' - \eta}{|\sin \phi|}\right) d\eta' \right| &\leq \left( \int_{\eta}^2 \ln^2(\eta') d\eta' \right)^{1/2} \left( \int_{\eta}^2 \exp\left(-\frac{2(\eta' - \eta)}{|\sin \phi|}\right) d\eta' \right)^{1/2} \\ &\leq \left( \int_0^2 \ln^2(\eta') d\eta' \right)^{1/2} \left( \int_{\eta}^2 \exp\left(-\frac{2(\eta' - \eta)}{|\sin \phi|}\right) d\eta' \right)^{1/2} \\ &\leq C \sqrt{|\sin \phi|}. \end{aligned} \quad (4.106)$$

Hence, we have

$$|II_4| \leq C(1 + |\ln(\epsilon)|) \sqrt{\delta_0} \|\mathcal{A}\|_{L^\infty L^\infty}. \quad (4.107)$$

Step 5: Synthesis.

Collecting all the terms in previous steps, we have proved

$$|II| \leq C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \quad (4.108)$$

$$+ \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right) + C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).$$

Therefore, we know

$$|\mathcal{A}|_{II} \leq \|S_{\mathcal{A}}\|_{L^\infty L^\infty} + \|p_{\mathcal{A}}\|_{L^\infty} + C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \quad (4.109)$$

$$+ \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right) + C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).$$

4.2.3. *Region III:  $\sin \phi < 0$  and  $|E(\eta, \phi, \psi)| \geq e^{-V(L)}$ .* We still ignore  $\psi$  dependence. Based on [27, Lemma 4.7, Lemma 4.8], we still have

$$|\mathcal{K}[p_{\mathcal{A}}]| \leq \|p_{\mathcal{A}}\|_{L^\infty}, \quad (4.110)$$

$$|\mathcal{T}[S_{\mathcal{A}}]| \leq \|S_{\mathcal{A}}\|_{L^\infty L^\infty}. \quad (4.111)$$

Hence, we only need to estimate  $III = \mathcal{T}[\tilde{\mathcal{A}}]$ . Note that  $|E(\eta, \phi, \psi)| \geq e^{-V(L, \psi)}$  implies

$$e^{-V(\eta, \psi)} \cos \phi \geq e^{-V(L, \psi)}. \quad (4.112)$$

Hence, based on Lemma 4.1, we can further deduce that

$$\cos \phi \geq e^{V(\eta, \psi) - V(L, \psi)} \geq e^{V(0, \psi) - V(L, \psi)} \geq \left(1 - \frac{\epsilon^{1-n}}{R'}\right). \quad (4.113)$$

Hence, we know

$$|\sin \phi| \leq \sqrt{1 - \left(1 - \frac{\epsilon^{1-n}}{R'}\right)^2} \leq \epsilon^{\frac{1}{2} - \frac{n}{2}}. \quad (4.114)$$

Hence, when  $\epsilon$  is sufficiently small, we always have

$$|\sin \phi| \leq \epsilon^{\frac{1}{2} - \frac{n}{2}} \leq \delta_0. \quad (4.115)$$

This means we do not need to bother with the estimate of  $\sin \phi \leq -\delta_0$  as Step 1 in estimating  $I$  and  $II$ . Since we can decompose

$$\begin{aligned} \mathcal{T}[\tilde{\mathcal{A}}] &= \int_0^\eta \frac{\tilde{\mathcal{A}}(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta^+, \eta', \psi} - G_{\eta^+, \eta, \psi}) d\eta' \\ &\quad \left( \int_\eta^{\eta^+} \frac{\tilde{\mathcal{A}}(\eta', \phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta^+, \eta', \psi} - G_{\eta^+, \eta, \psi}) d\eta' \right. \\ &\quad \left. + \int_\eta^{\eta^+} \frac{(\tilde{\mathcal{A}} + S_{\mathcal{A}})(\eta', \mathcal{R}\phi'(\eta'), \psi)}{\sin(\phi'(\eta'))} \exp(-G_{\eta', \eta, \psi}) d\eta' \right). \end{aligned} \quad (4.116)$$

Then the integral  $\int_0^\eta (\dots)$  is similar to the argument in Region I, and the integral  $\int_\eta^{\eta^+} (\dots)$  is similar to the argument in Region II. Hence, combining the methods in Region I and Region II, we can show the desired result, i.e.

$$|\mathcal{A}|_{III} \leq \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} + C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \quad (4.117)$$

$$+ C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right).$$

4.2.4. *Estimate of Normal Derivative.* Combining the analysis in these three regions, we have

$$\begin{aligned} |\mathcal{A}| &\leq \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} + C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \\ &\quad + \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right) + C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right). \end{aligned} \quad (4.118)$$

Taking supremum over all  $(\eta, \phi, \psi)$ , we have

$$\begin{aligned} \|\mathcal{A}\|_{L^\infty L^\infty} &\leq \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} + C(1 + |\ln(\epsilon)|)\sqrt{\delta_0}\|\mathcal{A}\|_{L^\infty L^\infty} + C\delta\|\mathcal{A}\|_{L^\infty L^\infty} \\ &\quad + \frac{C}{\delta_0^4} \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right) \\ &\quad + C(\delta) \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \right). \end{aligned} \quad (4.119)$$

Then we choose these constants to perform absorbing argument. First we choose  $0 < \delta \ll 1$  sufficiently small such that

$$C\delta \leq \frac{1}{4}. \quad (4.120)$$

Then we take  $\delta_0 = \delta |\ln(\epsilon)|^{-2}$  such that

$$C(1 + |\ln(\epsilon)|)\sqrt{\delta_0} \leq 2C\delta \leq \frac{1}{2}. \quad (4.121)$$

for  $\epsilon$  sufficiently small. Note that this mild decay of  $\delta_0$  with respect to  $\epsilon$  also justifies the assumption in Case III and the proof of Lemma 4.4 that

$$\epsilon^{\frac{1}{2} - \frac{n}{2}} \leq \frac{\delta_0}{2}, \quad (4.122)$$

for  $\epsilon$  sufficiently small. Here since  $\delta$  and  $C$  are independent of  $\epsilon$ , there is no circulant argument. Hence, we can absorb all the term related to  $\|\mathcal{A}\|_{L^\infty L^\infty}$  on the right-hand side of (4.119) to the left-hand side to obtain

$$\begin{aligned} \|\mathcal{A}\|_{L^\infty L^\infty} &\leq C \left( \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} \right) \\ &\quad + C |\ln(\epsilon)|^8 \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \end{aligned} \quad (4.123)$$

4.3. **Mild Formulation of Velocity Derivative.** Consider the general  $\epsilon$ -Milne problem for  $\mathcal{B} = \zeta \frac{\partial \mathcal{V}}{\partial \phi}$  as

$$\begin{cases} \sin \phi \frac{\partial \mathcal{B}}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \mathcal{B}}{\partial \phi} + \mathcal{B} = S_{\mathcal{B}}, \\ \mathcal{B}(0, \phi, \psi) = p_{\mathcal{B}}(\phi, \psi) \text{ for } \sin \phi > 0, \\ \mathcal{B}(L, \phi, \psi) = \mathcal{B}(L, \mathcal{R}\phi, \psi), \end{cases} \quad (4.124)$$

where  $p_{\mathcal{B}}$  and  $S_{\mathcal{B}}$  will be specified later. This is much simpler than normal derivative, since we do not have  $\tilde{\mathcal{B}}$ . Then by a direct argument that

$$|\mathcal{K}[p_{\mathcal{B}}]| \leq \|p_{\mathcal{B}}\|_{L^\infty}, \quad (4.125)$$

$$|\mathcal{T}[S_{\mathcal{B}}]| \leq \|S_{\mathcal{B}}\|_{L^\infty L^\infty}. \quad (4.126)$$

we can get the desired result.

**Lemma 4.8.** *We have*

$$\|\mathcal{B}\|_{L^\infty L^\infty} \leq \|p_{\mathcal{B}}\|_{L^\infty} + \|S_{\mathcal{B}}\|_{L^\infty L^\infty}. \quad (4.127)$$

**4.4. Estimate of Derivatives.** In this subsection, we combine above a priori estimates of normal and velocity derivatives.

