Slow motion for the 1D Swift–Hohenberg equation

Gurgen Hayrapetyan
Ohio University
Athens, OH, USA
email: hayrapet@ohio.edu

Matteo Rinaldi
Carnegie Mellon University
Pittsburgh, PA, USA
email: matteor@andrew.cmu.edu

April 8, 2016

Abstract

The goal of this paper is to study the behavior of certain solutions to the Swift–Hohenberg equation on a one–dimensional torus $\mathbb{T}$. Combining results from Γ–convergence and ODE theory, it is shown that solutions corresponding to initial data that is L^1–close to a jump function v, remain close to v for large time. This can be achieved by regarding the equation as the L^2–gradient flow of a second order energy functional, and obtaining asymptotic lower bounds on this energy in terms of the number of jumps of v.

1 Introduction, Motivation and Main Results

The fourth order partial differential equation

$$u_t = ru - (\bar{q}^2 + \partial_x^2)^2 u + f(u) \tag{1.1}$$

is a generalization of the Swift–Hohenberg equation introduced in 1977 by Swift and Hohenberg [36] as a model for the study of pattern formation, in connection with the Rayleigh–Bérnard convection, e.g. see [13],[27]. Among many different applications, the most famous ones in the literature are those in connection to pattern formation in vibrated granular materials [37], buckling of long elastic structures [24], Taylor–Couette flow [23], [32], and in the study of lasers [28]. Moreover, in recent years great attention has been paid to models of phase transitions in the study of pattern–formation in bilayer membranes, see e.g. [11] where the Swift–Hohenberg equation turns out to be the gradient flow of Ginzburg–Landau type energies, with respect to the right inner product structure.

Consider (1.1) on a periodic domain with a characteristic size $L = 1/\varepsilon$, where $0 < \varepsilon \ll 1$. Letting W be the primitive of $s \mapsto 2(f(s) + (r - \bar{q}^4)s)$, $q := 2\bar{q}^2$, and rescaling time and space by ε in (1.1) one arrives at the rescaled form

$$\begin{cases} u_t = -W'(u) - 2\varepsilon^2 q u_{xx} - 2\varepsilon^4 u_{xxxx} & x \in \mathbb{T}, t > 0, \\ u(x,0) = u_{0,\varepsilon}(x) & x \in \mathbb{T}, \end{cases} \tag{1.2}$$

where $\mathbb{T}$ is a one–dimensional torus. We assume that $W : \mathbb{R} \to [0, +\infty)$ is a double–well potential with phases supported at -1 and 1, and we study the long–time behavior of solutions when $q > 0$ is sufficiently small. In particular, due to the presence of the small parameter ε in (1.2) the solutions are expected to develop interfacial structure driven by the minima of the potential W. Equation (1.2) may be viewed as a gradient flow associated to a second order energy functional, and our main result consists of an asymptotic lower bound on the corresponding energy functional and the consequent bounds on the speed of evolution of the developed interfaces. Below we outline interfacial dynamics results for the lower order Allen–Cahn equation and its generalizations that provide much of the motivation for our analysis.

1.1 Allen–Cahn Equation and Generalizations to Higher Order

Equations displaying interfacial dynamics have been studied extensively in the last two decades. The prototypical example is the Allen–Cahn equation

$$u_t = \varepsilon^2 u_{xx} - W'(u), \quad x \in I,\, t > 0, \tag{1.3}$$

(as well as its higher dimensional analog) seen as the L^2–gradient flow of the energy

$$G_\varepsilon(u; I) := \int_I \left(\frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2}|u_x|^2\right) dx, \quad u \in H^1(I), \tag{1.4}$$

where $I \subset \mathbb{R}$ is an interval. The special gradient–flow structure of (1.3) has allowed its analysis by a wide variety of methods and techniques.

In particular, it has been shown for the Allen–Cahn equation (see [9] and the references therein) that if $\varepsilon \ll 1$ the evolution from a sufficiently regular initial data occurs in four main stages. In the first stage, the diffusion term $\varepsilon^2 u_{xx}$ can be ignored and the leading order dynamics are driven by the ε independent ordinary differential equation $u_t = -W'(u)$. This is the time-scale in which interfaces develop, i.e., regions in the space domain that separate almost constant solutions corresponding to the stable equilibria of the ordinary differential equation. This stage, referred to as the generation of interface, has been analyzed for the Allen–Cahn equation first in [16], and subsequently in [9], [10], [14], [35], and other papers.

As the regions separating unequal equilibria decrease in length, the spacial gradient necessarily increases, and after $O(|\ln \varepsilon|)$ time the dynamics are driven by a balance between the two terms on the right–hand side of (1.3). In particular, as shown in [9], after $O(\varepsilon^{-1})$ time the solution is exponentially close to the standing–wave profile

$$\Phi(x; p_1, \dots, p_n) := \pm \prod_n \phi\left(\frac{x - p_i}{\varepsilon}\right), \tag{1.5}$$

parametrized by the positions $p_1, \dots p_n$, where ϕ satisfies

$$\phi'' = W'(\phi), \quad \lim_{z \to \pm\infty} \phi(x) = \pm 1, \quad \phi(0) = 0. \tag{1.6}$$

The zeros $p_1(t), \dots, p_n(t)$ of Φ can be viewed as specifying the location of the interfaces. In particular, the residual $\varepsilon^2 \Phi_{xx} - W'(\Phi)$ is exponentially small and the corresponding third stage of the evolution proceeds on an exponentially slow time scale until two zeros of the solution of (1.3) u^ε collide and disappear as part of the fourth stage of the evolution.

The third stage, usually referred to as *Slow Motion* has been studied extensively. The most precise interface evolution results for the Allen–Cahn equation are given in [7], [8], [19], [20]. Specifically, the zeros of the solution u^ε are approximated by $\{p_i\}$, which at leading order move according to the evolution law

$$p_i' = \varepsilon S \left(\exp\left(-\mu \frac{p_{i+1} - p_i}{\varepsilon}\right) - \exp\left(-\mu \frac{p_i - p_{i-1}}{\varepsilon}\right)\right), \tag{1.7}$$

where $\mu = \sqrt{W''(\pm 1)}$, $S > 0$ is a constant depending only on W. The proof of this reduction involves invariant manifold theory and geometric analysis.

In [5] Bronsard and Kohn adopted a variational viewpoint to study the Allen–Cahn equation. While their method does not recover the evolution equation above, it does provide relatively simple energy arguments to obtain a bound on the speed of this evolution. In particular, Bronsard and Kohn first prove that for any $k > 0$ there exists a constant $c_k > 0$ such that, if $v \in H^1(I)$ is sufficiently close in L^1 norm to a step function taking values ± 1 and having exactly N jumps, and its energy satisfies

$$G_\varepsilon(v; I) \le N c_W + \varepsilon^k, \tag{1.8}$$

where $c_W = \int_{-1}^{1} \sqrt{2W(s)}ds$, then

$$G_\varepsilon(v; I) \geq Nc_W - c_k\varepsilon^k. \tag{1.9}$$

Using this energy estimate they prove that the solution u^ε of (1.3) with Dirichlet or Neumann boundary data, under the same conditions on the initial data $u_{0,\varepsilon}(x)$, satisfies

$$\lim_{\varepsilon\to 0} \sup_{0\leq t\leq \varepsilon^{-k}m} \int_I |u^\varepsilon(x,t) - v(x)|dx = 0, \tag{1.10}$$

for any $m > 0$. The limit in (1.10) may be viewed as providing an upper bound on the speed of the evolution of the transition layers of u^ε. Improvements of (1.9) have been obtained in [4] and [22]. In particular, it has recently been established in [4] that for a sequence $\{v_\varepsilon\} \subset H^1(\mathbb{T})$ converging to a step function taking values ± 1 and having exactly N jumps, the Allen–Cahn functional admits the following multiple order asymptotic expansion

$$\begin{aligned} G(v_\varepsilon; \mathbb{T}) = Nc_W &- 2\alpha_+\kappa_+^2 \sum_{k=1}^{N} \exp\left(-\alpha_+ \frac{d_k^\varepsilon}{\varepsilon}\right) - 2\alpha_-\kappa_-^2 \sum_{k=1}^{N} \exp\left(-\alpha_- \frac{d_k^\varepsilon}{\varepsilon}\right) \\ &+ \kappa_+^3\beta_+ \sum_{k=1}^{N} \exp\left(\frac{-3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}\right) + \kappa_-^3\beta_- \sum_{k=1}^{N} \exp\left(\frac{-3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}\right) \\ &+ o\left(\sum_{k=1}^{N} \exp\left(\frac{-3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}\right)\right) + o\left(\sum_{k=1}^{N} \exp\left(\frac{-3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}\right)\right) \end{aligned}$$

where $\alpha_\pm, \kappa_\pm, \beta_\pm$ are constants dependent on the potential W and d_k^ε is the distance between consecutive transition layers of v_ε. The gradient flow associated with the second order term in the above energy expansion gives, up to a multiplicative constant, the evolution equation (1.7), providing a crucial link between the variational and geometric approaches. Further insight into this connection can be seen as part of a general framework of Γ–convergence of gradient flows developed in [33].

In regards to extensions to higher–order functionals, the problem has been studied in [25] in connection with a family of higher order functionals of the form

$$\mathcal{H}(u) := \frac{1}{\varepsilon}\int_I \left(\sum_{k-1}^{n} \frac{\gamma_k\varepsilon^{2k}}{2}|u^{(k)}|^2 + W(u)\right) dx, \tag{1.11}$$

where $u^{(k)}$ stands for the k–th spatial derivative of u. Due to difficulties associated with higher order nature of the functional, in particular, the lack of exact solutions of the corresponding Euler–Lagrange equation, sharp bounds analogous to (1.1) have not been established. An important condition on $\mathcal{H}$ in [25] is

- Hypothesis 1: There exists constants $d_0, \eta > 0$ such that for every interval $I \subset \mathbb{R}$ with length $|I| \geq d_0$ and all $u \in H^n(I)$

$$\int_I \left(\sum_{k-1}^{n} \gamma_k|u^{(k)}|^2\right) dx \geq \eta \int_I \left(|u^{(n)}|^2 + |u'|^2\right) dx. \tag{1.12}$$

Under this hypothesis the authors prove that for any $u \in H^n(I)$ sufficiently close to a step function taking values ± 1 and having exactly N jumps,

$$\mathcal{H}_\varepsilon(u) \geq Nm_1 - C\exp\left(-\frac{d\lambda}{3\varepsilon}\right), \tag{1.13}$$

where λ is a constant satisfying $\lambda < |\text{Re}(\mu)|$, for all eigenvalues μ of the linearization of

$$\sum_{k=1}^{n}(-1)^k\gamma_k u^{(2k)} + W'(u) = 0 \tag{1.14}$$

at $(\pm 1, 0, \ldots, 0)$.

The initial value problem (1.2) can be seen as the L^2–gradient flow of the second order energy functional

$$E_\varepsilon(u;\mathbb{T}) := \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W(u) - \varepsilon q|u_x|^2 + \varepsilon^3|u_{xx}|^2\right) dx, \quad u \in H^2(\mathbb{T}) \tag{1.15}$$

and our main goals are the extension and the improvement of the bound (1.13) for this energy and, in turn, this will allow us to prove the slow motion of solutions of (1.2).

We note that the functional (1.15) does not satisfy Hypothesis 1 due to the negative term in the energy. We use recently established interpolation inequality (see [11] and [12]) to overcome this difficulty if $q > 0$ is sufficiently small. Moreover, in the proof of an energy estimate analogous to (1.13), see Theorem 1.1, we do not assume any closeness condition on the H^2 functions we work with, we instead make an assumptions on the zeros of such functions.

Furthermore, inspired by [4], our analysis relies on the use of a particular test function, and on the study of the solutions of the Euler–Lagrange equation associated to (1.15) via hyperbolic fixed point theory, in particular through the work of Sell [34]. Thanks to this approach we are able to improve the exponent in (1.13) and, consequently, obtain sharper bound on the speed of evolution for solutions of (1.2).

We recall that the Γ–convergence of the energy functional E_ε has been proved in [18] for the case $q = 0$, and in [11] and [12] when $q > 0$ is small. The asymptotic behavior of E_ε plays a crucial role in our analysis: we will use results from Γ–convergence, together with a careful analysis of the minimizers of the associated Euler–Lagrange equation, to study the speed of motion of solutions of (1.2).

To conclude, we remark that the situation in the higher dimensional setting is quite different: solutions of the higher dimensional version of (1.3) and other classical gradient flow–type equations have been studied by many different authors, see, e.g., [1], [2], [6], [15], [26], [31]. Due to the lack of results like (1.9), all of them use significantly different approaches to the one introduced in [5]. A more recent work, see [30], closes the gap by making use of a Γ–convergence result proved in [29] and doesn't assume any specific structure of the initial data.

1.2 Statement of Main Results

Theorem 1.1. *Let $\mathbb{T}$ be the one–dimensional unit torus, and let W satisfy the hypotheses* (2.1)–(2.4). *Let $\alpha_0 > 0$. Then there exist $q_0 > 0$ and $\varepsilon_0 > 0$, possibly dependent on α_0 and q_0, such that if $q < q_0$ and $w \in H^2(\mathbb{T})$ has at least N zeros, $\{x_k\}_{k=1}^N$, satisfying $\min_k |x_{k+1} - x_k| \geq \alpha_0$ then*

$$E_\varepsilon(w;\mathbb{T}) \geq Nm_1 - C\sum_{k=1}^{N}\exp\left(-\frac{d_k\gamma}{\varepsilon}\right), \tag{1.16}$$

for every $0 < \varepsilon < \varepsilon_0$, where $d_k = x_{k+1} - x_k$, $\gamma > 0$ is defined in (2.55) *and depends only on W, while $C > 0$ is independent of ε.*

We remark that a similar estimate can be obtained when the domain is an interval $I := (a_0, b_0)$, with (1.16) replaced with

$$E_\varepsilon(w; I) \geq N m_1 - C \sum_{k=0}^{N} \exp\left(-\frac{d_k \gamma}{\varepsilon}\right), \tag{1.17}$$

where $d_0 := x_1 - a_0$, $d_N := b_0 - x_N$.

