A chromaticity-brightness model for color images denoising in a Meyer's "u + v" framework

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Abstract

A variational model for imaging segmentation and denoising color images is proposed. The model combines Meyer's "u+v" decomposition with a chromaticity-brightness framework and is expressed by a minimization of energy integral functionals depending on a small parameter $\varepsilon > 0$. The asymptotic behavior as $\varepsilon \to 0^+$ is characterized, and convergence of infima, almost minimizers, and energies are established. In particular, an integral representation of the lower semicontinuous envelope, with respect to the L^1-norm, of functionals with linear growth and defined for maps taking values on a certain compact manifold is provided. This study escapes the realm of previous results since the underlying manifold has boundary, and the integrand and its recession function fail to satisfy hypotheses commonly assumed in the literature. The main tools are Γ-convergence and relaxation techniques.

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Key words: imaging denoising, color images, Meyer's G-norm, chromaticity-brightness decomposition, Γ-convergence, relaxation, BV functions, manifold constraints.

Contents

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1 Introduction and Main Results

An important problem in image processing is the restoration, or denoising, of a given "noisy" image. Deterioration of images may be caused by several factors, some of which occur in the process of acquisition (e.g., blur may derive from an incorrect lens adjustment or due to motion) or transmission. Variational PDE methods have proven to be successful in the restoration process, where the desired clean and sharp image is obtained as a minimizer of a certain energy functional. The energy functionals proposed in the literature share the common feature of taking into account a balance between a certain distance to the given noisy image, the so-called fidelity term, and a filter acting as a regularization of the image.

In the seminal work by Tikhonov and Arsenin [39], the fidelity term is expressed in terms of the L^2-distance to the noisy image, while the regularization term is given by the L^2-norm of the gradient. This model suffers from an important drawback, as an over smoothing is observed, and edges in images are not preserved. It turns out that the L^2-norm for the gradient allows the removal of noise, but penalizes too much the gradient near and on the edges of an image. The same observation applies to any L^p-norm, $p > 1$, and this suggests using instead the L^1-norm, as first noticed by Rudin, Osher, and Fatemi [37]. Precisely, representing by $\Omega \subset \mathbb{R}^2$ the image domain and by $u_0 : \Omega \to \mathbb{R}$ the observed noisy version of the true unknown image u, Rudin-Osher-Fatemi's model (the ROF model) aims at finding

$$\inf_{\substack{u \in W^{1,1}(\Omega) \\ u_0 - u \in L^2(\Omega)}} \left\{ \int_\Omega |\nabla u|\, \mathrm{d}x + \lambda \int_\Omega |u_0 - u|^2\, \mathrm{d}x \right\}$$

or, equivalently,

$$\min_{\substack{u \in BV(\Omega) \\ u_0 - u \in L^2(\Omega)}} \left\{ |Du|(\Omega) + \lambda \int_\Omega |u_0 - u|^2\, \mathrm{d}x \right\},$$

where $\lambda > 0$ is a tuning parameter and $BV(\Omega)$ is the space of functions of bounded variation in Ω. The ROF model, also known as the total variation model (TV model) since the filter used is the total variation of the image, searches functions u that best fit the data, measured in terms of the L^2 fidelity term, and whose gradient (total variation) is low so that noise is removed. It yields a decomposition of the type

$$u_0 = u + v, \tag{1.1}$$

where u is well-structured, aimed at modeling homogeneous regions, while v encodes noise or textures.

The ROF model removes noise while preserving edges, and it was extended to higher-order and vectorial settings to treat color images (see, for instance, [7, 16] for an overview). However, it leads to undesirable phenomena like blurring, stair-casing effect (see [7, 16]), and it may also fail to provide a good decomposition (1.1) of the given corrupted image as, for example, some pure geometric images (represented by characteristic functions) are treated as noise or textures (see [33]). The reasons pointed out in literature relate to both the fidelity term and the regularization term. In this paper, we will focus mainly on the former.

Meyer [33] showed that oscillating images are often treated as texture or noise. He proved that replacing the L^2-norm in the fidelity term by a certain G-norm leads to better decompositions. Accordingly, he suggested the model

$$\inf_{\substack{u \in BV(\Omega) \\ u - u_0 \in G(\Omega)}} \left\{ |Du|(\Omega) + \lambda \|u - u_0\|_{G(\Omega)} \right\}, \tag{1.2}$$

and we refer to Subsection 2.2 for a detailed description and main properties of the space $G(\Omega)$ established in [6, 33]. Meyer's model has motivated several contributions aiming at overcoming some numerical difficulties

2

posed by the structure of the G-norm (see, for instance, [8, 36, 40]). Finally, we mention that the infimum in (1.2) is a minimum, but the uniqueness of minimizers is still an open problem.

When dealing with color images, the general idea of the chromaticity-brightness approach is as follows: as before, $\Omega \subset \mathbb{R}^2$ denotes the image domain, while $u_0 : \Omega \to (\mathbb{R}_0^+)^3$ is the observed deteriorated image, represented in the RGB (red, green, blue) system and assumed to belong to $L^\infty(\Omega; \mathbb{R}^3)$. The brightness component, $(u_0)_b$, of u_0 measures the intensity of u_0, is defined by

$$(u_0)_b := |u_0|,$$

and is assumed to be different from zero a.e. in Ω. The chromaticity component, $(u_0)_c$, of u_0 is given by

$$(u_0)_c := \frac{u_0}{|u_0|} = \frac{u_0}{(u_0)_b},$$

which is well defined a.e. in Ω and takes values in S^2, the unit sphere in $\mathbb{R}^3$. It stores the color information of u_0. The function u_0 and its components are related by the identity $u_0 = (u_0)_b(u_0)_c$. The core of the chromaticity-brightness models is to restore these two components independently. Representing by u_b and u_c the restored brightness and chromaticity components, respectively, the restored imaged is given by $u := u_b u_c$.

Because $(u_0)_b$ behaves as a gray-scaled image, to restore this component we may use, for instance, one of the models previously mentioned. To restore the chromaticity component $(u_0)_c$, we adopt Kang and March's model [32] using weighted harmonic maps. To be precise, we consider the problem

$$\min_{u_c \in W^{1,2}(\Omega; S^2)} \left\{ \int_\Omega g(|\nabla u_b^\sigma|) |\nabla u_c|^2 \, \mathrm{d}x + \lambda \int_\Omega |u_c - (u_0)_c|^2 \, \mathrm{d}x \right\}, \tag{1.3}$$

where λ is a tuning parameter, u_0 is extended by zero outside Ω,

$$u_b^\sigma := G_\sigma * (u_0)_b, \quad G_\sigma(x) := \frac{A}{\sigma} e^{-\frac{|x|^2}{4\sigma}}, \ \ A > 0, \ \sigma > 0,$$

is a smooth regularization of $(u_0)_b$, and $g : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a non-increasing positive function satisfying $g(0) = 1$ and $\lim_{t \to +\infty} g(t) = 0$ (see also [19], and the references therein, for this choice of functions g). Examples of such functions g are

$$g(t) := \frac{1}{1 + \left(\frac{t}{a}\right)^2}, \quad g(t) := e^{-\left(\frac{t}{a}\right)^2}, \ \ a > 0.$$

The model proposed by Kang and March in [32] is aimed at image colorization. It is assumed that the brightness data is known everywhere in Ω, while the color data is only available in a subset D of Ω. Thus, in [32], the second integral (fidelity term) in (1.3) is taken over D (here we assume $D = \Omega$), and it forces the function u_c to be close to the chromaticity data in D. The first integral acts as a regularization functional and allows for sharp transitions of u_c across the edges of $(u_0)_b$ since the value of $g(|\nabla u_b^\sigma|)$ is close to zero in the regions where u_b^σ varies fast. To deal with the nonconvex S^2 constraint, in [32] the authors introduce a penalized version of the variational problem above, and convergence to the original variational problem as the penalty parameter tends to infinity is established. Numerical simulations are also performed.

A natural question that is not considered in [32] is the asymptotic behavior of the variational model (1.3) as σ tends to zero. Since $u_b^\sigma \in C^\infty(\overline{\Omega})$ for every $\sigma > 0$, it represents a smooth version of the brightness component and, therefore, some relevant information may not be encoded in the model (1.3). Furthermore, it avoids a compactness issue since, for fixed $\sigma > 0$, $\inf_\Omega g(|\nabla u_b^\sigma|) > 0$ and, therefore, every minimizing sequence for (1.3) is relatively compact in $W^{1,2}(\Omega; \mathbb{R}^3)$.

In this paper, we deal with the denoising problem for color images by considering a model that combines the strengths of Meyer's decomposition, adapted to color images, with the strengths of chromaticity-brightness models, which are preferred in literature as they are considered as reducing shadowing and providing better simulation results (see, for example, [17, 32, 38]). To be precise, we adopt Kang and March's brightness-chromaticity approach, but we avoid the smoothing step. A vectorial version of Meyer's model is

$$F_0(u) := |Du|(\Omega) + \lambda_0 \|u - u_0\|_{G(\Omega; \mathbb{R}^3)}, \quad u \in BV(\Omega; \mathbb{R}^3), \ u - u_0 \in G(\Omega; \mathbb{R}^3), \ \lambda_0 \in \mathbb{R}^+.$$

3

We treat the brightness component and the chromaticity component of u_0 separately. For the former, we use Meyer's model (for gray-scaled images), which leads to the functional

$$F_1(u_b) := |Du_b|(\Omega) + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)}, \quad u_b \in BV(\Omega), \ u_b - (u_0)_b \in G(\Omega), \ \lambda_b \in \mathbb{R}^+.$$

For the latter, we use Kang and March's model replacing u_b^σ by u_b assuming for the moment that $u_b \in W^{1,1}(\Omega)$, and thus we introduce the functional

$$F_2(u_c) := \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \, dx + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx, \quad u_c \in W^{1,2}(\Omega; S^2), \ \lambda_c \in \mathbb{R}^+.$$

To couple these two approaches, we set $u := u_b u_c$ and consider the minimization problem

$$\inf_{\substack{u_b \in W^{1,1}(\Omega), u_c \in W^{1,2}(\Omega;S^2), \\ u_b - (u_0)_b \in G(\Omega), u_0 - u_c u_b \in G(\Omega;\mathbb{R}^3)}} \left\{ F_0(u_b u_c) + F_1(u_b) + F_2(u_c) \right\};$$

that is,

$$\inf_{\substack{u_b \in W^{1,1}(\Omega), u_c \in W^{1,2}(\Omega;S^2), \\ u_b - (u_0)_b \in G(\Omega), u_0 - u_c u_b \in G(\Omega;\mathbb{R}^3)}} \left\{ \int_\Omega |\nabla(u_c u_b)| \, dx + \int_\Omega |\nabla u_b| \, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \, dx \right.$$

$$\left. + \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx \right\}. \quad (1.4)$$

As stated, problem (1.4) presents a lack of uniform estimates regarding the gradient of u_c. To overcome this issue, we will add a constraint for the brightness component $(u_0)_b$ and its test functions u_b, namely

$$(u_0)_b, \ u_b \in [\alpha, \beta] \text{ a.e. in } \Omega, \text{ for some } 0 < \alpha \leqslant \beta. \quad (1.5)$$

Under hypothesis (1.5), we have

$$\alpha \int_\Omega |\nabla u_c| \, dx \leqslant \int_\Omega |u_b \nabla u_c + u_c \otimes \nabla u_b| \, dx + \int_\Omega |u_c \otimes \nabla u_b| \, dx \leqslant \int_\Omega |\nabla(u_c u_b)| \, dx + \int_\Omega |\nabla u_b| \, dx.$$

Consequently, if

$$\{(u_b^n, u_c^n)\}_{n \in \mathbb{N}} \subset \left\{ (u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,2}(\Omega; S^2) : \ u_b - (u_0)_b \in G(\Omega), u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3) \right\}$$

is a infimizing sequence for (1.4), then, using the properties of the G- and BV-spaces, up to a (not relabeled) subsequence, we conclude that there exist $\bar{u}_b \in BV(\Omega; [\alpha, \beta])$ and $\bar{u}_c \in BV(\Omega; S^2)$ such that

$$u_b^n \overset{\star}{\rightharpoonup} \bar{u}_b \text{ weakly-}\star \text{ in } BV(\Omega), \quad u_c^n \overset{\star}{\rightharpoonup} \bar{u}_c \text{ weakly-}\star \text{ in } BV(\Omega;\mathbb{R}^3), \quad \text{as } n \to \infty, \quad (1.6)$$

$$\bar{u}_b - (u_0)_b \in G(\Omega), \quad \bar{u}_b \bar{u}_c - u_0 \in G(\Omega;\mathbb{R}^3),$$

$$\lim_{n \to +\infty} F^{fid}(u_b^n, u_c^n) = F^{fid}(\bar{u}_b, \bar{u}_c),$$

where

$$F^{fid}(u_b, u_c) := \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)} + \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx \quad (1.7)$$

is the sum of the three fidelity terms in (1.4). If it turned out that $\bar{u}_b \in W^{1,1}(\Omega; [\alpha, \beta])$ and $\bar{u}_c \in W^{1,1}(\Omega; S^2)$, then minimizers for (1.4) would exist provided that the functional given by the first three terms in (1.4) (the regularization terms) was sequential lower semicontinuous with respect to the convergences in (1.6). This sequential lower semicontinuity is intrinsically related to the problem of finding an integral representation for

$$\inf \left\{ \liminf_{n \to \infty} \int_\Omega h(u_b^n, u_c^n, \nabla u_b^n, \nabla u_c^n) \, dx : \ u_b^n \in W^{1,1}(\Omega; [\alpha, \beta]), u_b^n \rightharpoonup u_b \text{ weakly in } W^{1,1}(\Omega), \right.$$

$$\left. u_c^n \in W^{1,2}(\Omega; S^2), u_c^n \rightharpoonup u_c \text{ weakly in } W^{1,1}(\Omega;\mathbb{R}^3) \right\}, \quad (1.8)$$

with

$$h(r, s, \xi, \eta) := |\xi| + g(|\xi|)|\eta|^2 + |s \otimes \xi + r\eta|.$$

In general,

$$(\xi, \eta) \mapsto h(r, s, \xi, \eta) = |\xi| + g(|\xi|)|\eta|^2 + |s \otimes \xi + r\eta|$$

is not quasiconvex. Moreover, for $(r, s) \in [\alpha, \beta] \times S^2$, h satisfies the non-standard growth conditions

$$\frac{1}{C}(|\xi| + |\eta|) \leqslant h(r, s, \xi, \eta) \leqslant C(1 + |\xi| + |\eta|^2),$$

which leads us to a well-known, but poorly understood, gap problem (see [24, 25, 34] concerning the unconstrained setting). We also observe that the admissible sequences in (1.8) should satisfy in addition the restrictions $u_b^n - (u_0)_b \in G(\Omega)$ and $u_b^n u_c^n - u_0 \in G(\Omega; \mathbb{R}^3)$ or, equivalently (see Proposition 2.2),

$$\int_\Omega (u_b^n - (u_0)_b) \, dx = 0 \quad \text{and} \quad \int_\Omega (u_b^n u_c^n - u_0) \, dx = 0. \tag{1.9}$$

It turns out to be a challenging task to construct a recovery sequence that simultaneously satisfies the manifold constraint and the average restrictions.

In view of these considerations, to avoid the gap and to penalize deviations from average zero in (1.9), as a first approach to problem (1.4), we study the asymptotic behavior, as $\varepsilon \to 0^+$, of the problems

$$\inf_{(u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)} \left\{ F^{reg}(u_b, u_c) + F_\varepsilon^{fid}(u_b, u_c) \right\}, \tag{1.10}$$

where

$$F^{reg}(u_b, u_c) := \int_\Omega |\nabla u_b| \, dx + \int_\Omega g(|\nabla u_b|)|\nabla u_c| \, dx + \int_\Omega |\nabla(u_c u_b)| \, dx \tag{1.11}$$

and

$$\begin{aligned}
F_\varepsilon^{fid}(u_b, u_c) := & \lambda_v \left\| u_b u_c - u_0 - \fint_\Omega (u_b u_c - u_0) \, dx \right\|_{G(\Omega; \mathbb{R}^3)} + \frac{1}{\varepsilon} \left| \int_\Omega (u_b u_c - u_0) \, dx \right| \\
& + \lambda_b \left\| u_b - (u_0)_b - \fint_\Omega (u_b - (u_0)_b) \, dx \right\|_{G(\Omega)} + \frac{1}{\varepsilon} \left| \int_\Omega (u_b - (u_0)_b) \, dx \right| \\
& + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, dx.
\end{aligned} \tag{1.12}$$

The integrand involved in (1.11) satisfies standard growth conditions (see (1.14) below), and it remains relevant in terms of applications to imaging. Note that the term

$$\int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \, dx$$

in (1.4) can be viewed as a weighted version of Tikhonov and Asenin [39]'s regularization term mentioned at the beginning of this introduction, while in (1.10) we use instead a weighted version of ROF's regularization term (se also [15, 19]), namely

$$\int_\Omega g(|\nabla u_b|)|\nabla u_c| \, dx.$$

For small $\varepsilon > 0$, the functional F_ε^{fid} is a penalized version of the functional F^{fid} in (1.7) that, by means of the factor $\frac{1}{\varepsilon}$, penalizes sequences $\{(u_b^n, u_c^n)\}_{n \in \mathbb{N}}$ whose averages $\int_\Omega (u_b^n - (u_0)_b) \, dx$ and $\int_\Omega (u_b^n u_c^n - u_0) \, dx$ are far from zero. This penalization allows us to incorporate the G-norm and the G-restrictions in our model. As

we will see, in the limit as $\varepsilon \to 0^+$, we will recover the functional F^{fid} and limit pairs (u_b, u_c) will satisfy $u_b - (u_0)_b \in G(\Omega)$ and $u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3)$.

Before we state our main theorem, we introduce some notation. Regarding functions of bounded variation, we adopt the notations in [4], and we refer to Subsection 2.3 for more details. Let $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 2} \to [0, +\infty)$ be defined, for $(r, s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 2}$, by

$$f(r, s, \xi, \eta) := |\xi| + g(|\xi|)|\eta| + |r\eta + s \otimes \xi|, \tag{1.13}$$

where, as above, $g : [0, +\infty) \to (0, 1]$ is a non-increasing, Lipschitz continuous function satisfying $g(0) = 1$ and $\lim_{t \to +\infty} g(t) = 0$. Notice that

$$F^{reg}(u_b, u_c) = \int_\Omega f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x)) \, \mathrm{d}x.$$

For $s \in \overline{B(0,1)}$, we have $\frac{1}{2}|\xi| + \frac{|r|}{2}|\eta| \leqslant \frac{1}{2}|\xi| + \frac{1}{2}(|r\eta + s \otimes \xi| + |\xi|) \leqslant f(r, s, \xi, \eta)$, and so

$$\frac{1}{2}|\xi| + \frac{|r|}{2}|\eta| \leqslant f(r, s, \xi, \eta) \leqslant 2|\xi| + (1 + |r|)|\eta|, \tag{1.14}$$

where we used the fact that $g(\cdot) \leqslant 1$.

For $r \in \mathbb{R}$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$, where $T_s(S^2)$ is the tangential space to S^2 at s, we denote by $\mathcal{Q}_T f$ the tangential quasiconvex envelope of f; to be precise (see [18]),

$$\mathcal{Q}_T f(r, s, \xi, \eta) := \inf \left\{ \int_Q f(r, s, \xi + \nabla \varphi(y), \eta + \nabla \psi(y)) \, \mathrm{d}y : \ \varphi \in W_0^{1,\infty}(Q), \ \psi \in W_0^{1,\infty}(Q; T_s(S^2)) \right\}. \tag{1.15}$$

The recession function, f^∞, of f is the function defined, for $(r, s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 2}$, by

$$f^\infty(r, s, \xi, \eta) := \limsup_{t \to +\infty} \frac{f(r, s, t\xi, t\eta)}{t} = \limsup_{t \to +\infty} \left(|\xi| + g(t|\xi|)|\eta| + |r\eta + s \otimes \xi| \right)$$
$$= |\xi| + \chi_{\{0\}}(|\xi|)|\eta| + |r\eta + s \otimes \xi|, \tag{1.16}$$

where $\chi(t) := 1$ if $t = 0$ and $\chi(t) := 0$ if $t \in \mathbb{R} \backslash \{0\}$, because $g(0) = 1$ and $\lim_{t \to +\infty} g(t) = 0$. Note that

$$f^\infty(r, s, \xi, \eta) \leqslant (3 + \beta)|(\xi, \eta)| \tag{1.17}$$

for $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$. The recession function, $(Q_T f)^\infty$, of $Q_T f$ is the function defined, for $r \in \mathbb{R}$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$, by

$$(Q_T f)^\infty(r, s, \xi, \eta) := \limsup_{t \to +\infty} \frac{Q_T f(r, s, t\xi, t\eta)}{t}.$$

For $a, b \in [\alpha, \beta] \times S^2$ and $\nu \in S^1$, we set

$$K(a, b, \nu) := \inf \left\{ \int_{Q_\nu} f^\infty(\varphi(y), \psi(y), \nabla \varphi(y), \nabla \psi(y)) \, \mathrm{d}y : \ \vartheta = (\varphi, \psi) \in \mathcal{P}(a, b, \nu) \right\}$$
$$= \inf \left\{ \int_{Q_\nu} \left(|\nabla \varphi(y)| + |\nabla(\varphi \psi)(y)| + \chi_{\{0\}}(|\nabla \varphi|)|\nabla \psi| \right) \mathrm{d}y : \ \vartheta = (\varphi, \psi) \in \mathcal{P}(a, b, \nu) \right\}, \tag{1.18}$$

where Q_ν is the unit cube in $\mathbb{R}^2$ centered at the origin and with two faces orthogonal to ν, and

$$\mathcal{P}(a, b, \nu) := \left\{ \vartheta = (\varphi, \psi) \in W^{1,1}(Q_\nu; [\alpha, \beta] \times S^2) : \ \vartheta \text{ is 1-periodic in the orthogonal direction to } \nu, \right.$$
$$\left. \vartheta(y) = a \text{ if } y \cdot \nu = -\frac{1}{2}, \ \vartheta(y) = b \text{ if } y \cdot \nu = \frac{1}{2} \right\}. \tag{1.19}$$

Finally, we define the functional $F^{reg,sc^-} : BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2) \to \mathbb{R}$ as

$$F^{reg,sc^-}(u_b, u_c) := \int_\Omega \mathcal{Q}_T f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x)) \, dx$$
$$+ \int_{S_{(u_b,u_c)}} K\big((u_b, u_c)^+(x), (u_b, u_c)^-(x), \nu_{(u_b,u_c)}(x)\big) \, d\mathcal{H}^1(x)$$
$$+ \int_\Omega (\mathcal{Q}_T f)^\infty(\tilde{u}_b(x), \tilde{u}_c(x), W_b^c(x), W_c^c(x)) \, d|D^c(u_b, u_c)|(x), \quad (1.20)$$

where $\tilde{u}_b(x)$ and $\tilde{u}_c(x)$ are the approximate limits of u_b and u_c at x, respectively, and where W^c is the Radon-Nikodym derivative of $D^c(u_b, u_c)$ with respect to its total variation, W_b^c is the first row of W^c, and W_c^c is the 3×2 matrix obtained from W^c by erasing its first row. Our main result is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with Lipschitz boundary $\partial\Omega$, and let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ be two arbitrary sequences of positive numbers converging to zero. Let F^{reg}, $F^{fid}_{\varepsilon_n}$, F^{reg,sc^-}, and F^{fid} be the functionals introduced in (1.11), (1.12), (1.20), and (1.7), respectively, and let X be the set*

$$X := \big\{(u_b, u_c) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2) : u_b - (u_0)_b \in G(\Omega), \, u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3)\big\}. \quad (1.21)$$

Then,

$$\min_{(u_b,u_c)\in X} \big(F^{reg,sc^-}(u_b, u_c) + F^{fid}(u_b, u_c)\big) = \lim_{n\to\infty} \inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)} \big(F^{reg}(u_b, u_c) + F^{fid}_{\varepsilon_n}(u_b, u_c)\big).$$

Moreover, if for each $n \in \mathbb{N}$, $(u_b^n, u_c^n) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)$ is a δ_n-minimizer of the functional $\big(F^{reg} + F^{fid}_{\varepsilon_n}\big)$ in $W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)$, that is,

$$F^{reg}(u_b^n, u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n, u_c^n) \leqslant \inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)} \big(F^{reg}(u_b, u_c) + F^{fid}_{\varepsilon_n}(u_b, u_c)\big) + \delta_n,$$

then $\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}}$ is sequentially, relatively compact with respect to the weak-$\star$ convergence in $BV(\Omega) \times BV(\Omega; \mathbb{R}^3)$; and if (u_b, u_c) is a cluster point of $\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}}$, then $(u_b, u_c) \in X$ is a minimizer of $(F^{reg,sc^-} + F^{fid})$ in X and

$$F^{reg,sc^-}(u_b, u_c) + F^{fid}(u_b, u_c) = \lim_{n\to\infty} \big(F^{reg}(u_b^n, u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n, u_c^n)\big).$$

The proof of Theorem 1.1 relies on the following relaxation result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with Lipschitz boundary, let F^{reg} be given by (1.11), and consider the functional $F : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$ defined by*

$$F(u_b, u_c) := \begin{cases} F^{reg}(u_b, u_c) & \text{if } (u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2), \\ +\infty & \text{otherwise,} \end{cases}$$

for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$. Then the lower semicontinuous envelope of F, $\mathcal{F} : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$, defined by

$$\mathcal{F}(u_b, u_c) := \inf \Big\{ \liminf_{n\to+\infty} F(u_b^n, u_c^n) : \, n \in \mathbb{N}, \, (u_b^n, u_c^n) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3),$$
$$u_b^n \to u_b \text{ in } L^1(\Omega), \, u_c^n \to u_c \text{ in } L^1(\Omega; \mathbb{R}^3)\Big\},$$

has the integral representation

$$\mathcal{F}(u_b, u_c) = \begin{cases} F^{reg,sc^-}(u_b, u_c) & \text{if } (u_b, u_c) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2), \\ +\infty & \text{otherwise,} \end{cases} \quad (1.22)$$

for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$, where F^{reg,sc^-} is given by (1.20).

7

The relaxation result above falls within the general context of studying lower semicontinuity and/or finding integral representations for the lower semicontinuous envelope of functionals of the type

$$\int_\Omega f(x, u(x), \nabla u(x))\, \mathrm{d}x, \quad u \in W^{1,p}(\Omega; \mathcal{M}),$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded set, $p \in [1, +\infty)$, and $\mathcal{M} \subset \mathbb{R}^d$ is a (sufficiently) smooth, m-dimensional manifold. There is a vast literature in this framework (see, for instance, [2, 9, 10, 13, 14, 18, 23, 35, 41]), motivated, for example, by the study of equilibria for liquid crystals and magnetostrictive materials, where the class of admissible fields is constrained to take values on a certain manifold $\mathcal{M}$ (commonly, $\mathcal{M} = S^{d-1}$, the unit sphere in $\mathbb{R}^d$). As in [2, 9, 35], the key ingredients in the proof of Theorem 1.2 are the density of smooth functions in $W^{1,1}(\Omega; \mathcal{M})$ [11, 12, 29] and a projection technique introduced in [2, 30, 31]. However, new arguments are required as three main features of our problem prevent us from using immediately the relaxation results concerning the constraint case in the BV setting [2, 9, 35]: unlike [2, 35], our starting point cannot be a tangential quasiconvex function as the energy density considered here (see (1.13)) fail always to satisfy such condition (see Remark 4.2); and unlike the general setting in the literature, (i) our manifold, $\mathcal{M} = [\alpha, \beta] \times S^2$, has boundary, (ii) the recession function f^∞ in our case (see (1.16)) does not satisfy a hypothesis of the type $|f(r, s, \xi, \eta) - f^\infty(r, s, \xi, \eta)| \leqslant C(1 + |(\xi, \eta)|^{1-m})$ for some $C > 0$ and $m \in (0, 1)$ (for a.e. (r, s) and for all (ξ, η)). We anticipate that our arguments may be used to treat more general manifolds with boundary and more general integrands.

This paper is organized as follows. In Section 2, we collect the notation, we recall properties of the space G introduced by Meyer [33], and we also recall properties of functions of bounded variation and sets of finite perimeter. We also make some considerations on quasiconvexity. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we establish Theorem 1.2.

2 Notation and Preliminaries

2.1 Notation

Let $N, d \in \mathbb{N}$. If $x, y \in \mathbb{R}^N$, then $x \cdot y$ stands for the Euclidean inner product of x and y, and $|x| := \sqrt{x \cdot x}$ for the Euclidean norm of x. The space of $d \times N$-dimensional matrices is identified with $\mathbb{R}^{dN}$, and we write $\mathbb{R}^{d \times N}$. We define $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, $Q := (-\frac{1}{2}, \frac{1}{2})^N$, $Q(x, \delta) := x + \delta Q$, and $B(x, \delta) := \{y \in \mathbb{R}^N : |y - x| < \delta\}$, where $\delta > 0$. Given $\nu \in S^{N-1}$ and a rotation R_ν such that $R_\nu e_N = \nu$, we set $Q_\nu := R_\nu Q$, $Q_\nu(x, \delta) := x + \delta Q_\nu$, $B_\nu^+(x, \delta) := \{y \in B(x, \delta) : (y - x) \cdot \nu > 0\}$, and $B_\nu^-(x, \delta) := \{y \in B(x, \delta) : (y - x) \cdot \nu < 0\}$.

Let $\Omega \subset \mathbb{R}^N$ be an open set. We represent by $\mathcal{A}(\Omega)$ the family of all open subsets of Ω, and by $\mathcal{A}_\infty(\Omega)$ the family of all sets in $\mathcal{A}(\Omega)$ with Lipschitz boundary. The Borel σ-algebra on Ω is denoted by $\mathcal{B}(\Omega)$, and $\mathcal{M}(\Omega; \mathbb{R}^d)$ is the Banach space of all bounded Radon measures $\mu : \mathcal{B}(\Omega) \to \mathbb{R}^d$ endowed with the total variation norm $|\cdot|$. If $\mu \in \mathcal{M}(\Omega; \mathbb{R}_0^+)$ is a nonnegative Radon measure and $v : \Omega \to \mathbb{R}^d$ is a μ-measurable function, then $\fint_\Omega v(x)\, \mathrm{d}\mu(x)$ stands for $\frac{1}{\mu(\Omega)} \int_\Omega v(x)\, \mathrm{d}\mu(x)$. The N-dimensional Lebesgue measure is denoted by $\mathcal{L}^N$ and the $(N-1)$-dimensional Hausdorff measure is designated by $\mathcal{H}^{N-1}$. Also, "a.e. in Ω" stands for "almost everywhere in Ω with respect to $\mathcal{L}^N$".