**Theorem 4.9.** *Assume (3.9), (3.10), (4.2) and (4.3). The normal and velocity derivatives of  $\mathcal{V}$  are well-defined a.e. and satisfy*

$$\left\| \zeta \frac{\partial \mathcal{V}}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathcal{V}}{\partial \eta} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8. \quad (4.128)$$

*Proof.* Based on the analysis in [18], derivatives are a.e. well-defined. Collecting the estimates for  $\mathcal{A}$  and  $\mathcal{B}$  in Lemma 4.7 and Lemma 4.8, we have

$$\|\mathcal{A}\|_{L^\infty L^\infty} \leq C \left( \|p_{\mathcal{A}}\|_{L^\infty} + \|S_{\mathcal{A}}\|_{L^\infty L^\infty} \right) \quad (4.129)$$

$$+ C |\ln(\epsilon)|^8 \left( \|\mathcal{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right),$$

$$\|\mathcal{B}\|_{L^\infty L^\infty} \leq \|p_{\mathcal{B}}\|_{L^\infty} + \|S_{\mathcal{B}}\|_{L^\infty L^\infty}. \quad (4.130)$$

Taking derivatives on both sides of (4.8) and multiplying  $\zeta$ , based on Lemma 4.2, we have

$$p_{\mathcal{A}} = \epsilon \cos \phi \frac{\partial p}{\partial \phi} + p - \bar{\mathcal{V}}(0), \quad (4.131)$$

$$p_{\mathcal{B}} = \sin \phi \frac{\partial p}{\partial \phi}, \quad (4.132)$$

$$S_{\mathcal{A}} = \frac{\partial F}{\partial \eta} \mathcal{B} \cos \phi + \zeta \frac{\partial S}{\partial \eta}, \quad (4.133)$$

$$S_{\mathcal{B}} = \mathcal{A} \cos \phi + F \mathcal{B} \sin \phi + \zeta \frac{\partial S}{\partial \phi}. \quad (4.134)$$

Since  $|F(\eta)| + \left| \frac{\partial F}{\partial \eta} \right| \leq \epsilon$ , by absorbing  $\mathcal{A}$  and  $\mathcal{B}$  on the right-hand side of (4.129) and (4.130), we derive

$$\mathcal{A} \leq C |\ln(\epsilon)|^8, \quad (4.135)$$

$$\mathcal{B} \leq C |\ln(\epsilon)|^8. \quad (4.136)$$

□

**Theorem 4.10.** *Assume (3.9), (3.10), (4.2) and (4.3). The normal and velocity derivatives of  $\mathcal{V}$  are well-defined a.e. and satisfy*

$$\left\| e^{K_0 \eta} \zeta \frac{\partial \mathcal{V}}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| e^{K_0 \eta} \zeta \frac{\partial \mathcal{V}}{\partial \phi} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8. \quad (4.137)$$

*Proof.* This proof is almost identical to Theorem 4.9. The only difference is that  $S_{\mathcal{A}}$  is added by  $K_0 \mathcal{A} \sin \phi$  and  $S_{\mathcal{B}}$  added by  $K_0 \mathcal{B} \sin \phi$ . When  $K_0$  is sufficiently small, we can also absorb them into the left-hand side. Hence, this is obvious. □

**Corollary 4.11.** *Assume (3.9), (3.10), (4.2) and (4.3). We have*

$$\left\| e^{K_0 \eta} \sin \phi \frac{\partial \mathcal{V}}{\partial \eta} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8. \quad (4.138)$$

*Proof.* This is a natural result of Theorem 4.10 since  $\zeta(\eta, \phi, \psi) \geq |\sin \phi|$ . □

Now we pull  $\tau_i$  for  $i = 1, 2$  and  $\psi$  dependence back and study the tangential derivatives and velocity derivative.

**Theorem 4.12.** *Assume (3.9), (3.10), (4.2) and (4.3). We have for  $i = 1, 2$ ,*

$$\left\| e^{K_0 \eta} \frac{\partial \mathcal{V}}{\partial \tau_i} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8, \quad (4.139)$$

$$\left\| e^{K_0 \eta} \frac{\partial \mathcal{V}}{\partial \psi} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8 \quad (4.140)$$

*Proof.* Following a similar fashion in proof of Theorem 4.10, using iteration and characteristics, we can show  $\frac{\partial \mathcal{V}}{\partial \tau_i}$  is a.e. well-defined, so here we focus on the a priori estimate. Let  $\mathcal{W} = \frac{\partial \mathcal{V}}{\partial \tau_i}$ . Taking  $\tau_i$  derivative on both sides of (4.8), we have  $\mathcal{W}$  satisfies the equation

$$\left\{ \begin{array}{l} \sin \phi \frac{\partial \mathcal{W}}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \mathcal{W}}{\partial \phi} + \mathcal{W} - \bar{\mathcal{W}} = \frac{\partial S}{\partial \tau_i} + \epsilon \left( \frac{\partial_{\tau_i} R_1 \sin^2 \psi}{(R_1 - \epsilon \eta)^2} + \frac{\partial_{\tau_i} R_2 \cos^2 \psi}{(R_2 - \epsilon \eta)^2} \right) \left( \cos \phi \frac{\partial \mathcal{V}}{\partial \phi} \right), \\ \mathcal{W}(0, \phi, \psi) = \frac{\partial p}{\partial \tau_i}(\phi, \psi) \text{ for } \sin \phi > 0, \\ \mathcal{W}(L, \phi, \psi) = \mathcal{W}(L, \mathcal{R}\phi, \psi). \end{array} \right. \quad (4.141)$$

Our assumptions on  $S$  verify

$$\left\| e^{K_0 \eta} \frac{\partial S}{\partial \tau_i} \right\|_{L^\infty L^\infty} \leq C. \quad (4.142)$$

For  $\eta \in [0, L]$ , we have

$$\left\| \frac{\partial_{\tau_i} R_1 \sin^2 \psi}{(R_1 - \epsilon \eta)^2} + \frac{\partial_{\tau_i} R_2 \cos^2 \psi}{(R_2 - \epsilon \eta)^2} \right\|_{L^\infty L^\infty} \leq C, \quad (4.143)$$

and

$$C_1 \epsilon \leq F(\eta, \psi) \leq C_2 \epsilon. \quad (4.144)$$

Based on Corollary 4.11 and the equation (4.8), we know

$$\left\| e^{K_0 \eta} \left( \epsilon \cos \phi \frac{\partial \mathcal{V}}{\partial \phi} \right) \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8. \quad (4.145)$$

Therefore, the source term in the equation (4.141) is in  $L^\infty$  and decays exponentially. By Theorem 3.6, we have

$$\| e^{K_0 \eta} (\mathcal{W} - \mathcal{W}_L) \|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8, \quad (4.146)$$

for some constant  $\mathcal{W}_L$ . It is easy to see this  $\mathcal{W}_L$  must be zero due to decay of  $\mathcal{V}$ . Similarly, let  $\mathcal{W}' = \frac{\partial \mathcal{V}}{\partial \psi}$ .

Taking  $\psi$  derivative on both sides of (4.8), we have  $\mathcal{W}'$  satisfies the equation

$$\left\{ \begin{array}{l} \sin \phi \frac{\partial \mathcal{W}'}{\partial \eta} + F(\eta, \psi) \cos \phi \frac{\partial \mathcal{W}'}{\partial \phi} + \mathcal{W}' = \frac{\partial S}{\partial \psi} + \epsilon \left( \frac{\sin(2\psi)}{R_1 - \epsilon \eta} - \frac{\sin(2\psi)}{R_2 - \epsilon \eta} \right) \left( \cos \phi \frac{\partial \mathcal{V}}{\partial \phi} \right), \\ \mathcal{W}'(0, \phi, \psi) = \frac{\partial p}{\partial \psi}(\phi, \psi) \text{ for } \sin \phi > 0, \\ \mathcal{W}'(L, \phi, \psi) = \mathcal{W}'(L, \mathcal{R}\phi, \psi). \end{array} \right. \quad (4.147)$$

We may directly estimate

$$\left\| \frac{\sin(2\psi)}{R_1 - \epsilon \eta} - \frac{\sin(2\psi)}{R_2 - \epsilon \eta} \right\|_{L^\infty L^\infty} \leq C, \quad (4.148)$$

which means the source term is in  $L^\infty$  and decays exponentially. This equation does not involve  $\bar{\mathcal{W}}'$  term, which makes it even simpler. Hence, we get

$$\| e^{K_0 \eta} \mathcal{W}' \|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8, \quad (4.149)$$

for some constant  $\mathcal{W}'_L$ . Naturally we have  $\mathcal{W}'_L$  must be zero.  $\square$

We finally come to the  $\epsilon$ -Milne problem with diffusive boundary.

**Theorem 4.13.** *Assume (3.9), (3.10), (4.2) and (4.3). There exists  $K_0 > 0$  such that the unique solution  $f(\eta, \phi, \psi)$  to the  $\epsilon$ -Milne problem (3.100) with the normalization condition (3.104) satisfies for  $i = 1, 2$ ,*

$$\left\| e^{K_0 \eta} \frac{\partial(f - f_L)}{\partial \tau_i} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8, \quad (4.150)$$

$$\left\| e^{K_0 \eta} \frac{\partial(f - f_L)}{\partial \psi} \right\|_{L^\infty L^\infty} \leq C |\ln(\epsilon)|^8 \quad (4.151)$$

## APPENDIX A. REMAINDER ESTIMATE

In this section, we consider the remainder equation for  $u(\vec{x}, \vec{w})$  as

$$\begin{cases} \epsilon \vec{w} \cdot \nabla_{\vec{x}} u + u - \bar{u} &= f(\vec{x}, \vec{w}) \text{ in } \Omega, \\ u(\vec{x}_0, \vec{w}) &= \mathcal{P}[u](\vec{x}_0) + h(\vec{x}_0, \vec{w}) \text{ for } \vec{w} \cdot \vec{\nu} < 0 \text{ and } \vec{x}_0 \in \partial\Omega, \end{cases} \quad (\text{A.1})$$

where

$$\bar{u}(\vec{x}) = \frac{1}{4\pi} \int_{\mathcal{S}^2} u(\vec{x}, \vec{w}) d\vec{w}, \quad (\text{A.2})$$

$$\mathcal{P}[u](\vec{x}_0) = \frac{1}{4\pi} \int_{\vec{w} \cdot \vec{\nu} > 0} u(\vec{x}_0, \vec{w}) (\vec{w} \cdot \vec{\nu}) d\vec{w}, \quad (\text{A.3})$$