Remark 1.2. We highlight the fact that we are *not* requiring the function w of Theorem 1.1 to be L^1–close to a jump function, in contrast with [4], [5], [22], [25]. On the other hand, it is easy to show that if w is L^1–close to a jump function v taking values ± 1, then there exists an $\alpha_0 > 0$ with the property that the zeros of w are at least $\alpha_0 > 0$ apart, as in the statement of Theorem 1.1.

The energy estimate above is a crucial ingredient to prove slow motion of solutions of (1.2), when the initial data is close in the L^1 norm to a BV function, as in [5], [22], [25]. In particular, we will consider regular solutions of (1.2), whose existence is proved in the Appendix, see Theorem 4.1. Our analysis yields the following result.

Theorem 1.3. *Let $v \in BV(\mathbb{T}; \{\pm 1\})$ be a function with $N(v) \neq 0$ jumps at $x_k(v)$, for $k = 1, \dots, N(v)$, and let $q_0 > 0$ be as in Theorem 1.1. Let $d := \min_k |x_{k+1}(v) - x_k(v)|$. Then there exist $\varepsilon_0, \delta_0 > 0$ with $d - 4\delta_0 > 0$ such that, if u^ε is a solution of (1.2) with $u^\varepsilon \in L^2((0,\infty); H^4(\mathbb{T}))$, $u^\varepsilon_t \in L^2((0,\infty); H^2(\mathbb{T}))$ and initial data $u_{0,\varepsilon} \in H^2(\mathbb{T})$ satisfying*

$$||u_{0,\varepsilon} - v||_{L^1(\mathbb{T})} \leq \delta \tag{1.18}$$

for $0 < \delta < \delta_0$ and

$$E_\varepsilon(u_0; \mathbb{T}) \leq E_0(v; \mathbb{T}) + \frac{1}{h(\varepsilon)}, \tag{1.19}$$

for all $0 < \varepsilon < \varepsilon_0$ and for some function $h : (0, \infty) \to (0, \infty)$, then for all $q < q_0$,

$$\lim_{\varepsilon \to 0^+} \left\{ \sup_{0 \leq t \leq T_\varepsilon} \int_{\mathbb{T}} |u^\varepsilon(x,t) - u_{0,\varepsilon}(x)| dx \right\} = 0, \tag{1.20}$$

where

$$T_\varepsilon := \delta^2 \min\{h(\varepsilon), \exp((d - 4\delta)\gamma/\varepsilon)\}.$$

Remark 1.4. If $h(\varepsilon) = \exp(d\gamma/\varepsilon)$, then

$$T_\varepsilon = \delta \exp((d - 4\delta)\gamma/\varepsilon)$$

which is consistent with the estimates obtained in [22] and [25]. On the other hand, we remark that our Theorem 1.3 provides more general results.

Remark 1.5. To the best of our knowledge, only recently some regularity results for the Swift–Hohenberg equation have been proved, see [21]. In the statement of Theorem 1.3 we *assume* that the solutions are sufficiently regular. In the Appendix we prove existence of solutions (though with weaker regularity) using De Giorgi's technique of Minimizing Movements (see Theorem 4.1).

1.3 Outline of the Proof

A key step in proving the energy inequality (1.16) is a bound from below by the energy of an appropriately chosen test function. Given $w \in H^2(\mathbb{T})$ satisfying the assumptions of Theorem 1.1, we follow [4] to construct this test function by gluing together minimizers of the energy on each subinterval $I_k := [x_k, x_{k+1}]$, where the admissible class now consists of $H^2(I_k)$ functions that equal zero at the endpoints of I_k. Thus,

$$E_\varepsilon(w; \mathbb{T}) \geq \sum_k E_\varepsilon(\hat{w}_k; I_k), \tag{1.21}$$

where $\hat{w}_k$ also solves a fourth order Euler–Lagrange equation corresponding to the energy functional.

This initial energy inequality has several key advantages. First, it assumes no assumptions about closeness of w to a step function taking values ± 1. The required estimates can be *proved* for $\hat{w}_k$. Secondly, the additional property that $\hat{w}_k$ solves a fourth order ODE on the whole subinterval is key in obtaining a sharper lower bound than the one established in [25]. Specifically, in the middle of each subinterval I_k, we can show that the minimizer $\hat{w}_k = \pm 1 + O(\exp(-\gamma(x_{k+1} - x_k)/2\varepsilon))$, where the exponent γ is related to the linearization of the Euler–Lagrange equation. In fact, obtaining this bound is the central contribution of this paper, starting from Corollary 2.3 and culminating in Proposition 2.7. The proofs of Lemmas 2.4 and 2.5, which give the initial crude estimates on the 'closeness' of $\hat{w}_k$ to ± 1, follow the ideas of [25] supplemented by the use of the interpolation inequality given in Lemma 2.2 and the use of $\hat{w}_k$ instead of the original function w. A point of departure is Lemma 2.6, in which the use of a Hartman–Grobman type theorem (see Theorem 5.4, from [34]), combined with the extra information on $\hat{w}_k$ and the analysis of the linearized problem, allow us to obtain sharper exponential decay estimate.

Once these bounds on $\hat{w}_k$ are obtained, we show that its energy is larger than the energy of the 'optimal profile' connecting the zeros of $\hat{w}_k$ with ± 1 and having energy $m_1/2$. This is accomplished in the proof of Theorem 1.1.

In the remainder of the paper, we use the energy lower bound to obtain slow motion results in Section 3. Finally, in the Appendix we present a proof of existence of solutions for equation (1.2) in the more general case of a bounded domain $\Omega \subset \mathbb{R}^n$, along with partial regularity results for the solutions themselves.

2 Preliminaries and Assumptions

Throughout this paper we will work with a double–well potential $W : \mathbb{R} \to [0, \infty)$ satisfying

$$W \in C^5(\mathbb{R}),\ W(s) = W(-s),\ \text{for all } s \in \mathbb{R}; \tag{2.1}$$

$$W(s) > 0,\ \text{for } s \geq 0, s \neq 1; \tag{2.2}$$

$$W(1) = W'(1) = 0; \tag{2.3}$$

$$\text{there exists } 0 < c_W \leq 1 \text{ such that } W(s) \geq c_W |s-1|^2,\ \text{for } s \geq 0. \tag{2.4}$$

A prototype for W is given by

$$W(s) := \frac{1}{4}(s^2 - 1)^2. \tag{2.5}$$

2.1 Γ–convergence and Interpolation Inequalities

In this section we recall some properties of the energy

$$E_\varepsilon(u; \Omega) := \int_\Omega \left(\frac{1}{\varepsilon} W(u) - \varepsilon q |\nabla u|^2 + \varepsilon^3 |\nabla^2 u|^2 \right) dx, \tag{2.6}$$

in the more general setting where Ω is a bounded open set of $\mathbb{R}^n$ with C^1 boundary, $q > 0$ is a small parameter, and W is a double–well potential, as in (2.5). In [11] Chermisi, Dal Maso, Fonseca and Leoni proved that the sequence of functionals $\mathscr{E}_\varepsilon : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\mathscr{E}_\varepsilon(u) := \begin{cases} E_\varepsilon(u; \Omega) & \text{if } u \in H^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H^2(\Omega), \end{cases}$$

Γ–converges as $\varepsilon \to 0^+$ to the functional $\mathscr{E}_0 : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$\mathscr{E}_0(u) := \begin{cases} m_n \mathrm{Per}_\Omega(\{u = 1\}) & \text{if } u \in BV(\Omega; \{-1, +1\}), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega; \{-1, +1\}), \end{cases}$$

where

$$m_n := \inf \{E_\varepsilon(u; Q) : 0 < \varepsilon \le 1, u \in \mathcal{A}_n\},$$

$Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$, and

$$\begin{aligned} \mathcal{A}_n := \Big\{ u \in & H^2_{\mathrm{loc}}(\mathbb{R}^n),\ u(x) = -1 \text{ near } x \cdot e_n = -\frac{1}{2}, \\ & u(x) = 1 \text{ near } x \cdot e_n = \frac{1}{2}, \\ & u(x) = u(x + e_i) \text{ for all } x \in \mathbb{R}^n, i = 1, \dots, n-1 \Big\}. \end{aligned}$$

We define the one–dimensional rescaled energy

$$E(v; A) := \int_A \left(W(v) - q(v')^2 + (v'')^2\right) dx, \tag{2.7}$$

and we introduce the set of admissible functions

$$\mathcal{A} := \left\{ v \in H^2_{\mathrm{loc}}(\mathbb{R}) : v(x) = -1 \text{ near } x = a, v(x) = 1 \text{ near } x = b \right\}. \tag{2.8}$$

We note that it was proved in [11], Section 5.1, that

$$m_1 = \inf \left\{ E(v; \mathbb{R}) : v \in H^2_{\mathrm{loc}}(\mathbb{R}), \lim_{x \to \pm\infty} v(x) = \pm 1 \right\}, \tag{2.9}$$

so that in dimension $n = 1$ we have

$$\mathscr{E}_0(u) = \begin{cases} N m_1 & \text{if } u \in BV((a, b); \{-1, +1\}) \\ +\infty & \text{if } u \in L^2((a, b)) \setminus BV((a, b); \{-1, +1\}), \end{cases}$$

where N is the number of jumps of the function u. We further define

$$\begin{aligned} m_\pm &:= \inf \left\{ E(u; \mathbb{R}^+) : u \in H^2_{\mathrm{loc}}(\mathbb{R}^+), \lim_{x \to \infty} u(x) = \pm 1, \quad u(0) = 0 \right\} \\ &= \inf \left\{ E(u; \mathbb{R}^-) : u \in H^2_{\mathrm{loc}}(\mathbb{R}^-), \lim_{x \to -\infty} u(x) = \pm 1, \quad u(0) = 0 \right\} \end{aligned} \tag{2.10}$$

and remark that in our case of symmetric potential W, $m_+ = m_- = m_1/2$. One of the key tools to prove the Γ–convergence result is the following nonlinear interpolation inequality, see e.g. Theorem 3.4 in [11].

Lemma 2.1. *Let Ω be a bounded open set of $\mathbb{R}^n$ with C^1 boundary, and assume that W satisfies (2.1)–(2.4). Then there exists a constant $q^* > 0$, independent of Ω, such that for every $-\infty < q < q^*/N$ there exists $\varepsilon_0 = \varepsilon_0(\Omega, q) > 0$ such that*

$$q\varepsilon^2 \int_\Omega |\nabla u|^2 dx \le \int_\Omega W(u) dx + \varepsilon^4 \int_\Omega |\nabla^2 u|^2 dx$$

for every $\varepsilon \in (0, \varepsilon_0)$ and every $u \in H^2(\Omega)$.

In particular, in the one dimensional setting, we will often use the following nonrescaled version of the previous result, see Lemma 3.1 in [12].

Lemma 2.2. *Let W be a continuous potential satisfying* (2.2)–(2.4)*. Let $I \subset \mathbb{R}$ be an open, bounded interval. Then there exists a constant $q^* > 0$ such that*

$$q^* \int_I (u')^2 dx \leq \frac{1}{\mathcal{L}^1(I)^2} \int_I W(u) dx + \mathcal{L}^1(I)^2 \int_I (u'')^2 dx$$

for every $u \in H^2(I)$.

Corollary 2.3. *Let W and q^* be as in Lemma 2.2. Then there exist $\sigma > 0$ such that for every open interval I, every $0 < \varepsilon \leq \mathcal{L}^1(I)$, and every $-\infty < q \leq q^*/4$,*

$$q\varepsilon^2 \int_I (u')^2 dx \leq \int_I \left(W(u) + \varepsilon^4 (u'')^2\right) dx, \tag{2.11}$$

and

$$E_\varepsilon(u; I) \geq \sigma \int_I \left(W(u) + \varepsilon^2 (u')^2 + \varepsilon^4 (u'')^2\right) dx. \tag{2.12}$$

for all $u \in H^2_{loc}(I)$.

Proof. Let $I = (a, b)$ and $u \in H^2((a, b))$. We change variables $v(y) := u(\varepsilon x)$, subdivide the resulting rescaled domain $I_\varepsilon = (a/\varepsilon, b/\varepsilon)$ into $\left[\frac{b-a}{\varepsilon}\right]$ subintervals, I_ε^k, of length between 1/2 and 2 (since $0 < \varepsilon \leq b - a$) and use Lemma 2.2 to obtain

$$\begin{aligned}\frac{q^*}{4} \int_a^b (u')^2 dx &= \frac{q^*}{4\varepsilon} \int_{a/\varepsilon}^{b/\varepsilon} (v')^2 dy = \frac{1}{4\varepsilon} \sum_k q^* \int_{I_\varepsilon^k} (v')^2 dy \leq \frac{1}{4\varepsilon} \sum_k \int_{I_\varepsilon^k} \left(4W(v) + 4(v'')^2\right) dy \\ &= \frac{1}{\varepsilon} \int_{a/\varepsilon}^{b/\varepsilon} \left(W(v) + (v'')^2\right) dy = \int_a^b (W(u) + \varepsilon^3 (u'')^2) dx.\end{aligned} \tag{2.13}$$

Since $q \leq q^*/4$, (2.11) easily follows. To prove (2.12) we follow closely the strategy used in the proof of Theorem 1.1 of [11] and proceed as follows. Fix $\sigma \in (0, 1)$ sufficiently small so that $(q + \sigma)/(1 - \sigma) < q^*/4$. Then,

$$\begin{aligned}\int_a^b (W(u) - q\varepsilon^2 (u')^2 + \varepsilon^4 (u'')^2) dx = (1 - \sigma) \int_a^b \left(W(u) - \frac{q + \sigma}{1 - \sigma} \varepsilon^2 (u')^2 + \varepsilon^4 (u'')^2\right) dx \\ + \sigma \int_a^b \left(W(u) + \varepsilon^2 (u')^2 + \varepsilon^4 (u'')^2\right) dx,\end{aligned} \tag{2.14}$$

and (2.12) follows since by (2.13) the first term on the right-hand side of (2.14) is nonnegative.

