GEOMETRIC CORRECTION IN DIFFUSIVE LIMIT OF NEUTRON TRANSPORT EQUATION IN 2D CONVEX DOMAINS

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Abstract. Consider the steady neutron transport equation with diffusive boundary condition. In [17] and [18], it was discovered that geometric correction is necessary for the Milne problem of Knudsen-layer construction in a disk or annulus. In this paper, we establish diffusive limit for a 2D convex domain. Our contribution relies on novel weighted $W^{1,\infty}$ estimates for the Milne problem with geometric correction in the presence of a convex domain, as well as an $L^{2m}-L^{\infty}$ framework which yields stronger remainder estimates.
Keywords: Geometric correction; $W^{1,\infty}$ estimates; $L^{2m}-L^{\infty}$ framework.

Contents

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1. Introduction

1.1. **Problem Formulation.** We consider the steady neutron transport equation in a two-dimensional bounded convex domain with diffusive boundary. In the spacial domain $\vec{x} = (x_1, x_2) \in \Omega$ where $\partial\Omega \in C^3$ and the velocity domain $\vec{w} = (w_1, w_2) \in \mathcal{S}^1$, the neutron density $u^\epsilon(\vec{x}, \vec{w})$ satisfies

$$\begin{cases} \epsilon\vec{w} \cdot \nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon &= 0 \ \text{ in } \ \Omega, \\ u^\epsilon(\vec{x}_0, \vec{w}) &= \mathcal{P}[u^\epsilon](\vec{x}_0) + \epsilon g(\vec{x}_0, \vec{w}) \ \text{ for } \ \vec{w} \cdot \vec{\nu} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\bar{u}^\epsilon(\vec{x}) = \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon(\vec{x}, \vec{w}) \mathrm{d}\vec{w}, \tag{1.2}$$

$$\mathcal{P}[u^\epsilon](\vec{x}_0) = \frac{1}{2} \int_{\vec{w}\cdot\vec{\nu}>0} u^\epsilon(\vec{x}_0, \vec{w})(\vec{w} \cdot \vec{\nu}) \mathrm{d}\vec{w}, \tag{1.3}$$

$\vec{\nu}$ is the outward unit normal vector, with the Knudsen number $0 < \epsilon << 1$. Also, u^ϵ satisfies the normalization condition

$$\int_{\Omega\times\mathcal{S}^1} u^\epsilon(\vec{x}, \vec{w}) \mathrm{d}\vec{w}\mathrm{d}\vec{x} = 0, \tag{1.4}$$

and g satisfies the compatibility condition

$$\int_{\partial\Omega} \int_{\vec{w}\cdot\vec{\nu}<0} g(\vec{x}_0, \vec{w})(\vec{w} \cdot \vec{\nu}) \mathrm{d}\vec{w}\mathrm{d}\vec{x}_0 = 0. \tag{1.5}$$

We intend to study the behavior of u^ϵ as $\epsilon \to 0$.

Based on the flow direction, we can divide the boundary $\Gamma = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega\}$ into the in-flow boundary Γ^-, the out-flow boundary Γ^+ and the grazing set Γ^0 as

$$\Gamma^- = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{\nu} < 0\} \tag{1.6}$$

$$\Gamma^+ = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{\nu} > 0\} \tag{1.7}$$

$$\Gamma^0 = \{(\vec{x}, \vec{w}) : \vec{x} \in \partial\Omega, \vec{w} \cdot \vec{\nu} = 0\} \tag{1.8}$$

It is easy to see $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$. Hence, the boundary condition is only given for Γ^-.

1.2. **Main Result.**

Theorem 1.1. *Assume $g(\vec{x}_0, \vec{w}) \in C^2(\Gamma^-)$ satisfying (1.5). Then for the steady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(\vec{x}, \vec{w}) \in L^\infty(\Omega \times \mathcal{S}^1)$ satisfying (1.4). Moreover, for any $0 < \delta << 1$, the solution obeys the estimate*

$$\|u^\epsilon(\vec{x}, \vec{w}) - U_0(\vec{x})\|_{L^\infty(\Omega\times\mathcal{S}^1)} \le C(\delta, \Omega)\epsilon^{1-\delta}, \tag{1.9}$$

where $U_0(\vec{x})$ satisfies

$$\begin{cases} \Delta_x U_0 &= 0 \ \ in \ \ \Omega, \\ \dfrac{\partial U_0}{\partial \vec{\nu}} &= \dfrac{1}{\pi} \displaystyle\int_{\vec{w}\cdot\vec{\nu}<0} g(\vec{x}, \vec{w}) \left|\vec{w} \cdot \vec{\nu}\right| \mathrm{d}\vec{w} \ \ on \ \ \partial\Omega, \\ \displaystyle\int_\Omega U_0(\vec{x}) \mathrm{d}\vec{x} &= 0, \end{cases} \tag{1.10}$$

in which $C(\delta, \Omega) > 0$ denotes a constant that depends on δ and Ω.

1.3. **Background and Methods.** Diffusive limit, or more general hydrodynamic limit, plays a key role in connecting kinetic theory and fluid mechanics. Since 1960s, this type of problems have been extensively studied in many different settings: steady or unsteady, linear or nonlinear, strong solution or weak solution, etc. We refer to the references [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16] for more details. Among all these variations, one of the simplest but most important models - steady neutron transport equation with one-speed velocity in bounded domains, where the boundary layer effect shows up, has long been believed to be satisfactorily solved since Bensoussan, Lions and Papanicolaou published their remarkable paper [1] in 1979.

Unfortunately, their results are shown to be false due to lack of regularity for the classical Milne problem in [17] and [18]. A new approach with geometric correction to the Milne problem has been developed to ensure regularity in the cases of disk and annulus in [17] and [18]. However, this new method fails to treat more general domains.

Consider the boundary layer expansion with geometric correction

$$\mathscr{U}^\epsilon(\eta,\tau,\phi)=\mathscr{U}_0^\epsilon(\eta,\tau,\phi)+\epsilon\mathscr{U}_1^\epsilon(\eta,\tau,\phi), \tag{1.11}$$

where η denotes the rescaled normal variable, τ the tangential variable and ϕ the velocity variable defined in (2.41), (2.46), (2.51), and (2.56). Thanks to the diffusive boundary condition, we always have $\mathscr{U}_0^\epsilon=0$. As [17] stated, the boundary layer must formally satisfy

$$\sin\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\eta}+\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\phi}+\mathscr{U}_1^\epsilon-\bar{\mathscr{U}}_1^\epsilon=0, \tag{1.12}$$

where R_κ is the radius of curvature at boundary.

In the absence of the geometric correction $\dfrac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\phi}$ as in [1], the key tangential derivative $\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\tau}$ is not bounded for such a classical Milne problem. Therefore, the expansion breaks down. In the case when R_κ is constant, as in [17] and [18], $\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\tau}$ is smooth, since the tangential derivative commutes with the equation. On the other hand, when R_κ is a function of τ, then $\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\tau}$ relates to the normal derivative $\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\eta}$, whose boundedness had remained open.

Our main contribution is to show $\dfrac{\partial\mathscr{U}_1^\epsilon}{\partial\tau}$ is bounded when R_κ is not a constant for a general convex domain. Our proof is intricate and lies on the weighted L^∞ estimates for the normal derivative. We use careful analysis along the characteristic curves in the presence of non-local averaging $\bar{\mathscr{U}}_1^\epsilon$ over ϕ. The convexity and invariant kinetic distance $\zeta(\eta,\tau,\phi)$ defined in (3.30), plays the crucial role. Our paper marks an important first step towards the study of diffusive expansions of neutron transport equations and other kinetic equations with boundary layer correction.

Moreover, we have to improve the remainder estimate to avoid higher-order expansion. This is done by a new L^{2m}-L^∞ framework. The main idea is to introduce a special test function in the weak formulation to treat kernel and non-kernel parts separately, and further improve the L^∞ estimate by a modified double Duhamel's principle. The proof relies on a delicate analysis using interpolation and Young's inequality.

Applying these two new techniques, we successfully obtain the diffusive limit of neutron transport equation in a convex domain with diffusive boundary.

1.4. **Notation and Structure.** Throughout this paper, unless specified, $C>0$ denotes a universal constant which does not depend on the data and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z.

Our paper is organized as follows: in Section 2, we present the asymptotic analysis of the equation (1.1); in Section 3, we prove the weighted L^∞ estimates of derivatives in ϵ-Milne problem with geometric correction; in Section 4, we prove the improved L^∞ estimate of remainder equation; finally, in Section 5, we prove the diffusive limit, i.e. Theorem 1.1.

2. Asymptotic Analysis

2.1. **Interior Expansion.** We define the interior expansion as follows:

$$U(\vec{x},\vec{w})\sim U_0(\vec{x},\vec{w})+\epsilon U_1(\vec{x},\vec{w})+\epsilon^2U_2(\vec{x},\vec{w}), \tag{2.1}$$

where U_k can be determined by comparing the order of ϵ by plugging (2.1) into the equation (1.1). Thus we have

$$U_0 - \bar{U}_0 = 0, \tag{2.2}$$
$$U_1 - \bar{U}_1 = -\vec{w}\cdot\nabla_x U_0, \tag{2.3}$$
$$U_2 - \bar{U}_2 = -\vec{w}\cdot\nabla_x U_1. \tag{2.4}$$

Plugging (2.2) into (2.3), we obtain

$$U_1 = \bar{U}_1 - \vec{w}\cdot\nabla_x \bar{U}_0. \tag{2.5}$$

Plugging (2.5) into (2.4), we get

$$U_2 - \bar{U}_2 = -\vec{w}\cdot\nabla_x(\bar{U}_1 - \vec{w}\cdot\nabla_x\bar{U}_0) = -\vec{w}\cdot\nabla_x\bar{U}_1 + \vec{w}^2\Delta_x\bar{U}_0 + 2w_1w_2\partial_{x_1x_2}\bar{U}_0. \tag{2.6}$$

Integrating (2.6) over $\vec{w}\in\mathcal{S}^1$, we achieve the final form

$$\Delta_x\bar{U}_0 = 0. \tag{2.7}$$

which further implies $U_0(\vec{x},\vec{w})$ satisfies the equation

$$\begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x\bar{U}_0 &= 0. \end{cases} \tag{2.8}$$

In a similar fashion, for $k=1,2$, U_k satisfies

$$\begin{cases} U_k &= \bar{U}_k - \vec{w}\cdot\nabla_x U_{k-1}, \\ \Delta_x\bar{U}_k &= -\displaystyle\int_{\mathcal{S}^1}\vec{w}\cdot\nabla_x U_{k-1}\mathrm{d}\vec{w}. \end{cases} \tag{2.9}$$

It is easy to see $\bar{U}_k$ satisfies an elliptic equation. However, the boundary condition of $\bar{U}_k$ is unknown at this stage, since generally U_k does not necessarily satisfy the diffusive boundary condition of (1.1). Therefore, we have to resort to boundary layer.

2.2. **Local Coordinate System.** Basically, we use two types of coordinate systems: Cartesian coordinate system for interior solution, which is stated above, and local coordinate system in a neighborhood of the boundary for boundary layer.

Assume the Cartesian coordinate system is $\vec{x}=(x_1,x_2)$. Using polar coordinates system $(r,\theta)\in[0,\infty)\times[-\pi,\pi)$ and choosing pole in Ω, we assume $\vec{x}_0\in\partial\Omega$ is

$$\begin{cases} x_{1,0} &= r(\theta)\cos\theta, \\ x_{2,0} &= r(\theta)\sin\theta, \end{cases} \tag{2.10}$$

where $r(\theta)>0$ is a given function. Our local coordinate system is similar to polar coordinate system, but varies to satisfy the specific requirement.

In the domain near the boundary, for each θ, we have the outward unit normal vector

$$\vec{\nu} = \left(\frac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}, \frac{r(\theta)\sin\theta - r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}\right). \tag{2.11}$$

We can determine each point $\vec{x}\in\bar{\Omega}$ as $\vec{x}=\vec{x}_0-\mu\vec{\nu}$ where μ is the distance to boundary point $\vec{x}_0$. In detail, this means

$$\begin{cases} x_1 &= r(\theta)\cos\theta + \mu\dfrac{-r(\theta)\cos\theta - r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}, \\ x_2 &= r(\theta)\sin\theta + \mu\dfrac{-r(\theta)\sin\theta + r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}, \end{cases} \tag{2.12}$$

where $r'(\theta)=\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$. It is easy to see that $\mu=0$ denotes the boundary $\partial\Omega$ and $\mu>0$ denotes the interior of Ω. (μ,θ) is the desired local coordinate system.

By chain rule, for any $u = u(x_1, x_2)$ we have

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x_1} + \frac{\partial u}{\partial \mu}\frac{\partial \mu}{\partial x_1}, \tag{2.13}$$

$$\frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x_2} + \frac{\partial u}{\partial \mu}\frac{\partial \mu}{\partial x_2}. \tag{2.14}$$

Hence, in order to perform the substitution $(x_1, x_2) \to (\mu, \theta)$, the key step is to calculate $\frac{\partial \theta}{\partial x_1}$, $\frac{\partial \mu}{\partial x_1}$, $\frac{\partial \theta}{\partial x_2}$ and $\frac{\partial \mu}{\partial x_2}$ in terms of μ and θ. For simplicity, we may denote the transform (2.12) as follows.

$$\begin{cases} x_1 &= a(\theta) + \mu A(\theta), \\ x_2 &= b(\theta) + \mu B(\theta), \end{cases} \tag{2.15}$$

where

$$a = r\cos\theta, \quad A = \frac{-r\cos\theta - r'\sin\theta}{(r^2 + (r')^2)^{1/2}}, \tag{2.16}$$

$$b = r\sin\theta, \quad B = \frac{-r\sin\theta + r'\cos\theta}{(r^2 + (r')^2)^{1/2}}. \tag{2.17}$$

Taking x_1 and x_2 derivative in (2.15) reveals that

$$(a' + \mu A')\frac{\partial \theta}{\partial x_1} + A\frac{\partial \mu}{\partial x_1} = 1, \tag{2.18}$$

$$(b' + \mu B')\frac{\partial \theta}{\partial x_1} + B\frac{\partial \mu}{\partial x_1} = 0, \tag{2.19}$$

$$(a' + \mu A')\frac{\partial \theta}{\partial x_2} + A\frac{\partial \mu}{\partial x_2} = 0, \tag{2.20}$$

$$(b' + \mu B')\frac{\partial \theta}{\partial x_2} + B\frac{\partial \mu}{\partial x_2} = 1, \tag{2.21}$$

where the superscript $'$ denotes the derivative with respect to θ. The detailed expression is as follows:

$$a' = r'\cos\theta - r\sin\theta, \tag{2.22}$$

$$b' = r'\sin\theta + r\cos\theta, \tag{2.23}$$

$$A' = \frac{r^3\sin\theta - 2(r')^3\cos\theta - r''r^2\sin\theta + 2r(r')^2\sin\theta - r'r^2\cos\theta + rr'r''\cos\theta}{(r^2 + (r')^2))^{3/2}}, \tag{2.24}$$

$$B' = \frac{-r^3\cos\theta - 2(r')^3\sin\theta - 2r(r')^2\cos\theta - r^2r'\sin\theta + r^2r''\cos\theta + rr'r''\sin\theta}{(r^2 + (r')^2))^{3/2}}. \tag{2.25}$$

Then we can solve the linear system (2.18) to (2.21) by Cramer's rule as

$$\frac{\partial \theta}{\partial x_1} = \frac{\begin{vmatrix} 1 & A \\ 0 & B \end{vmatrix}}{\begin{vmatrix} a' + \mu A' & A \\ b' + \mu B' & B \end{vmatrix}} = \frac{B}{C}, \qquad \frac{\partial \mu}{\partial x_1} = \frac{\begin{vmatrix} a' + \mu A' & 1 \\ b' + \mu B' & 0 \end{vmatrix}}{\begin{vmatrix} a' + \mu A' & A \\ b' + \mu B' & B \end{vmatrix}} = -\frac{b' + \mu B'}{C}, \tag{2.26}$$

$$\frac{\partial \theta}{\partial x_2} = \frac{\begin{vmatrix} 0 & A \\ 1 & B \end{vmatrix}}{\begin{vmatrix} b' + \mu B' & A \\ b' + \mu B' & B \end{vmatrix}} = -\frac{A}{C}, \qquad \frac{\partial \mu}{\partial x_2} = \frac{\begin{vmatrix} a' + \mu A' & 0 \\ b' + \mu B' & 1 \end{vmatrix}}{\begin{vmatrix} a' + \mu A' & A \\ b' + \mu B' & B \end{vmatrix}} = \frac{a' + \mu A'}{C}, \tag{2.27}$$

where C denotes the determinant of the system, which is also the Jacobian of the transform $(x_1, x_2) \to (\mu, \theta)$ as

$$C = \begin{vmatrix} a' + \mu A' & A \\ b' + \mu B' & B \end{vmatrix}. \tag{2.28}$$

Then a direct calculation reveals that

$$C=(r^2+(r')^2)^{1/2}+\mu\frac{rr''-r^2-2r'^2}{(r^2+r'^2)},$$

and

$$\frac{\partial\theta}{\partial x_1}=\frac{(-r\sin\theta+r'\cos\theta)(r^2+r'^2)^{1/2}}{(r^2+r'^2)^{3/2}+\mu(rr''-r^2-2r'^2)}, \tag{2.29}$$

$$\frac{\partial\theta}{\partial x_2}=\frac{(r\cos\theta+r'\sin\theta)(r^2+r'^2)^{1/2}}{(r^2+r'^2)^{3/2}+\mu(rr''-r^2-2r'^2)}, \tag{2.30}$$

$$\begin{aligned}\frac{\partial\mu}{\partial x_1}=&\frac{-(r\cos\theta+r'\sin\theta)(r^2+r'^2)}{(r^2+r'^2)^{3/2}+\mu(rr''-r^2-2r'^2)}\\&-\mu\frac{-r^3\cos\theta-2r'^3\sin\theta-2rr'^2\cos\theta-r^2r'\sin\theta+r^2r''\cos\theta+rr'r''\sin\theta}{(r^2+r'^2)^2+\mu(rr''-r^2-2r'^2)(r^2+r'^2)^{1/2}},\end{aligned} \tag{2.31}$$

$$\begin{aligned}\frac{\partial\mu}{\partial x_2}=&\frac{(-r\sin\theta+r'\cos\theta)(r^2+r'^2)}{(r^2+r'^2)^{3/2}+\mu(rr''-r^2-2r'^2)}\\&+\mu\frac{r^3\sin\theta-2r'^3\cos\theta-r''r^2\sin\theta+2rr'^2\sin\theta-r'r^2\cos\theta+rr'r''\cos\theta}{(r^2+r'^2)^2+\mu(rr''-r^2-2r'^2)(r^2+r'^2)^{1/2}}.\end{aligned} \tag{2.32}$$

A further simplification shows that, we may denote above relation as follows:

$$\frac{\partial\theta}{\partial x_1}=\frac{MP}{P^3+Q\mu},\quad \frac{\partial\mu}{\partial x_1}=-\frac{N}{P}, \tag{2.33}$$

$$\frac{\partial\theta}{\partial x_2}=\frac{NP}{P^3+Q\mu},\quad \frac{\partial\mu}{\partial x_2}=\frac{M}{P}, \tag{2.34}$$

where

$$P=(r^2+r'^2)^{1/2}, \tag{2.35}$$

$$Q=rr''-r^2-2r'^2, \tag{2.36}$$

$$M=-r\sin\theta+r'\cos\theta, \tag{2.37}$$

$$N=r\cos\theta+r'\sin\theta. \tag{2.38}$$

Therefore, by (2.13) and (2.14), noting the fact that for C^2 convex domain, the curvature

$$\kappa(\theta)=\frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{3/2}}, \tag{2.39}$$

and radius of curvature

$$R_\kappa(\theta)=\frac{1}{\kappa(\theta)}=\frac{(r^2+r'^2)^{3/2}}{r^2+2r'^2-rr''}, \tag{2.40}$$

we define substitutions as follows:

Substitution 1:
Let $u^\epsilon(x_1,x_2,w_1,w_2)\to u^\epsilon(\mu,\theta,w_1,w_2)$ with $(\mu,\theta,w_1,w_2)\in[0,R_{\min})\times[-\pi,\pi)\times\mathcal{S}^1$ for $R_{\min}=\min_\theta R_\kappa$ as

$$\begin{cases} x_1 &= r(\theta)\cos\theta+\mu\dfrac{-r(\theta)\cos\theta-r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}},\\ x_2 &= r(\theta)\sin\theta+\mu\dfrac{-r(\theta)\sin\theta+r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}},\end{cases} \tag{2.41}$$

and then the equation (1.1) is transformed into

$$\begin{cases} \epsilon\left(w_1\dfrac{-r\cos\theta-r'\sin\theta}{(r^2+r'^2)^{1/2}}+w_2\dfrac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)^{1/2}}\right)\dfrac{\partial u^\epsilon}{\partial\mu} \\ +\epsilon\left(w_1\dfrac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)(1-\kappa\mu)}+w_2\dfrac{r\cos\theta+r'\sin\theta}{(r^2+r'^2)(1-\kappa\mu)}\right)\dfrac{\partial u^\epsilon}{\partial\theta}+u^\epsilon-\bar{u}^\epsilon=0, \\ u^\epsilon(0,\theta,\vec{w})=\mathcal{P}[u^\epsilon](0,\theta)+\epsilon g(\theta,\vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{\nu}<0, \end{cases} \tag{2.42}$$

where

$$\vec{w}\cdot\vec{\nu}=w_1\frac{-r\cos\theta-r'\sin\theta}{(r^2+r'^2)^{1/2}}+w_2\frac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)^{1/2}}, \tag{2.43}$$

and

$$\mathcal{P}[u^\epsilon](0,\theta)=\frac{1}{2}\int_{\vec{w}\cdot\vec{\nu}>0}u^\epsilon(0,\theta,\vec{w})(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}, \tag{2.44}$$

in a neighborhood of the boundary.

Noting the fact that

$$\left(\frac{M}{P}\right)^2+\left(\frac{N}{P}\right)^2=\left(\frac{-r\cos\theta-r'\sin\theta}{(r^2+r'^2)^{1/2}}\right)^2+\left(\frac{-r\sin\theta+r'\cos\theta}{(r^2+r'^2)^{1/2}}\right)^2=1, \tag{2.45}$$

we can further simplify (2.42).

Substitution 2:
Let $u^\epsilon(\mu,\theta,w_1,w_2)\to u^\epsilon(\mu,\tau,w_1,w_2)$ with $(\mu,\tau,w_1,w_2)\in[0,R_{\min})\times[-\pi,\pi)\times\mathcal{S}^1$ as

$$\begin{cases} \sin\tau &= \dfrac{r\sin\theta-r'\cos\theta}{(r^2+r'^2)^{1/2}}, \\ \cos\tau &= \dfrac{r\cos\theta+r'\sin\theta}{(r^2+r'^2)^{1/2}}, \end{cases} \tag{2.46}$$

which implies

$$\frac{\mathrm{d}\tau}{\mathrm{d}\theta}=\kappa(r^2+r'^2)^{1/2}>0, \tag{2.47}$$

and then the equation (1.1) is transformed into

$$\begin{cases} -\epsilon\left(w_1\cos\tau+w_2\sin\tau\right)\dfrac{\partial u^\epsilon}{\partial\mu}-\dfrac{\epsilon}{R_\kappa-\mu}\left(w_1\sin\tau-w_2\cos\tau\right)\dfrac{\partial u^\epsilon}{\partial\tau}+u^\epsilon-\bar{u}^\epsilon=0, \\ u^\epsilon(0,\tau,\vec{w})=\mathcal{P}[u^\epsilon](0,\tau)+\epsilon g(\tau,\vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{\nu}<0, \end{cases} \tag{2.48}$$

where

$$\vec{w}\cdot\vec{\nu}=w_1\cos\tau+w_2\sin\tau, \tag{2.49}$$

and

$$\mathcal{P}[u^\epsilon](0,\tau)=\frac{1}{2}\int_{\vec{w}\cdot\vec{\nu}>0}u^\epsilon(0,\tau,\vec{w})(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}, \tag{2.50}$$

in a neighborhood of the boundary. Note that here since τ denotes the angle of normal vector, the domain of τ is the same as θ, i.e. $[-\pi,\pi)$.

2.3. **Boundary Layer Expansion with Geometric Correction.** In order to define boundary layer, we need several more substitutions:

Substitution 3:
We further make the scaling transform for $u^\epsilon(\mu,\tau,w_1,w_2)\to u^\epsilon(\eta,\tau,w_1,w_2)$ with $(\eta,\tau,w_1,w_2)\in[0,R_{\min}/\epsilon)\times[-\pi,\pi)\times\mathcal{S}^1$ as

$$\left\{\begin{array}{rcl}\eta&=&\mu/\epsilon,\\ \tau&=&\tau,\\ w_1&=&w_1,\\ w_2&=&w_2,\end{array}\right. \tag{2.51}$$

which implies

$$\frac{\partial u^\epsilon}{\partial\mu}=\frac{1}{\epsilon}\frac{\partial u^\epsilon}{\partial\eta}. \tag{2.52}$$

Then the equation (1.1) is transformed into

$$\left\{\begin{array}{l}-\Big(w_1\cos\tau+w_2\sin\tau\Big)\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{R_\kappa-\epsilon\eta}\Big(w_1\sin\tau-w_2\cos\tau\Big)\dfrac{\partial u^\epsilon}{\partial\tau}+u^\epsilon-\bar{u}^\epsilon=0,\\ \\ u^\epsilon(0,\tau,\vec{w})=\mathcal{P}[u^\epsilon](0,\tau)+\epsilon g(\tau,\vec{w})\quad\text{for}\quad\vec{w}\cdot\vec{\nu}<0,\end{array}\right. \tag{2.53}$$

where

$$\vec{w}\cdot\vec{\nu}=w_1\cos\tau+w_2\sin\tau, \tag{2.54}$$

and

$$\mathcal{P}[u^\epsilon](0,\tau)=\frac{1}{2}\int_{\vec{w}\cdot\vec{\nu}>0}u^\epsilon(0,\tau,\vec{w})(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}. \tag{2.55}$$

Substitution 4:
Define the velocity substitution for $u^\epsilon(\eta,\tau,w_1,w_2)\to u^\epsilon(\eta,\tau,\xi)$ with $(\eta,\tau,\xi)\in[0,R_{\min}/\epsilon)\times[-\pi,\pi)\times[-\pi,\pi)$ as

$$\left\{\begin{array}{rcl}\eta&=&\eta\\ \tau&=&\tau\\ w_1&=&-\sin\xi\\ w_2&=&-\cos\xi\end{array}\right. \tag{2.56}$$

We have the succinct form of the equation (1.1) as

$$\left\{\begin{array}{l}\sin(\tau+\xi)\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{R_\kappa-\epsilon\xi}\cos(\tau+\xi)\dfrac{\partial u^\epsilon}{\partial\tau}+u^\epsilon-\bar{u}^\epsilon=0,\\ \\ u^\epsilon(0,\tau,\xi)=\mathcal{P}[u^\epsilon](0,\tau)+\epsilon g(\tau,\xi)\quad\text{for}\quad\sin(\tau+\xi)>0,\end{array}\right. \tag{2.57}$$

where

$$\mathcal{P}[u^\epsilon](0,\tau)=-\frac{1}{2}\int_{\sin(\tau+\xi)<0}u^\epsilon(0,\tau,\xi)\sin(\tau+\xi)\mathrm{d}\xi \tag{2.58}$$

Substitution 5:
Finally, we make the key rotational substitution for $u^\epsilon(\eta,\tau,\xi)\to u^\epsilon(\eta,\tau,\phi)$ with $(\eta,\tau,\phi)\in[0,R_{\min}/\epsilon)\times[-\pi,\pi)\times[-\pi,\pi)$ as

$$\left\{\begin{array}{rcl}\eta&=&\eta\\ \tau&=&\tau\\ \phi&=&\tau+\xi\end{array}\right. \tag{2.59}$$

and achieve the form

$$\begin{cases} \sin\phi\dfrac{\partial u^\epsilon}{\partial\eta} - \dfrac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\left(\dfrac{\partial u^\epsilon}{\partial\phi}+\dfrac{\partial u^\epsilon}{\partial\tau}\right) + u^\epsilon - \bar{u}^\epsilon = 0 \\ u^\epsilon(0,\tau,\phi) = \mathcal{P}[u^\epsilon](0,\tau) + \epsilon g(\tau,\phi) \quad \text{for} \quad \sin\phi > 0 \end{cases} \tag{2.60}$$

where

$$\mathcal{P}[u^\epsilon](0,\tau) = -\frac{1}{2}\int_{\sin\phi<0} u^\epsilon(0,\tau,\phi)\sin\phi \mathrm{d}\phi \tag{2.61}$$

This step is trying to compensate the variants of normal vector ν along the boundary. A bi-product of such substitution is that we decompose the tangential derivative and introduce the velocity derivative.