$\vec{\nu}$  is the outward unit normal vector, with the Knudsen number  $0 < \epsilon \ll 1$ . To guarantee uniqueness, we need the normalization condition

$$\int_{\Omega \times \mathcal{S}^2} u(\vec{x}, \vec{w}) d\vec{w} d\vec{x} = 0. \quad (\text{A.4})$$

Also, the data  $f$  and  $h$  satisfy the compatibility condition

$$\int_{\Omega \times \mathcal{S}^2} f(\vec{x}, \vec{w}) d\vec{w} d\vec{x} + \epsilon \int_{\partial\Omega} \int_{\vec{w} \cdot \vec{\nu} < 0} h(\vec{x}_0, \vec{w}) (\vec{w} \cdot \vec{\nu}) d\vec{w} d\vec{x}_0 = 0. \quad (\text{A.5})$$

Based on the flow direction, we can divide the boundary  $\Gamma = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega\}$  into the in-flow boundary  $\Gamma^-$ , the out-flow boundary  $\Gamma^+$  and the grazing set  $\Gamma^0$  as

$$\Gamma^- = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{\nu} < 0\} \quad (\text{A.6})$$

$$\Gamma^+ = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{\nu} > 0\} \quad (\text{A.7})$$

$$\Gamma^0 = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{\nu} = 0\} \quad (\text{A.8})$$

It is easy to see  $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$ . Hence, the boundary condition is only given for  $\Gamma^-$ . We define the  $L^p$  norm with  $1 \leq p < \infty$  and  $L^\infty$  norms in  $\Omega \times \mathcal{S}^2$  as usual:

$$\|f\|_{L^p(\Omega \times \mathcal{S}^2)} = \left( \int_{\Omega} \int_{\mathcal{S}^2} |f(\vec{x}, \vec{w})|^p d\vec{w} d\vec{x} \right)^{1/p}, \quad (\text{A.9})$$

$$\|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} = \sup_{(\vec{x}, \vec{w}) \in \Omega \times \mathcal{S}^2} |f(\vec{x}, \vec{w})|. \quad (\text{A.10})$$

Define the  $L^p$  norm with  $1 \leq p < \infty$  and  $L^\infty$  norms on the boundary as follows:

$$\|f\|_{L^p(\Gamma)} = \left( \iint_{\Gamma} |f(\vec{x}, \vec{w})|^p |\vec{w} \cdot \vec{\nu}| d\vec{w} d\vec{x} \right)^{1/p}, \quad (\text{A.11})$$

$$\|f\|_{L^p(\Gamma^\pm)} = \left( \iint_{\Gamma^\pm} |f(\vec{x}, \vec{w})|^p |\vec{w} \cdot \vec{\nu}| d\vec{w} d\vec{x} \right)^{1/p}, \quad (\text{A.12})$$

$$\|f\|_{L^\infty(\Gamma)} = \sup_{(\vec{x}, \vec{w}) \in \Gamma} |f(\vec{x}, \vec{w})|, \quad (\text{A.13})$$

$$\|f\|_{L^\infty(\Gamma^\pm)} = \sup_{(\vec{x}, \vec{w}) \in \Gamma^\pm} |f(\vec{x}, \vec{w})|. \quad (\text{A.14})$$

The direct application of energy method as in [27] and [28], we may obtain

**Lemma A.1.** *Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  and  $h(\vec{x}_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the transport equation (A.1), there exists a unique solution  $u(\vec{x}, \vec{w}) \in L^2(\Omega \times \mathcal{S}^2)$  satisfying*

$$\frac{1}{\epsilon} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + \|u\|_{L^2(\Omega \times \mathcal{S}^2)} \leq C \left( \frac{1}{\epsilon^2} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|h\|_{L^2(\Gamma^-)} \right), \quad (\text{A.15})$$

Based on classical  $L^2 - L^\infty$  framework, we are able to show

$$\|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C(\Omega) \left( \frac{1}{\epsilon^{\frac{3}{2}}} \|u\|_{L^2(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)} \right). \quad (\text{A.16})$$

Therefore, it is natural to deduce  $L^\infty$  estimate.



**Theorem A.2.** *Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  and  $h(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then the solution  $u(\vec{x}, \vec{w})$  to the transport equation (A.1) satisfies*

$$\|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C(\Omega) \left( \frac{1}{\epsilon^{\frac{7}{2}}} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{\frac{5}{2}}} \|h\|_{L^2(\Gamma^-)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)} \right).$$

However, the estimates here is not strong enough to close the diffusive limit, so we must further improve them.

**A.1.  $L^{2m}$  Estimate.** In this subsection, we try to improve previous estimates. In the following, we assume  $m$  is an integer and let  $o(1)$  denote a sufficiently small constant.

**Theorem A.3.** *Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  and  $h(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for  $\frac{3}{2} \leq m \leq 3$ ,  $u(\vec{x}, \vec{w})$  satisfies*

$$\begin{aligned} & \frac{1}{\epsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)} + \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} \\ & \leq C \left( o(1)\epsilon^{\frac{3}{2m}} \|u\|_{L^\infty(\Gamma^+)} + \frac{1}{\epsilon} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^2} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|h\|_{L^2(\Gamma^-)} + \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} \right). \end{aligned} \quad (\text{A.1.1})$$

*Proof.* We divide the proof into several steps:

Step 1: Kernel Estimate.

Applying Green's identity to the equation (A.1). Then for any  $\phi \in L^2(\Omega \times \mathcal{S}^2)$  satisfying  $\vec{w} \cdot \nabla_x \phi \in L^2(\Omega \times \mathcal{S}^2)$  and  $\phi \in L^2(\Gamma)$ , we have

$$\epsilon \int_{\Gamma} u \phi d\gamma - \epsilon \iint_{\Omega \times \mathcal{S}^2} (\vec{w} \cdot \nabla_x \phi) u + \iint_{\Omega \times \mathcal{S}^2} (u - \bar{u}) \phi = \iint_{\Omega \times \mathcal{S}^2} f \phi. \quad (\text{A.1.2})$$

Our goal is to choose a particular test function  $\phi$ . We first construct an auxiliary function  $\zeta$ . Naturally  $u \in L^\infty(\Omega \times \mathcal{S}^2)$  implies  $\bar{u} \in L^{2m}(\Omega)$  which further leads to  $(\bar{u})^{2m-1} \in L^{\frac{2m}{2m-1}}(\Omega)$ . We define  $\zeta(\vec{x})$  on  $\Omega$  satisfying

$$\begin{cases} \Delta \zeta &= (\bar{u})^{2m-1} - \frac{1}{|\Omega|} \int_{\Omega} (\bar{u})^{2m-1} d\vec{x} \text{ in } \Omega, \\ \frac{\partial \zeta}{\partial \vec{\nu}} &= 0 \text{ on } \partial\Omega. \end{cases} \quad (\text{A.1.3})$$

In the bounded domain  $\Omega$ , based on the standard elliptic estimate, we have a unique  $\zeta$  satisfying

$$\|\zeta\|_{W^{2, \frac{2m}{2m-1}}(\Omega)} \leq C \|(\bar{u})^{2m-1}\|_{L^{\frac{2m}{2m-1}}(\Omega)} = C \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}, \quad (\text{A.1.4})$$

and

$$\int_{\Omega} \zeta(\vec{x}) d\vec{x} = 0. \quad (\text{A.1.5})$$

We plug the test function

$$\phi = -\vec{w} \cdot \nabla_x \zeta \quad (\text{A.1.6})$$

into the weak formulation (A.1.2) and estimate each term there. By Sobolev embedding theorem, we have for  $1 \leq m \leq 3$ ,

$$\|\phi\|_{L^2(\Omega)} \leq C \|\zeta\|_{H^1(\Omega)} \leq C \|\zeta\|_{W^{2, \frac{2m}{2m-1}}(\Omega)} \leq C \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}, \quad (\text{A.1.7})$$

$$\|\phi\|_{L^{\frac{2m}{2m-1}}(\Omega)} \leq C \|\zeta\|_{W^{1, \frac{2m}{2m-1}}(\Omega)} \leq C \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \quad (\text{A.1.8})$$

Easily we can decompose

$$-\epsilon \iint_{\Omega \times \mathcal{S}^2} (\vec{w} \cdot \nabla_x \phi) u_\lambda = -\epsilon \iint_{\Omega \times \mathcal{S}^2} (\vec{w} \cdot \nabla_x \phi) \bar{u}_\lambda - \epsilon \iint_{\Omega \times \mathcal{S}^2} (\vec{w} \cdot \nabla_x \phi) (u_\lambda - \bar{u}_\lambda). \quad (\text{A.1.9})$$

We estimate the two term on the right-hand side of (A.1.9) separately. By (A.1.3) and (A.1.6), we have

$$\begin{aligned}
-\epsilon \iint_{\Omega \times \mathcal{S}^2} (\vec{w} \cdot \nabla_x \phi) \bar{u} &= \epsilon \iint_{\Omega \times \mathcal{S}^2} \bar{u} \left( w_1(w_1 \partial_{11} \zeta + w_2 \partial_{12} \zeta) + w_2(w_1 \partial_{12} \zeta + w_2 \partial_{22} \zeta) \right) \quad (\text{A.1.10}) \\
&= \epsilon \iint_{\Omega \times \mathcal{S}^2} \bar{u} \left( w_1^2 \partial_{11} \zeta + w_2^2 \partial_{22} \zeta \right) \\
&= 2\epsilon \pi \int_{\Omega} \bar{u} (\partial_{11} \zeta + \partial_{22} \zeta) \\
&= \epsilon \|\bar{u}\|_{L^{2m}(\Omega)}^{2m}.
\end{aligned}$$