□

The following lemmas established for a generalization of the Modica–Mortola Functional in [25] will be useful to prove our main result. While our energy does not satisfy the assumptions of [25], their argument is easily extended to our case with the help of the interpolation inequality (2.12). In particular, Lemma 2.4, shows that an H^2 function with a uniformly bounded energy, necessarily takes values close to $\{\pm 1\}$ and has small derivatives, except on a set of measure $O(\varepsilon)$ and Lemma 2.5 gives a characterization of the global minimizers for the energy $E(\cdot, \cdot)$, defined in (2.7), subject to small boundary conditions.

Lemma 2.4. *Let I be an open interval, $M > 0$ and $0 < \delta < 1$. Then there exists a constant $C_1 > 0$ such that for any $0 < \varepsilon \leq \mathcal{L}^1(I)$ and every $u \in H^2(I)$ with $E_\varepsilon(u; I) \leq M$ the following property holds: there is a measurable set $J \subset I$ with $\mathcal{L}^1(J) \leq C_1 \varepsilon$ such that*

$$\operatorname{dist}(u(x), \{\pm 1\}) < \delta \quad \textit{and} \quad |\varepsilon u'(x)| < \delta \quad \textit{and}$$

hold for all $x \in I \setminus J$, where dist *denotes the usual distance between a point and a set.*

Proof. By (2.12), for every $0 < \varepsilon \leq \mathcal{L}^1(I)$ and $u \in H^2(I)$,

$$\begin{aligned}\int_I \left(W(u) - q\varepsilon^2(u')^2 + \varepsilon^4(u'')^2\right) dx &\geq \sigma \int_I \left(W(u) + \varepsilon^2(u')^2 + \varepsilon^4(u'')^2\right) dx \\ &\geq \sigma \int_I W(u)dx.\end{aligned} \tag{2.15}$$

We now let $J_0 := \{x \in I : \operatorname{dist}(u(x), \{\pm 1\}) \geq \delta\}$ and from the definition of W we have $c := \inf\{W(s) : \operatorname{dist}(s, \{\pm 1\}) \geq \delta\} > 0$. Then (2.15) implies

$$M \geq E_\varepsilon(u; I) \geq \frac{\sigma}{\varepsilon} \int_I W(u)dx \geq \frac{c\sigma}{\varepsilon} \mathcal{L}^1(J_0),$$

and therefore

$$\mathcal{L}^1(J_0) \leq \frac{M\varepsilon}{c\sigma}.$$

Similarly, setting $J_1 := \{x \in I : |\varepsilon u'(x)| \geq \delta\}$, (2.15) yields the estimates

$$M \geq E_\varepsilon(u; I) \geq \frac{\sigma}{\varepsilon} \int_I \varepsilon^2(u')^2 dx \geq \frac{\sigma\delta^2}{\varepsilon} \mathcal{L}^1(J_1) \tag{2.16}$$

and consequently

$$\mathcal{L}^1(J_1) \leq \frac{M\varepsilon}{\sigma\delta^2}.$$

Setting $J := J_0 \cup J_1$ yields the desired result. □

Lemma 2.5. *Let $I := (a, b)$ be an open interval and $W \in C^2$ satisfy (2.2)–(2.4). Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ define*

$$\mathcal{M}^{\pm}_{\alpha,\beta} := \{v \in H^2(I) : v(a) = \pm 1 + \alpha_0,\ v'(a) = \alpha_1,\ v(b) = \pm 1 + \beta_0,\ v'(b) = \beta_1\}. \tag{2.17}$$

Then there exist constants $\delta_0, C > 0$ such that the following holds. If $\mathcal{L}^1(I) > 1$ and $||\alpha||, ||\beta|| \leq \delta < \delta_0$ then the functional $E(\cdot; I)$ defined in (2.7) has a global minimizer $v_\pm$ on $\mathcal{M}^{\pm}_{\alpha,\beta}$. This minimizer $v_\pm$ solves the Euler–Lagrange equation, and satisfies the estimates

$$||v_\pm \pm 1||_{L^\infty(I)} \leq C\delta, \tag{2.18}$$

$$||v_\pm^{(k)}|_{L^2(I)} \leq C\delta \text{ for } k = 1, \ldots, 4. \tag{2.19}$$

$$||v_\pm^{(k)}||_{L^\infty(I)} \leq C\delta \text{ for } k = 1, \ldots, 3. \tag{2.20}$$

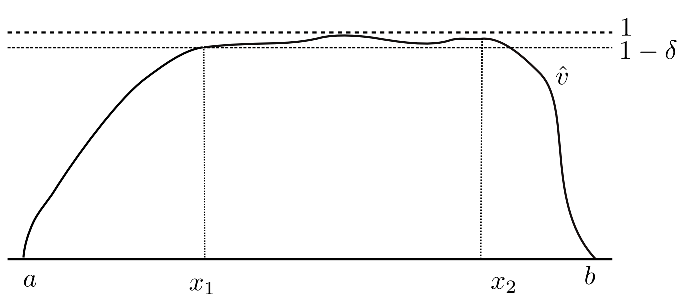

Figure 1: If $\hat{v}$ is close to 1 at x_1 and x_2, then it stays close in between.

Proof. We prove the proposition when $s = -1$, the $s = 1$ case being identical. We divide the proof into several steps. Moreover, we simplify the notation used for the L^p norms when the domain of integration will be clear from the context.

Step 1. Fix $\delta > 0$. We claim that there exists $C_1 > 0$ such that if $||\alpha||, ||\beta|| \leq \delta$, then

$$\inf_{\mathcal{M}^-_{\alpha,\beta}} E(\cdot; I) \leq C_1\delta^2. \tag{2.21}$$

To show this we note that, if $\varphi_0, \varphi_1 \in C^\infty(\mathbb{R})$ satisfy $\varphi_i(x) = 0$ for all $x \geq 1/2$, with $\varphi_0(0) = 1, \varphi_0'(0) = 0, \varphi_1(0) = 0$, and $\varphi_1'(0) = 1$, then the function

$$\phi(x) := -1 + \alpha_0\varphi_0(x-a) + \alpha_1\varphi_1(x-a) + \beta_0\varphi_0(b-x) - \beta_1\varphi_1(b-x),\ x \in (a,b), \tag{2.22}$$

belongs to $\mathcal{M}^-_{\alpha,\beta}$. Using ϕ as a test function, (2.21) follows from Taylor's formula for W and the facts that $W(\pm 1) = W'(\pm 1) = 0$ and $W \in C^2(\mathbb{R})$.

Step 2. Fix $0 < \delta < 1$. We will show that there exists $C_2 > 0$ such that for every $v \in \mathcal{M}^-_{\alpha,\beta}$, with $v \leq 0$ on I and $||\alpha||, ||\beta|| \leq \delta$ we have

$$E(v; I) \geq C_2||v+1||^2_{L^\infty}. \tag{2.23}$$

Suppose that $|v(x)+1| \geq ||v+1||_\infty/2$ for all $x \in I$. Using (2.4) and (2.12) with $\varepsilon = 1$ we have,

$$E(v; I) \geq \sigma\int_I W(v)dx \geq \sigma c_W \int_I |v+1|^2 \geq \mathcal{L}^1(I)\frac{\sigma}{4}c_W||v+1||^2_{L^\infty} \tag{2.24}$$

Otherwise, there are points $x_0, x_1 \in \bar{I}$ satisfying

$$|v(x_0)+1| = \frac{||v+1||_\infty}{2} \ \text{ and } \ |v(x_1)+1| = ||v+1||_\infty,$$

in which case, again by (2.4), (2.12) and Young's Inequality

$$\begin{aligned} E(v; I) &\geq \sigma\int_I \left(W(v) + |v'|^2\right)dx \geq 2\sigma\int_I \sqrt{W(v)}|v'| \\ &\geq 2c_W\sigma\left|\int_{x_0}^{x_1} |v+1|v'dx\right| \\ &= c_W\sigma\left((v+1)^2(x_1) - (v+1)^2(x_0)\right) = \frac{\sigma}{2}c_W||v+1||_{L^\infty} \end{aligned}$$

and this proves (2.23).

Step 3. We claim that there exists $\delta_0 > 0$ and $C_3 = C_3(\delta_0) > 0$ such that if $||\alpha||, ||\beta|| \leq \delta < \delta_0$ and $v \in \mathcal{M}^-_{\alpha,\beta}$, with $E(v; I) \leq 2\inf_{\mathcal{M}^-_{\alpha,\beta}} E$, then

$$||v+1||_{L^\infty} \leq C_3\delta. \tag{2.25}$$

By taking $0 < \delta < 1$ sufficiently small, we may assume that $v \leq 0$ on I. Indeed, since $v(a) = -1 + \alpha_0 \leq -1 + \delta < 0$, if $v(x) > 0$ for some x, then necessarily there exists x_1 such that $v(x_1) = 0$, and so by (2.4),

$$\begin{aligned} E(v; I) &\geq \sigma\int_I \left(W(v) + |v'|^2\right)dx \geq 2\sigma\int_a^{x_1} \sqrt{W(v)}|v'| \\ &\geq 2C_W\sigma\left|\int_a^{x_1} |v+1|v'dx\right| \\ &= \sigma C_W\left((v+1)^2(x_1) - (v+1)^2(a)\right) \\ &\geq \sigma(1 - |\alpha_0|^2), \end{aligned}$$

which contradicts Step 1 for δ sufficiently small. Hence, Steps 1 and 2 imply (2.25).

Step 4. Finally, (2.12) with $\varepsilon = 1$ and standard compactness and lower semicontinuity arguments imply the existence of minimizer v_- of $E(\cdot; I)$ and since by previous step $v_- \leq 0$ for $\delta < \delta_0$ and

$$||v_- + 1||_{L^2}^2 \leq \mathcal{L}^1(I)||v_- + 1||_{L^\infty}^2 \leq C\delta^2, \tag{2.26}$$

for some $C > 0$, again using (2.12) along with (2.21) yields

$$||v_-^{(k)}||_{L^2} \leq C\delta, \text{ for } k = 1, 2.$$

Furthermore, since W is C^2, from (2.26) and the Mean Value Theorem we have

$$W'(v_-) = W'(v_-) - W'(-1) \leq \max_{0 \leq \xi \leq 2} W''(\xi)(v_- + 1). \tag{2.27}$$

The Euler–Lagrange equation

$$2v_-^{(iv)} + 2qv_-'' + W'(v_-) = 0,$$

the L^∞ bound from Step 3 and (2.27) imply

$$||v_-^{(iv)}||_{L^2} \leq |q|||v_-''||_{L^2} + \frac{1}{2}||W'(v_-)||_{L^2} \leq |q|||v_-''||_{L^2} + C||v_- + 1||_{L^2} \leq C\delta \tag{2.28}$$

for some $C > 0$.

The energy bound (2.21) and standard interpolation inequalities (e.g., see Theorem 6.4 in [17]) imply (2.18), (2.19), (2.20). □

2.2 The Euler–Lagrange Equation

In this section we further analyze the behavior of the minimizers of the energy E_ε with the aid of the corresponding Euler-Lagrange equation, and we prove our main result, Theorem 1.1.

Lemma 2.6. *Consider the ordinary differential equation*

$$x' = F(x), \tag{2.29}$$

where $F : \mathbb{R}^4 \to \mathbb{R}^4$ is a C^4 mapping satisfying $F(x_0) = 0$ for some $x_0 \in \mathbb{R}^4$. Assume $DF(x_0)$ has four eigenvalues $\pm\gamma \pm \delta i$, where $\gamma > 0$ and $\delta \in \mathbb{R}$. Then for $0 < \lambda \leq \gamma$ there exist a constant $C(\gamma, \delta) > 0$, $T_0(\gamma, \delta) > 0$ and $R > 0$ such that for all $T > T_0$, if $x : [0, T] \to B(x_0, R)$ is a solution of (2.29)*, then the inequality*

$$|x(t) - x_0| \leq C(\gamma, \delta) \exp(-\lambda T/2) \tag{2.30}$$

holds for all $t \in \left[\frac{\lambda T}{2\gamma}, T - \frac{\lambda T}{2\gamma}\right]$. In particular, if $\gamma = \lambda$,

$$|x(T/2) - x_0| \leq C(\gamma, \delta) \exp(-\gamma T/2). \tag{2.31}$$

Proof. Changing variables if necessary, we may assume, without loss of generality, that $x_0 = 0$. Let $A := DF(0)$. By an extension of the Hartman–Grobman Theorem (see, e.g. [34] and Lemma 5.5 in the Appendix), there exist two open neighborhoods of 0, $V_1, V_2 \subset \mathbb{R}^4$, and a diffeomorphism $h : V_1 \to V_2$ of class C^1, with $h(0) = 0$, such that if $x(t) \in V_1$ for all $t \in [0, T]$ then the funciton $y(t) := h(x(t)), t \in [0, T]$ is a solution of the linearized system

$$y' = Ay. \tag{2.32}$$

Let $R > 0$ be so small that $\overline{B(0,R)} \subset V_1$, and define $V := h(B(0,R))$. Then V is bounded and since $h(0) = 0$, there exists $L > 0$ such that $V \subset B(0,L)$. Hence if $x(t) \in B(0,R)$ for all $t \in [0,T]$, then $y(t) \in B(0,L)$ for all $t \in [0,T]$.