Let $\mathcal{M}$ be an m-dimensional manifold, $m \in \mathbb{N}$, in $\mathbb{R}^d$. The tangent space of $\mathcal{M}$ at $z \in \mathcal{M}$ is represented by $T_z(\mathcal{M})$. Given a Banach space $X(\Omega; \mathbb{R}^d)$ of functions $\vartheta : \Omega \to \mathbb{R}^d$, we denote by $X(\Omega; \mathcal{M})$ the set

$$X(\Omega; \mathcal{M}) := \big\{\vartheta \in X(\Omega; \mathbb{R}^d) : \vartheta(\cdot) \in \mathcal{M} \text{ a.e. in } \Omega\big\}.$$

To simplify the notation, if $\mathcal{M}_1$ is an m_1-dimensional manifold in $\mathbb{R}^{d_1}$, $\mathcal{M}_2$ is an m_2-dimensional manifold in $\mathbb{R}^{d_2}$, $u \in X(\Omega; \mathcal{M}_1)$, $v \in X(\Omega; \mathcal{M}_2)$, and $w := (u, v)$, we write $w \in X(\Omega; \mathcal{M}_1 \times \mathcal{M}_2)$.

2.2 Meyer's G-space

In this section, we recall the definition of the space $G(\Omega)$ introduced in [33] for $\Omega = \mathbb{R}^2$ and generalized in [6] for bounded domains $\Omega \subset \mathbb{R}^2$. The vectorial case, which allows modeling textures in color images, has been

treated in [22]. Below we collect the main properties of the space G used in this paper. For the proofs and for more considerations on the space G, we refer to [6, 22, 33].

Definition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with Lipschitz boundary, and let $d \in \mathbb{N}$. We define*

$$G(\Omega; \mathbb{R}^d) := \left\{ v \in L^2(\Omega; \mathbb{R}^d) \colon v_i = \operatorname{div} \xi_i, \; \xi \in L^\infty(\Omega; (\mathbb{R}^2)^d), \; \xi_i \cdot n = 0 \text{ on } \partial\Omega, \; i \in \{1, ..., d\} \right\},$$

where n is the outward unit normal to $\partial\Omega$. We endow $G(\Omega; \mathbb{R}^d)$ with the norm

$$\|v\|_{G(\Omega; \mathbb{R}^d)} := \inf \left\{ \|\xi\|_{L^\infty(\mathbb{R}^2; (\mathbb{R}^2)^d)} \colon v_i = \operatorname{div} \xi_i, \; \xi_i \cdot n = 0 \text{ on } \partial\Omega, \; i \in \{1, ..., d\} \right\}.$$

$G(\Omega; \mathbb{R}^d)$ is a Banach space, and when $N = 2$, it admits the following characterization.

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with Lipschitz boundary. Then,*

$$G(\Omega; \mathbb{R}^d) = \left\{ v \in L^2(\Omega; \mathbb{R}^d) \colon \int_\Omega v(x) \, \mathrm{d}x = 0 \right\}.$$

The topology induced by the G-norm is coarser than the one induced by the L^2-norm as there are sequences that converge to zero in the G-norm but not in the L^2-norm. More generally, the following result shows that the G-norm is well adapted to capture oscillations of a function in an energy minimization method.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with Lipschitz boundary and let $p > 2$. If $\{v_n\}_{n \in \mathbb{N}} \subset G(\Omega; \mathbb{R}^d)$ is such that $v_n \rightharpoonup 0$ weakly in $L^p(\Omega; \mathbb{R}^d)$ as $n \to \infty$, then $\lim_{n \to \infty} \|v_n\|_{G(\Omega; \mathbb{R}^d)} = 0$.*

2.3 The Space BV of Functions of Bounded Variation and Sets of Finite Perimeter

We will adopt the notations of [4] regarding functions of bounded variation and sets of finite perimeter.

In what follows, $N, d \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^N$ is an open set. Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function such that

$$\int_{\mathbb{R}^N} \rho(x) \, \mathrm{d}x = 1, \quad \operatorname{supp} \rho = \overline{B(0, 1)}, \quad \rho(x) = \rho(-x) \text{ for all } x \in \mathbb{R}^N.$$

For $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^d)$ and $\delta > 0$, we set

$$\rho_\delta(x) := \frac{1}{\delta^N} \rho\left(\frac{x}{\delta}\right), \; x \in \mathbb{R}^N, \tag{2.1}$$

and

$$u_\delta(x) := (u * \rho_\delta)(x) = \int_\Omega u(y) \rho_\delta(x - y) \, \mathrm{d}y, \; x \in \Omega_\delta := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \delta\}. \tag{2.2}$$

We observe that $\operatorname{supp} \rho_\delta \subset \overline{B(0, \delta)}$, and we recall that $u_\delta \in C^\infty(\Omega_\delta)$ and $\|u_\delta\|_{L^1(\Omega_\delta; \mathbb{R}^d)} \leqslant \|u\|_{L^1(\Omega; \mathbb{R}^d)}$.

Definition 2.4. *Let $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^d)$.*

(a) *We define the set A_u as the set of points $x \in \Omega$ for which there exists a vector $z \in \mathbb{R}^d$ such that*

$$\lim_{\delta \to 0^+} \fint_{B(x, \delta)} |u(y) - z| \, \mathrm{d}y = 0, \tag{2.3}$$

in which case we say that u has an approximate limit at x, and the vector z, uniquely determined by (2.3), is represented by $\tilde{u}(x)$. The set $S_u := \Omega \setminus A_u$ is called the approximate discontinuity set. We say that u is approximately continuous at x if $x \in A_u$ and $\tilde{u}(x) = u(x)$, i.e., x is a Lebesgue point of u.

9

(b) We define the set of approximate jump points of u, represented by J_u, as the set of points $x \in \Omega$ for which there exist vectors $a, b \in \mathbb{R}^d$, $a \neq b$, and $\nu \in S^{N-1}$ such that

$$\lim_{\delta \to 0^+} \fint_{B_\nu^+(x,\delta)} |u(y) - a| \, \mathrm{d}y = 0, \quad \lim_{\delta \to 0^+} \fint_{B_\nu^-(x,\delta)} |u(y) - b| \, \mathrm{d}y = 0. \tag{2.4}$$

A point $x \in J_u$ is called an approximate jump point of u; the associated triplet (a, b, ν), uniquely determined by (2.4) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

(c) We say that u is approximately differentiable at $x \in A_u$ if there exists a $d \times N$ matrix L such that

$$\lim_{\delta \to 0^+} \fint_{B(x,\delta)} \frac{|u(y) - \tilde{u}(x) - L(y - x)|}{\delta} \, \mathrm{d}y = 0, \tag{2.5}$$

in which case we denote the matrix L, uniquely determined by (2.5), by $\nabla u(x)$. The set of approximate differentiability points is denoted by $\mathcal{D}_u$.

Remark 2.5. The set A_u does not depend on the representative in the equivalent class of u, i.e., if $v = u$ $\mathcal{L}^N$-a.e. in Ω then $A_v = A_u =: A$ and $\tilde{v}(x) = \tilde{u}(x)$ for all $x \in A$. In contrast, the property of being approximately continuous at x depends on the value of u at x, thus on the representative in the equivalent class of u.

The proof of the following result may be found in [4, Prop. 3.64, Prop. 3.69, Prop. 3.71].

Proposition 2.6. Let $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$, let $u_\delta \in C^\infty(\Omega_\delta)$ be given by (2.2), let $\phi : \mathbb{R}^d \to \mathbb{R}^m$ be a Lipschitz map, and let $v := \phi \circ u$. Then

(a) i) S_u is a $\mathcal{L}^N$-negligible Borel set and $\tilde{u} : A_u \to \mathbb{R}^d$ is a Borel function, coinciding $\mathcal{L}^N$-a.e. in A_u with u;

ii) $\lim_{\delta \to 0^+} u_\delta(x) = \tilde{u}(x)$ for all $x \in A_u$;

iii) $S_v \subset S_u$ and $\tilde{v}(x) = \phi(\tilde{u}(x))$ for all $x \in A_u$.

(b) i) J_u is a Borel subset of S_u and there exist Borel functions $(u^+, u^-, \nu_u) : J_u \to \mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$ such that (2.4) holds for all $x \in J_u$;

ii) $\lim_{\delta \to 0^+} u_\delta(x) = \frac{u^+(x) + u^-(x)}{2}$ for all $x \in J_u$;

iii) if $x \in J_u$, then $x \in J_v$ if and only if $\phi(u^+(x)) \neq \phi(u^-(x))$, in which case $(v^+(x), v^-(x), \nu_v(x)) = (\phi(u^+(x)), \phi(u^-(x)), \nu_u(x))$; otherwise, $x \in A_v$ and $\tilde{v}(x) = \phi(u^+(x)) = \phi(u^-(x))$.

(c) i) $\mathcal{D}_u$ is a Borel subset and $\nabla u : \mathcal{D}_u \to \mathbb{R}^{d \times N}$ is a Borel map;

ii) if $x \in \mathcal{D}_u$ and, in addition, φ has linear growth at infinity and is differentiable at $\tilde{u}(x)$, then v is approximately differentiable at x and $\nabla v(x) = \nabla \phi(\tilde{u}(x)) \nabla u(x)$.

A function $u : \Omega \to \mathbb{R}^d$ is said to be a function of *bounded variation*, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if $u \in L^1(\Omega; \mathbb{R}^d)$ and its distributional derivative, Du, belongs to $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$; that is, if there exists a measure $Du \in \mathcal{M}(\Omega; \mathbb{R}^{d \times N})$ such that for all $\phi \in C_c(\Omega)$, $j \in \{1, \cdots, d\}$ and $i \in \{1, \cdots, N\}$, one has

$$\int_\Omega u_j(x) \frac{\partial \phi}{\partial x_i}(x) \, \mathrm{d}x = -\int_\Omega \phi(x) \, \mathrm{d}D_i u_j(x),$$

where $u = (u_1, \cdots, u_d)$ and $Du_j = (D_1 u_j, \cdots, D_N u_j)$. The space $BV(\Omega; \mathbb{R}^d)$ is a Banach space when endowed with the norm $\|u\|_{BV(\Omega; \mathbb{R}^d)} := \|u\|_{L^1(\Omega; \mathbb{R}^d)} + |Du|(\Omega)$.

We recall that $\{u_j\}_{j \in \mathbb{N}} \subset BV(\Omega; \mathbb{R}^d)$ is said to weakly-$\star$ converge in $BV(\Omega; \mathbb{R}^d)$ to some $u \in BV(\Omega; \mathbb{R}^d)$ if $u_j \to u$ (strongly) in $L^1(\Omega; \mathbb{R}^d)$ and $Du_j \overset{\star}{\rightharpoonup} Du$ weakly-$\star$ in $\mathcal{M}(\Omega; \mathbb{R}^{d \times N})$.

The proof of the following result may be found in [4, Prop. 3.7] and [5, Lemma 4.5].

Lemma 2.7. *Let $u \in BV(\Omega; \mathbb{R}^d)$ and let $u_\delta \in C^\infty(\Omega_\delta)$ be given by (2.2). Then,*

i) $u_\delta \xrightarrow{\star} u$ weakly-$\star$ in $BV(\Omega'; \mathbb{R}^d)$ as $\delta \to 0^+$, for all $\Omega' \subset\subset \Omega$. Moreover, if $\mathcal{L}^N(\partial\Omega') = 0$, then

$$\lim_{\delta \to 0^+} |Du_\delta|(\Omega') = \lim_{\delta \to 0^+} \int_{\Omega'} |\nabla u_\delta(x)| \, \mathrm{d}x = |Du|(\Omega');$$

ii)

$$\int_{B(x_0,\varepsilon)} \mathfrak{h}(x)|\nabla u_\delta(x)| \, \mathrm{d}x \leqslant \int_{B(x_0,\varepsilon+\delta)} (\mathfrak{h} * \rho_\delta)(x) \, \mathrm{d}|Du|(x)$$

whenever $\mathrm{dist}(x_0, \partial\Omega) > \varepsilon + \delta$ and $\mathfrak{h}$ is a nonnegative Borel function;

iii)

$$\lim_{\delta \to 0^+} \int_{B(x_0,\varepsilon)} \theta(\nabla u_\delta(x)) \, \mathrm{d}x = \int_{B(x_0,\varepsilon)} \theta\Big(\frac{\mathrm{d}Du}{\mathrm{d}|Du|}(x)\Big) \, \mathrm{d}|Du|(x)$$

for every positively 1-homogeneous continuous function θ and for every $\varepsilon \in (0, \mathrm{dist}(x_0, \partial\Omega))$ such that $|Du|(\partial B(x_0, \varepsilon)) = 0$;

iv)

$$\lim_{\delta \to 0^+} (|u_\delta - u| * \rho_\delta)(x) = 0 \text{ for all } x \in A_u$$

if, in addition, $u \in L^\infty(\Omega; \mathbb{R}^d)$.

In what follows, $Du = D^a u + D^s u$ is the Radon-Nikodym decomposition of Du in absolutely continuous and singular parts with respect to $\mathcal{L}^N_{\lfloor \Omega}$. The proof of the following results may be found in [4, Rmk. 3.93, Thm. 3.83, Thm. 3.78].

Lemma 2.8. *Let $u_1, u_2 \in BV(\Omega; \mathbb{R}^d)$ and $A := \{x \in A_{u_1} \cap A_{u_2} : \tilde{u}_1(x) = \tilde{u}_2(x)\}$. Then $Du_1\lfloor_A = Du_2\lfloor_A$.*

Theorem 2.9. *Let $u \in BV(\Omega; \mathbb{R}^d)$. Then,*

(a) u is approximately differentiable at $\mathcal{L}^N$-a.e. point of Ω, and the approximate differential ∇u is the density of the absolutely continuous part of Du with respect to $\mathcal{L}^N_{\lfloor \Omega}$; that is, $D^a u = \nabla u \mathcal{L}^N_{\lfloor \Omega}$;

(b) the set S_u is countably $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. Moreover, $Du_{\lfloor J_u} = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}_{\lfloor J_u}$.

Definition 2.10. *Given $u \in BV(\Omega; \mathbb{R}^d)$, the measures*

$$D^j u := D^s u_{\lfloor J_u} \text{ and } D^c u := D^s u_{\lfloor A_u}$$

are called the jump part of the derivative and the Cantor part of the derivative, respectively. The sum $D^a u + D^c u$ is called the diffuse part of the derivative and is denoted by $\tilde{D}u$.

Remark 2.11. It can be proved that $D^j u = Du_{\lfloor J_u}$ (see [4, Prop. 3.92]). In view of Definition 2.10 and Theorem 2.9, we have the following decompositions for Du:

$$Du = D^a u + D^s u = \nabla u \mathcal{L}^N_{\lfloor \Omega} + (u^+ - u^-) \otimes \nu_u \mathcal{H}^{N-1}_{\lfloor J_u} + D^c u = \tilde{D}u + D^c u.$$

The next result is due to G. Alberti (see [1]).

Theorem 2.12. *If $u \in BV(\Omega; \mathbb{R}^d)$ and $Du = h|Du|$, then h has rank one for $(|D^j u| + |D^c u|)$-a.e. point of Ω.*

We now state a result regarding the chain rule in BV, which proof may be found in [4, Thm. 3.96].

Theorem 2.13. *Let $u \in BV(\Omega; \mathbb{R}^d)$ and $\phi \in C^1(\mathbb{R}^d; \mathbb{R}^m)$ be a Lipschitz function satisfying $\phi(0) = 0$ if $\mathcal{L}^N(\Omega) = +\infty$. Then $v := \phi \circ u$ belongs to $BV(\Omega; \mathbb{R}^m)$, and*

$$\tilde{D}v = \nabla\phi(u)\nabla u \mathcal{L}^N_{\lfloor \Omega} + \nabla\phi(\tilde{u})D^c u = \nabla\phi(\tilde{u})\tilde{D}u, \qquad D^j v = \left(\phi(u^+) - \phi(u^-)\right) \otimes \nu_u \mathcal{H}^{N-1}_{\lfloor J_u}. \tag{2.6}$$

As a consequence of Lebesgue-Besicovitch Differentiation Theorem, we have the following.

Theorem 2.14. *If μ is a nonnegative Radon measure and if $v \in L^1_{\mathrm{loc}}(\Omega, \mu; \mathbb{R}^d)$, then*

$$\lim_{\epsilon \to 0^+} \fint_{x_0 + \epsilon C} |v(y) - v(x_0)| \, \mathrm{d}\mu(y) = 0$$

for μ-a.e. $x_0 \in \Omega$ and for every bounded, convex, open set C containing the origin.

In the remainder of this subsection Ω denotes an open subset of $\mathbb{R}^N$ and E a $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$.

Definition 2.15. *The perimeter of E in Ω is represented by $\mathrm{Per}_\Omega(E)$ and defined by*

$$\mathrm{Per}_\Omega(E) := \sup \left\{ \int_E \mathrm{div}\,\varphi \, \mathrm{d}x \colon \varphi \in C^1_c(\Omega; \mathbb{R}^N), \, \|\varphi\|_{L^\infty(\Omega)} \leqslant 1 \right\}.$$

We say that E is a set of finite perimeter in Ω if $\mathrm{Per}_\Omega(E) < +\infty$.

The proof of the following result may be found in [4, Thm. 3.36].

Theorem 2.16. *Assume that E is a set of finite perimeter in Ω. Then the distributional derivative of χ_E, $D\chi_E$, belongs to $\mathcal{M}(\Omega; \mathbb{R}^N)$ and $|D\chi_E|(\Omega) = \mathrm{Per}_\Omega(E)$. Moreover, the following generalized Gauss–Green formula holds*

$$\int_E \mathrm{div}\,\varphi \, \mathrm{d}x = - \int_\Omega \nu_E \cdot \varphi \, \mathrm{d}|D\chi_E|, \quad \text{for all } \varphi \in C^1_c(\Omega; \mathbb{R}^N),$$

where $D\chi_E = \nu_E |D\chi_E|$ is the polar decomposition of $D\chi_E$.

Definition 2.17. *Let E be a set of finite perimeter in Ω. The reduced boundary of E, denoted by $\mathcal{F}^*E$, is the set of all points $x \in \Omega$ such that for all $\epsilon > 0$,*

$$|D\chi_E|(B(x, \epsilon) \cap \Omega) > 0,$$

and such that the limit

$$\nu_E(x) := \lim_{\epsilon \to 0^+} \frac{D\chi_E(B(x, \epsilon))}{|D\chi_E|(B(x, \epsilon))}$$

exists in $\mathbb{R}^N$ and satisfies $|\nu_E(x)| = 1$. The function $\nu_E : \mathcal{F}^*E \to S^{N-1}$ is called the generalised inner normal to E.

Definition 2.18. *Given $t \in [0, 1]$, we represent by E^t the set of all points where E has density t, i.e.,*

$$E^t := \left\{ x \in \mathbb{R}^N \colon \lim_{\epsilon \to 0^+} \frac{\mathcal{L}^N(E \cap B(x, \epsilon))}{\mathcal{L}^N(B(x, \epsilon))} = t \right\}.$$

The set $\partial^* E := \mathbb{R}^N \backslash (E^0 \cup E^1)$ is called the essential boundary of E.

The proof of the following theorem may be found in [4, Thm. 3.59, Thm. 3.61, Example 3.68].

Theorem 2.19. *Let E be a set of finite perimeter in Ω.*

(i) *(De Giorgi) The set $\mathcal{F}^*E$ is contained, up to $\mathcal{H}^{N-1}$ negligible sets, in a countable union of C^1 hypersurfaces, and*

$$D\chi_E = \nu_E \, \mathcal{H}^{N-1} \llcorner_{\mathcal{F}^*E}, \quad |D\chi_E| = \mathcal{H}^{N-1} \llcorner_{\mathcal{F}^*E},$$

where ν_E is the generalised inner normal to E.

(ii) *(Federer) It holds*

$$\mathcal{F}^*E \subset E^{1/2} \subset \partial^*E, \quad \mathcal{H}^{N-1}(\Omega \backslash (E^0 \cup \mathcal{F}^*E \cup E^1)) = 0.$$

In particular, E has density either 0 or $1/2$ or 1 at $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and $\mathcal{H}^{N-1}$-a.e. $x \in \partial^*E \cap \Omega$ belongs to $\mathcal{F}^*E$.

(iii) *Setting $u := \chi_E$, then $u \in BV(\Omega)$ with $S_u = \partial^*E \cap \Omega$, $\mathcal{F}^*E \subset J_u \subset E^{1/2}$, and $\{u^+(x), u^-(x)\} = \{0,1\}$ for all $x \in J_u$.*

Remark 2.20. Another property of sets of finite perimeter in $\mathbb{R}^N$, which is due to De Giorgi [21], is the following. If E is a set of finite perimeter in Ω, then there exists a sequence of open sets $\{E_n\}_{n \in \mathbb{N}}$ such that each set ∂E_n is contained in a finite number of hyperplanes and

$$\lim_{n \to \infty} \mathcal{L}^N(E_n \Delta E) = 0, \quad \lim_{n \to \infty} |D\chi_{E_n}|(\Omega) = |D\chi_E|(\Omega).$$

2.4 Quasiconvex Functions

We say that a Borel function $h : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to \mathbb{R}$ is quasiconvex if for all $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$, $\varphi \in W_0^{1,\infty}(Q)$, and $\psi \in W_0^{1,\infty}(Q; \mathbb{R}^d)$, we have

$$h(\xi, \eta) \leqslant \int_Q h(\xi + \nabla\varphi(x), \eta + \nabla\psi(x)) \, \mathrm{d}x.$$

Remark 2.21. Consider the mapping that to each matrix $A = (a_{ij})_{1 \leqslant i \leqslant d+1, 1 \leqslant j \leqslant N} \in \mathbb{R}^{(d+1) \times N}$ associates the pair $(\xi_A, \eta_A) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$, where $\xi_A := (a_{(d+1)j})_{1 \leqslant j \leqslant N}$ is the last row of A, and $\eta_A := (a_{ij})_{1 \leqslant i \leqslant d, 1 \leqslant j \leqslant N}$ is obtained from A by erasing its last row. Then, to a Borel function $h : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to \mathbb{R}$ we may associate the Borel function $\bar{h} : \mathbb{R}^{(d+1) \times N} \to \mathbb{R}$ defined by

$$\bar{h}(A) := h(\xi_A, \eta_A), \quad A \in \mathbb{R}^{(d+1) \times N}.$$

In this setting, h is a quasiconvex function if and only if $\bar{h}$ is a quasiconvex function in the *usual* sense; that is, for all $A \in \mathbb{R}^{(d+1) \times N}$ and $\vartheta \in W_0^{1,\infty}(Q; \mathbb{R}^{(d+1)})$,

$$\bar{h}(A) \leqslant \int_Q \bar{h}(A + \nabla\vartheta(x)) \, \mathrm{d}x.$$

In view of Remark 2.21 and well-know results concerning the *usual* notion of quasiconvexity, if $h : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to \mathbb{R}$ is a quasiconvex for which there exists a positive constant C such that for all $(\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$,

$$0 \leqslant h(\xi, \eta) \leqslant C(1 + |(\xi, \eta)|),$$

then h is Lipschitz; i.e., there exists a constant $L > 0$, only depending on C, N, and d, such that for all $(\xi, \eta), (\xi', \eta') \in \mathbb{R}^N \times \mathbb{R}^{d \times N}$,

$$|h(\xi, \eta) - h(\xi', \eta')| \leqslant L|(\xi, \eta) - (\xi', \eta')|. \tag{2.7}$$

Remark 2.22. We finish this subsection by noting that for all $(r, s) \in [\alpha, \beta] \times S^2$, the function $f(r, s, \cdot, \cdot)$ introduced in (1.13) is not quasiconvex in $\mathbb{R}^2 \times \mathbb{R}^{3 \times 2}$. In fact, if it were, then so would be its recession function $f^\infty(r, s, \cdot, \cdot)$ introduced in (1.16) (see [26, Rmk. 2.2]). In turn, by (2.7), $f^\infty(r, s, \cdot, \cdot)$ would be continuous in $\mathbb{R}^2 \times \mathbb{R}^{3 \times 2}$. However, taking $\xi_n \in \mathbb{R}^2 \backslash \{0\}$ and $\eta \in \mathbb{R}^{3 \times 2} \backslash \{0\}$ such that $\xi_n \to 0$ as $n \to \infty$, we conclude that $\lim_{n \to \infty} f^\infty(r, s, \xi_n, \eta) = |r\eta| \neq |r\eta| + |\eta| = f^\infty(r, s, 0, \eta)$. Thus, neither $f(r, s, \cdot, \cdot)$ nor $f^\infty(r, s, \cdot, \cdot)$ are quasiconvex functions.

3 Proof of Theorem 1.1

This subsection is devoted to the proof of Theorem 1.1, under the assumption that Theorem 1.2 holds. We start by observing that there are admissible fields as introduced in (1.21).

Lemma 3.1. *The set*

$$X = \left\{ (u_b, u_c) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2) \colon u_b - (u_0)_b \in G(\Omega), \ u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3) \right\}$$

is nonempty.

Proof. Recall that a.e. in Ω,

$$(u_0)_b = |u_0| \in [\alpha, \beta], \quad (u_0)_c = \frac{u_0}{|u_0|} = \frac{u_0}{(u_0)_b} \in S^2, \quad u_0 = (u_0)_b (u_0)_c,$$

and $0 < \alpha \leqslant \beta$.

Let $c_0 := \fint_\Omega (u_0)_b \, dx$ and set

$$u_b(x) := c_0 \quad \text{for all } x \in \Omega.$$

Clearly, $u_b \in BV(\Omega; [\alpha, \beta])$ and $\int_\Omega u_b \, dx = \int_\Omega (u_0)_b \, dx$; since $u_b - (u_0)_b \in L^\infty(\Omega) \subset L^2(\Omega)$, it follows that $u_b - (u_0)_b \in G(\Omega)$ by Proposition 2.2.

Because

$$\left| \fint_\Omega (u_0)_b (u_0)_c \, dx \right| \leqslant \fint_\Omega (u_0)_b \, dx = c_0,$$

we have $\fint_\Omega (u_0)_b (u_0)_c \, dx \in \overline{B(0, c_0)} \subset \mathbb{R}^3$; thus, there exist $\theta \in [0, 1]$ and $s_1, s_2 \in \partial B(0, c_0)$ such that

$$\fint_\Omega (u_0)_b (u_0)_c \, dx = \theta s_1 + (1 - \theta) s_2.$$

Let $\{\Omega_1, \Omega_2\}$ be a Lipschitz partition of Ω satisfying $\mathcal{L}^2(\Omega_1) = \theta \mathcal{L}^2(\Omega)$, $\mathcal{L}^2(\Omega_2) = (1 - \theta)\mathcal{L}^2(\Omega)$, and consider the function u_c defined, for $x \in \Omega$, by

$$u_c(x) := \begin{cases} \dfrac{s_1}{c_0} & \text{if } x \in \Omega_1, \\[2mm] \dfrac{s_2}{c_0} & \text{if } x \in \Omega_2. \end{cases}$$

Then, $u_c \in BV(\Omega; S^2)$, $u_b u_c - u_0 \in L^\infty(\Omega; \mathbb{R}^3)$, and $\int_\Omega u_b u_c \, dx = \int_\Omega (u_0)_b (u_0)_c \, dx = \int_\Omega u_0 \, dx$. Thus, $u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3)$, and this completes the proof. $\square$

Proof of Theorem 1.1. Fix sequences $\{\varepsilon_n\}_{n \in \mathbb{N}}$ and $\{\delta_n\}_{n \in \mathbb{N}}$ as in the statement. Let G_n, $G_0 : L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \to [0, +\infty]$ be the functionals defined, for $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$, by

$$G_n(u_b, u_c) := \begin{cases} F^{reg}(u_b, u_c) + F^{fid}_{\varepsilon_n}(u_b, u_c) & \text{if } (u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2), \\ +\infty & \text{otherwise}, \end{cases}$$

and

$$G_0(u_b, u_c) := \begin{cases} F^{reg, sc^-}(u_b, u_c) + F^{fid}(u_b, u_c) & \text{if } (u_b, u_c) \in X, \\ +\infty & \text{otherwise}, \end{cases}$$

respectively.

We claim that $\{G_n\}_{n\in\mathbb{N}}$ Γ-converges to G_0 in $L^1(\Omega) \times L^1(\Omega;\mathbb{R}^3)$. Invoking [20, Prop. 8.1], this claim follows from Steps 1 and 2 below.