In the second equality, above cross terms vanish due to the symmetry of the integral over  $\mathcal{S}^2$ . On the other hand, for the second term in (A.1.9), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned}
-\epsilon \iint_{\Omega \times \mathcal{S}^2} (\vec{w} \cdot \nabla_x \phi)(u - \bar{u}) &\leq C\epsilon \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} \|\nabla_x \phi\|_{L^{\frac{2m}{2m-1}}(\Omega)} \quad (\text{A.1.11}) \\
&\leq C\epsilon \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} \|\zeta\|_{W^{2, \frac{2m}{2m-1}}(\Omega)} \\
&\leq C\epsilon \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}.
\end{aligned}$$

Based on (A.1.4), (A.1.7), (A.1.8), Sobolev embedding theorem and the trace theorem, we have

$$\|\nabla_x \zeta\|_{L^{\frac{4m}{4m-3}}(\Gamma)} \leq C \|\nabla_x \zeta\|_{W^{\frac{1}{2m}, \frac{2m}{2m-1}}(\Gamma)} \leq C \|\nabla_x \zeta\|_{W^{1, \frac{2m}{2m-1}}(\Omega)} \leq C \|\zeta\|_{W^{2, \frac{2m}{2m-1}}(\Omega)} \leq C \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \quad (\text{A.1.12})$$

We may also decompose  $\vec{w} = (\vec{w} \cdot \vec{\nu})\vec{\nu} + \vec{w}_{\perp}$  to obtain

$$\begin{aligned}
\epsilon \int_{\Gamma} u \phi d\gamma &= \epsilon \int_{\Gamma} u (\vec{w} \cdot \nabla_x \zeta) d\gamma \quad (\text{A.1.13}) \\
&= \epsilon \int_{\Gamma} u (\vec{\nu} \cdot \nabla_x \zeta) (\vec{w} \cdot \vec{\nu}) d\gamma + \epsilon \int_{\Gamma} u (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma \\
&= \epsilon \int_{\Gamma} u (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma.
\end{aligned}$$

Based on (A.1.4), (A.1.8) and Hölder's inequality, we have

$$\begin{aligned}
\epsilon \int_{\Gamma} u \phi d\gamma &= \epsilon \int_{\Gamma} u (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma \quad (\text{A.1.14}) \\
&= \epsilon \int_{\Gamma} \mathcal{P}[u] (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma + \epsilon \int_{\Gamma^+} (1 - \mathcal{P})[u] (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma + \epsilon \int_{\Gamma^-} h (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma \\
&= \epsilon \int_{\Gamma^+} (1 - \mathcal{P})[u] (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma + \epsilon \int_{\Gamma^-} h (\vec{w}_{\perp} \cdot \nabla_x \zeta) d\gamma \\
&\leq C\epsilon \|\nabla_x \zeta\|_{L^{\frac{4m}{4m-3}}(\Gamma)} \left( \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)} + \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} \right) \\
&\leq C\epsilon \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^{2m-1} \left( \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)} + \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} \right).
\end{aligned}$$

Hence, we know

$$\epsilon \int_{\Gamma} u \phi d\gamma \leq C\epsilon \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^{2m-1} \left( \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)} + \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} \right). \quad (\text{A.1.15})$$

Also, we have

$$\iint_{\Omega \times \mathcal{S}^2} (u - \bar{u}) \phi \leq C \|\phi\|_{L^2(\Omega \times \mathcal{S}^2)} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} \leq C \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}, \quad (\text{A.1.16})$$

$$\iint_{\Omega \times \mathcal{S}^2} f \phi \leq C \|\phi\|_{L^2(\Omega \times \mathcal{S}^2)} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} \leq C \|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1} \|f\|_{L^2(\Omega \times \mathcal{S}^2)}. \quad (\text{A.1.17})$$

Collecting terms in (A.1.10), (A.1.11), (A.1.15), (A.1.16) and (A.1.17), we obtain

$$\begin{aligned} \epsilon \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} &\leq C \left( \epsilon \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} + \|f\|_{L^2(\Omega \times \mathcal{S}^2)} \right. \\ &\quad \left. + \epsilon \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)} + \epsilon \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} \right), \end{aligned} \quad (\text{A.1.18})$$

Step 2: Energy Estimate.

In the weak formulation (A.1.2), we may take the test function  $\phi = u$  to get the energy estimate

$$\frac{1}{2} \epsilon \int_{\Gamma} |u|^2 d\gamma + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 = \iint_{\Omega \times \mathcal{S}^2} f u. \quad (\text{A.1.19})$$

Hence, by  $L^2$  estimate as in Theorem A.1 and [18], this naturally implies

$$\epsilon \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \leq \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + \iint_{\Omega \times \mathcal{S}^2} f u + \|h\|_{L^2(\Gamma^-)}^2. \quad (\text{A.1.20})$$

On the other hand, we can square on both sides of (A.1.18) to obtain

$$\begin{aligned} \epsilon^2 \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 &\leq C \left( \epsilon^2 \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \right. \\ &\quad \left. + \epsilon^2 \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)}^2 + \epsilon^2 \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)}^2 \right), \end{aligned} \quad (\text{A.1.21})$$

Multiplying a sufficiently small constant on both sides of (A.1.21) and adding it to (A.1.20) to absorb  $\|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2$  and  $\epsilon^2 \|\bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2$ , we deduce

$$\begin{aligned} &\epsilon \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + \epsilon^2 \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \\ &\leq C \left( \epsilon^2 \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 + \epsilon^2 \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)}^2 \right. \\ &\quad \left. + \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + \iint_{\Omega \times \mathcal{S}^2} f u + \|h\|_{L^2(\Gamma^-)}^2 + \epsilon^2 \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)}^2 \right). \end{aligned} \quad (\text{A.1.22})$$

By interpolation estimate and Young's inequality, for  $\frac{3}{2} \leq m \leq 3$ , we have

$$\begin{aligned} \|(1 - \mathcal{P})[u]\|_{L^{\frac{4m}{3}}(\Gamma^+)} &\leq \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^{\frac{3}{2m}} \|(1 - \mathcal{P})[u]\|_{L^\infty(\Gamma^+)}^{\frac{2m-3}{2m}} \\ &= \left( \frac{1}{\epsilon^{\frac{6m-9}{4m^2}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^{\frac{3}{2m}} \right) \left( \epsilon^{\frac{6m-9}{4m^2}} \|(1 - \mathcal{P})[u]\|_{L^\infty(\Gamma^+)}^{\frac{2m-3}{2m}} \right) \\ &\leq C \left( \frac{1}{\epsilon^{\frac{6m-9}{4m^2}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^{\frac{3}{2m}} \right)^{\frac{2m}{3}} + o(1) \left( \epsilon^{\frac{6m-9}{4m^2}} \|(1 - \mathcal{P})[u]\|_{L^\infty(\Gamma^+)}^{\frac{2m-3}{2m}} \right)^{\frac{2m}{2m-3}} \\ &\leq \frac{C}{\epsilon^{\frac{2m-3}{2m}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)} + o(1) \epsilon^{\frac{3}{2m}} \|(1 - \mathcal{P})[u]\|_{L^\infty(\Gamma^+)} \\ &\leq \frac{C}{\epsilon^{\frac{2m-3}{2m}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)} + o(1) \epsilon^{\frac{3}{2m}} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}. \end{aligned} \quad (\text{A.1.23})$$

Similarly, we have

$$\begin{aligned} \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} &\leq \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^{\frac{1}{m}} \|u - \bar{u}\|_{L^\infty(\Omega \times \mathcal{S}^2)}^{\frac{m-1}{m}} \\ &= \left( \frac{1}{\epsilon^{\frac{3m-3}{2m^2}}} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^{\frac{1}{m}} \right) \left( \epsilon^{\frac{3m-3}{2m^2}} \|u - \bar{u}\|_{L^\infty(\Omega \times \mathcal{S}^2)}^{\frac{m-1}{m}} \right) \\ &\leq C \left( \frac{1}{\epsilon^{\frac{3m-3}{2m^2}}} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^{\frac{1}{m}} \right)^m + o(1) \left( \epsilon^{\frac{3m-3}{2m^2}} \|u - \bar{u}\|_{L^\infty(\Omega \times \mathcal{S}^2)}^{\frac{m-1}{m}} \right)^{\frac{m}{m-1}} \\ &\leq \frac{C}{\epsilon^{\frac{3m-3}{2m}}} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} + o(1) \epsilon^{\frac{3}{2m}} \|u - \bar{u}\|_{L^\infty(\Omega \times \mathcal{S}^2)}. \end{aligned} \quad (\text{A.1.24})$$

We need this extra  $\epsilon^{\frac{3}{2m}}$  for the convenience of  $L^\infty$  estimate. Then we know for sufficiently small  $\epsilon$  and  $\frac{3}{2} \leq m \leq 3$ ,

$$\epsilon^2 \|(1 - \mathcal{P})[u]\|_{L^m(\Gamma^+)}^2 \leq C\epsilon^{2 - \frac{2m-3}{m}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Gamma^+)}^2 \quad (\text{A.1.25})$$

$$\leq \epsilon^{\frac{3}{m}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Gamma^+)}^2, \quad (\text{A.1.26})$$

$$\leq o(1)\epsilon \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Gamma^+)}^2.$$