We define the boundary layer expansion as follows:

$$\mathscr{U}^\epsilon(\eta,\tau,\phi) \sim \mathscr{U}_0^\epsilon(\eta,\tau,\phi) + \epsilon\mathscr{U}_1^\epsilon(\eta,\tau,\phi), \tag{2.62}$$

where $\mathscr{U}_k^\epsilon$ can be determined by comparing the order of ϵ via plugging (2.62) into the equation (2.60). Thus, in a neighborhood of the boundary, we have

$$\sin\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\eta} + \frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\phi} + \mathscr{U}_0^\epsilon - \bar{\mathscr{U}}_0^\epsilon = 0, \tag{2.63}$$

$$\sin\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\eta} + \frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1^\epsilon}{\partial\phi} + \mathscr{U}_1^\epsilon - \bar{\mathscr{U}}_1^\epsilon = \frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\tau}, \tag{2.64}$$

where

$$\bar{\mathscr{U}}_k^\epsilon(\eta,\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{U}_k^\epsilon(\eta,\tau,\phi)\mathrm{d}\phi. \tag{2.65}$$

2.4. **Matching of Interior Solution and Boundary Layer.** The bridge between interior solution and boundary layer is the boundary condition of (1.1), so we first consider the boundary expansion:

$$(U_0 + \mathscr{U}_0^\epsilon) = \mathcal{P}[U_0 + \mathscr{U}_0^\epsilon], \tag{2.66}$$

$$(U_1 + \mathscr{U}_1^\epsilon) = \mathcal{P}[U_1 + \mathscr{U}_1^\epsilon] + g. \tag{2.67}$$

Noting the fact that $\bar{U}_k = \mathcal{P}[\bar{U}_k]$, we can simplify above conditions as follows:

$$\mathscr{U}_0^\epsilon = \mathcal{P}[\mathscr{U}_0^\epsilon], \tag{2.68}$$

$$\mathscr{U}_1^\epsilon = \mathcal{P}[\mathscr{U}_1^\epsilon] + (\vec{w}\cdot U_0 - \mathcal{P}[\vec{w}\cdot U_0]) + g. \tag{2.69}$$

The construction of U_k and $\mathscr{U}_k^\epsilon$ is as follows:

Step 0: Preliminaries.
Assume the cut-off functions $\psi_0 \in C^\infty[0,\infty)$ is defined as

$$\psi_0(y) = \begin{cases} 1 & 0 \le y \le \dfrac{1}{2}, \\ 0 & \dfrac{3}{4} \le y < \infty. \end{cases} \tag{2.70}$$

Also, define the force as

$$F(\epsilon;\eta,\tau) = -\frac{\epsilon}{R_\kappa(\tau)-\epsilon\eta}, \tag{2.71}$$

and the length for ϵ-Milne problem as $L = \epsilon^{-1/2}$. For $\phi \in [-\pi,\pi]$, denote $R\phi = -\phi$.

Step 1: Construction of $\mathscr{U}_0^\epsilon$.

Define the zeroth order boundary layer as

$$\begin{cases} \mathscr{U}_0^\epsilon(\eta,\tau,\phi) &=& \psi_0(\epsilon^{1/2}\eta)\Big(f_0^\epsilon(\eta,\tau,\phi)-f_{0,L}^\epsilon(\tau)\Big), \\ \sin\phi\dfrac{\partial f_0^\epsilon}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial f_0^\epsilon}{\partial\phi}+f_0^\epsilon-\bar{f}_0^\epsilon &=& 0, \\ f_0^\epsilon(0,\tau,\phi) &=& \mathcal{P}[f_0^\epsilon](0,\tau) \quad \text{for} \quad \sin\phi>0, \\ f_0^\epsilon(L,\tau,\phi) &=& f_0^\epsilon(L,\tau,R\phi), \end{cases} \tag{2.72}$$

with

$$\mathcal{P}[f_0^\epsilon](0,\tau)=0, \tag{2.73}$$

and $f_{0,L}^\epsilon$ is defined as in Theorem 3.6. Thus, we have $\mathscr{U}_0^\epsilon$ is well-defined. It is obvious to see $f_0^\epsilon=f_{0,L}^\epsilon=0$ is the only solution, which means $\mathscr{U}_0^\epsilon=0$.

Step 2: Construction of $\mathscr{U}_1^\epsilon$ and U_0.
Define the first order boundary layer as

$$\begin{cases} \mathscr{U}_1^\epsilon(\eta,\tau,\phi) &=& \psi_0(\epsilon^{1/2}\eta)\Big(f_1^\epsilon(\eta,\tau,\phi)-f_{1,L}^\epsilon(\tau)\Big), \\ \sin\phi\dfrac{\partial f_1^\epsilon}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial f_1^\epsilon}{\partial\phi}+f_1^\epsilon-\bar{f}_1^\epsilon &=& \dfrac{1}{R_\kappa-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_0^\epsilon}{\partial\tau}, \\ f_1^\epsilon(0,\tau,\phi) &=& \mathcal{P}[f_1^\epsilon](0,\tau)+g_1(\tau,\phi) \quad \text{for} \quad \sin\phi>0, \\ f_1^\epsilon(L,\tau,\phi) &=& f_1^\epsilon(L,\tau,R\phi), \end{cases} \tag{2.74}$$

with

$$\mathcal{P}[f_1^\epsilon](0,\tau)=0, \tag{2.75}$$

and $f_{1,L}^\epsilon$ is defined as in Theorem 3.6, where

$$g_1(\vec{x}_0,\vec{w})=\vec{w}\cdot\nabla_x U_0(\vec{x}_0)-\mathcal{P}[\vec{w}\cdot\nabla_x U_0(\vec{x}_0)]+g(\vec{x}_0,\vec{w}), \tag{2.76}$$

with $\vec{x}_0$ and $(0,\tau)$ denoting the same boundary point, and

$$\vec{w}=(-\sin(\phi-\tau),-\cos(\phi-\tau)), \tag{2.77}$$

$$\vec{\nu}=(\cos\tau,\sin\tau). \tag{2.78}$$

To solve (2.74), the data must satisfy the compatibility condition (3.307) as

$$\begin{aligned} &\int_{\sin\phi>0}\Big(g+\vec{w}\cdot\nabla_x U_0(\vec{x}_0)-\mathcal{P}[\vec{w}\cdot\nabla_x U_0(\vec{x}_0)]\Big)\sin\phi\mathrm{d}\phi+\int_0^L\int_{-\pi}^{\pi}\mathrm{e}^{-V(s)}\frac{1}{1-\epsilon s}\cos\phi\frac{\partial\mathscr{U}_0^\epsilon}{\partial\tau}(s,\tau,\phi)\mathrm{d}\phi\mathrm{d}s \\ &\qquad =0, \end{aligned} \tag{2.79}$$

where $\dfrac{\partial V}{\partial\eta}=-F$ and $V(0)=0$. Note the fact

$$\begin{aligned} &\int_{\sin\phi>0}\Big(\vec{w}\cdot\nabla_x U_0(\vec{x}_0)-\mathcal{P}[\vec{w}\cdot\nabla_x U_0(\vec{x}_0)]\Big)\sin\phi\mathrm{d}\phi \\ =&\int_{\sin\phi>0}(\vec{w}\cdot\nabla_x U_0(\vec{x}_0))\sin\phi\mathrm{d}\phi-2\mathcal{P}[\vec{w}\cdot\nabla_x U_0(\vec{x}_0)] \\ =&\int_{\sin\phi>0}(\vec{w}\cdot\nabla_x U_0(\vec{x}_0))\sin\phi\mathrm{d}\phi+\int_{\sin\phi<0}(\vec{w}\cdot\nabla_x U_0(\vec{x}_0))\sin\phi\mathrm{d}\phi \\ =&\int_{-\pi}^{\pi}(\vec{w}\cdot\nabla_x U_0(\vec{x}_0))\sin\phi\mathrm{d}\phi \\ =&-\pi\nabla_x\bar{U}_0(\vec{x}_0)\cdot\vec{\nu}=-\pi\frac{\partial\bar{U}_0(\vec{x}_0)}{\partial\vec{\nu}}. \end{aligned} \tag{2.80}$$

We can simplify the compatibility condition as follows:

$$\int_{\sin\phi>0} g(\phi)\sin\phi \mathrm{d}\phi - \pi\frac{\partial \bar{U}_0(\vec{x}_0)}{\partial \vec{\nu}} + \int_0^L\int_{-\pi}^{\pi} \mathrm{e}^{-V(s)}\frac{1}{1-\epsilon s}\cos\phi\frac{\partial \mathscr{U}_0^\epsilon}{\partial \tau}(s,\tau,\phi)\mathrm{d}\phi\mathrm{d}s = 0. \tag{2.81}$$

Then we have

$$\begin{aligned}\frac{\partial \bar{U}_0(\vec{x}_0)}{\partial \vec{\nu}} &= \frac{1}{\pi}\int_{\sin\phi>0} g(\tau,\phi)\sin\phi\mathrm{d}\phi + \frac{1}{\pi}\int_0^L\int_{-\pi}^{\pi}\mathrm{e}^{-V(s)}\frac{1}{1-\epsilon s}\cos\phi\frac{\partial \mathscr{U}_0^\epsilon}{\partial\tau}(s,\tau,\phi)\mathrm{d}\phi\mathrm{d}s \\ &= \frac{1}{\pi}\int_{\sin\phi>0} g(\tau,\phi)\sin\phi\mathrm{d}\phi.\end{aligned} \tag{2.82}$$

Hence, we define the zeroth order interior solution $U_0(\vec{x},\vec{w})$ as

$$\begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x \bar{U}_0 &= 0 \ \text{ in } \ \Omega, \\ \dfrac{\partial \bar{U}_0}{\partial \vec{\nu}} &= \dfrac{1}{\pi}\displaystyle\int_{\sin\phi>0} g(\tau,\phi)\sin\phi\mathrm{d}\phi \ \text{ on } \ \partial\Omega, \\ \displaystyle\int_\Omega U_0(\vec{x})\mathrm{d}\vec{x} &= 0. \end{cases} \tag{2.83}$$

Step 3: Construction of U_1.
We do not expand the boundary layer to $\mathscr{U}_2^\epsilon$ and just terminate at $\mathscr{U}_1^\epsilon$. Then we define the first order interior solution $U_1(\vec{x})$ as

$$\begin{cases} U_1 &= \bar{U}_1 - \vec{w}\cdot\nabla_x U_0, \\ \Delta_x \bar{U}_1 &= -\displaystyle\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_0)\mathrm{d}\vec{w} \ \text{ in } \ \Omega, \\ \dfrac{\partial \bar{U}_1}{\partial \vec{\nu}} &= -\displaystyle\int_\Omega\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_0)\mathrm{d}\vec{w}\mathrm{d}\vec{x} \ \text{ on } \ \partial\Omega, \\ \displaystyle\int_\Omega \bar{U}_1(\vec{x})\mathrm{d}\vec{x} &= \displaystyle\int_\Omega\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_0 - \mathscr{U}_1^\epsilon)\mathrm{d}\vec{w}\mathrm{d}\vec{x}. \end{cases} \tag{2.84}$$

Note that here we only require the trivial boundary condition since we cannot resort to the compatibility condition in ϵ-Milne problem with geometric correction. Based on [17], this might lead to $O(\epsilon^2)$ error to the boundary approximation. Thanks to the improved remainder estimate, this error is acceptable.

Step 4: Construction of U_2.
By a similar fashion, we define the second order interior solution as

$$\begin{cases} U_2 &= \bar{U}_2 - \vec{w}\cdot\nabla_x U_1, \\ \Delta_x \bar{U}_2 &= -\displaystyle\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_1)\mathrm{d}\vec{w} \ \text{ in } \ \Omega, \\ \dfrac{\partial \bar{U}_2}{\partial \vec{\nu}} &= -\displaystyle\int_\Omega\int_{\mathcal{S}^1}(\vec{w}\cdot\nabla_x U_1)\mathrm{d}\vec{w}\mathrm{d}\vec{x} \ \text{ on } \ \partial\Omega, \\ \displaystyle\int_\Omega \bar{U}_2(\vec{x})\mathrm{d}\vec{x} &= \displaystyle\int_\Omega\int_{\mathcal{S}^1}\vec{w}\cdot\nabla_x U_1\mathrm{d}\vec{w}\mathrm{d}\vec{x}. \end{cases} \tag{2.85}$$

As the case of U_1, we might have $O(\epsilon^3)$ error in this step due to the trivial boundary data. However, it will not affect the diffusive limit.

3. Regularity of ϵ-Milne Problem with Geometric Correction

We consider the ϵ-Milne problem with geometric correction for $f^\epsilon(\eta,\tau,\phi)$ in the domain $(\eta,\tau,\phi)\in[0,L]\times[-\pi,\pi)\times[-\pi,\pi)$ where $L=\epsilon^{-1/2}$ as

$$\begin{cases} \sin\phi\dfrac{\partial f^\epsilon}{\partial\eta} + F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial f^\epsilon}{\partial\phi} + f^\epsilon - \bar{f}^\epsilon &= S^\epsilon(\eta,\tau,\phi), \\ f^\epsilon(0,\tau,\phi) &= \mathcal{P}[f^\epsilon](0,\tau) + h^\epsilon(\tau,\phi) \ \text{ for } \ \sin\phi>0, \\ f^\epsilon(L,\tau,\phi) &= f^\epsilon(L,\tau,R\phi), \end{cases} \tag{3.1}$$

where $R\phi = -\phi$,

$$\mathcal{P}[f^\epsilon](0,\tau) = -\frac{1}{2}\int_{\sin\phi<0} f^\epsilon(0,\tau,\phi)\sin\phi \mathrm{d}\phi, \tag{3.2}$$

and

$$F(\epsilon;\eta,\tau) = -\frac{\epsilon}{R_\kappa(\tau)-\epsilon\eta}. \tag{3.3}$$

In this section, for convenience, we temporarily ignore the superscript on ϵ. We define the norms in the space $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$ as follows:

$$\|f(\tau)\|_{L^2L^2} = \left(\int_0^L\int_{-\pi}^{\pi}|f(\eta,\tau,\phi)|^2\,\mathrm{d}\phi\mathrm{d}\eta\right)^{1/2}, \tag{3.4}$$

$$\|f(\tau)\|_{L^\infty L^\infty} = \sup_{(\eta,\phi)\in[0,L]\times[-\pi,\pi)}|f(\eta,\tau,\phi)|, \tag{3.5}$$

Similarly, we can define the norm at in-flow boundary as

$$\|f(0,\tau)\|_{L^2} = \left(\int_{\sin\phi>0}|f(0,\tau,\phi)|^2\,\mathrm{d}\phi\right)^{1/2}, \tag{3.6}$$

$$\|f(0,\tau)\|_{L^\infty} = \sup_{\sin\phi>0}|f(0,\tau,\phi)|, \tag{3.7}$$

Also define

$$\langle f,g\rangle_\phi(\eta,\tau) = \int_{-\pi}^{\pi} f(\eta,\tau,\phi)g(\eta,\tau,\phi)\mathrm{d}\phi \tag{3.8}$$

as the L^2 inner product in ϕ. We further assume

$$\|h(\tau)\|_{L^\infty} + \left\|\frac{\partial h}{\partial\phi}(\tau)\right\|_{L^\infty} + \left\|\frac{\partial h}{\partial\tau}(\tau)\right\|_{L^\infty} \le C, \tag{3.9}$$

and

$$\|S(\tau)\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\eta}(\tau)\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\phi}(\tau)\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\tau}(\tau)\right\|_{L^\infty L^\infty} \le C\mathrm{e}^{-K\eta}, \tag{3.10}$$

for $C>0$ and $K>0$ uniform in ϵ and τ.

As in [17, Section 6], in order to study problem with diffusive boundary, we first need to study the ϵ-Milne problem with in-flow boundary for $f(\eta,\tau,\phi)$ in the domain $(\eta,\tau,\phi)\in[0,L]\times[-\pi,\pi)\times[-\pi,\pi)$ as

$$\begin{cases} \sin\phi\dfrac{\partial f}{\partial\eta} + F(\eta,\tau)\cos\phi\dfrac{\partial f}{\partial\phi} + f - \bar{f} &= S(\eta,\tau,\phi), \\ f(0,\tau,\phi) &= h(\tau,\phi) \quad \text{for} \quad \sin\phi>0, \\ f(L,\tau,\phi) &= f(L,\tau,R\phi). \end{cases} \tag{3.11}$$

Define a potential function $V(\epsilon;\eta,\tau)$ satisfying $V(\epsilon;0,\tau)=0$ and $\dfrac{\partial V}{\partial\eta} = -F(\epsilon;\eta,\tau)$.

Lemma 3.1. *We have* $\mathrm{e}^{-V(\epsilon;0,\tau)} = 1$ *and*

$$\mathrm{e}^{-V(\epsilon;L,\tau)} = 1 - \frac{\epsilon^{1/2}}{R_\kappa(\tau)}. \tag{3.12}$$

Proof. We directly compute

$$V(\epsilon;\eta,\tau) = \ln\left(\frac{R_\kappa(\tau)}{R_\kappa(\tau)-\epsilon\eta}\right), \tag{3.13}$$

and

$$\mathrm{e}^{-V(\epsilon;\eta,\tau)} = \frac{R_\kappa(\tau)-\epsilon\eta}{R_\kappa(\tau)}. \tag{3.14}$$

Hence, our result naturally follows. □

In the following, we will temporarily ignore ϵ dependence. Note that all the estimates are uniform in ϵ, which further means uniform in L. From now on, let C denote a finite universal constant that is independent of ϵ and τ.

3.1. **Well-Posedness and Decay.** Since most of the results can be obtained via obvious modifications of [17, Section 4], we will only state the main theorems without proofs.

3.1.1. L^2 *Estimates.* We may decompose the solution

$$f(\eta,\tau,\phi) = q(\eta,\tau) + r(\eta,\tau,\phi), \tag{3.15}$$

where the hydrodynamical part q is in the null space of the operator $f - \bar{f}$, and the microscopic part r is the orthogonal complement, i.e.

$$q(\eta,\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta,\tau,\phi)\mathrm{d}\phi \quad r(\eta,\tau,\phi) = f(\eta,\tau,\phi) - q(\eta,\tau). \tag{3.16}$$

Lemma 3.2. *Assume (3.9) and (3.10) hold. Then there exists a unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.11), satisfying*

$$\|r(\tau)\|_{L^2L^2} \le C, \tag{3.17}$$

$$\langle \sin\phi, r\rangle_\phi(\eta,\tau) = -\int_\eta^L \mathrm{e}^{V(\eta,\tau)-V(y,\tau)}\bar{S}(y,\tau)\mathrm{d}y. \tag{3.18}$$

Also for

$$f_L(\tau) = q_L(\tau) = \frac{\langle \sin^2\phi, f\rangle_\phi(L,\tau)}{\|\sin\phi\|_{L^2}^2}. \tag{3.19}$$

we have

$$|q_L(\tau)| \le C, \tag{3.20}$$

$$\|q(\eta,\tau) - q_L(\tau)\|_{L^2} \le C\bigg(\|r(\eta,\tau)\|_{L^2} + \int_\eta^L |F(y,\tau)|\,\|r(y,\tau)\|_{L^2}\,\mathrm{d}y + \int_\eta^L \|S(y,\tau)\|_{L^\infty}\,\mathrm{d}y\bigg), \tag{3.21}$$

$$\|q(\tau) - q_L(\tau)\|_{L^2L^2} \le C. \tag{3.22}$$

Theorem 3.3. *Assume (3.9) and (3.10) hold. For the ϵ-Milne problem (3.11), there exists a unique solution $f(\eta,\tau,\phi)$ satisfying the estimates*

$$\|f(\eta,\tau,\phi) - f_L(\tau)\|_{L^2L^2} \le C \tag{3.23}$$

for some real number $f_L(\tau)$ satisfying

$$|f_L(\tau)| \le C. \tag{3.24}$$

3.1.2. L^∞ *Estimates.*

Lemma 3.4. *Assume (3.9) and (3.10) hold. The unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.11) satisfies*

$$\|f(\eta,\tau,\phi) - f_L(\tau)\|_{L^\infty L^\infty} \le C\bigg(1 + \|f(\eta,\tau,\phi) - f_L(\tau)\|_{L^2L^2}\bigg). \tag{3.25}$$

Theorem 3.5. *Assume (3.9) and (3.10) hold. The unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.11) satisfies*

$$\|f(\eta,\tau,\phi) - f_L(\tau)\|_{L^\infty L^\infty} \le C. \tag{3.26}$$

3.1.3. *Exponential Decay.*

Theorem 3.6. *Assume (3.9) and (3.10) hold. There exists $K_0 > 0$ such that the unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.11) satisfies*

$$\bigg\| \mathrm{e}^{K_0\eta}\bigg(f(\eta,\tau,\phi) - f_L(\tau)\bigg)\bigg\|_{L^\infty L^\infty} \le C. \tag{3.27}$$

3.2. **Preliminaries.** It is easy to see $\mathscr{V}(\eta,\tau,\phi)=f(\eta,\tau,\phi)-f_L(\tau)$ satisfies the equation

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{V}}{\partial\eta}+F(\eta,\tau)\cos\phi\dfrac{\partial\mathscr{V}}{\partial\phi}+\mathscr{V}-\bar{\mathscr{V}} &= S(\eta,\tau,\phi), \\ \mathscr{V}(0,\tau,\phi) &= p(\tau,\phi) \ \text{ for } \ \sin\phi>0, \\ \mathscr{V}(L,\tau,\phi) &= \mathscr{V}(L,\tau,R\phi). \end{cases} \tag{3.28}$$

where

$$p(\tau,\phi)=h(\tau,\phi)-f_L(\tau). \tag{3.29}$$

We intend to estimate the normal, tangential and velocity derivative. This idea is motivated by [4]. From now on, without specification, we temporarily ignore the dependence on τ and all the estimates are uniform in τ. Define a distance function $\zeta(\eta,\phi)$ as

$$\zeta(\eta,\phi)=\left(1-\left(\mathrm{e}^{-V(\eta)}\cos\phi\right)^2\right)^{1/2}. \tag{3.30}$$

Note that the closer (η,ϕ) is to the grazing set which satisfies $\eta=0$ and $\sin\phi=0$, the smaller ζ is. In particular, at grazing set, $\zeta=0$. Also, we have $0\le\zeta\le 1$.

Lemma 3.7. *We have*

$$\sin\phi\frac{\partial\zeta}{\partial\eta}+F(\eta)\cos\phi\frac{\partial\zeta}{\partial\phi}=0. \tag{3.31}$$

Proof. We may directly compute

$$\frac{\partial\zeta}{\partial\eta}=\frac{1}{2}\left(1-\left(\mathrm{e}^{-V(\eta)}\cos\phi\right)^2\right)^{-1/2}\Big(-2\mathrm{e}^{-2V(\eta)}\cos^2\phi\Big)F(\eta)=-\frac{\mathrm{e}^{-2V(\eta)}F(\eta)\cos^2\phi}{\zeta}, \tag{3.32}$$

$$\frac{\partial\zeta}{\partial\phi}=\frac{1}{2}\left(1-\left(\mathrm{e}^{-V(\eta)}\cos\phi\right)^2\right)^{-1/2}\Big(-2\mathrm{e}^{-2V(\eta)}\cos\phi\Big)(-\sin\phi)=\frac{\mathrm{e}^{-2V(\eta)}\cos\phi\sin\phi}{\zeta}. \tag{3.33}$$

Hence, we know

$$\sin\phi\frac{\partial\zeta}{\partial\eta}+F(\eta)\cos\phi\frac{\partial\zeta}{\partial\phi}=\frac{-\sin\phi\Big(\mathrm{e}^{-2V(\eta)}F(\eta)\cos^2\phi\Big)+F(\eta)\cos\phi\Big(\mathrm{e}^{-2V(\eta)}\cos\phi\sin\phi\Big)}{\zeta}=0. \tag{3.34}$$

□

3.3. **Direct Estimates along Characteristics.** In this subsection, we will prove some preliminary estimates that are based on the characteristics of $\mathscr{V}$ itself instead of the derivative. Here, we have two formulations of the equation (3.28) along the characteristics:

- Formulation I: η is the principal variable, $\phi=\phi(\eta)$, and the equation can be rewritten as

$$\sin\phi\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\eta}+\mathscr{V}=S+\bar{\mathscr{V}}. \tag{3.35}$$

- Formulation II: ϕ is the principal variable, $\eta=\eta(\phi)$ and the equation can be rewritten as

$$F(\eta)\cos\phi\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\phi}+\mathscr{V}=S+\bar{\mathscr{V}}. \tag{3.36}$$

These two formulations are equivalent and can be applied to different regions of the domain. Define the energy as follows:

$$E(\eta,\phi)=\mathrm{e}^{-V(\eta)}\cos\phi. \tag{3.37}$$

Along the characteristics, this energy is conserved. In the following, let $0<\delta_0<<1$ be a small quantity.

Lemma 3.8. *Assume* $\|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \phi}\right\|_{L^\infty L^\infty} \le C$. *For* $\sin\phi > \delta_0$, *we have*

$$\left|\sin\phi \frac{\partial \mathscr{V}}{\partial \eta}(\eta,\phi)\right| \le C\left(1+\frac{1}{\delta_0^3}\right). \tag{3.38}$$

Proof. Based on the ϵ-Milne problem (3.28), we only need to show

$$\left|F(\eta)\cos\phi \frac{\partial \mathscr{V}}{\partial \phi}(\eta,\phi)\right| \le C\left(1+\frac{1}{\delta_0^3}\right). \tag{3.39}$$

We use Formulation I to rewrite the equation along the characteristics as

$$\mathscr{V}(\eta,\phi) = \exp\left(-G_{\eta,0}\right)\left(p(\phi'(0)) + \int_0^\eta \frac{(S+\bar{\mathscr{V}})(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp\left(G_{\eta',0}\right)\mathrm{d}\eta'\right). \tag{3.40}$$

where $\phi'(\eta') = \phi'(\eta';\eta,\phi)$ satisfying (η',ϕ') and (η,ϕ) are on the same characteristic with $\sin\phi' \ge 0$, and

$$G_{t,s} = \int_s^t \frac{1}{\sin\phi'(\xi)}\mathrm{d}\xi. \tag{3.41}$$

for any $s,t \ge 0$. Note that

$$\frac{\partial G_{t,s}}{\partial \phi} = \int_s^t \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\phi'(\xi)}\right)\mathrm{d}\xi = -\int_s^t \frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial \phi'(\xi)}{\partial \phi}\mathrm{d}\xi. \tag{3.42}$$

Taking ϕ derivative on both sides of (3.40), we have

$$\frac{\partial \mathscr{V}}{\partial \phi} = J = J_1 + J_2 + J_3 + J_4, \tag{3.43}$$

where

$$\begin{aligned} J_1 =& \exp\left(-G_{\eta,0}\right)\left(\int_0^\eta \frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\phi}\mathrm{d}\xi\right)\left(p(\phi'(0))\right. \\ &\left. + \int_0^\eta \frac{(S+\bar{\mathscr{V}})(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp\left(G_{\eta',0}\right)\mathrm{d}\eta'\right), \end{aligned} \tag{3.44}$$

$$J_2 = \exp\left(-G_{\eta,0}\right)\frac{\partial p(\phi'(0))}{\partial\phi}, \tag{3.45}$$

$$\begin{aligned} J_3 =& \exp\left(-G_{\eta,0}\right)\left(\int_0^\eta (S+\bar{\mathscr{V}})(\eta',\phi'(\eta'))\exp\left(G_{\eta',0}\right)\right. \\ &\left.\left(-\frac{1}{\sin\phi'(\eta')}\int_0^{\eta'}\frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\phi}\mathrm{d}\xi - \frac{\cos\phi'(\eta')}{\sin\phi_*^2(\eta')}\frac{\partial\phi'(\eta')}{\partial\phi}\right)\mathrm{d}\eta'\right), \end{aligned} \tag{3.46}$$

$$J_4 = \exp\left(-G_{\eta,0}\right)\int_0^\eta \frac{1}{\sin\phi'(\eta')}\frac{\partial S(\eta',\phi'(\eta'))}{\partial\phi}\exp\left(G_{\eta',0}\right)\mathrm{d}\eta'. \tag{3.47}$$

Then we divide the proof into several steps:

Step 1: Estimate of J_1.
We can directly compute

$$J_1 = \mathscr{V}\left(\int_0^\eta \frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\phi}\mathrm{d}\xi\right). \tag{3.48}$$

Since

$$E(\eta,\phi) = \mathrm{e}^{-V(\eta)}\cos\phi = \mathrm{e}^{-V(\xi)}\cos\phi'(\xi), \tag{3.49}$$

when taking ϕ derivative on both sides of (3.49), we obtain

$$\frac{\partial\phi'(\xi)}{\partial\phi} = \frac{\sin\phi}{\sin\phi'(\xi)}\mathrm{e}^{V(\xi)-V(\eta)}. \tag{3.50}$$

Hence, we have

$$\int_0^\eta \frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\phi}\mathrm{d}\xi = \int_0^\eta \frac{\cos\phi'(\xi)\sin\phi}{\sin\phi'^3(\xi)}\mathrm{e}^{V(\xi)-V(\eta)}\mathrm{d}\xi. \tag{3.51}$$

Since $\sin\phi' \geq \sin\phi \geq \delta_0$, we naturally have

$$\left|\int_0^\eta \frac{\cos\phi'(\xi)\sin\phi}{\sin\phi'^3(\xi)}\mathrm{e}^{V(\xi)-V(\eta)}\mathrm{d}\xi\right| \leq \frac{C\eta}{\delta_0^3}. \tag{3.52}$$

Since $\mathscr{V}$ decays exponentially, we obtain

$$|J_1| \leq \mathrm{e}^{-K_0\eta}\frac{C\eta}{\delta_0^3} \leq \frac{C}{\delta_0^3}. \tag{3.53}$$

Step 2: Estimate of J_2.
For J_2, we can estimate

$$|J_2| = \left|\exp\left(-G_{\eta,0}\right)\frac{\partial p(\phi'(0))}{\partial\phi}\right| \leq \left|\frac{\partial p(\phi'(0))}{\partial\phi}\right|, \tag{3.54}$$

since for any $\xi \in [0,\eta]$,

$$\frac{1}{\sin\phi'(\xi)} \geq 1. \tag{3.55}$$

We may directly compute

$$\mathrm{e}^{-V(\eta)}\cos\phi = \mathrm{e}^{-V(0)}\cos\phi'(0) = \cos\phi'(0). \tag{3.56}$$