<u>Step 1.</u> *(liminf inequality)* Fix $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega;\mathbb{R}^3)$, and let $\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}} \subset L^1(\Omega) \times L^1(\Omega;\mathbb{R}^3)$ be an arbitrary sequence converging to (u_b, u_c) in $L^1(\Omega) \times L^1(\Omega;\mathbb{R}^3)$. We claim that

$$G_0(u_b, u_c) \leqslant \liminf_{n\to\infty} G_n(u_b^n, u_c^n). \tag{3.1}$$

Without loss of generality, we assume that the limit inferior on the right-hand side of (3.1) is a limit and is finite, with $\sup_{n\in\mathbb{N}} G_n(u_b^n, u_c^n) < +\infty$. Then, $\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}} \subset W^{1,1}(\Omega;[\alpha,\beta]) \times W^{1,1}(\Omega;S^2)$ and there exists a positive constant, C, independent of $n \in \mathbb{N}$, such that

$$C \geqslant F^{reg}(u_b^n, u_c^n) = \int_\Omega f(u_b^n, u_c^n, \nabla u_b^n, \nabla u_c^n)\,\mathrm{d}x \geqslant \frac{1}{2}\int_\Omega |\nabla u_b^n|\,\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |\nabla u_c^n|\,\mathrm{d}x,$$

$$C \geqslant F^{fid}_{\varepsilon_n}(u_b^n, u_c^n) \geqslant \frac{1}{\varepsilon_n}\left|\int_\Omega (u_b^n u_c^n - u_0)\,\mathrm{d}x\right| + \frac{1}{\varepsilon_n}\left|\int_\Omega (u_b^n - (u_0)_b)\,\mathrm{d}x\right|,$$

where we used (1.14) together with the fact that $u_b^n \geqslant \alpha$ and $u_c^n \in S^2$ a.e. in Ω. Consequently,

$$u_b^n \overset{\star}{\rightharpoonup} u_b \text{ weakly-} \star \text{ in } BV(\Omega), \quad u_c^n \overset{\star}{\rightharpoonup} u_c \text{ weakly-} \star \text{ in } BV(\Omega;\mathbb{R}^3), \quad \text{as } n \to \infty,$$
$$\lim_{n\to\infty}\int_\Omega u_b^n u_c^n\,\mathrm{d}x = \int_\Omega u_0\,\mathrm{d}x, \quad \lim_{n\to\infty}\int_\Omega u_b^n\,\mathrm{d}x = \int_\Omega (u_0)_b\,\mathrm{d}x, \tag{3.2}$$

and, up to a (not relabeled) subsequence,

$$u_b^n \to u_b \text{ a.e. in } \Omega, \quad u_c^n \to u_c \text{ a.e. in } \Omega, \quad \text{as } n \to \infty.$$

These two last convergences yield $u_b \in [\alpha,\beta]$ and $u_c \in S^2$ a.e. in Ω. By Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_\Omega u_b^n u_c^n\,\mathrm{d}x = \int_\Omega u_b u_c\,\mathrm{d}x, \quad \lim_{n\to\infty}\int_\Omega u_b^n\,\mathrm{d}x = \int_\Omega u_b\,\mathrm{d}x,$$

which, together with (3.2) and in view of Proposition 2.2, implies $(u_b, u_c) \in X$. Furthermore, by Theorem 1.2,

$$F^{reg,sc^-}(u_b, u_c) \leqslant \liminf_{n\to\infty} F^{reg}(u_b^n, u_c^n). \tag{3.3}$$

Finally, we prove that

$$F^{fid}(u_b, u_c) \leqslant \liminf_{n\to\infty} F^{fid}_{\varepsilon_n}(u_b^n, u_c^n), \tag{3.4}$$

which, together with (3.3), yields (3.1).

The sequence $\{u_b^n u_c^n - u_0 - \fint_\Omega (u_b^n u_c^n - u_0)\,\mathrm{d}x\}_{n\in\mathbb{N}} \subset G(\Omega;\mathbb{R}^3)$ converges a.e. in Ω to $u_b u_c - u_0$ and is bounded in $L^\infty(\Omega;\mathbb{R}^3)$; hence, the convergence holds weakly in $L^p(\Omega)$ for any $p > 2$. Consequently, by Proposition 2.3, we have

$$\lim_{n\to\infty}\left\|u_b^n u_c^n - u_0 - \fint_\Omega (u_b^n u_c^n - u_0)\,\mathrm{d}x\right\|_{G(\Omega;\mathbb{R}^3)} = \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)}.$$

Similarly,

$$\lim_{n\to\infty}\left\|u_b^n - (u_0)_b - \fint_\Omega (u_b^n - (u_0)_b)\,\mathrm{d}x\right\|_{G(\Omega)} = \|u_b - (u_0)_b\|_{G(\Omega)}.$$

Moreover, because $u_c^n, (u_0)_c \in S^2$ and $u_c^n \to u_c$ a.e. in Ω, as $n \to \infty$,

$$\lim_{n\to\infty}\int_\Omega |u_c^n - (u_0)_c|^2\,\mathrm{d}x = \int_\Omega |u_c - (u_0)_c|^2\,\mathrm{d}x.$$

15

Thus,

$$\liminf_{n\to\infty} F_{\varepsilon_n}^{fid}(u_b^n, u_c^n) \geqslant \liminf_{n\to\infty} \left(\lambda_v \left\| u_b^n u_c^n - u_0 - \fint_\Omega (u_b^n u_c^n - u_0)\, \mathrm{d}x \right\|_{G(\Omega;\mathbb{R}^3)} \right.$$
$$\left. + \lambda_b \left\| u_b^n - (u_0)_b - \fint_\Omega (u_b^n - (u_0)_b)\, \mathrm{d}x \right\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c^n - (u_0)_c|^2\, \mathrm{d}x \right)$$
$$= F^{fid}(u_b, u_c),$$

which proves (3.4). This concludes Step 1.

Step 2. (limsup inequality) Fix $(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$. We claim that there exists a sequence $\{(u_b^n, u_c^n)\}_{n\in\mathbb{N}} \subset L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$ converging to (u_b, u_c) in $L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$ that satisfies

$$G_0(u_b, u_c) \geqslant \limsup_{n\to\infty} G_n(u_b^n, u_c^n). \tag{3.5}$$

Without loss of generality, we may assume that $(u_b, u_c) \in X$. By Theorem 1.2, and recalling the bounds (1.14) for f (see also (1.11)), we can find a sequence $\{(u_b^j, u_c^j)\}_{j\in\mathbb{N}} \subset W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)$ such that

$$u_b^j \overset{\star}{\rightharpoonup} u_b \text{ weakly-}\star \text{ in } BV(\Omega), \quad u_c^j \overset{\star}{\rightharpoonup} u_c \text{ and } u_b^j u_c^j \overset{\star}{\rightharpoonup} u_b u_c \text{ weakly-}\star \text{ in } BV(\Omega; \mathbb{R}^3), \quad \text{as } j \to \infty;$$
$$u_b^j \to u_b, \quad u_c^j \to u_c, \text{ and } u_b^j u_c^j \to u_b u_c \text{ a.e. in } \Omega, \quad \text{as } j \to \infty;$$
$$F^{reg,sc^-}(u_b, u_c) = \lim_{j\to\infty} F^{reg}(u_b^j, u_c^j). \tag{3.6}$$

In particular, arguing as in Step 1,

$$F^{fid}(u_b, u_c) = \lim_{j\to\infty} \left(\lambda_v \left\| u_b^j u_c^j - u_0 - \fint_\Omega (u_b^j u_c^j - u_0)\, \mathrm{d}x \right\|_{G(\Omega;\mathbb{R}^3)} \right.$$
$$\left. + \lambda_b \left\| u_b^j - (u_0)_b - \fint_\Omega (u_b^j - (u_0)_b)\, \mathrm{d}x \right\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c^j - (u_0)_c|^2\, \mathrm{d}x \right). \tag{3.7}$$

Moreover, recalling Proposition 2.2 and the fact that $(u_b, u_c) \in X$,

$$\lim_{j\to\infty} \int_\Omega u_b^j u_c^j\, \mathrm{d}x = \int_\Omega u_b u_c\, \mathrm{d}x = \int_\Omega u_0\, \mathrm{d}x, \quad \lim_{j\to\infty} \int_\Omega u_b^j\, \mathrm{d}x = \int_\Omega u_b\, \mathrm{d}x = \int_\Omega (u_0)_b\, \mathrm{d}x.$$

Hence, we can find a subsequence $j_n \preceq j$ such that

$$\left| \int_\Omega u_b^{j_n} u_c^{j_n}\, \mathrm{d}x - \int_\Omega u_0\, \mathrm{d}x \right| \leqslant \varepsilon_n^2, \quad \left| \int_\Omega u_b^{j_n}\, \mathrm{d}x - \int_\Omega (u_0)_b\, \mathrm{d}x \right| \leqslant \varepsilon_n^2. \tag{3.8}$$

From (3.6), (3.7), and (3.8), we obtain (3.5) for $\{(u_b^{j_n}, u_c^{j_n})\}_{n\in\mathbb{N}}$, which concludes Step 2.

We now observe that $\{G_n\}_{n\in\mathbb{N}}$ is equi-coercive in $L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$. In fact, arguing as in Step 1 above, given $C \in \mathbb{R}$ we can find a positive constant $c = c(C, \alpha)$ such that for all $n \in \mathbb{N}$,

$$\left\{ (u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3) \colon G_n(u_b, u_c) \leqslant C \right\}$$
$$\subset \left\{ (u_b, u_c) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega; \mathbb{R}^3) \colon \|(u_b, u_c)\|_{W^{1,1}(\Omega) \times W^{1,1}(\Omega; \mathbb{R}^3)} \leqslant c \right\}, \tag{3.9}$$

which, together with the compact injection of $W^{1,1}(\Omega) \times W^{1,1}(\Omega; \mathbb{R}^3)$ into $L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$, yields the conclusion.

We have just proved that $\{G_n\}_{n\in\mathbb{N}}$ is an equi-coercive sequence that Γ-converges to G_0 in $L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$. Therefore, by [20, Thm 7.8, Cor. 7.20], we have

$$\min_{(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)} G_0(u_b, u_c) = \lim_{n\to\infty} \inf_{(u_b, u_c) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)} G_n(u_b, u_c). \tag{3.10}$$

16

Note that by Lemma 3.1 and (1.14), the minimum on the left-hand side of (3.10) is finite, and (3.10) is equivalent to saying that

$$\min_{(u_b, u_c) \in X} \left(F^{reg, sc^-}(u_b, u_c) + F^{fid}(u_b, u_c) \right) = \lim_{n \to \infty} \inf_{(u_b, u_c) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)} \left(F^{reg}(u_b, u_c) + F^{fid}_{\varepsilon_n}(u_b, u_c) \right).$$

(3.11)

Let $(u_b^n, u_c^n) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)$ be a δ_n-minimizer of the functional $\left(F^{reg} + F^{fid}_{\varepsilon_n} \right)$ in $W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2)$. Observe that (u_b^n, u_c^n) is also a δ_n-minimizer of G_n in $L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$, $G_n(u_b^n, u_c^n) = F^{reg}(u_b^n, u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n, u_c^n)$, and, by (3.11),

$$\min_{(u_b, u_c) \in X} \left(F^{reg, sc^-}(u_b, u_c) + F^{fid}(u_b, u_c) \right) = \lim_{n \to \infty} \left(F^{reg}(u_b^n, u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n, u_c^n) \right).$$

(3.12)

Using (3.9) and the fact that the minimum on the left-hand side of (3.12) is finite, we deduce that $\{(u_b^n, u_c^n)\}_{n \in \mathbb{N}}$ is sequentially, relatively compact with respect to the weak-$\star$ convergence in $BV(\Omega) \times BV(\Omega; \mathbb{R}^3)$ and its cluster points belong to X. Let $(\bar{u}_b, \bar{u}_c) \in X$ be a cluster point of $\{(u_b^n, u_c^n)\}_{n \in \mathbb{N}}$. Then, by Step 1 and (3.12),

$$F^{reg, sc^-}(\bar{u}_b, \bar{u}_c) + F^{fid}(\bar{u}_b, \bar{u}_c) = G_0(\bar{u}_b, \bar{u}_c) \leqslant \liminf_{n \to \infty} G_n(u_b^n, u_c^n) = \liminf_{n \to \infty} \left(F^{reg}(u_b^n, u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n, u_c^n) \right)$$

$$= \min_{(u_b, u_c) \in X} \left(F^{reg, sc^-}(u_b, u_c) + F^{fid}(u_b, u_c) \right) \leqslant F^{reg, sc^-}(\bar{u}_b, \bar{u}_c) + F^{fid}(\bar{u}_b, \bar{u}_c).$$

Thus, $(\bar{u}_b, \bar{u}_c)$ is a minimizer of $(F^{reg, sc^-} + F^{fid})$ in X and $F^{reg, sc^-}(\bar{u}_b, \bar{u}_c) + F^{fid}(\bar{u}_b, \bar{u}_c) = \lim_{n \to \infty} \left(F^{reg}(u_b^n, u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n, u_c^n) \right)$. This concludes the proof. $\qquad\square$

4 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2 and is organized as follows. In Subsection 4.1, we state some properties concerning the densities $Q_T f$ and K characterizing the functional in (1.20). In Subsection 4.2, we collect several auxiliary results, which will be used to establish the integral representation for $\mathcal{F}$ stated in Theorem 1.2. A lower bound for the latter is proved in Subsection 4.3 and an upper bound in Subsection 4.4.

To simplify the notation, throughout the present section we will drop the indices b and c, referring to brightness and chromaticity, respectively, and we replace u_b by u, u_c by v, W_b^c by W_u^c, and W_c^c by W_v^c. Also, we recall that $\Omega \subset \mathbb{R}^2$ is an open, bounded set with Lipschitz boundary $\partial\Omega$.

4.1 Properties of $Q_T f$ and K

We start this subsection by proving some properties of $\mathcal{Q}_T f$ (see (1.15)). Given $s \in S^2$ and $\eta \in \mathbb{R}^{3 \times 2}$ (respectively, $\eta \in \mathbb{R}^3$), set (cf. [18])

$$P_s \eta := (\mathbb{I}_{3 \times 3} - s \otimes s)\eta,$$

(4.1)

which defines a projection of $\mathbb{R}^{3 \times 2}$ onto $[T_s(S^2)]^2$ (respectively, of $\mathbb{R}^3$ onto $T_s(S^2)$). Note that if $\eta \in T_s(S^2) \cup [T_s(S^2)]^2$, then $P_s \eta = \eta$ and $|P_s \eta| \leqslant \sqrt{2}|\eta|$. Let $\tilde{f} : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 2} \to [0, \infty)$ be the function defined, for $(r, s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 2}$, by

$$\tilde{f}(r, s, \xi, \eta) := \begin{cases} f(\tilde{r}, \tilde{s}, \xi, P_{\tilde{s}} \eta) \, \phi(|s|) & \text{if } s \in \mathbb{R}^3 \backslash \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

(4.2)

where

$$\tilde{r} := \begin{cases} \alpha & \text{if } r \leqslant \alpha, \\ r & \text{if } \alpha \leqslant r \leqslant \beta, \\ \beta & \text{if } r \geqslant \beta, \end{cases} \qquad \tilde{s} := \frac{s}{|s|},$$

(4.3)

17

and $\phi \in C^\infty(\mathbb{R}; [0, 1])$ is a cut-off function such that

$$\phi(t) = 1 \text{ if } t \geqslant 1, \qquad \phi(t) = 0 \text{ if } t \leqslant \frac{3}{4}. \tag{4.4}$$

Note that for all $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$, we have that

$$\tilde{f}(r, s, \xi, \eta) = f(r, s, \xi, \eta). \tag{4.5}$$

We observe also that $\tilde{f}_{|[\alpha,\beta] \times S^2 \times \mathbb{R}^2 \times \mathbb{R}^{3\times2}}$ plays the role of the function introduced in [18, (1.4)] and, as stated next, an analogous result to [18, Prop. 2.2 (ii)] providing an alternative characterization of $\mathcal{Q}_T f$ holds.

Lemma 4.1.   *For all $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$, we have that*

$$\mathcal{Q}_T f(r, s, \xi, \eta) = \mathcal{Q}\tilde{f}(r, s, \xi, \eta), \tag{4.6}$$

where

$$\mathcal{Q}\tilde{f}(r, s, \xi, \eta) := \inf \left\{ \int_Q \tilde{f}(r, s, \xi + \nabla\varphi(y), \eta + \nabla\psi(y)) \, \mathrm{d}y \colon \varphi \in W_0^{1,\infty}(Q), \ \psi \in W_0^{1,\infty}(Q; \mathbb{R}^3) \right\}.$$

Proof. Fix $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$.

Let $\varphi \in W_0^{1,\infty}(Q)$ and $\psi \in W_0^{1,\infty}(Q; T_s(S^2))$ be given. Then, in particular, $\psi \in W_0^{1,\infty}(Q; \mathbb{R}^3)$ and $P_s \circ \nabla\psi = \nabla\psi$; hence

$$\int_Q f(r, s, \xi + \nabla\varphi, \eta + \nabla\psi) \, \mathrm{d}y = \int_Q f(r, s, \xi + \nabla\varphi, P_s \circ (\eta + \nabla\psi)) \, \mathrm{d}y$$

$$= \int_Q \tilde{f}(r, s, \xi + \nabla\varphi, \eta + \nabla\psi) \, \mathrm{d}y \geqslant \mathcal{Q}\tilde{f}(r, s, \xi, \eta).$$

Taking the infimum over all admissible ψ and φ, we get $\mathcal{Q}_T f(r, s, \xi, \eta) \geqslant \mathcal{Q}\tilde{f}(r, s, \xi, \eta)$.

Conversely, let $\varphi \in W_0^{1,\infty}(Q)$ and $\bar{\psi} \in W_0^{1,\infty}(Q; \mathbb{R}^3)$ be given. Define $\psi := P_s \circ \bar{\psi}$. Then, $\psi \in W_0^{1,\infty}(Q; T_s(S^2))$ and $\nabla\psi = P_s \circ \nabla\bar{\psi}$. Thus,

$$\int_Q \tilde{f}(r, s, \xi + \nabla\varphi, \eta + \nabla\bar{\psi}) \, \mathrm{d}y = \int_Q f(r, s, \xi + \nabla\varphi, P_s \circ (\eta + \nabla\bar{\psi})) \, \mathrm{d}y$$

$$= \int_Q f(r, s, \xi + \nabla\varphi, \eta + \nabla\psi) \, \mathrm{d}y \geqslant \mathcal{Q}_T f(r, s, \xi, \eta).$$

Taking the infimum over all such φ and $\bar{\psi}$, we get $\mathcal{Q}\tilde{f}(r, s, \xi, \eta) \geqslant \mathcal{Q}_T f(r, s, \xi, \eta)$, which concludes the proof of (4.6). $\qquad\square$

Remark 4.2.   Arguing exactly as at the end of Subsection 2.4, there does not exist $(r, s) \in [\alpha, \beta] \times S^2$ for which

$$(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^{3\times2} \mapsto \tilde{f}(r, s, \xi, \eta) \in \mathbb{R}^+$$

is quasiconvex. Consequently, given $(r, s) \in [\alpha, \beta] \times S^2$,

$$(\xi, \eta) \in \mathbb{R}^2 \times [T_s(S^2)]^2 \mapsto f(r, s, \xi, \eta) \in \mathbb{R}^+$$

is not tangential quasiconvex; that is, there exists $(\bar{\xi}, \bar{\eta}) \in \mathbb{R}^2 \times [T_s(S^2)]^2$ such that $f(r, s, \bar{\xi}, \bar{\eta}) \neq \mathcal{Q}_T f(r, s, \bar{\xi}, \bar{\eta})$. In fact, if $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^{3\times2}$ and $(\varphi, \psi) \in W_0^{1,\infty}(Q) \times W_0^{1,\infty}(Q; \mathbb{R}^3)$ are such that $\tilde{f}(r, s, \xi, \eta) > \int_Q \tilde{f}(r, s, \xi + \nabla\varphi(y), \eta + \nabla\psi(y)) \, \mathrm{d}y$, with $(r, s) \in [\alpha, \beta] \times S^2$, then $(\bar{\xi}, \bar{\eta}) := (\xi, P_s \eta) \in \mathbb{R}^2 \times [T_s(S^2)]^2$ and $(\bar{\varphi}, \bar{\psi}) := (\varphi, P_s \circ \psi) \in W_0^{1,\infty}(Q) \times W_0^{1,\infty}(Q; T_s(S^2))$ are such that $f(r, s, \bar{\xi}, \bar{\eta}) > \int_Q f(r, s, \bar{\xi} + \nabla\bar{\varphi}(y), \bar{\eta} + \nabla\bar{\psi}(y)) \, \mathrm{d}y$.

We now establish some properties of $\tilde{f}$, $\mathcal{Q}\tilde{f}$, and $(\mathcal{Q}\tilde{f})^\infty$ that will be useful in what follows, where

$$(\mathcal{Q}\tilde{f})^\infty(r,s,\xi,\eta) := \limsup_{t\to+\infty} \frac{\mathcal{Q}\tilde{f}(r,s,t\xi,t\eta)}{t}$$

for $(r,s,\xi,\eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$. We first observe that since the application $s \in \mathbb{R}^3 \mapsto s \otimes s \in \mathbb{R}^{3\times 3}$ is locally Lipschitz, there exists a positive constant, $c_\otimes$, such that for all $s, \bar{s} \in \overline{B(0,1)}$, it holds

$$|s \otimes s - \bar{s} \otimes \bar{s}| \leqslant c_\otimes |s - \bar{s}|. \tag{4.7}$$

Lemma 4.3. *For all $(r,s,\xi,\eta) \in [\alpha,\beta] \times S^2 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$, we have that*

$$\frac{1}{2}|\xi| + \frac{\alpha}{2}|P_s\eta| \leqslant \tilde{f}(r,s,\xi,\eta) \leqslant 2|\xi| + \sqrt{2}(1+\beta)|\eta|, \tag{4.8}$$

$$\frac{1}{2}|\xi| + \frac{\alpha}{2}|P_s\eta| \leqslant \mathcal{Q}\tilde{f}(r,s,\xi,\eta) \leqslant 2|\xi| + \sqrt{2}(1+\beta)|\eta|. \tag{4.9}$$

Moreover, there exists a positive constant, c, depending only on α, β, $c_\otimes$, and $Lip(g)$, such that for all r, $\bar{r} \in [\alpha,\beta]$, s, $\bar{s} \in S^2$, ξ, $\bar{\xi} \in \mathbb{R}^2$, $\eta \in T_s(S^2)$, and $\bar{\eta} \in T_{\bar{s}}(S^2)$, one has

$$\mathcal{Q}\tilde{f}(r,s,\xi,\eta) \leqslant \mathcal{Q}\tilde{f}(\bar{r},\bar{s},\xi,\eta) + c(|r-\bar{r}| + |s-\bar{s}|)(|\xi| + |\eta|), \tag{4.10}$$

$$\left| \mathcal{Q}\tilde{f}(r,s,\xi,\eta) - \mathcal{Q}\tilde{f}(\bar{r},\bar{s},\bar{\xi},\bar{\eta}) \right|$$
$$\leqslant c|\eta - \bar{\eta}| + c(|r-\bar{r}| + |s-\bar{s}| + |\xi-\bar{\xi}|)(1 + |\xi| + |\bar{\xi}| + |\eta| + |\bar{\eta}|), \tag{4.11}$$

$$(\mathcal{Q}\tilde{f})^\infty(r,s,\xi,\eta) \leqslant (\mathcal{Q}\tilde{f})^\infty(\bar{r},\bar{s},\xi,\eta) + c(|r-\bar{r}| + |s-\bar{s}|)(|\xi| + |\bar{\eta}|). \tag{4.12}$$

Proof. Fix $(r,s,\xi,\eta) \in [\alpha,\beta] \times S^2 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$. We have that

$$\tilde{f}(r,s,\xi,\eta) = |\xi| + g(|\xi|)|P_s\eta| + |rP_s\eta + s \otimes \xi| \leqslant |\xi| + \sqrt{2}|\eta| + \sqrt{2}|r||\eta| + |\xi| \leqslant 2|\xi| + \sqrt{2}(1+\beta)|\eta|,$$

where we used the fact that $g \leqslant 1$. On the other hand,

$$\alpha|P_s\eta| \leqslant |rP_s\eta + s \otimes \xi| + |s \otimes \xi| = |rP_s\eta + s \otimes \xi| + |\xi|,$$

which, together with the fact that $g > 0$, yields

$$\tilde{f}(r,s,\xi,\eta) \geqslant |\xi| + |rP_s\eta + s \otimes \xi| \geqslant \frac{\alpha}{2}|P_s\eta| + \frac{1}{2}|\xi|.$$

This concludes the proof of (4.8). Then, (4.9) follows from (4.8) taking into account that the lower and upper bounds for $\tilde{f}$ in (4.8) are quasiconvex functions (with respect to the pair (ξ,η)).

Next, we establish (4.10)–(4.12). Let r, $\bar{r} \in [\alpha,\beta]$, s, $\bar{s} \in S^2$, ξ, $\bar{\xi} \in \mathbb{R}^2$, $\tilde{\eta} \in \mathbb{R}^{3\times 2}$, $\eta \in T_s(S^2)$, and $\bar{\eta} \in T_{\bar{s}}(S^2)$ be given. To simplify the notation, c represents a positive constant that depends only on α, β, $c_\otimes$, and $Lip(g)$ and whose value may change from one instance to another. We divide the proof into three steps.

Step 1. We show that

$$\left| \mathcal{Q}\tilde{f}(r,s,\xi,P_s\tilde{\eta}) - \mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\tilde{\eta}) \right| \leqslant c|s-\bar{s}||\tilde{\eta}|. \tag{4.13}$$

Fix $\varepsilon > 0$, and let $\varphi_\varepsilon \in W_0^{1,\infty}(Q)$ and $\psi_\varepsilon \in W_0^{1,\infty}(Q;\mathbb{R}^3)$ be such that

$$\mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\tilde{\eta}) + \varepsilon \geqslant \int_Q \tilde{f}(r,s,\xi + \nabla\varphi_\varepsilon, P_{\bar{s}}\tilde{\eta} + \nabla\psi_\varepsilon) \, dy.$$

Using the facts that $0 < g(\cdot) \leqslant 1$, $0 < \alpha \leqslant r \leqslant \beta$, the linearity of $P_s\cdot$, and the estimate $|P_s\eta| \leqslant \sqrt{2}|\eta|$, in this order, we get

$$\mathcal{Q}\tilde{f}(r,s,\xi,P_s\tilde{\eta}) - \mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\tilde{\eta})$$

19

$$\leqslant \int_Q \tilde{f}(r,s,\xi+\nabla\varphi_\varepsilon, P_s\tilde{\eta}+\nabla\psi_\varepsilon)\,\mathrm{d}y - \int_Q \tilde{f}(r,s,\xi+\nabla\varphi_\varepsilon, P_{\bar{s}}\tilde{\eta}+\nabla\psi_\varepsilon)\,\mathrm{d}y + \varepsilon$$

$$= \int_Q \Big[f(r,s,\xi+\nabla\varphi_\varepsilon, P_s\circ(P_s\tilde{\eta}+\nabla\psi_\varepsilon)) - f(r,s,\xi+\nabla\varphi_\varepsilon, P_s\circ(P_{\bar{s}}\tilde{\eta}+\nabla\psi_\varepsilon)) \Big]\,\mathrm{d}y + \varepsilon$$

$$\leqslant \int_Q (1+\beta)\big|P_s\circ(P_s\tilde{\eta}+\nabla\psi_\varepsilon)) - P_s\circ(P_{\bar{s}}\tilde{\eta}+\nabla\psi_\varepsilon))\big|\,\mathrm{d}y + \varepsilon = \int_Q (1+\beta)\big|P_s(P_s\tilde{\eta} - P_{\bar{s}}\tilde{\eta})\big|\,\mathrm{d}y + \varepsilon$$

$$\leqslant \sqrt{2}(1+\beta)|P_s\tilde{\eta} - P_{\bar{s}}\tilde{\eta}| + \varepsilon = \sqrt{2}(1+\beta)|(s\otimes s - \bar{s}\otimes\bar{s})\tilde{\eta}| + \varepsilon \leqslant \sqrt{2}c_\otimes(1+\beta)|s-\bar{s}||\tilde{\eta}| + \varepsilon.$$

Letting $\varepsilon \to 0^+$ first and then interchanging the roles of s and $\bar{s}$, we conclude (4.13).

Step 2. We establish (4.10) and (4.12).

By Step 1, applied to $\tilde{\eta} := \eta = P_s\eta$,

$$\mathcal{Q}\tilde{f}(r,s,\xi,\eta) \leqslant \mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\eta) + c|s-\bar{s}||\eta|. \tag{4.14}$$

Next, we estimate $\mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\eta)$ in terms of $\mathcal{Q}\tilde{f}(\bar{r},\bar{s},\xi,P_{\bar{s}}\eta)$. Using (4.6), for all $\varepsilon > 0$, we can find $\varphi_\varepsilon \in W_0^{1,\infty}(Q)$ and $\psi_\varepsilon \in W_0^{1,\infty}(Q;T_{\bar{s}}(S^2))$ such that

$$\mathcal{Q}\tilde{f}(\bar{r},\bar{s},\xi,P_{\bar{s}}\eta) + \varepsilon = \mathcal{Q}_T f(\bar{r},\bar{s},\xi,P_{\bar{s}}\eta) + \varepsilon \geqslant \int_Q f(\bar{r},\bar{s},\xi+\nabla\varphi_\varepsilon, P_{\bar{s}}\eta+\nabla\psi_\varepsilon)\,\mathrm{d}y$$

$$= \int_Q \tilde{f}(\bar{r},\bar{s},\xi+\nabla\varphi_\varepsilon, P_{\bar{s}}\eta+\nabla\psi_\varepsilon)\,\mathrm{d}y,$$

where in the last equality we used the fact that $P_{\bar{s}}\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon) = P_{\bar{s}}\eta+\nabla\psi_\varepsilon$. In particular, in view of (4.8), (4.9), and the inequality $|P_{\bar{s}}\eta| \leqslant \sqrt{2}|\eta|$, we get

$$2|\xi| + \sqrt{2}(1+\beta)|\eta| + \varepsilon \geqslant \int_Q \frac{1}{2}|\xi+\nabla\varphi_\varepsilon| + \frac{\alpha}{2}|P_{\bar{s}}\eta+\nabla\psi_\varepsilon|\,\mathrm{d}y.$$

Thus,

$$\max\left\{ \int_Q |\xi+\nabla\varphi_\varepsilon|\,\mathrm{d}y, \int_Q |P_{\bar{s}}\eta+\nabla\psi_\varepsilon|\,\mathrm{d}y \right\} \leqslant c(|\xi|+|\eta|+\varepsilon). \tag{4.15}$$

Moreover, using the fact that $0 < g(\cdot) \leqslant 1$, the estimates $|P_s\eta - P_{\bar{s}}\eta| \leqslant c_\otimes|s-\bar{s}||\eta|$ and (4.15), and the identity $P_{\bar{s}}\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon) = P_{\bar{s}}\eta+\nabla\psi_\varepsilon$, in this order, we obtain

$$\mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\eta) - \mathcal{Q}\tilde{f}(\bar{r},\bar{s},\xi,P_{\bar{s}}\eta)$$

$$\leqslant \int_Q \tilde{f}(r,s,\xi+\nabla\varphi_\varepsilon, P_{\bar{s}}\eta+\nabla\psi_\varepsilon)\,\mathrm{d}y - \int_Q \tilde{f}(\bar{r},\bar{s},\xi+\nabla\varphi_\varepsilon, P_{\bar{s}}\eta+\nabla\psi_\varepsilon)\,\mathrm{d}y + \varepsilon$$

$$= \int_Q f(r,s,\xi+\nabla\varphi_\varepsilon, P_s\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon)) - f(\bar{r},\bar{s},\xi+\nabla\varphi_\varepsilon, P_{\bar{s}}\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon))\,\mathrm{d}y + \varepsilon$$

$$\leqslant \int_Q c_\otimes|s-\bar{s}||P_{\bar{s}}\eta+\nabla\psi_\varepsilon| + \big|rP_s\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon)) - \bar{r}P_{\bar{s}}\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon)) + (s-\bar{s})\otimes(\xi+\nabla\varphi_\varepsilon)\big|\,\mathrm{d}y + \varepsilon$$

$$\leqslant c(c_\otimes+1)|s-\bar{s}|(|\xi|+|\eta|+\varepsilon) + \int_Q \big|rP_s\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon)) \mp rP_{\bar{s}}\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon))$$

$$-\bar{r}P_{\bar{s}}\circ(P_{\bar{s}}\eta+\nabla\psi_\varepsilon))\big|\,\mathrm{d}y + \varepsilon$$

$$\leqslant c|s-\bar{s}|(|\xi|+|\eta|+\varepsilon) + c|s-\bar{s}|\int_Q |P_{\bar{s}}\eta+\nabla\psi_\varepsilon|\,\mathrm{d}y + |r-\bar{r}|\int_Q |P_{\bar{s}}\eta+\nabla\psi_\varepsilon|\,\mathrm{d}y + \varepsilon$$

$$\leqslant c(|r-\bar{r}|+|s-\bar{s}|)(|\xi|+|\eta|+\varepsilon) + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we conclude that

$$\mathcal{Q}\tilde{f}(r,s,\xi,P_{\bar{s}}\eta) \leqslant \mathcal{Q}\tilde{f}(\bar{r},\bar{s},\xi,P_{\bar{s}}\eta) + c(|r-\bar{r}|+|s-\bar{s}|)(|\xi|+|\eta|). \tag{4.16}$$

20

Finally, interchanging the roles of (r,s) and $(\bar{r}, \bar{s})$, we conclude (4.10).