Since the eigenvalues of A are all distinct, the solution of (2.32) has the form

$$y(t) = c_1 v_1 \exp\left((-\gamma - \delta i)t\right) + c_2 v_2 \exp\left((-\gamma + \delta i)t\right) + c_3 v_3 \exp\left((\gamma - \delta i)t\right) + c_4 v_4 \exp\left((-\gamma - \delta i)t\right),$$

where $c_1, \ldots, c_4$ are complex valued constants and $\{v_i\} \subset \mathbb{C}^4$ is a linearly independent set of eigenvectors of A. Letting $P = [v_1, v_2, v_3, v_4]$ be the matrix of eigenvectors of A, we write the above solution as

$$y(t) = P[c_1 \exp\left((-\gamma - \delta i)t\right), c_2 \exp\left((-\gamma + \delta i)t\right), c_3 \exp\left((\gamma - \delta i)t\right), c_4 \exp\left((\gamma - \delta i)t\right)]^{\mathrm{Tr}}, \tag{2.33}$$

where the superscript Tr denotes the transpose of a matrix. Since $y(t) \in B(0,L)$ for all $t \in [0,T]$,

$$\begin{aligned} |[c_1 \exp\left((-\gamma - \delta i)t\right), c_2 \exp\left((-\gamma + \delta i)t\right), c_3 \exp\left((\gamma - \delta i)t\right), c_4 \exp\left((\gamma - \delta i)t\right)]|^2 &\le ||P^{-1}||^2 |y(t)|^2 \\ &\le L^2 ||P^{-1}||^2, \end{aligned}$$

where $||P^{-1}||$ is the operator norm of P^{-1}. In particular,

$$|c_1|^2 \le L^2 ||P^{-1}||^2 \exp(2\gamma t), \quad |c_2|^2 \le L^2 ||P^{-1}||^2 \exp(2\gamma t), \tag{2.34}$$

$$|c_3|^2 \le L^2 ||P^{-1}||^2 \exp(-2\gamma t), \quad |c_4|^2 \le L^2 ||P^{-1}||^2 \exp(-2\gamma t), \tag{2.35}$$

for all $t \in [0,T]$. Setting $t = 0$ and $t = T$ in the first and second row respectively we obtain bounds on the constants $c_1, \ldots, c_4$,

$$|c_1| \le L||P^{-1}||, \quad |c_2| \le L||P^{-1}||, \tag{2.36}$$

$$|c_3| \le L||P^{-1}|| \exp(-\gamma T), \quad |c_4| \le L||P^{-1}|| \exp(-\gamma T). \tag{2.37}$$

Using the resulting bounds in (2.33) yields

$$\begin{aligned} \exp(\lambda T)|y(t)|^2 &\le \exp(\lambda T)||P||^2 \left(|c_1|^2 \exp(-2\gamma t) + |c_2|^2 \exp(-2\gamma t) + |c_3|^2 \exp(2\gamma t) + |c_4|^2 \exp(2\gamma t)\right) \\ &\le 4L^2 ||P||^2 ||P^{-1}||^2, \end{aligned}$$

provided

$$\lambda T - 2\gamma t \le 0 \quad \text{and} \quad \lambda T - 2\gamma T + 2\gamma t \le 0.$$

Both of these conditions are satisfied as long as

$$t \in \left[\frac{\lambda T}{2\gamma}, T - \frac{\lambda T}{2\gamma}\right] =: [t_1, t_2].$$

Hence for $t \in [t_1, t_2]$,

$$|y(t)|^2 \le 4L^2 ||P||^2 ||P^{-1}||^2 \exp(-\lambda T).$$

In particular, if T is sufficiently large (depending only on γ, δ, and V_2), there exists a compact set E such that $y(t) \in E \subset V_2$ for all $t \in [t_1, t_2]$. Since h^{-1} is C^1 and $h(0) = 0$, by the Mean Value Theorem,

$$|x(t)| = |h^{-1}(y(t))| \le \sup_{s \in E} |\nabla h^{-1}(s)||y(t)| \le C_{\gamma,\delta} \exp(-\lambda T/2) \tag{2.38}$$

for all $t \in [t_1, t_2]$, where $C_{\gamma,\delta} := L \sup_{s \in E} |\nabla h^{-1}(s)| ||P|| ||P^{-1}||$. ☐

For a given open interval I and a subinterval $(y_1, y_2) \subset I$ we define

$$\mathcal{M} := \left\{ w \in H^2((y_1, y_2)) : w(y_1) = 0, \ w(y_2) = 0 \right\}. \tag{2.39}$$

Proposition 2.7. *Let $\varepsilon_0 > 0$ and let $\hat{w}_\varepsilon$ be a global minimizer of $E_\varepsilon(\cdot;(y_1,y_2))$ on $\mathcal{M}$ satisfying*

$$E_\varepsilon(\hat{w}_\varepsilon;(y_1,y_2)) \le M, \tag{2.40}$$

for all $\varepsilon < \varepsilon_0$. Then $\hat{w}_\varepsilon$ solves the Euler–Lagrange equation

$$2\varepsilon^4 \hat{w}_\varepsilon^{(iv)} + 2q\varepsilon^2 \hat{w}_\varepsilon'' + W'(\hat{w}_\varepsilon) = 0, \tag{2.41}$$

with additional natural boundary conditions $\hat{w}_\varepsilon''(y_1) = \hat{w}_\varepsilon''(y_2) = 0$, and for all $\varepsilon < \varepsilon_0$ satisfies the estimates

$$\operatorname{dist}(\hat{w}_\varepsilon((y_1+y_2)/2), \{\pm 1\}) \le C_M \exp\left(-\frac{d\gamma}{2\varepsilon}\right), \tag{2.42}$$

$$|\hat{w}_\varepsilon^{(m)}((y_1+y_2)/2)| \le C_M \exp\left(-\frac{d\gamma}{2\varepsilon}\right), \quad m = 1, \dots 3, \tag{2.43}$$

where $d := y_2 - y_1$ and $C_M > 0$ is a positive constant dependent only on M, q and the potential W.

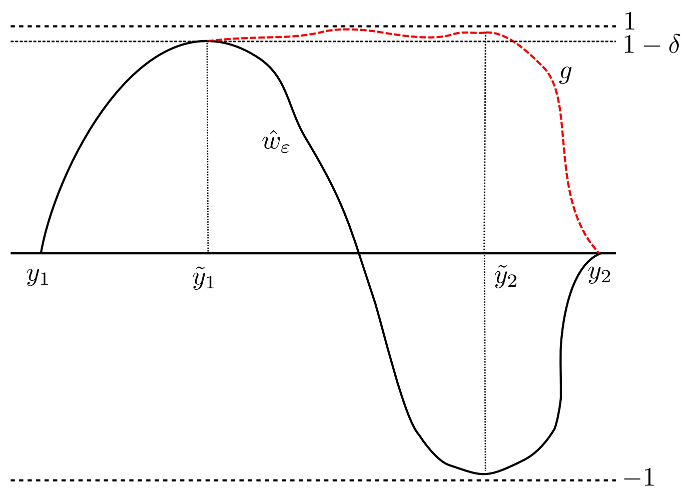

Figure 2: The contradiction argument.

Proof. Fix $\delta > 0$ to be chosen later. We first observe that, due to the upper bound (2.40) and Lemma 2.4, there exists $c = c(\delta, M) > 0$ and points $\tilde{y}_1 \in (y_1, y_1 + c\varepsilon)$ and $\tilde{y}_2 \in (y_2 - c\varepsilon, y_2)$ such that

$$\operatorname{dist}(\hat{w}_\varepsilon(\tilde{y}_1), \{\pm 1\}) < \delta, \quad |\varepsilon \hat{w}_\varepsilon'(\tilde{y}_1)| < \delta, \tag{2.44}$$

$$\operatorname{dist}(\hat{w}_\varepsilon(\tilde{y}_2), \{\pm 1\}) < \delta, \quad |\varepsilon \hat{w}_\varepsilon'(\tilde{y}_2)| < \delta. \tag{2.45}$$

In addition, we claim that since $\hat{w}_\varepsilon$ is a minimizer, at $\tilde{y}_1$ and $\tilde{y}_2$ its value is near the same well of W, i.e., we may assume without loss of generality that

$$|\hat{w}_\varepsilon(\tilde{y}_1) - 1| < \delta, \quad |\hat{w}_\varepsilon(\tilde{y}_2) - 1| < \delta. \tag{2.46}$$

As a matter of fact, if this was not the case and for example

$$|\hat{w}_\varepsilon(\tilde{y}_1) - 1| < \delta, \quad |\hat{w}_\varepsilon(\tilde{y}_2) + 1| < \delta. \tag{2.47}$$

then consider

$$g(x) := \begin{cases} \hat{w}_\varepsilon(x), & y_1 \le x \le \tilde{y}_1, \\ \phi(x), & \tilde{y}_1 \le x \le \tilde{y}_2, \\ -\hat{w}_\varepsilon(x), & \tilde{y}_2 \le x \le y_2, \end{cases}$$

where

$$\begin{aligned} \phi(x) := 1 &+ (\hat{w}_\varepsilon(\tilde{y}_1) - 1)\varphi_0(x - \tilde{y}_1) + \hat{w}'_\varepsilon(\tilde{y}_1)\varphi_1(x - \tilde{y}_1) \\ &+ (-\hat{w}_\varepsilon(\tilde{y}_2) - 1)\varphi_0(\tilde{y}_2 - x) + \hat{w}'_\varepsilon(\tilde{y}_2)\varphi_1(\tilde{y}_2 - x), \end{aligned} \tag{2.48}$$

and φ_0, φ_1 satisfy

$$\begin{gathered} \varphi_j \in C^\infty(\mathbb{R}),\ \varphi_j(x) = 0 \text{ for all } x \ge (y_2 - y_1)/2, \\ \varphi_0(0) = 1,\ \varphi'_0(0) = 0,\ \varphi_1(0) = 0,\ \varphi'_1(0) = 1. \end{gathered}$$

It is easy to see that

$$\begin{aligned} \phi(\tilde{y}_1) &= \hat{w}_\varepsilon(\tilde{y}_1), \quad & \phi'(\tilde{y}_1) &= \hat{w}'_\varepsilon(\tilde{y}_1), \\ \phi(\tilde{y}_2) &= -\hat{w}_\varepsilon(\tilde{y}_2), \quad & \phi'(\tilde{y}_2) &= -\hat{w}'_\varepsilon(\tilde{y}_2), \end{aligned}$$

and consequently $g \in H^2((y_1, y_2))$. Obtaining ϕ' from (2.48) and using (2.47), we get

$$||\phi'||^2_{L^\infty(\tilde{y}_1,\tilde{y}_2)} \le c(||\varphi'_0||^2_{L^\infty(\mathbb{R})} + ||\varphi'_1||^2_{L^\infty(\mathbb{R})})\delta^2,$$

where $c > 0$ is a constant and we notice that

$$\int_{\tilde{y}_1}^{\tilde{y}_2} |\phi'|^2 dx \le c(y_2 - y_1)(||\varphi'_0||^2_{L^\infty(\mathbb{R})} + ||\varphi'_1||^2_{L^\infty(\mathbb{R})})\delta^2.$$

Similarly, an analogous bound for ϕ'' can be derived. Additionally, using Taylor's formula for W and the facts that $W(\pm 1) = W'(\pm 1) = 0$ and $W \in C^2(\mathbb{R})$, it follows that

$$E_\varepsilon(\phi; (\tilde{y}_1, \tilde{y}_2)) \le \xi_1 \delta^2, \tag{2.49}$$

where ξ_1 only depends on y_1 and y_2, which do not depend on δ, while interpolation inequality of Corollary 2.3 yields for δ sufficiently small

$$\begin{aligned} E_\varepsilon(\hat{w}_\varepsilon; (\tilde{y}_1, \tilde{y}_2)) &= \int_{\tilde{y}_1}^{\tilde{y}_2} \left(\frac{1}{\varepsilon} W(\hat{w}_\varepsilon) - q\varepsilon|\hat{w}'_\varepsilon|^2 + \varepsilon^3|\hat{w}''_\varepsilon|^2\right) dx \ge \sigma \int_{\tilde{y}_1}^{\tilde{y}_2} \left(\frac{1}{\varepsilon} W(\hat{w}_\varepsilon) + \varepsilon|\hat{w}'_\varepsilon|^2\right) dx \\ &\ge \sigma \int_{\tilde{y}_1}^{\tilde{y}_2} \sqrt{W(\hat{w}_\varepsilon)}\hat{w}'_\varepsilon dx = \sigma \int_{\hat{w}_\varepsilon(\tilde{y}_1)}^{\hat{w}_\varepsilon(\tilde{y}_2)} \sqrt{W(s)} ds \ge \sigma \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{W(s)} ds =: \sigma\xi_2 > 0. \end{aligned}$$

In turn, from (2.49), possibly choosing δ even smaller we get a contradiction with the fact that $\hat{w}_\varepsilon$ is a minimizer.