Taking ϕ derivative on both sides, we get

$$\frac{\partial\phi'(0)}{\partial\phi} = \frac{\sin\phi}{\sin\phi'(0)}\mathrm{e}^{-V(\eta)}, \tag{3.57}$$

which implies

$$\left|\frac{\partial p(\phi'(0))}{\partial\phi}\right| \leq \|p\|_{W^{1,\infty}}\left|\frac{\partial\phi'(0)}{\partial\phi}\right| \leq (C+\|h\|_{W^{1,\infty}})\left|\frac{\partial\phi'(0)}{\partial\phi}\right| \leq C\mathrm{e}^{-V(\eta)} \leq C. \tag{3.58}$$

Hence, we have shown

$$|J_2| \leq C. \tag{3.59}$$

Step 3: Estimate of J_3.
Similar to the estimate of J_1, we have

$$|J_3| \leq C\left|\int_0^\eta \exp\left(-G_{\eta',\eta}\right)\left(\frac{C(\eta'+1)}{\delta_0^3}\right)\mathrm{d}\eta'\right| \leq \frac{C(\eta^2+1)}{\delta_0^3}. \tag{3.60}$$

Considering $F(\eta) \leq \epsilon$ and $\eta \leq L = \epsilon^{-1/2}$, we have

$$|J_3| \leq \left|\frac{1}{F(\eta)}\right|\frac{C}{\delta_0^3}. \tag{3.61}$$

Step 4: Estimate of J_4.
Similar to the estimate of J_1, we have

$$|J_4| \leq \frac{C}{\delta_0}\left|\int_0^\eta \exp\left(-G_{\eta',\eta}\right)\mathrm{d}\eta'\right| \leq C. \tag{3.62}$$

Step 5: Synthesis.
In summary, we have

$$|J| \leq \frac{C}{\delta_0^3}\left(1+\left|\frac{1}{F(\eta)}\right|\right). \tag{3.63}$$

which implies

$$\left|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right|\le C\bigg(1+\frac{1}{\delta_0^3}\bigg), \tag{3.64}$$

and further

$$\left|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right|\le C\bigg(1+\frac{1}{\delta_0^3}\bigg). \tag{3.65}$$

□

Lemma 3.9. *Assume* $\|S\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}\le C$. *For* $\sin\phi<0$ *with* $|E(\eta,\phi)|\le \mathrm{e}^{-V(L)}$, *if it satisfies* $\min_{\phi'}\sin\phi'\ge\delta_0$ *where* (η',ϕ') *are on the same characteristics as* (η,ϕ) *with* $\sin\phi'\ge 0$, *then we have*

$$\left|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}(\eta,\phi)\right|\le C\bigg(1+\frac{1}{\delta_0^3}\bigg). \tag{3.66}$$

Proof. We use Formulation I to rewrite the equation along the characteristics as

$$\begin{aligned}\mathscr{V}(\eta,\phi)=&\,p(\phi'(0))\exp(-G_{L,0}-G_{L,\eta})\\&+\int_0^L\frac{(S+\bar{\mathscr{V}})(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta'\\&+\int_\eta^L\frac{(S+\bar{\mathscr{V}})(\eta',R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.\end{aligned} \tag{3.67}$$

Then taking η derivative on both sides of (3.67) yields

$$\frac{\partial\mathscr{V}}{\partial\eta}=JJ=JJ_1+JJ_2+JJ_3+JJ_4+JJ_5+JJ_6+JJ_7, \tag{3.68}$$

where

$$JJ_1=-\frac{(S+\bar{\mathscr{V}})(\eta,\phi)}{\sin\phi}, \tag{3.69}$$

$$\begin{aligned}JJ_2=&\int_\eta^L\frac{(S+\bar{\mathscr{V}})(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta',\eta})\\&\bigg(\frac{\cos\phi'(\eta')}{\sin\phi'(\eta')}\frac{\partial\phi'(\eta')}{\partial\eta}-\int_\eta^{\eta'}\frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\eta}\mathrm{d}\xi\bigg)\mathrm{d}\eta',\end{aligned} \tag{3.70}$$

$$JJ_3=\int_\eta^L\frac{\partial S(\eta',\phi'(\eta'))}{\partial\eta}\frac{1}{\sin\phi'(\eta')}\exp(-G_{\eta',\eta})\mathrm{d}\eta', \tag{3.71}$$

$$JJ_4=\frac{\partial p(\phi'(0))}{\partial\eta}\exp(-G_{L,0}-G_{L,\eta}), \tag{3.72}$$

$$JJ_5=p(\phi'(0))\exp(-G_{L,0}-G_{L,\eta})\bigg(-\int_0^L\frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\eta}\mathrm{d}\xi-\int_\eta^L\frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\eta}\mathrm{d}\xi\bigg), \tag{3.73}$$

$$\begin{aligned}JJ_6=&\int_0^L\frac{(S+\bar{\mathscr{V}})(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{L,\eta'}-G_{L,\eta})\\&\bigg(\frac{\cos\phi'(\eta')}{\sin\phi'(\eta')}\frac{\partial\phi'(\eta')}{\partial\eta}-\int_{\eta'}^L\frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\eta}\mathrm{d}\xi-\int_\eta^L\frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\eta}\mathrm{d}\xi\bigg)\mathrm{d}\eta',\end{aligned} \tag{3.74}$$

$$JJ_7=\int_0^L\frac{\partial S(\eta',\phi'(\eta'))}{\partial\eta}\frac{1}{\sin\phi'(\eta')}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta'. \tag{3.75}$$

We divide the proof into several steps:

Step 1: Estimate of JJ_1.
This is obvious, since S and $\bar{\mathscr{V}}$ are bounded. We have

$$|JJ_1| \le \frac{C}{\delta_0} \tag{3.76}$$

Step 2: Estimate of JJ_2.
We may directly compute

$$\frac{\partial \phi'(\eta')}{\partial \eta} = -\frac{F(\eta)\cos\phi'(\eta')}{\sin\phi'(\eta')} \tag{3.77}$$

Hence, we have

$$\frac{\cos\phi'(\eta')}{\sin^2\phi'(\eta')}\frac{\partial\phi'(\eta')}{\partial\eta} = -\frac{F(\eta)\cos^2\phi'(\eta')}{\sin^3\phi'(\eta')} \tag{3.78}$$

Since $\sin\phi'(\eta') \ge \delta_0$, we know

$$\left|\frac{\cos\phi'(\eta')}{\sin^2\phi'(\eta')}\frac{\partial\phi'(\eta')}{\partial\eta}\right| \le \frac{\epsilon}{\delta_0^3} \tag{3.79}$$

Thus, we obtain

$$\left|\int_\eta^{\eta'} \frac{\cos\phi'(\xi)}{\sin\phi'^2(\xi)}\frac{\partial\phi'(\xi)}{\partial\eta}\mathrm{d}\xi\right| \le \frac{\epsilon(\eta'-\eta)}{\delta_0^3}. \tag{3.80}$$

Also, it is easy to see

$$\exp(-G_{\eta',\eta}) \le \exp\Big(-(\eta'-\eta)\Big). \tag{3.81}$$

Since

$$\sin\phi \ge \sin\phi'(\xi) \ge \delta_0, \tag{3.82}$$

we directly obtain

$$|JJ_2| \le \frac{C\epsilon}{\delta_0^3} \tag{3.83}$$

Step 3: Estimate of JJ_3.
We compute

$$\frac{\partial S(\eta',\phi'(\eta'))}{\partial\eta} = \partial_2 S\frac{\partial\phi'(\eta')}{\partial\eta} = -\partial_2 S\frac{F(\eta)\cos\phi'(\eta')}{\sin\phi'(\eta')}, \tag{3.84}$$

where ∂_2 denotes derivative with respect to the second argument in $S(\eta,\phi)$. Hence, this implies

$$\left|\frac{\partial S(\eta',\phi'(\eta'))}{\partial\eta}\right| \le \frac{\epsilon}{\delta_0}. \tag{3.85}$$

Thus, we have

$$|JJ_3| \le \frac{\epsilon}{\delta_0}. \tag{3.86}$$

Step 4: Estimate of JJ_4.
Since

$$\mathrm{e}^{-V(0)}\cos\phi'(0) = \mathrm{e}^{-V(\eta)}\cos\phi, \tag{3.87}$$

we have

$$\frac{\partial\phi'(0)}{\partial\eta} = \frac{\cos\phi'(0)F(\eta)}{\sin\phi'(0)} \tag{3.88}$$

Thus, we know

$$\left|\frac{\partial p(\phi'(0))}{\partial\eta}\right| \le \|p\|_{W^{1,\infty}}\left|\frac{\partial\phi'(0)}{\partial\eta}\right| \le \frac{C\epsilon}{\delta_0}. \tag{3.89}$$

Hence, we have

$$|JJ_4| \le \frac{C\epsilon}{\delta_0}. \tag{3.90}$$

Step 5: Estimate of JJ_5, JJ_6 and JJ_7.
This is basically identical to the estimate of JJ_2. We have

$$|JJ_5| + |JJ_6| + |JJ_7| \le \frac{C\epsilon}{\delta_0^3} \tag{3.91}$$

Step 6: Synthesis.
Summarizing all above, we have

$$\left|\sin\phi \frac{\partial \mathscr{V}}{\partial \eta}(\eta,\phi)\right| \le C\left(1 + \frac{\epsilon}{\delta_0^3}\right). \tag{3.92}$$

□

Lemma 3.10. *Assume* $\|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty} \le C$ *and* $\left|\frac{\partial \bar{\mathscr{V}}}{\partial \eta}\right| \le C(1 + |\ln(\epsilon)| + |\ln(\eta)|)$. *For* $\sin\phi \le 0$ *and* $|E(\eta,\phi)| \ge \mathrm{e}^{-V(L)}$, *we have*

$$\left|\sin\phi \frac{\partial \mathscr{V}}{\partial \eta}(\eta,\phi)\right| \le C(1 + |\ln(\epsilon)|). \tag{3.93}$$

Proof. We use Formulation II to rewrite the equation as

$$\mathscr{V}(\eta,\phi) = p(\phi_*(0))\exp(-H_{\phi,\phi_*(0)}) + \int_{\phi_*(0)}^{\phi} \frac{(S+\bar{\mathscr{V}})(\eta_*(\phi_*),\phi_*)}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*})\mathrm{d}\phi_* \tag{3.94}$$

where

$$H_{\phi,\phi_*} = \int_{\phi_*}^{\phi} \frac{1}{F(\eta_*(\varpi))\cos\varpi}\mathrm{d}\varpi. \tag{3.95}$$

with $(\eta_*(\phi_*),\phi_*)$, $(0,\phi_*(0))$ and (η,ϕ) are on the same characteristics. Then taking η derivative on both sides of (3.94) to obtain

$$\frac{\partial \mathscr{V}}{\partial \eta}(\eta,\phi) = JJJ = JJJ_1 + JJJ_2 + JJJ_3 + JJJ_4 + JJJ_5, \tag{3.96}$$

where

$$JJJ_1 = \frac{\partial p(\phi_*(0))}{\partial \eta}\exp(-H_{\phi,\phi_*(0)}) \tag{3.97}$$

$$JJJ_2 = p(\phi_*(0))\exp(-H_{\phi,\phi_*(0)})\bigg(-\frac{\partial\phi_*(0)}{\partial\eta}\frac{1}{F(0)\cos\phi_*(0)} - \int_{\phi_*(0)}^{\phi}\frac{F'(\eta_*(\varpi))}{F^2(\eta_*(\varpi))\cos\varpi}\frac{\partial\eta_*(\varpi)}{\partial\eta}\mathrm{d}\varpi\bigg) \tag{3.98}$$

$$JJJ_3 = -\frac{(S+\bar{\mathscr{V}})(0,\phi_*(0))}{F(0)\cos\phi_*(0)}\exp(-H_{\phi,\phi_*(0)})\frac{\partial\phi_*(0)}{\partial\eta} \tag{3.99}$$

$$JJJ_4 = \int_{\phi_*(0)}^{\phi}\frac{(S+\bar{\mathscr{V}})(\eta_*(\phi_*),\phi_*)}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*}) \tag{3.100}$$
$$\bigg(-\frac{F'(\eta_*(\phi_*)}{F^2(\eta_*(\phi_*))\cos\phi_*}\frac{\partial\eta_*(\phi_*)}{\partial\eta} - \int_{\phi_*}^{\phi}\frac{F'(\eta_*(\varpi))}{F^2(\eta_*(\varpi))\cos\varpi}\frac{\partial\eta_*(\varpi)}{\partial\eta}\mathrm{d}\varpi\bigg)\mathrm{d}\phi_*,$$

$$JJJ_5 = \int_{\phi_*(0)}^{\phi}\frac{\partial(S+\bar{\mathscr{V}})(\eta_*(\phi_*),\phi_*)}{\partial\eta}\frac{1}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*})\mathrm{d}\phi_*. \tag{3.101}$$

Then we divide the proof into several steps:

Step 1: Estimate of JJJ_1.
Since we know

$$\mathrm{e}^{-V(0)}\cos\phi_*(0)=\mathrm{e}^{-V(\eta)}\cos\phi, \tag{3.102}$$

taking η derivative on both sides of (3.102) implies

$$\frac{\partial\phi_*(0)}{\partial\eta}=-\frac{\mathrm{e}^{V(0)-V(\eta)}F(\eta)\cos\phi}{\sin\phi_*(0)}. \tag{3.103}$$

which further yields

$$\left|\frac{\partial\phi_*(0)}{\partial\eta}\right|\le\left|\frac{\mathrm{e}^{V(0)-V(\eta)}F(\eta)\cos\phi}{\sin\phi}\right|\le\left|\frac{C\epsilon}{\sin\phi}\right|, \tag{3.104}$$

due to

$$|\sin\phi_*(0)|\ge|\sin\phi|. \tag{3.105}$$

Hence, we have

$$\left|\frac{\partial p(\phi_*(0))}{\partial\eta}\right|\le\|p\|_{W^{1,\infty}}\left|\frac{\partial\phi_*(0)}{\partial\eta}\right|\le\left|\frac{C\epsilon}{\sin\phi}\right|. \tag{3.106}$$

Also, for $\sin\phi\le\frac{1}{2}$ and $\eta\in[0,L]$, we always have

$$-H_{\phi,\phi_*(0)}=-\int_{\phi_*(0)}^{\phi}\frac{1}{F(\eta_*(\varpi))\cos\varpi}\mathrm{d}\varpi\le 0, \tag{3.107}$$

which further yields

$$\exp(-H_{\phi,\phi_*(0)})\le 1. \tag{3.108}$$

Then combining (3.106) and (3.108), we have

$$|JJJ_1|\le\left|\frac{C\epsilon}{\sin\phi}\right|. \tag{3.109}$$

Step 2: Estimate of JJJ_2.
Based on the results in Step 1, we can easily verify

$$|p(\phi_*(0))|\le C, \tag{3.110}$$

$$\left|\exp(-H_{\phi,\phi_*(0)})\right|\le 1, \tag{3.111}$$

$$\left|-\frac{\partial\phi_*(0)}{\partial\eta}\right|\le\left|\frac{C\epsilon}{\sin\phi}\right|. \tag{3.112}$$

Then since

$$\mathrm{e}^{-V(\eta_*(\phi_*))}\cos\phi_*=\mathrm{e}^{-V(\eta)}\cos\phi, \tag{3.113}$$

taking η derivative on both sides implies

$$\frac{\partial\eta_*(\phi_*)}{\partial\eta}=\frac{F(\eta)\cos\phi}{F(\eta_*(\phi_*))\cos\phi_*}\mathrm{e}^{V(\eta_*(\phi_*))-V(\eta)}=\frac{F(\eta)}{F(\eta_*(\phi_*))}. \tag{3.114}$$

Considering $0\le\eta^*\le L$, we may directly obtain

$$\left|\frac{F(\eta)}{F(\eta_*(\phi_*))}\right|\le C, \tag{3.115}$$

which further leads to

$$\left|\frac{\partial\eta_*(\phi_*)}{\partial\eta}\right|\le C. \tag{3.116}$$

On the other hand, for $\sin\phi\le 0$ with $|E(\eta,\phi)|\ge\mathrm{e}^{-V(L)}$, we know

$$\left|\mathrm{e}^{-V(\eta_*(\phi_*))}\cos\phi_*\right|=\left|\mathrm{e}^{-V(\eta)}\cos\phi\right|\ge\mathrm{e}^{-V(L)}, \tag{3.117}$$

which implies

$$|\cos\phi_*| \geq \mathrm{e}^{V(\eta_*(\phi_*))-V(L)} \geq \mathrm{e}^{V(0)-V(L)} \geq C_0 > 0. \tag{3.118}$$

In total, we have shown

$$|\cos\phi_*| \geq C_0 > 0, \tag{3.119}$$

which naturally yields

$$\left|\frac{1}{F(0)\cos\phi_*(0)}\right| \leq \frac{C}{\epsilon}. \tag{3.120}$$

Also, we have

$$\left|\frac{F'(\eta_*(\varpi))}{F^2(\eta_*(\varpi))}\right| = 1. \tag{3.121}$$

Hence, we have

$$|JJJ_2| \leq \left|\frac{C}{\sin\phi}\right|. \tag{3.122}$$

Step 3: Estimate of JJJ_3.
We may directly estimate

$$|JJJ_3| \leq \left|-\frac{(S+\bar{\mathscr{V}})(0,\phi_*(0))}{F(0)\cos\phi_*(0)}\right| \left|\exp(-H_{\phi,\phi_*(0)})\right| \left|\frac{\partial\phi_*(0)}{\partial\eta}\right| \leq \frac{C}{\epsilon}\cdot 1\cdot\left|\frac{C\epsilon}{\sin\phi}\right| \leq \left|\frac{C}{\sin\phi}\right|, \tag{3.123}$$

based on the estimates from Step 1. Therefore, we have proved

$$|JJJ_3| \leq \left|\frac{C}{\sin\phi}\right|. \tag{3.124}$$

Step 4: Estimate of JJJ_4.
Using estimates in Step 1 and Step 2, we have

$$\begin{aligned} &\left|-\frac{F'(\eta_*(\phi_*)}{F^2(\eta_*(\phi_*))\cos\phi_*}\frac{\partial\eta_*(\phi_*)}{\partial\eta} - \int_{\phi_*}^{\phi}\frac{F'(\eta_*(\varpi))}{F^2(\eta_*(\varpi))\cos\varpi}\frac{\partial\eta_*(\varpi)}{\partial\eta}\mathrm{d}\varpi\right| \\ =\ &\left|\frac{1}{\cos\phi_*}\frac{\partial\eta_*(\phi_*)}{\partial\eta}\right| + \left|\int_{\phi_*}^{\phi}\frac{1}{\cos\varpi}\frac{\partial\eta_*(\varpi)}{\partial\eta}\mathrm{d}\varpi\right| \leq C. \end{aligned} \tag{3.125}$$

Then we know

$$|JJJ_4| \leq C\left|\int_{\phi_*(0)}^{\phi}\frac{1}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*})\mathrm{d}\phi_*\right| = \left|\exp(-H_{\phi,\phi_*(0)})-1\right| \leq C. \tag{3.126}$$

Step 5: Estimate of JJJ_5.
We decompose JJJ_5 as

$$\begin{aligned} JJJ_5 =\ &\int_{\phi_*(0)}^{\phi}\frac{\partial S(\eta_*(\phi_*),\phi_*)}{\partial\eta}\frac{1}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*})\mathrm{d}\phi_* \\ &+\int_{\phi_*(0)}^{\phi}\frac{\partial\bar{\mathscr{V}}(\eta_*(\phi_*))}{\partial\eta}\frac{1}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*})\mathrm{d}\phi_* \\ =\ &JJJ_{5,1}+JJJ_{5,2}. \end{aligned} \tag{3.127}$$

We may direct estimate

$$|JJJ_{5,1}| \leq C\left|\int_{\phi_*(0)}^{\phi}\frac{1}{F(\eta_*(\phi_*))\cos\phi_*}\exp(-H_{\phi,\phi_*})\mathrm{d}\phi_*\right| = C\left|\exp(-H_{\phi,\phi_*(0)})-1\right| \leq C. \tag{3.128}$$

Note that we cannot estimate $JJJ_{5,2}$ as above since $\dfrac{\partial\bar{\mathscr{V}}(\eta_*(\phi_*))}{\partial\eta}$ involves derivative of $\bar{\mathscr{V}}$ in the normal direction, which might contain singularity when approaching the boundary. Fortunately, this term lies in the

integral along the characteristics, so we may substitute the principle variable $\phi_* \to \eta_*$ back into formulation I as $(\eta_*, \phi_*(\eta_*)))$. This implies the substitution for derivative

$$\frac{\partial}{\partial \eta} \to \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \eta_*}\frac{\partial \eta_*}{\partial \eta} = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}. \tag{3.129}$$

Hence, for η^+ denoting the intersection of characteristics with $\sin\phi = 0$, we know

$$\begin{aligned} JJJ_{5,2} =& \int_0^{\eta^+} \left(\frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta} + \frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}\right)\frac{1}{F(\eta_*)\cos(\phi_*(\eta_*))}\exp(-H_{\phi,\phi_*})\frac{\partial \phi_*}{\partial \eta_*}\mathrm{d}\eta_* \\ &+ \int_\eta^{\eta^+} \left(\frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta} + \frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}\right)\frac{1}{F(\eta_*)\cos(\phi_*(\eta_*))}\exp(-H_{\phi,\phi_*})\frac{\partial \phi_*}{\partial \eta_*}\mathrm{d}\eta_* \\ =& \int_0^{\eta^+} \frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}\frac{1}{F(\eta_*)\cos(\phi_*(\eta_*))}\exp(-H_{\phi,\phi_*})\frac{\partial \phi_*}{\partial \eta_*}\mathrm{d}\eta_* \\ &+ \int_\eta^{\eta^+} \frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}\frac{1}{F(\eta_*)\cos(\phi_*(\eta_*))}\exp(-H_{\phi,\phi_*})\frac{\partial \phi_*}{\partial \eta_*}\mathrm{d}\eta_* \\ =& \int_0^{\eta^+} \frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}\frac{1}{\sin(\phi_*(\eta_*))}\exp(-H_{\phi,\phi_*})\mathrm{d}\eta_*, \\ & \int_\eta^{\eta^+} \frac{\partial \bar{\mathscr{V}}(\eta_*)}{\partial \eta_*}\frac{F(\eta)}{F(\eta_*)}\frac{1}{\sin(\phi_*(\eta_*))}\exp(-H_{\phi,\phi_*})\mathrm{d}\eta_*, \end{aligned} \tag{3.130}$$

where

$$\frac{\partial \phi_*}{\partial \eta_*} = \frac{F(\eta_*)\cos(\phi_*(\eta_*))}{\sin(\phi_*(\eta_*))}. \tag{3.131}$$

Since

$$\left|\frac{\partial \bar{\mathscr{V}}}{\partial \eta}\right| \le C(1 + |\ln(\epsilon)| + |\ln(\eta)|), \tag{3.132}$$

and $\sin\phi_* \sim \sqrt{\epsilon(\eta_* - \eta^+)}$, we know above integral is finite, i.e.

$$|JJJ_{5,2}| \le C(1 + |\ln(\epsilon)|). \tag{3.133}$$

Therefore, we know

$$|JJJ_5| \le C(1 + |\ln(\epsilon)|). \tag{3.134}$$

Step 6: Synthesis.
In summary, we have shown

$$|JJJ| \le C(1 + |\ln(\epsilon)|) + \left|\frac{C}{\sin\phi}\right|. \tag{3.135}$$

which implies

$$\left|\sin\phi\frac{\partial \mathscr{V}}{\partial \eta}(\eta, \phi)\right| \le C(1 + |\ln(\epsilon)|). \tag{3.136}$$

□

Remark 3.11. *Estimates in Lemma 3.8, Lemma 3.9 and Lemma 3.10 can provide pointwise bounds of derivatives. However, they are not uniform estimates due to presence of δ_0 and $\ln(\epsilon)$. We need weighted L^∞ estimates of derivatives to close the proof. Also, the estimate $\left|\frac{\partial \bar{\mathscr{V}}}{\partial \eta}\right| \le C(1 + |\ln(\epsilon)| + |\ln(\eta)|)$ are not known a priori, so we need an iteration argument.*

3.4. **Mild Formulation of Normal Derivative.** In this and next subsection, we will prove stronger a priori estimates of derivatives. Consider the ϵ-transport problem for $\mathscr{A} = \zeta \dfrac{\partial \mathscr{V}}{\partial \eta}$ as

$$\begin{cases} \sin\phi \dfrac{\partial \mathscr{A}}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial \mathscr{A}}{\partial \phi} + \mathscr{A} &= \tilde{\mathscr{A}} + S_{\mathscr{A}}, \\ \mathscr{A}(0,\phi) &= p_{\mathscr{A}}(\phi) \quad \text{for} \quad \sin\phi > 0, \\ \mathscr{A}(L,\phi) &= \mathscr{A}(L, R\phi), \end{cases} \tag{3.137}$$

where $p_{\mathscr{A}}$ and $S_{\mathscr{A}}$ will be specified later with

$$\tilde{\mathscr{A}}(\eta,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\zeta(\eta,\phi)}{\zeta(\eta,\phi_*)} \mathscr{A}(\eta,\phi_*)\mathrm{d}\phi_*. \tag{3.138}$$

Lemma 3.12. *We have*

$$\begin{aligned} \|\mathscr{A}\|_{L^\infty L^\infty} \le C\bigg(\|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty} \bigg) \\ + C\,|\ln(\epsilon)|^8 \left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right). \end{aligned} \tag{3.139}$$

The rest of this subsection will be devoted to the proof of this lemma. We first introduce some notation. Define the energy as before

$$E(\eta,\phi) = \mathrm{e}^{-V(\eta)}\cos\phi. \tag{3.140}$$

Along the characteristics, where this energy is conserved and ζ is a constant, the equation can be simplified as follows:

$$\sin\phi \frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\eta} + \mathscr{A} = \tilde{\mathscr{A}} + S_{\mathscr{A}}. \tag{3.141}$$

An implicit function $\eta^+(\eta,\phi)$ can be determined through

$$|E(\eta,\phi)| = \mathrm{e}^{-V(\eta^+)}. \tag{3.142}$$

which means (η^+, ϕ_0) with $\sin\phi_0 = 0$ is on the same characteristics as (η,ϕ). Define the quantities for $0 \le \eta' \le \eta^+$ as follows:

$$\phi'(\phi,\eta,\eta') = \cos^{-1}(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi), \tag{3.143}$$

$$R\phi'(\phi,\eta,\eta') = -\cos^{-1}(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi) = -\phi'(\phi,\eta,\eta'), \tag{3.144}$$

where the inverse trigonometric function can be defined single-valued in the domain $[0,\pi)$ and the quantities are always well-defined due to the monotonicity of V. Note that $\sin\phi' \ge 0$, even if $\sin\phi < 0$. Finally we put

$$G_{\eta,\eta'}(\phi) = \int_{\eta'}^{\eta} \frac{1}{\sin(\phi'(\phi,\eta,\xi))}\mathrm{d}\xi. \tag{3.145}$$

Similar to ϵ-Milne problem, we can define the solution along the characteristics as follows:

$$\mathscr{A}(\eta,\phi) = \mathcal{K}[p_{\mathscr{A}}] + \mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}], \tag{3.146}$$

where

Region I:
For $\sin\phi > 0$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}(\phi'(0))\exp(-G_{\eta,0}) \tag{3.147}$$

$$\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = \int_0^{\eta} \frac{(\tilde{\mathscr{A}} + S_{\mathscr{A}})(\eta', \phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{3.148}$$

Region II:

For $\sin\phi<0$ and $|E(\eta,\phi)|\le \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p_{\mathscr{A}}]=p_{\mathscr{A}}(\phi'(0))\exp(-G_{L,0}-G_{L,\eta}) \tag{3.149}$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]=&\int_0^L\frac{(\tilde{\mathscr{A}}+S)(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta'\\&+\int_\eta^L\frac{(\tilde{\mathscr{A}}+S)(\eta',R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.\end{aligned} \tag{3.150}$$

Region III:
For $\sin\phi<0$ and $|E(\eta,\phi)|\ge \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p_{\mathscr{A}}]=p_{\mathscr{A}}(\phi'(\phi,\eta,0))\exp(-G_{\eta^+,0}-G_{\eta^+,\eta}) \tag{3.151}$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]=&\int_0^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta'\\&+\int_\eta^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.\end{aligned} \tag{3.152}$$

Then we need to estimate $\mathcal{K}[p_{\mathscr{A}}]$ and $\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]$ in each region. We assume $0<\delta<<1$ and $0<\delta_0<<1$ are small quantities which will be determined later.