Similarly, we have

$$\epsilon^2 \|u - \bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 \leq \epsilon^{2 - \frac{3m-3}{m}} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}^2 \quad (\text{A.1.27})$$

$$\leq \epsilon^{\frac{3}{m} - 1} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}^2, \quad (\text{A.1.28})$$

$$\leq o(1) \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}^2.$$

By (A.1.20), we can absorb  $\|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}$  and  $\epsilon \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}$  into left-hand side to obtain

$$\epsilon \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + \epsilon^2 \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \quad (\text{A.1.29})$$

$$\leq C \left( o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}^2 + \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + \iint_{\Omega \times \mathcal{S}^2} fu + \|h\|_{L^2(\Gamma^-)}^2 + \epsilon^2 \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)}^2 \right).$$

We can decompose

$$\iint_{\Omega \times \mathcal{S}^2} fu = \iint_{\Omega \times \mathcal{S}^2} f\bar{u} + \iint_{\Omega \times \mathcal{S}^2} f(u - \bar{u}). \quad (\text{A.1.30})$$

Hölder's inequality and Cauchy's inequality imply

$$\iint_{\Omega \times \mathcal{S}^2} f\bar{u} \leq \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} \leq \frac{C}{\epsilon^2} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)}^2 + o(1)\epsilon^2 \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2, \quad (\text{A.1.31})$$

and

$$\iint_{\Omega \times \mathcal{S}^2} f(u - \bar{u}) \leq C \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + o(1) \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2. \quad (\text{A.1.32})$$

Hence, absorbing  $\epsilon^2 \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2$  and  $\|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2$  into left-hand side of (A.1.29), we get

$$\epsilon \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)}^2 + \epsilon^2 \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}^2 + \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)}^2 \quad (\text{A.1.33})$$

$$\leq C \left( o(1)\epsilon^{2 + \frac{3}{m}} \|u\|_{L^\infty(\Gamma^+)}^2 + \|f\|_{L^2(\Omega \times \mathcal{S}^2)}^2 + \frac{1}{\epsilon^2} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)}^2 + \|h\|_{L^2(\Gamma^-)}^2 + \epsilon^2 \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)}^2 \right),$$

which implies

$$\frac{1}{\epsilon^{\frac{1}{2}}} \|(1 - \mathcal{P})[u]\|_{L^2(\Gamma^+)} + \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|u - \bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} \quad (\text{A.1.34})$$

$$\leq C \left( o(1)\epsilon^{\frac{3}{2m}} \|u\|_{L^\infty(\Gamma^+)} + \frac{1}{\epsilon} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^2} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon} \|h\|_{L^2(\Gamma^-)} + \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} \right).$$

□

**A.2.  $L^\infty$  Estimate.** In this subsection, we prove the  $L^\infty$  estimate. We consider the characteristics that reflect several times on the boundary.

**Definition A.4.** (*Stochastic Cycle*) For fixed point  $(t, \vec{x}, \vec{w})$  with  $(\vec{x}, \vec{w}) \notin \Gamma^0$ , let  $(t_0, \vec{x}_0, \vec{w}_0) = (0, \vec{x}, \vec{w})$ . For  $\vec{w}_{k+1}$  such that  $\vec{w}_{k+1} \cdot \vec{v}(\vec{x}_{k+1}) > 0$ , define the  $(k+1)$ -component of the back-time cycle as

$$(t_{k+1}, \vec{x}_{k+1}, \vec{w}_{k+1}) = (t_k + t_b(\vec{x}_k, \vec{w}_k), \vec{x}_b(\vec{x}_k, \vec{w}_k), \vec{w}_{k+1}) \quad (\text{A.2.1})$$

where

$$t_b(\vec{x}, \vec{w}) = \inf\{t > 0 : \vec{x} - \epsilon t \vec{w} \notin \Omega\} \quad (\text{A.2.2})$$

$$x_b(\vec{x}, \vec{w}) = \vec{x} - \epsilon t_b(\vec{x}, \vec{w}) \vec{w} \notin \Omega \quad (\text{A.2.3})$$

Set

$$X_{cl}(s; t, \vec{x}, \vec{w}) = \sum_k \mathbf{1}_{t_{k+1} \leq s < t_k} \left( \vec{x}_k - \epsilon(t_k - s) \vec{w}_k \right) \quad (\text{A.2.4})$$

$$W_{cl}(s; t, \vec{x}, \vec{w}) = \sum_k \mathbf{1}_{t_{k+1} \leq s < t_k} \vec{w}_k \quad (\text{A.2.5})$$

Define  $\mu_{k+1} = \{\vec{w} \in \mathcal{S}^2 : \vec{w} \cdot \vec{\nu}(\vec{x}_{k+1}) > 0\}$ , and let the iterated integral for  $k \geq 2$  be defined as

$$\int_{\prod_{j=1}^{k-1} \mu_j} \prod_{j=1}^{k-1} d\sigma_j = \int_{\mu_1} \dots \left( \int_{\mu_{k-1}} d\sigma_{k-1} \right) \dots d\sigma_1 \quad (\text{A.2.6})$$

where  $d\sigma_j = (\vec{\nu}(\vec{x}_j) \cdot \vec{w}) d\vec{w}$  is a probability measure.

**Lemma A.5.** For  $T_0 > 0$  sufficiently large, there exists constants  $C_1, C_2 > 0$  independent of  $T_0$ , such that for  $k = C_1 T_0^{5/4}$ ,

$$\int_{\prod_{j=1}^{k-1} \mu_j} \mathbf{1}_{t_k(t, \vec{x}, \vec{w}, \vec{w}_1, \dots, \vec{w}_{k-1}) < T_0} \prod_{j=1}^{k-1} d\sigma_j \leq \left( \frac{1}{2} \right)^{C_2 T_0^{5/4}} \quad (\text{A.2.7})$$

*Proof.* See [14, Lemma 4.1]. □

**Theorem A.6.** Assume  $f(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  and  $h(x_0, \vec{w}) \in L^\infty(\Gamma^-)$ . Then for the steady neutron transport equation (A.1), there exists a unique solution  $u(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  satisfying

$$\begin{aligned} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} &\leq C \left( \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{2+\frac{3}{2m}}} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} \right) \\ &\quad + \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{3}{2m}}} \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}. \end{aligned} \quad (\text{A.2.8})$$

*Proof.* We divide the proof into several steps:

Step 1: Mild formulation.

We rewrite the equation (A.1) along the characteristics as

$$\begin{aligned} u(\vec{x}, \vec{w}) &= h(\vec{x} - \epsilon t_1 \vec{w}, \vec{w}) e^{-t_1} + \mathcal{P}[u](\vec{x} - \epsilon t_1 \vec{w}, \vec{w}) e^{-t_1} \\ &\quad + \int_0^{t_1} f(\vec{x} - \epsilon(t_1 - s_1) \vec{w}, \vec{w}) e^{-(t_1 - s_1)} ds_1 + \int_0^{t_1} \bar{u}(\vec{x} - \epsilon(t_1 - s_1) \vec{w}) e^{-(t_1 - s_1)} ds_1. \end{aligned} \quad (\text{A.2.9})$$

Note that here  $\mathcal{P}[u]$  is an integral over  $\mu_1$  at  $\vec{x}_1$ , using stochastic cycle, we may rewrite it again along the characteristics to  $\vec{x}_2$ . This process can continue to arbitrary  $\vec{x}_k$ . Then we get

$$\begin{aligned} u(\vec{x}, \vec{w}) &= e^{-t_1} H + \sum_{l=1}^{k-1} \left( \int_{\prod_{j=1}^l \mu_j} e^{-t_{l+1}} G \prod_{j=1}^l d\sigma_j \right) + \sum_{l=1}^{k-1} \left( \int_{\prod_{j=1}^l \mu_j} e^{-t_{l+1}} \mathcal{P}[u](\vec{x}_k, \vec{w}_{k-1}) \prod_{j=1}^l d\sigma_j \right) \\ &= I + II + III. \end{aligned} \quad (\text{A.2.10})$$

where

$$\begin{aligned} H &= h(\vec{x} - \epsilon t_1 \vec{w}, \vec{w}) \\ &\quad + \int_0^{t_1} f(\vec{x} - \epsilon(t_1 - s_1) \vec{w}, \vec{w}) e^{s_1} ds_1 + \int_0^{t_1} \bar{u}(\vec{x} - \epsilon(t_1 - s_1) \vec{w}) e^{s_1} ds_1, \end{aligned} \quad (\text{A.2.11})$$

$$\begin{aligned} G &= h(\vec{x}_l - \epsilon t_{l+1} \vec{w}_l, \vec{w}_l) \\ &\quad + \int_0^{t_l} f(\vec{x}_l - \epsilon(t_{l+1} - s_{l+1}) \vec{w}_l, \vec{w}_l) e^{s_{l+1}} ds_{l+1} + \int_0^{t_l} \bar{u}(\vec{x}_l - \epsilon(t_{l+1} - s_{l+1}) \vec{w}_l) e^{s_{l+1}} ds_{l+1}. \end{aligned} \quad (\text{A.2.12})$$

We need to estimate each term on the right-hand side of (A.2.10).

Step 2: Estimate of mild formulation.