Since $\hat{w}_\varepsilon$ is a minimizer of $E_\varepsilon(\cdot; (y_1, y_2))$, it follows from standard arguments that it satisfies the Euler–Lagrange equation (2.41). We change variables $z = \frac{x - y_1}{\varepsilon}$ and define $\hat{v}(z) := \hat{w}_\varepsilon(x)$. Observe that

$$E_\varepsilon(\hat{w}_\varepsilon; (y_1, y_2)) = E(\hat{v}; (0, d/\varepsilon)) \tag{2.50}$$

and the rescaled minimizer $\hat{v}$ satisfies the Euler–Lagrange equation

$$2\hat{v}^{(iv)} + 2q\hat{v}'' + W'(\hat{v}) = 0, \quad \hat{v}''(0) = \hat{v}''(d/\varepsilon) = 0. \tag{2.51}$$

We now apply Lemma 2.5 on the interval $\left(\frac{\tilde{y}_1 - y_1}{\varepsilon}, \frac{\tilde{y}_2 - y_1}{\varepsilon}\right)$ with

$$\alpha_0 := \hat{w}_\varepsilon(\tilde{y}_1) = \hat{v}\left(\frac{\tilde{y}_1 - y_1}{\varepsilon}\right), \quad \alpha_1 := \varepsilon\hat{w}'_\varepsilon(\tilde{y}_1) = \hat{v}'\left(\frac{\tilde{y}_1 - y_1}{\varepsilon}\right), \tag{2.52}$$

$$\beta_0 := \hat{w}_\varepsilon(\tilde{y}_2) = \hat{v}\left(\frac{\tilde{y}_2 - y_1}{\varepsilon}\right), \quad \beta_1 := \varepsilon \hat{w}_\varepsilon'(\tilde{y}_2) = \hat{v}'\left(\frac{\tilde{y}_2 - y_1}{\varepsilon}\right). \tag{2.53}$$

The resulting minimizer agrees with $\hat{v}$ on this interval and given $R > 0$, for δ sufficiently small the bounds (2.44) and (2.45) imply that

$$\chi := [\hat{v} - 1, \hat{v}', \hat{v}'', \hat{v}'''] \in B(0, R).$$

Using the notation $\chi = [\chi_1, \chi_2, \chi_3, \chi_4]$, we rewrite (2.51) in the system form

$$\chi' = F(\chi) \tag{2.54}$$

where

$$F(\chi) = \begin{bmatrix} \chi_2 \\ \chi_3 \\ \chi_4 \\ -\frac{1}{2}W'(\chi_1) - q\chi_2 \end{bmatrix}$$

and the Jacobian of F at 0 is given by

$$DF(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{2}W''(1) & 0 & -q & 0 \end{bmatrix}.$$

The eigenvalues of $DF(0)$ are the roots of the characteristic polynomial

$$2r^4 + 2qr^2 + W''(1) = 0.$$

In particular,

$$r^2 = \frac{-2q \pm \sqrt{4q^2 - 8W''(1)}}{4},$$

and since $q > 0$ is small, the expression under the square root is negative. We write

$$\begin{cases} r^2 = \dfrac{-2q + \sqrt{4q^2 - 8W''(1)}}{4}, \\ r^2 = \dfrac{-2q - \sqrt{4q^2 - 8W''(1)}}{4} \end{cases}$$

and let r_1, r_2 be the roots of the first equation, r_3, r_4 those of the second one. We recall that

$$\sqrt{a + ib} = \pm(\gamma + i\delta),$$

for

$$\gamma = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}, \quad \delta = \operatorname{sgn}(b)\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}.$$

In the case of r_1, we write

$$r_1 = \left(-\frac{q}{2} + i\frac{\sqrt{2W''(1) - q^2}}{2}\right)^{1/2},$$

and a simple calculation shows that

$$\gamma = \frac{1}{2}\left(-q + \sqrt{2W''(1)}\right)^{1/2}, \quad \delta = \frac{1}{2}\left(q + \sqrt{2W''(1)}\right)^{1/2}. \tag{2.55}$$

Similarly, one can show that

$$\begin{cases} r_1 = \gamma + i\delta, \\ r_2 = -r_1, \\ r_3 = \gamma - i\delta, \\ r_4 = -r_3, \end{cases} \tag{2.56}$$

Applying Lemma 2.6 on the interval $\left(c, \frac{y_2-y_1-c\varepsilon}{\varepsilon}\right) \subset \left(\frac{\tilde{y}_1-y_1}{\varepsilon}, \frac{\tilde{y}_2-y_1}{\varepsilon}\right)$ yields

$$\left|\varphi\left(\frac{y_2-y_1}{2\varepsilon}\right)\right| \le C(\gamma,\delta)\exp\left(-\gamma\frac{y_2-y_1-2c\varepsilon}{2\varepsilon}\right) \le C(\gamma,\delta)\exp\left(\gamma c - \gamma\frac{d}{2\varepsilon}\right) \tag{2.57}$$

and (2.42), (2.43) follow from definition of φ and the fact that

$$\hat{w}\left(\frac{y_1+y_2}{2}\right) = \hat{v}\left(\frac{y_2-y_1}{2\varepsilon}\right). \tag{2.58}$$

□

Proof of Theorem 1.1. Without loss of generality we can assume that $N(v) \ge 2$ and define

$$\mathcal{M}_k := \{w \in H^2((x_k, x_{k+1})) : w(x_k) = 0,\ w(x_{k+1}) = 0\}. \tag{2.59}$$

We define $\hat{w}_k \in H^2((x_k, x_{k+1}))$, for $1 \le k \le N$, to be the minimizer of $E_\varepsilon(\cdot, (x_k, x_{k+1}))$ over $\mathcal{M}_k$. We also let $\hat{w}_0 := \hat{w}_N$. In turn, $\hat{w}_k$ solves the Euler–Lagrange equation (2.41) with

$$\hat{w}_k(x_k) = \hat{w}_k(x_{k+1}) = 0.$$

Define $d_k := x_{k+1} - x_k$ for $k = 1, \dots, N(v)$ and

$$I_k^-(x_k) := \left(x_k - \frac{d_{k-1}}{2}, x_k\right) \quad \text{and} \quad I_k^+(x_k) := \left(x_k, x_k + \frac{d_k}{2}\right).$$

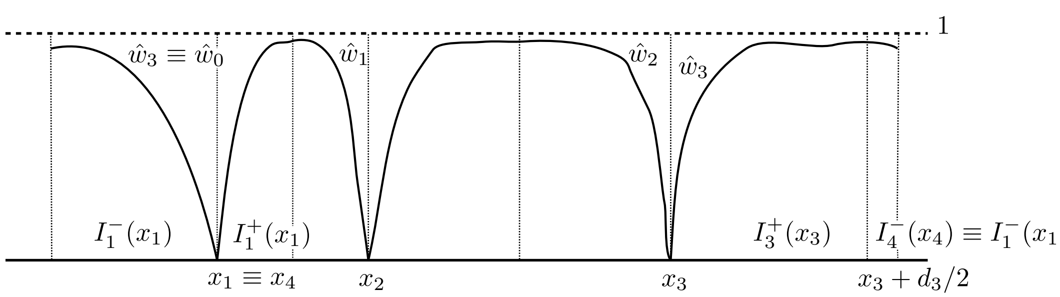

Figure 3: $\hat{w}_k$ and $I_k^\pm$

From the minimality of $\hat{w}_k$, we have

$$\begin{aligned} E_\varepsilon(w;\mathbb{T}) &= \sum_{k=1}^{N(v)} E_\varepsilon(w;(x_k,x_{k+1})) \ge \sum_{k=1}^{N(v)} E_\varepsilon(\hat{w}_k;(x_k,x_{k+1})) \\ &= \sum_{k=1}^{N(v)} E_\varepsilon(\hat{w}_{k-1}; I_k^-(x_k)) + E_\varepsilon(\hat{w}_k; I_k^+(x_k)), \end{aligned} \tag{2.60}$$

where in the last equality we have used the fact that $x_{N+1} := x_1$. To complete the proof, it remains to show that

$$E_\varepsilon(\hat{w}_{k-1}; I_k^-(x_k)) \geq \frac{m_1}{2} - C\exp\left(-\frac{d_{k-1}\gamma}{\varepsilon}\right) \tag{2.61}$$

and

$$E_\varepsilon(\hat{w}_k; I_k^+(x_k)) \geq \frac{m_1}{2} - C\exp\left(-\frac{d_k\gamma}{\varepsilon}\right). \tag{2.62}$$

We will only prove (2.62), the proof of the first inequality being analogous. Applying the change of variables $z := \frac{x-x_k}{\varepsilon}$ gives

$$\begin{aligned} E_\varepsilon(\hat{w}_k; I_k^+(x_k)) &= \int_{\frac{1}{\varepsilon}I_k^+(0)} \left(W(\hat{w}_k(x_k+\varepsilon z)) - \varepsilon q|\hat{w}_k'(x_k+\varepsilon z)|^2 + \varepsilon^3|\hat{w}_k''(x_k+\varepsilon z)|^2\right)\varepsilon dz \\ &= \int_{\frac{1}{\varepsilon}I_k^+(0)} \left(W(\hat{v}_k(z)) - q|\hat{v}_k'(z)|^2 + |\hat{v}_k''(z)|^2\right)dz = E\left(\hat{v}_k; \frac{1}{\varepsilon}I_k^+(0)\right), \end{aligned}$$

where $E(\cdot;\cdot)$ is the rescaled functional defined in (2.7) and

$$\hat{v}_k(z) := \hat{w}_k(x) \text{ on each } \frac{1}{\varepsilon}I_k^+(0).$$

In addition, we notice that $\hat{v}_k(0) = \hat{w}_k(x_k) = 0$ for $1 \leq k \leq N$ and Proposition 2.7, together with the change of variables we performed, gives

$$|\hat{v}_k(d_k/2\varepsilon) - s_k| = |\hat{w}_k((x_k + x_{k+1})/2) - s_k| \leq C_f \exp\left(-\frac{d_k\gamma}{2\varepsilon}\right) \tag{2.63}$$

and

$$|\hat{v}_k'(d_k/2\varepsilon)| = |\hat{w}_k'((x_k + x_{k+1})/2)| \leq C_f \exp\left(-\frac{d_k\gamma}{2\varepsilon}\right), \tag{2.64}$$

where s_k is equal to either 1 or -1. We claim that

$$E\left(\hat{v}_k; \frac{1}{\varepsilon}I_k^{\varepsilon,+}(0)\right) \geq \frac{m_1}{2} - E\left(\eta_k; \mathbb{R}^+\right) \tag{2.65}$$

where

$$\begin{aligned} \eta_k(x) := s_k &+ (\hat{v}_k(d_k/2\varepsilon) - s_k)\exp(-\gamma x)\cos(\delta x) \\ &+ \frac{\hat{v}_k'(d_k/2\varepsilon) + \gamma(\hat{v}_k(d_k/2\varepsilon) - s_k)}{\delta}\exp(-\gamma x)\sin(\delta x). \end{aligned} \tag{2.66}$$

Indeed, let $\theta_\varepsilon^+ \in H^2_{loc}(\mathbb{R}^+)$ be the function that coincides with $\hat{v}_k$ on $\frac{1}{\varepsilon}I_k^{\varepsilon,+}(0)$ and $\eta_k^+ := \eta_k(\cdot - d_k/2\varepsilon)$ on $\mathbb{R}^+ \setminus \frac{1}{\varepsilon}I_k^{\varepsilon,+}(0)$. Then,

$$E(\theta_\varepsilon^+; \mathbb{R}^+) \geq m_1/2,$$

and in turn (2.65) follows. We now want to find an upper bound for $E(\eta_k; \mathbb{R}^+)$, for ε small enough. The bounds (2.63), (2.64) and the definition of η_k imply that there exists a constant $C > 0$ such that

$$|\eta_k(x) - s_+| + |\eta_k'(x)| + |\eta_k''(x)| \leq C\exp\left(-\frac{d_k\gamma}{2\varepsilon}\right)\exp(-\gamma x) \quad \text{for all } x > 0 \tag{2.67}$$

and consequently

$$\begin{aligned} E(\eta_k; \mathbb{R}^+) &= \int_0^\infty W(\eta_k) - q|\eta_k'|^2 + |\eta_k''|^2 dx \\ &= \int_0^\infty \frac{W''(s_+)}{2}(\eta_k - s_+)^2 - q|\eta_k'|^2 + |\eta_k''|^2 + O((\eta^k - s_+)^3)dx \\ &\leq C\exp\left(-\frac{d_k\gamma}{\varepsilon}\right)\int_0^\infty \exp(-2\gamma x)dx \leq C\exp\left(-\frac{d_k\gamma}{\varepsilon}\right). \end{aligned}$$

□

3 Slow Motion Dynamics: Proof of Theorem 1.3

Proof of Theorem 1.3. Fix $0 < \delta < \min\{1, d/8\}$. We recall that by definition of E_ε

$$E_\varepsilon(u^\varepsilon(\cdot,t);\mathbb{T}) = \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W(u^\varepsilon) - \varepsilon q|u_x^\varepsilon|^2 + \varepsilon^3|u_{xx}^\varepsilon|^2\right) dx.$$

Integrating by parts and using the regularity of the solution u^ε and equation (1.2) gives

$$\begin{aligned}\frac{d}{dt}E_\varepsilon(u^\varepsilon(\cdot,t);\mathbb{T}) &= \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W'(u^\varepsilon)u_t^\varepsilon - 2\varepsilon q u_x^\varepsilon u_{xt}^\varepsilon + 2\varepsilon^3 u_{xx}^\varepsilon u_{xxt}^\varepsilon\right) dx \\ &= \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W'(u^\varepsilon)u_t^\varepsilon + 2\varepsilon q u_{xx} u_t^\varepsilon + 2\varepsilon^3 u_{xxxx}^\varepsilon u_t^\varepsilon\right) dx \\ &= -\int_{\mathbb{T}} |u_t^\varepsilon|^2 dx.\end{aligned}$$

It follows that for every $T > 0$,

$$E_\varepsilon(u_{0,\varepsilon};\mathbb{T}) - E_\varepsilon(u^\varepsilon(\cdot,T);\mathbb{T}) = \frac{1}{\varepsilon}\int_0^T \int_{\mathbb{T}} |u_t^\varepsilon|^2 dxdt. \tag{3.1}$$

Suppose there exists T_ε such that

$$\int_0^{T_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon| dxdt \le \delta \tag{3.2}$$

Then,

$$\int_{\mathbb{T}} |u_{0,\varepsilon} - u^\varepsilon(\cdot,T_\varepsilon)|dx = \int_{\mathbb{T}} \left|\int_0^{T_\varepsilon} u_t^\varepsilon dt\right| dx \le \int_{\mathbb{T}}\int_0^{T_\varepsilon} |u_t^\varepsilon| dtdx \le \delta \tag{3.3}$$

and using (1.18) and the triangle inequality

$$||u^\varepsilon(\cdot,T_\varepsilon) - v||_{L^1(\mathbb{T})} \le 2\delta. \tag{3.4}$$

We claim that $u^\varepsilon(\cdot,T_\varepsilon)$ has at least N_ε zeros, $\{x_k^\varepsilon\}_{k=1}^{N_\varepsilon}$ that satisfy $\min_k |x_{k+1}^\varepsilon - x_k^\varepsilon| \ge d - 4\delta$.