Property (4.12) follows from (4.10) and the definition of $(\mathcal{Q}\tilde{f})^\infty$.

Step 3. We show that (4.11) holds true.

Arguing as in the previous steps, we have

$$|Q\tilde{f}(\bar{r}, \bar{s}, \xi, \eta) - Q\tilde{f}(\bar{r}, \bar{s}, \bar{\xi}, \eta)| \leqslant c|\xi - \bar{\xi}|(1 + |\xi| + |\bar{\xi}| + |\eta|)$$

and

$$|Q\tilde{f}(\bar{r}, \bar{s}, \bar{\xi}, \eta) - Q\tilde{f}(\bar{r}, \bar{s}, \bar{\xi}, \bar{\eta})| \leqslant c|\eta - \bar{\eta}|.$$

Using these two estimates together with (4.10), we obtain (4.11). $\qquad\square$

Next, we show that for each $\varepsilon > 0$, the function H_ε defined by

$$H_\varepsilon(r,s,\xi,\eta) := \mathcal{Q}\tilde{f}(r,s,\xi,\eta) + \varepsilon(|\xi| + |\eta|), \quad (r,s,\xi,\eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}, \tag{4.17}$$

satisfies hypotheses (H1)–(H4) of [26]. These integrands will play an important role in the proof of the lower bound for $\mathcal{F}$.

Proposition 4.4. *The function $H_\varepsilon : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \to [0, +\infty)$ defined in (4.17) satisfies the following conditions:*

   (i) H_ε is continuous;

   (ii) $H_\varepsilon(r, s, \cdot, \cdot)$ is quasiconvex for all $(r,s) \in \mathbb{R} \times \mathbb{R}^3$;

   (iii) there exists a positive constant, C, depending only on β, such that for all $0 < \varepsilon \leqslant 1$ and $(r,s,\xi,\eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$,

$$\varepsilon(|\xi| + |\eta|) \leqslant H_\varepsilon(r,s,\xi,\eta) \leqslant C(|\xi| + |\eta|);$$

   (iv) for every compact set $\mathfrak{V} \subset \mathbb{R} \times \mathbb{R}^3$, there exists a positive constant, $C_{\mathfrak{V}}$, depending only on $\mathfrak{V}$, such that for all $(r,s,\xi,\eta), (\bar{r}, \bar{s}, \xi, \eta) \in \mathfrak{V} \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$,

$$|H_\varepsilon(r,s,\xi,\eta) - H_\varepsilon(\bar{r}, \bar{s}, \xi, \eta)| \leqslant C_{\mathfrak{V}}(|r - \bar{r}| + |s - \bar{s}|)(1 + |\xi| + |\eta|).$$

Proof. Conditions *(ii)* and *(iii)* follow from the definition of H_ε and from (4.9). To deduce *(i)* and *(iv)* it suffices to observe that on the one hand,

$$\mathcal{Q}\tilde{f}(r,s,\xi,\eta) = 0$$

for all $(r,s,\xi,\eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$ such that $|s| \leqslant \frac{3}{4}$ in view of the definition of $\tilde{f}$ (see (4.2) and (4.4)). On the other hand, if $(r,s,\xi,\eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$ is such that $|s| > \frac{1}{4}$, then

$$\mathcal{Q}\tilde{f}(r,s,\xi,\eta) = \phi(|s|)\mathcal{Q}\tilde{f}(\tilde{r}, \tilde{s}, \xi, \eta) = \phi(|s|)\mathcal{Q}\tilde{f}(\tilde{r}, \tilde{s}, \xi, P_{\tilde{s}}\eta).$$

Hence, to conclude, it suffices to use the local Lipschitz continuity in $\mathbb{R} \times (\mathbb{R}^3 \backslash \{0\})$ of $(\tilde{r}, \tilde{s})$ defined in (4.3) as function of (r,s), (4.11), the estimates $|P_{\tilde{s}}\eta - P_{\tilde{\tilde{s}}}\eta| \leqslant c_\otimes |\tilde{s} - \tilde{\tilde{s}}||\eta|$ and $|P_{\tilde{s}}\eta| \leqslant \sqrt{2}|\eta|$, the Lipschitz continuity of ϕ, and (4.9). $\qquad\square$

We now turn our attention to the *jump integrand K* defined by (1.18)–(1.19). We prove that an analogous result to [26, Lemma 2.15] (see also [2, Lemma 4.1]) holds even though our functions f and f^∞ do not satisfy some of the hypotheses assumed in [2, 26], such as quasiconvexity.

Lemma 4.5. *Let $K : ([\alpha, \beta] \times S^2) \times ([\alpha, \beta] \times S^2) \times S^1 \to [0, \infty)$ be the function defined by (1.18)–(1.19). Then,*

21

(a) *There exists a positive constant, C, such that for all a, a', b, $b' \in [\alpha, \beta] \times S^2$, and $\nu \in S^1$, it holds*

$$|K(a, b, \nu) - K(a', b', \nu)| \leqslant C(|a - a'| + |b - b'|).$$

(b) *For all a, $b \in [\alpha, \beta] \times S^2$, the map $\nu \in S^1 \mapsto K(a, b, \nu)$ is upper semicontinuous.*

(c) *K is upper semicontinuous in $([\alpha, \beta] \times S^2) \times ([\alpha, \beta] \times S^2) \times S^1$.*

(d) *There is a positive constant, C, such that for all a, $b \in [\alpha, \beta] \times S^2$ and $\nu \in S^1$, we have*

$$K(a, b, \nu) \leqslant C|a - b|.$$

Proof. (a) We start by proving that there is a positive constant, C, such that for all $a = (r_1, s_1)$, $b = (r_2, s_2) \in [\alpha, \beta] \times S^2$, one has

$$d_{[\alpha, \beta] \times S^2}(a, b) \leqslant C|a - b|, \tag{4.18}$$

where

$$d_{[\alpha, \beta] \times S^2}(a, b) := \inf \left\{ \int_0^1 |\gamma'(t)| \, dt \colon \gamma \in W^{1,1}((0,1); [\alpha, \beta] \times S^2),\ \gamma(0) = a,\ \gamma(1) = b \right\}$$

is the geodesic distance between a and b on $[\alpha, \beta] \times S^2$.

We claim that to prove (4.18) it suffices to prove that there is a positive constant, C, independent of s_1 and s_2, such that

$$d_{S^2}(s_1, s_2) \leqslant C|s_1 - s_2|, \tag{4.19}$$

where $d_{S^2}(s_1, s_2) := \inf\{\int_0^1 |\gamma'(t)| \, dt \colon \gamma \in W^{1,1}((0,1); S^2),\ \gamma(0) = s_1,\ \gamma(1) = s_2\}$ is the geodesic distance between s_1 and s_2 on S^2. Indeed, let $\gamma \in W^{1,1}((0,1); S^2)$ be such that $\gamma(0) = s_1$ and $\gamma(1) = s_2$. Then, $\bar{\gamma} : [0,1] \to [\alpha, \beta] \times S^2$ defined by $\bar{\gamma}(t) := ((1-t)r_1 + tr_2, \gamma(t))$, $t \in [0,1]$, belongs to $W^{1,1}((0,1); [\alpha, \beta] \times S^2)$ and satisfies $\bar{\gamma}(0) = a$ and $\bar{\gamma}(1) = b$. Moreover,

$$d_{[\alpha, \beta] \times S^2}(a, b) \leqslant \int_0^1 |\bar{\gamma}'(t)| \, dt \leqslant |r_1 - r_2| + \int_0^1 |\gamma'(t)| \, dt.$$

Thus, taking the infimum over all $\gamma \in W^{1,1}((0,1); S^2)$ with $\gamma(0) = s_1$ and $\gamma(1) = s_2$ in this estimate, (4.18) follows from (4.19).

To prove (4.19), we show first that if $|s_1 - s_2| \leqslant \frac{1}{2}$, then $d_{S^2}(s_1, s_2) \leqslant 4|s_1 - s_2|$. To prove this implication, assume that $|s_1 - s_2| \leqslant \frac{1}{2}$, and let

$$\gamma(t) := \frac{(1-t)s_1 + ts_2}{|(1-t)s_1 + ts_2|}$$

for $t \in [0,1]$. Note that $|(1-t)s_1 + ts_2| = |s_1 - t(s_1 - s_2)| \geqslant 1 - |s_1 - s_2| \geqslant \frac{1}{2}$. Moreover, γ is an admissible parameterization for $d_{S^2}(s_1, s_2)$. Hence,

$$d_{S^2}(s_1, s_2) \leqslant \int_0^1 |\gamma'(t)| \, dt \leqslant \int_0^1 \frac{2|s_1 - s_2|}{|(1-t)s_1 + ts_2|} \, dt \leqslant 4|s_1 - s_2|.$$

Therefore, if the claim (4.19) would fail, then for all $n \in \mathbb{N}$, there would exist s_1^n, $s_2^n \in S^2$, $s_1^n \neq s_2^n$, such that $d_{S^2}(s_1^n, s_2^n) > n|s_1^n - s_2^n|$. Then, because $d_{S^2}(s_1^n, s_2^n) \leqslant \pi$, we would have $|s_1^n - s_2^n| \leqslant \frac{1}{2}$ for all $n \in \mathbb{N}$ sufficiently large. In turn, by the implication proved above, for all such $n \in \mathbb{N}$, we would also have to have $d_{S^2}(s_1^n, s_2^n) \leqslant 4|s_1^n - s_2^n|$. We are thus led to a contradiction. Hence, (4.19) holds, and so does (4.18).

Note that the infimum defining $d_{[\alpha, \beta] \times S^2}(a, b)$ does not change if instead of the interval $[0, 1]$ we consider any interval $[t_1, t_2] \subset \mathbb{R}$ with $t_1 < t_2$ as the domain of the parameterizations γ. Fix $\varepsilon > 0$, and let γ_1,

$\gamma_2 \in W^{1,1}((\frac{1}{4}, \frac{1}{2}); [\alpha, \beta] \times S^2)$ be such that

$$\gamma_1\left(\frac{1}{4}\right) = b, \quad \gamma_1\left(\frac{1}{2}\right) = b', \quad \int_{\frac{1}{4}}^{\frac{1}{2}} |\gamma_1'(t)| \, dt - \varepsilon \leqslant d_{[\alpha, \beta] \times S^2}(b, b') \leqslant C|b - b'|,$$

$$\gamma_2\left(\frac{1}{4}\right) = a, \quad \gamma_2\left(\frac{1}{2}\right) = a', \quad \int_{\frac{1}{4}}^{\frac{1}{2}} |\gamma_2'(t)| \, dt - \varepsilon \leqslant d_{[\alpha, \beta] \times S^2}(a, a') \leqslant C|a - a'|. \tag{4.20}$$

Let $\vartheta = (\varphi, \psi) \in \mathcal{P}(a, b, \nu)$, and define $\vartheta^* = (\varphi^*, \psi^*) \in \mathcal{P}(a', b', \nu)$ by setting, for $y \in Q_\nu$,

$$\vartheta^*(y) := \begin{cases} \gamma_1(y \cdot \nu) & \text{if } \frac{1}{4} < y \cdot \nu < \frac{1}{2}, \\ \vartheta(2y) & \text{if } |y \cdot \nu| < \frac{1}{4}, \\ \gamma_2(-y \cdot \nu) & \text{if } -\frac{1}{2} < y \cdot \nu < -\frac{1}{4}. \end{cases}$$

Denoting by $\nu_1 \in S^1$ a fixed vector such that $\{\nu_1, \nu\}$ is an orthonormal basis of $\mathbb{R}^2$, we have that

$$\begin{aligned}
K(a', b', \nu) &\leqslant \int_{Q_\nu} f^\infty(\vartheta^*(y), \nabla\vartheta^*(y)) \, dy \\
&= \int_{|y \cdot \nu_1| < \frac{1}{2}} \int_{|y \cdot \nu| < \frac{1}{4}} f^\infty(\vartheta(2y), 2\nabla\vartheta(2y)) \, dy \\
&\quad + \int_{|y \cdot \nu_1| < \frac{1}{2}} \int_{\frac{1}{4} < y \cdot \nu < \frac{1}{2}} f^\infty(\gamma_1(y \cdot \nu), \gamma_1'(y \cdot \nu) \otimes \nu) \, dy \\
&\quad + \int_{|y \cdot \nu_1| < \frac{1}{2}} \int_{-\frac{1}{2} < y \cdot \nu < -\frac{1}{4}} f^\infty(\gamma_2(-y \cdot \nu), -\gamma_2'(-y \cdot \nu) \otimes \nu) \, dy.
\end{aligned}$$

Hence, using (1.17), (4.20), the 1-homogeneity of $f^\infty(r, s, \cdot, \cdot)$, and the 1-periodicity of ϑ in the ν_1-direction, we have

$$\begin{aligned}
K(a', b', \nu) &\leqslant \frac{1}{2} \int_{|y \cdot \nu_1| < 1} \int_{|y \cdot \nu| < \frac{1}{2}} f^\infty(\vartheta(y), \nabla\vartheta(y)) \, dy + (3 + \beta)\big(C|b - b'| + C|a - a'| + 2\varepsilon\big) \\
&= \int_{Q_\nu} f^\infty(\vartheta(y), \nabla\vartheta(y)) \, dy + (3 + \beta)\big(C|b - b'| + C|a - a'| + 2\varepsilon\big).
\end{aligned}$$

Letting $\varepsilon \to 0^+$ and taking the infimum over $\vartheta = (\varphi, \psi) \in \mathcal{P}(a, b, \nu)$, we deduce that

$$K(a, b, \nu) \leqslant K(a', b', \nu) + C(3 + \beta)\big(|b - b'| + |a - a'|\big).$$

Interchanging the roles between (a, b) and (a', b'), assertion *(a)* follows.

(b) Let $\nu_n, \nu \in S^1$, $n \in \mathbb{N}$, be such that $\lim_{n \to \infty} |\nu_n - \nu| = 0$. Fix $\varepsilon > 0$, and let $\vartheta = (\varphi, \psi) \in \mathcal{P}(a, b, \nu)$ be such that

$$\int_{Q_\nu} f^\infty(\vartheta(y), \nabla\vartheta(y)) \, dy \leqslant K(a, b, \nu) + \varepsilon.$$

Let R be a rotation such that $Re_2 = \nu$, and choose rotations R_n with $\lim_{n \to \infty} |R_n - R| = 0$ and $R_n e_2 = \nu_n$. Define $\vartheta_n \in \mathcal{P}(a, b, \nu_n)$ by setting

$$\vartheta_n(y) := \vartheta(RR_n^T y) \quad \text{for } y \in Q_{\nu_n}.$$

Then,

$$\begin{aligned}
K(a, b, \nu_n) &\leqslant \int_{Q_{\nu_n}} f^\infty(\vartheta_n(y), \nabla\vartheta_n(y)) \, dy = \int_{Q_{\nu_n}} f^\infty(\vartheta(RR_n^T y), \nabla\vartheta(RR_n^T y) RR_n^T) \, dy \\
&= \int_{Q_\nu} f^\infty(\vartheta(z), \nabla\vartheta(z) RR_n^T) \, dz.
\end{aligned}$$

Since f^∞ is upper semicontinuous, in view of (1.17) and Fatou's Lemma, we obtain

$$\limsup_{n\to\infty} K(a,b,\nu_n) \leqslant \int_{Q_\nu} \limsup_{n\to\infty} f^\infty(\vartheta(z), \nabla\vartheta(z)RR_n^T)\,\mathrm{d}z \leqslant \int_{Q_\nu} f^\infty(\vartheta(z), \nabla\vartheta(z))\,\mathrm{d}z \leqslant K(a,b,\nu) + \varepsilon.$$

To conclude, let $\varepsilon \to 0^+$.

(c) It follows from *(a)* and *(b)*.

(d) Fix $\varepsilon > 0$, and let $\gamma \in W^{1,1}((-\frac{1}{2}, \frac{1}{2}); [\alpha, \beta] \times S^2)$ be such that

$$\gamma\left(-\frac{1}{2}\right) = a, \quad \gamma\left(\frac{1}{2}\right) = b, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |\gamma'(t)|\,\mathrm{d}t - \varepsilon \leqslant d_{[\alpha,\beta]\times S^2}(a,b) \leqslant C|a-b|, \tag{4.21}$$

where C is the constant in (4.18), and define $\vartheta(y) := \gamma(y \cdot \nu)$ for $y \in Q_\nu$. Then, $\vartheta \in \mathcal{P}(a,b,\nu)$ and, arguing as in *(a)*,

$$K(a,b,\nu) \leqslant \int_{Q_\nu} f^\infty(\vartheta(y), \nabla\vartheta(y))\,\mathrm{d}y = \int_{Q_\nu} f^\infty(\gamma(y\cdot\nu), \gamma'(y\cdot\nu)\otimes\nu)\,\mathrm{d}y \leqslant (3+\beta)\int_{-\frac{1}{2}}^{\frac{1}{2}} |\gamma'(t)|\,\mathrm{d}t.$$

This estimate, together with (4.21), yields the conclusion. $\qquad\square$

4.2 Auxiliary Lemmas

As in [2], given $y \in B(0, \frac{1}{2}) \subset \mathbb{R}^3$, we define the projection function $\pi_y \colon \overline{B(0,1)}\backslash\{y\} \to S^2$ by setting

$$\pi_y(s) := y + \frac{-y\cdot(s-y) + \sqrt{(y\cdot(s-y))^2 + |s-y|^2(1-|y|^2)}}{|s-y|^2}(s-y),$$

which projects each $s \in \overline{B(0,1)}\backslash\{y\}$ onto S^2 along the direction $s-y$. We have that

$$\pi_{y|S^2} = \mathrm{Id}_{S^2}, \quad \nabla\pi_y(s) = \mathbb{I}_{3\times 3} + (s-y)\otimes\frac{1}{|s-y|^2}\left(\frac{|y|^2-1}{s\cdot y-1} - 2\right)s \text{ if } s\in S^2. \tag{4.22}$$

Note that by (4.22), if $s \in S^2$ and $w \in T_s(S^2)$, then

$$\nabla\pi_y(s)w = w. \tag{4.23}$$

Furthermore, there exists a positive constant, $\bar{C}$, independent of $y \in B(0,\frac{1}{2})$, such that for all $s \in \overline{B(0,1)}\backslash\{y\}$, we have

$$|\nabla\pi_y(s)| \leqslant \frac{\bar{C}}{|s-y|}, \qquad |\nabla^2\pi_y(s)| \leqslant \frac{\bar{C}}{|s-y|^2}. \tag{4.24}$$

Consequently, there exists a positive constant, $\overline{C}$, independent of $y \in B(0,\frac{1}{2})$, such that for all $s_1, s_2 \in \{s \in \overline{B(0,1)}\colon \mathrm{dist}(s,S^2) \leqslant \frac{1}{4}\}$, we have

$$|\pi_y(s_1) - \pi_y(s_2)| \leqslant \overline{C}|s_1-s_2|, \qquad |\nabla\pi_y(s_1) - \nabla\pi_y(s_2)| \leqslant \overline{C}|s_1-s_2|. \tag{4.25}$$

The following result holds (see also [2, Lem. 5.2 and Lem. 6.1]).

Lemma 4.6. *Let $A \in \mathcal{A}(\Omega)$, let $v \in W^{1,1}(A; \overline{B(0,1)}) \cap C^\infty(A; \mathbb{R}^3)$, and let A' be an open subset of A. Then, there exists $y \in B(0,\frac{1}{2})$, depending on v and A', such that $\pi_y \circ v \in W^{1,1}(A'; S^2) \cap C^\infty(A; S^2)$ and*

$$\int_{A'} |\nabla(\pi_y \circ v)|\,\mathrm{d}x \leqslant C_\star \int_{A'} |\nabla v|\,\mathrm{d}x, \tag{4.26}$$

where $C_\star$ is a positive constant independent of A, A', v, and y.

Proof. Since $v : A \subset \mathbb{R}^2 \to \mathbb{R}^3$ is smooth, *Mini-Sard*'s Theorem (see [28]) yields $\mathcal{L}^3(v(A)) = 0$. In particular, setting $G := \{y \in B(0, \frac{1}{2}) : \text{there exists } x \in A \text{ such that } v(x) = y\}$, then $\mathcal{L}^3(G) = 0$. Moreover, for all $y \in B(0, \frac{1}{2}) \backslash G$, the function $\pi_y \circ v$ belongs to $C^\infty(A; S^2)$ and, by Fubini's Theorem and the first estimate in (4.24),

$$
\int_{B(0,\frac{1}{2})} \int_{A'} |\nabla(\pi_y(v(x)))| \, \mathrm{d}x \, \mathrm{d}y = \int_{B(0,\frac{1}{2})} \int_{A'} |(\nabla \pi_y)(v(x)) \nabla v(x)| \, \mathrm{d}x \, \mathrm{d}y
$$
$$
\leqslant \bar{C} \int_{A'} \left(|\nabla v(x)| \int_{B(0,\frac{1}{2})} \frac{1}{|v(x) - y|} \, \mathrm{d}y \right) \mathrm{d}x. \tag{4.27}
$$

For fixed $x \in A'$, use the change of variables $z = y - v(x)$ to get

$$
\int_{B(0,\frac{1}{2})} \frac{1}{|v(x) - y|} \, \mathrm{d}y = \int_{B(-v(x),\frac{1}{2})} \frac{1}{|z|} \, \mathrm{d}z \leqslant \int_{B(0,\frac{3}{2})} \frac{1}{|z|} \, \mathrm{d}z =: c_1 \in \mathbb{R}, \tag{4.28}
$$

where we used the fact that $\|v\|_{L^\infty(A)} \leqslant 1$. From (4.27) and (4.28), we conclude that

$$
\int_{B(0,\frac{1}{2})} \int_{A'} |\nabla(\pi_y(v(x)))| \, \mathrm{d}x \, \mathrm{d}y \leqslant c_1 \bar{C} \int_{A'} |\nabla v(x)| \, \mathrm{d}x.
$$

Consequently, we can find $y \in B(0, \frac{1}{2}) \backslash G$ such that (4.26) holds with $C_\star := c_1 \bar{C} / \mathcal{L}^3(B(0, \frac{1}{2}))$. Finally, we observe that for such y, we have $\pi_y \circ v \in W^{1,1}(A'; S^2) \cap C^\infty(A; S^2)$. $\qquad \square$

Lemma 4.7. *Let $A \in \mathcal{A}_\infty(\Omega)$ and $w = (u, v) \in BV(A; [\alpha, \beta] \times S^2)$. Then, there exists a sequence $\{\bar{w}_n\}_{n \in \mathbb{N}} \subset W^{1,1}(A; [\alpha, \beta] \times S^2) \cap C^\infty(A; \mathbb{R} \times \mathbb{R}^3)$ such that $\bar{w}_n = w$ on ∂A for all $n \in \mathbb{N}$, $\lim_{n \to \infty} \|\bar{w}_n - w\|_{L^1(A; \mathbb{R} \times \mathbb{R}^3)} = 0$, and $\limsup_{n \to \infty} \int_A |\nabla \bar{w}_n(x)| \, \mathrm{d}x \leqslant \widetilde{C} |Dw|(A)$, where $\widetilde{C}$ is a positive constant only depending on $\bar{C}$, $\overline{C}$, and $C_\star$.*

Proof. Because $w = (u, v)$ takes values on $[\alpha, \beta] \times S^2$, its mollification (see (2.2)) takes values on $[\alpha, \beta] \times \overline{B(0,1)}$. Thus, by [27, Thm. 2.17, Rmk. 1.18] (see also [27, Rmk. 2.12]), there exists a sequence $\{w_n\}_{n \in \mathbb{N}} \subset W^{1,1}(A; [\alpha, \beta] \times \overline{B(0,1)}) \cap C^\infty(A; \mathbb{R} \times \mathbb{R}^3)$ such that

$$
w_n = w \text{ on } \partial A \text{ for all } n \in \mathbb{N}, \quad w_n \overset{\star}{\rightharpoonup} w \text{ weakly-} \star \text{ in } BV(A; \mathbb{R} \times \mathbb{R}^3) \text{ as } n \to \infty,
$$
$$
\lim_{n \to \infty} \int_A |\nabla w_n(x)| \, \mathrm{d}x = |Dw|(A). \tag{4.29}
$$

We write $w_n(\cdot) = (u_n(\cdot), v_n(\cdot)) \in [\alpha, \beta] \times \overline{B(0,1)}$ $\mathcal{L}^2$-a.e. in A. Fix $\delta_0 \in (0, \frac{1}{4})$, and set $A_n := \{x \in A : \text{dist}(v_n(x), S^2) > \delta_0\}$. By Lemma 4.6 applied to A, v_n, and A_n, we can find $y_n \in B(0, \frac{1}{2})$ such that $\pi_{y_n} \circ v_n \in C^\infty(A; S^2)$ and

$$
\int_{A_n} |\nabla(\pi_{y_n} \circ v_n)| \, \mathrm{d}x \leqslant C_\star \int_{A_n} |\nabla v_n| \, \mathrm{d}x.
$$

Using the first estimate in (4.24), we obtain

$$
\int_{A \backslash A_n} |\nabla(\pi_{y_n} \circ v_n)| \, \mathrm{d}x \leqslant \int_{A \backslash A_n} \frac{\bar{C}}{|v_n - y_n|} |\nabla v_n| \, \mathrm{d}x \leqslant 4\bar{C} \int_{A \backslash A_n} |\nabla v_n| \, \mathrm{d}x,
$$

and so

$$
\int_A |\nabla(\pi_{y_n} \circ v_n)| \, \mathrm{d}x \leqslant \max\{C_\star, 4\bar{C}\} \int_A |\nabla v_n| \, \mathrm{d}x. \tag{4.30}
$$

Setting $\bar{w}_n := (u_n, \pi_{y_n} \circ v_n)$, we have that $\{\bar{w}_n\}_{n \in \mathbb{N}} \subset W^{1,1}(A; [\alpha, \beta] \times S^2) \cap C^\infty(A; \mathbb{R} \times \mathbb{R}^3)$ is a bounded sequence in $W^{1,1}(A; [\alpha, \beta] \times S^2)$. Moreover, using (4.22), the first estimate in (4.25), and the fact that $v(\cdot) \in S^2$ (so that $\pi_{y_n} \circ v = v$) for $\mathcal{L}^2$-a.e. in A, and the estimate $|\pi_{y_n} \circ v_n - v| \leqslant 2 \leqslant 2/\delta_0 |v_n - v|$ in A_n, we obtain

$$
\int_A |\pi_{y_n} \circ v_n - v| \, \mathrm{d}x = \int_{A_n} |\pi_{y_n} \circ v_n - v| \, \mathrm{d}x + \int_{A \backslash A_n} |\pi_{y_n} \circ v_n - \pi_{y_n} \circ v| \, \mathrm{d}x
$$

25

$$\leqslant 2\mathcal{L}^2(A_n) + \overline{C}\int_{A\setminus A_n}|v_n - v|\,\mathrm{d}x \leqslant \frac{2}{\delta_0}\int_{A_n}|v_n - v|\,\mathrm{d}x + \overline{C}\int_{A\setminus A_n}|v_n - v|\,\mathrm{d}x$$

$$\leqslant \max\left\{\frac{2}{\delta_0}, \overline{C}\right\}\int_A |v_n - v|\,\mathrm{d}x. \tag{4.31}$$

In view of (4.29)–(4.31), we conclude that $\{\bar{w}_n\}_{n\in\mathbb{N}}$ satisfies the requirements stated in Lemma 4.7. $\qquad\square$

Remark 4.8. If $A \in \mathcal{A}(\Omega)$ is of the form $A = A_1\setminus\overline{A_0}$, where $A_1 \in \mathcal{A}(\Omega)$, $A_0 \in \mathcal{A}_\infty(\Omega)$, and $A_0 \subset\subset A_1$, then a simple adaptation of the proof above yields the existence of a sequence as in Lemma 4.7 with the trace condition only holding on ∂A_0; that is, the trace condition becomes "$\bar{w}_n = w$ on ∂A_0'.

The next lemma is a simplified version of a result proved in [12] (see also [2, Thm. 2.2]), which will be useful in the subsequent slicing result.