3.4.1. *Region I:* $\sin\phi>0$. Based on [17, Lemma 4.7, Lemma 4.8], we can directly obtain

$$|\mathcal{K}[p_{\mathscr{A}}]|\ \le\ \|p_{\mathscr{A}}\|_{L^\infty}\,, \tag{3.153}$$

$$|\mathcal{T}[S_{\mathscr{A}}]|\ \le \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \tag{3.154}$$

Hence, we only need to estimate $I=\mathcal{T}[\tilde{\mathscr{A}}]$. We divide it into several steps:

Step 0: Preliminaries.
We have

$$E(\eta',\phi')=\frac{R_\kappa-\epsilon\eta'}{R_\kappa}\cos\phi'. \tag{3.155}$$

We can directly obtain

$$\begin{aligned}\zeta(\eta',\phi')=&\frac{1}{R_\kappa}\sqrt{R_\kappa^2-\Big((R_\kappa-\epsilon\eta')\cos\phi'\Big)^2}=\frac{1}{R_\kappa}\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2+(R_\kappa-\epsilon\eta')^2\sin^2\phi'},\\ \le&\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}+\sqrt{(R_\kappa-\epsilon\eta')^2\sin^2\phi'}\le C\Big(\sqrt{\epsilon\eta'}+\sin\phi'\Big),\end{aligned} \tag{3.156}$$

and

$$\zeta(\eta',\phi')\ge\frac{1}{R_\kappa}\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}\ge C\sqrt{\epsilon\eta'}. \tag{3.157}$$

Also, we know for $0\le\eta'\le\eta$,

$$\sin\phi'=\sqrt{1-\cos^2\phi'}=\sqrt{1-\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}\right)^2\cos^2\phi} \tag{3.158}$$

$$=\frac{\sqrt{(R_\kappa-\epsilon\eta')^2\sin^2\phi+(2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta-\epsilon\eta')\cos^2\phi}}{R_\kappa-\epsilon\eta'}. \tag{3.159}$$

Since

$$0\le(2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta-\epsilon\eta')\cos^2\phi\le 2R_\kappa\epsilon(\eta-\eta'), \tag{3.160}$$

we have

$$\sin\phi\le\sin\phi'\le 2\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}, \tag{3.161}$$

which means

$$\frac{1}{2\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}}\le\frac{1}{\sin\phi'}\le\frac{1}{\sin\phi}. \tag{3.162}$$

Therefore,

$$\begin{aligned}-\int_{\eta'}^{\eta}\frac{1}{\sin\phi'(y)}\mathrm{d}y\le&-\int_{\eta'}^{\eta}\frac{1}{2\sqrt{\sin^2\phi+\epsilon(\eta-y)}}\mathrm{d}y\\=&\frac{1}{\epsilon}\bigg(\sin\phi-\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}\bigg)\\=&-\frac{\eta-\eta'}{\sin\phi+\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}}\\\le&-\frac{\eta-\eta'}{2\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}}.\end{aligned} \tag{3.163}$$

Define a cut-off function $\chi\in C^\infty[-\pi,\pi]$ satisfying

$$\chi(\phi)=\begin{cases}1 & \text{for}\quad|\sin\phi|\le\delta,\\0 & \text{for}\quad|\sin\phi|\ge2\delta,\end{cases} \tag{3.164}$$

In the following, we will divide the estimate of I into several cases based on the value of $\sin\phi$, $\sin\phi'$, $\epsilon\eta'$ and $\epsilon(\eta-\eta')$. Let $\mathbf{1}$ denote the indicator function. We write

$$\begin{aligned}I=&\int_0^\eta\mathbf{1}_{\{\sin\phi\ge\delta_0\}}+\int_0^\eta\mathbf{1}_{\{0\le\sin\phi\le\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)<1\}}+\int_0^\eta\mathbf{1}_{\{0\le\sin\phi\le\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\ge\sin\phi'\}}\\&+\int_0^\eta\mathbf{1}_{\{0\le\sin\phi\le\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\le\sin\phi'\}}\mathbf{1}_{\{\sin^2\phi\le\epsilon(\eta-\eta')\}}\\&+\int_0^\eta\mathbf{1}_{\{0\le\sin\phi\le\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\le\sin\phi'\}}\mathbf{1}_{\{\sin^2\phi\ge\epsilon(\eta-\eta')\}}\\=&I_1+I_2+I_3+I_4+I_5.\end{aligned} \tag{3.165}$$

Step 1: Estimate of I_1 for $\sin\phi\ge\delta_0$.
Based on Lemma 3.8, we know

$$\left|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right|\le C\bigg(1+\frac{1}{\delta_0^3}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\left\|\frac{\partial h}{\partial\phi}\right\|_{L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg). \tag{3.166}$$

Hence, we have

$$|I_1|\le C\left|\frac{\partial\mathscr{V}}{\partial\eta}\right|\le\frac{C}{\delta_0^4}\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\left\|\frac{\partial h}{\partial\phi}\right\|_{L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg). \tag{3.167}$$

Step 2: Estimate of I_2 for $0\le\sin\phi\le\delta_0$ and $\chi(\phi_*)<1$.
We have

$$\begin{aligned}I_2=&\frac{1}{2\pi}\int_0^\eta\bigg(\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}(1-\chi(\phi_*))\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\bigg)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\\=&\frac{1}{2\pi}\int_0^\eta\bigg(\int_{-\pi}^{\pi}\zeta(\eta',\phi')(1-\chi(\phi_*))\frac{\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\mathrm{d}\phi_*\bigg)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'.\end{aligned} \tag{3.168}$$

Based on the ϵ-Milne problem of $\mathscr{V}$ as

$$\sin\phi_*\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\eta'}+F(\eta')\cos\phi_*\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\phi_*}+\mathscr{V}(\eta',\phi_*)-\bar{\mathscr{V}}(\eta')=S(\eta',\phi_*), \tag{3.169}$$

we have

$$\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \eta'} = -\frac{1}{\sin\phi_*}\bigg(F(\eta')\cos\phi_*\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \phi_*} + \mathscr{V}(\eta',\phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta',\phi_*)\bigg) \tag{3.170}$$

Hence, we have

$$\begin{aligned}\tilde{\mathscr{A}} =& \int_{-\pi}^{\pi}\zeta(\eta',\phi')(1-\chi(\phi_*))\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \eta'}\mathrm{d}\phi_* \\ =& -\int_{-\pi}^{\pi}\zeta(\eta',\phi')(1-\chi(\phi_*))\frac{1}{\sin\phi_*}\bigg(\mathscr{V}(\eta',\phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta',\phi_*)\bigg)\mathrm{d}\phi_* \\ &-\int_{-\pi}^{\pi}\zeta(\eta',\phi')(1-\chi(\phi_*))\frac{1}{\sin\phi_*}F(\eta')\cos\phi_*\frac{\partial \mathscr{V}(\eta',\phi_*)}{\partial \phi_*}\mathrm{d}\phi_* \\ =& \tilde{\mathscr{A}_1} + \tilde{\mathscr{A}_2}.\end{aligned} \tag{3.171}$$

We may directly obtain

$$\begin{aligned}\Big|\tilde{\mathscr{A}_1}\Big| \le& \int_{-\pi}^{\pi}\zeta(\eta',\phi')(1-\chi(\phi_*))\frac{1}{\sin\phi_*}\bigg(\mathscr{V}(\eta',\phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta',\phi_*)\bigg)\mathrm{d}\phi_* \\ \le& \frac{R_\kappa}{\delta}\bigg|\int_{-\pi}^{\pi}\bigg(\mathscr{V}(\eta',\phi_*) - \bar{\mathscr{V}}(\eta') - S(\eta',\phi_*)\bigg)\mathrm{d}\phi_*\bigg| \\ \le& C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.172}$$

On the other hand, an integration by parts yields

$$\tilde{\mathscr{A}_2} = \int_{-\pi}^{\pi}\frac{\partial}{\partial\phi_*}\bigg(\zeta(\eta',\phi')(1-\chi(\phi_*))\frac{1}{\sin\phi_*}F(\eta')\cos\phi_*\bigg)\mathscr{V}(\eta',\phi_*)\mathrm{d}\phi_*, \tag{3.173}$$

which further implies

$$\Big|\tilde{\mathscr{A}_2}\Big| \le \frac{C\epsilon}{\delta^2}\|\mathscr{V}\|_{L^\infty L^\infty} \le C(\delta)\|\mathscr{V}\|_{L^\infty L^\infty}. \tag{3.174}$$

Since we can use substitution to show

$$\int_0^{\eta}\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \le 1, \tag{3.175}$$

we have

$$\begin{aligned}|I_2| \le& C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\bigg)\int_0^{\eta}\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ \le& C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.176}$$

Step 3: Estimate of I_3 for $0 \le \sin\phi \le \delta_0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \ge \sin\phi'$.
Based on (3.156), this implies

$$\zeta(\eta',\phi') \le C\sqrt{\epsilon\eta'}.$$

Then combining this with (3.157), we can directly obtain

$$\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\chi(\phi_*)\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_* \le C\int_{-\delta}^{\delta}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_* \le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{3.177}$$

Hence, we have

$$|I_3| \le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{3.178}$$

Step 4: Estimate of I_4 for $0 \le \sin\phi \le \delta_0$, $\chi(\phi_*) = 1$, $\sqrt{\epsilon\eta'} \le \sin\phi'$ and $\sin^2\phi \le \epsilon(\eta-\eta')$.
Based on (3.156), this implies

$$\zeta(\eta',\phi') \le C\sin\phi'. \tag{3.179}$$

Based on (3.163), we have

$$-G_{\eta,\eta'} = -\int_{\eta'}^{\eta} \frac{1}{\sin\phi'(y)}\mathrm{d}y \le -\frac{\eta-\eta'}{2\sqrt{\epsilon(\eta-\eta')}} \le -C\sqrt{\frac{\eta-\eta'}{\epsilon}}. \tag{3.180}$$

Hence, we know

$$\begin{aligned} |I_4| &\le C\int_0^{\eta}\left(\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\chi(\phi_*)\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ &\le C\int_0^{\eta}\left(\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\right)\frac{\zeta(\eta',\phi')}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ &\le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\frac{1}{\sqrt{\epsilon\eta'}}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ &\le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\frac{1}{\sqrt{\epsilon\eta'}}\exp\Big(-C\sqrt{\frac{\eta-\eta'}{\epsilon}}\Big)\mathrm{d}\eta' \end{aligned} \tag{3.181}$$

Define $z = \dfrac{\eta'}{\epsilon}$, which implies $\mathrm{d}\eta' = \epsilon\mathrm{d}z$. Substituting this into above integral, we have

$$\begin{aligned} |I_4| &\le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta/\epsilon}\frac{1}{\sqrt{z}}\exp\Big(-C\sqrt{\frac{\eta}{\epsilon}-z}\Big)\mathrm{d}z \\ &= C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\left(\int_0^1\frac{1}{\sqrt{z}}\exp\Big(-C\sqrt{\frac{\eta}{\epsilon}-z}\Big)\mathrm{d}z + \int_1^{\eta/\epsilon}\frac{1}{\sqrt{z}}\exp\Big(-C\sqrt{\frac{\eta}{\epsilon}-z}\Big)\mathrm{d}z\right). \end{aligned} \tag{3.182}$$

We can estimate these two terms separately.

$$\int_0^1\frac{1}{\sqrt{z}}\exp\Big(-C\sqrt{\frac{\eta}{\epsilon}-z}\Big)\mathrm{d}z \le \int_0^1\frac{1}{\sqrt{z}}\mathrm{d}z = 2. \tag{3.183}$$

$$\int_1^{\eta/\epsilon}\frac{1}{\sqrt{z}}\exp\Big(-C\sqrt{\frac{\eta}{\epsilon}-z}\Big)\mathrm{d}z \le \int_1^{\eta/\epsilon}\exp\Big(-C\sqrt{\frac{\eta}{\epsilon}-z}\Big)\mathrm{d}z \overset{t^2=\frac{\eta}{\epsilon}-z}{\le} 2\int_0^{\infty} t\mathrm{e}^{-Ct}\mathrm{d}t < \infty. \tag{3.184}$$

Hence, we know

$$|I_4| \le C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{3.185}$$

Step 5: Estimate of I_5 for $0 \le \sin\phi \le \delta_0$, $\chi(\phi_*) = 1$, $\sqrt{\epsilon\eta'} \le \sin\phi'$ and $\sin^2\phi \ge \epsilon(\eta-\eta')$.
Based on (3.156), this implies

$$\zeta(\eta',\phi') \le C\sin\phi'.$$

Based on (3.163), we have

$$-G_{\eta,\eta'} = -\int_{\eta'}^{\eta}\frac{1}{\sin\phi'(y)}\mathrm{d}y \le -\frac{C(\eta-\eta')}{\sin\phi}. \tag{3.186}$$

Hence, we have

$$|I_5| \le C\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\left(\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_*\right)\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta' \tag{3.187}$$

Here, we use a different way to estimate the inner integral. We use substitution to find

$$\begin{aligned}\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_* &= \int_{-\delta}^{\delta}\frac{1}{\left(R_\kappa^2-(R_\kappa-\epsilon\eta')^2\cos\phi_*^2\right)^{1/2}}\mathrm{d}\phi_* \\ &\overset{\sin\phi_*\ small}{\leq} C\int_{-\delta}^{\delta}\frac{\cos\phi_*}{\left(R_\kappa^2-(R_\kappa-\epsilon\eta')^2\cos\phi_*^2\right)^{1/2}}\mathrm{d}\phi_* \\ &= C\int_{-\delta}^{\delta}\frac{\cos\phi_*}{\left(R_\kappa^2-(R_\kappa-\epsilon\eta')^2+(R_\kappa-\epsilon\eta')^2\sin\phi_*^2\right)^{1/2}}\mathrm{d}\phi_* \\ &\overset{y=\sin\phi_*}{=} C\int_{-\delta}^{\delta}\frac{1}{\left(R_\kappa^2-(R_\kappa-\epsilon\eta')^2+(R_\kappa-\epsilon\eta')^2y^2\right)^{1/2}}\mathrm{d}y.\end{aligned} \tag{3.188}$$

Define

$$p=\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}=\sqrt{2R_\kappa\epsilon\eta'-\epsilon^2\eta'^2}\leq C\sqrt{\epsilon\eta'}, \tag{3.189}$$

$$q=R_\kappa-\epsilon\eta'\geq C, \tag{3.190}$$

$$r=\frac{p}{q}\leq C\sqrt{\epsilon\eta'}. \tag{3.191}$$

Then we have

$$\begin{aligned}\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_* &\leq C\int_{-\delta}^{\delta}\frac{1}{(p^2+q^2y^2)^{1/2}}\mathrm{d}y \\ &\leq C\int_{-2}^{2}\frac{1}{(p^2+q^2y^2)^{1/2}}\mathrm{d}y\leq C\int_{-2}^{2}\frac{1}{(r^2+y^2)^{1/2}}\mathrm{d}y \\ &\leq C\int_{0}^{2}\frac{1}{(r^2+y^2)^{1/2}}\mathrm{d}y=\Big(\ln(y+\sqrt{r^2+y^2})-\ln(r)\Big)\Big|_0^2 \\ &\leq C\Big(\ln(2+\sqrt{r^2+4})-\ln r\Big)\leq C\Big(1+\ln(r)\Big) \\ &\leq C\Big(1+|\ln(\epsilon)|+|\ln(\eta')|\Big).\end{aligned} \tag{3.192}$$

Hence, we know

$$|I_5|\leq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^\eta\Big(1+|\ln(\epsilon)|+|\ln(\eta')|\Big)\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta' \tag{3.193}$$

We may directly compute

$$\left|\int_0^\eta\Big(1+|\ln(\epsilon)|\Big)\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right|\leq C\sin\phi(1+|\ln(\epsilon)|). \tag{3.194}$$

Hence, we only need to estimate

$$\left|\int_0^\eta|\ln(\eta')|\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right|. \tag{3.195}$$

If $\eta\leq 2$, using Cauchy's inequality, we have

$$\begin{aligned}\left|\int_0^\eta|\ln(\eta')|\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right| &\leq \left(\int_0^\eta\ln^2(\eta')\mathrm{d}\eta'\right)^{1/2}\left(\int_0^\eta\exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right)^{1/2} \\ &\leq \left(\int_0^2\ln^2(\eta')\mathrm{d}\eta'\right)^{1/2}\left(\int_0^\eta\exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right)^{1/2} \\ &\leq \sqrt{\sin\phi}.\end{aligned} \tag{3.196}$$

If $\eta \geq 2$, we decompose and apply Cauchy's inequality to obtain

$$\begin{aligned}
&\left|\int_0^\eta |\ln(\eta')| \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) \mathrm{d}\eta'\right| \qquad (3.197)\\
\leq &\left|\int_0^2 |\ln(\eta')| \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) \mathrm{d}\eta'\right| + \left|\int_2^\eta \ln(\eta') \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) \mathrm{d}\eta'\right| \\
\leq &\left(\int_0^2 \ln^2(\eta')\mathrm{d}\eta'\right)^{1/2}\left(\int_0^2 \exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right) \mathrm{d}\eta'\right)^{1/2} + \ln(2)\left|\int_2^\eta \exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right) \mathrm{d}\eta'\right| \\
\leq &C\left(\sqrt{\sin\phi} + \sin\phi\right) \leq C\sqrt{\sin\phi}.
\end{aligned}$$

Hence, we have

$$|I_5| \leq C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}. \qquad (3.198)$$

Step 6: Synthesis.
Collecting all the terms in previous steps, we have proved

$$\begin{aligned}
|I| \leq &C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty} + C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \qquad (3.199)\\
&+ \frac{C}{\delta_0^4}\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\|\frac{\partial p}{\partial \phi}\right\|_{L^\infty} + \left\|\frac{\partial S}{\partial \phi}\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}\right) + C(\delta)\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\right).
\end{aligned}$$

Therefore, we know

$$\begin{aligned}
|\mathscr{A}|_I \leq &\|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty} + C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty} + C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \qquad (3.200)\\
&+ \frac{C}{\delta_0^4}\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\|\frac{\partial p}{\partial \phi}\right\|_{L^\infty} + \left\|\frac{\partial S}{\partial \phi}\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}\right) + C(\delta)\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\right).
\end{aligned}$$

3.4.2. *Region II:* $\sin\phi < 0$ *and* $|E(\eta,\phi)| \leq \mathrm{e}^{-V(L)}$.

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}(\phi'(0))\exp(-G_{L,0} - G_{L,\eta}) \qquad (3.201)$$

$$\begin{aligned}
\mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = &\int_0^L \frac{(\tilde{\mathscr{A}} + S)(\eta', \phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' \qquad (3.202)\\
&+ \int_\eta^L \frac{(\tilde{\mathscr{A}} + S)(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.
\end{aligned}$$

Based on [17, Lemma 4.7, Lemma 4.8], we can directly obtain

$$|\mathcal{K}[p_{\mathscr{A}}]| \leq \|p_{\mathscr{A}}\|_{L^\infty}, \qquad (3.203)$$

$$|\mathcal{T}[S_{\mathscr{A}}]| \leq \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \qquad (3.204)$$

Hence, we only need to estimate $II = \mathcal{T}[\tilde{\mathscr{A}}]$. In particular, we can decompose

$$\begin{aligned}
\mathcal{T}[\tilde{\mathscr{A}}] = &\int_0^L \frac{\tilde{\mathscr{A}}(\eta', \phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' + \int_\eta^L \frac{\tilde{\mathscr{A}}(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta' \qquad (3.205)\\
= &\int_0^\eta \frac{\tilde{\mathscr{A}}(\eta', \phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' \\
&+ \int_\eta^L \frac{\tilde{\mathscr{A}}(\eta', \phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' + \int_\eta^L \frac{\tilde{\mathscr{A}}(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.
\end{aligned}$$

The integral $\int_0^\eta \cdots$ can be estimated as in Region I, so we only need to estimate the integral $\int_\eta^L \cdots$. Also, noting that fact that

$$\exp(-G_{L,\eta'} - G_{L,\eta}) \leq \exp(-G_{\eta',\eta}), \qquad (3.206)$$

we only need to estimate

$$\int_\eta^L \frac{\tilde{\mathscr{A}}(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{\eta',\eta})\mathrm{d}\eta'. \tag{3.207}$$

Here the proof is almost identical to that in Region I, so we only point out the key differences.

Step 0: Preliminaries.
We need to update one key result. For $0 \le \eta \le \eta'$,

$$\begin{aligned}\sin\phi' = \sqrt{1-\cos^2\phi'} &= \sqrt{1-\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}\right)^2\cos^2\phi} \qquad (3.208)\\ &= \frac{\sqrt{(R_\kappa-\epsilon\eta')^2\sin^2\phi + (2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta'-\epsilon\eta)\cos^2\phi}}{R_\kappa-\epsilon\eta'}\\ &\le |\sin\phi|. \qquad (3.209)\end{aligned}$$

Then we have

$$-\int_\eta^{\eta'} \frac{1}{\sin\phi'(y)}\mathrm{d}y \le -\frac{\eta'-\eta}{|\sin\phi|}. \tag{3.210}$$

In the following, we will divide the estimate of II into several cases based on the value of $\sin\phi$, $\sin\phi'$ and $\epsilon\eta'$. We write

$$\begin{aligned}II = &\int_\eta^L \mathbf{1}_{\{\sin\phi\le-\delta_0\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0\le\sin\phi\le0\}}\mathbf{1}_{\{\chi(\phi_*)<1\}} \qquad (3.211)\\ &+\int_\eta^L \mathbf{1}_{\{-\delta_0\le\sin\phi\le0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\ge\sin\phi'\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0\le\sin\phi\le0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\le\sin\phi'\}}\\ &= II_1+II_2+II_3+II_4.\end{aligned}$$

Step 1: Estimate of II_1 for $\sin\phi \le -\delta_0$.
We first estimate $\sin\phi'$. Along the characteristics, we know

$$\mathrm{e}^{-V(\eta')}\cos\phi' = \mathrm{e}^{-V(\eta)}\cos\phi, \tag{3.212}$$

which implies

$$\cos\phi' = \mathrm{e}^{V(\eta')-V(\eta)}\cos\phi \le \mathrm{e}^{V(L)-V(0)}\cos\phi = \mathrm{e}^{V(L)-V(0)}\sqrt{1-\delta_0^2}. \tag{3.213}$$

Based on Lemma 3.1, we can further deduce that

$$\cos\phi' \le \left(1-\frac{\epsilon^{1/2}}{R_\kappa}\right)^{-1}\sqrt{1-\delta_0^2}. \tag{3.214}$$

Then we have

$$\sin\phi' \ge \sqrt{1-\left(1-\frac{\epsilon^{1/2}}{R_\kappa}\right)^{-2}(1-\delta_0^2)} \ge \delta_0 - \epsilon^{1/4} > \frac{\delta_0}{2}, \tag{3.215}$$

when ϵ is sufficiently small. Based on Lemma 3.9, we know

$$\left|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right| \le C\left(1+\frac{1}{\delta_0^3}\right)\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\right). \tag{3.216}$$

Hence, we have

$$|II_1| \le \frac{1}{|\sin\phi|}\left|\frac{\partial\mathscr{V}}{\partial\eta}\right| \le \frac{C}{\delta_0^4}\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\right). \tag{3.217}$$

Step 2: Estimate of II_2 for $-\delta_0 \le \sin\phi \le 0$ and $\chi(\phi_*) < 1$.
This is similar to the estimate of I_2 based on the integral

$$\int_\eta^L \frac{1}{\sin\phi'} \exp(-G_{\eta',\eta}) \mathrm{d}\eta' \le 1. \tag{3.218}$$

Then we have

$$|II_2| \le C(\delta)\Big(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \Big). \tag{3.219}$$

Step 3: Estimate of II_3 for $-\delta_0 \le \sin\phi \le 0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \ge \sin\phi'$.
This is identical to the estimate of I_4, we have

$$|II_3| \le C\delta \|\mathscr{A}\|_{L^\infty L^\infty}. \tag{3.220}$$

Step 4: Estimate of II_4 for $-\delta_0 \le \sin\phi \le 0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \le \sin\phi'$.
This step is different. We do not need to further decompose the cases. Based on (3.210), we have,

$$-G_{\eta,\eta'} \le -\frac{\eta'-\eta}{|\sin\phi|}. \tag{3.221}$$

Then following the same argument in estimating I_5, we obtain

$$|II_4| \le C\|\mathscr{A}\|_{L^\infty L^\infty} \int_\eta^L \Big(1 + |\ln(\epsilon)| + |\ln(\eta')|\Big) \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \tag{3.222}$$

If $\eta \ge 2$, we directly obtain

$$\begin{aligned} \left| \int_\eta^L |\ln(\eta')| \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right| &\le \left| \int_2^L \ln(\eta') \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right| \\ &\le \ln(2) \left| \int_2^L \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right| \\ &\le C\sqrt{|\sin\phi|}. \end{aligned} \tag{3.223}$$

If $\eta \le 2$, we decompose as

$$\begin{aligned} &\left| \int_\eta^L |\ln(\eta')| \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right| \\ \le &\left| \int_\eta^2 |\ln(\eta')| \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right| + \left| \int_2^L |\ln(\eta')| \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right|. \end{aligned} \tag{3.224}$$

The second term is identical to the estimate in $\eta \ge 2$. We apply Cauchy's inequality to the first term

$$\begin{aligned} \left| \int_\eta^2 |\ln(\eta')| \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right) \mathrm{d}\eta' \right| &\le \left(\int_\eta^2 \ln^2(\eta') \mathrm{d}\eta' \right)^{1/2} \left(\int_\eta^2 \exp\left(-\frac{2(\eta'-\eta)}{|\sin\phi|}\right) \mathrm{d}\eta' \right)^{1/2} \\ &\le \left(\int_0^2 \ln^2(\eta') \mathrm{d}\eta' \right)^{1/2} \left(\int_\eta^2 \exp\left(-\frac{2(\eta'-\eta)}{|\sin\phi|}\right) \mathrm{d}\eta' \right)^{1/2} \\ &\le C\sqrt{|\sin\phi|}. \end{aligned} \tag{3.225}$$

Hence, we have

$$|II_4| \le C(1 + |\ln(\epsilon)|)\sqrt{\delta_0} \|\mathscr{A}\|_{L^\infty L^\infty}. \tag{3.226}$$

Step 5: Synthesis.

Collecting all the terms in previous steps, we have proved

$$\begin{aligned}|II| \leq\ & C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}+C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \\ &+\frac{C}{\delta_0^4}\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg)+C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.227}$$

Therefore, we know

$$\begin{aligned}|\mathscr{A}|_{II} \leq\ & \|S_{\mathscr{A}}\|_{L^\infty L^\infty}+\|p_{\mathscr{A}}\|_{L^\infty}+C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}+C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \\ &+\frac{C}{\delta_0^4}\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg)+C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.228}$$

3.4.3. *Region III:* $\sin\phi<0$ *and* $|E(\eta,\phi)|\geq \mathrm{e}^{-V(L)}$. Based on [17, Lemma 4.7, Lemma 4.8], we still have

$$|\mathcal{K}[p_{\mathscr{A}}]| \leq \|p_{\mathscr{A}}\|_{L^\infty}, \tag{3.229}$$

$$|\mathcal{T}[S_{\mathscr{A}}]| \leq \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \tag{3.230}$$

Hence, we only need to estimate $III=\mathcal{T}[\tilde{\mathscr{A}}]$. Note that $|E(\eta,\phi)|\geq \mathrm{e}^{-V(L)}$ implies

$$\mathrm{e}^{-V(\eta)}\cos\phi\geq \mathrm{e}^{-V(L)}. \tag{3.231}$$

Hence, based on Lemma 3.1, we can further deduce that

$$\cos\phi\geq \mathrm{e}^{V(\eta)-V(L)}\geq \mathrm{e}^{V(0)-V_\infty}\geq\left(1-\frac{\epsilon^{1/2}}{R_\kappa}\right). \tag{3.232}$$

Hence, we know

$$|\sin\phi|\leq\sqrt{1-\left(1-\frac{\epsilon^{1/2}}{R_\kappa}\right)^2}\leq\epsilon^{1/4}. \tag{3.233}$$

Hence, when ϵ is sufficiently small, we always have

$$|\sin\phi|\leq\epsilon^{1/4}\leq\delta_0. \tag{3.234}$$

This means we do not need to bother with the estimate of $\sin\phi\leq-\delta_0$ as Step 1 in estimating I and II. Since we can decompose

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}}]=\ & \int_0^\eta\frac{\tilde{\mathscr{A}}(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta' \\ &\bigg(\int_\eta^{\eta^+}\frac{\tilde{\mathscr{A}}(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta'+\int_\eta^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})(\eta',R\phi'(\eta'))}{\sin(\phi'(\eta'))}\exp(-G_{\eta',\eta})\mathrm{d}\eta'\bigg).\end{aligned} \tag{3.235}$$

Then the integral $\displaystyle\int_0^\eta(\cdots)$ is similar to the argument in Region I, and the integral $\displaystyle\int_\eta^{\eta^+}(\cdots)$ is similar to the argument in Region II. Hence, combining the methods in Region I and Region II, we can show the desired result, i.e.

$$\begin{aligned}|\mathscr{A}|_{III} \leq\ & \|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}+C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}+C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \\ &+C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.236}$$

3.4.4. *Estimate of Normal Derivative.* Combining the analysis in these three regions, we have

$$\begin{aligned}|\mathscr{A}| \leq\ & \|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}+C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}+C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \\ &+\frac{C}{\delta_0^4}\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg)+C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.237}$$

Taking supremum over all (η,ϕ), we have

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \le\ & \|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty} + C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty} + C\delta\|\mathscr{A}\|_{L^\infty L^\infty} \\ &+ \frac{C}{\delta_0^4}\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg) \\ &+ C(\delta)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{3.238}$$

Then we choose these constants to perform absorbing argument. First we choose $0<\delta<<1$ sufficiently small such that

$$C\delta \le \frac{1}{4}. \tag{3.239}$$

Then we take $\delta_0 = \delta\,|\ln(\epsilon)|^{-2}$ such that

$$C(1+|\ln(\epsilon)|)\sqrt{\delta_0} \le 2C\delta \le \frac{1}{2}. \tag{3.240}$$

for ϵ sufficiently small. Note that this mild decay of δ_0 with respect to ϵ also justifies the assumption in Case III and the proof of Lemma 3.9 that

$$\epsilon^{1/4} \le \frac{\delta_0}{2}, \tag{3.241}$$

for ϵ sufficiently small. Here since δ and C are independent of ϵ, there is no circulant argument. Hence, we can absorb all the term related to $\|\mathscr{A}\|_{L^\infty L^\infty}$ on the right-hand side of (3.238) to the left-hand side to obtain

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \le C\bigg(&\|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg) \\ &+ C\,|\ln(\epsilon)|^8\left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\right).\end{aligned} \tag{3.242}$$

3.5. **Mild Formulation of Velocity Derivative.** Consider the general ϵ-Milne problem for $\mathscr{B} = \zeta\dfrac{\partial\mathscr{V}}{\partial\phi}$ as

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{B}}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial\mathscr{B}}{\partial\phi} + \mathscr{B} &= S_{\mathscr{B}}, \\ \mathscr{B}(0,\phi) &= p_{\mathscr{B}}(\phi) \quad \text{for} \quad \sin\phi > 0, \\ \mathscr{B}(L,\phi) &= \mathscr{B}(L,R\phi), \end{cases} \tag{3.243}$$

where $p_{\mathscr{B}}$ and $S_{\mathscr{B}}$ will be specified later. This is much simpler than normal derivative, since we do not have $\tilde{\mathscr{B}}$. Then by a direct argument that

$$|\mathcal{K}[p_{\mathscr{B}}]| \le \|p_{\mathscr{B}}\|_{L^\infty}, \tag{3.244}$$

$$|\mathcal{T}[S_{\mathscr{B}}]| \le \|S_{\mathscr{B}}\|_{L^\infty L^\infty}. \tag{3.245}$$

we can get the desired result.