We first consider  $III$ . We may decompose it as

$$\begin{aligned}
III &= \sum_{l=1}^{k-1} \int_{\prod_{j=1}^l} \mathcal{P}[u](\vec{x}_k, \vec{w}_{k-1}) e^{-t_{l+1}} \prod_{j=1}^l d\sigma_j \\
&= \sum_{l=1}^{k-1} \int_{\prod_{j=1}^l} \mathbf{1}_{t_k \leq T_0} \mathcal{P}[u](\vec{x}_k, \vec{w}_{k-1}) e^{-t_{l+1}} \prod_{j=1}^l d\sigma_j \\
&\quad + \sum_{l=1}^{k-1} \int_{\prod_{j=1}^l} \mathbf{1}_{t_k \geq T_0} \mathcal{P}[u](\vec{x}_k, \vec{w}_{k-1}) e^{-t_{l+1}} \prod_{j=1}^l d\sigma_j, \\
&= III_1 + III_2,
\end{aligned} \tag{A.2.13}$$

where  $T_0 > 0$  is defined as in Lemma A.5. Then we take  $k = C_1 T_0^{5/4}$ . By Lemma A.5, we deduce

$$|III_1| \leq C \left(\frac{1}{2}\right)^{C_2 T_0^{5/4}} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}. \tag{A.2.14}$$

Also, we may directly estimate

$$|III_2| \leq C e^{-T_0} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}. \tag{A.2.15}$$

Then taking  $T_0$  sufficiently large, we know

$$|III| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}, \tag{A.2.16}$$

for  $\delta > 0$  small. On the other hand, we may directly estimate the terms in  $I$  and  $II$  related to  $h$  and  $f$ , which we denote as  $I_1$  and  $II_1$ . For fixed  $T$ , it is easy to see

$$|I_1| + |II_1| \leq \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)}. \tag{A.2.17}$$

Step 3: Estimate of  $\bar{u}$  term.

The most troubling terms are related to  $\bar{u}$ . Here, we use the trick as in [14] and [27]. Collecting the results in (A.2.16) and (A.2.17), we obtain

$$\begin{aligned}
|u| &\leq A + \left| \int_0^{t_1} \bar{u}(\vec{x} - \epsilon(t_1 - s_1)\vec{w}) e^{-(t_1 - s_1)} ds_1 \right| \\
&\quad + \left| \sum_{l=1}^{k-1} \left( \int_{\prod_{j=1}^l} \left( \int_0^{t_l} \bar{u}(\vec{x}_l - \epsilon(t_{l+1} - s_{l+1})\vec{w}_l) e^{-(t_{l+1} - s_{l+1})} ds_{l+1} \right) \prod_{j=1}^l d\sigma_j \right) \right|, \\
&= A + I_2 + II_2,
\end{aligned} \tag{A.2.18}$$

where

$$A = \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)} + \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}. \tag{A.2.19}$$

By definition, we know

$$|I_2| = \left| \int_0^{t_1} \left( \int_{\mathcal{S}^2} u(\vec{x} - \epsilon(t_1 - s_1)\vec{w}, \vec{w}_{s_1}) d\vec{w}_{s_1} \right) e^{-(t_1 - s_1)} ds_1 \right|, \tag{A.2.20}$$

where  $\vec{w}_{s_1} \in \mathcal{S}^2$  is a dummy variable. Then we can utilize the mild formulation (A.2.10) to rewrite  $u(\vec{x} - \epsilon(t_1 - s_1)\vec{w}, \vec{w}_{s_1})$  along the characteristics. We denote the stochastic cycle as  $(t'_k, \vec{x}'_k, \vec{w}'_k)$  correspondingly

and  $(t'_0, \vec{x}'_0, \vec{w}'_0) = (0, \vec{x} - \epsilon(t_1 - s_1)\vec{w}, \vec{w}_{s_1})$ . Then

$$\begin{aligned}
|I_2| &\leq \left| \int_0^{t_1} \left( \int_{\mathcal{S}^2} \text{Ad}\vec{w}_{s_1} \right) e^{-(t_1-s_1)} ds_1 \right| \\
&\quad + \left| \int_0^{t_1} \left( \int_{\mathcal{S}^2} \int_0^{t'_1} \bar{u}(\vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1}) e^{-(t'_1-s'_1)} ds'_1 d\vec{w}_{s_1} \right) e^{-(t_1-s_1)} ds_1 \right| \\
&\quad + \left| \int_0^{t_1} \left( \int_{\mathcal{S}^2} \sum_{l'=1}^{k-1} \int_{\Pi'_{j'=1}} \left( \int_0^{t'_{l'}} \bar{u}(\vec{x}'_{l'} - \epsilon(t'_{l'+1} - s'_{l'+1})\vec{w}_{l'}) e^{-(t'_{l'+1}-s'_{l'+1})} ds'_{l'+1} \right) \prod_{j'=1}^{l'} d\sigma_{j'} d\vec{w}_{s_1} \right) \right. \\
&\quad \left. e^{-(t_1-s_1)} ds_1 \right|, \\
&= |I_{2,1}| + |I_{2,2}| + |I_{2,3}|.
\end{aligned} \tag{A.2.21}$$

It is obvious that

$$\begin{aligned}
|I_{2,1}| &= \left| \int_0^{t_1} \left( \int_{\mathcal{S}^2} \text{Ad}\vec{w}_{s_1} \right) e^{-(t_1-s_1)} ds_1 \right| \leq A \\
&\leq \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)} + \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}.
\end{aligned} \tag{A.2.22}$$

Then by definition, we know

$$|I_{2,2}| = \left| \int_0^{t_1} \left( \int_{\mathcal{S}^2} \int_0^{t'_1} \bar{u}(\vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1}) e^{-(t'_1-s'_1)} ds'_1 d\vec{w}_{s_1} \right) e^{-(t_1-s_1)} ds_1 \right|. \tag{A.2.23}$$

We may decompose this integral

$$\int_0^{t_1} \int_{\mathcal{S}^2} \int_0^{t'_1} = \int_0^{t_1} \int_{\mathcal{S}^2} \int_{t'_1-s'_1 \leq \delta} + \int_0^{t_1} \int_{\mathcal{S}^2} \int_{t'_1-s'_1 \geq \delta} = I_{2,2,1} + I_{2,2,2}. \tag{A.2.24}$$

For  $I_{2,2,1}$ , since the integral is defined in the small domain  $[t'_1 - \delta, t'_1]$ , it is easy to see

$$|I_{2,2,1}| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}. \tag{A.2.25}$$

For  $I_{2,2,2}$ , applying Hölder's inequality, we get

$$\begin{aligned}
|I_{2,2,2}| &\leq \left| \int_0^{t_1} \int_{\mathcal{S}^2} \int_{t'_1-s'_1 \geq \delta} \bar{u}(\vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1}) e^{-(t'_1-s'_1)} e^{-(t_1-s_1)} ds'_1 d\vec{w}_{s_1} ds_1 \right| \\
&\leq \left( \int_0^{t_1} \int_{\mathcal{S}^2} \int_{t'_1-s'_1 \geq \delta} e^{-(t'_1-s'_1)} e^{-(t_1-s_1)} ds'_1 d\vec{w}_{s_1} ds_1 \right)^{\frac{2m-1}{2m}} \\
&\quad \left( \int_0^{t_1} \int_{\mathcal{S}^2} \int_{t'_1-s'_1 \geq \delta} \mathbf{1}_{\vec{x}' - \epsilon(t'_1-s'_1)\vec{w}_{s_1} \in \Omega} |\bar{u}|^{2m} (\vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1}) e^{-(t'_1-s'_1)} e^{-(t_1-s_1)} ds'_1 d\vec{w}_{s_1} ds_1 \right)^{\frac{1}{2m}} \\
&\leq \left( \int_0^{t_1} \int_{\mathcal{S}^2} \int_{t'_1-s'_1 \geq \delta} \mathbf{1}_{\vec{x}' - \epsilon(t'_1-s'_1)\vec{w}_{s_1} \in \Omega} |\bar{u}|^{2m} (\vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1}) e^{-(t'_1-s'_1)} e^{-(t_1-s_1)} ds'_1 d\vec{w}_{s_1} ds_1 \right)^{\frac{1}{2m}}.
\end{aligned} \tag{A.2.26}$$

Since  $\vec{w}_{s_1} \in \mathcal{S}^2$ , we can express it as  $(\sin \phi \cos \psi, \sin \phi \sin \psi, \cos \phi)$ . Then considering  $\vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1} \in \bar{\Omega}$ , we apply the substitution  $(\phi, \psi, s'_1) \rightarrow (y_1, y_2, y_3)$  as

$$\vec{y} = \vec{x}' - \epsilon(t'_1 - s'_1)\vec{w}_{s_1}, \tag{A.2.27}$$

whose Jacobian is

$$\left| \frac{\partial(y_1, y_2, y_3)}{\partial(\phi, \psi, s'_1)} \right| = \left\| \begin{array}{ccc} -\epsilon(t'_1 - s'_1) \cos \phi \cos \psi & \epsilon(t'_1 - s'_1) \sin \phi \sin \psi & \epsilon \sin \phi \cos \psi \\ -\epsilon(t'_1 - s'_1) \cos \phi \sin \psi & -\epsilon(t'_1 - s'_1) \sin \phi \cos \psi & \epsilon \sin \phi \sin \psi \\ \epsilon(t'_1 - s'_1) \sin \phi & 0 & \epsilon \cos \phi \end{array} \right\| \tag{A.2.28}$$

$$= \epsilon^3 (t'_1 - s'_1)^2 \sin \phi. \tag{A.2.29}$$

Hence, we can further decompose  $I_{2,2,2}$  into  $|\sin \phi| \leq \delta$  and  $|\sin \phi| \geq \delta$ . In the first part, the integral can be bounded by  $\delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}$ . Then for the second part, we have

$$\left| \frac{\partial(y_1, y_2, y_3)}{\partial(\phi, \psi, r')} \right| \geq \epsilon^3 \delta^3. \quad (\text{A.2.30})$$

Therefore, we know

$$|I_{2,2,2}| \leq \frac{1}{\epsilon^{\frac{3}{2m}} \delta^{\frac{3}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}. \quad (\text{A.2.31})$$

Hence, we have shown

$$|I_{2,2}| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{\frac{3}{2m}} \delta^{\frac{3}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}. \quad (\text{A.2.32})$$

After a similar but tedious computation, we can show

$$|I_{2,3}| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{\frac{3}{2m}} \delta^{\frac{3}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)}. \quad (\text{A.2.33})$$

Hence, we have proved

$$|I_2| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{\frac{3}{2m}} \delta^{\frac{3}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)}. \quad (\text{A.2.34})$$

In a similar fashion, we can show

$$|II_2| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{\frac{3}{2m}} \delta^{\frac{3}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)}. \quad (\text{A.2.35})$$

Step 4: Synthesis.