Indeed, consider x_k, the k–th jump point of v. Since the distance between jump points of v is at least d and $\delta \le d/8$, we know that v is constant on $(x_k - 2\delta, x_k)$ and on $(x_k, x_k + 2\delta)$ and may assume without loss of generality that its value is equal to 1 on $(x_k - 2\delta, x_k)$ and to -1 on $(x_k, x_k + 2\delta)$. It follows from (3.4) that $u^\varepsilon(\cdot,T_\varepsilon)$ must take a positive value somewhere on $(x_k - 2\delta, x_k)$ and a negative value on $(x_k, x_k + 2\delta)$. Hence, there exists a zero $x_k^\varepsilon \in (x_k - 2\delta, x_k + 2\delta)$ of $u^\varepsilon(\cdot,T_\varepsilon)$.

Applying Hölder inequality, (1.18), (3.1), and Theorem 1.1 yields

$$\begin{aligned}\frac{1}{T_\varepsilon}\left(\int_0^{T_\varepsilon}\int_{\mathbb{T}} |u_t^\varepsilon| dxdt\right)^2 &\le \int_0^{T_\varepsilon}\int_{\mathbb{T}} |u_t^\varepsilon|^2 dxdt \\ &= \varepsilon\left(E_\varepsilon(u_{0,\varepsilon};\mathbb{T}) - E_\varepsilon(u^\varepsilon(\cdot,T_\varepsilon);\mathbb{T})\right) \\ &\le \varepsilon\left(E_0(v;\mathbb{T}) + \frac{1}{h(\varepsilon)} - m_1 N_\varepsilon + C\sum_{k=1}^{N_\varepsilon}\exp\left(-\frac{(x_{k+1}^\varepsilon - x_k^\varepsilon)\gamma}{\varepsilon}\right)\right) \qquad (3.5) \\ &\le \varepsilon\left(E_0(v;\mathbb{T}) + \frac{1}{h(\varepsilon)} - E_0(v;\mathbb{T}) + C\exp\left(-\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right) \\ &= \varepsilon\left(\frac{1}{h(\varepsilon)} + C\exp\left(-\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right)\end{aligned}$$

and as a consequence,

$$T_\varepsilon \geq \frac{1}{C\varepsilon}\left[\frac{1}{h(\varepsilon)} + \exp\left(-(d-4\delta)\gamma/\varepsilon\right)\right]^{-1}\left(\int_0^{T_\varepsilon}\int_{\mathbb{T}}|u_t^\varepsilon|dxdt\right)^2. \tag{3.6}$$

Following the ideas of [22], we prove the existence of T_ε as in (3.2) by dividing the analysis into two cases: first assume that

$$\int_0^\infty\int_{\mathbb{T}}|u_t^\varepsilon|dxdt > \delta.$$

Since by (3.1) with T replaced by any $S > 0$,

$$\int_0^S\int_{\mathbb{T}}|u_t^\varepsilon|^2dxdt \leq \varepsilon E_\varepsilon(u_{0,\varepsilon};\mathbb{T}) < \infty$$

we can choose T_ε such that

$$\int_0^{T_\varepsilon}\int_{\mathbb{T}}|u_t^\varepsilon|dxdt = \delta, \tag{3.7}$$

and thanks to (3.7), equation (3.6) gives

$$T_\varepsilon \geq \frac{\delta^2}{C\varepsilon\left[\frac{1}{h(\varepsilon)} + \exp\left(-(d-4\delta)\gamma/\varepsilon\right)\right]} \geq \frac{\delta^2}{2C\varepsilon}\min\{h(\varepsilon), \exp((d-4\delta)\gamma/\varepsilon)\} =: \Lambda_\varepsilon.$$

In turn, (3.2) is satisfied and (3.5) yields

$$\int_0^{\Lambda_\varepsilon}\int_{\mathbb{T}}|u_t^\varepsilon|^2dxdt \leq C\varepsilon\left[\frac{1}{h(\varepsilon)} + \exp\left(-(d-4\delta)\gamma/\varepsilon\right)\right]. \tag{3.8}$$

On the other hand, if

$$\int_0^\infty\int_{\mathbb{T}}|u_t^\varepsilon|dxdt \leq \delta,$$

then (3.2) holds true for all $T > 0$ and again (3.8) follows. To conclude the proof note that for ε sufficiently small

$$s_\varepsilon := \delta^2\min\{h(\varepsilon), \exp((d-4\delta)\gamma/\varepsilon)\} \leq \Lambda_\varepsilon$$

and Hölder's inequality together with (3.8) yield

$$\begin{aligned}
&\sup_{0\leq t\leq s_\varepsilon}\int_{\mathbb{T}}|u^\varepsilon(x,t) - u_{0,\varepsilon}(x)|dx \leq \int_0^{s_\varepsilon}\int_{\mathbb{T}}|u_t^\varepsilon|dxdt \\
&\leq \left(\min\left\{h(\varepsilon), \exp\left(\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right\}\int_0^{s_\varepsilon}\int_{\mathbb{T}}|u_t^\varepsilon|^2dxdt\right)^{1/2} \\
&\leq C\left(\min\left\{h(\varepsilon), \exp\left(\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right\}\varepsilon\delta^2\left[\frac{1}{h(\varepsilon)} + \exp\left(-\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right]\right)^{1/2} \\
&\leq C\sqrt{\varepsilon}\delta.
\end{aligned}$$

Letting $\varepsilon \to 0^+$ gives (1.20). □

4 Existence of Solutions Via Minimizing Movements

We now turn to the existence and regularity of solutions for (1.2) in the more general case of an open, bounded domain $\Omega \subset \mathbb{R}^d$. We notice that the same proof carries over in the case of the one–dimensional torus $\Omega = \mathbb{T}$, that is, when we deal with periodic Dirichlet boundary conditions, which is the framework in which we have analyzed slow motion of solutions of (1.2).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be an open bounded set with C^2 boundary, let $u_0 \in H^2(\Omega)$ and the real valued function $z \mapsto W(z)$ be a double–well potential satisfying hypotheses (2.1)–(2.4). Then for every $T > 0$ there exists a weak solution $u^\varepsilon \in L^\infty((0,T); H^2(\Omega))$ in the sense of (4.25), with $u_t^\varepsilon \in L^2((0,T); L^2(\Omega))$ of*

$$\begin{cases} u_t = -\frac{1}{\varepsilon}W'(u) - 2\varepsilon q \Delta u - 2\varepsilon^3 \Delta^2 u & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.1}$$

such that

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx + \int_0^t \int_\Omega \frac{1}{\varepsilon} W'(u(x,s))dxds.$$

Moreover, the following estimates hold

$$\int_0^T \int_\Omega |u_t(x,t)|^2 dxdt \leq M_\varepsilon \sigma^{-1},$$

$$\int_\Omega |\nabla u(x,t)|^2 dx \leq 3M_\varepsilon \sigma^{-1},$$

$$\int_\Omega |\nabla^2 u(x,t)|^2 dx \leq 3M_\varepsilon \sigma^{-1},$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$, where $\sigma \in (0,1)$ and

$$M_\varepsilon := 2\int_\Omega \left(\frac{1}{\varepsilon} W(u_0) + \varepsilon |\nabla u_0|^2 + \varepsilon^3 |\nabla^2 u_0|^2 \right) dx. \tag{4.2}$$

Proof. **Step 1.** For $\ell \in \mathbb{N}$ we set $\tau := T/\ell$ and subdivide the interval $(0,T)$ into ℓ subintervals of length τ,

$$\tau_0 := 0 < \tau_1 < \ldots < \tau_\ell := T,$$

where $\tau_n := n\tau$ for $n = 1, \ldots, \ell$. For every $n = 1, \ldots, \ell$, we let $u_n \in H^2(\Omega)$ be a solution of the minimization problem

$$\min_{v \in H^2(\Omega)} J_{\varepsilon,n}(v;\Omega),$$

where

$$\begin{aligned} J_{\varepsilon,n}(v;\Omega) &:= \int_\Omega \left(\frac{1}{\varepsilon} W(v) - \varepsilon q |\nabla v|^2 + \varepsilon^3 |\nabla^2 v|^2 \right) dx + \frac{1}{2\tau} \int_\Omega (v - u_{n-1})^2 dx \\ &= E_\varepsilon(v;\Omega) + \frac{1}{2\tau} \int_\Omega (v - u_{n-1})^2 dx. \end{aligned}$$

In order to prove the existence of u_n, we begin by showing that J_n is non–negative and coercive in $H^2(\Omega)$. We fix $q^* > 0$ such that the interpolation inequality Lemma 2.1 holds in Ω, namely

$$k\varepsilon^2 \int_\Omega |\nabla u|^2 dx \leq \int_\Omega \left[W(u) + \varepsilon^4 |\nabla^2 u|^2 \right] dx, \ -\infty < k \leq q^*,$$

and we let $\sigma \in (0,1)$ be such that $(q+\sigma)/(1-\sigma) < q^*$, so that we can write

$$\begin{aligned} W(u) - q^2\varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2 = (1-\sigma)\left(W(u) - \frac{q+\sigma}{1-\sigma}\varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2\right) \\ + \sigma(W(u) + \varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2), \end{aligned} \tag{4.3}$$

and in turn $J_{\varepsilon,n}$ is non–negative. Then by (2.4), and using the fact that $c_W \le 1$, we obtain

$$E_\varepsilon(u;\Omega) \ge \sigma c_W \int_\Omega \left((|u|-1)^2 + \varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2\right) dx. \tag{4.4}$$

The above chain of inequalities implies that

$$J_{\varepsilon,n}(u;\Omega) = E_\varepsilon(u;\Omega) + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 dx \to \infty \quad \text{as } ||u||_{H^2(\Omega)} \to \infty,$$

and hence J_ε is coercive in $H^2(\Omega)$.

We now let $m_n := \inf_{v \in H^2(\Omega)} J_{\varepsilon,n}(v;\Omega)$, and consider a minimizing sequence $\{v_k\} \subset H^2(\Omega)$ satisfying

$$m_n \le J_{\varepsilon,n}(v_k;\Omega) \le m_n + \frac{1}{k},$$

so that

$$\lim_{k\to\infty} J_{\varepsilon,n}(v_k;\Omega) = m_n.$$

It follows from (4.4) that $\{v_k\}$ is bounded in $H^2(\Omega)$, and hence there exist a subsequence of $\{v_k\}$ (not relabeled) and some $u_n \in H^2(\Omega)$ such that

$$\begin{aligned} v_k &\to u_n && \text{in } L^2(\Omega), \\ v_k &\to u_n && \text{pointwise a.e. in } \Omega, \\ \nabla v_k &\to \nabla u_n && \text{in } L^2(\Omega), \\ \nabla^2 v_k &\rightharpoonup \nabla^2 u_n && \text{in } L^2(\Omega). \end{aligned}$$

We claim that the above convergences imply that $J_{\varepsilon,n}(u_n;\Omega) = m_n$. Indeed, by Fatou's Lemma and lower semicontinuity of L^2 norm with respect to weak convergence, we have

$$m_n = \liminf_{k\to\infty} J_{\varepsilon,n}(v_k;\Omega) \ge J_n(u_n) \ge m_n.$$

It follows that for all $w \in H^2(\Omega)$ and all $t \in \mathbb{R}$,

$$J_{\varepsilon,n}(u_n;\Omega) \le J_{\varepsilon,n}(u_n + tw;\Omega),$$

and hence the real valued function $\omega(t) := J_{\varepsilon,n}(u_n + tw;\Omega)$ has a minimum at $t = 0$, so that $\omega'(0) = 0$. Standard arguments show that for every $w \in H^2(\Omega)$,

$$\begin{aligned} 0 = &\int_\Omega \left(\frac{1}{\varepsilon}W'(u_n)w - 2\varepsilon q \nabla u_n \cdot \nabla w + 2\varepsilon^3 \nabla^2 u_n \cdot \nabla^2 w\right) \\ &+ \frac{1}{\tau}\int_\Omega (u_n - u_{n-1})w, \end{aligned} \tag{4.5}$$

where $W'(u_n)w$ is well–defined by the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ for $d \le 3$, and $\nabla^2 u_n \cdot \nabla^2 w = \sum_{i,j} \frac{\partial^2 u_n}{\partial x_i \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_j}$ is the Fröbenius inner product. In particular, this shows that u_n is a weak solution of the equation

$$-\frac{1}{\varepsilon}W'(u_n) - 2\varepsilon q \Delta u_n - 2\varepsilon^3 \Delta^2 u_n = \frac{1}{\tau}(u_n - u_{n-1}) \quad \text{in } \Omega.$$

Since Ω has finite measure, choosing $w = 1$ in (4.5) gives

$$0 = \int_\Omega \frac{1}{\varepsilon} W'(u_n) dx + \frac{1}{\tau} \int_\Omega (u_n - u_{n-1}) dx.$$

Step 2: Apriori bounds. For $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$, $n = 1, \ldots, \ell$, we define

$$u^\tau(x,t) := u_n(x) + (t - \tau_n) \frac{u_n(x) - u_{n-1}(x)}{\tau}. \tag{4.6}$$

The goal of this step is to find apriori bounds on u^τ.