Lemma 3.13. *We have*

$$\|\mathscr{B}\|_{L^\infty L^\infty} \le \|p_{\mathscr{B}}\|_{L^\infty} + \|S_{\mathscr{B}}\|_{L^\infty L^\infty}. \tag{3.246}$$

3.6. **A Priori Estimate of Derivatives.** In this subsection, we combine above a priori estimates of normal and velocity derivatives.

Theorem 3.14. *We have*

$$\left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} \le C\,|\ln(\epsilon)|^8. \tag{3.247}$$

Proof. Collecting the estimates for $\mathscr{A}$ and $\mathscr{B}$ in Lemma 3.12 and Lemma 3.13, we have

$$\|\mathscr{A}\|_{L^\infty L^\infty} \le C\Big(\|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty} \Big) \tag{3.248}$$
$$+ C\left|\ln(\epsilon)\right|^8 \left(\|\mathscr{V}\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} \right),$$
$$\|\mathscr{B}\|_{L^\infty L^\infty} \le \|p_{\mathscr{B}}\|_{L^\infty} + \|S_{\mathscr{B}}\|_{L^\infty L^\infty}. \tag{3.249}$$

Taking derivatives on both sides of (3.28) and multiplying ζ, based on Lemma 3.7, we have

$$p_{\mathscr{A}} = \epsilon \cos\phi \frac{\partial p}{\partial \phi} + p - \bar{\mathscr{V}}(0), \tag{3.250}$$
$$p_{\mathscr{B}} = \sin\phi \frac{\partial p}{\partial \phi}, \tag{3.251}$$
$$S_{\mathscr{A}} = \frac{\partial F}{\partial \eta} \mathscr{B} \cos\phi + \zeta \frac{\partial S}{\partial \eta}, \tag{3.252}$$
$$S_{\mathscr{B}} = \mathscr{A} \cos\phi + F\mathscr{B} \sin\phi + \zeta \frac{\partial S}{\partial \phi}. \tag{3.253}$$

Since $|F(\eta)| + \left| \frac{\partial F}{\partial \eta} \right| \le \epsilon$, by absorbing $\mathscr{A}$ and $\mathscr{B}$ on the right-hand side of (3.248) and (3.249), we derive

$$\mathscr{A} \le C\left|\ln(\epsilon)\right|^8, \tag{3.254}$$
$$\mathscr{B} \le C\left|\ln(\epsilon)\right|^8. \tag{3.255}$$

□

Theorem 3.15. *For $K_0 > 0$ sufficiently small, we have*

$$\left\| \mathrm{e}^{K_0\eta} \zeta \frac{\partial \mathscr{V}}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \mathrm{e}^{K_0\eta} \zeta \frac{\partial \mathscr{V}}{\partial \eta} \right\|_{L^\infty L^\infty} \le C\left|\ln(\epsilon)\right|^8. \tag{3.256}$$

Proof. This proof is almost identical to Theorem 3.14. The only difference is that $S_{\mathscr{A}}$ is added by $K_0 \mathscr{A} \sin\phi$ and $S_{\mathscr{B}}$ added by $K_0 \mathscr{B} \sin\phi$. When K_0 is sufficiently small, we can also absorb them into the left-hand side. Hence, this is obvious. □

3.7. **Continuity and Regularity.** So far, all the estimates are a priori. Hence, we first need to confirm the derivatives are well-defined by an iteration argument. We start from the continuity of solution. Consider the ϵ-transport equation for $\mathscr{V}$ as

$$\begin{cases} \sin\phi \dfrac{\partial \mathscr{V}}{\partial \eta} + F(\eta)\cos\phi \dfrac{\partial \mathscr{V}}{\partial \phi} + \mathscr{V} &= H, \\ \mathscr{V}(0,\phi) &= p(\phi) \quad \text{for} \quad \sin\phi > 0, \\ f(L,\phi) &= f(L, R\phi). \end{cases} \tag{3.257}$$

Lemma 3.16. *Assume H is continuous in $[0,L]\times[-\pi,\pi)$. Then we have $\mathscr{V}$ is continuous in $[0,L]\times[-\pi,\pi)$.*

Proof. As before, we can define the solution along the characteristics as follows:

$$\mathscr{V}(\eta,\phi) = \mathcal{K}[p] + \mathcal{T}[H], \tag{3.258}$$

where

Region I:
For $\sin\phi > 0$,

$$\mathcal{K}[p] = p(\phi'(0)) \exp(-G_{\eta,0}), \tag{3.259}$$
$$\mathcal{T}[H] = \int_0^\eta \frac{H(\eta', \phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{\eta,\eta'}) \mathrm{d}\eta'. \tag{3.260}$$

Region II:
For $\sin\phi < 0$ and $|E(\eta,\phi)| \le \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p] = p(\phi'(0))\exp(-G_{L,0} - G_{L,\eta})$$
$$\mathcal{T}[H] = \bigg(\int_0^L \frac{H(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' + \int_\eta^L \frac{H(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \bigg).$$

Region III:
For $\sin\phi < 0$ and $|E(\eta,\phi)| \ge \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p] = p(\phi'(0))\exp(-G_{\eta^+,0} - G_{\eta^+,\eta})$$
$$\mathcal{T}[H] = \bigg(\int_0^{\eta^+} \frac{H(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\mathrm{d}\eta' + \int_\eta^{\eta^+} \frac{H(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \bigg).$$

When (η,ϕ) does not touch the boundary of each cases, we can directly use above mild formulation to see the continuity. Hence, we concentrate on the separatrix between these regions. We divide the proof into several steps:

Step 1: Separatrix between Region I and Case II.
In our formulation, there is no intersection between these two regions, so we do not need to worry about it.

Step 2: Separatrix between Region II and Region III.
The separatrix is the curve satisfying $|E(\eta,\phi)| = \mathrm{e}^{-V(L)}$. We have in Region II:

$$\mathcal{K}[p] = p(\phi'(0))\exp(-G_{L,0} - G_{L,\eta})$$
$$\mathcal{T}[H] = \bigg(\int_0^L \frac{H(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{L,\eta'} - G_{L,\eta})\mathrm{d}\eta' + \int_\eta^L \frac{H(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \bigg).$$

and in Region III:

$$\mathcal{K}[p] = p(\phi'(0))\exp(-G_{\eta^+,0} - G_{\eta^+,\eta})$$
$$\mathcal{T}[H] = \bigg(\int_0^{\eta^+} \frac{H(\eta',\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\mathrm{d}\eta' + \int_\eta^{\eta^+} \frac{H(\eta', R\phi'(\eta'))}{\sin(\phi'(\eta'))} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \bigg).$$

Since we know $\eta^+ = L$ on this curve, above two formulations give exactly the same formula. Hence, it is continuous.

Step 3: Separatrix between Region I and Region III.
This is actually the segment of line $(\eta, 0)$ for $0 < \eta < L$.

Direction 1: Approaching from Region I.
Consider $(\eta_*,\phi_*) \to (\eta, 0)$. Assume (η_*,ϕ_*) and (η',ϕ') are on the same characteristics. Then we have

$$\mathcal{K}[p] = p(\phi'(0))\exp(-G_{\eta_*,0}) \tag{3.261}$$
$$\mathcal{T}[H] = \int_0^{\eta_*} \frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')} \exp(-G_{\eta_*,\eta'})\mathrm{d}\eta'. \tag{3.262}$$

We can directly take limit $(\eta_*,\phi_*) \to (\eta, 0)$ and obtain

$$\mathcal{K}[p] \to p(\phi'(\eta,0;0))\exp\bigg(-\int_0^\eta \frac{1}{\sin\phi'(y)}\mathrm{d}y \bigg), \tag{3.263}$$
$$\mathcal{T}[H] \to \int_0^\eta \frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')} \exp\bigg(-\int_{\eta'}^\eta \frac{1}{\sin\phi'(y)}\mathrm{d}y \bigg)\mathrm{d}\eta'. \tag{3.264}$$

Here, we cannot further simplify these quantities.

Direction 2: Approaching from Region III.
Consider $(\eta_*,\phi_*)\to(\eta,0)$. Assume (η_*,ϕ_*) and (η',ϕ') are on the same characteristics. Then we have

$$\mathcal{K}[p]=p(\phi'(0))\exp(-G_{\eta^+,0}-G_{\eta^+,\eta_*}) \tag{3.265}$$

$$\begin{aligned}\mathcal{T}[H]=&\int_0^{\eta^+}\frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta_*})\mathrm{d}\eta' \\ &+\int_{\eta_*}^{\eta^+}\frac{H(\eta',R\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta',\eta_*})\mathrm{d}\eta'.\end{aligned} \tag{3.266}$$

In this region, we always have $\eta^+<L$ and

$$\mathrm{e}^{-V(\eta^+)}=\mathrm{e}^{-V(\eta_*)}\cos\phi_*. \tag{3.267}$$

Also, it is easy to see

$$\exp(-G_{\eta^+,\eta_*})\le\mathrm{e}^0=1. \tag{3.268}$$

Hence, considering

$$\mathrm{e}^{-V(\eta^+)}=\mathrm{e}^{-V(y)}\cos\phi'(y), \tag{3.269}$$

we have

$$\begin{aligned}-\int_{\eta_*}^{\eta^+}\frac{1}{\sin\phi'(y)}\mathrm{d}y=&-\int_{\eta_*}^{\eta^+}\frac{1}{\sqrt{1-\mathrm{e}^{2V(y)-2V(\eta^+)}}}\mathrm{d}y \\ =&-\int_{\eta_*}^{\eta^+}\frac{R_\kappa-\epsilon y}{\sqrt{(R_\kappa-\epsilon y)^2-(R_\kappa-\epsilon\eta^+)^2}}\mathrm{d}y \\ =&-\int_{\eta_*}^{\eta^+}\frac{R_\kappa-\epsilon y}{\sqrt{\epsilon(\eta^+-y)(2R_\kappa-\epsilon y-\epsilon\eta^+)}}\mathrm{d}y \\ \ge&-C\int_{\eta_*}^{\eta^+}\frac{1}{\sqrt{\epsilon(\eta^+-y)}}\mathrm{d}y \\ =&-2C\sqrt{\frac{\eta^+-\eta_*}{\epsilon}}.\end{aligned} \tag{3.270}$$

Therefore, we know

$$\exp(-G_{\eta^+,\eta_*})=\exp\Big(-\int_{\eta_*}^{\eta^+}\frac{1}{\sin\phi'(y)}\mathrm{d}y\Big)\ge\exp\Big(-2C\sqrt{\frac{\eta^+-\eta_*}{\epsilon}}\Big). \tag{3.271}$$

When $\phi_*\to0$, since

$$\mathrm{e}^{-V(\eta^+)}=\mathrm{e}^{-V(\eta_*)}\cos\phi_*, \tag{3.272}$$

we have $\eta^+\to\eta_*$, which further implies

$$\exp(-G_{\eta^+,\eta_*})\to\mathrm{e}^0=1. \tag{3.273}$$

Then we apply such result to $\mathcal{K}[p]$ to obtain when $(\eta_*,\phi_*)\to(\eta,0)$

$$\mathcal{K}[p]=p(\phi'(0))\exp(-G_{\eta^+,0}-G_{\eta^+,\eta_*})\to p(\phi'(0))\exp(-G_{\eta,0}). \tag{3.274}$$

On the other hand, we consider $\mathcal{T}[H]$. We directly obtain

$$\int_0^{\eta^+}\frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta_*})\mathrm{d}\eta'\to\int_0^{\eta}\frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{3.275}$$

Also, we know

$$\left|\int_{\eta_*}^{\eta^+}\frac{H(\eta',R\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta',\eta_*})\mathrm{d}\eta'\right| \le \|H\|_{L^\infty L^\infty}\left|\int_{\eta_*}^{\eta^+}\frac{1}{\sin\phi'(\eta')}\exp(-G_{\eta',\eta_*})\mathrm{d}\eta'\right| \tag{3.276}$$
$$\le \|H\|_{L^\infty L^\infty}\left|\exp(-G_{\eta',\eta_*})\Big|_{\eta_*}^{\eta^+}\right|$$
$$= \|H\|_{L^\infty L^\infty}\left|\exp(-G_{\eta^+,\eta_*})-\mathrm{e}^0\right|\to 0.$$

Therefore, we have

$$\mathcal{T}[H]\to\int_0^\eta\frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta_*,\eta'})\mathrm{d}\eta'. \tag{3.277}$$

Synthesis:
Summarizing above two cases, we always have

$$\mathcal{K}[p]\to p(\phi'(0))\exp(-G_{\eta,0}), \tag{3.278}$$
$$\mathcal{T}[H]\to\int_0^\eta\frac{H(\eta',\phi'(\eta'))}{\sin\phi'(\eta')}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{3.279}$$

Hence, the solution is continuous.

Step 4: Triple Point $(L,0)$.
This is the only point that three cases can be applied simultaneously. However, based on previous analysis, we know at this point, Case II and Case III provides exactly the same formula. Also, Case I and Case III is equivalent when taking limit $(\eta_*,\phi_*)\to(\eta,0)$. Then this point is also continuous. □

Theorem 3.17. *The normal and velocity derivatives of $\mathscr{V}$ are well-defined a.e. and satisfy*

$$\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty}\le C\left|\ln(\epsilon)\right|^8. \tag{3.280}$$

Proof. Based on the a priori estimate, it suffices to show the derivatives are well-defined. Consider the iteration of penalized ϵ-Milne problem for $\{\mathscr{V}_\lambda^m\}_{m=0}^\infty$ with $\mathscr{V}_\lambda^0=0$ and for $m\ge1$

$$\begin{cases}\sin\phi\dfrac{\partial\mathscr{V}_\lambda^m}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{V}_\lambda^m}{\partial\phi}+(1+\lambda)\mathscr{V}_\lambda^m-\bar{\mathscr{V}}_\lambda^{m-1} &=& S(\eta,\phi),\\ \mathscr{V}_\lambda^m(0,\phi) &=& p(\phi)\ \text{ for }\ \sin\phi>0,\\ \mathscr{V}_\lambda^m(L,\phi) &=& \mathscr{V}_\lambda^m(L,R\phi).\end{cases} \tag{3.281}$$

Here we require $\lambda>0$. We divide the proof into several steps:

Step 1: $m\to\infty$ convergence.
Tracking along the characteristics, as we have shown in ϵ-Milne problem of [17, Section 4], we have $\mathscr{V}_\lambda^m\in L^\infty([0,L]\times[-\pi,\pi))$. Hence, it is easy to see each $\mathscr{V}_\lambda^m$ is uniquely determined. Define $\mathscr{Z}^m=\mathscr{V}_\lambda^m-\mathscr{V}_\lambda^{m-1}$ for $m\ge1$. Then $\mathscr{Z}^m$ satisfies the equation

$$\begin{cases}\sin\phi\dfrac{\partial\mathscr{Z}^m}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{Z}^m}{\partial\phi}+(1+\lambda)\mathscr{Z}^m-\bar{\mathscr{Z}}^{m-1} &=& 0,\\ \mathscr{Z}^m(0,\phi) &=& 0\ \text{ for }\ \sin\phi>0,\\ \mathscr{Z}^m(L,\phi) &=& \mathscr{Z}^m(L,R\phi).\end{cases} \tag{3.282}$$

Based on [17, Section 4], we know

$$\|\mathscr{Z}^m\|_{L^\infty L^\infty}\le\frac{1}{1+\lambda}\|\mathscr{Z}^{m-1}\|_{L^\infty L^\infty}\le\left(\frac{1}{1+\lambda}\right)^{m-1}\|\mathscr{Z}^1\|_{L^\infty L^\infty}. \tag{3.283}$$

Since $\mathscr{V}_\lambda^0=0$, we have $\mathscr{Z}^1=\mathscr{V}_\lambda^1$. Applying Lemma 3.16 for $H=0$, we know $\mathscr{Z}^1$ is continuous. Using the proofs of Lemma 3.8, Lemma 3.9 and Lemma 3.10 with

$$\mathscr{V}=\mathscr{V}_\lambda^1,\quad\bar{\mathscr{V}}=0,\quad p=0, \tag{3.284}$$

we get $\dfrac{\partial \mathscr{Z}^1}{\partial \eta}$ and $\dfrac{\partial \mathscr{Z}^1}{\partial \phi}$ are a.e. well-defined. However, the estimates from these lemmas are not strong enough to show the convergence of this iteration. Then we can use estimates of ϵ-Milne problem and the proofs of Lemma 3.12 and Lemma 3.13 with

$$\mathscr{A} = \zeta \frac{\partial \mathscr{V}_\lambda^1}{\partial \eta}, \quad \tilde{\mathscr{A}} = 0, \quad p_{\mathscr{A}} = \cos\phi \frac{\partial p}{\partial \phi} + \mathscr{V}_\lambda^1(0,\phi), \quad S_{\mathscr{A}} = \zeta \frac{\partial S}{\partial \eta}, \tag{3.285}$$

$$\mathscr{B} = \zeta \frac{\partial \mathscr{V}_\lambda^1}{\partial \phi}, \quad p_{\mathscr{B}} = \frac{\partial p}{\partial \phi}, \quad S_{\mathscr{B}} = \zeta \frac{\partial S}{\partial \phi}, \tag{3.286}$$

to see

$$\begin{aligned} &\left\| \zeta \frac{\partial \mathscr{Z}^1}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^1}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \mathscr{Z}^1 \right\|_{L^\infty L^\infty} \\ &\le C \left(\|S\|_{L^\infty L^\infty} + \|p\|_{L^\infty} + \left\| \frac{\partial S}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial S}{\partial \phi} \right\|_{L^\infty L^\infty} + \left\| \frac{\partial p}{\partial \phi} \right\|_{L^\infty} \right). \end{aligned} \tag{3.287}$$

and further by direct integration over ϕ,

$$\left| \frac{\partial \bar{\mathscr{Z}}^1}{\partial \eta} \right| \le C \left| \ln(\epsilon) \right|^8 (1 + |\ln(\epsilon)| + |\ln(\eta)|). \tag{3.288}$$

Note that here the extra λ will not affect the result. Similarly, for each $m > 1$, $\bar{\mathscr{Z}}^{m-1}$ can be regarded as known. Applying Lemma 3.16 for $H = \bar{\mathscr{Z}}^{m-1}$, we know $\mathscr{Z}^m$ is continuous. Then we use the proofs of Lemma 3.8, Lemma 3.9 and Lemma 3.10 with

$$\mathscr{V} = \mathscr{Z}^m, \quad S + \bar{\mathscr{V}} = \mathscr{Z}^{m-1}, \quad p = 0, \tag{3.289}$$

to confirm the derivatives $\dfrac{\partial \mathscr{Z}^m}{\partial \eta}$ and $\dfrac{\partial \mathscr{Z}^m}{\partial \phi}$ are a.e. well-defined. Then we utilize the proofs of Lemma 3.12 and Lemma 3.13 with

$$\mathscr{A} = \zeta \frac{\partial \mathscr{Z}^m}{\partial \eta}, \quad \tilde{\mathscr{A}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\zeta(\eta,\phi)}{\zeta(\eta,\phi_*)} \frac{\partial \mathscr{Z}^{m-1}(\eta,\phi_*)}{\partial \eta} \mathrm{d}\phi_*, \quad p_{\mathscr{A}} = \mathscr{Z}^m(0,\phi) - \bar{\mathscr{Z}}^{m-1}, \quad S_{\mathscr{A}} = 0, \tag{3.290}$$

$$\mathscr{B} = \zeta \frac{\partial \mathscr{Z}^m}{\partial \phi}, \quad p_{\mathscr{B}} = 0, \quad S_{\mathscr{B}} = 0, \tag{3.291}$$

to show

$$\left\| \zeta \frac{\partial \mathscr{Z}^m}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^m}{\partial \phi} \right\|_{L^\infty L^\infty} \le \delta \left(\left\| \zeta \frac{\partial \mathscr{Z}^{m-1}}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^{m-1}}{\partial \phi} \right\|_{L^\infty L^\infty} \right) + C \left| \ln(\epsilon) \right|^8 \left\| \mathscr{Z}^m \right\|_{L^\infty L^\infty}. \tag{3.292}$$

for $0 < \delta << 1$ and

$$\left| \frac{\partial \bar{\mathscr{Z}}^m}{\partial \eta} \right| \le C \left| \ln(\epsilon) \right|^8 (1 + |\ln(\epsilon)| + |\ln(\eta)|). \tag{3.293}$$

Therefore, combining (3.283) and (3.292), for fixed $\delta \le \dfrac{1}{1+\lambda}$, we have

$$\begin{aligned} &\left\| \zeta \frac{\partial \mathscr{Z}^m}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^m}{\partial \phi} \right\|_{L^\infty L^\infty} \\ &\le \delta^{m-1} \left(\left\| \zeta \frac{\partial \mathscr{Z}^1}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^1}{\partial \phi} \right\|_{L^\infty L^\infty} \right) + Cm \left| \ln(\epsilon) \right|^8 \left(\frac{1}{1+\lambda} \right)^{m-1} \left\| \mathscr{Z}^1 \right\|_{L^\infty L^\infty} \\ &\le Cm \left| \ln(\epsilon) \right|^8 \left(\frac{1}{1+\lambda} \right)^{m-1} \left(\left\| \mathscr{Z}^1 \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^1}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^1}{\partial \phi} \right\|_{L^\infty L^\infty} \right). \end{aligned} \tag{3.294}$$

For fixed ϵ and $\lambda > 0$, when $m \to \infty$, we know

$$\left\| \mathscr{Z}^m \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^m}{\partial \eta} \right\|_{L^\infty L^\infty} + \left\| \zeta \frac{\partial \mathscr{Z}^m}{\partial \phi} \right\|_{L^\infty L^\infty} \to 0, \tag{3.295}$$

and further for any $N > 1$,

$$\sum_{k=m}^{m+N}\left(\|\mathscr{Z}^m\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{Z}^m}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{Z}^m}{\partial\phi}\right\|_{L^\infty L^\infty}\right)\to 0. \tag{3.296}$$

Hence, $\mathscr{V}_\lambda^m$ is a Cauchy sequence. Thus we have $\mathscr{V}_\lambda^m\to\mathscr{V}_\lambda$ strongly which satisfies

$$\begin{aligned}&\|\mathscr{V}_\lambda\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{V}_\lambda}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{V}_\lambda}{\partial\phi}\right\|_{L^\infty L^\infty}\\ \le&\sum_{k=1}^{\infty}\left(\|\mathscr{Z}^m\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{Z}^m}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{Z}^m}{\partial\phi}\right\|_{L^\infty L^\infty}\right)\\ \le&\frac{C}{\lambda}\left|\ln(\epsilon)\right|^8\left\|\mathscr{Z}^1\right\|_{L^\infty L^\infty}\\ \le&\frac{C}{\lambda}\left|\ln(\epsilon)\right|^8\left(\|S\|_{L^\infty L^\infty}+\|p\|_{L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial h}{\partial\phi}\right\|_{L^\infty}\right).\end{aligned} \tag{3.297}$$

Hence, we know $\dfrac{\partial\mathscr{V}_\lambda}{\partial\eta}$ and $\dfrac{\partial\mathscr{V}_\lambda}{\partial\phi}$ are a.e. well-defined.

Step 2: $\lambda\to 0$ convergence.
We know $\mathscr{V}_\lambda$ satisfies the equation

$$\begin{cases}\sin\phi\dfrac{\partial\mathscr{V}_\lambda}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{V}_\lambda}{\partial\phi}+(1+\lambda)\mathscr{V}_\lambda-\bar{\mathscr{V}_\lambda} &=& S(\eta,\phi),\\ \mathscr{V}_\lambda(0,\phi) &=& p(\phi)\ \text{ for }\ \sin\phi>0,\\ \mathscr{V}_\lambda(L,\phi) &=& \mathscr{V}_\lambda(L,R\phi).\end{cases} \tag{3.298}$$

Since its derivatives are a.e. well-defined, we can use the proof of Lemma 3.12 and Lemma 3.13 to show

$$\begin{aligned}&\|\mathscr{V}_\lambda\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{V}_\lambda}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial\mathscr{V}_\lambda}{\partial\phi}\right\|_{L^\infty L^\infty}\\ \le&C\left|\ln(\epsilon)\right|^8\left(\|S\|_{L^\infty L^\infty}+\|p\|_{L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}\right),\end{aligned} \tag{3.299}$$

which is uniform in λ. Then we can define weak-$*$ limit $\mathscr{V}_\lambda\to\mathscr{V}$ in weighted $W^{1,\infty}$, up to extracting a subsequence as $\lambda\to 0$. Also, the analysis of ϵ-Milne problem in [17, Section 4] reveals that $\mathscr{V}_\lambda\to\mathscr{V}$ weakly in L^2L^2 as $\lambda\to 0$. Hence, $\dfrac{\partial\mathscr{V}}{\partial\eta}$ and $\dfrac{\partial\mathscr{V}}{\partial\phi}$ are a.e. well-defined. Therefore, we can apply the a priori estimates in Theorem 3.14 and Theorem 3.15 to obtain the desired result. □

Corollary 3.18. *We have*

$$\left\|\mathrm{e}^{K_0\eta}\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}(\eta,\phi)\right\|_{L^\infty L^\infty}\le C\left|\ln(\epsilon)\right|^8. \tag{3.300}$$

Proof. This is a natural result of Theorem 3.17 since $\zeta(\eta,\phi)\ge|\sin\phi|$. □

Now we pull τ dependence back and study the tangential derivative.

Theorem 3.19. *We have*

$$\left\|\mathrm{e}^{K_0\eta}\frac{\partial\mathscr{V}}{\partial\tau}(\eta,\tau,\phi)\right\|_{L^\infty L^\infty}\le C\left|\ln(\epsilon)\right|^8. \tag{3.301}$$

Proof. Following a similar fashion in proof of Lemma 3.17, using iteration and characteristics, we can show $\dfrac{\partial\mathscr{V}}{\partial\tau}$ is a.e. well-defined, so here we focus on the a priori estimate. Let $\mathscr{W}=\dfrac{\partial\mathscr{V}}{\partial\tau}$. Taking τ derivative on

both sides of (3.28), we have $\mathscr{W}$ satisfies the equation

$$\begin{cases} \sin\phi\dfrac{\partial \mathscr{W}}{\partial \eta}+F(\eta)\cos\phi\dfrac{\partial \mathscr{W}}{\partial \phi}+\mathscr{W}-\bar{\mathscr{W}} &=& \dfrac{\partial S}{\partial \tau}(\eta,\tau,\phi)+\dfrac{R_\kappa'(\tau)}{R_\kappa(\tau)-\epsilon\eta}\left(F(\eta)\cos\phi\dfrac{\partial \mathscr{V}}{\partial \phi}\right), \\ \mathscr{W}(0,\tau,\phi) &=& \dfrac{\partial p}{\partial \tau}(\tau,\phi) \quad \text{for} \quad \sin\phi>0, \\ \mathscr{W}(L,\tau,\phi) &=& \mathscr{W}(L,\tau,R\phi), \end{cases} \tag{3.302}$$

where R_κ' is the τ derivative of R_κ. Our assumptions on S verify

$$\left\|\mathrm{e}^{K_0\eta}\frac{\partial S}{\partial \tau}(\eta,\tau,\phi)\right\|_{L^\infty L^\infty}\le C. \tag{3.303}$$

For $\eta\in[0,L]$, we have

$$\frac{R_\kappa'(\tau)}{R_\kappa(\tau)-\epsilon\eta}\le C\max_\tau R_\kappa'(\tau)\le C. \tag{3.304}$$

Based on Corollary 3.18 and the equation (3.28), we know

$$\left\|\mathrm{e}^{K_0\eta}\left(F(\eta)\cos\phi\frac{\partial \mathscr{V}}{\partial \phi}\right)(\eta,\tau,\phi)\right\|_{L^\infty L^\infty}\le C\left|\ln(\epsilon)\right|^8. \tag{3.305}$$

Therefore, the source term in the equation (3.302) is in L^∞ and decays exponentially. By Theorem 3.6, we have

$$\left\|\mathrm{e}^{K_0\eta}\mathscr{W}(\eta,\tau,\phi)\right\|_{L^\infty L^\infty}\le C\left|\ln(\epsilon)\right|^8, \tag{3.306}$$

which is the desired estimate. □

3.8. **Diffusive Boundary.** In this subsection, we come back to the ϵ-Milne problem with diffusive boundary. In [17, Section 6], it has been proved that

Lemma 3.20. *In order for the equation (3.1) to have a solution $f(\eta,\tau,\phi)\in L^\infty([0,L]\times[-\pi,\pi)\times[-\pi,\pi))$, the boundary data h and the source term S must satisfy the compatibility condition*

$$\int_{\sin\phi>0}h(\tau,\phi)\sin\phi\mathrm{d}\phi+\int_0^L\int_{-\pi}^{\pi}\mathrm{e}^{-V(s)}S(s,\tau,\phi)\mathrm{d}\phi\mathrm{d}s=0. \tag{3.307}$$

In particular, if $S=0$, then the compatibility condition reduces to

$$\int_{\sin\phi>0}h(\tau,\phi)\sin\phi\mathrm{d}\phi=0. \tag{3.308}$$

It is easy to see if f is a solution to (3.1), then $f+C$ is also a solution for any constant C. Hence, in order to obtain a unique solution, we need a normalization condition

$$\mathcal{P}[f](0,\tau)=0. \tag{3.309}$$

The following lemma in [17, Section 6] tells us the problem (3.1) can be reduced to the ϵ-Milne problem with in-flow boundary (3.11).