Summarizing all above, we have shown

$$\begin{aligned} |u| &\leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \frac{1}{\delta^{\frac{3}{2m}} \epsilon^{\frac{3}{2m}}} \|\bar{u}\|_{L^2(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)} \\ &\leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \frac{1}{\delta^{\frac{3}{2m}} \epsilon^{\frac{3}{2m}}} \|u\|_{L^2(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|h\|_{L^\infty(\Gamma^-)}. \end{aligned} \quad (\text{A.2.36})$$

Since  $(\vec{x}, \vec{w})$  are arbitrary and  $\delta$  is small, taking supremum on both sides and applying Lemma A.1, we have

$$\|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C \left( \frac{1}{\epsilon^{\frac{3}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} + \|g\|_{L^\infty(\Gamma^-)} \right). \quad (\text{A.2.37})$$

Considering Theorem A.3, we obtain

$$\begin{aligned} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} &\leq C \left( \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{2+\frac{3}{2m}}} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} \right. \\ &\quad \left. + \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{3}{2m}}} \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)} \right) + o(1) \|u\|_{L^\infty(\Gamma^+)}. \end{aligned} \quad (\text{A.2.38})$$

Absorbing  $\|u\|_{L^\infty(\Omega \times \mathcal{S}^2)}$  into the left-hand side, we obtain

$$\begin{aligned} \|u\|_{L^\infty(\Omega \times \mathcal{S}^2)} &\leq C \left( \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|f\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{2+\frac{3}{2m}}} \|f\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} + \|f\|_{L^\infty(\Omega \times \mathcal{S}^2)} \right. \\ &\quad \left. + \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{3}{2m}}} \|h\|_{L^{\frac{4m}{3}}(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)} \right). \end{aligned} \quad (\text{A.2.39})$$

This is the desired estimate.  $\square$



## APPENDIX B. DIFFUSIVE LIMIT

**Corollary B.1.** *Assume  $g(\vec{x}_0, \vec{w}) \in C^2(\Gamma^-)$  satisfying (1.5). Then for the steady neutron transport equation (1.1), there exists a unique solution  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$  satisfying (1.4). Moreover, for any  $0 < \delta < 1$ , the solution obeys the estimate*

$$\|u^\epsilon - U_0^\epsilon\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C(\delta, \Omega)\epsilon^{\frac{1}{3}-\delta}, \quad (\text{B.1})$$

where  $U_0^\epsilon$  is defined in (2.53).

*Proof.* Based on Theorem A.6, we know there exists a unique  $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^2)$ , so we focus on the diffusive limit. We can divide the proof into several steps:

Step 1: Remainder definitions.

We define the remainder as

$$R = u^\epsilon - \sum_{k=0}^2 \epsilon^k U_k^\epsilon - \sum_{k=0}^1 \epsilon^k \mathcal{W}_k^\epsilon = u^\epsilon - Q - \mathcal{Q}, \quad (\text{B.2})$$

where

$$Q = U_0^\epsilon + \epsilon U_1^\epsilon + \epsilon^2 U_2^\epsilon, \quad (\text{B.3})$$

$$\mathcal{Q} = \mathcal{W}_0^\epsilon + \epsilon \mathcal{W}_1^\epsilon. \quad (\text{B.4})$$

Noting the equation (2.30) is equivalent to the equation (1.1), we write  $\mathcal{L}$  to denote the neutron transport operator as follows:

$$\begin{aligned} \mathcal{L}[u] &= \epsilon \vec{w} \cdot \nabla_x u + u - \bar{u} \\ &= \sin \phi \frac{\partial u}{\partial \eta} - \epsilon \left( \frac{\sin^2 \psi}{R_1 - \epsilon \eta} + \frac{\cos^2 \psi}{R_2 - \epsilon \eta} \right) \cos \phi \frac{\partial u}{\partial \phi} + u - \bar{u} + G[u], \end{aligned} \quad (\text{B.5})$$

where

$$\begin{aligned} G[u] &= \epsilon \left( \frac{\cos \phi \sin \psi}{P_1(1 - \epsilon \kappa_1 \eta)} \frac{\partial u}{\partial \tau_1} + \frac{\cos \phi \cos \psi}{P_2(1 - \epsilon \kappa_2 \eta)} \frac{\partial u}{\partial \tau_2} \right) \\ &\quad + \epsilon \left( \frac{\sin \psi}{1 - \epsilon \kappa_1 \eta} (\vec{t}_2 \times (\partial_{21} \vec{r} \times \vec{t}_2)) \cdot \vec{t}_2 + \frac{\cos \psi}{1 - \epsilon \kappa_2 \eta} (\vec{t}_1 \times (\partial_{12} \vec{r} \times \vec{t}_1)) \cdot \vec{t}_1 \right) \frac{\cos \phi}{P_1 P_2} \frac{\partial u}{\partial \psi}. \end{aligned} \quad (\text{B.6})$$

Step 2: Estimates of  $\mathcal{L}[Q]$ .

The interior contribution can be estimated as

$$\mathcal{L}[Q] = \epsilon \vec{w} \cdot \nabla_x Q + Q - \bar{Q} = \epsilon^3 \vec{w} \cdot \nabla_x U_2^\epsilon. \quad (\text{B.7})$$

Based on classical elliptic estimates, we have

$$\|\mathcal{L}[Q]\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq \|\epsilon^3 \vec{w} \cdot \nabla_x U_2^\epsilon\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C \epsilon^3 \|\nabla_x U_2^\epsilon\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C \epsilon^3. \quad (\text{B.8})$$

This implies

$$\|\mathcal{L}[Q]\|_{L^2(\Omega \times \mathcal{S}^2)} \leq C \epsilon^3, \quad (\text{B.9})$$

$$\|\mathcal{L}[Q]\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} \leq C \epsilon^3, \quad (\text{B.10})$$

$$\|\mathcal{L}[Q]\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C \epsilon^3. \quad (\text{B.11})$$

Step 3: Estimates of  $\mathcal{L}\mathcal{Q}$ .

Since  $\mathcal{W}_0^\epsilon = 0$ , we only need to estimate  $\mathcal{W}_1^\epsilon = (f_1^\epsilon - f_{1,L}^\epsilon) \cdot \psi_0 = \mathcal{V} \psi_0$  where  $f_1^\epsilon(\eta, \tau_1, \tau_2, \phi, \psi)$  solves the

$\epsilon$ -Milne problem and  $\mathcal{V} = f_1^\epsilon - f_{1,L}^\epsilon$ . The boundary layer contribution can be estimated as

$$\begin{aligned}
\mathcal{L}[\epsilon\mathcal{U}_1^\epsilon] &= \sin\phi \frac{\partial(\epsilon\mathcal{U}_1^\epsilon)}{\partial\eta} - \epsilon \left( \frac{\sin^2\psi}{R_1 - \epsilon\eta} + \frac{\cos^2\psi}{R_2 - \epsilon\eta} \right) \cos\phi \frac{\partial(\epsilon\mathcal{U}_1^\epsilon)}{\partial\phi} + (\epsilon\mathcal{U}_1^\epsilon) - (\epsilon\bar{\mathcal{U}}\epsilon_1) + G[\epsilon\mathcal{U}_1^\epsilon] \\
&= \epsilon \left( \sin\phi \left( \psi_0 \frac{\partial\mathcal{V}}{\partial\eta} + \mathcal{V} \frac{\partial\psi_0}{\partial\eta} \right) - \epsilon\psi_0 \left( \frac{\sin^2\psi}{R_1 - \epsilon\eta} + \frac{\cos^2\psi}{R_2 - \epsilon\eta} \right) \cos\phi \frac{\partial\mathcal{V}}{\partial\phi} + \frac{\partial\mathcal{V}}{\partial\tau} + \psi_0\mathcal{V} - \psi_0\bar{\mathcal{V}} + \psi_0G[\mathcal{V}] \right) \\
&= \epsilon\psi_0 \left( \sin\phi \frac{\partial\mathcal{V}}{\partial\eta} - \epsilon \left( \frac{\sin^2\psi}{R_1 - \epsilon\eta} + \frac{\cos^2\psi}{R_2 - \epsilon\eta} \right) \cos\phi \frac{\partial\mathcal{V}}{\partial\phi} + \mathcal{V} - \bar{\mathcal{V}} \right) + \epsilon \left( \sin\phi \frac{\partial\psi_0}{\partial\eta} \mathcal{V} - \epsilon\psi_0G[\mathcal{V}] \right) \\
&= \epsilon \left( \sin\phi \frac{\partial\psi_0}{\partial\eta} \mathcal{V} - \epsilon\psi_0G[\mathcal{V}] \right).
\end{aligned} \tag{B.12}$$