Since $J_{\varepsilon,n}(u_n;\Omega) = m_n$, it follows that $J_{\varepsilon,n}(u_n;\Omega) \le J_{\varepsilon,n}(u_{n-1};\Omega)$, which implies

$$\frac{1}{2\tau}\int_\Omega (u_n - u_{n-1})^2 dx \le \int_\Omega \left(\frac{1}{\varepsilon}(W(u_{n-1}) - W(u_n)) - \varepsilon q(|\nabla u_{n-1}|^2 - |\nabla u_n|^2)\right) dx + \int_\Omega \varepsilon^3 (|\nabla^2 u_{n-1}|^2 - |\nabla^2 u_n|^2) dx.$$

Summing over $n = 1, \ldots, \ell$, we get

$$\frac{1}{2\tau}\sum_{n=1}^{\ell}\int_\Omega (u_n - u_{n-1})^2 dx \le \int_\Omega \left(\frac{1}{\varepsilon}(W(u_0) - W(u_\ell)) - \varepsilon q(|\nabla u_0|^2 - |\nabla u_\ell|^2)\right) dx + \int_\Omega \varepsilon^3 (|\nabla^2 u_0|^2 - |\nabla^2 u_\ell|^2) dx. \tag{4.7}$$

By the interpolation inequality in Lemma 2.1,

$$\int_\Omega \left(\frac{1}{\varepsilon} W(u_\ell) - \varepsilon q |\nabla u_\ell|^2 + \varepsilon^3 |\nabla^2 u_\ell|^2\right) dx \ge \sigma \int_\Omega \left(\frac{1}{\varepsilon} W(u_\ell) + \varepsilon |\nabla u_\ell|^2 + \varepsilon^3 |\nabla^2 u_\ell|^2\right) dx,$$

where $\sigma \in (0,1)$ was chosen above. Thus, the previous inequalities imply

$$\frac{1}{2\tau}\sum_{n=1}^{\ell}\int_\Omega (u_n - u_{n-1})^2 \, dx + \sigma \int_\Omega \left(\frac{1}{\varepsilon} W(u_\ell) + \varepsilon |\nabla u_\ell|^2 + \varepsilon^3 |\nabla^2 u_\ell|^2\right) dx \le \int_\Omega \left(\frac{1}{\varepsilon} W(u_0) + \varepsilon |\nabla u_0|^2 + \varepsilon^3 |\nabla^2 u_0|^2\right) dx = \frac{M_\varepsilon}{2}, \tag{4.8}$$

see (4.2). By (4.6), for every $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$,

$$\begin{aligned} u_t^\tau(x,t) &= \frac{u_n(x) - u_{n-1}(x)}{\tau}, \\ \nabla u^\tau(x,t) &= \nabla u_n(x) + (t - \tau_n) \frac{\nabla u_n(x) - \nabla u_{n-1}(x)}{\tau}, \\ \nabla^2 u^\tau(x,t) &= \nabla^2 u_n(x) + (t - \tau_n) \frac{\nabla^2 u_n(x) - \nabla^2 u_{n-1}(x)}{\tau}, \end{aligned} \tag{4.9}$$

so that by (4.8) we have

$$\frac{1}{2}\int_{\Omega_T} (u_t^\tau(x,t))^2 \, dxdt + \sigma \int_\Omega \left(\frac{1}{\varepsilon} W(u_\ell) + \varepsilon |\nabla u_\ell|^2 + \varepsilon^3 |\nabla^2 u_\ell|^2\right) dx \le \frac{M_\varepsilon}{2} \tag{4.10}$$

which implies

$$\int_{\Omega_T} (u_t^\tau(x,t))^2 \, dxdt \le M_\varepsilon, \tag{4.11}$$

for every $\tau > 0$. Since u^τ is absolutely continuous, for every $0 \le t_1 < t_2 \le T$,

$$\begin{aligned}\int_\Omega (u^\tau(x,t_2) - u^\tau(x,t_1))^2\,dx &= \int_\Omega \left(\int_{t_1}^{t_2} u_t^\tau(x,t)dt\right)^2 dx \\ &\le (t_2 - t_1)\int_{\Omega_T} (u_t^\tau(x,t))^2\,dxdt \\ &\le M_\varepsilon(t_2 - t_1).\end{aligned} \tag{4.12}$$

Taking $t_1 = 0$ and noticing that $u^\tau(x,0) = u_0(x)$, we get

$$\int_\Omega (u^\tau(x,t) - u_0)^2\,dx \le M_\varepsilon t \tag{4.13}$$

for every $\tau > 0$ and all $t \in (0,T)$. In turn, by convexity of the function $z \mapsto z^2$,

$$\int_\Omega (u^\tau(x,t))^2\,dx \le 2M_\varepsilon t + 2\int_\Omega u_0^2(x)dx \tag{4.14}$$

for every $\tau > 0$ and all $t \in (0,T)$.

Moreover, by (4.9), for $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$,

$$\begin{aligned}|\nabla u^\tau(x,t)| &\le 2|\nabla u_n(x)| + |\nabla u_{n-1}(x)|,\\ |\nabla^2 u^\tau(x,t)| &\le 2|\nabla^2 u_n(x)| + |\nabla^2 u_{n-1}(x)|,\end{aligned}$$

and by (4.8) and arbitrariness of ℓ we get

$$\int_\Omega |\nabla u^\tau(x,t)|^2 dx \le \frac{3M_\varepsilon}{\sigma}, \quad \int_\Omega |\nabla^2 u^\tau(x,t)|^2 dx \le \frac{3M_\varepsilon}{\sigma}. \tag{4.15}$$

Step 3: Convergence as $\tau \to 0^+$. In the previous step we have shown that $\{u^\tau\}$ is bounded in $L^2((0,T);H^2(\Omega))$ and $\{u_t^\tau\}$ is bounded in $L^2((0,T);L^2(\Omega))$. Since these spaces are reflexive, there exist a subsequence of $\{u^\tau\}$ (not relabeled) and u such that $u^\tau \rightharpoonup u$ in $L^2((0,T);H^2(\Omega))$ and in $H^1((0,T);L^2(\Omega))$. Using the fact that the embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega)$ and $H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact, it follows by the compactness theorem of Aubin and Lions (see e.g. [3]) and a diagonal argument, that, up to a further subsequence, $u^\tau \to u$ in $L^2((0,T);L^2(\Omega))$. In turn, for $\mathcal{L}^1$ a.e. $t \in (0,T)$ we have that $u^\tau(\cdot,t) \to u(\cdot,t)$ in $L^2(\Omega)$. We are now ready to let $\ell \to \infty$, or equivalently, $\tau \to 0^+$ in (4.11), (4.13), (4.15), and deduce the corresponding apriori bounds.

Step 4: u is a weak solution of the Swift–Hohenberg equation.
We let $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n)$, $n = 1, \ldots, \ell$, and define

$$\tilde{u}^\tau(x,t) := u_n(x). \tag{4.16}$$

We claim that $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T);H^2(\Omega))$ as $\tau \to 0^+$.

Given $t \in (0,T]$, we find n such that $t \in (\tau_{n-1}, \tau_n]$ and we notice that

$$\tilde{u}^\tau(x,t) - u^\tau(x,t) = u_n(x) - u^\tau(x,t) = u^\tau(x,\tau_n) - u^\tau(x,t).$$

By (4.12),

$$\begin{aligned}\int_\Omega |\tilde{u}^\tau(x,t) - u^\tau(x,t)|^2 dx &= \int_\Omega |u^\tau(x,\tau_{n-1}) - u^\tau(x,t)|^2 dx \\ &\le M_\varepsilon(t - \tau_{n-1}) \le M_\varepsilon \tau \to 0,\end{aligned} \tag{4.17}$$

as $\tau \to 0^+$. This shows that $\tilde{u}^\tau(\cdot,t) - u^\tau(\cdot,t) \to 0$ in $L^2(\Omega)$ as $\tau \to 0^+$. Moreover, given $\phi \in L^2(\Omega \times (0,T))$, we have

$$\begin{aligned}\int_{\Omega_T} \tilde{u}^\tau(x,t)\phi(x,t)dxdt &= \int_{\Omega_T} (\tilde{u}^\tau(x,t) - u^\tau(x,t))\phi(x,t)dxdt \\ &\quad + \int_{\Omega_T} u^\tau(x,t)\phi(x,t)dxdt.\end{aligned} \tag{4.18}$$

By Hölder's inequality and (4.17), the first integral on the right-hand side of (4.18) converges to zero. Using the fact that $u^\tau \rightharpoonup u$ in $L^2((0,T);H^2(\Omega))$ in the second integral, we deduce that $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T);L^2(\Omega))$.
Moreover, by (4.15) and the fact that $\tilde{u}^\tau(x,t) = u^\tau(x,\tau_n)$ for $t \in (\tau_{n-1},\tau_n]$,

$$\int_\Omega |\nabla \tilde{u}^\tau(x,t)|^2 dx \le \frac{3M_\varepsilon}{\sigma}, \quad \int_\Omega |\nabla^2 \tilde{u}^\tau(x,t)|^2 dx \le \frac{3M_\varepsilon}{\sigma}, \tag{4.19}$$

for all $\tau > 0$ and all $t \in (0,T)$. Hence, up to a subsequence, $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T);H^2(\Omega))$. Furthermore, by (4.5), for every $w \in L^2((0,T);H^2(\Omega))$,

$$\begin{aligned}0 &= \int_\Omega \left(\frac{1}{\varepsilon}W'(\tilde{u}^\tau(x,t))w - 2\varepsilon q \nabla \tilde{u}^\tau(x,t)\cdot \nabla w + 2\varepsilon^3 \nabla^2 \tilde{u}^\tau(x,t)\cdot \nabla^2 w\right) dx \\ &\quad + \int_\Omega u_t^\tau(x,t) w \ dx.\end{aligned}$$

Integrating in time over (t_1,t_2) gives

$$\begin{aligned}0 &= \int_{t_1}^{t_2}\int_\Omega \left(\frac{1}{\varepsilon}W'(\tilde{u}^\tau(x,t))w - 2\varepsilon q \nabla \tilde{u}^\tau(x,t)\cdot \nabla w + 2\varepsilon^3 \nabla^2 \tilde{u}^\tau(x,t)\cdot \nabla^2 w\right) dxdt \\ &\quad + \int_{t_1}^{t_2}\int_\Omega u_t^\tau w \ dxdt.\end{aligned}$$

We note that from (4.8) we have

$$\begin{aligned}\int_\Omega (u_n - u_0)^2 \ dx &= \int_\Omega (u_n - u_{n-1} + u_{n-1} - \ldots + u_1 - u_0)^2 \ dx \\ &\le \ell \sum_{k=1}^{\ell} \int_\Omega (u_k - u_{k-1})^2 \ dx \le \ell\tau M_\varepsilon = TM_\varepsilon\end{aligned}$$

where we have used the convexity of the function $z \mapsto z^2$ and the fact that $\tau = T/\ell$, and this implies

$$\int_\Omega |u_n|^2 \ dx \le C \tag{4.20}$$

for some constant $C > 0$. Moreover, arguing as in (4.7), it follows that

$$\int_\Omega \left(\frac{1}{\varepsilon}W(u_n) - \varepsilon q|\nabla u_n|^2 + \varepsilon^3|\nabla^2 u_\ell|^2\right) dx \le \int_\Omega \left(\frac{1}{\varepsilon}W(u_0) - \varepsilon q|\nabla u_0|^2 + \varepsilon^3|\nabla^2 u_0|^2\right) dx \le \frac{M_\varepsilon}{2}$$

for all $n \in \{0,\ldots,\ell\}$, and in turn, by the interpolation inequality in Lemma 2.1,

$$\int_\Omega |\nabla u_n|^2 dx \le C \quad \text{and} \quad \int_\Omega |\nabla^2 u_n|^2 dx \le C \tag{4.21}$$

for some constant $C > 0$ and for all $n \in \{0, \dots, \ell\}$. Using (4.20), (4.21) and the Sobolev embedding theorem, we have

$$||u_n||_{L^\infty(\Omega)} \le C||u_n||_{H^2(\Omega)} \le C, \tag{4.22}$$

where $C > 0$ changes from side to side. By the Mean Value Theorem, (4.22), and the fact that W is C^2, we deduce

$$\begin{aligned}\int_\Omega (W'(\tilde{u}^\tau(x,t)) - W'(u(x,t)))w dx &\le \max_{-C^\star \le \xi \le C^\star} |W''(\xi)| \int_\Omega |\tilde{u}(x,t) - u(x,t)||w| dx \\ &\le C \int_\Omega |\tilde{u}(x,t) - u(x,t)||w| dx.\end{aligned} \tag{4.23}$$

Letting $\tau \to 0^+$ and using the facts that $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T); H^2(\Omega))$, $u^\tau \rightharpoonup u$ in $H^1((0,T); L^2(\Omega))$ we get

$$\begin{aligned}0 = &\int_{t_1}^{t_2} \int_\Omega \left(\frac{1}{\varepsilon} W'(u(x,t))w - 2\varepsilon q \nabla u(x,t) \cdot \nabla w + 2\varepsilon^3 \nabla^2 u(x,t) \cdot \nabla^2 w \right) dxdt \\ &+ \int_{t_1}^{t_2} \int_\Omega u_t(x,t) w \; dxdt.\end{aligned} \tag{4.24}$$

In particular, let $\{w_k\} \subset H^2(\Omega)$ be dense. Using the fact that $u(\cdot,t) \in H^2(\Omega)$ and $\frac{\partial u}{\partial t} \in L^2(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, by the arbitrariness of t_1 and t_2, we find that

$$\begin{aligned}0 = &\int_\Omega \left(\frac{1}{\varepsilon} W'(u(x,t))w_k - 2\varepsilon q \nabla u(x,t) \cdot \nabla w_k + 2\varepsilon^3 \nabla^2 u(x,t) \cdot \nabla^2 w_k \right) dx \\ &+ \int_\Omega u_t(x,t) w_k \; dx\end{aligned}$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$, where the measure–zero set depends on k. Since $\{w_k\}$ is countable, we can find a set $E \subset (0,T)$ with $\mathcal{L}^1(E) = 0$ such that the previous equality holds for all $t \in (0,T) \setminus E$ and all k.