Lemma 3.21. *If the boundary data h and S satisfy the compatibility condition (3.307), then the solution f to the ϵ-Milne problem (3.11) with in-flow boundary as $f=h$ on $\sin\phi>0$ is also a solution to the ϵ-Milne problem (3.1) with diffusive boundary, which satisfies the normalization condition (3.309). Furthermore, this is the unique solution to (3.1) among the functions satisfying (3.309) and $\|f(\eta,\tau,\phi)-f_L(\tau)\|_{L^2L^2}\le C$.*

In summary, based on above analysis, we can utilize the known result for ϵ-Milne problem (3.11) to obtain the desired results of the solution to the ϵ-Milne problem (3.1).

Theorem 3.22. *There exists a unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.1) with the normalization condition (3.309) satisfying*

$$\|f(\eta,\tau,\phi)-f_L(\tau)\|_{L^2L^2}\le C. \tag{3.310}$$

Theorem 3.23. *The unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.1) with the normalization condition (3.309) satisfying*

$$\|f(\eta,\tau,\phi)-f_L(\tau)\|_{L^\infty L^\infty}\le C. \tag{3.311}$$

Theorem 3.24. *There exists $K_0>0$ such that the solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.1) with the normalization condition (3.309) satisfies*

$$\left\|\mathrm{e}^{K_0\eta}\Big(f(\eta,\tau,\phi)-f_L(\tau)\Big)\right\|_{L^\infty L^\infty}\le C. \tag{3.312}$$

Theorem 3.25. *There exists $K_0>0$ such that the unique solution $f(\eta,\tau,\phi)$ to the ϵ-Milne problem (3.1) with the normalization condition (3.309) satisfies*

$$\left\|\mathrm{e}^{K_0\eta}\frac{\partial(f-f_L)}{\partial\tau}(\eta,\tau,\phi)\right\|_{L^\infty L^\infty}\le C\left|\ln(\epsilon)\right|^8. \tag{3.313}$$

4. Remainder Estimate

In this section, we consider the remainder equation for $u(\vec{x},\vec{w})$ as

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x u+u-\bar{u} &= f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u(\vec{x}_0,\vec{w}) &= \mathcal{P}[u](\vec{x}_0)+h(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{cases} \tag{4.1}$$

where

$$\bar{u}(\vec{x})=\frac{1}{2\pi}\int_{\mathcal{S}^1}u(\vec{x},\vec{w})\mathrm{d}\vec{w}, \tag{4.2}$$

$$\mathcal{P}[u](\vec{x}_0)=\frac{1}{2}\int_{\vec{w}\cdot\vec{\nu}>0}u(\vec{x}_0,\vec{w})(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}, \tag{4.3}$$

$\vec{\nu}$ is the outward unit normal vector, with the Knudsen number $0<\epsilon<<1$. To guarantee uniqueness, we need the normalization condition

$$\int_{\Omega\times\mathcal{S}^1}u(\vec{x},\vec{w})\mathrm{d}\vec{w}\mathrm{d}\vec{x}=0. \tag{4.4}$$

Also, the data f and h satisfy the compatibility condition

$$\int_{\Omega\times\mathcal{S}^1}f(\vec{x},\vec{w})\mathrm{d}\vec{w}\mathrm{d}\vec{x}+\epsilon\int_{\partial\Omega}\int_{\vec{w}\cdot\vec{\nu}<0}h(\vec{x}_0,\vec{w})(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}\mathrm{d}\vec{x}_0=0. \tag{4.5}$$

We define the L^p norm with $1\le p<\infty$ and L^∞ norms in $\Omega\times\mathcal{S}^1$ as usual:

$$\|f\|_{L^p(\Omega\times\mathcal{S}^1)} = \left(\int_\Omega\int_{\mathcal{S}^1}|f(\vec{x},\vec{w})|^p\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\right)^{1/p}, \tag{4.6}$$

$$\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} = \sup_{(\vec{x},\vec{w})\in\Omega\times\mathcal{S}^1}|f(\vec{x},\vec{w})|. \tag{4.7}$$

Define the L^p norm with $1\le p<\infty$ and L^∞ norms on the boundary as follows:

$$\|f\|_{L^p(\Gamma)} = \left(\iint_\Gamma|f(\vec{x},\vec{w})|^p\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\right)^{1/p}, \tag{4.8}$$

$$\|f\|_{L^p(\Gamma^\pm)} = \left(\iint_{\Gamma^\pm}|f(\vec{x},\vec{w})|^p\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\right)^{1/p}, \tag{4.9}$$

$$\|f\|_{L^\infty(\Gamma)} = \sup_{(\vec{x},\vec{w})\in\Gamma}|f(\vec{x},\vec{w})|, \tag{4.10}$$

$$\|f\|_{L^\infty(\Gamma^\pm)} = \sup_{(\vec{x},\vec{w})\in\Gamma^\pm}|f(\vec{x},\vec{w})|. \tag{4.11}$$

In particular, we denote $\mathrm{d}\gamma=(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}\mathrm{d}\vec{x}_0$ on the boundary.

4.1. **Preliminaries.** In order to show the L^∞ estimates of the equation (4.1), we start with some preparations with the transport equation.

Lemma 4.1. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $h(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then for the transport equation*

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x u+u &= f(\vec{x},\vec{w}) \ \ in \ \ \Omega \\ u(\vec{x}_0,\vec{w}) &= h(\vec{x}_0,\vec{w}) \ \ for \ \ \vec{x}_0\in\partial\Omega \ \ and \ \vec{w}\cdot\vec{\nu}<0, \end{cases} \tag{4.12}$$

there exists a unique solution $u(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ satisfying

$$\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|h\|_{L^\infty(\Gamma^-)} \tag{4.13}$$

Proof. The characteristics $(X(s),W(s))$ of the equation (4.12) which goes through $(\vec{x},\vec{w})$ is defined by

$$\begin{cases} \dfrac{\mathrm{d}X(s)}{\mathrm{d}s}=\epsilon W(s), \quad \dfrac{\mathrm{d}W(s)}{\mathrm{d}s}=0, \\ (X(0),W(0))=(\vec{x},\vec{w}). \end{cases} \tag{4.14}$$

which implies

$$X(s)=\vec{x}+(\epsilon\vec{w})s, \quad W(s)=\vec{w}. \tag{4.15}$$

Along the characteristics, the equation (4.12) takes the form

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s}+u &= f(\vec{x},\vec{w}) \ \text{ in } \ \Omega \\ u(\vec{x}_b,\vec{w}) &= h(\vec{x}_b,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0, \end{cases} \tag{4.16}$$

where

$$t_b(\vec{x},\vec{w})=\inf\{t\ge0:\vec{x}-\epsilon t\vec{w}\in\partial\Omega\}, \tag{4.17}$$

$$x_b(\vec{x},\vec{w})=\vec{x}-\epsilon t_b\vec{w}. \tag{4.18}$$

We rewrite the equation (4.12) along the characteristics as

$$u(\vec{x},\vec{w})=h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}+\int_0^{t_b}f(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})\mathrm{e}^{-(t_b-s)}\mathrm{d}s. \tag{4.19}$$

The existence and uniqueness directly follows from above formulation. Also, we have

$$\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le\|h\|_{L^\infty(\Gamma^-)}+\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.20}$$

Hence, our desired result is obvious. □

4.2. L^2 **Estimate.** In this subsection, we start from the preliminary equation (4.12) and take $\bar{u}$ and $\mathcal{P}[u]$ into consideration.

Lemma 4.2. *Define the near-grazing set of Γ^+ or Γ^- as*

$$\Gamma_\pm^\delta=\left\{(\vec{x},\vec{w})\in\Gamma^\pm:|\vec{\nu}(\vec{x})\cdot\vec{w}|\le\delta\right\}. \tag{4.21}$$

Then

$$\left\|u\mathbf{1}_{\Gamma^\pm\setminus\Gamma_\pm^\delta}\right\|_{L^1(\Gamma^\pm)}\le C(\delta)\bigg(\|u\|_{L^1(\Omega\times\mathcal{S}^1)}+\|\vec{w}\cdot\nabla_x u\|_{L^1(\Omega\times\mathcal{S}^1)}\bigg). \tag{4.22}$$

Proof. See the proof of [3, Lemma 2.1]. □

Lemma 4.3. *(Green's Identity) Assume $u(\vec{x},\vec{w}),\ v(\vec{x},\vec{w})\in L^2(\Omega\times\mathcal{S}^1)$ and $\vec{w}\cdot\nabla_x u,\ \vec{w}\cdot\nabla_x v\in L^2(\Omega\times\mathcal{S}^1)$ with $u,\ v\in L^2(\Gamma)$. Then*

$$\iint_{\Omega\times\mathcal{S}^1}\bigg((\vec{w}\cdot\nabla_x u)v+(\vec{w}\cdot\nabla_x u)v\bigg)\mathrm{d}\vec{x}\mathrm{d}\vec{w}=\int_\Gamma uv\mathrm{d}\gamma. \tag{4.23}$$

Proof. See the proof of [2, Chapter 9] and [3]. □

Lemma 4.4. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $h(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then for the transport equation (4.1), there exists a unique solution $u(\vec{x},\vec{w})\in L^2(\Omega\times\mathcal{S}^1)$ satisfying*

$$\frac{1}{\epsilon}\left\|(1-\mathcal{P})[u]\right\|^2_{L^2(\Gamma^+)}+\|u\|_{L^2(\Omega\times\mathcal{S}^1)}\le C\bigg(\frac{1}{\epsilon^2}\|f\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\|h\|_{L^2(\Gamma^-)}\bigg), \tag{4.24}$$

Proof. We divide the proof into several steps:

Step 1: Penalized equation.
We first consider the penalized equation

$$\begin{cases}\epsilon\vec{w}\cdot\nabla_x u_{j,\lambda}+(1+\lambda)u_{j,\lambda}-\bar{u}_{j,\lambda} &= f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u_{j,\lambda}(\vec{x}_0,\vec{w}) &= \left(1-\frac{1}{j}\right)\mathcal{P}[u_{j,\lambda}](\vec{x}_0)+h(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega,\end{cases} \tag{4.25}$$

for $\lambda>0$, $j\in\mathbb{N}$ and $j\ge\dfrac{2}{\lambda}$. We iteratively construct an approximating sequence $\{u^k_{j,\lambda}\}_{k=0}^\infty$ where $u^0_{j,\lambda}=0$ and

$$\begin{cases}\epsilon\vec{w}\cdot\nabla_x u^k_{j,\lambda}+(1+\lambda)u^k_{j,\lambda}-\bar{u}^{k-1}_{j,\lambda} &= f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u^k_{j,\lambda}(\vec{x}_0,\vec{w}) &= \left(1-\frac{1}{j}\right)\mathcal{P}[u^{k-1}_{j,\lambda}](\vec{x}_0)+h(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{x}_0\in\partial\Omega \ \text{ and } \vec{w}\cdot\vec{\nu}<0.\end{cases} \tag{4.26}$$

By Lemma 4.1, this sequence is well-defined and $\left\|u^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}<\infty$. We rewrite equation (4.26) along the characteristics as

$$\begin{aligned}u^k_{j,\lambda}(\vec{x},\vec{w})=&\left(h+\left(1-\frac{1}{j}\right)\mathcal{P}[u^{k-1}_{j,\lambda}]\right)(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)t_b}\\ &+\int_0^{t_b}(f+\bar{u}^{k-1}_{j,\lambda})(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s.\end{aligned} \tag{4.27}$$

We define the difference $v^k=u^k_{j,\lambda}-u^{k-1}_{j,\lambda}$ for $k\ge1$. Then $v^k_{j,\lambda}$ satisfies

$$v^{k+1}_{j,\lambda}(\vec{x},\vec{w})=\left(1-\frac{1}{j}\right)\mathcal{P}[v^k_{j,\lambda}](\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)t_b}+\int_0^{t_b}\bar{v}^k_{j,\lambda}(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s \tag{4.28}$$

Since $\left\|\bar{v}^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}$ and $\left\|\mathcal{P}[v^k_{j,\lambda}]\right\|_{L^\infty(\Gamma^+)}\le\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}$, we can directly estimate

$$\begin{aligned}\left\|v^{k+1}_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}&\le\mathrm{e}^{-(1+\lambda)t_b}\left(1-\frac{1}{j}\right)\left\|v^{k+1}_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\int_0^{t_b}\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s \qquad (4.29)\\ &\le\mathrm{e}^{-(1+\lambda)t_b}\left(1-\frac{1}{j}\right)\left\|v^{k+1}_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{1+\lambda}(1-\mathrm{e}^{-(1+\lambda)t_b})\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\\ &\le\left(1-\frac{1}{j}\right)\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)},\end{aligned}$$

since $j\ge\dfrac{2}{\lambda}$. Hence, we naturally have

$$\left\|v^{k+1}_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le\left(1-\frac{1}{j}\right)\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.30}$$

Thus, this is a contraction iteration. Considering $v^1_{j,\lambda}=u^1_{j,\lambda}$, we have

$$\left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le\left(1-\frac{1}{j}\right)^{k-1}\left\|u^1_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.31}$$

for $k \geq 1$. Therefore, $u^k_{j,\lambda}$ converges strongly in L^∞ to the limiting solution $u_{j,\lambda}$ satisfying

$$\|u_{j,\lambda}\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \sum_{k=1}^{\infty} \left\|v^k_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq j \left\|u^1_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.32}$$

Since $u^1_{j,\lambda}$ satisfies the equation

$$u^1_{j,\lambda}(\vec{x},\vec{w}) = h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)t_b} + \int_0^{t_b} f(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s.$$

Based on Lemma 4.1, we can directly estimate

$$\left\|u^1_{j,\lambda}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|h\|_{L^\infty(\Gamma^-)}. \tag{4.33}$$

Combining (4.32) and (4.33), we can naturally obtain the existence and the estimate

$$\|u_{j,\lambda}\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq j\bigg(\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|h\|_{L^\infty(\Gamma^-)} \bigg). \tag{4.34}$$

This justify the well-posedness of $u_{j,\lambda}$. Note that when $\lambda \to 0$ or $j \to \infty$, this estimate blows up. Hence, we have to find a uniform estimate in λ and j.

Step 2: Energy Estimate of $u_{\lambda,j}$.
Multiplying $u_{j,\lambda}$ on both sides of (4.25) and integrating over $\Omega\times\mathcal{S}^1$, by Lemma 4.3, we get the energy estimate

$$\frac{1}{2}\epsilon\int_\Gamma |u_{j,\lambda}|^2\,\mathrm{d}\gamma + \lambda\|u_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \|u_{j,\lambda}-\bar{u}_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} = \iint_{\Omega\times\mathcal{S}^1} f u_{j,\lambda}. \tag{4.35}$$

A direct computation shows

$$\begin{aligned}
&\frac{1}{2}\epsilon\int_\Gamma |u_{j,\lambda}|^2\,\mathrm{d}\gamma \qquad (4.36)\\
&= \frac{1}{2}\epsilon\|u_{j,\lambda}\|^2_{L^2(\Gamma^+)} - \frac{1}{2}\epsilon\left\|\left(1-\frac{1}{j}\right)\mathcal{P}[u_{j,\lambda}] + h\right\|^2_{L^2(\Gamma^-)}\\
&= \frac{1}{2}\epsilon\bigg(\|u_{j,\lambda}\|^2_{L^2(\Gamma^+)} - \left\|\left(1-\frac{1}{j}\right)\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}\bigg) - \frac{1}{2}\epsilon\|h\|^2_{L^2(\Gamma^-)} - \epsilon\left(1-\frac{1}{j}\right)\int_{\Gamma^-} h\mathcal{P}[u_{j,\lambda}]\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\gamma.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{2}\epsilon\bigg(\|u_{j,\lambda}\|^2_{L^2(\Gamma^+)} - \left\|\left(1-\frac{1}{j}\right)\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}\bigg) + \lambda\|u_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \|u_{j,\lambda}-\bar{u}_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} \qquad (4.37)\\
&= \iint_{\Omega\times\mathcal{S}^1} f u_{j,\lambda} + \frac{1}{2}\epsilon\|h\|^2_{L^2(\Gamma^-)} + \epsilon\left(1-\frac{1}{j}\right)\int_{\Gamma^-} h\mathcal{P}[u_{j,\lambda}]\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\gamma.
\end{aligned}$$

Noting the fact that

$$\epsilon\bigg(\|u_{j,\lambda}\|^2_{L^2(\Gamma^+)} - \|\mathcal{P}[u_{j,\lambda}]\|^2_{L^2(\Gamma^-)}\bigg) = \epsilon\|(1-\mathcal{P})[u_{j,\lambda}]\|^2_{L^2(\Gamma^+)}, \tag{4.38}$$

we deduce

$$\begin{aligned}
&\frac{1}{2}\epsilon\|(1-\mathcal{P})[u_{j,\lambda}]\|^2_{L^2(\Gamma^+)} + \lambda\|u_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \|u_{j,\lambda}-\bar{u}_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} \qquad (4.39)\\
&\leq \iint_{\Omega\times\mathcal{S}^1} f u_{j,\lambda} + \frac{1}{2}\epsilon\|h\|^2_{L^2(\Gamma^-)} + \epsilon\int_{\Gamma^-} h\mathcal{P}[u_{j,\lambda}]\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\gamma.
\end{aligned}$$

Applying Cauchy's inequality, we obtain for $\eta > 0$ sufficiently small,

$$\epsilon\int_{\Gamma^-} h\mathcal{P}[u_{j,\lambda}]\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\gamma \leq \frac{4}{\eta}\|h\|^2_{L^2(\Gamma^-)} + \epsilon^2\eta\|\mathcal{P}[u_{j,\lambda}]\|^2_{L^2(\Gamma^-)}, \tag{4.40}$$

which further implies

$$\begin{aligned}&\frac{1}{2}\epsilon\left\|(1-\mathcal{P})[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^+)}+\lambda\left\|u_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|u_{j,\lambda}-\bar{u}_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}\\ \leq&\iint_{\Omega\times\mathcal{S}^1}fu_{j,\lambda}+\left(1+\frac{4}{\eta}\right)\left\|h\right\|^2_{L^2(\Gamma^-)}+\epsilon^2\eta\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}.\end{aligned}\tag{4.41}$$

Now the only difficulty is $\epsilon^2\eta\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}$, which we cannot bound directly.

Step 3: Estimate of $\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}$.
Multiplying $u_{j,\lambda}$ on both sides of (4.25), we have

$$\frac{1}{2}\epsilon\vec{w}\cdot\nabla_x(u^2_{j,\lambda})=-\lambda u^2_{j,\lambda}-u_{j,\lambda}(u_{j,\lambda}-\bar{u}_{j,\lambda})+fu_{j,\lambda}.\tag{4.42}$$

Taking absolute value on both sides of (4.42) and integrating over $\Omega\times\mathcal{S}^1$, we get

$$\left\|\vec{w}\cdot\nabla_x(u^2_{j,\lambda})\right\|_{L^1(\Omega\times\mathcal{S}^1)}\leq\frac{2\lambda}{\epsilon}\left\|u_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{2}{\epsilon}\left\|u_{j,\lambda}-\bar{u}_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{2}{\epsilon}\iint_{\Omega\times\mathcal{S}^1}fu_{j,\lambda}.\tag{4.43}$$

Based on (4.41), we can further obtain

$$\left\|\vec{w}\cdot\nabla_x(u^2_{j,\lambda})\right\|_{L^1(\Omega\times\mathcal{S}^1)}\leq\frac{1}{\epsilon}\left(1+\frac{4}{\eta}\right)\left\|h\right\|^2_{L^2(\Gamma^-)}+\epsilon\eta\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}+\frac{4}{\epsilon}\iint_{\Omega\times\mathcal{S}^1}fu.\tag{4.44}$$

Hence, by Lemma 4.2, (4.41) and (4.44), we know for given $\delta>0$

$$\begin{aligned}\left\|u^2_{j,\lambda}\mathbf{1}_{\Gamma^\pm\setminus\Gamma^\delta_\pm}\right\|_{L^1(\Gamma^\pm)}&\leq C(\delta)\Big(\left\|u_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|\vec{w}\cdot\nabla_x(u^2_{j,\lambda})\right\|_{L^1(\Omega\times\mathcal{S}^1)}\Big)\\ &\leq C(\delta)\bigg(\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)\left(1+\frac{1}{\eta}\right)\left\|h\right\|^2_{L^2(\Gamma^-)}+\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)(\epsilon^2\eta)\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}\\ &\quad+\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)\iint_{\Omega\times\mathcal{S}^1}fu_{j,\lambda}\bigg).\end{aligned}\tag{4.45}$$

Noting the fact that

$$\left\|\mathcal{P}[u_{j,\lambda}\mathbf{1}_{\Gamma_+\setminus\Gamma^\delta_+}]\right\|_{L^2(\Gamma^-)}\leq\left\|u_{j,\lambda}\mathbf{1}_{\Gamma_+\setminus\Gamma^\delta_+}\right\|_{L^2(\Gamma^-)},\tag{4.46}$$

and for δ sufficiently small, we have

$$\left\|\mathcal{P}[u_{j,\lambda}\mathbf{1}_{\Gamma_+\setminus\Gamma^\delta_+}]\right\|_{L^2(\Gamma^-)}\geq\frac{1}{2}\left\|\mathcal{P}[u_{j,\lambda}]\right\|_{L^2(\Gamma^-)}.\tag{4.47}$$

Combining with (4.45), we naturally obtain

$$\begin{aligned}\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}&\leq2\left\|\mathcal{P}[u_{j,\lambda}\mathbf{1}_{\Gamma_+\setminus\Gamma^\delta_+}]\right\|^2_{L^2(\Gamma^-)}\leq2\left\|u_{j,\lambda}\mathbf{1}_{\Gamma_+\setminus\Gamma^\delta_+}\right\|^2_{L^2(\Gamma^-)}\\ &\leq C(\delta)\bigg(\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)\left(1+\frac{1}{\eta}\right)\left\|h\right\|^2_{L^2(\Gamma^-)}+\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)(\epsilon^2\eta)\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}\\ &\quad+\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)\iint_{\Omega\times\mathcal{S}^1}fu_{j,\lambda}\bigg).\end{aligned}\tag{4.48}$$

For fixed λ, taking $\eta>0$ sufficiently small, we obtain

$$\left\|\mathcal{P}[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^-)}\leq C\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)\bigg(\left\|h\right\|^2_{L^2(\Gamma^-)}+\iint_{\Omega\times\mathcal{S}^1}fu_{j,\lambda}\bigg).\tag{4.49}$$

Plugging (4.49) into (4.41), we deduce

$$\begin{aligned}&\frac{1}{2}\epsilon\left\|(1-\mathcal{P})[u_{j,\lambda}]\right\|^2_{L^2(\Gamma^+)}+\lambda\left\|u_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|u_{j,\lambda}-\bar{u}_{j,\lambda}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}\\ \leq&C\epsilon^2\left(\frac{1}{\lambda}+\frac{1}{\epsilon}\right)\bigg(\iint_{\Omega\times\mathcal{S}^1}fu_{j,\lambda}+\left\|h\right\|^2_{L^2(\Gamma^-)}\bigg).\end{aligned}\tag{4.50}$$

Step 4: Limit $j \to \infty$.
Naturally, based on (4.50), we deduce

$$\|u_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} \le \frac{C}{\lambda}\Big(\frac{1}{\lambda}+\frac{1}{\epsilon}\Big)\Big(\iint_{\Omega\times\mathcal{S}^1} f u_{j,\lambda} + \|h\|^2_{L^2(\Gamma^-)}\Big). \tag{4.51}$$

Applying Cauchy's inequality, we obtain for $C_0 > 0$ sufficiently small

$$\frac{C}{\lambda}\Big(\frac{1}{\lambda}+\frac{1}{\epsilon}\Big)\iint_{\Omega\times\mathcal{S}^1} f u_{j,\lambda} \le \frac{4(\lambda+\epsilon)^2}{C_0\lambda^4\epsilon^2}\|f\|^2_{L^2(\Omega\times\mathcal{S}^1)} + C_0\|u_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)}. \tag{4.52}$$

Combining (4.51) and (4.52), we obtain

$$\|u_{j,\lambda}\|^2_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\frac{1}{\lambda^6\epsilon^2}\|f\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\lambda^2\epsilon}\|h\|^2_{L^2(\Gamma^-)}\Big), \tag{4.53}$$

which further implies

$$\|u_{j,\lambda}\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\frac{1}{\lambda^3\epsilon}\|f\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\lambda\epsilon^{1/2}}\|h\|_{L^2(\Gamma^-)}\Big). \tag{4.54}$$

Since this estimate is uniform in j, we may take weak limit $u_{j,\lambda} \rightharpoonup u_\lambda$ in $L^2(\Omega\times\mathcal{S}^1)$ as $j\to\infty$ up to a subsequence. By Lemma 4.3 and weak lower semi-continuity, there exists a unique solution u_λ to the equation

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x u_\lambda + (1+\lambda)u_\lambda - \bar{u}_\lambda &= f(\vec{x},\vec{w}) \ \text{ in } \ \Omega, \\ u_\lambda(\vec{x}_0,\vec{w}) &= \mathcal{P}[u_\lambda](\vec{x}_0) + h(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{cases} \tag{4.55}$$

and satisfies

$$\|u_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\frac{1}{\lambda^3\epsilon}\|f\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\lambda\epsilon^{1/2}}\|h\|_{L^2(\Gamma^-)}\Big). \tag{4.56}$$

However, this estimate still blows up when $\lambda\to 0$, so we need to find a uniform estimate for u_λ.