Lemma 4.9. *Let Ω be an open subset of $\mathbb{R}^2$. The space $W^{1,1}(\Omega; S^2) \cap C^\infty(\Omega; S^2)$ is dense in $W^{1,1}(\Omega; S^2)$ with respect to the $W^{1,1}(\Omega; \mathbb{R}^3)$-norm.*

Lemma 4.10. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded set and $\mathfrak{h} : [\alpha,\beta] \times S^2 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2} \to [0,+\infty)$ be an upper semicontinuous function satisfying, for some $C > 0$ and for all $(r,s,\xi,\eta) \in [\alpha,\beta] \times S^2 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$,*

$$0 \leqslant \mathfrak{h}(r,s,\xi,\eta) \leqslant C(1 + |\xi| + |\eta|). \tag{4.32}$$

Let $A \in \mathcal{A}_\infty(\Omega)$, $w = (u,v) \in BV(A; [\alpha,\beta] \times S^2)$, and $w_n = (u_n, v_n) \in W^{1,1}(A; [\alpha,\beta] \times S^2)$, $n \in \mathbb{N}$, be such that $\lim_{n\to\infty}\|w_n - w\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)} = 0$. Then, for all $n \in \mathbb{N}$, there exists $\tilde{w}_n = (\tilde{u}_n, \tilde{v}_n) \in W^{1,1}(A; [\alpha,\beta] \times S^2)$ satisfying

$$\lim_{n\to\infty}\|\tilde{w}_n - w\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \tilde{w}_n = w \text{ on } \partial A, \tag{4.33}$$

$$\limsup_{n\to\infty}\int_A \mathfrak{h}(\tilde{u}_n, \tilde{v}_n, \nabla\tilde{u}_n, \nabla\tilde{v}_n)\,\mathrm{d}x \leqslant \liminf_{n\to\infty}\int_A \mathfrak{h}(u_n, v_n, \nabla u_n, \nabla v_n)\,\mathrm{d}x. \tag{4.34}$$

Proof. In view of the hypotheses on $\mathfrak{h}$, by Fatou's Lemma, Lemma 4.9, and using a diagonalization argument, we may assume that the component v_n of w_n belongs to $W^{1,1}(A; S^2) \cap C^\infty(A; S^2)$.

Extracting a subsequence, if needed, we may assume without loss of generality that the limit inferior on the right-hand side of (4.34) is a limit. By Lemma 4.7, there exists a sequence $\{\bar{w}_n\}_{n\in\mathbb{N}} = \{(\bar{u}_n, \bar{v}_n)\}_{n\in\mathbb{N}} \subset W^{1,1}(A; [\alpha,\beta] \times S^2) \cap C^\infty(A; \mathbb{R} \times \mathbb{R}^3)$ such that

$$\bar{w}_n = w \text{ on } \partial A \text{ for all } n \in \mathbb{N}, \quad \bar{w}_n \overset{\star}{\rightharpoonup} w \text{ weakly-}\star \text{ in } BV(A; \mathbb{R} \times \mathbb{R}^3) \text{ as } n \to \infty. \tag{4.35}$$

For each $n \in \mathbb{N}$, let $\kappa_n := \dfrac{a_n}{b_n}$, where

$$a_n := \sqrt{\|u_n - \bar{u}_n\|_{L^1(A)} + \|v_n - \bar{v}_n\|_{L^1(A;\mathbb{R}^3)}},$$

$$b_n := n\left[\!\!\left[1 + \|\nabla u_n\|_{L^1(A;\mathbb{R}^2)} + \|\nabla\bar{u}_n\|_{L^1(A;\mathbb{R}^2)} + \|\nabla v_n\|_{L^1(A;\mathbb{R}^{3\times 2})} + \|\nabla\bar{v}_n\|_{L^1(A;\mathbb{R}^{3\times 2})}\right]\!\!\right],$$

with $[\![b]\!]$ denoting the integer part of b. Clearly, $\kappa_n \to 0^+$ as $n \to \infty$. For $i \in \{1, \cdots, b_n\}$, define

$$A_{n,0} := \{x \in A\colon \operatorname{dist}(x, \partial A) > a_n\}, \quad A_{n,i} := \{x \in A\colon \operatorname{dist}(x, \partial A) > a_n - i\kappa_n\}.$$

We have that $A_{n,0} \subset A_{n,1} \subset \cdots \subset A_{n,b_n}$ and, for all n large enough, $A_{n,0} \neq \emptyset$ since $a_n \to 0$ as $n \to \infty$. Fix any such n, and let $\varphi_i \in C_c^\infty(\mathbb{R}^2; [0,1])$ be a cut-off function such that $\varphi_i = 1$ in $A_{n,i-1}$, $\varphi_i = 0$ in $\mathbb{R}^2\setminus A_{n,i}$, and $\|\nabla\varphi_i\|_\infty \leqslant \frac{c}{\kappa_n}$, being c a positive constant independent of i and n, and set

$$w_n^i = (u_n^i, v_n^i) := \varphi_i(u_n, v_n) + (1 - \varphi_i)(\bar{u}_n, \bar{v}_n) = \varphi_i w_n + (1 - \varphi_i)\bar{w}_n.$$

We have that $w_n^i \in W^{1,1}(A; [\alpha,\beta] \times \overline{B(0,1)})$, $v_n^i \in W^{1,1}(A; \overline{B(0,1)}) \cap C^\infty(A; \mathbb{R}^3)$,

$$w_n^i = w_n \text{ in } A_{n,i-1}, \quad w_n^i = \bar{w}_n \text{ on } A\setminus A_{n,i}, \quad \lim_{n\to\infty}\|w_n^i - w\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)} = 0, \tag{4.36}$$

26

and, since $\nabla w_n^i = \varphi_i \nabla w_n + (1 - \varphi_i) \nabla \bar{w}_n + (w_n - \bar{w}_n) \otimes \nabla \varphi_i$,

$$\int_{A_{n,i} \setminus \overline{A}_{n,i-1}} |\nabla w_n^i| \, \mathrm{d}x \leqslant \int_{A_{n,i} \setminus \overline{A}_{n,i-1}} \left(|\nabla w_n| + |\nabla \bar{w}_n| + \frac{c}{\kappa_n} |w_n - \bar{w}_n| \right) \mathrm{d}x. \tag{4.37}$$

We now apply Lemma 4.6 to A, v_n^i, and $A_{n,i} \setminus \overline{A}_{n,i-1}$, to find a point $y_n^i \in B(0, \frac{1}{2})$ such that $\pi_{y_n^i} \circ v_n^i \in W^{1,1}(A_{n,i} \setminus \overline{A}_{n,i-1}; S^2) \cap C^\infty(A; S^2)$ and

$$\int_{A_{n,i} \setminus \overline{A}_{n,i-1}} |\nabla(\pi_{y_n^i} \circ v_n^i)| \, \mathrm{d}x \leqslant C_\star \int_{A_{n,i} \setminus \overline{A}_{n,i-1}} |\nabla v_n^i| \, \mathrm{d}x. \tag{4.38}$$

In view of (4.36) and (4.22), since v_n and $\bar{v}_n$ take values in S^2 $\mathcal{L}^2$-a.e. in A, we get

$$\begin{aligned}
\int_{A_{n,i-1}} |\nabla(\pi_{y_n^i} \circ v_n^i)| \, \mathrm{d}x &= \int_{A_{n,i-1}} |\nabla(\pi_{y_n^i} \circ v_n)| \, \mathrm{d}x = \int_{A_{n,i-1}} |\nabla v_n| \, \mathrm{d}x, \\
\int_{A \setminus A_{n,i}} |\nabla(\pi_{y_n^i} \circ v_n^i)| \, \mathrm{d}x &= \int_{A \setminus A_{n,i}} |\nabla(\pi_{y_n^i} \circ \bar{v}_n)| \, \mathrm{d}x = \int_{A \setminus A_{n,i}} |\nabla \bar{v}_n| \, \mathrm{d}x.
\end{aligned} \tag{4.39}$$

Thus, (4.38) and (4.39) yield $\tilde{w}_n^i = (\tilde{u}_n^i, \tilde{v}_n^i) := (u_n^i, \pi_{y_n^i} \circ v_n^i) \in W^{1,1}(A; [\alpha, \beta] \times S^2)$. Moreover, the first condition in (4.35), (4.22), and the second condition in (4.36) ensure that

$$\tilde{w}_n^i = w \text{ on } \partial A. \tag{4.40}$$

Next, we prove that for fixed i,

$$\lim_{n \to \infty} \|\tilde{w}_n^i - w\|_{L^1(A; \mathbb{R} \times \mathbb{R}^3)} = 0. \tag{4.41}$$

We have

$$\int_A |u_n^i - u| \, \mathrm{d}x = \int_A |\varphi_i \, u_n + (1 - \varphi_i) \, \bar{u}_n - \varphi_i \, u - (1 - \varphi_i) \, u| \, \mathrm{d}x \leqslant \int_A \left(|u_n - u| + |\bar{u}_n - u| \right) \mathrm{d}x$$

and, arguing as in (4.31) with A_n replaced by the open set $\{x \in A : \operatorname{dist}(v_n^i, S^2) > \delta_0\}$, where $\delta_0 \in (0, \frac{1}{4})$ is fixed,

$$\int_A |\pi_{y_n^i} \circ v_n^i - v| \, \mathrm{d}x \leqslant \max \left\{ \frac{2}{\delta_0}, \overline{C} \right\} \int_A |v_n^i - v| \, \mathrm{d}x \leqslant \max \left\{ \frac{2}{\delta_0}, \overline{C} \right\} \int_A \left(|v_n - v| + |\bar{v}_n - v| \right) \mathrm{d}x.$$

This yields (4.41) because $\{u_n\}_{n \in \mathbb{N}}$ and $\{\bar{u}_n\}_{n \in \mathbb{N}}$ are sequences converging to u in $L^1(A)$, while $\{v_n\}_{n \in \mathbb{N}}$ and $\{\bar{v}_n\}_{n \in \mathbb{N}}$ are sequences converging to v in $L^1(A; \mathbb{R}^3)$.

We now estimate the functional evaluated at $\tilde{w}_n^i$. Using the bounds in (4.32), (4.38), and (4.37), in this order, we deduce that

$$\begin{aligned}
\int_A &\mathfrak{h}(\tilde{u}_n^i, \tilde{v}_n^i, \nabla \tilde{u}_n^i, \nabla \tilde{v}_n^i) \, \mathrm{d}x \\
&\leqslant \int_{\overline{A}_{n,i-1}} \mathfrak{h}(u_n, v_n, \nabla u_n, \nabla v_n) \, \mathrm{d}x + \int_{A_{n,i} \setminus \overline{A}_{n,i-1}} C(1 + |\nabla \tilde{u}_n^i| + |\nabla \tilde{v}_n^i|) \, \mathrm{d}x \\
&\quad + \int_{A \setminus A_{n,i}} C(1 + |\nabla \bar{u}_n| + |\nabla \bar{v}_n|) \, \mathrm{d}x \\
&\leqslant \int_A \mathfrak{h}(u_n, v_n, \nabla u_n, \nabla v_n) \, \mathrm{d}x + \tilde{C} \int_{A_{n,i} \setminus \overline{A}_{n,i-1}} \left(1 + |\nabla w_n| + |\nabla \bar{w}_n| + \frac{c}{\kappa_n} |w_n - \bar{w}_n| \right) \mathrm{d}x \\
&\quad + 2C \int_{A \setminus A_{n,0}} (1 + |\nabla \bar{w}_n|) \, \mathrm{d}x,
\end{aligned} \tag{4.42}$$

where $\tilde{C}$ is a positive constant only depending on C and $C_\star$. Furthermore, using the definition of κ_n,

$$\frac{1}{b_n} \sum_{i=1}^{b_n} \int_{A_{n,i} \setminus \overline{A}_{n,i-1}} \left(1 + |\nabla w_n| + |\nabla \bar{w}_n| + \frac{c}{\kappa_n} |w_n - \bar{w}_n| \right) \mathrm{d}x$$

27

$$\leqslant \frac{1}{b_n} \int_A \left(1 + |\nabla w_n| + |\nabla \bar{w}_n| + \frac{c}{\kappa_n}|w_n - \bar{w}_n|\right) \mathrm{d}x \leqslant \frac{\mathcal{L}^2(A)}{b_n} + \frac{1}{n} + c\|w_n - \bar{w}_n\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)}^{\frac{1}{2}}.$$

Thus, there exists $i_n \in \{1, \cdots, b_n\}$ such that

$$\int_{A_{n,i_n}\backslash\overline{A}_{n,i_n-1}} \left(1 + |\nabla w_n| + |\nabla \bar{w}_n| + \frac{c}{\kappa_n}|w_n - \bar{w}_n|\right) \mathrm{d}x$$
$$\leqslant \frac{\mathcal{L}^2(A)}{b_n} + \frac{1}{n} + c\|w_n - \bar{w}_n\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)}^{\frac{1}{2}} = o(1) \text{ as } n \to \infty. \tag{4.43}$$

Fixing $j \in \mathbb{N}$ and defining $\tilde{A}_j := \{x \in A: \mathrm{dist}(x, \partial A) > 1/j\}$, we have $A\backslash\tilde{A}_j \supset A\backslash A_{n,0}$ for all n large enough because $a_n \to 0$ as $n \to \infty$. Hence, using the fact that $A\backslash\tilde{A}_j$ is a closed subset of A and $D\bar{w}_n \overset{\star}{\rightharpoonup} Dw$ weakly-$\star$ in $\mathcal{M}(A;\mathbb{R}^2 \times \mathbb{R}^{3\times2})$, we get, for fixed j,

$$\limsup_{n\to\infty} \int_{A\backslash A_{n,0}} (1 + |\nabla \bar{w}_n|)\,\mathrm{d}x \leqslant \limsup_{n\to\infty} \int_{A\backslash\tilde{A}_j} (1 + |\nabla \bar{w}_n|)\,\mathrm{d}x \leqslant \mathcal{L}^2(A\backslash\tilde{A}_j) + |Dw|(A\backslash\tilde{A}_j). \tag{4.44}$$

Observing that $\{A\backslash\tilde{A}_j\}_{j\in\mathbb{N}}$ is a decreasing sequence of $(\mathcal{L}^2 + |Dw|)$-finite measure sets whose intersection is the empty set, letting $j \to \infty$ in (4.44), we conclude that

$$\lim_{n\to\infty} \int_{A\backslash A_{n,0}} (1 + |\nabla \bar{w}_n|)\,\mathrm{d}x = 0. \tag{4.45}$$

Finally, setting $\tilde{w}_n := \tilde{w}_n^{i_n}$, in view of (4.40)–(4.43) and (4.45), the sequence $\{\tilde{w}_n\}_{n\in\mathbb{N}} \subset W^{1,1}(A;[\alpha,\beta]\times S^2)$ satisfies (4.33) and (4.34). $\square$

Remark 4.11. If $A \in \mathcal{A}(\Omega)$ is of the form $A = A_1\backslash\overline{A}_0$, where $A_1 \in \mathcal{A}(\Omega)$, $A_0 \in \mathcal{A}_\infty(\Omega)$, and $A_0 \subset\subset A_1$, then in view of Remark 4.8, Lemma 4.10 holds for all such open sets A as long as we replace the trace condition in (4.33) by "$\bar{w}_n = w$ on ∂A_0".

Lemma 4.12. *For every $(u, v) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2)$, the set function*

$$A \in \mathcal{A}(\Omega) \mapsto \mathcal{F}(u, v; A) := \inf\Big\{\liminf_{n\to+\infty} \int_A f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x))\,\mathrm{d}x :$$
$$n \in \mathbb{N}, (u_n, v_n) \in W^{1,1}(A;[\alpha,\beta]) \times W^{1,1}(A;S^2),$$
$$u_n \to u \text{ in } L^1(A), v_n \to v \text{ in } L^1(A;\mathbb{R}^3)\Big\} \tag{4.46}$$

is the restriction of a Radon measure on Ω to $\mathcal{A}(\Omega)$.

Proof. Fix $w = (u, v) \in BV(\Omega;[\alpha,\beta]\times S^2)$. Using the bounds (1.14) and a diagonalization argument, we can find a sequence $\{w_n\}_{n\in\mathbb{N}} = \{(u_n, v_n)\}_{n\in\mathbb{N}} \subset W^{1,1}(\Omega;[\alpha,\beta]\times S^2)$ converging to w in $L^1(\Omega;\mathbb{R}\times\mathbb{R}^3)$ such that

$$\mathcal{F}(w;\Omega) = \lim_{n\to\infty} \int_\Omega f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x))\,\mathrm{d}x,$$
$$\mu_n := f(u_n, v_n, \nabla u_n, \nabla v_n)\mathcal{L}^2_{\llcorner\Omega} \overset{\star}{\rightharpoonup} \mu \text{ weakly-}\star \text{ in } \mathcal{M}(\Omega)$$

for some nonnegative Radon measure $\mu \in \mathcal{M}(\Omega)$.

We claim that for all $A \in \mathcal{A}(\Omega)$,

$$\mathcal{F}(w; A) = \mu(A). \tag{4.47}$$

We will proceed in three steps.

Step1. We prove that for all $A \in \mathcal{A}(\Omega)$,

$$\mathcal{F}(w; A) \leqslant (3 + \beta)\widetilde{C}|Dw|(A), \tag{4.48}$$

where $\widetilde{C}$ is a positive constant only depending on $\bar{C}$ and $C_\star$.

Arguing as in Lemma 4.7, we can find a sequence $\{\bar{w}_n\}_{n\in\mathbb{N}} = \{(\bar{u}_n, \bar{v}_n)\}_{n\in\mathbb{N}} \subset W^{1,1}(A; [\alpha, \beta] \times S^2) \cap C^\infty(A; \mathbb{R} \times \mathbb{R}^3)$ such that $\lim_{n\to\infty} \|\bar{w}_n - w\|_{L^1(A;\mathbb{R}\times\mathbb{R}^3)} = 0$ and $\limsup_{n\to\infty} \int_A |\nabla\bar{w}_n(x)|\, \mathrm{d}x \leqslant \widetilde{C}|Dw|(A)$, where $\widetilde{C}$ is a positive constant only depending on $\bar{C}$ and $C_\star$. Then, by (1.14),

$$\mathcal{F}(w; A) \leqslant \liminf_{n\to\infty} \int_A f(\bar{u}_n(x), \bar{v}_n(x), \nabla\bar{u}_n(x), \nabla\bar{v}_n(x))\, \mathrm{d}x$$
$$\leqslant \liminf_{n\to\infty} \int_A (3+\beta)|\nabla\bar{w}_n(x)|\, \mathrm{d}x \leqslant (3+\beta)\widetilde{C}|Dw|(A).$$

<u>Step 2.</u> We claim that for all $A_1, A_2, A_3 \in \mathcal{A}(\Omega)$ such that $A_1 \subset\subset A_2 \subset A_3$, the following inequality holds

$$\mathcal{F}(w; A_3) \leqslant \mathcal{F}(w; A_2) + \mathcal{F}(w; A_3\backslash\overline{A}_1). \tag{4.49}$$

Let $U \in \mathcal{A}_\infty(\Omega)$ be such that $A_1 \subset\subset U \subset\subset A_2$. Let $\{w_n^1\}_{n\in\mathbb{N}} = \{(u_n^1, v_n^1)\}_{n\in\mathbb{N}} \subset W^{1,1}(A_3\backslash\overline{A}_1; [\alpha, \beta] \times S^2)$ and $\{w_n^2\}_{n\in\mathbb{N}} = \{(u_n^2, v_n^2)\}_{n\in\mathbb{N}} \subset W^{1,1}(A_2; [\alpha, \beta] \times S^2)$ be sequences converging to w in $L^1(A_3\backslash\overline{A}_1; \mathbb{R} \times \mathbb{R}^3)$ and $L^1(A_2; \mathbb{R} \times \mathbb{R}^3)$, respectively, and such that

$$\mathcal{F}(w; A_3\backslash\overline{A}_1) = \lim_{n\to\infty} \int_{A_3\backslash\overline{A}_1} f(u_n^1(x), v_n^1(x), \nabla u_n^1(x), \nabla v_n^1(x))\, \mathrm{d}x,$$
$$\mathcal{F}(w; A_2) = \lim_{n\to\infty} \int_{A_2} f(u_n^2(x), v_n^2(x), \nabla u_n^2(x), \nabla v_n^2(x))\, \mathrm{d}x. \tag{4.50}$$

In view of Lemma 4.10 and Remark 4.11, we can find sequences $\{\tilde{w}_n^1\}_{n\in\mathbb{N}} = \{(\tilde{u}_n^1, \tilde{v}_n^1)\}_{n\in\mathbb{N}} \subset W^{1,1}(A_3\backslash\overline{U}; [\alpha, \beta] \times S^2)$ and $\{\tilde{w}_n^2\}_{n\in\mathbb{N}} = \{(\tilde{u}_n^2, \tilde{v}_n^2)\}_{n\in\mathbb{N}} \subset W^{1,1}(U; [\alpha, \beta] \times S^2)$ converging to w in $L^1(A_3\backslash\overline{U}; \mathbb{R} \times \mathbb{R}^3)$ and $L^1(U; \mathbb{R} \times \mathbb{R}^3)$, respectively, and such that

$$\tilde{w}_n^1 = w \text{ on } \partial U, \quad \tilde{w}_n^2 = w \text{ on } \partial U,$$
$$\limsup_{n\to\infty} \int_{A_3\backslash\overline{U}} f(u_n^1, v_n^1, \nabla u_n^1, \nabla v_n^1)\, \mathrm{d}x \geqslant \limsup_{n\to\infty} \int_{A_3\backslash\overline{U}} f(\tilde{u}_n^1, \tilde{v}_n^1, \nabla\tilde{u}_n^1, \nabla\tilde{v}_n^1)\, \mathrm{d}x,$$
$$\limsup_{n\to\infty} \int_U f(u_n^2, v_n^2, \nabla u_n^2, \nabla v_n^2)\, \mathrm{d}x \geqslant \limsup_{n\to\infty} \int_U f(\tilde{u}_n^2, \tilde{v}_n^2, \nabla\tilde{u}_n^2, \nabla\tilde{v}_n^2)\, \mathrm{d}x. \tag{4.51}$$

Define for $n \in \mathbb{N}$,

$$\tilde{w}_n := \begin{cases} \tilde{w}_n^1 & \text{in } A_3\backslash\overline{U}, \\ \tilde{w}_n^2 & \text{in } U. \end{cases}$$

Then, $\tilde{w}_n = (\tilde{u}_n, \tilde{v}_n) \in W^{1,1}(A_3; [\alpha, \beta] \times S^2)$ and $\{\tilde{w}_n\}_{n\in\mathbb{N}}$ is a sequence converging to w in $L^1(A_3; \mathbb{R} \times \mathbb{R}^3)$. Moreover, using (4.50) and (4.51), together with the set inclusions $A_3\backslash\overline{U} \subset A_3\backslash\overline{A}_1$ and $U \subset A_2$ and the non-negativeness of f,

$$\mathcal{F}(w; A_3) \leqslant \liminf_{n\to\infty} \int_{A_3} f(\tilde{u}_n, \tilde{v}_n, \nabla\tilde{u}_n, \nabla\tilde{v}_n)\, \mathrm{d}x$$
$$= \liminf_{n\to\infty} \left(\int_{A_3\backslash\overline{U}} f(\tilde{u}_n^1, \tilde{v}_n^1, \nabla\tilde{u}_n^1, \nabla\tilde{v}_n^1)\, \mathrm{d}x + \int_U f(\tilde{u}_n^2, \tilde{v}_n^2, \nabla\tilde{u}_n^2, \nabla\tilde{v}_n^2)\, \mathrm{d}x \right)$$
$$\leqslant \limsup_{n\to\infty} \int_{A_3\backslash\overline{U}} f(\tilde{u}_n^1, \tilde{v}_n^1, \nabla\tilde{u}_n^1, \nabla\tilde{v}_n^1)\, \mathrm{d}x + \limsup_{n\to\infty} \int_U f(\tilde{u}_n^2, \tilde{v}_n^2, \nabla\tilde{u}_n^2, \nabla\tilde{v}_n^2)\, \mathrm{d}x$$
$$\leqslant \mathcal{F}(w; A_3\backslash\overline{A}_1) + \mathcal{F}(w; A_2),$$

which concludes Step 2.

<u>Step 3.</u> We establish (4.47). Fix $A \in \mathcal{A}(\Omega)$.

Substep 3.1. We prove that $\mathcal{F}(w; A) \leqslant \mu(A)$.

Using the upper semicontinuity of the weak-$\star$ convergence in $\mathcal{M}(\Omega)$ with respect to compact sets and the fact that $\{w_n\}_{n\in\mathbb{N}}$ is an admissible sequence for $\mathcal{F}(w; A)$, we conclude that

$$\mathcal{F}(w; A) \leqslant \liminf_{n\to\infty} \int_A f(u_n, v_n, \nabla u_n, \nabla v_n)\, dx = \liminf_{n\to\infty} \mu_n(A) \leqslant \limsup_{n\to\infty} \mu_n(\overline{A}) \leqslant \mu(\overline{A}). \tag{4.52}$$

Fix $\varepsilon > 0$ and let A'_ε, $A''_\varepsilon \in \mathcal{A}(\Omega)$ be such that $A'_\varepsilon \subset\subset A''_\varepsilon \subset\subset A$ and $|Dw|(A\backslash\overline{A}'_\varepsilon) < \varepsilon/\tilde{c}$, where $\tilde{c} := (3 + \beta)\widetilde{C}$. Using (4.49), (4.48), and (4.52), in this order, we obtain

$$\mathcal{F}(w; A) \leqslant \mathcal{F}(w; A''_\varepsilon) + \mathcal{F}(w; A\backslash\overline{A}'_\varepsilon) \leqslant \mathcal{F}(w; A''_\varepsilon) + \varepsilon \leqslant \mu(\overline{A}''_\varepsilon) + \varepsilon \leqslant \mu(A) + \varepsilon,$$

from which Substep 3.1 follows by letting $\varepsilon \to 0^+$.

Substep 3.2. We prove that $\mathcal{F}(w; A) \geqslant \mu(A)$.

Fix $\varepsilon > 0$, and let $A_\varepsilon \in \mathcal{A}(\Omega)$ be such that $A_\varepsilon \subset\subset A$ and $\mu(A\backslash\overline{A}_\varepsilon) < \varepsilon$. Using the equality $\mathcal{F}(w; \Omega) = \mu(\Omega)$, from Substep 3.1 (applied to $\Omega\backslash\overline{A}_\varepsilon$) and Step 2 (applied to $A_\varepsilon \subset\subset A \subset \Omega$), it follows that

$$\mu(A) = \mu(A\backslash\overline{A}_\varepsilon) + \mu(\overline{A}_\varepsilon) < \varepsilon + \mu(\overline{A}_\varepsilon) = \varepsilon + \mu(\Omega) - \mu(\Omega\backslash\overline{A}_\varepsilon)$$
$$\leqslant \varepsilon + \mathcal{F}(w; \Omega) - \mathcal{F}(w; \Omega\backslash\overline{A}_\varepsilon) \leqslant \varepsilon + \mathcal{F}(w; A).$$

Letting $\varepsilon \to 0^+$, we conclude the proof of Substep 3.2 as well as of Lemma 4.12. $\qquad\square$

Lemma 4.13. *Let $w \in BV(\Omega; [\alpha, \beta] \times S^2)$. Then*[1],

(a) $\tilde{w}(x) \in [\alpha, \beta] \times S^2$ *for all* $x \in A_w = \Omega\backslash S_w$;

(b) $w^\pm(x) \in [\alpha, \beta] \times S^2$ *for all* $x \in J_w$;

(c) $\nabla w(x) \in \left[T_{w(x)}([\alpha, \beta] \times S^2)\right]^2$ *for* $\mathcal{L}^2$*-a.e.* $x \in \Omega$;

(d) $W^c(x) := \dfrac{dD^c w}{d|D^c w|}(x) \in \left[T_{\tilde{w}(x)}([\alpha, \beta] \times S^2)\right]^2$ *for* $|D^c w|$*-a.e.* $x \in \Omega$.

Proof. We start by proving *(a)* and *(b)*. Let $x_0 \in A_w$. Because $w(\cdot) \in [\alpha, \beta] \times S^2$ $\mathcal{L}^2$-a.e. in Ω, we have $|w(\cdot) - \tilde{w}(x_0)| \geqslant \text{dist}(\tilde{w}(x_0), [\alpha, \beta] \times S^2)$ $\mathcal{L}^2$-a.e. in Ω, and so

$$0 = \lim_{\epsilon\to 0^+} \fint_{B(x_0,\epsilon)} |w(y) - \tilde{w}(x_0)|\, dy \geqslant \text{dist}(\tilde{w}(x_0), [\alpha, \beta] \times S^2).$$

This implies that $\tilde{w}(x_0) \in [\alpha, \beta] \times S^2$. Similarly, if $x_0 \in J_w$, then

$$0 = \lim_{\epsilon\to 0^+} \fint_{B^\pm_{\nu_w(x_0)}(x_0,\epsilon)} |w(y) - w^\pm(x_0)|\, dy \geqslant \text{dist}(w^\pm(x_0), [\alpha, \beta] \times S^2),$$

from which we conclude that $w^\pm(x_0) \in [\alpha, \beta] \times S^2$.

In order to prove *(c)* and *(d)*, we fix an open, bounded subset U of $\mathbb{R} \times \mathbb{R}^3$ such that $U \supset [\alpha, \beta] \times S^2$, and we consider a cut-off function $\theta \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3; [0, 1])$ satisfying $\text{supp}\,\theta \subset U$ and $\theta(r, s) = 1$ for all $(r, s) \in [\alpha, \beta] \times S^2$. Finally, we define $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ by setting $\phi(r, s) := \theta(r, s)(|s|^2 - 1)$. Then, ϕ belongs to $C_c^1(\mathbb{R} \times \mathbb{R}^3)$ and

$$\frac{\partial\phi}{\partial r}(r, s) = \frac{\partial\theta}{\partial r}(r, s)(|s|^2 - 1), \quad \frac{\partial\phi}{\partial s}(r, s) = \frac{\partial\theta}{\partial s}(r, s)(|s|^2 - 1) + 2\theta(r, s)s.$$

Hence, if $(r, s) \in [\alpha, \beta] \times S^2$ and $h = (h_1, h') \in \mathbb{R} \times \mathbb{R}^3$, then

$$\nabla\phi(r, s) \cdot h = 0 \Leftrightarrow h_1 \in \mathbb{R} \ \wedge \ h' \cdot s = 0 \Leftrightarrow h \in T_{(r,s)}([\alpha, \beta] \times S^2). \tag{4.53}$$

---

[1]We refer to Subsection 2.3 for the notation concerning BV functions.