Since $u(\cdot,t) \in H^2(\Omega)$, then $u(\cdot,t) \in L^\infty(\Omega)$ and, again by Mean Value Theorem and the fact that W is C^2, it follows that $W'(u(\cdot,t)) \in L^2(\Omega)$. This, together with the density of $\{w_k\}$ in $H^2(\Omega)$, and the fact that $u_t \in L^2(\Omega)$ for $t \in (0,T) \setminus E$, implies that

$$\begin{aligned}0 = &\int_\Omega \left(\frac{1}{\varepsilon} W'(u(x,t))w - 2\varepsilon q \nabla u(x,t) \cdot \nabla w + 2\varepsilon^3 \nabla^2 u(x,t) \cdot \nabla^2 w \right) dx \\ &+ \int_\Omega u_t(x,t) w \; dx\end{aligned} \tag{4.25}$$

for all $t \in (0,T) \setminus E$ and all $w \in H^2(\Omega)$. Hence u is a weak solution of equation (4.1) and since Ω has finite measure, taking $w = 1$ leads to

$$0 = \int_\Omega \frac{1}{\varepsilon} W'(u(x,t)) dx + \int_\Omega u_t(x,t) \; dx, \tag{4.26}$$

which implies

$$\int_\Omega u(x,t) dx = \int_\Omega u_0(x) dx + \int_0^t \int_\Omega \frac{1}{\varepsilon} W'(u(x,s)) dxds.$$

□

5 Appendix

5.1 Smooth Linearization Near the Hyperbolic Fixed Point

In the proof of Lemma 2.6 we use the fact that in a sufficiently small neighborhood of the fixed point x_0 of the system (2.29), F admits a C^1 linearization. This variant of the classical Hartman–Grobman Theorem is based on the concept of Q–smoothness of the Jacobian matrix $DF(x_0)$ introduced in [34]. Following [34], we define

$$\gamma(\lambda;m) := \lambda - \sum_{i=1}^{4} m_i r_i, \text{ for } \lambda \in \mathbb{C},\ m_i \in \mathbb{N}_0, \tag{5.1}$$

where r_i are the eigenvalues in (2.56).

Definition 5.1. A matrix A is said to satisfy the *Sternberg condition of order N, $N \geq 2$*, if

$$\gamma(\lambda;m) \neq 0, \text{ for all } \lambda \in \Sigma(A), \text{and for all } m \text{ such that } 2 \leq |m| \leq N, \tag{5.2}$$

where $|m| := \sum m_i$. We will say that A satisfies the *strong Sternberg condition of order N*, if A satisfies (5.2) and

$$\text{Re}\gamma(\lambda;m) \neq 0, \tag{5.3}$$

for all $\lambda \in \Sigma(A)$ and all m such that $|m| = N$.

Definition 5.2. Let $\Sigma^+(A)$ and $\Sigma^-(A)$ be the set of eigenvalues of A having positive and negative real part respectively. A is said to be *strictly hyperbolic* if

$$\Sigma^+(A) \neq \emptyset, \quad \Sigma^-(A) \neq \emptyset.$$

The *spectral spread* of A is defined by

$$\rho^j := \frac{\max\{|\text{Re}\lambda| : \lambda \in \Sigma^j(A)\}}{\min\{|\text{Re}\lambda| : \lambda \in \Sigma^j(A)\}},$$

for $j = \pm$.

Definition 5.3. Let $Q \in \mathbb{N}$ and A be hyperbolic. The *Q–smoothness* of A is the largest integer $K \geq 0$ such that

(i) $Q - K\rho^- \geq 0$, if $\Sigma^+(A) = \emptyset$;

(ii) $Q - K\rho^+ \geq 0$, if $\Sigma^-(A) = \emptyset$;

(iii) there exist $M, N \in \mathbb{N}$ with $Q = M + N$ and $M - K\rho^+ \geq 0$, $N - K\rho^- \geq 0$, when A is strictly hyperbolic.

The following theorem is proved in [34] (Theorem 1, page 4).

Theorem 5.4. *Let X be a finite dimensional Banach space. Let $Q \geq 2$ be an integer. Assume G is of class C^{3Q} on $U \subset X$ with $0 \in U$, where $D^pG(0) = 0$ for $p = 0, 1$. Let A be strictly hyperbolic and assume it satisfies the strong Sternberg condition of order Q. Then*

$$x' = Ax + G(x) \tag{5.4}$$

admits a C^K–linearization, where K is the Q–smoothness of A. In other words, there exists a C^K–diffeomorphism between solutions of (5.4) and solutions of its linear part.

In fact, as remarked in [34], in the case of A strictly hyperbolic it suffices to assume that G is of class $C^{Q+\max(M,N)+K}$. In the remainder, we show that under the assumptions of Lemma 2.6, the matrix $DF(0)$ satisfies the strong Sternberg condition of order $N = 2$ and the 2-smoothness of $DF(0)$ is $K = 1$.

Lemma 5.5. *Consider the ordinary differential equation*

$$x' = F(x), \tag{5.5}$$

where F is a C^4 mapping $\mathbb{R}^4 \to \mathbb{R}^4$ satisfying $F(0) = 0$. Assume the linearization $DF(0)$ has four eigenvalues $\pm\gamma \pm \delta i$, where $\gamma \geq \lambda > 0$. Then, the matrix $DF(0)$ satisfies the strong Sternberg condition of order $N = 2$. Moreover, the Q–smoothness of $DF(0)$ is $K = 1$, and (2.29) admits a C^1–linearization around the hyperbolic fixed point 0.

Proof. We write (5.5) as

$$x' = DF(0)x + G, \tag{5.6}$$

where $G(x) := F(x) - DF(0)x$ is of class C^4, $G(0) = F(0) = 0$, $DG(0) = DF(0) - DF(0) = 0$ and show that (5.2) and (5.3) hold, for $N = 2$. Recalling (5.1), we have

$$\gamma(r_1; m) = (1 - m_1)r_1 - m_2 r_2 - m_3 r_3 - m_4 r_4, \tag{5.7}$$

where $|m| = \sum_{i=1}^{4} m_i = 2$ and $r_1 := \gamma + \delta i, r_2 := \gamma - \delta i, r_3 := -\gamma + \delta i, r_4 := -\gamma - \delta i$ are the eigenvalues of $DF(0)$. Assume, for the sake of contradiction, that $\mathrm{Re}\gamma(r_1; m) = 0$ with $|m| = 2$. Setting the real part of (5.7) to 0 and recalling $|m| = 2$, we have

$$\begin{cases} 1 - m_1 - m_2 + m_3 + m_4 = 0, \\ m_1 + m_2 + m_3 + m_4 = 2, \end{cases} \tag{5.8}$$

Adding the two equations and dividing by two, one has

$$m_3 + m_4 = 1/2, \tag{5.9}$$

a contradiction since m_3 and m_4 are integers. A similar argument for any $\lambda \in \Sigma(Df(0))$ shows that (5.3) and (5.2) hold for the matrix $Df(0)$, and $N = 2$.

It remains to show that the 2-smoothness of $DF(0)$ is $K = 1$. Since $|\mathrm{Re}\lambda| = \gamma$, for all $\lambda \in \Sigma(Df(0))$, then the spectral radius of $Df(0)$ is $\rho^i = 1$, for $i = \pm$. Being $Df(0)$ is strictly hyperbolic, we are in case (iii) of Definition 5.3 and $Q = 2$ implies $M = N = 1$. In turn, the largest integer K that satisfies

$$\begin{cases} M - K\rho^+ = 1 - K \geq 0, \\ N - K\rho^- = 1 - K \geq 0, \end{cases} \tag{5.10}$$

is $K = 1$, which is then the 2–smoothness of $Df(0)$. We now apply Theorem 5.4 with $Q = 2$ and $A = DF(0)$ to conclude that (5.6) admits a C^1–linearization. □

6 Acknowledgments

The authors would like to thank Irene Fonseca and Giovanni Leoni for the fruitful discussions and preliminary readings of the manuscript. They would also like to thank the Center for Nonlinear Analysis at Carnegie Mellon University, where part of the research was developed. The results contained in this manuscript fulfill part of the Ph.D. requirements of the second author, whose research was partially supported by the awards DMS 0905778 and DMS 1412095.

References

[1] N. D. Alikakos, L. Bronsard, and G. Fusco. Slow motion in the gradient theory of phase transitions via energy and spectrum. *Calc. Var. Partial Differential Equations*, 6(1):39–66, 1998.

[2] N. D. Alikakos and G. Fusco. Slow dynamics for the Cahn-Hilliard equation in higher space dimensions. I. Spectral estimates. *Comm. Partial Differential Equations*, 19(9-10):1397–1447, 1994.

[3] J.-P. Aubin. Un théorème de compacité. *C. R. Acad. Sci. Paris*, 256:5042–5044, 1963.

[4] G. Bellettini, A.-H. Nayam, and M. Novaga. Γ-type estimates for the one-dimensional Allen-Cahn's action. *Asymptot. Anal.*, 94(1-2):161–185, 2015.

[5] L. Bronsard and R. V. Kohn. On the slowness of phase boundary motion in one space dimension. *Comm. Pure Appl. Math.*, 43(8):983–997, 1990.

[6] L. Bronsard and R. V. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Differential Equations*, 90(2):211–237, 1991.

[7] J. Carr and R. L. Pego. Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$. *Comm. Pure Appl. Math.*, 42(5):523–576, 1989.

[8] J. Carr and R. L. Pego. Very slow phase separation in one dimension. In *PDEs and continuum models of phase transitions (Nice, 1988)*, volume 344 of *Lecture Notes in Phys.*, pages 216–226. Springer, Berlin, 1989.

[9] X. Chen. Generation and propagation of interfaces for reaction-diffusion equations. *J. Differential Equations*, 96(1):116–141, 1992.

[10] X.-Y. Chen. Dynamics of interfaces in reaction diffusion systems. *Hiroshima Math. J.*, 21(1):47–83, 1991.

[11] M. Chermisi, G. Dal Maso, I. Fonseca, and G. Leoni. Singular perturbation models in phase transitions for second-order materials. *Indiana Univ. Math. J.*, 60(2):367–409, 2011.

[12] M. Cicalese, E. N. Spadaro, and C. I. Zeppieri. Asymptotic analysis of a second-order singular perturbation model for phase transitions. *Calc. Var. Partial Differential Equations*, 41(1-2):127–150, 2011.

[13] M. C. Cross and P. C. C. Hohenberg. Pattern formation outside of equilibrium. *Rev. Mod. Phys.*, 65:851–1112, Jul 1993.

[14] P. de Mottoni and M. Schatzman. Development of interfaces in $\mathbf{R}^N$. *Proc. Roy. Soc. Edinburgh Sect. A*, 116(3-4):207–220, 1990.

[15] S.-I. Ei and E. Yanagida. Slow dynamics of interfaces in the Allen-Cahn equation on a strip-like domain. *SIAM J. Math. Anal.*, 29(3):555–595 (electronic), 1998.

[16] P. C. Fife and L. Hsiao. The generation and propagation of internal layers. *Nonlinear Anal.*, 12(1):19–41, 1988.

[17] I. Fonseca, N. Fusco, G. Leoni, and M. Morini. Motion of elastic thin films by anisotropic surface diffusion with curvature regularization. *Arch. Ration. Mech. Anal.*, 205(2):425–466, 2012.

[18] I. Fonseca and C. Mantegazza. Second order singular perturbation models for phase transitions. *SIAM J. Math. Anal.*, 31(5):1121–1143 (electronic), 2000.

[19] G. Fusco. A geometric approach to the dynamics of $u_t = \epsilon^2 u_{xx} + f(u)$ for small ϵ. In *Problems involving change of type (Stuttgart, 1988)*, volume 359 of *Lecture Notes in Phys.*, pages 53–73. Springer, Berlin, 1990.

[20] G. Fusco and J. K. Hale. Slow-motion manifolds, dormant instability, and singular perturbations. *J. Dynam. Differential Equations*, 1(1):75–94, 1989.

[21] A. Giorgini. On the Swift-Hohenberg equation with slow and fast dynamics: well-posedness and long-time behavior. *Commun. Pure Appl. Anal.*, 15(1):219–241, 2016.

[22] C. P. Grant. Slow motion in one-dimensional Cahn-Morral systems. *SIAM J. Math. Anal.*, 26(1):21–34, 1995.

[23] P. C. Hohenberg and J. B. Swift. Effects of additive noise at the onset of Rayleigh-Bénard convection. *Phys. Rev. A*, 46:4773–4785, Oct 1992.

[24] G. W. Hunt, M. A. Peletier, A. R. Champneys, P. D. Woods, M. A. Wadee, C. J. Budd, and G. J. Lord. Cellular buckling in long structures. *Nonlinear Dynam.*, 21(1):3–29, 2000. The theme of solitary waves and localization phenomena in elastic structures.

[25] W. D. Kalies, R. C. A. M. VanderVorst, and T. Wanner. Slow motion in higher-order systems and Γ-convergence in one space dimension. *Nonlinear Anal.*, 44(1, Ser. A: Theory Methods):33–57, 2001.

[26] M. Kowalczyk. Exponentially slow dynamics and interfaces intersecting the boundary. *J. Differential Equations*, 138(1):55–85, 1997.

[27] M. Kuwamura. The phase dynamics method with applications to the Swift-Hohenberg equation. *J. Dynam. Differential Equations*, 6(1):185–225, 1994.

[28] J. Lega, J. V. Moloney, and A. C. Newell. Swift–Hohenberg Equation for Lasers. *Phys. Rev. Lett.*, 73:2978–2981, Nov 1994.

[29] G. Leoni and R. Murray. Second-Order Γ-limit for the Cahn–Hilliard Functional. *Arch. Ration. Mech. Anal.*, 219(3):1383–1451, 2016.

[30] R. Murray and M. Rinaldi. Slow motion for the nonlocal Allen–Cahn in n-dimensions. *Submitted*, 2015.

[31] F. Otto and M. G. Reznikoff. Slow motion of gradient flows. *J. Differential Equations*, 237(2):372–420, 2007.

[32] Y. Pomeau and P. Manneville. Wavelength selection in cellular flows. *Physics Letters A*, 75(4):296 – 298, 1980.

[33] E. Sandier and S. Serfaty. Gamma-convergence of gradient flows with applications to Ginzburg-Landau. *Comm. Pure Appl. Math.*, 57(12):1627–1672, 2004.

[34] G. R. Sell. Smooth linearization near a fixed point. *Amer. J. Math.*, 107(5):1035–1091, 1985.

[35] H. M. Soner. Ginzburg-Landau equation and motion by mean curvature. I. Convergence. *J. Geom. Anal.*, 7(3):437–475, 1997.

[36] J. Swift and P. C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Phys. Rev. A*, 15:319–328, Jan 1977.

[37] P. B. Umbanhowar, F. Melo, and H. L. Swinney. Localized excitations in a vertically vibrated granular layer. *Nature*, 382(6594):793–796, 08 1996.