Step 5: Kernel Estimate.
Applying Lemma 4.3 to the equation (4.55). Then for any $\phi\in L^2(\Omega\times\mathcal{S}^1)$ satisfying $\vec{w}\cdot\nabla_x\phi\in L^2(\Omega\times\mathcal{S}^1)$ and $\phi\in L^2(\Gamma)$, we have

$$\lambda\iint_{\Omega\times\mathcal{S}^1} u_\lambda\phi + \epsilon\int_\Gamma u_\lambda\phi \mathrm{d}\gamma - \epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u_\lambda + \iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi = \iint_{\Omega\times\mathcal{S}^1} f\phi. \tag{4.57}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function ζ. Since $u_\lambda\in L^2(\Omega\times\mathcal{S}^1)$, it naturally implies $\bar{u}_\lambda\in L^2(\Omega)$. We define $\zeta(\vec{x})$ on Ω satisfying

$$\begin{cases} \Delta\zeta &= \bar{u}_\lambda \ \text{ in } \ \Omega, \\ \dfrac{\partial\zeta}{\partial\vec{\nu}} &= 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.58}$$

A direct integration over $\Omega\times\mathcal{S}^1$ in (4.55) implies

$$\int_{\Omega\times\mathcal{S}^1} u_\lambda(\vec{x},\vec{w})\mathrm{d}\vec{w}\mathrm{d}\vec{x} = 0. \tag{4.59}$$

Hence, in the bounded domain Ω, based on the standard elliptic estimate, there exists $\zeta\in H^2(\Omega)$ such that

$$\|\zeta\|_{H^2(\Omega)} \le C\|\bar{u}_\lambda\|_{L^2(\Omega)}, \tag{4.60}$$

and

$$\int_\Omega \zeta(\vec{x})\mathrm{d}\vec{x} = 0. \tag{4.61}$$

We plug the test function

$$\phi = -\vec{w}\cdot\nabla_x\zeta \tag{4.62}$$

into the weak formulation (4.57) and estimate each term there. Naturally, we have

$$\|\phi\|_{L^2(\Omega)} \le C\,\|\zeta\|_{H^1(\Omega)} \le C\,\|\bar{u}_\lambda\|_{L^2(\Omega)}\,. \tag{4.63}$$

Easily we can decompose

$$-\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)u_\lambda = -\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)\bar{u}_\lambda - \epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)(u_\lambda-\bar{u}_\lambda). \tag{4.64}$$

We estimate the two term on the right-hand side separately. By (4.58) and (4.62), we have

$$\begin{aligned} -\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)\bar{u}_\lambda &= \epsilon \iint_{\Omega\times\mathcal{S}^1} \bar{u}_\lambda\Big(w_1(w_1\partial_{11}\zeta + w_2\partial_{12}\zeta) + w_2(w_1\partial_{12}\zeta + w_2\partial_{22}\zeta)\Big) \\ &= \epsilon \iint_{\Omega\times\mathcal{S}^1} \bar{u}_\lambda\Big(w_1^2\partial_{11}\zeta + w_2^2\partial_{22}\zeta\Big) \\ &= 2\epsilon\pi \int_\Omega \bar{u}_\lambda(\partial_{11}\zeta+\partial_{22}\zeta) \\ &= 2\epsilon\pi\,\|\bar{u}_\lambda\|^2_{L^2(\Omega)} \\ &= \epsilon\,\|\bar{u}_\lambda\|^2_{L^2(\Omega\times\mathcal{S}^1)}\,. \end{aligned} \tag{4.65}$$

In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathcal{S}^1$. On the other hand, for the second term in (4.64), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned} -\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)(u_\lambda-\bar{u}_\lambda) &\le C\epsilon\,\|u_\lambda-\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,\|\zeta\|_{H^2(\Omega)} \\ &\le C\epsilon\,\|u_\lambda-\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,\|\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,. \end{aligned} \tag{4.66}$$

We may decompose $\vec{w} = (\vec{w}\cdot\vec{\nu})\vec{\nu} + \vec{w}_\perp$ where $\vec{w}_\perp$ denotes the tangential component and combine with (4.58) to obtain

$$\begin{aligned} \epsilon\int_\Gamma u_\lambda\phi\,\mathrm{d}\gamma &= \epsilon\int_\Gamma u_\lambda(\vec{w}\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_\Gamma u_\lambda(\vec{\nu}\cdot\nabla_x\zeta)(\vec{w}\cdot\vec{\nu})\mathrm{d}\gamma + \epsilon\int_\Gamma u_\lambda(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_\Gamma u_\lambda(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma. \end{aligned} \tag{4.67}$$

Based on (4.60), (4.63), the boundary condition of the penalized neutron transport equation (4.55), the trace theorem, Hölder's inequality and the elliptic estimate, we have

$$\begin{aligned} \epsilon\int_\Gamma u_\lambda\phi\,\mathrm{d}\gamma &= \epsilon\int_\Gamma u_\lambda(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_\Gamma \mathcal{P}[u_\lambda](\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma + \epsilon\int_{\Gamma^+}(1-\mathcal{P})[u_\lambda](\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma + \epsilon\int_{\Gamma^-} h(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_{\Gamma^+}(1-\mathcal{P})[u_\lambda](\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma + \epsilon\int_{\Gamma^-} h(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &\le \epsilon\,\|\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\Big(\,\|(1-\mathcal{P})[u_\lambda]\|_{L^2(\Gamma^+)} + \|h\|_{L^2(\Gamma^-)}\Big). \end{aligned} \tag{4.68}$$

Also, we obtain

$$\begin{aligned} \lambda\iint_{\Omega\times\mathcal{S}^1} u_\lambda\phi = \lambda\iint_{\Omega\times\mathcal{S}^1}\bar{u}_\lambda\phi + \lambda\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi &= \lambda\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi \\ &\le C\lambda\,\|\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,\|u_\lambda-\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,, \end{aligned} \tag{4.69}$$

$$\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi \le C\,\|\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,\|u_\lambda-\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,, \tag{4.70}$$

$$\iint_{\Omega\times\mathcal{S}^1} f\phi \le C\,\|\bar{u}_\lambda\|_{L^2(\Omega\times\mathcal{S}^1)}\,\|f\|_{L^2(\Omega\times\mathcal{S}^1)}\,. \tag{4.71}$$

Collecting terms in (4.65), (4.66), (4.68), (4.69), (4.70) and (4.71), we obtain

$$\epsilon \left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big((1+\epsilon+\lambda)\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon\left\|u_\lambda\right\|_{L^2(\Gamma^+)} + \left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon\left\|(1-\mathcal{P})[u_\lambda]\right\|_{L^2(\Gamma^+)} + \epsilon\left\|h\right\|_{L^2(\Gamma^-)}\Big), \tag{4.72}$$

When $0\le\lambda<1$ and $0<\epsilon<1$, we get the desired uniform estimate with respect to λ as

$$\epsilon \left\|\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon\left\|u_\lambda\right\|_{L^2(\Gamma^+)} + \left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon\left\|(1-\mathcal{P})[u_\lambda]\right\|_{L^2(\Gamma^+)} + \epsilon\left\|h\right\|_{L^2(\Gamma^-)}\Big), \tag{4.73}$$

Step 6: Limit $\lambda\to 0$.
In the weak formulation (4.57), we may take the test function $\phi=u_\lambda$ to get the energy estimate

$$\lambda\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{2}\epsilon\int_\Gamma |u_\lambda|^2\,\mathrm{d}\gamma + \left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} = \iint_{\Omega\times\mathcal{S}^1} f u_\lambda. \tag{4.74}$$

Similar to (4.41), we have

$$\begin{aligned} &\frac{1}{2}\epsilon\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)} + \lambda\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \\ \le\ & \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \left(1+\frac{4}{\eta}\right)\left\|h\right\|^2_{L^2(\Gamma^-)} + \epsilon^2\eta\left\|\mathcal{P}[u_\lambda]\right\|^2_{L^2(\Gamma^-)}. \end{aligned} \tag{4.75}$$

Also, based on Step 3, we know

$$\begin{aligned} \left\|\mathcal{P}[u_\lambda]\right\|^2_{L^2(\Gamma^-)} &\le C\Big(\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\|\vec{w}\cdot\nabla_x(u_\lambda^2)\right\|_{L^1(\Omega\times\mathcal{S}^1)}\Big) \\ &\le C\Big(\left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\|\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon}\Big(1+\frac{1}{\eta}\Big)\left\|h\right\|^2_{L^2(\Gamma^-)} + \frac{1}{\epsilon}\iint_{\Omega\times\mathcal{S}^1} f u\Big). \end{aligned} \tag{4.76}$$

Hence, combining (4.75) and (4.76), we know

$$\epsilon\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)} + \left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\epsilon^2\eta\left\|\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \left\|h\right\|^2_{L^2(\Gamma^-)}\Big). \tag{4.77}$$

On the other hand, we can square on both sides of (4.73) to obtain

$$\epsilon^2\left\|\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon^2\left\|u_\lambda\right\|^2_{L^2(\Gamma^+)} + \left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \epsilon^2\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)} + \epsilon^2\left\|h\right\|^2_{L^2(\Gamma^-)}\Big). \tag{4.78}$$

Taking η sufficiently small, multiplying a sufficiently small constant on both sides of (4.78) and adding it to (4.77) to absorb $\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)}$, $\epsilon^2\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$ and $\left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$, we deduce

$$\begin{aligned} &\epsilon\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)} + \epsilon^2\left\|\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\|u_\lambda-\bar{u}_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \\ \le\ & C\Big(\left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \left\|h\right\|^2_{L^2(\Gamma^-)}\Big). \end{aligned} \tag{4.79}$$

Hence, we have

$$\epsilon\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)} + \epsilon^2\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \le C\Big(\left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} f u_\lambda + \left\|h\right\|^2_{L^2(\Gamma^-)}\Big). \tag{4.80}$$

A simple application of Cauchy's inequality leads to

$$\iint_{\Omega\times\mathcal{S}^1} f u_\lambda \le \frac{1}{4C\epsilon^2}\left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + C\epsilon^2\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}. \tag{4.81}$$

Taking C sufficiently small, we can divide (4.80) by ϵ^2 to obtain

$$\frac{1}{\epsilon}\left\|(1-\mathcal{P})[u_\lambda]\right\|^2_{L^2(\Gamma^+)}+\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}\leq C\bigg(\frac{1}{\epsilon^4}\left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}+\frac{1}{\epsilon^2}\left\|h\right\|^2_{L^2(\Gamma^-)}\bigg). \tag{4.82}$$

Since above estimate does not depend on λ, it gives a uniform estimate for the penalized neutron transport equation (4.55). Thus, we may take weak limit $u_\lambda \rightharpoonup u$ in $L^2(\Omega\times\mathcal{S}^1)$ as $\lambda\to 0$ up to a subsequence. By weak lower semi-continuity, we have u also satisfies the estimate (4.82). Hence, in the weak formulation (4.57), we can take $\lambda\to 0$ to deduce that u satisfies equation (4.1). Also $u_\lambda - u$ satisfies the equation

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x(u_\lambda-u)+(u_\lambda-u)-(\bar{u}_\lambda-\bar{u}) &= -\lambda u_\lambda \ \text{ in } \ \Omega, \\ (u_\lambda-u)(\vec{x}_0,\vec{w}) &= \mathcal{P}[u_\lambda-u](\vec{x}_0) \ \text{ for } \ \vec{x}_0\in\partial\Omega \ \text{ and } \vec{w}\cdot\vec{n}<0. \end{cases} \tag{4.83}$$

By a similar argument as above, we can achieve

$$\left\|u_\lambda-u\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}\leq C\bigg(\frac{\lambda}{\epsilon^4}\left\|u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^2)}\bigg). \tag{4.84}$$

When $\lambda\to 0$, the right-hand side approaches zero, which implies the convergence is actually in the strong sense. The uniqueness easily follows from the energy estimates.

□

4.3. L^∞ **Estimate - First Round.** In this subsection, we prove the L^∞ estimate. We consider the characteristics that reflect several times on the boundary.

Definition 4.5. *(Stochastic Cycle) For fixed point $(t,\vec{x},\vec{w})$ with $(\vec{x},\vec{w})\notin\Gamma^0$, let $(t_0,\vec{x}_0,\vec{w}_0)=(0,\vec{x},\vec{w})$. For $\vec{w}_{k+1}$ such that $\vec{w}_{k+1}\cdot\vec{\nu}(\vec{x}_{k+1})>0$, define the $(k+1)$-component of the back-time cycle as*

$$(t_{k+1},\vec{x}_{k+1},\vec{w}_{k+1})=(t_k+t_b(\vec{x}_k,\vec{w}_k),\vec{x}_b(\vec{x}_k,\vec{w}_k),\vec{w}_{k+1}) \tag{4.85}$$

where

$$t_b(\vec{x},\vec{w})=\inf\{t>0:\vec{x}-\epsilon t\vec{w}\notin\Omega\} \tag{4.86}$$

$$x_b(\vec{x},\vec{w})=\vec{x}-\epsilon t_b(\vec{x},\vec{w})\vec{w}\notin\Omega \tag{4.87}$$

Set

$$X_{cl}(s;t,\vec{x},\vec{w})=\sum_k \mathbf{1}_{t_{k+1}\leq s<t_k}\Big(\vec{x}_k-\epsilon(t_k-s)\vec{w}_k\Big) \tag{4.88}$$

$$W_{cl}(s;t,\vec{x},\vec{w})=\sum_k \mathbf{1}_{t_{k+1}\leq s<t_k}\vec{w}_k \tag{4.89}$$

Define $\mu_{k+1}=\{\vec{w}\in\mathcal{S}^1:\vec{w}\cdot\vec{\nu}(\vec{x}_{k+1})>0\}$, and let the iterated integral for $k\geq 2$ be defined as

$$\int_{\prod_{k=1}^{k-1}\mu_j}\prod_{j=1}^{k-1}\mathrm{d}\sigma_j=\int_{\mu_1}\dots\bigg(\int_{\mu_{k-1}}\mathrm{d}\sigma_{k-1}\bigg)\dots\mathrm{d}\sigma_1 \tag{4.90}$$

where $\mathrm{d}\sigma_j=(\vec{\nu}(\vec{x}_j)\cdot\vec{w})\mathrm{d}\vec{w}$ is a probability measure.

Lemma 4.6. *For $T_0>0$ sufficiently large, there exists constants $C_1,C_2>0$ independent of T_0, such that for $k=C_1T_0^{5/4}$,*

$$\int_{\prod_{j=1}^{k-1}\mu_j}\mathbf{1}_{t_k(t,\vec{x},\vec{w},\vec{w}_1,\dots,\vec{w}_{k-1})<T_0}\prod_{j=1}^{k-1}\mathrm{d}\sigma_j\leq\bigg(\frac{1}{2}\bigg)^{C_2T_0^{5/4}} \tag{4.91}$$

Proof. See [3, Lemma 4.1]. □

Theorem 4.7. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $h(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then the solution $u(\vec{x},\vec{w})$ to the transport equation (4.1) satisfies*

$$\left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\leq C(\Omega)\bigg(\frac{1}{\epsilon^3}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^2}\left\|h\right\|_{L^2(\Gamma^-)}+\left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg).$$

Proof. We divide the proof into several steps:

Step 1: Mild formulation.
We rewrite the equation (4.1) along the characteristics as

$$u(\vec{x},\vec{w}) = h(\vec{x}-\epsilon t_1\vec{w},\vec{w})\mathrm{e}^{-t_1} + \mathcal{P}[u](\vec{x}-\epsilon t_1\vec{w},\vec{w})\mathrm{e}^{-t_1} + \int_0^{t_1} f(\vec{x}-\epsilon(t_1-s_1)\vec{w},\vec{w})\mathrm{e}^{-(t_1-s_1)}\mathrm{d}s_1 + \int_0^{t_1}\bar{u}(\vec{x}-\epsilon(t_1-s_1)\vec{w})\mathrm{e}^{-(t_1-s_1)}\mathrm{d}s_1. \tag{4.92}$$

Note that here $\mathcal{P}[u]$ is an integral over μ_1 at $\vec{x}_1$, using stochastic cycle, we may rewrite it again along the characteristics to $\vec{x}_2$. This process can continue to arbitrary $\vec{x}_k$. Then we get

$$\begin{aligned} u(\vec{x},\vec{w}) &= \mathrm{e}^{-t_1}H + \sum_{l=1}^{k-1}\Big(\int_{\prod_{j=1}^{l}} \mathrm{e}^{-t_{l+1}}G\prod_{j=1}^{l}\mathrm{d}\sigma_j\Big) + \sum_{l=1}^{k-1}\Big(\int_{\prod_{j=1}^{l}} \mathrm{e}^{-t_{l+1}}\mathcal{P}[u](\vec{x}_k,\vec{w}_{k-1})\prod_{j=1}^{l}\mathrm{d}\sigma_j\Big) \\ &= I + II + III. \end{aligned} \tag{4.93}$$

where

$$\begin{aligned} H &= h(\vec{x}-\epsilon t_1\vec{w},\vec{w}) \\ &\quad + \int_0^{t_1} f(\vec{x}-\epsilon(t_1-s_1)\vec{w},\vec{w})\mathrm{e}^{s_1}\mathrm{d}s_1 + \int_0^{t_1}\bar{u}(\vec{x}-\epsilon(t_1-s_1)\vec{w})\mathrm{e}^{s_1}\mathrm{d}s_1, \end{aligned} \tag{4.94}$$

$$\begin{aligned} G &= h(\vec{x}_l-\epsilon t_{l+1}\vec{w}_l,\vec{w}_l) \\ &\quad + \int_0^{t_l} f(\vec{x}_l-\epsilon(t_{l+1}-s_{l+1})\vec{w}_l,\vec{w}_l)\mathrm{e}^{s_{l+1}}\mathrm{d}s_{l+1} + \int_0^{t_l}\bar{u}(\vec{x}_l-\epsilon(t_{l+1}-s_{l+1})\vec{w}_l)\mathrm{e}^{s_{l+1}}\mathrm{d}s_{l+1}. \end{aligned} \tag{4.95}$$

We need to estimate each term on the right-hand side of (4.93).

Step 2: Estimate of mild formulation.
We first consider III. We may decompose it as

$$\begin{aligned} III &= \sum_{l=1}^{k-1}\int_{\prod_{j=1}^{l}} \mathcal{P}[u](\vec{x}_k,\vec{w}_{k-1})\mathrm{e}^{-t_{l+1}}\prod_{j=1}^{l}\mathrm{d}\sigma_j \\ &= \sum_{l=1}^{k-1}\int_{\prod_{j=1}^{l}} \mathbf{1}_{t_k\le T_0}\mathcal{P}[u](\vec{x}_k,\vec{w}_{k-1})\mathrm{e}^{-t_{l+1}}\prod_{j=1}^{l}\mathrm{d}\sigma_j \\ &\quad + \sum_{l=1}^{k-1}\int_{\prod_{j=1}^{l}} \mathbf{1}_{t_k\ge T_0}\mathcal{P}[u](\vec{x}_k,\vec{w}_{k-1})\mathrm{e}^{-t_{l+1}}\prod_{j=1}^{l}\mathrm{d}\sigma_j, \\ &= III_1 + III_2, \end{aligned} \tag{4.96}$$

where $T_0>0$ is defined as in Lemma 4.6. Then we take $k = C_1T_0^{5/4}$. By Lemma 4.6, we deduce

$$|III_1| \le C\left(\frac{1}{2}\right)^{C_2T_0^{5/4}} \|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.97}$$

Also, we may directly estimate

$$|III_2| \le C\mathrm{e}^{-T_0}\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}. \tag{4.98}$$

Then taking T_0 sufficiently large, we know

$$|III| \le \delta\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}, \tag{4.99}$$

for $\delta>0$ small. On the other hand, we may directly estimate the terms in I and II related to h and f, which we denote as I_1 and II_1. For fixed T, it is easy to see

$$|I_1| + |II_1| \le \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|h\|_{L^\infty(\Gamma^-)}. \tag{4.100}$$

Step 3: Estimate of $\bar{u}$ term.

The most troubling terms are related to $\bar{u}$. Here, we use the trick as in [3] and [18]. Collecting the results in (4.99) and (4.100), we obtain

$$\begin{aligned} |u| \leq & A + \left| \int_0^{t_1} \bar{u}(\vec{x} - \epsilon(t_1 - s_1)\vec{w}) \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right| \\ & + \left| \sum_{l=1}^{k-1} \left(\int_{\prod_{j=1}^{l}} \left(\int_0^{t_l} \bar{u}(\vec{x}_l - \epsilon(t_{l+1} - s_{l+1})\vec{w}_l) \mathrm{e}^{-(t_{l+1} - s_{l+1})} \mathrm{d}s_{l+1} \right) \prod_{j=1}^{l} \mathrm{d}\sigma_j \right) \right|, \\ = & A + I_2 + II_2, \end{aligned} \tag{4.101}$$

where

$$A = \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|h\|_{L^\infty(\Gamma^-)} + \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^1)}. \tag{4.102}$$

By definition, we know

$$|I_2| = \left| \int_0^{t_1} \left(\int_{\mathcal{S}^1} u(\vec{x} - \epsilon(t_1 - s_1)\vec{w}, \vec{w}_{s_1}) \mathrm{d}\vec{w}_{s_1} \right) \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right|, \tag{4.103}$$

where $\vec{w}_{s_1} \in \mathcal{S}^1$ is a dummy variable. Then we can utilize the mild formulation (4.93) to rewrite $u(\vec{x} - \epsilon(t_1 - s_1)\vec{w}, \vec{w}_{s_1})$ along the characteristics. We denote the stochastic cycle as $(t_k', \vec{x}_k', \vec{w}_k')$ correspondingly and $(t_0', \vec{x}_0', \vec{w}_0') = (0, \vec{x} - \epsilon(t_1 - s_1)\vec{w}, \vec{w}_{s_1})$. Then

$$\begin{aligned} |I_2| \leq & \left| \int_0^{t_1} \left(\int_{\mathcal{S}^1} A \mathrm{d}\vec{w}_{s_1} \right) \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right| \\ & + \left| \int_0^{t_1} \left(\int_{\mathcal{S}^1} \int_0^{t_1'} \bar{u}(\vec{x}' - \epsilon(t_1' - s_1')\vec{w}_{s_1}) \mathrm{e}^{-(t_1' - s_1')} \mathrm{d}s_1' \mathrm{d}\vec{w}_{s_1} \right) \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right| \\ & + \left| \int_0^{t_1} \left(\int_{\mathcal{S}^1} \sum_{l'=1}^{k-1} \int_{\prod_{j'=1}^{l'}} \left(\int_0^{t_{l'}'} \bar{u}(\vec{x}_{l'} - \epsilon(t_{l'+1}' - s_{l'+1}')\vec{w}_{l'}) \mathrm{e}^{-(t_{l'+1}' - s_{l'+1}')} \mathrm{d}s_{l'+1}' \right) \prod_{j'=1}^{l'} \mathrm{d}\sigma_{j'} \mathrm{d}\vec{w}_{s_1} \right) \right. \\ & \left. \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right|, \\ = & |I_{2,1}| + |I_{2,2}| + |I_{2,3}|. \end{aligned} \tag{4.104}$$

It is obvious that

$$\begin{aligned} |I_{2,1}| = & \left| \int_0^{t_1} \left(\int_{\mathcal{S}^1} A \mathrm{d}\vec{w}_{s_1} \right) \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right| \leq A \\ \leq & \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|h\|_{L^\infty(\Gamma^-)} + \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^1)}. \end{aligned} \tag{4.105}$$

Then by definition, we know

$$|I_{2,2}| = \left| \int_0^{t_1} \left(\int_{\mathcal{S}^1} \int_0^{t_1'} \bar{u}(\vec{x}' - \epsilon(t_1' - s_1')\vec{w}_{s_1}) \mathrm{e}^{-(t_1' - s_1')} \mathrm{d}s_1' \mathrm{d}\vec{w}_{s_1} \right) \mathrm{e}^{-(t_1 - s_1)} \mathrm{d}s_1 \right|. \tag{4.106}$$

We may decompose this integral

$$\int_0^{t_1} \int_{\mathcal{S}^1} \int_0^{t_1'} = \int_0^{t_1} \int_{\mathcal{S}^1} \int_{t_1' - s_1' \leq \delta} + \int_0^{t_1} \int_{\mathcal{S}^1} \int_{t_1' - s_1' \geq \delta} = I_{2,2,1} + I_{2,2,2}. \tag{4.107}$$

For $I_{2,2,1}$, since the integral is defined in the small domain $[t_1' - \delta, t_1']$, it is easy to see

$$|I_{2,2,1}| \leq \delta \|u\|_{L^\infty(\Omega \times \mathcal{S}^1)}. \tag{4.108}$$

For $I_{2,2,2}$, applying Hölder's inequality, we get

$$
\begin{aligned}
|I_{2,2,2}| \leq & \left| \int_0^{t_1} \int_{\mathcal{S}^1} \int_{t_1'-s_1'\geq\delta} \bar{u}(\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}) \mathrm{e}^{-(t_1'-s_1')} \mathrm{e}^{-(t_1-s_1)} \mathrm{d}s_1' \mathrm{d}\vec{w}_{s_1} \mathrm{d}s_1 \right| \qquad (4.109)\\
\leq & \left(\int_0^{t_1} \int_{\mathcal{S}^1} \int_{t_1'-s_1'\geq\delta} \mathrm{e}^{-(t_1'-s_1')} \mathrm{e}^{-(t_1-s_1)} \mathrm{d}s_1' \mathrm{d}\vec{w}_{s_1} \mathrm{d}s_1 \right)^{1/2} \\
& \left(\int_0^{t_1} \int_{\mathcal{S}^1} \int_{t_1'-s_1'\geq\delta} \mathbf{1}_{\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}\in\Omega} \left|\bar{u}\right|^2 (\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}) \mathrm{e}^{-(t_1'-s_1')} \mathrm{e}^{-(t_1-s_1)} \mathrm{d}s_1' \mathrm{d}\vec{w}_{s_1} \mathrm{d}s_1 \right)^{1/2} \\
\leq & \left(\int_0^{t_1} \int_{\mathcal{S}^1} \int_{t_1'-s_1'\geq\delta} \mathbf{1}_{\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}\in\Omega} \left|\bar{u}\right|^2 (\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}) \mathrm{e}^{-(t_1'-s_1')} \mathrm{e}^{-(t_1-s_1)} \mathrm{d}s_1' \mathrm{d}\vec{w}_{s_1} \mathrm{d}s_1 \right)^{1/2}.
\end{aligned}
$$

Since $\vec{w}_{s_1} \in \mathcal{S}^1$, we can express it as $(\cos\phi, \sin\phi)$. Then considering $\vec{x}' - \epsilon(t_1' - s_1')\vec{w}_{s_1} \in \bar{\Omega}$, we apply the substitution $(\phi, s_1') \to (y_1, y_2)$ as

$$
\vec{y} = \vec{x}' - \epsilon(t_1' - s_1')\vec{w}_{s_1}, \qquad (4.110)
$$

whose Jacobian is

$$
\left| \frac{\partial(y_1, y_2)}{\partial(\phi, r')} \right| = \left\| \begin{array}{cc} \epsilon(t_1'-s_1')\sin\phi & \cos\phi \\ -\epsilon(t_1'-s_1')\cos\phi & \sin\phi \end{array} \right\| = \epsilon^2(t_1'-s_1') \geq \epsilon^2\delta. \qquad (4.111)
$$

Therefore, we know

$$
|I_{2,2,2}| \leq \frac{1}{\epsilon\delta^{\frac{1}{2}}} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)}. \qquad (4.112)
$$

Therefore, we have shown

$$
|I_{2,2}| \leq \delta \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{\delta^{\frac{1}{2}}\epsilon} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)}. \qquad (4.113)
$$

After a similar but tedious computation, we can show

$$
|I_{2,3}| \leq \delta \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{\delta^{\frac{1}{2}}\epsilon} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)}. \qquad (4.114)
$$

Hence, we have proved

$$
|I_2| \leq \delta \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{\delta^{\frac{1}{2}}\epsilon} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|h\right\|_{L^\infty(\Gamma^-)}. \qquad (4.115)
$$

In a similar fashion, we can show

$$
|II_2| \leq \delta \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{\delta^{\frac{1}{2}}\epsilon} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|h\right\|_{L^\infty(\Gamma^-)}. \qquad (4.116)
$$

Step 4: Synthesis.
Summarizing all above, we have shown

$$
\begin{aligned}
|u| \leq & \delta \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{\delta^{\frac{1}{2}}\epsilon} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|h\right\|_{L^\infty(\Gamma^-)} \qquad (4.117)\\
\leq & \delta \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \frac{1}{\delta^{\frac{1}{2}}\epsilon} \left\|u\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|h\right\|_{L^\infty(\Gamma^-)}.
\end{aligned}
$$

Since $(\vec{x}, \vec{w})$ are arbitrary and δ is small, taking supremum on both sides and applying Lemma 4.4, we have

$$
\begin{aligned}
\left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq & C(\Omega)\left(\frac{1}{\epsilon} \left\|u\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|h\right\|_{L^\infty(\Gamma^-)} \right) \qquad (4.118)\\
\leq & C(\Omega)\left(\frac{1}{\epsilon^3} \left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon^2} \left\|h\right\|_{L^2(\Gamma^-)} + \left\|f\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \left\|h\right\|_{L^\infty(\Gamma^-)} \right).
\end{aligned}
$$

This is the desired result. □

4.4. L^{2m} **Estimate.** In this subsection, we try to improve previous estimates. In the following, we assume $m > 2$ is an integer and let $o(1)$ denote a sufficiently small constant.