Moreover, by Theorem 2.13, we have that $\phi \circ w \in BV(\Omega)$ and (see (2.6))

$$D(\phi \circ w) = \nabla\phi(w)\nabla w \mathcal{L}^N + \left(\phi(w^+) - \varphi(w^-)\right) \otimes \nu_w \mathcal{H}^{N-1}\lfloor_{J_w} + \nabla\phi(\tilde{w})D^c w$$
$$= \nabla\phi(w)\nabla w \mathcal{L}^N + \nabla\phi(\tilde{w})W^c|D^c w|,$$

where we also used *(b)* together with the fact that $\phi(r,s) = 0$ for $s \in S^2$. Similarly, since $w(\cdot) \in [\alpha, \beta] \times S^2$ for $\mathcal{L}^2$-a.e. in Ω, it follows that $\phi \circ w = 0$ for $\mathcal{L}^2$-a.e. in Ω. Thus, $D(\phi \circ w) \equiv 0$ and, because $\mathcal{L}^2$ and $|D^c w|$ are mutually singular measures, we conclude that

$$\nabla\phi(w(x))\nabla w(x) = 0 \text{ for } \mathcal{L}^2\text{-a.e. } x \in \Omega, \qquad \nabla\phi(\tilde{w}(x))W^c(x) = 0 \text{ for } |D^c w|\text{-a.e. } x \in \Omega;$$

that is,

$$\nabla\phi(w(x)) \cdot (\nabla w(x), 0) = 0 \text{ for } \mathcal{L}^2\text{-a.e. } x \in \Omega, \qquad \nabla\phi(\tilde{w}(x)) \cdot (0, W^c(x)) = 0 \text{ for } |D^c w|\text{-a.e. } x \in \Omega,$$

which, together with (4.53), yields *(c)* and *(d)*. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4.3 On the Lower Bound for $\mathcal{F}$

Let G denote the function on the right-hand side of (1.22). We claim that

$$\mathcal{F}(u,v) \geqslant G(u,v)$$

for all $(u,v) \in L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$ or, equivalently,

$$\liminf_{n \to +\infty} F(u_n, v_n) \geqslant G(u,v) \tag{4.54}$$

whenever $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$ is a sequence converging to (u,v) in $L^1(\Omega) \times L^1(\Omega; \mathbb{R}^3)$. To prove (4.54), we may assume without loss of generality that

$$\liminf_{n \to +\infty} F(u_n, v_n) = \lim_{n \to +\infty} F(u_n, v_n) \in \mathbb{R}_0^+, \tag{4.55}$$

and for all $n \in \mathbb{N}$,

$$(u_n, v_n) \in W^{1,1}(\Omega; [\alpha, \beta]) \times W^{1,1}(\Omega; S^2).$$

In particular,

$$F(u_n, v_n) = \int_\Omega \left(|\nabla u_n| + g(|\nabla u_n|)|\nabla v_n| + |\nabla(u_n v_n)|\right) \mathrm{d}x = \int_\Omega f(u_n, v_n, \nabla u_n, \nabla v_n)\,\mathrm{d}x \leqslant C, \tag{4.56}$$

for some positive constant C independent of n. Hence,

$$\int_\Omega \left(|\nabla u_n| + |\nabla(u_n v_n)|\right) \mathrm{d}x \leqslant C, \tag{4.57}$$

and, in turn,

$$\alpha \int_\Omega |\nabla v_n|\,\mathrm{d}x \leqslant \int_\Omega |u_n \nabla v_n + v_n \otimes \nabla u_n|\,\mathrm{d}x + \int_\Omega |v_n \otimes \nabla u_n|\,\mathrm{d}x \leqslant C. \tag{4.58}$$

Thus, up to the extraction of a subsequence (not relabeled), we have

$$u_n \overset{\star}{\rightharpoonup} u \text{ weakly-}\star \text{ in } BV(\Omega) \text{ and } v_n \overset{\star}{\rightharpoonup} v \text{ weakly-}\star \text{ in } BV(\Omega; \mathbb{R}^3) \text{ as } n \to +\infty;$$
$$u(x) \in [\alpha, \beta] \text{ and } v(x) \in S^2 \text{ for } \mathcal{L}^2\text{-a.e. } x \in \Omega;$$
$$\mu_n := f(u_n, v_n, \nabla u_n, \nabla v_n)\mathcal{L}^2_{\lfloor \Omega} \overset{\star}{\rightharpoonup} \mu \text{ weakly-}\star \text{ in } \mathcal{M}(\Omega)$$

31

for some nonnegative finite Radon measure $\mu \in \mathcal{M}(\Omega)$. In view of the Radon-Nikodym Theorem, we can decompose μ into a sum of four mutually singular, nonnegative finite Radon measures as follows:

$$\mu = \mu_a \mathcal{L}^2_{\lfloor \Omega} + \mu_c |D^c(u,v)| + \mu_j |(u,v)^+ - (u,v)^-|\mathcal{H}^1_{\lfloor J_{(u,v)}} + \mu_s.$$

We claim that

$$\mu_a(x_0) \geqslant \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)) \quad \text{for } \mathcal{L}^2\text{-a.e. } x_0 \in \Omega; \tag{4.59}$$

$$\mu_c(x_0) \geqslant (\mathcal{Q}_T f)^\infty \big(\tilde{u}(x_0), \tilde{v}(x_0), W_u^c(x_0), W_v^c(x_0)\big) \quad \text{for } |D^c(u,v)|\text{-a.e. } x_0 \in \Omega; \tag{4.60}$$

$$\mu_j(x_0) \geqslant \frac{1}{|(u,v)^+(x_0) - (u,v)^-(x_0)|} K\big((u,v)^+(x_0), (u,v)^-(x_0), \nu_{(u,v)}(x_0)\big)$$
$$\text{for } |(u,v)^+ - (u,v)^-|\mathcal{H}^1_{\lfloor J_{(u,v)}}\text{-a.e. } x_0 \in \Omega. \tag{4.61}$$

Assume that (4.59), (4.60), and (4.61) hold, and let $\{\phi_k\}_{k \in \mathbb{N}} \subset C_0^\infty(\Omega; [0,1])$ be an increasing sequence of smooth cut-off functions such that $\sup_{k \in \mathbb{N}} \phi_k(x) = 1$ for all $x \in \Omega$. Then, using the convergence $\mu_n \overset{\star}{\rightharpoonup} \mu$ weakly-$\star$ in $\mathcal{M}(\Omega)$,

$$\lim_{n \to +\infty} \int_\Omega f(u_n, v_n, \nabla u_n, \nabla v_n) \, dx \geqslant \liminf_{n \to +\infty} \int_\Omega \phi_k(x) f(u_n, v_n, \nabla u_n, \nabla v_n) \, dx = \int_\Omega \phi_k(x) \, d\mu$$

$$\geqslant \int_\Omega \phi_k(x) \mathcal{Q}_T f(u(x), v(x), \nabla u(x), \nabla v(x)) \, dx$$

$$+ \int_{S_{(u,v)}} \phi_k(x) K\big((u,v)^+(x), (u,v)^-(x), \nu_{(u,v)}(x)\big) \, d\mathcal{H}^1(x)$$

$$+ \int_\Omega \phi_k(x) (\mathcal{Q}_T f)^\infty \big(\tilde{u}(x), \tilde{v}(x), W_u^c(x), W_v^c(x)\big) \, d|D^c(u,v)|(x). \tag{4.62}$$

In view of Lebesgue's Monotone Convergence Theorem, (4.55), and (4.56), letting $k \to +\infty$ in (4.62), we obtain (4.54).

We start by proving (4.59) and (4.60). Let H_ε be the function defined in (4.17), and let $A \in \mathcal{A}(\Omega)$. Because H_ε satisfies conditions (H1)–(H4) of [26] by Proposition 4.4, we have

$$\liminf_{n \to +\infty} \int_A H_\varepsilon(u_n, v_n, \nabla u_n, \nabla v_n) \, dx$$

$$\geqslant \int_A H_\varepsilon(u, v, \nabla u, \nabla v) \, dx + \int_A (H_\varepsilon)^\infty(\tilde{u}, \tilde{v}, W_u^c, W_v^c) \, d|D^c(u,v)|(x)$$

$$= \int_A \mathcal{Q}\tilde{f}(u, v, \nabla u, \nabla v) \, dx + \int_A (\mathcal{Q}\tilde{f})^\infty(\tilde{u}, \tilde{v}, W_u^c, W_v^c) \, d|D^c(u,v)|(x) + O(\varepsilon) \tag{4.63}$$

as $\varepsilon \to 0^+$, where in the last equality we also used the identity

$$(H_\varepsilon)^\infty(r, s, \xi, \eta) = (\mathcal{Q}\tilde{f})^\infty(r, s, \xi, \eta) + \varepsilon(|\xi| + |\eta|).$$

Recalling that for $(r, s) \in [\alpha, \beta] \times S^2$, $T_{(r,s)}([\alpha, \beta] \times S^2) = \mathbb{R} \times T_s(S^2)$, from Lemma 4.13, (4.5), (4.57), and (4.58), we conclude that

$$\liminf_{n \to +\infty} \int_A f(u_n, v_n, \nabla u_n, \nabla v_n) \, dx = \liminf_{n \to +\infty} \int_A \tilde{f}(u_n, v_n, \nabla u_n, \nabla v_n) \, dx$$

$$\geqslant \liminf_{n \to +\infty} \int_A \mathcal{Q}\tilde{f}(u_n, v_n, \nabla u_n, \nabla v_n) \, dx \geqslant \liminf_{n \to +\infty} \int_A H_\varepsilon(u_n, v_n, \nabla u_n, \nabla v_n) \, dx - \varepsilon C.$$

These estimates and (4.63) entail

$$\liminf_{n \to +\infty} \int_A f(u_n, v_n, \nabla u_n, \nabla v_n) \, dx \geqslant \int_A \mathcal{Q}\tilde{f}(u, v, \nabla u, \nabla v) \, dx + \int_A (\mathcal{Q}\tilde{f})^\infty(\tilde{u}, \tilde{v}, W_u^c, W_v^c) \, d|D^c(u,v)|(x).$$

$$\tag{4.64}$$

Since $\mathcal{L}^N_{\lfloor \Omega}$ and $|D^c(u,v)|_{\lfloor \Omega}$ are mutually singular, (4.59) and (4.60) are a consequence of (4.64).

We now establish (4.61). We start by recalling that if $\nu \in S^1$, then Q_ν denotes the unit cube in $\mathbb{R}^2$ centered at the origin and with two faces orthogonal to ν. We set

$$Q_\nu^+ := \{x \in Q_\nu : \; x \cdot \nu > 0\}, \qquad Q_\nu^- := \{x \in Q_\nu : \; x \cdot \nu < 0\},$$

and, for $x_0 \in \mathbb{R}^2$ and $\epsilon > 0$,

$$Q_\nu(x_0, \epsilon) := x_0 + \varepsilon Q_\nu, \qquad Q_\nu^\pm(x_0, \epsilon) := x_0 + \varepsilon Q_\nu^\pm.$$

To simplify the notation, we further set $w := (u,v)$ and $w_n := (u_n, v_n)$, $n \in \mathbb{N}$. Let $x_0 \in J_w$ be such that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon^2} \int_{Q_{\nu_w(x_0)}^\pm(x_0,\epsilon)} |w(x) - w^\pm(x_0)| \, \mathrm{d}x = 0, \tag{4.65}$$

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_{S_w \cap Q_{\nu_w(x_0)}(x_0,\epsilon)} |w^+(x) - w^-(x)| \, \mathrm{d}\mathcal{H}^1(x) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} |w^+ - w^-| \mathcal{H}^1_{\lfloor (S_w \cap Q_{\nu_w(x_0)}(x_0,\epsilon))}$$

$$= |w^+(x_0) - w^-(x_0)|, \tag{4.66}$$

$$\mu_j(x_0) = \lim_{\epsilon \to 0^+} \frac{\mu(Q_{\nu_w(x_0)}(x_0,\epsilon))}{|w^+ - w^-| \mathcal{H}^1_{\lfloor (S_w \cap Q_{\nu_w(x_0)}(x_0,\epsilon))}} \in \mathbb{R}. \tag{4.67}$$

In view of Proposition 2.6, Theorem 2.14, and Besicovitch Differentiation Theorem, (4.65)–(4.67) hold for $|w^+ - w^-| \mathcal{H}^1_{\lfloor J_w}$-a.e. $x_0 \in \Omega$.

Let $\{\epsilon_i\}_{i \in \mathbb{N}}$ be a sequence of positive numbers converging to zero such that the boundary of each $Q_{\nu_w(x_0)}(x_0, \epsilon_i)$ has zero μ–measure. Using (4.66), (4.67), and the weak-$\star$ convergence $\mu_n \overset{\star}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, we obtain

$$|w^+(x_0) - w^-(x_0)| \mu_j(x_0) = \lim_{i \to +\infty} \frac{1}{\epsilon_i} \int_{Q_{\nu_w(x_0)}(x_0,\epsilon_i)} \mathrm{d}\mu$$

$$= \lim_{i \to +\infty} \lim_{n \to +\infty} \frac{1}{\epsilon_i} \int_{Q_{\nu_w(x_0)}(x_0,\epsilon_i)} f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x)) \, \mathrm{d}x$$

$$= \lim_{i \to +\infty} \lim_{n \to +\infty} \int_{Q_{\nu_w(x_0)}} \epsilon_i \, f\Big(u_{n,\epsilon_i}(y), v_{n,\epsilon_i}(y), \frac{1}{\epsilon_i}\nabla u_{n,\epsilon_i}(y), \frac{1}{\epsilon_i}\nabla v_{n,\epsilon_i}(y)\Big) \, \mathrm{d}y$$

$$= \lim_{i \to +\infty} \lim_{n \to +\infty} \int_{Q_{\nu_w(x_0)}} \Big[|\nabla u_{n,\epsilon_i}(y)| + |\nabla(u_{n,\epsilon_i} v_{n,\epsilon_i})(y)| + g\Big(\frac{1}{\epsilon_i}|\nabla u_{n,\epsilon_i}(y)|\Big)|\nabla v_{n,\epsilon_i}(y)|\Big] \, \mathrm{d}y, \quad (4.68)$$

where

$$u_{n,\epsilon_i}(y) := u_n(x_0 + \epsilon_i y), \quad v_{n,\epsilon_i}(y) := v_n(x_0 + \epsilon_i y), \quad y \in Q_{\nu_w(x_0)}.$$

Setting $w_{n,\epsilon_i} := (u_{n,\epsilon_i}, v_{n,\epsilon_i})$, we have that $w_{n,\epsilon_i} \in W^{1,1}(Q_{\nu_w(x_0)}; [\alpha, \beta] \times S^2)$, and, in view of (4.65),

$$\lim_{i \to +\infty} \lim_{n \to +\infty} \int_{Q_{\nu_w(x_0)}} |w_{n,\epsilon_i}(y) - w_0(y)| \, \mathrm{d}y = 0, \tag{4.69}$$

where

$$w_0(y) := \begin{cases} w^+(x_0) & \text{if } y \cdot \nu_w(x_0) \geqslant 0, \\ w^-(x_0) & \text{if } y \cdot \nu_w(x_0) < 0. \end{cases}$$

By a standard diagonalization argument, from (4.68), (4.69), and Lemma 4.10, we can construct a sequence $\{\bar{w}_k\}_{k \in \mathbb{N}} = \{(\bar{u}_k, \bar{v}_k)\}_{k \in \mathbb{N}} \subset W^{1,1}(Q_{\nu_w(x_0)}; [\alpha, \beta] \times S^2)$ such that $\bar{w}_k = w_0$ on $\partial Q_{\nu_w}(x_0)$ for all $k \in \mathbb{N}$,

$$\lim_{k \to +\infty} \|\bar{w}_k - w_0\|_{L^1(Q_{\nu_w(x_0)}; \mathbb{R} \times \mathbb{R}^3)} = 0,$$

33

and

$$\left| w^+(x_0) - w^-(x_0) \right| \mu_j(x_0) \geqslant \limsup_{k \to +\infty} \int_{Q_{\nu_w(x_0)}} \left[|\nabla \bar u_k(y)| + |\nabla (\bar u_k \bar v_k)(y)| + g\left(\frac{1}{\epsilon_{i_k}} |\nabla \bar u_k(y)| \right) |\nabla \bar v_k(y)| \right] \mathrm{d}y. \tag{4.70}$$

Because

$$\int_{Q_{\nu_w(x_0)}} g\left(\frac{1}{\epsilon_{i_k}} |\nabla \bar u_k(y)| \right) |\nabla \bar v_k(y)| \, \mathrm{d}y \geqslant \int_{\{y \in Q_{\nu_w(x_0)} : \nabla \bar u_k(y) = 0\}} |\nabla \bar v_k(y)| \, \mathrm{d}y$$

$$= \int_{Q_{\nu_w(x_0)}} \chi_{\{0\}}(|\nabla \bar u_k(y)|) |\nabla \bar v_k(y)| \, \mathrm{d}y,$$

from (4.70) and (1.18), and since $(\bar u_k, \bar v_k) \in \mathcal{P}((u,v)^+(x_0), (u,v)^-(x_0), \nu_{(u,v)}(x_0))$ for all $k \in \mathbb{N}$, we obtain (4.61).

4.4 On the Upper Bound for $\mathcal{F}$

We identify the Radon measure on Ω given by Lemma 4.12 with its restriction $\mathcal{F}(u,v;\cdot)$ to $\mathcal{A}(\Omega)$ introduced in (4.46). In view of (4.48) and [5, Step 1 of Prop. 4.4], we have that $\mathcal{F}(u,v;\cdot)$ is local in $\mathcal{B}(\Omega)$ in the following sense:

$$\mathcal{F}(u,v;B) = \mathcal{F}(u',v';B)$$

for all $B \in \mathcal{B}(\Omega)$ and $w := (u,v)$, $w' := (u',v') \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$ such that

$$B \subset S_w \cap S_{w'}, \qquad (w^+(x), w^-(x), \nu_w(x)) \sim (w'^+(x), w'^-(x), \nu_{w'}(x)) \quad \text{for all } x \in B,$$

where

$$(a, b, \nu) \sim (a', b', \nu') \Leftrightarrow (a = a' \wedge b = b' \wedge \nu = \nu') \vee (b = a' \wedge a = b' \wedge \nu = -\nu'). \tag{4.71}$$

Lemma 4.14. *Let $(u,v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$. Then, for $\mathcal{L}^2$-a.e. $x_0 \in \Omega$, we have*

$$\frac{\mathrm{d}\mathcal{F}(u,v;\cdot)}{\mathrm{d}\mathcal{L}^2}(x_0) \leqslant \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)). \tag{4.72}$$

Proof. Let $x_0 \in \Omega$ be such that

$$\alpha \leqslant u(x_0) \leqslant \beta, \tag{4.73}$$

$$|v(x_0)| = 1, \quad \nabla v(x_0) \in [T_{v(x_0)}(S^2)]^2, \tag{4.74}$$

$$\frac{\mathrm{d}\mathcal{F}(u,v;\cdot)}{\mathrm{d}\mathcal{L}^2}(x_0) \text{ exists and is finite}, \tag{4.75}$$

$$\lim_{\epsilon \to 0^+} \fint_{B(x_0, \epsilon)} |u(x) - u(x_0)| \, \mathrm{d}x = 0, \tag{4.76}$$

$$\lim_{\epsilon \to 0^+} \fint_{B(x_0, \epsilon)} |\nabla u(x) - \nabla u(x_0)| \, \mathrm{d}x = 0, \tag{4.77}$$

$$\lim_{\epsilon \to 0^+} \fint_{B(x_0, \epsilon)} |v(x) - v(x_0)| \, \mathrm{d}x = 0, \tag{4.78}$$

$$\lim_{\epsilon \to 0^+} \fint_{B(x_0, \epsilon)} |\nabla v(x) - \nabla v(x_0)| \, \mathrm{d}x = 0, \tag{4.79}$$

$$\lim_{\varepsilon \to 0^+} \frac{|D^s u|(B(x_0, \epsilon))}{\mathcal{L}^2(B(x_0, \epsilon))} = 0, \tag{4.80}$$

$$\lim_{\varepsilon \to 0^+} \frac{|D^s v|(B(x_0, \epsilon))}{\mathcal{L}^2(B(x_0, \epsilon))} = 0. \tag{4.81}$$

34

We observe that (4.73)–(4.81) hold for $\mathcal{L}^2$-a.e. $x_0 \in \Omega$.

Fix $\varepsilon > 0$. Let $\varphi_\varepsilon \in W_0^{1,\infty}(Q)$ and $\psi_\varepsilon \in W_0^{1,\infty}(Q; T_{v(x_0)}(S^2))$, extended by periodicity to the whole $\mathbb{R}^2$, be such that

$$\mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)) + \varepsilon \geqslant \int_Q f(u(x_0), v(x_0), \nabla u(x_0) + \nabla \varphi_\varepsilon(y), \nabla v(x_0) + \nabla \psi_\varepsilon(y)) \, dy. \quad (4.82)$$

For each $n \in \mathbb{N}$ and $\varepsilon > 0$, consider the function $\Phi_{n,\varepsilon} : \mathbb{R} \to \mathbb{R}$ defined by

$$\Phi_{n,\varepsilon}(r) := \frac{n(\beta - \alpha)r + (\beta + \alpha)\|\varphi_\varepsilon\|_\infty}{n(\beta - \alpha) + 2\|\varphi_\varepsilon\|_\infty}.$$

Then, $\Phi_{n,\varepsilon} \in C^\infty(\mathbb{R})$, $0 < \Phi'_{n,\varepsilon}(r) \leqslant 1$ for all $r \in \mathbb{R}$, and $\{\Phi'_{n,\varepsilon}\}_{n \in \mathbb{N}}$ converges uniformly to 1 in $\mathbb{R}$. Observe that

$$\Phi_{n,\varepsilon}\big|_{[\alpha - \|\varphi_\varepsilon\|_\infty/n, \beta + \|\varphi_\varepsilon\|_\infty/n]} : \left[\alpha - \frac{\|\varphi_\varepsilon\|_\infty}{n}, \beta + \frac{\|\varphi_\varepsilon\|_\infty}{n}\right] \to [\alpha, \beta] \quad (4.83)$$

defines a projection of $\left[\alpha - \|\varphi_\varepsilon\|_\infty/n, \beta + \|\varphi_\varepsilon\|_\infty/n\right]$ onto $[\alpha, \beta]$.

Fix $\delta_0 \in (0, 1/2)$, and consider the nearest point projection $\Pi : y \in B(v(x_0), \delta_0) \mapsto \frac{y}{|y|} \in S^2$ of $B(v(x_0), \delta_0)$ onto S^2, which defines a C^∞ mapping. Let

$$a_\varepsilon := \max\left\{2 + 2|\nabla u(x_0)| + \|\nabla \varphi_\varepsilon\|_\infty, (\|\nabla \Pi\|_\infty + 1)(2 + 2|\nabla v(x_0)| + \|\nabla \psi_\varepsilon\|_\infty)\right\},$$
$$b_\varepsilon := 1 + |\nabla v(x_0)| + \|\nabla \psi_\varepsilon\|_\infty.$$

In view of the continuity properties of f and the regularity of Π, we can find $\ell_\varepsilon \in (0, 1)$ such that

$$|\xi_1|, |\xi_2|, |\eta_1|, |\eta_2| \leqslant a_\varepsilon, \; |\xi_1 - \xi_2|, |\eta_1 - \eta_2| \leqslant \ell_\varepsilon \Rightarrow |f(u(x_0), v(x_0), \xi_1, \eta_1) - f(u(x_0), v(x_0), \xi_2, \eta_2)| \leqslant \varepsilon, \quad (4.84)$$

and there exists $\delta_\varepsilon \in (0, \delta_0)$ such that

$$s_1, s_2 \in B(v(x_0), \delta_\varepsilon) \Rightarrow |\nabla \Pi(s_1) - \nabla \Pi(s_2)| \leqslant \frac{\ell_\varepsilon}{2b_\varepsilon}. \quad (4.85)$$

Let $\{\varsigma_k\}_{k \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $B(x_0, 2\varsigma_k) \subset \Omega$ and $|Du|(\partial B(x_0, \varsigma_k)) = |Dv|(\partial B(x_0, \varsigma_k)) = 0$ for all $k \in \mathbb{N}$. Let $\{\rho_n\}_{n \in \mathbb{N}}$ be the sequence of standard mollifiers defined in (2.1) for $\delta = 1/n$. Choose $n_0 = n_0(x_0) \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $B(x_0, 2\varsigma_1) \subset \{x \in \Omega : \text{dist}(x, \partial \Omega) > 1/n\}$. For $n \geqslant n_0$, we define (see (2.2))

$$u_n(x) := u * \rho_n, \quad v_n := v * \rho_n.$$

Then (see Lemma 2.7-i)), for all $k \in \mathbb{N}$, $u_n \in W^{1,1}(B(x_0, \varsigma_k); [\alpha, \beta]) \cap C^\infty(\overline{B(x_0, \varsigma_k)})$ converges strictly to u in $BV(B(x_0, \varsigma_k))$, and $v_n \in W^{1,1}(B(x_0, \varsigma_k); \overline{B(0, 1)}) \cap C^\infty(\overline{B(x_0, \varsigma_k)}; \mathbb{R}^3)$ converges strictly to v in $BV(B(x_0, \varsigma_k); \mathbb{R}^3)$ as $n \to \infty$; i.e., for all $k \in \mathbb{N}$, $u_n \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(B(x_0, \varsigma_k))$, $v_n \overset{\star}{\rightharpoonup} v$ weakly-$\star$ in $BV(B(x_0, \varsigma_k); \mathbb{R}^3)$, as $n \to \infty$, and

$$\lim_{n \to \infty} \int_{B(x_0, \varsigma_k)} |\nabla u_n| \, dx = |Du|(B(x_0, \varsigma_k)), \qquad \lim_{n \to \infty} \int_{B(x_0, \varsigma_k)} |\nabla v_n| \, dx = |Dv|(B(x_0, \varsigma_k)). \quad (4.86)$$

Without loss of generality, we assume that $n_0 = 1$. Also, in what follows, C_ε represents a positive constant depending on ε but independent of n and k and whose value may change from one instance to another.

<u>Step 1.</u> We construct admissible sequences for $\mathcal{F}(u, v; B(x_0, \varsigma_k))$.

Let $v_{n,k} := \pi_{y_{n,k}} \circ v_n \in W^{1,1}(B(x_0, \varsigma_k); S^2) \cap C^\infty(\overline{B(x_0, \varsigma_k)}; \mathbb{R}^3)$, where $y_{n,k} \in B(0, 1/2)$ is given by Lemma 4.6 applied to $B(x_0; \varsigma_k)$, v_n, and $A_{n,k}^\varepsilon := \{x \in B(x_0, \varsigma_k) : \text{dist}(v_n(x), S^2) > \delta_\varepsilon/2\}$, so that

$$\int_{A_{n,k}^\varepsilon} |\nabla v_{n,k}(x)| \, dx \leqslant C_\star \int_{A_{n,k}^\varepsilon} |\nabla v_n(x)| \, dx. \quad (4.87)$$

Because $\frac{\delta_\varepsilon}{2} \leqslant \frac{1}{4}$, $|v_n(x) - y_{n,k}| \geqslant \frac{1}{4}$ whenever $x \in B(x_0, \varsigma_k) \backslash A^\varepsilon_{n,k}$. Thus, by (4.24), for $x \in B(x_0, \varsigma_k) \backslash A^\varepsilon_{n,k}$, we have

$$|\nabla v_{n,k}(x)| = |\nabla \pi_{y_{n,k}}(v_n(x)) \nabla v_n(x)| \leqslant 4\bar{C}|\nabla v_n(x)|. \tag{4.88}$$

Moreover, by (4.23) and (4.74), $\nabla \pi_{y_{n,k}}(v(x_0)) \nabla v(x_0) = \nabla v(x_0)$. Hence, (4.24), (4.25), and the inequality $|v(x_0) - y_{n,k}| \geqslant \frac{1}{2}$ yield, for $x \in B(x_0, \varsigma_k) \backslash A^\varepsilon_{n,k}$ and $\mathcal{C} := \max\{2\overline{C} + 2\bar{C}, \overline{C}|\nabla v(x_0)|\}$,

$$
\begin{aligned}
&|\nabla v_{n,k}(x) - \nabla v(x_0)| \\
&\leqslant |\nabla \pi_{y_{n,k}}(v_n(x)) \nabla v_n(x) - \nabla \pi_{y_{n,k}}(v(x_0)) \nabla v_n(x)| + |\nabla \pi_{y_{n,k}}(v(x_0)) \nabla v_n(x) - \nabla \pi_{y_{n,k}}(v(x_0)) \nabla v(x_0)| \\
&\leqslant \overline{C}|v_n(x) - v(x_0)||\nabla v_n(x)| + 2\bar{C}|\nabla v_n(x) - \nabla v(x_0)| \\
&\leqslant \mathcal{C}\big(|\nabla v_n(x) - \nabla v(x_0)| + |v_n(x) - v(x_0)|\big).
\end{aligned} \tag{4.89}
$$

We claim that

$$\lim_{n\to\infty} \int_{B(x_0, \varsigma_k)} |v_{n,k}(x) - v(x)|\, dx = 0. \tag{4.90}$$

In fact, using the condition $v(x) \in S^2$ for $\mathcal{L}^2$-a.e. $x \in \Omega$ and the convergence $v_n \to v$ in $L^1(B(x_0, \varsigma_k); \mathbb{R}^3)$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} \mathcal{L}^2(A^\varepsilon_{n,k}) \leqslant \limsup_{n\to\infty} \frac{2}{\delta_\varepsilon} \int_{A^\varepsilon_{n,k}} \mathrm{dist}(v_n(x), S^2)\, dx \leqslant \limsup_{n\to\infty} \frac{2}{\delta_\varepsilon} \int_{B(x_0, \varsigma_k)} |v_n(x) - v(x)|\, dx = 0,$$

and so

$$\lim_{n\to\infty} \mathcal{L}^2(A^\varepsilon_{n,k}) = 0. \tag{4.91}$$

Observing that

$$
\begin{aligned}
\int_{B(x_0, \varsigma_k)} |v_{n,k}(x) - v(x)|\, dx &= \int_{A^\varepsilon_{n,k}} |v_{n,k}(x) - v(x)|\, dx + \int_{B(x_0, \varsigma_k) \backslash A^\varepsilon_{n,k}} |v_{n,k}(x) - v(x)|\, dx \\
&\leqslant 2\mathcal{L}^2(A^\varepsilon_{n,k}) + \overline{C} \int_{B(x_0, \varsigma_k)} |v_n(x) - v(x)|\, dx,
\end{aligned}
$$

where we used (4.22) and (4.25), invoking again the convergence $v_n \to v$ in $L^1(B(x_0, \varsigma_k); \mathbb{R}^3)$ as $n \to \infty$ and (4.91), we obtain (4.90).