Since  $\psi_0 = 1$  when  $\eta \leq R_{\min}/(4\epsilon^n)$ , the effective region of  $\partial_\eta\psi_0$  is  $\eta \geq R_{\min}/(4\epsilon^n)$  which is further and further from the origin as  $\epsilon \rightarrow 0$ . By Theorem 3.11, the first term in (B.12) can be bounded as

$$\left\| \epsilon \sin\phi \frac{\partial\psi_0}{\partial\eta} \mathcal{V} \right\|_{L^\infty(\Omega \times S^2)} \leq C\epsilon e^{-\frac{K_0}{\epsilon^n}} \leq C\epsilon^3. \tag{B.13}$$

Then we turn to the crucial estimate in the second term of (B.12), by Theorem 4.13, we have

$$\begin{aligned}
\|\epsilon\psi_0G[\mathcal{V}]\|_{L^\infty(\Omega \times S^2)} &\leq C\epsilon^2 \left( \left\| \frac{\partial\mathcal{V}}{\partial\tau_1} \right\|_{L^\infty(\Omega \times S^2)} + \left\| \frac{\partial\mathcal{V}}{\partial\tau_2} \right\|_{L^\infty(\Omega \times S^2)} + \left\| \frac{\partial\mathcal{V}}{\partial\psi} \right\|_{L^\infty(\Omega \times S^2)} \right) \\
&\leq C\epsilon^2 |\ln(\epsilon)|^8.
\end{aligned} \tag{B.14}$$

Also, the exponential decay of  $\frac{\partial\mathcal{V}}{\partial\tau}$  by Theorem 4.13 and the rescaling  $\eta = \mu/\epsilon$  implies

$$\begin{aligned}
\|\epsilon\psi_0G[\mathcal{V}]\|_{L^2(\Omega \times S^2)} &\leq \epsilon^2 \left( \left\| \frac{\partial\mathcal{V}}{\partial\tau_1} \right\|_{L^2(\Omega \times S^2)} + \left\| \frac{\partial\mathcal{V}}{\partial\tau_2} \right\|_{L^2(\Omega \times S^2)} + \left\| \frac{\partial\mathcal{V}}{\partial\psi} \right\|_{L^2(\Omega \times S^2)} \right) \\
&\leq \epsilon^2 \left( \int_{-\pi}^{\pi} \int_0^1 (1-\mu) \left( \left\| \frac{\partial\mathcal{V}}{\partial\tau_1}(\mu, \tau) \right\|_{L^\infty}^2 + \left\| \frac{\partial\mathcal{V}}{\partial\tau_2}(\mu, \tau) \right\|_{L^\infty}^2 + \left\| \frac{\partial\mathcal{V}}{\partial\psi}(\mu, \tau) \right\|_{L^\infty}^2 \right) d\mu d\tau \right)^{1/2} \\
&\leq \epsilon^{\frac{5}{2}} \left( \int_{-\pi}^{\pi} \int_0^{1/\epsilon} (1-\epsilon\eta) \left( \left\| \frac{\partial\mathcal{V}}{\partial\tau_1}(\eta, \tau) \right\|_{L^\infty}^2 + \left\| \frac{\partial\mathcal{V}}{\partial\tau_2}(\eta, \tau) \right\|_{L^\infty}^2 + \left\| \frac{\partial\mathcal{V}}{\partial\psi}(\eta, \tau) \right\|_{L^\infty}^2 \right) d\eta d\tau \right)^{1/2} \\
&\leq C\epsilon^{\frac{5}{2}} \left( \int_{-\pi}^{\pi} \int_0^{1/\epsilon} e^{-2K_0\eta} |\ln(\epsilon)|^{16} d\eta d\tau \right)^{1/2} \\
&\leq C\epsilon^{\frac{5}{2}} |\ln(\epsilon)|^8.
\end{aligned} \tag{B.15}$$

Similarly, we have

$$\|\epsilon\psi_0G[\mathcal{V}]\|_{L^{\frac{2m}{2m-1}}(\Omega \times S^2)} \leq C\epsilon^{3-\frac{1}{2m}} |\ln(\epsilon)|^8. \tag{B.16}$$

In total, we have

$$\|\mathcal{L}[\mathcal{Q}]\|_{L^2(\Omega \times S^2)} \leq C\epsilon^{\frac{5}{2}} |\ln(\epsilon)|^8, \tag{B.17}$$

$$\|\mathcal{L}[\mathcal{Q}]\|_{L^{\frac{2m}{2m-1}}(\Omega \times S^2)} \leq C\epsilon^{3-\frac{1}{2m}} |\ln(\epsilon)|^8, \tag{B.18}$$

$$\|\mathcal{L}[\mathcal{Q}]\|_{L^\infty(\Omega \times S^2)} \leq C\epsilon^2 |\ln(\epsilon)|^8. \tag{B.19}$$

Step 4: Diffusive Limit.

In summary, since  $\mathcal{L}[u^\epsilon] = 0$ , collecting estimates in Step 2 and Step 3, we can prove

$$\|\mathcal{L}[R]\|_{L^2(\Omega \times \mathcal{S}^2)} \leq C\epsilon^{\frac{5}{2}} |\ln(\epsilon)|^8, \quad (\text{B.20})$$

$$\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} \leq C\epsilon^{3-\frac{1}{2m}} |\ln(\epsilon)|^8, \quad (\text{B.21})$$

$$\|\mathcal{L}[R]\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C\epsilon^2 |\ln(\epsilon)|^8. \quad (\text{B.22})$$

Also, based on our construction, it is easy to see

$$R - \mathcal{P}[R] = -\epsilon^2(\vec{w} \cdot \nabla_x U_1^\epsilon - \mathcal{P}[\vec{w} \cdot \nabla_x U_1^\epsilon]), \quad (\text{B.23})$$

which further implies

$$\|R - \mathcal{P}[R]\|_{L^2(\Gamma^-)} \leq C\epsilon^2, \quad (\text{B.24})$$

$$\|R - \mathcal{P}[R]\|_{L^m(\Gamma^-)} \leq C\epsilon^2, \quad (\text{B.25})$$

$$\|R - \mathcal{P}[R]\|_{L^\infty(\Gamma^-)} \leq C\epsilon^2 \quad (\text{B.26})$$

Hence, the remainder  $R$  satisfies the equation

$$\begin{cases} \epsilon \vec{w} \cdot \nabla_x R + R - \bar{R} = \mathcal{L}[R] & \text{for } \vec{x} \in \Omega, \\ R - \mathcal{P}[R] = R - \mathcal{P}[R] & \text{for } \vec{w} \cdot \vec{\nu} < 0 \text{ and } \vec{x}_0 \in \partial\Omega. \end{cases} \quad (\text{B.27})$$

It is easy to verify  $R$  satisfies the normalization condition (A.4) and the data satisfies the compatibility condition (A.5). By Theorem A.6, we have for  $2 < m \leq 3$ ,

$$\begin{aligned} \|R\|_{L^\infty(\Omega \times \mathcal{S}^2)} &\leq C \left( \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|\mathcal{L}[R]\|_{L^2(\Omega \times \mathcal{S}^2)} + \frac{1}{\epsilon^{2+\frac{3}{2m}}} \|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega \times \mathcal{S}^2)} + \|\mathcal{L}[R]\|_{L^\infty(\Omega \times \mathcal{S}^2)} \right. \\ &\quad \left. + \frac{1}{\epsilon^{1+\frac{3}{2m}}} \|R - \mathcal{P}[R]\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{3}{2m}}} \|R - \mathcal{P}[R]\|_{L^{\frac{4m}{3}}(\Gamma^-)} + \|R - \mathcal{P}[R]\|_{L^\infty(\Gamma^-)} \right) \\ &\leq C \left( \frac{1}{\epsilon^{1+\frac{3}{2m}}} \left( \epsilon^{\frac{5}{2}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{2+\frac{3}{2m}}} \left( \epsilon^{3-\frac{1}{2m}} |\ln(\epsilon)|^8 \right) + \left( \epsilon^2 |\ln(\epsilon)|^8 \right) \right. \\ &\quad \left. + \frac{1}{\epsilon^{1+\frac{3}{2m}}} (\epsilon^2) + \frac{1}{\epsilon^{\frac{3}{2m}}} (\epsilon^2) + (\epsilon^2) \right) \\ &\leq C\epsilon^{1-\frac{2}{m}} |\ln(\epsilon)|^8 \leq C\epsilon^{\frac{1}{3}-\delta} \end{aligned} \quad (\text{B.29})$$

Note that the constant  $C$  might depend on  $m$  and thus depend on  $\delta$ . Since it is easy to see

$$\left\| \sum_{k=1}^2 \epsilon^k U_k^\epsilon + \sum_{k=0}^1 \epsilon^k \mathcal{U}_k^\epsilon \right\|_{L^\infty(\Omega \times \mathcal{S}^2)} \leq C\epsilon, \quad (\text{B.30})$$

our result naturally follows. This completes the proof of diffusive limit.  $\square$

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