Theorem 4.8. *Assume $f(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathcal{S}^1)$ and $h(x_0,\vec{w}) \in L^\infty(\Gamma^-)$. Then $u(\vec{x},\vec{w})$ satisfies*

$$\begin{aligned}&\frac{1}{\epsilon^{\frac{1}{2}}}\left\|(1-\mathcal{P})[u]\right\|_{L^2(\Gamma^+)} + \left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon}\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} \\ \le C\bigg(&o(1)\epsilon^{\frac{1}{m}}\left\|u\right\|_{L^\infty(\Gamma^+)} + \frac{1}{\epsilon}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon^2}\left\|f\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon}\left\|h\right\|_{L^2(\Gamma^-)} + \left\|h\right\|_{L^m(\Gamma^-)}\bigg).\end{aligned} \tag{4.119}$$

Proof. We divide the proof into several steps:

Step 1: Kernel Estimate.
Applying Green's identity to the equation (4.1). Then for any $\phi \in L^2(\Omega\times\mathcal{S}^1)$ satisfying $\vec{w}\cdot\nabla_x\phi \in L^2(\Omega\times\mathcal{S}^1)$ and $\phi \in L^2(\Gamma)$, we have

$$\epsilon\int_\Gamma u\phi \mathrm{d}\gamma - \epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u + \iint_{\Omega\times\mathcal{S}^1}(u-\bar{u})\phi = \iint_{\Omega\times\mathcal{S}^1} f\phi. \tag{4.120}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function ζ. Naturally $u \in L^\infty(\Omega\times\mathcal{S}^1)$ implies $\bar{u} \in L^{2m}(\Omega)$ which further leads to $(\bar{u})^{2m-1} \in L^{\frac{2m}{2m-1}}(\Omega)$. We define $\zeta(\vec{x})$ on Ω satisfying

$$\begin{cases} \Delta\zeta &= (\bar{u})^{2m-1} - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega (\bar{u})^{2m-1}\mathrm{d}\vec{x} \ \text{ in } \ \Omega, \\ \dfrac{\partial\zeta}{\partial\vec{\nu}} &= 0 \ \text{ on } \ \partial\Omega. \end{cases} \tag{4.121}$$

In the bounded domain Ω, based on the standard elliptic estimate, we have a unique ζ satisfying

$$\left\|\zeta\right\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \le C\left\|(\bar{u})^{2m-1}\right\|_{L^{\frac{2m}{2m-1}}(\Omega)} = C\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}, \tag{4.122}$$

and

$$\int_\Omega \zeta(\vec{x})\mathrm{d}\vec{x} = 0. \tag{4.123}$$

We plug the test function

$$\phi = -\vec{w}\cdot\nabla_x\zeta \tag{4.124}$$

into the weak formulation (4.120) and estimate each term there. By Sobolev embedding theorem, we have

$$\left\|\phi\right\|_{L^2(\Omega)} \le C\left\|\zeta\right\|_{H^1(\Omega)} \le C\left\|\zeta\right\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \le C\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}, \tag{4.125}$$

$$\left\|\phi\right\|_{L^{\frac{2m}{2m-1}}(\Omega)} \le C\left\|\zeta\right\|_{W^{1,\frac{2m}{2m-1}}(\Omega)} \le C\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}. \tag{4.126}$$

Easily we can decompose

$$-\epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u_\lambda = -\epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)\bar{u}_\lambda - \epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)(u_\lambda-\bar{u}_\lambda). \tag{4.127}$$

We estimate the two term on the right-hand side of (4.127) separately. By (4.121) and (4.124), we have

$$\begin{aligned}-\epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)\bar{u} &= \epsilon\iint_{\Omega\times\mathcal{S}^1}\bar{u}\Big(w_1(w_1\partial_{11}\zeta + w_2\partial_{12}\zeta) + w_2(w_1\partial_{12}\zeta + w_2\partial_{22}\zeta)\Big) \\ &= \epsilon\iint_{\Omega\times\mathcal{S}^1}\bar{u}\Big(w_1^2\partial_{11}\zeta + w_2^2\partial_{22}\zeta\Big) \\ &= 2\epsilon\pi\int_\Omega \bar{u}(\partial_{11}\zeta + \partial_{22}\zeta) \\ &= \epsilon\left\|\bar{u}\right\|^{2m}_{L^{2m}(\Omega)}.\end{aligned} \tag{4.128}$$

In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathcal{S}^1$. On the other hand, for the second term in (4.127), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned} -\epsilon \iint_{\Omega\times\mathcal{S}^1} (\vec{w}\cdot\nabla_x\phi)(u-\bar{u}) &\le C\epsilon \left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} \left\|\nabla_x\phi\right\|_{L^{\frac{2m}{2m-1}}(\Omega)} \\ &\le C\epsilon \left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} \left\|\zeta\right\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \\ &\le C\epsilon \left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} \left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}. \end{aligned} \tag{4.129}$$

Based on (4.122), (4.125), (4.126), Sobolev embedding theorem and the trace theorem, we have

$$\left\|\nabla_x\zeta\right\|_{L^{\frac{m}{m-1}}(\Gamma)} \le C\left\|\nabla_x\zeta\right\|_{W^{\frac{1}{2m},\frac{2m}{2m-1}}(\Gamma)} \le C\left\|\nabla_x\zeta\right\|_{W^{1,\frac{2m}{2m-1}}(\Omega)} \le C\left\|\zeta\right\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \le C\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}. \tag{4.130}$$

We may also decompose $\vec{w} = (\vec{w}\cdot\vec{\nu})\vec{\nu} + \vec{w}_\perp$ to obtain

$$\begin{aligned} \epsilon\int_\Gamma u\phi \mathrm{d}\gamma &= \epsilon\int_\Gamma u(\vec{w}\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_\Gamma u(\vec{\nu}\cdot\nabla_x\zeta)(\vec{w}\cdot\vec{\nu})\mathrm{d}\gamma + \epsilon\int_\Gamma u(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_\Gamma u(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma. \end{aligned} \tag{4.131}$$

Based on (4.122), (4.126) and Hölder's inequality, we have

$$\begin{aligned} \epsilon\int_\Gamma u\phi \mathrm{d}\gamma &= \epsilon\int_\Gamma u(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_\Gamma \mathcal{P}[u](\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma + \epsilon\int_{\Gamma^+}(1-\mathcal{P})[u](\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma + \epsilon\int_{\Gamma^-} h(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &= \epsilon\int_{\Gamma^+}(1-\mathcal{P})[u](\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma + \epsilon\int_{\Gamma^-} h(\vec{w}_\perp\cdot\nabla_x\zeta)\mathrm{d}\gamma \\ &\le C\epsilon\left\|\nabla_x\zeta\right\|_{L^{\frac{m}{m-1}}(\Gamma)}\left(\left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} + \left\|h\right\|_{L^m(\Gamma^-)}\right) \\ &\le C\epsilon\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega\times\mathcal{S}^1)}\left(\left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} + \left\|h\right\|_{L^m(\Gamma^-)}\right). \end{aligned} \tag{4.132}$$

Hence, we know

$$\epsilon\int_\Gamma u\phi \mathrm{d}\gamma \le C\epsilon\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega\times\mathcal{S}^1)}\left(\left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} + \left\|h\right\|_{L^m(\Gamma^-)}\right). \tag{4.133}$$

Also, we have

$$\iint_{\Omega\times\mathcal{S}^1}(u-\bar{u})\phi \le C\left\|\phi\right\|_{L^2(\Omega\times\mathcal{S}^1)}\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)}, \tag{4.134}$$

$$\iint_{\Omega\times\mathcal{S}^1} f\phi \le C\left\|\phi\right\|_{L^2(\Omega\times\mathcal{S}^1)}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\left\|\bar{u}\right\|^{2m-1}_{L^{2m}(\Omega)}\left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)}. \tag{4.135}$$

Collecting terms in (4.128), (4.129), (4.133), (4.134) and (4.135), we obtain

$$\begin{aligned} \epsilon\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} \le C\Big(&\epsilon\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} + \left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|_{L^2(\Omega\times\mathcal{S}^1)} \\ &+ \epsilon\left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} + \epsilon\left\|h\right\|_{L^m(\Gamma^-)}\Big), \end{aligned} \tag{4.136}$$

Step 2: Energy Estimate.
In the weak formulation (4.120), we may take the test function $\phi = u$ to get the energy estimate

$$\frac{1}{2}\epsilon\int_\Gamma |u|^2\,\mathrm{d}\gamma + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} = \iint_{\Omega\times\mathcal{S}^1} fu. \tag{4.137}$$

Hence, by (4.77), this naturally implies

$$\epsilon \left\|(1-\mathcal{P})[u]\right\|^2_{L^2(\Gamma^+)} + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \le o(1)\epsilon^2 \left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} fu + \left\|h\right\|^2_{L^2(\Gamma^-)}. \tag{4.138}$$

On the other hand, we can square on both sides of (4.136) to obtain

$$\begin{aligned}\epsilon^2 \left\|\bar{u}\right\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)} \le C\Bigg(&\epsilon^2 \left\|u-\bar{u}\right\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)} + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \\ &+ \epsilon^2 \left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} + \epsilon^2 \left\|h\right\|^2_{L^m(\Gamma^-)}\Bigg),\end{aligned} \tag{4.139}$$

Multiplying a sufficiently small constant on both sides of (4.139) and adding it to (4.138) to absorb $\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$ and $\epsilon^2 \left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$, we deduce

$$\begin{aligned}&\epsilon \left\|(1-\mathcal{P})[u]\right\|^2_{L^2(\Gamma^+)} + \epsilon^2 \left\|\bar{u}\right\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)} + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} \\ \le C\Bigg(&\epsilon^2 \left\|u-\bar{u}\right\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)} + \epsilon^2 \left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} \\ &+ \left\|f\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + \iint_{\Omega\times\mathcal{S}^1} fu + \left\|h\right\|^2_{L^2(\Gamma^-)} + \epsilon^2 \left\|h\right\|^2_{L^m(\Gamma^-)}\Bigg).\end{aligned} \tag{4.140}$$

By interpolation estimate and Young's inequality, we have

$$\begin{aligned}\left\|(1-\mathcal{P})[u]\right\|_{L^m(\Gamma^+)} &\le \left\|(1-\mathcal{P})[u]\right\|^{\frac{2}{m}}_{L^2(\Gamma^+)} \left\|(1-\mathcal{P})[u]\right\|^{\frac{m-2}{m}}_{L^\infty(\Gamma^+)} \\ &= \left(\frac{1}{\epsilon^{\frac{m-2}{m^2}}} \left\|(1-\mathcal{P})[u]\right\|^{\frac{2}{m}}_{L^2(\Gamma^+)}\right)\left(\epsilon^{\frac{m-2}{m^2}} \left\|(1-\mathcal{P})[u]\right\|^{\frac{m-2}{m}}_{L^\infty(\Gamma^+)}\right) \\ &\le C\left(\frac{1}{\epsilon^{\frac{m-2}{m^2}}} \left\|(1-\mathcal{P})[u]\right\|^{\frac{2}{m}}_{L^2(\Gamma^+)}\right)^{\frac{m}{2}} + o(1)\left(\epsilon^{\frac{m-2}{m^2}} \left\|(1-\mathcal{P})[u]\right\|^{\frac{m-2}{m}}_{L^\infty(\Gamma^+)}\right)^{\frac{m}{m-2}} \\ &\le \frac{C}{\epsilon^{\frac{m-2}{2m}}} \left\|(1-\mathcal{P})[u]\right\|_{L^2(\Gamma^+)} + o(1)\epsilon^{\frac{1}{m}} \left\|(1-\mathcal{P})[u]\right\|_{L^\infty(\Gamma^+)} \\ &\le \frac{C}{\epsilon^{\frac{m-2}{2m}}} \left\|(1-\mathcal{P})[u]\right\|_{L^2(\Gamma^+)} + o(1)\epsilon^{\frac{1}{m}} \left\|u\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{4.141}$$

Similarly, we have

$$\begin{aligned}\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathcal{S}^1)} &\le \left\|u-\bar{u}\right\|^{\frac{1}{m}}_{L^2(\Omega\times\mathcal{S}^1)} \left\|u-\bar{u}\right\|^{\frac{m-1}{m}}_{L^\infty(\Omega\times\mathcal{S}^1)} \\ &= \left(\frac{1}{\epsilon^{\frac{m-1}{m^2}}} \left\|u-\bar{u}\right\|^{\frac{1}{m}}_{L^2(\Omega\times\mathcal{S}^1)}\right)\left(\epsilon^{\frac{m-1}{m^2}} \left\|u-\bar{u}\right\|^{\frac{m-1}{m}}_{L^\infty(\Omega\times\mathcal{S}^1)}\right) \\ &\le C\left(\frac{1}{\epsilon^{\frac{m-1}{m^2}}} \left\|u-\bar{u}\right\|^{\frac{1}{m}}_{L^2(\Omega\times\mathcal{S}^1)}\right)^{m} + o(1)\left(\epsilon^{\frac{m-1}{m^2}} \left\|u-\bar{u}\right\|^{\frac{m-1}{m}}_{L^\infty(\Omega\times\mathcal{S}^1)}\right)^{\frac{m}{m-1}} \\ &\le \frac{C}{\epsilon^{\frac{m-1}{m}}} \left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathcal{S}^1)} + o(1)\epsilon^{\frac{1}{m}} \left\|u-\bar{u}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{4.142}$$

We need this extra $\epsilon^{\frac{1}{m}}$ for the convenience of L^∞ estimate. Then we know for sufficiently small ϵ,

$$\begin{aligned}\epsilon^2 \left\|(1-\mathcal{P})[u]\right\|^2_{L^m(\Gamma^+)} &\le C\epsilon^{2-\frac{m-2}{m}} \left\|(1-\mathcal{P})[u]\right\|^2_{L^2(\Gamma^+)} + o(1)\epsilon^{2+\frac{2}{m}} \left\|u\right\|^2_{L^\infty(\Gamma^+)} \\ &\le o(1)\epsilon \left\|(1-\mathcal{P})[u]\right\|^2_{L^2(\Gamma^+)} + o(1)\epsilon^{2+\frac{2}{m}} \left\|u\right\|^2_{L^\infty(\Gamma^+)}.\end{aligned} \tag{4.143}$$

Similarly, we have

$$\begin{aligned}\epsilon^2 \left\|u-\bar{u}\right\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)} &\le \epsilon^{2-\frac{2m-2}{m}} \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + o(1)\epsilon^{2+\frac{2}{m}} \left\|u\right\|^2_{L^\infty(\Omega\times\mathcal{S}^1)} \\ &\le o(1) \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathcal{S}^1)} + o(1)\epsilon^{2+\frac{2}{m}} \left\|u\right\|^2_{L^\infty(\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{4.144}$$

In (4.42), we can absorb $\|u-\bar{u}\|_{L^2(\Omega\times\mathcal{S}^1)}$ and $\epsilon\|(1-\mathcal{P})[u]\|^2_{L^2(\Gamma^+)}$ into left-hand side to obtain

$$\begin{aligned}&\epsilon\|(1-\mathcal{P})[u]\|^2_{L^2(\Gamma^+)}+\epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)}+\|u-\bar{u}\|^2_{L^2(\Omega\times\mathcal{S}^1)}\\ &\le C\bigg(o(1)\epsilon^{2+\frac{2}{m}}\|u\|^2_{L^\infty(\Omega\times\mathcal{S}^1)}+\|f\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\iint_{\Omega\times\mathcal{S}^1}fu+\|h\|^2_{L^2(\Gamma^-)}+\epsilon^2\|h\|^2_{L^m(\Gamma^-)}\bigg).\end{aligned}\tag{4.145}$$

We can decompose

$$\iint_{\Omega\times\mathcal{S}^1}fu=\iint_{\Omega\times\mathcal{S}^1}f\bar{u}+\iint_{\Omega\times\mathcal{S}^1}f(u-\bar{u}).\tag{4.146}$$

Hölder's inequality and Cauchy's inequality imply

$$\iint_{\Omega\times\mathcal{S}^1}f\bar{u}\le\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)}\|\bar{u}\|_{L^{2m}(\Omega\times\mathcal{S}^1)}\le\frac{C}{\epsilon^2}\|f\|^2_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)}+o(1)\epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)},\tag{4.147}$$

and

$$\iint_{\Omega\times\mathcal{S}^1}f(u-\bar{u})\le C\|f\|^2_{L^2(\Omega\times\mathcal{S}^1)}+o(1)\|u-\bar{u}\|^2_{L^2(\Omega\times\mathcal{S}^1)}.\tag{4.148}$$

Hence, absorbing $\epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)}$ and $\|u-\bar{u}\|^2_{L^2(\Omega\times\mathcal{S}^1)}$ into left-hand side of (4.145), we get

$$\begin{aligned}&\epsilon\|(1-\mathcal{P})[u]\|^2_{L^2(\Gamma^+)}+\epsilon^2\|\bar{u}\|^2_{L^{2m}(\Omega\times\mathcal{S}^1)}+\|u-\bar{u}\|^2_{L^2(\Omega\times\mathcal{S}^1)}\\ &\le C\bigg(o(1)\epsilon^{2+\frac{2}{m}}\|u\|^2_{L^\infty(\Gamma^+)}+\|f\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^2}\|f\|^2_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)}+\|h\|^2_{L^2(\Gamma^-)}+\epsilon^2\|h\|^2_{L^m(\Gamma^-)}\bigg),\end{aligned}\tag{4.149}$$

which implies

$$\begin{aligned}&\frac{1}{\epsilon^{\frac12}}\|(1-\mathcal{P})[u]\|_{L^2(\Gamma^+)}+\|\bar{u}\|_{L^{2m}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\|u-\bar{u}\|_{L^2(\Omega\times\mathcal{S}^1)}\\ &\le C\bigg(o(1)\epsilon^{\frac{1}{m}}\|u\|_{L^\infty(\Gamma^+)}+\frac{1}{\epsilon}\|f\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^2}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\|h\|_{L^2(\Gamma^-)}+\|h\|_{L^m(\Gamma^-)}\bigg).\end{aligned}\tag{4.150}$$

□

4.5. L^∞ Estimate - Second Round.

Theorem 4.9. *Assume $f(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $h(x_0,\vec{w})\in L^\infty(\Gamma^-)$. Then for the steady neutron transport equation (4.1), there exists a unique solution $u(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ satisfying*

$$\begin{aligned}\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\le C\bigg(&\frac{1}{\epsilon^{1+\frac1m}}\|f\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{2+\frac1m}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)}+\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}\\ &+\frac{1}{\epsilon^{1+\frac1m}}\|h\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac1m}}\|h\|_{L^m(\Gamma^-)}+\|h\|_{L^\infty(\Gamma^-)}\bigg).\end{aligned}\tag{4.151}$$

Proof. Following the argument in the proof of Theorem 4.7, by double Duhamel's principle along the characteristics, the key step is to utilize L^{2m} estimates as

$$\begin{aligned}|I_{2,2,2}|&\le\left|\int_0^{t_1}\int_{\mathcal{S}^1}\int_{t_1'-s_1'\ge\delta}\bar{u}(\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1})\mathrm{e}^{-(t_1'-s_1')}\mathrm{e}^{-(t_1-s_1)}\mathrm{d}s_1'\mathrm{d}\vec{w}_{s_1}\mathrm{d}s_1\right|\\ &\le\bigg(\int_0^{t_1}\int_{\mathcal{S}^1}\int_{t_1'-s_1'\ge\delta}\mathrm{e}^{-(t_1'-s_1')}\mathrm{e}^{-(t_1-s_1)}\mathrm{d}s_1'\mathrm{d}\vec{w}_{s_1}\mathrm{d}s_1\bigg)^{\frac{2m-1}{2m}}\\ &\quad\bigg(\int_0^{t_1}\int_{\mathcal{S}^1}\int_{t_1'-s_1'\ge\delta}\mathbf{1}_{\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}\in\Omega}|\bar{u}|^{2m}(\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1})\mathrm{e}^{-(t_1'-s_1')}\mathrm{e}^{-(t_1-s_1)}\mathrm{d}s_1'\mathrm{d}\vec{w}_{s_1}\mathrm{d}s_1\bigg)^{\frac{1}{2m}}\\ &\le\bigg(\int_0^{t_1}\int_{\mathcal{S}^1}\int_{t_1'-s_1'\ge\delta}\mathbf{1}_{\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1}\in\Omega}|\bar{u}|^{2m}(\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1})\mathrm{e}^{-(t_1'-s_1')}\mathrm{e}^{-(t_1-s_1)}\mathrm{d}s_1'\mathrm{d}\vec{w}_{s_1}\mathrm{d}s_1\bigg)^{\frac{1}{2m}}.\end{aligned}\tag{4.152}$$

Then using the same substitution $(\phi,s_1')\to(y_1,y_2)$ as

$$\vec{y}=\vec{x}'-\epsilon(t_1'-s_1')\vec{w}_{s_1},\tag{4.153}$$

whose Jacobian is larger than $\epsilon^2\delta$, we know

$$|I_{2,2,2}| \leq \frac{1}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}} \|\bar{u}\|_{L^{2m}(\Omega\times\mathcal{S}^1)}. \tag{4.154}$$

Therefore, we can show

$$\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\bigg(\frac{1}{\epsilon^{\frac{1}{m}}}\|\bar{u}\|_{L^{2m}(\Omega\times\mathcal{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|g\|_{L^\infty(\Gamma^-)}\bigg). \tag{4.155}$$

Considering Theorem 4.8, we obtain

$$\begin{aligned}\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\bigg(&\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} \\ &+ \frac{1}{\epsilon^{1+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg) + o(1)\|u\|_{L^\infty(\Gamma^+)}.\end{aligned} \tag{4.156}$$

Absorbing $\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}$ into the left-hand side, we obtain

$$\begin{aligned}\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\bigg(&\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathcal{S}^1)} \\ &+ \frac{1}{\epsilon^{1+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg).\end{aligned} \tag{4.157}$$

□

5. Diffusive Limit

Theorem 5.1. *Assume $g(\vec{x}_0,\vec{w}) \in C^2(\Gamma^-)$ satisfying (1.5). Then for the steady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathcal{S}^1)$ satisfying (1.4). Moreover, for any $0<\delta<<1$, the solution obeys the estimate*

$$\|u^\epsilon - U_0\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C(\delta,\Omega)\epsilon^{1-\delta}, \tag{5.1}$$

where U_0 is defined in (2.83).

Proof. Based on Theorem 4.9, we know there exists a unique $u^\epsilon(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathcal{S}^1)$, so we focus on the diffusive limit. We can divide the proof into several steps:

Step 1: Remainder definitions.
We define the remainder as

$$R = u^\epsilon - \sum_{k=0}^{2}\epsilon^k U_k - \sum_{k=0}^{1}\epsilon^k \mathscr{U}_k^\epsilon = u^\epsilon - Q - \mathscr{Q}, \tag{5.2}$$

where

$$Q = U_0 + \epsilon U_1 + \epsilon^2 U_2, \tag{5.3}$$

$$\mathscr{Q} = \mathscr{U}_0^\epsilon + \epsilon\mathscr{U}_1^\epsilon. \tag{5.4}$$

Noting the equation (2.60) is equivalent to the equation (1.1), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned}\mathcal{L}[u] &= \epsilon\vec{w}\cdot\nabla_x u + u - \bar{u} \\ &= \sin\phi\frac{\partial u}{\partial\eta} - \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\bigg(\frac{\partial u}{\partial\phi} + \frac{\partial u}{\partial\tau}\bigg) + u - \bar{u}.\end{aligned} \tag{5.5}$$

Step 2: Estimates of $\mathcal{L}[Q]$.
The interior contribution can be estimated as

$$\mathcal{L}[Q] = \epsilon\vec{w}\cdot\nabla_x Q + Q - \bar{Q} = \epsilon^3\vec{w}\cdot\nabla_x U_2. \tag{5.6}$$

Based on classical elliptic estimates, we have

$$\|\mathcal{L}[Q]\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq \big\|\epsilon^3\vec{w}\cdot\nabla_x U_2\big\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\epsilon^3\|\nabla_x U_2\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\epsilon^3. \tag{5.7}$$

This implies

$$\|\mathcal{L}[Q]\|_{L^2(\Omega\times\mathcal{S}^1)} \le C\epsilon^3, \tag{5.8}$$

$$\|\mathcal{L}[Q]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} \le C\epsilon^3, \tag{5.9}$$

$$\|\mathcal{L}[Q]\|_{L^\infty(\Omega\times\mathcal{S}^1)} \le C\epsilon^3. \tag{5.10}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}$.
Since $\mathscr{U}_0^\epsilon = 0$, we only need to estimate $\mathscr{U}_1^\epsilon = (f_1^\epsilon - f_{1,L}^\epsilon)\cdot\psi_0 = \mathscr{V}\psi_0$ where $f_1^\epsilon(\eta,\tau,\phi)$ solves the ϵ-Milne problem and $\mathscr{V} = f_1^\epsilon - f_{1,L}^\epsilon$. The boundary layer contribution can be estimated as

$$\begin{aligned}
\mathcal{L}[\epsilon\mathscr{U}_1^\epsilon] &= \sin\phi\frac{\partial(\epsilon\mathscr{U}_1^\epsilon)}{\partial\eta} - \frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\left(\frac{\partial(\epsilon\mathscr{U}_1^\epsilon)}{\partial\phi} + \frac{\partial(\epsilon\mathscr{U}_1^\epsilon)}{\partial\tau}\right) + (\epsilon\mathscr{U}_1^\epsilon) - (\epsilon\bar{\mathscr{U}}^\epsilon{}_1) \\
&= \epsilon\left(\sin\phi\left(\psi_0\frac{\partial\mathscr{V}}{\partial\eta} + \mathscr{V}\frac{\partial\psi_0}{\partial\eta}\right) - \frac{\psi_0\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\left(\frac{\partial\mathscr{V}}{\partial\phi} + \frac{\partial\mathscr{V}}{\partial\tau}\right) + \psi_0\mathscr{V} - \psi_0\bar{\mathscr{V}}\right) \\
&= \epsilon\psi_0\left(\sin\phi\frac{\partial\mathscr{V}}{\partial\eta} - \frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}}{\partial\phi} + \mathscr{V} - \bar{\mathscr{V}}\right) + \epsilon\left(\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V} - \frac{\psi_0\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}}{\partial\tau}\right) \\
&= \epsilon\left(\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V} - \frac{\psi_0\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}}{\partial\tau}\right).
\end{aligned} \tag{5.11}$$

Since $\psi_0 = 1$ when $\eta \le R_{\min}/(4\epsilon^{1/2})$, the effective region of $\partial_\eta\psi_0$ is $\eta \ge R_{\min}/(4\epsilon^{1/2})$ which is further and further from the origin as $\epsilon\to 0$. By Theorem 3.6, the first term in (5.11) can be bounded as

$$\left\|\epsilon\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \le C\epsilon\mathrm{e}^{-\frac{K_0}{\epsilon^{1/2}}} \le C\epsilon^3. \tag{5.12}$$

Then we turn to the crucial estimate in the second term of (5.11), by Theorem 3.25, we have

$$\left\|-\epsilon\frac{\psi_0\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \le C\epsilon^2\left\|\frac{\partial\mathscr{V}}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \le C\epsilon^2\left|\ln(\epsilon)\right|^8. \tag{5.13}$$

Also, the exponential decay of $\dfrac{\partial\mathscr{V}}{\partial\tau}$ by Theorem 3.25 and the rescaling $\eta = \mu/\epsilon$ implies

$$\begin{aligned}
\left\|-\epsilon\frac{\psi_0\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}}{\partial\tau}\right\|_{L^2(\Omega\times\mathcal{S}^1)} &\le \epsilon^2\left\|\frac{\partial\mathscr{V}}{\partial\tau}\right\|_{L^2(\Omega\times\mathcal{S}^1)} \\
&\le \epsilon^2\left(\int_{-\pi}^{\pi}\int_0^1(1-\mu)\left\|\frac{\partial\mathscr{V}}{\partial\tau}(\mu,\tau)\right\|_{L^\infty}^2\mathrm{d}\mu\mathrm{d}\tau\right)^{1/2} \\
&\le \epsilon^{\frac{5}{2}}\left(\int_{-\pi}^{\pi}\int_0^{1/\epsilon}(1-\epsilon\eta)\left\|\frac{\partial\mathscr{V}}{\partial\tau}(\eta,\tau)\right\|_{L^\infty}^2\mathrm{d}\eta\mathrm{d}\tau\right)^{1/2} \\
&\le C\epsilon^{\frac{5}{2}}\left(\int_{-\pi}^{\pi}\int_0^{1/\epsilon}\mathrm{e}^{-2K_0\eta}\left|\ln(\epsilon)\right|^{16}\mathrm{d}\eta\mathrm{d}\tau\right)^{1/2} \\
&\le C\epsilon^{\frac{5}{2}}\left|\ln(\epsilon)\right|^8.
\end{aligned} \tag{5.14}$$

Similarly, we have

$$\left\|-\epsilon\frac{\psi_0\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}}{\partial\tau}\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} \le C\epsilon^{3-\frac{1}{2m}}\left|\ln(\epsilon)\right|^8. \tag{5.15}$$

In total, we have

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^2(\Omega\times\mathcal{S}^1)} \leq C\epsilon^{\frac{5}{2}} |\ln(\epsilon)|^8, \tag{5.16}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} \leq C\epsilon^{3-\frac{1}{2m}} |\ln(\epsilon)|^8, \tag{5.17}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\epsilon^2 |\ln(\epsilon)|^8. \tag{5.18}$$

Step 4: Diffusive Limit.
In summary, since $\mathcal{L}[u^\epsilon] = 0$, collecting estimates in Step 2 and Step 3, we can prove

$$\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathcal{S}^1)} \leq C\epsilon^{\frac{5}{2}} |\ln(\epsilon)|^8, \tag{5.19}$$

$$\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} \leq C\epsilon^{3-\frac{1}{2m}} |\ln(\epsilon)|^8, \tag{5.20}$$

$$\|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\epsilon^2 |\ln(\epsilon)|^8. \tag{5.21}$$

Also, based on our construction, it is easy to see

$$R - \mathcal{P}[R] = -\epsilon^2(\vec{w}\cdot\nabla_x U_1 - \mathcal{P}[\vec{w}\cdot\nabla_x U_1]), \tag{5.22}$$

which further implies

$$\|R - \mathcal{P}[R]\|_{L^2(\Gamma^-)} \leq C\epsilon^2, \tag{5.23}$$

$$\|R - \mathcal{P}[R]\|_{L^m(\Gamma^-)} \leq C\epsilon^2, \tag{5.24}$$

$$\|R - \mathcal{P}[R]\|_{L^\infty(\Gamma^-)} \leq C\epsilon^2 \tag{5.25}$$

Hence, the remainder R satisfies the equation

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x R + R - \bar{R} &= \mathcal{L}[R] \ \text{ for } \ \vec{x}\in\Omega, \\ R - \mathcal{P}[R] &= R - \mathcal{P}[R] \ \text{ for } \ \vec{w}\cdot\vec{\nu} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega. \end{cases} \tag{5.26}$$

It is easy to verify R satisfies the normalization condition (4.4) and the data satisfies the compatibility condition (4.5). By Theorem 4.9, we have for m sufficiently large,

$$\begin{aligned} \|R\|_{L^\infty(\Omega\times\mathcal{S}^1)} &\leq C\bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}}\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathcal{S}^1)} + \frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathcal{S}^1)} + \|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathcal{S}^1)} \\ &\quad + \frac{1}{\epsilon^{1+\frac{1}{m}}}\|R - \mathcal{P}[R]\|_{L^2(\Gamma^-)} + \frac{1}{\epsilon^{\frac{1}{m}}}\|R - \mathcal{P}[R]\|_{L^m(\Gamma^-)} + \|R - \mathcal{P}[R]\|_{L^\infty(\Gamma^-)}\bigg), \end{aligned} \tag{5.27}$$

$$\begin{aligned} &\leq C\bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}}\left(\epsilon^{\frac{5}{2}}|\ln(\epsilon)|^8\right) + \frac{1}{\epsilon^{2+\frac{1}{m}}}\left(\epsilon^{3-\frac{1}{2m}}|\ln(\epsilon)|^8\right) + \left(\epsilon^2|\ln(\epsilon)|^8\right) \\ &\quad + \frac{1}{\epsilon^{1+\frac{1}{m}}}(\epsilon^2) + \frac{1}{\epsilon^{\frac{1}{m}}}(\epsilon^2) + (\epsilon^2)\bigg) \\ &\leq C\epsilon^{1-\frac{3}{2m}}|\ln(\epsilon)|^8 \leq C\epsilon^{1-\delta} \end{aligned} \tag{5.28}$$

Note that the constant C might depend on m and thus depend on δ. Since it is easy to see

$$\left\|\sum_{k=1}^{2}\epsilon^k U_k + \sum_{k=0}^{1}\epsilon^k\mathscr{U}_k^\epsilon\right\|_{L^\infty(\Omega\times\mathcal{S}^1)} \leq C\epsilon, \tag{5.29}$$

our result naturally follows. This completes the proof of main theorem. □

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