Let $\zeta_1 \in C^\infty_c(\mathbb{R}; [0,1])$ and $\zeta_2 \in C^\infty_c(\mathbb{R}^3; [0,1])$ be cut-off functions such that $\|\zeta'_1\|_\infty \leqslant 2/\delta_\varepsilon$, $\|\nabla \zeta_2\|_\infty \leqslant 2/\delta_\varepsilon$, and

$$
\begin{aligned}
\zeta_1(r) = 1 \text{ if } r \in \Big(-\frac{\delta_\varepsilon}{4}, \frac{\delta_\varepsilon}{4} \Big), \quad &\zeta_1(r) = 0 \text{ if } r \notin \Big(-\frac{\delta_\varepsilon}{2}, \frac{\delta_\varepsilon}{2} \Big), \\
\zeta_2(s) = 1 \text{ if } s \in B\Big(0, \frac{\delta_\varepsilon}{4} \Big), \quad &\zeta_2(s) = 0 \text{ if } s \notin B\Big(0, \frac{\delta_\varepsilon}{2} \Big).
\end{aligned}
$$

Set

$$u^\varepsilon_{n,k}(x) := u_n(x) + \frac{1}{n}\zeta_1(u_n(x) - u(x_0))\, \varphi_\varepsilon(nx), \quad x \in B(x_0, \varsigma_k),$$

$$v^\varepsilon_{n,k}(x) := v_{n,k}(x) + \frac{1}{n}\zeta_2(v_{n,k}(x) - v(x_0))\, \psi_\varepsilon(nx), \quad x \in B(x_0, \varsigma_k).$$

Finally, for $n \in \mathbb{N}$ such that $n > \frac{2}{\delta_\varepsilon}\max\big\{\|\varphi_\varepsilon\|_\infty, \|\psi_\varepsilon\|_\infty\big\}$ and for $x \in B(x_0, \varsigma_k)$, define

$$\bar{u}^\varepsilon_{n,k}(x) := \Phi_{n,\varepsilon}(u^\varepsilon_{n,k}(x))$$

and

$$\bar{v}_{n,k}^{\varepsilon}(x) := \begin{cases} v_{n,k}(x) & \text{if } |v_{n,k}(x) - v(x_0)| \geqslant \dfrac{\delta_{\varepsilon}}{2}, \\[2mm] \Pi(v_{n,k}^{\varepsilon}(x)) & \text{if } |v_{n,k}(x) - v(x_0)| < \dfrac{\delta_{\varepsilon}}{2}. \end{cases}$$

By (4.83), we have $\bar{u}_{n,k}^{\varepsilon} \in W^{1,1}(B(x_0, \varsigma_k); [\alpha, \beta])$ and $\bar{v}_{n,k}^{\varepsilon} \in W^{1,1}(B(x_0, \varsigma_k); S^2)$. We claim that

$$\begin{aligned} &\{\bar{u}_{n,k}^{\varepsilon}\}_{n \in \mathbb{N}} \text{ weakly-}\star \text{ converges to } u \text{ in } BV(B(x_0, \varsigma_k)), \\ &\{\bar{v}_{n,k}^{\varepsilon}\}_{n \in \mathbb{N}} \text{ weakly-}\star \text{ converges to } v \text{ in } BV(B(x_0, \varsigma_k); \mathbb{R}^3). \end{aligned} \tag{4.92}$$

In fact, we have

$$\begin{aligned} \int_{B(x_0,\varsigma_k)} |\bar{u}_{n,k}^{\varepsilon}(x) - u(x)| \, \mathrm{d}x &= \int_{B(x_0,\varsigma_k)} \left| \frac{n(\beta - \alpha)u_{n,k}^{\varepsilon}(x) + (\beta + \alpha)\|\varphi_{\varepsilon}\|_{\infty}}{n(\beta - \alpha) + 2\|\varphi_{\varepsilon}\|_{\infty}} - u(x) \right| \mathrm{d}x \\ &\leqslant \frac{n(\beta - \alpha)}{n(\beta - \alpha) + 2\|\varphi_{\varepsilon}\|_{\infty}} \int_{B(x_0,\varsigma_k)} |u_{n,k}^{\varepsilon}(x) - u(x)| \, \mathrm{d}x + \left| \frac{n(\beta - \alpha)}{n(\beta - \alpha) + 2\|\varphi_{\varepsilon}\|_{\infty}} - 1 \right| \int_{B(x_0,\varsigma_k)} |u(x)| \, \mathrm{d}x \\ &\quad + \frac{(\beta + \alpha)\|\varphi_{\varepsilon}\|_{\infty}}{n(\beta - \alpha) + 2\|\varphi_{\varepsilon}\|_{\infty}} \mathcal{L}^2(B(x_0, \varsigma_k)) \\ &\leqslant \int_{B(x_0,\varsigma_k)} |u_n(x) - u(x)| \, \mathrm{d}x + \frac{\|\varphi_{\varepsilon}\|_{\infty}}{n} \mathcal{L}^2(B(x_0, \varsigma_k)) + \frac{2\|\varphi_{\varepsilon}\|_{\infty}}{n(\beta - \alpha) + 2\|\varphi_{\varepsilon}\|_{\infty}} \int_{B(x_0,\varsigma_k)} |u(x)| \, \mathrm{d}x \\ &\quad + \frac{(\beta + \alpha)\|\varphi_{\varepsilon}\|_{\infty}}{n(\beta - \alpha) + 2\|\varphi_{\varepsilon}\|_{\infty}} \mathcal{L}^2(B(x_0, \varsigma_k)). \end{aligned}$$

Letting $n \to \infty$, taking into account that $\{u_n\}_{n \in \mathbb{N}}$ converges to u in $L^1(B(x_0, \varsigma_k))$, we conclude that $\{\bar{u}_{n,k}^{\varepsilon}\}_{n \in \mathbb{N}}$ converges to u in $L^1(B(x_0, \varsigma_k))$. Furthermore, using the fact that $0 < \Phi'_{n,\varepsilon}(r) \leqslant 1$ for all $r \in \mathbb{R}$, we obtain for $\mathcal{L}^2$-a.e. $x \in B(x_0, \varsigma_k)$,

$$\begin{aligned} |\nabla \bar{u}_{n,k}^{\varepsilon}(x)| &= \left| \Phi'_{n,\varepsilon}(u_{n,k}^{\varepsilon}(x)) \nabla u_{n,k}^{\varepsilon}(x) \right| \\ &\leqslant \left| \nabla u_n(x) + \zeta_1(u_n(x) - u(x_0))\nabla \varphi_{\varepsilon}(nx) + \frac{1}{n}\varphi_{\varepsilon}(nx)\zeta_1'(u_n(x) - u(x_0))\nabla u_n(x) \right| \\ &\leqslant |\nabla u_n(x)| + \|\nabla \varphi_{\varepsilon}\|_{\infty} + \frac{\|\varphi_{\varepsilon}\|_{\infty}}{n}\|\zeta_1'\|_{\infty}|\nabla u_n(x)| \\ &\leqslant 2|\nabla u_n(x)| + \|\nabla \varphi_{\varepsilon}\|_{\infty}, \end{aligned} \tag{4.93}$$

provided that $n > 2\|\varphi_{\varepsilon}\|_{\infty}/\delta_{\varepsilon}$, where we used the fact that $\|\zeta_1'\|_{\infty} \leqslant 2/\delta_{\varepsilon}$. This, together with (4.86) and the convergence in $L^1(B(x_0, \varsigma_k))$ proved above, allows us to conclude that $\{\bar{u}_{n,k}^{\varepsilon}\}_{n \in \mathbb{N}}$ is a bounded sequence in $W^{1,1}(B(x_0, \varsigma_k))$ weakly-$\star$ converging to u in $BV(B(x_0, \varsigma_k))$. We observe further that in view of (4.93) we have that

$$|\nabla \bar{u}_{n,k}^{\varepsilon}(x)| \leqslant C_{\varepsilon}(1 + |\nabla u_n(x) - \nabla u(x_0)|) \tag{4.94}$$

for $\mathcal{L}^2$-a.e. $x \in B(x_0, \varsigma_k)$.

We now turn to the sequence $\{\bar{v}_{n,k}^{\varepsilon}\}_{n \in \mathbb{N}}$. We start by proving that $\bar{v}_{n,k}^{\varepsilon} \to v$ in $L^1(B(x_0, \varsigma_k); \mathbb{R}^3)$ as $n \to \infty$. Because $\Pi(v_{n,k}(\cdot)) = v_{n,k}(\cdot)$ in $\{x \in B(x_0, \varsigma_k) : |v_{n,k}(x) - v(x_0)| < \delta_{\varepsilon}/2\}$, it follows that

$$\begin{aligned} \int_{B(x_0,\varsigma_k)} |\bar{v}_{n,k}^{\varepsilon}(x) - v(x)| \, \mathrm{d}x &= \int_{\{x \in B(x_0,\varsigma_k):|v_{n,k}(x)-v(x_0)| \geqslant \frac{\delta_{\varepsilon}}{2}\}} |v_{n,k}(x) - v(x)| \, \mathrm{d}x \\ &\quad + \int_{\{x \in B(x_0,\varsigma_k):|v_{n,k}(x)-v(x_0)| < \frac{\delta_{\varepsilon}}{2}\}} |\Pi(v_{n,k}^{\varepsilon}(x)) - \Pi(v_{n,k}(x) + \Pi(v_{n,k}(x)) - v(x)| \, \mathrm{d}x \\ &\leqslant \int_{B(x_0,\varsigma_k)} |v_{n,k}(x) - v(x)| \, \mathrm{d}x + \|\Pi\|_{1,\infty} \int_{B(x_0,\varsigma_k)} |v_{n,k}^{\varepsilon}(x) - v_{n,k}(x)| \, \mathrm{d}x \end{aligned}$$

37

$$\leqslant \int_{B(x_0,\varsigma_k)} |v_{n,k}(x) - v(x)|\, \mathrm{d}x + \|\Pi\|_{1,\infty}\frac{\|\psi_\varepsilon\|_\infty}{n}\mathcal{L}^2(B(x_0,\varsigma_k)),$$

which, together with (4.90), implies the convergence in $L^1(B(x_0,\varsigma_k);\mathbb{R}^3)$ of $\{\bar{v}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ to v. To estimate the sequence $\{\nabla \bar{v}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$, we observe that if $|v_{n,k}(x) - v(x_0)| > \delta_\varepsilon/2$, then

$$|\nabla \bar{v}_{n,k}^\varepsilon(x)| = |\nabla v_{n,k}(x)|. \tag{4.95}$$

If $|v_{n,k}(x) - v(x_0)| < \delta_\varepsilon/2$, then, arguing as in (4.93), for $n > 2\|\psi_\varepsilon\|_\infty/\delta_\varepsilon$, we have

$$|\nabla \bar{v}_{n,k}^\varepsilon(x)| = \left|\nabla\Pi(v_{n,k}^\varepsilon(x))\nabla v_{n,k}^\varepsilon(x)\right|$$

$$\leqslant \|\nabla\Pi\|_\infty\left|\nabla v_{n,k}(x) + \zeta_2(v_{n,k}(x) - v(x_0))\nabla\psi_\varepsilon(nx) + \frac{1}{n}\psi_\varepsilon(nx)\otimes\nabla\zeta_2(v_{n,k}(x) - v(x_0))\nabla v_{n,k}(x)\right|$$

$$\leqslant \|\nabla\Pi\|_\infty\big(2|\nabla v_{n,k}(x)| + \|\nabla\psi_\varepsilon\|_\infty\big). \tag{4.96}$$

Hence, in view of (4.87), (4.88), (4.86), and the convergence in $L^1(B(x_0,\varsigma_k);\mathbb{R}^3)$ proved above, we infer that $\{\bar{v}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,1}(B(x_0,\varsigma_k);\mathbb{R}^3)$ weakly-$\star$ converging to v in $BV(B(x_0,\varsigma_k);\mathbb{R}^3)$. We observe further that from (4.95) and (4.96), we get

$$|\nabla \bar{v}_{n,k}^\varepsilon(x)| \leqslant C_\varepsilon(1 + |\nabla v_{n,k}(x) - \nabla v(x_0)|) \tag{4.97}$$

for $\mathcal{L}^2$-a.e. $x \in B(x_0,\varsigma_k)$.

We have just proved that $\{\bar{u}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ and $\{\bar{v}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ are admissible sequences for $\mathcal{F}(u,v;B(x_0;\varsigma_k))$, which concludes Step 1.

<u>Step 2.</u> We prove that the sequences $\{\bar{u}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ and $\{\bar{v}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ constructed in Step 1 satisfy

$$\frac{\mathrm{d}\mathcal{F}(u,v;\cdot)}{\mathrm{d}\mathcal{L}^2}(x_0) \leqslant \limsup_{k\to\infty}\limsup_{n\to\infty}\fint_{B(x_0,\varsigma_k)} f(u(x_0), v(x_0), \nabla\bar{u}_{n,k}^\varepsilon(x), \nabla\bar{v}_{n,k}^\varepsilon(x))\, \mathrm{d}x. \tag{4.98}$$

By Step 1, (4.46), and (4.75), we obtain

$$\frac{\mathrm{d}\mathcal{F}(u,v;\cdot)}{\mathrm{d}\mathcal{L}^2}(x_0) = \lim_{k\to\infty}\frac{\mathcal{F}(u,v;B(x_0,\varsigma_k))}{\mathcal{L}^2(B(x_0,\varsigma_k))}$$

$$\leqslant \liminf_{k\to\infty}\liminf_{n\to\infty}\fint_{B(x_0,\varsigma_k)} f(\bar{u}_{n,k}^\varepsilon(x), \bar{v}_{n,k}^\varepsilon(x), \nabla\bar{u}_{n,k}^\varepsilon(x), \nabla\bar{v}_{n,k}^\varepsilon(x))\, \mathrm{d}x.$$

We claim that

$$\liminf_{k\to\infty}\liminf_{n\to\infty}\fint_{B(x_0,\varsigma_k)} f(\bar{u}_{n,k}^\varepsilon(x), \bar{v}_{n,k}^\varepsilon(x), \nabla\bar{u}_{n,k}^\varepsilon(x), \nabla\bar{v}_{n,k}^\varepsilon(x))\, \mathrm{d}x$$

$$\leqslant \limsup_{k\to\infty}\limsup_{n\to\infty}\fint_{B(x_0,\varsigma_k)} f(u(x_0), v(x_0), \nabla\bar{u}_{n,k}^\varepsilon(x), \nabla\bar{v}_{n,k}^\varepsilon(x))\, \mathrm{d}x, \tag{4.99}$$

from which (4.98) follows. Using the definition of f (see (1.13)), (4.94), and (4.97), we get

$$\int_{B(x_0,\varsigma_k)}\left|f(\bar{u}_{n,k}^\varepsilon(x), \bar{v}_{n,k}^\varepsilon(x), \nabla\bar{u}_{n,k}^\varepsilon(x), \nabla\bar{v}_{n,k}^\varepsilon(x)) - f(u(x_0), v(x_0), \nabla\bar{u}_{n,k}^\varepsilon(x), \nabla\bar{v}_{n,k}^\varepsilon(x))\right|\mathrm{d}x$$

$$\leqslant \int_{B(x_0,\varsigma_k)}\big(|\bar{u}_{n,k}^\varepsilon(x) - u(x_0)||\nabla\bar{v}_{n,k}^\varepsilon(x)| + |\bar{v}_{n,k}^\varepsilon(x) - v(x_0)||\nabla\bar{u}_{n,k}^\varepsilon(x)|\big)\ \mathrm{d}x$$

$$\leqslant C_\varepsilon\bigg[\int_{B(x_0,\varsigma_k)}\big(|\bar{u}_{n,k}^\varepsilon(x) - u(x_0)| + |\bar{v}_{n,k}^\varepsilon(x) - v(x_0)|\big)\ \mathrm{d}x$$

$$+ \int_{B(x_0,\varsigma_k)}|\bar{u}_{n,k}^\varepsilon(x) - u(x_0)||\nabla v_{n,k}(x)|\, \mathrm{d}x + \int_{B(x_0,\varsigma_k)}|\bar{v}_{n,k}^\varepsilon(x) - v(x_0)||\nabla u_n(x)|\, \mathrm{d}x\bigg]. \tag{4.100}$$

By (4.92), (4.76), and (4.78), we deduce that

$$\lim_{k\to\infty}\lim_{n\to\infty}\fint_{B(x_0,\varsigma_k)}\left(|\bar{u}_{n,k}^\varepsilon(x)-u(x_0)|+|\bar{v}_{n,k}^\varepsilon(x)-v(x_0)|\right)\mathrm{d}x$$

$$=\lim_{k\to\infty}\fint_{B(x_0,\varsigma_k)}\left(|u(x)-u(x_0)|+|v(x)-v(x_0)|\right)\mathrm{d}x=0. \qquad (4.101)$$

We now estimate the last two integrals in (4.100). Since $|\bar{u}_{n,k}^\varepsilon(\cdot)-u(x_0)|\leqslant 2\beta$, (4.87), and (4.88), we obtain

$$\int_{B(x_0,\varsigma_k)}|\bar{u}_{n,k}^\varepsilon(x)-u(x_0)||\nabla v_{n,k}(x)|\,\mathrm{d}x$$

$$\leqslant 2\beta\int_{A_{n,k}^\varepsilon}|\nabla v_{n,k}(x)|\,\mathrm{d}x+4\bar{C}\int_{B(x_0,\varsigma_k)\setminus A_{n,k}^\varepsilon}|\bar{u}_{n,k}^\varepsilon(x)-u(x_0)||\nabla v_n(x)|\,\mathrm{d}x$$

$$\leqslant 2\beta C_\star\int_{A_{n,k}^\varepsilon}\left(|\nabla v_n(x)-\nabla v(x_0)|+|\nabla v(x_0)|\right)\mathrm{d}x$$

$$+4\bar{C}\int_{B(x_0,\varsigma_k)\setminus A_{n,k}^\varepsilon}\left(2\beta|\nabla v_n(x)-\nabla v(x_0)|+|\bar{u}_{n,k}^\varepsilon(x)-u(x_0)||\nabla v(x_0)|\right)\mathrm{d}x$$

$$\leqslant \tilde{C}\left(\int_{B(x_0,\varsigma_k)}|\nabla v_n(x)-\nabla v(x_0)|\,\mathrm{d}x+\mathcal{L}^2(A_{n,k}^\varepsilon)+\int_{B(x_0,\varsigma_k)}|\bar{u}_{n,k}^\varepsilon(x)-u(x_0)|\,\mathrm{d}x\right), \qquad (4.102)$$

where $\tilde{C}:=\max\{2\beta(C_\star+4\bar{C}),2\beta C_\star|\nabla v(x_0)|,4\bar{C}|\nabla v(x_0)|\}$. Because $v_n=v*\rho_n$ and $|Dv|(\partial B(x_0,\varsigma_k))=0$, we have

$$\lim_{n\to\infty}\int_{B(x_0,\varsigma_k)}|\nabla v_n(x)-\nabla v(x_0)|\,\mathrm{d}x=\int_{B(x_0,\varsigma_k)}|\nabla v(x)-\nabla v(x_0)|\,\mathrm{d}x+|D^s v|(B(x_0,\varsigma_k)). \qquad (4.103)$$

Recalling that $\{\bar{u}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ converges to u in $L^1(B(x_0,\varsigma_k))$ (see (4.92)), from (4.102), (4.103), (4.91), (4.79), (4.81), and (4.76), we deduce that

$$\lim_{k\to\infty}\lim_{n\to\infty}\fint_{B(x_0,\varsigma_k)}|\bar{u}_{n,k}^\varepsilon(x)-u(x_0)||\nabla v_{n,k}(x)|\,\mathrm{d}x=0. \qquad (4.104)$$

Finally, we estimate the last integral in (4.100). We have that

$$\int_{B(x_0,\varsigma_k)}|\bar{v}_{n,k}^\varepsilon(x)-v(x_0)||\nabla u_n(x)|\,\mathrm{d}x$$

$$\leqslant 2\int_{B(x_0,\varsigma_k)}|\nabla u_n(x)-\nabla u(x_0)|\,\mathrm{d}x+\int_{B(x_0,\varsigma_k)}|\bar{v}_{n,k}^\varepsilon(x)-v(x_0)||\nabla u(x_0)|\,\mathrm{d}x. \qquad (4.105)$$

Arguing as above, an equality for u similar to that in (4.103) holds; that is,

$$\lim_{n\to\infty}\int_{B(x_0,\varsigma_k)}|\nabla u_n(x)-\nabla u(x_0)|\,\mathrm{d}x=\int_{B(x_0,\varsigma_k)}|\nabla u(x)-\nabla u(x_0)|\,\mathrm{d}x+|D^s u|(B(x_0,\varsigma_k)). \qquad (4.106)$$

Recalling that $\{\bar{v}_{n,k}^\varepsilon\}_{n\in\mathbb{N}}$ converges to v in $L^1(B(x_0,\varsigma_k);\mathbb{R}^3)$ (see (4.92)), from (4.105), (4.106), (4.77), (4.80), and (4.78), we deduce that

$$\lim_{k\to\infty}\lim_{n\to\infty}\fint_{B(x_0,\varsigma_k)}|\bar{v}_{n,k}^\varepsilon(x)-v(x_0)||\nabla u_n(x)|\,\mathrm{d}x=0. \qquad (4.107)$$

Hence, (4.99) follows from (4.100), (4.101), (4.104), and (4.107).

<u>Step 3.</u> We conclude the proof of Lemma 4.14.

Set

$$z_n^\varepsilon(x):=\nabla u(x_0)\,x+\frac{1}{n}\varphi_\varepsilon(nx), \quad w_n^\varepsilon(x):=\nabla v(x_0)\,x+\frac{1}{n}\psi_\varepsilon(nx),$$

39

and observe that

$$|\nabla z_n^\varepsilon(x)| \leqslant |\nabla u(x_0)| + \|\nabla\varphi_\varepsilon\|_\infty \leqslant a_\varepsilon, \quad |\nabla w_n^\varepsilon(x)| \leqslant |\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty \leqslant a_\varepsilon.$$

Additionally, let

$$\gamma_\varepsilon := \frac{\ell_\varepsilon}{2(\|\nabla\Pi\|_\infty + 1)},$$

and define

$$B_u := \left\{ x \in B(x_0, \varsigma_k) \colon |u_n(x) - u(x_0)| < \frac{\delta_\varepsilon}{4} \right\}, \qquad B_{\nabla u} := \left\{ x \in B(x_0, \varsigma_k) \colon |\nabla u_n(x) - \nabla u(x_0)| < \gamma_\varepsilon \right\},$$

$$B_v := \left\{ x \in B(x_0, \varsigma_k) \colon |v_{n,k}(x) - v(x_0)| < \frac{\delta_\varepsilon}{4} \right\}, \qquad B_{\nabla v} := \left\{ x \in B(x_0, \varsigma_k) \colon |\nabla v_{n,k}(x) - \nabla v(x_0)| < \gamma_\varepsilon \right\}.$$

If $x \in B_u \cap B_{\nabla u} \cap B_v \cap B_{\nabla v}$, then $\zeta_1(u_n(x) - u(x_0)) = 1$, $\zeta_2(v_{n,k}(x) - v(x_0)) = 1$, and, by (4.93) and (4.96), for $n > 2/\delta_\varepsilon \max\{\|\varphi_\varepsilon\|_\infty, \|\psi_\varepsilon\|_\infty\}$, we have

$$|\nabla \bar{u}_{n,k}^\varepsilon(x)| \leqslant 2\gamma_\varepsilon + 2|\nabla u(x_0)| + \|\nabla\varphi_\varepsilon\|_\infty \leqslant a_\varepsilon,$$
$$|\nabla \bar{v}_{n,k}^\varepsilon(x)| \leqslant \|\nabla\Pi\|_\infty \big(2\gamma_\varepsilon + 2|\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty\big) \leqslant a_\varepsilon.$$

Moreover, using the fact that $\Phi'_{n,\varepsilon}(r) = n(\beta - \alpha)/(n(\beta - \alpha) + 2\|\varphi_\varepsilon\|_\infty) \in (0,1]$ for all $r \in \mathbb{R}$, we get

$$\begin{aligned}
|\nabla \bar{u}_{n,k}^\varepsilon(x) - \nabla z_n^\varepsilon(x)| &= \left| \Phi'_{n,\varepsilon}\Big(u_n(x) + \frac{1}{n}\varphi_\varepsilon(nx)\Big)\big(\nabla u_n(x) \mp \nabla u(x_0) + \nabla\varphi_\varepsilon(nx)\big) - \big(\nabla u(x_0) + \nabla\varphi_\varepsilon(nx)\big) \right| \\
&\leqslant |\nabla u_n(x) - \nabla u(x_0)| + \left| \Phi'_{n,\varepsilon}\Big(u_n(x) + \frac{1}{n}\varphi_\varepsilon(nx)\Big) - 1 \right| |\nabla u(x_0) + \nabla\varphi_\varepsilon(nx)| \\
&\leqslant \gamma_\varepsilon + \frac{2\|\varphi_\varepsilon\|_\infty}{n(\beta - \alpha) + 2\|\varphi_\varepsilon\|_\infty}\big(|\nabla u(x_0)| + \|\nabla\varphi_\varepsilon\|_\infty\big) \\
&\leqslant \frac{\ell_\varepsilon}{2} + \frac{\ell_\varepsilon}{2} = \ell_\varepsilon,
\end{aligned}$$

provided that

$$n > \frac{2\|\varphi_\varepsilon\|_\infty\big(2|\nabla u(x_0)| + 2\|\nabla\varphi_\varepsilon\|_\infty - \ell_\varepsilon\big)}{(\beta - \alpha)\ell_\varepsilon}.$$

Next, using the second condition in (4.74) and the equality $\nabla\Pi(v(x_0))\nabla\psi_\varepsilon(\cdot) = \nabla\psi_\varepsilon(\cdot)$, which holds for $\mathcal{L}^2$-a.e. in $\mathbb{R}^2$ since $\nabla\psi_\varepsilon(\cdot) \in [T_{v(x_0)}(S^2)]^2$ for $\mathcal{L}^2$-a.e. in $\mathbb{R}^2$, we obtain

$$\begin{aligned}
&|\nabla \bar{v}_{n,k}^\varepsilon(x) - \nabla w_n^\varepsilon(x)| \\
&= \left| \nabla\Pi\Big(v_{n,k}(x) + \frac{1}{n}\psi_\varepsilon(nx)\Big)\big(\nabla v_{n,k}(x) - \nabla v(x_0) + \nabla v(x_0) + \nabla\psi_\varepsilon(nx)\big) - \big(\nabla v(x_0) + \nabla\psi_\varepsilon(nx)\big) \right| \\
&\leqslant \|\nabla\Pi\|_\infty |\nabla v_{n,k}(x) - \nabla v(x_0)| + \left| \nabla\Pi\Big(v_{n,k}(x) + \frac{1}{n}\psi_\varepsilon(nx)\Big) - \nabla\Pi(v(x_0)) \right| |\nabla v(x_0) + \nabla\psi_\varepsilon(nx)| \\
&\leqslant \|\nabla\Pi\|_\infty \gamma_\varepsilon + \frac{\ell_\varepsilon}{2b_\varepsilon}\big(|\nabla v(x_0)| + \|\nabla\psi_\varepsilon\|_\infty\big) \leqslant \frac{\ell_\varepsilon}{2} + \frac{\ell_\varepsilon}{2} = \ell_\varepsilon,
\end{aligned}$$

provided that $n > 2\|\psi_\varepsilon\|_\infty/\delta_\varepsilon$, because for all such $n \in \mathbb{N}$, we have $|v_{n,k}(x) + \frac{1}{n}\psi_\varepsilon(nx) - v(x_0)| \leqslant \delta_\varepsilon/4 + \delta_\varepsilon/2 < \delta_\varepsilon$, and so (4.85) applies.

Thus, using (4.84), Riemann–Lebesgue's Lemma, and (4.82), in this order, we conclude that

$$\begin{aligned}
&\limsup_{k\to\infty}\limsup_{n\to\infty} \frac{1}{\mathcal{L}^2(B(x_0,\varsigma_k))} \int_{B_u \cap B_{\nabla u} \cap B_v \cap B_{\nabla v}} f\big(u(x_0), v(x_0), \nabla \bar{u}_{n,k}^\varepsilon(x), \nabla \bar{v}_{n,k}^\varepsilon(x)\big)\, \mathrm{d}x \\
&\qquad \leqslant \limsup_{k\to\infty}\limsup_{n\to\infty} \fint_{B(x_0,\varsigma_k)} f\big(u(x_0), v(x_0), \nabla u(x_0) + \nabla\varphi_\varepsilon(nx), \nabla v(x_0) + \nabla\psi_\varepsilon(nx)\big)\, \mathrm{d}x + \varepsilon
\end{aligned}$$

40

$$= \int_Q f\big(u(x_0), v(x_0), \nabla u(x_0) + \nabla \varphi_\varepsilon(y), \nabla v(x_0) + \nabla \psi_\varepsilon(y)\big) \, \mathrm{d}y + \varepsilon$$
$$\leqslant \mathcal{Q}_T f\big(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)\big) + 2\varepsilon. \tag{4.108}$$

Next, we observe that from (4.87) and (4.89), we have, for $c := \max\{C_\star + 1, \mathcal{C}, (C_\star + 1)|\nabla v(x_0)|\}$,

$$\fint_{B(x_0, \varsigma_k)} |\nabla v_{n,k}(x) - \nabla v(x_0)| \, \mathrm{d}x \leqslant (C_\star + 1) \fint_{A_{n,k}^\varepsilon} \big(|\nabla v_n(x) - \nabla v(x_0)| + |\nabla v(x_0)|\big) \, \mathrm{d}x$$
$$+ \mathcal{C}\bigg(\fint_{B(x_0, \varsigma_k) \setminus A_{n,k}^\varepsilon} \big(|\nabla v_n(x) - \nabla v(x_0)| + |v_n(x) - v(x_0)|\big) \mathrm{d}x \bigg)$$
$$\leqslant c\bigg(\fint_{B(x_0, \varsigma_k)} \big(|\nabla v_n(x) - \nabla v(x_0)| + |v_n(x) - v(x_0)|\big) \mathrm{d}x + \mathcal{L}^2(A_{n,k}^\varepsilon) \bigg). \tag{4.109}$$

Note also that

$$\frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k) \setminus B_u} 1 \, \mathrm{d}x \leqslant \frac{4}{\delta_\varepsilon} \fint_{B(x_0, \varsigma_k)} |u_n(x) - u(x_0)| \, \mathrm{d}x, \tag{4.110}$$

and, by (1.14), (4.94), and (4.97),

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k) \setminus B_u} f(u(x_0), v(x_0), \nabla \bar{u}_{n,k}^\varepsilon(x), \nabla \bar{v}_{n,k}^\varepsilon(x)) \, \mathrm{d}x$$
$$\leqslant (3 + \beta) \limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k) \setminus B_u} \big(|\nabla \bar{u}_{n,k}^\varepsilon(x)| + |\nabla \bar{v}_{n,k}^\varepsilon(x)|\big) \, \mathrm{d}x$$
$$\leqslant C_\varepsilon \limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k) \setminus B_u} \big(1 + |\nabla u_n(x) - \nabla u(x_0)| + |\nabla v_{n,k}(x) - \nabla v(x_0)|\big) \, \mathrm{d}x. \tag{4.111}$$

Plugging in (4.109) and (4.110) in (4.111), from the convergences $u_n \to u$ in $L^1(B(x_0, \varsigma_k))$ and $v_n \to v$ in $L^1(B(x_0, \varsigma_k); \mathbb{R}^3)$, as $n \to \infty$, and from (4.91), (4.103), (4.106), (4.76), (4.77), (4.79), (4.80), and (4.81), it follows that

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k) \setminus B_u} f(u(x_0), v(x_0), \nabla \bar{u}_{n,k}^\varepsilon(x), \nabla \bar{v}_{n,k}^\varepsilon(x)) \, \mathrm{d}x = 0. \tag{4.112}$$

Similarly, using in addition (4.90) and (4.78),

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k) \setminus B_v} f(u(x_0), v(x_0), \nabla \bar{u}_{n,k}^\varepsilon(x), \nabla \bar{v}_{n,k}^\varepsilon(x)) \, \mathrm{d}x = 0. \tag{4.113}$$

Also, since

$$\fint_{B(x_0, \varsigma_k) \setminus B_{\nabla u}} 1 \, \mathrm{d}x \leqslant \frac{1}{\gamma_\varepsilon} \fint_{B(x_0, \varsigma_k)} |\nabla u_n(x) - \nabla u(x_0)| \, \mathrm{d}x,$$
$$\fint_{B(x_0, \varsigma_k) \setminus B_{\nabla v}} 1 \, \mathrm{d}x \leqslant \frac{1}{\gamma_\varepsilon} \fint_{B(x_0, \varsigma_k)} |\nabla v_{n,k}(x) - \nabla v(x_0)| \, \mathrm{d}x,$$

we have

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{(B_u \cap B_v) \setminus B_{\nabla u}} f(u(x_0), v(x_0), \nabla \bar{u}_{n,k}^\varepsilon(x), \nabla \bar{v}_{n,k}^\varepsilon(x)) \, \mathrm{d}x = 0 \tag{4.114}$$

and

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\mathcal{L}^2(B(x_0, \varsigma_k))} \int_{(B_u \cap B_v) \setminus B_{\nabla v}} f(u(x_0), v(x_0), \nabla \bar{u}_{n,k}^\varepsilon(x), \nabla \bar{v}_{n,k}^\varepsilon(x)) \, \mathrm{d}x = 0. \tag{4.115}$$

Finally, owing to (4.98), (4.108), and (4.112)–(4.115), we conclude that

$$\frac{\mathrm{d}\mathcal{F}(u, v; \cdot)}{\mathrm{d}\mathcal{L}^2}(x_0) \leqslant \mathcal{Q}_T f(u(x_0), v(x_0), \nabla u(x_0), \nabla v(x_0)) + 2\varepsilon,$$

and (4.72) follows by letting $\varepsilon \to 0^+$. $\qquad \square$

Lemma 4.15. *The infimum in* (4.46) *does not change if we replace f by $\mathcal{Q}_T f$.*

Proof. For $(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$ and $A \in \mathcal{A}(\Omega)$, set

$$
\mathcal{QF}(u, v; A) := \inf \Big\{ \liminf_{n \to +\infty} \int_A \mathcal{Q}_T f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x)) \, dx :
$$

$$
n \in \mathbb{N}, \ (u_n, v_n) \in W^{1,1}(A; [\alpha, \beta]) \times W^{1,1}(A; S^2), \ u_n \to u \text{ in } L^1(A), v_n \to v \text{ in } L^1(A; \mathbb{R}^3) \Big\}.
$$

The inequality $\mathcal{QF}(u, v; A) \leqslant \mathcal{F}(u, v; A)$ follows from the fact that $\mathcal{Q}_T f \leqslant f$. To prove the converse inequality, let $(\bar{u}, \bar{v}) \in W^{1,1}(A; [\alpha, \beta]) \times W^{1,1}(A; S^2)$. Using the growth conditions (1.14) satisfied by f, we conclude that

$$
\mathcal{F}(\bar{u}, \bar{v}; A) \leqslant \int_A f(\bar{u}(x), \bar{v}(x), \nabla \bar{u}(x), \nabla \bar{v}(x)) \, dx \leqslant 2 \int_A |\nabla \bar{u}(x)| \, dx + (1 + \beta) \int_A |\nabla \bar{v}(x)| \, dx,
$$

which proves that $A \in \mathcal{A}(\Omega) \mapsto \mathcal{F}(\bar{u}, \bar{v}; A)$ is absolutely continuous with respect to the Lebesgue measure. Hence, by Lemma 4.14, we conclude that

$$
\mathcal{F}(\bar{u}, \bar{v}; A) \leqslant \int_A \mathcal{Q}_T f(\bar{u}(x), \bar{v}(x), \nabla \bar{u}(x), \nabla \bar{v}(x)) \, dx. \tag{4.116}
$$

Fix $(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$. Let $\{(u_n, v_n)\}_{n \in \mathbb{N}} \subset W^{1,1}(A; [\alpha, \beta]) \times W^{1,1}(A; S^2)$ be such that $u_n \to u$ in $L^1(A)$ and $v_n \to v$ in $L^1(A; \mathbb{R}^3)$. The sequential lower semicontinuity of $\mathcal{F}(\cdot, \cdot; A)$ with respect to the strong convergence in $L^1(A) \times L^1(A; \mathbb{R}^3)$, together with (4.116), yields

$$
\mathcal{F}(u, v; A) \leqslant \liminf_{n \to \infty} \mathcal{F}(u_n, v_n; A) \leqslant \liminf_{n \to \infty} \int_A \mathcal{Q}_T f(u_n(x), v_n(x), \nabla u_n(x), \nabla v_n(x)) \, dx.
$$

Taking the infimum over all admissible sequences, we conclude that $\mathcal{F}(u, v; A) \leqslant \mathcal{QF}(u, v; A)$. $\qquad \square$

Lemma 4.16. *Fix $(u, v) \in BV(\Omega; [\alpha, \beta]) \times BV(\Omega; S^2)$. Then,*

$$
\frac{d\mathcal{F}(u, v; \cdot)}{d|D^c(u, v)|}(x_0) \leqslant (\mathcal{Q}_T f)^{\infty}\big(\tilde{u}(x_0), \tilde{v}(x_0), W_u^c(x_0), W_v^c(x_0)\big) \tag{4.117}
$$

for $|D^c(u, v)|$-a.e. $x_0 \in \Omega$.

Proof. Set $w := (u, v)$, and define $\nu := |Dw| - |D^c w|$. Let $x_0 \in \Omega$ be such that

$$
\tilde{w}(x_0) = (\tilde{u}(x_0), \tilde{v}(x_0)) \in [\alpha, \beta] \times S^2, \quad W_v^c(x_0) \in [T_{\tilde{v}(x_0)}(S^2)]^2, \tag{4.118}
$$

$$
\frac{d\mathcal{F}(w; \cdot)}{d|D^c w|}(x_0) \text{ exists and is finite,} \tag{4.119}
$$

$$
\lim_{\epsilon \to 0^+} \frac{\nu(B(x_0, \epsilon))}{|D^c w|(B(x_0, \epsilon))} = 0, \tag{4.120}
$$

$$
\lim_{\epsilon \to 0^+} \fint_{B(x_0, \epsilon)} |\tilde{w}(x) - \tilde{w}(x_0)| \, d|D^c w|(x) = 0, \tag{4.121}
$$

$$
\lim_{\epsilon \to 0^+} \fint_{B(x_0, \epsilon)} |W^c(x) - W^c(x_0)| \, d|D^c w|(x) = 0, \tag{4.122}
$$

$$
\lim_{\epsilon \to 0^+} \fint_{B(x_0, \varsigma_k)} \Big| (\mathcal{Q}\tilde{f})^{\infty}(\tilde{u}(x), \tilde{v}(x), W_u^c(x), W_v^c(x))
$$

$$
- (\mathcal{Q}\tilde{f})^{\infty}(\tilde{u}(x_0), \tilde{v}(x_0), W_u^c(x_0), W_v^c(x_0)) \Big| \, d|D^c w|(x) = 0. \tag{4.123}
$$

We observe that (4.118)–(4.123) hold for $|D^c w|$-a.e. $x_0 \in \Omega$.

As in the proof of Lemma 4.14, let $\{\varsigma_k\}_{k \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $B(x_0, 2\varsigma_k) \subset \Omega$ and $|Du|(\partial B(x_0, \varsigma_k)) = |Dv|(\partial B(x_0, \varsigma_k)) = 0 = |Dw|(\partial B(x_0, \varsigma_k))$ for all $k \in \mathbb{N}$. Let $\{\rho_n\}_{n \in \mathbb{N}}$

be the sequence of standard mollifiers defined in (2.1) for $\delta = 1/n$. Without loss of generality, we may assume that for all $n \in \mathbb{N}$, we have $B(x_0, 2\varsigma_1) \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 1/n\}$, and thus define (see (2.2))

$$u_n(x) := u * \rho_n, \quad v_n := v * \rho_n, \quad w_n := w * \rho_n = (u_n, v_n).$$

Then, $u_n \in W^{1,1}(B(x_0, \varsigma_k); [\alpha, \beta]) \cap C^\infty(\overline{B(x_0, \varsigma_k)})$, $v_n \in W^{1,1}(B(x_0, \varsigma_k); \overline{B(0,1)}) \cap C^\infty(\overline{B(x_0, \varsigma_k)}; \mathbb{R}^3)$, $u_n \xrightarrow{\star} u$ weakly-$\star$ in $BV(B(x_0, \varsigma_k))$, and $v_n \xrightarrow{\star} v$ weakly-$\star$ in $BV(B(x_0, \varsigma_k); \mathbb{R}^3)$, as $n \to \infty$, for all $k \in \mathbb{N}$.

Fix $\delta \in (0, 1/4)$. For $n, k \in \mathbb{N}$, consider the function $v_{n,k} := \pi_{y_{n,k}} \circ v_n \in W^{1,1}(B(x_0, \varsigma_k); S^2) \cap C^\infty(\overline{B(x_0, \varsigma_k)}; \mathbb{R}^3)$ with $y_{n,k} \in B(0, 1/2)$ given by Lemma 4.6 applied to $B(x_0; \varsigma_k)$, v_n, and $A_{n,k} := \{x \in B(x_0, \varsigma_k) : \operatorname{dist}(v_n(x), S^2) > \delta\}$. Then, arguing as in Step 1 of the proof of Lemma 4.14, for all $n, k \in \mathbb{N}$ and $x \in B(x_0, \varsigma_k) \backslash A_{n,k}$,

$$\lim_{n\to\infty} \mathcal{L}^2(A_{n,k}) = 0, \quad \int_{A_{n,k}} |\nabla v_{n,k}(x)| \, \mathrm{d}x \leqslant C_\star \int_{A_{n,k}} |\nabla v_n(x)| \, \mathrm{d}x, \tag{4.124}$$

$$|\nabla v_{n,k}(x)| \leqslant 4\bar{C} |\nabla v_n(x)|, \quad |v_{n,k}(x) - \tilde{v}(x_0)| \leqslant \overline{C} |v_n(x) - \tilde{v}(x_0)|. \tag{4.125}$$

Step 1. We prove that

$$\frac{\mathrm{d}\mathcal{F}(w; \cdot)}{\mathrm{d}|D^c w|}(x_0)$$
$$\leqslant \limsup_{k\to\infty} \limsup_{n\to\infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \left(\int_{B(x_0, \varsigma_k)} \mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), \nabla u_n(x), \nabla v_{n,k}(x)) \, \mathrm{d}x + c\, I_{n,k} \right), \tag{4.126}$$

where c is the constant in (4.10) and

$$I_{n,k} := \int_{B(x_0, \varsigma_k)} \big(|u_n(x) - \tilde{u}(x_0)| + |v_{n,k}(x) - \tilde{v}(x_0)| \big) \big(|\nabla u_n(x)| + |\nabla v_{n,k}(x)| \big) \, \mathrm{d}x. \tag{4.127}$$

The sequence $\{(u_n, v_{n,k})\}_{n \in \mathbb{N}}$ is admissible for $\mathcal{F}(w; B(x_0; \varsigma_k))$, thus, in view of (4.119) and Lemma 4.15, we have

$$\frac{\mathrm{d}\mathcal{F}(w; \cdot)}{\mathrm{d}|D^c w|}(x_0) = \lim_{k\to\infty} \frac{\mathcal{F}(w; B(x_0, \varsigma_k))}{|D^c w|(B(x_0, \varsigma_k))}$$
$$\leqslant \liminf_{k\to\infty} \liminf_{n\to\infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} \mathcal{Q}_T f(u_n(x), v_{n,k}(x), \nabla u_n(x), \nabla v_{n,k}(x)) \, \mathrm{d}x.$$

Observing that $\nabla v_{n,k}(\cdot) \in [T_{v_{n,k}(\cdot)}(S^2)]^2$ $\mathcal{L}^2$-a.e. in $B(x_0; \varsigma_k)$ and using (4.118), then (4.6) and (4.10) applied to $r = u_{n,k}(x)$, $\bar{r} = \tilde{u}(x_0)$, $s = v_{n,k}(x)$, $\bar{s} = \tilde{v}(x_0)$, $\xi = \nabla u_n$, and $\eta = \nabla v_{n,k}$ entails

$$\mathcal{Q}_T f(u_n(x), v_{n,k}(x), \nabla u_n(x), \nabla v_{n,k}(x)) = \mathcal{Q}\tilde{f}(u_n(x), v_{n,k}(x), \nabla u_n(x), \nabla v_{n,k}(x))$$
$$\leqslant \mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), \nabla u_n(x), \nabla v_{n,k}(x))$$
$$+ c\big(|u_n(x) - \tilde{u}(x_0)| + |v_{n,k}(x) - \tilde{v}(x_0)| \big) \big(|\nabla u_n(x)| + |\nabla v_{n,k}(x)| \big),$$

from which (4.126) follows.

Step 2. We prove that

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} I_{n,k} = 0, \tag{4.128}$$

where $I_{n,k}$ is the integral defined in (4.127).

In this step we will denote by $\mathcal{C}_\delta$ any positive constant only depending on δ, β, $C_\star$, and $\bar{C}$. By (4.125), recalling that $v_{n,k}(\cdot)$, $\tilde{v}(x_0) \in S^2$, $u_n(\cdot)$, $\tilde{u}(x_0) \in [\alpha, \beta]$, and using Lemma 2.7-ii) applied to u and $\mathfrak{h}(\cdot) := |u_n(\cdot) - \tilde{u}(x_0)| + |v_n(\cdot) - \tilde{v}(x_0)|$, we have

$$\int_{B(x_0, \varsigma_k)} \big(|u_n(x) - \tilde{u}(x_0)| + |v_{n,k}(x) - \tilde{v}(x_0)| \big) |\nabla u_n(x)| \, \mathrm{d}x$$

43

$$\leqslant (2\beta + 2) \int_{A_{n,k}} |\nabla u_n(x)|\,\mathrm{d}x + \int_{B(x_0,\varsigma_k)\setminus A_{n,k}} \big(|u_n(x) - \tilde{u}(x_0)| + \overline{C}|v_n(x) - \tilde{v}(x_0)|\big)|\nabla u_n(x)|\,\mathrm{d}x$$

$$\leqslant \frac{2\beta + 2}{\delta} \int_{A_{n,k}} |v_n(x) - \tilde{v}(x_0)||\nabla u_n(x)|\,\mathrm{d}x$$

$$+ \int_{B(x_0,\varsigma_k)\setminus A_{n,k}} \big(|u_n(x) - \tilde{u}(x_0)| + \overline{C}|v_n(x) - \tilde{v}(x_0)|\big)|\nabla u_n(x)|\,\mathrm{d}x$$

$$\leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k)} \big(|u_n(x) - \tilde{u}(x_0)| + |v_n(x) - \tilde{v}(x_0)|\big)|\nabla u_n(x)|\,\mathrm{d}x$$

$$\leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k+\frac{1}{n})} \Big[(|u_n - \tilde{u}(x_0)| * \rho_n)(x) + (|v_n - \tilde{v}(x_0)| * \rho_n)(x)\Big]\,\mathrm{d}|Du|(x)$$

$$\leqslant 2\mathcal{C}_\delta \int_{B(x_0,\varsigma_k+\frac{1}{n})} (|w_n - \tilde{w}(x_0)| * \rho_n)(x)\,\mathrm{d}|Du|(x).$$

Similarly,

$$\int_{B(x_0,\varsigma_k)} \big(|u_n(x) - \tilde{u}(x_0)| + |v_{n,k}(x) - \tilde{v}(x_0)|\big)|\nabla v_{n,k}(x)|\,\mathrm{d}x$$

$$\leqslant (2\beta + 2)C_\star \int_{A_{n,k}} |\nabla v_n(x)|\,\mathrm{d}x + 4\bar{C} \int_{B(x_0,\varsigma_k)\setminus A_{n,k}} \big(|u_n(x) - \tilde{u}(x_0)| + \overline{C}|v_n(x) - \tilde{v}(x_0)|\big)|\nabla v_n(x)|\,\mathrm{d}x$$

$$\leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k)} \big(|u_n(x) - \tilde{u}(x_0)| + |v_n(x) - \tilde{v}(x_0)|\big)|\nabla v_n(x)|\,\mathrm{d}x$$

$$\leqslant 2\mathcal{C}_\delta \int_{B(x_0,\varsigma_k+\frac{1}{n})} (|w_n - \tilde{w}(x_0)| * \rho_n)(x)\,\mathrm{d}|Dv|(x). \tag{4.129}$$

Hence, since $|Du| + |Dv| \leqslant 2|Dw|$ in $\mathcal{B}(\Omega)$, we deduce that

$$I_{n,k} \leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k+\frac{1}{n})} (|w_n - \tilde{w}(x_0)| * \rho_n)(x)\,\mathrm{d}|Dw|(x).$$

Using the estimate $\||w_n - \tilde{w}(x_0)| * \rho_n\|_{L^\infty(B(x_0,\varsigma_k+\frac{1}{n}))} \leqslant 2(\beta + 1)$, we obtain

$$I_{n,k} \leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k+\frac{1}{n})\setminus S_w} (|w_n - \tilde{w}(x_0)| * \rho_n)(x)\,\mathrm{d}|Dw|(x) + \mathcal{C}_\delta|Dw|\Big(B\Big(x_0,\varsigma_k + \frac{1}{n}\Big) \cap S_w\Big)$$

$$\leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k+\frac{1}{n})\setminus S_w} \big((|w_n - w| * \rho_n)(x) + (|w - \tilde{w}(x_0)| * \rho_n)(x)\big)\,\mathrm{d}|Dw|(x)$$

$$+ \mathcal{C}_\delta|Dw|\Big(B\Big(x_0,\varsigma_k + \frac{1}{n}\Big) \cap S_w\Big).$$

Define $\bar{w}(\cdot) := |w(\cdot) - \tilde{w}(x_0)|$. By Proposition 2.6 *(a)-ii)* and *(a)-iii)* applied to $\bar{w}$ and to w, respectively, we conclude that

$$\lim_{n\to\infty} (|w - \tilde{w}(x_0)| * \rho_n)(x) = \tilde{\bar{w}}(x) = |\tilde{w}(x) - \tilde{w}(x_0)| \quad \text{for all } x \in A_w = \Omega\setminus S_w,$$

while in view of Lemma 2.7-*iv)* applied to w,

$$\lim_{n\to\infty} (|w_n - w| * \rho_n)(x) = 0 \quad \text{for all } x \in A_w = \Omega\setminus S_w.$$

These last two limits, together with Lebesgue's Dominated and Monotone Convergence Theorems, yield

$$\limsup_{n\to\infty} I_{n,k} \leqslant \mathcal{C}_\delta \int_{\overline{B(x_0,\varsigma_k)}\setminus S_w} |\tilde{w}(x) - \tilde{w}(x_0)|\,\mathrm{d}|Dw|(x) + \mathcal{C}_\delta|Dw|\big(\overline{B(x_0,\varsigma_k)} \cap S_w\big)$$

$$\leqslant \mathcal{C}_\delta \int_{B(x_0,\varsigma_k)} |\tilde{w}(x) - \tilde{w}(x_0)|\,\mathrm{d}|D^c w|(x) + \mathcal{C}_\delta \nu(B(x_0,\varsigma_k)) + \mathcal{C}_\delta|Dw|(B(x_0,\varsigma_k) \cap S_w),$$

44

where we used the fact that $|Dw|(\partial B(x_0, \varsigma_k)) = 0$ and the equality $|Dw| = |D^c w| + \nu$. We observe that $|Dw|(B(x_0, \varsigma_k) \cap S_w) = \nu(B(x_0, \varsigma_k) \cap S_w) + |D^c w|(B(x_0, \varsigma_k) \cap S_w) = \nu(B(x_0, \varsigma_k) \cap S_w) \leqslant \nu(B(x_0, \varsigma_k))$; hence, by (4.121) and (4.120),

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} I_{n,k}$$
$$\leqslant \mathcal{C}_\delta \limsup_{k \to \infty} \left(\fint_{B(x_0, \varsigma_k)} |\tilde{w}(x) - \tilde{w}(x_0)| \, \mathrm{d}|D^c w|(x) + \frac{\nu(B(x_0, \varsigma_k))}{|D^c w|(B(x_0, \varsigma_k))} \right) = 0,$$

and we conclude (4.128).

Step 3. We show that

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} \mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), \nabla u_n(x), \nabla v_{n,k}(x)) \, \mathrm{d}x$$
$$\leqslant (\mathcal{Q}_T f)^\infty (\tilde{u}(x_0), \tilde{v}(x_0), W_u^c(x_0), W_v^c(x_0)). \tag{4.130}$$

As in [3, Prop. 4.2], we define a function $\mathfrak{z} : \mathbb{R}^{4 \times 2} \to [0, +\infty)$ by setting

$$\mathfrak{z}(\zeta) := \sup_{t > 0} \frac{\mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), t\xi, t\eta)}{t}, \quad \zeta \in \mathbb{R}^{4 \times 2},$$

where ξ is the first row of ζ and η is the 3×2 matrix obtained from ζ by erasing its first row. Observe that for all $r \in \mathbb{R}$, $s \in \mathbb{R}^3$, $\mathcal{Q}\tilde{f}(r, s, 0, 0) = 0$ since $\mathcal{Q}\tilde{f} \leqslant \tilde{f}$ and $\tilde{f}(r, s, 0, 0) = 0$ by (4.2), (4.1), and (1.13). Note also that

$$\mathfrak{z}(\zeta) \geqslant \mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), \xi, \eta) \quad \text{for all } \zeta \in \mathbb{R}^{4 \times 2}. \tag{4.131}$$

Moreover (cf. [3, Prop. 4.2]), $\mathfrak{z}$ is a positively 1-homogeneous quasiconvex function satisfying (2.7) and the rank-one convexity of $\mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), \cdot, \cdot)$ implies that

$$\mathfrak{z}(\zeta) = (\mathcal{Q}\tilde{f})^\infty (\tilde{u}(x_0), \tilde{v}(x_0), \xi, \eta) \quad \text{for all } \zeta \in \mathbb{R}^{4 \times 2}, \; \mathrm{rank}(\zeta) \leqslant 1. \tag{4.132}$$

In view of (4.131) and (2.7), we have

$$\int_{B(x_0, \varsigma_k)} \mathcal{Q}\tilde{f}(\tilde{u}(x_0), \tilde{v}(x_0), \nabla u_n(x), \nabla v_{n,k}(x)) \, \mathrm{d}x$$
$$\leqslant \int_{B(x_0, \varsigma_k)} \mathfrak{z}(\nabla w_n(x)) \, \mathrm{d}x + L \int_{B(x_0, \varsigma_k)} |\nabla v_n(x) - \nabla v_{n,k}(x)| \, \mathrm{d}x. \tag{4.133}$$

We claim that

$$\limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} \mathfrak{z}(\nabla w_n(x)) \, \mathrm{d}x \leqslant (\mathcal{Q}_T f)^\infty (\tilde{u}(x_0), \tilde{v}(x_0), W_u^c(x_0), W_v^c(x_0)) \tag{4.134}$$

and that

$$\lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} |\nabla v_n(x) - \nabla v_{n,k}(x)| \, \mathrm{d}x = 0, \tag{4.135}$$

which, together with (4.133), yield (4.130).

We start by proving (4.134). By Lemma 2.7-iii), we obtain

$$\lim_{n \to \infty} \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} \mathfrak{z}(\nabla w_n(x)) \, \mathrm{d}x = \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} \mathfrak{z}\left(\frac{\mathrm{d}Dw}{\mathrm{d}|Dw|}(x) \right) \mathrm{d}|Dw|(x)$$
$$\leqslant \frac{1}{|D^c w|(B(x_0, \varsigma_k))} \int_{B(x_0, \varsigma_k)} \mathfrak{z}(W^c(x)) \, \mathrm{d}|D^c w|(x) + (2 + \sqrt{2}(1 + \beta)) \frac{\nu(B(x_0, \varsigma_k))}{|D^c w|(B(x_0, \varsigma_k))},$$

45

where we also used (4.9). In view of Theorem 2.12, (4.132), (4.120), and (4.118), in this order, we have

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{1}{|D^c w|(B(x_0,\varsigma_k))}\int_{B(x_0,\varsigma_k)}\mathfrak{z}(\nabla w_n(x))\,\mathrm{d}x$$

$$\leqslant\limsup_{k\to\infty}\fint_{B(x_0,\varsigma_k)}(\mathcal{Q}\tilde{f})^\infty(\tilde{u}(x_0),\tilde{v}(x_0),W_u^c(x),W_v^c(x))\,\mathrm{d}|D^c w|(x)$$

$$\leqslant(\mathcal{Q}\tilde{f})^\infty(\tilde{u}(x_0),\tilde{v}(x_0),W_u^c(x_0),W_v^c(x_0))$$

$$+\lim_{k\to\infty}\fint_{B(x_0,\varsigma_k)}\big|(\mathcal{Q}\tilde{f})^\infty(\tilde{u}(x_0),\tilde{v}(x_0),W_u^c(x_0),W_v^c(x_0))$$

$$-(\mathcal{Q}\tilde{f})^\infty(\tilde{u}(x),\tilde{v}(x),W_u^c(x),W_v^c(x))\big|\,\mathrm{d}|D^c w|(x)$$

$$+\limsup_{k\to\infty}\fint_{B(x_0,\varsigma_k)}\big|(\mathcal{Q}\tilde{f})^\infty(\tilde{u}(x),\tilde{v}(x),W_u^c(x),W_v^c(x))$$

$$-(\mathcal{Q}\tilde{f})^\infty(\tilde{u}(x_0),\tilde{v}(x_0),W_u^c(x),W_v^c(x))\big|\,\mathrm{d}|D^c w|(x)$$

$$\leqslant(\mathcal{Q}_T f)^\infty(\tilde{u}(x_0),\tilde{v}(x_0),W_u^c(x_0),W_v^c(x_0))$$

$$+c\limsup_{k\to\infty}\fint_{B(x_0,\varsigma_k)}\big(|\tilde{u}(x)-\tilde{u}(x_0)|+|\tilde{v}(x)-\tilde{v}(x_0)|\big)\big(|W_u^c(x)|+|W_v^c(x)|\big)\,\mathrm{d}|D^c w|(x),\quad(4.136)$$

where in the last inequality we used (4.6), together with Lemma 4.13 and the definition of the recession functions of $\mathcal{Q}_T f$ and $\mathcal{Q}\tilde{f}$, (4.123), and (4.12). Furthermore,

$$\limsup_{k\to\infty}\fint_{B(x_0,\varsigma_k)}\big(|\tilde{u}(x)-\tilde{u}(x_0)|+|\tilde{v}(x)-\tilde{v}(x_0)|\big)\big(|W_u^c(x)|+|W_v^c(x)|\big)\,\mathrm{d}|D^c w|(x)$$

$$\leqslant 4\limsup_{k\to\infty}\fint_{B(x_0,\varsigma_k)}\big[|W^c(x_0)||\tilde{w}(x)-\tilde{w}(x_0)|+(\beta+1)|W^c(x)-W^c(x_0)|\big]\,\mathrm{d}|D^c w|(x),$$

which, together with (4.136), (4.121), and (4.122), entails (4.134).

Finally, we establish (4.135). Arguing as in Step 2, using (4.124), (4.125), and the second estimate in (4.25) applied to $y=y_{n,k}$, $s_1=v_n(x)$ (for $x\in B(x_0,\varsigma_k)\backslash A_{n,k}$), and $s_2=\tilde{v}(x_0)$, and recalling that $\nabla v_{n,k}(\cdot)=\nabla\pi_{y_{n,k}}(v_n(\cdot))\nabla v_n(\cdot)$, we obtain

$$\int_{B(x_0,\varsigma_k)}|\nabla v_n(x)-\nabla v_{n,k}(x)|\,\mathrm{d}x$$

$$\leqslant\frac{1+C_\star}{\delta}\int_{A_{n,k}}|v_n(x)-\tilde{v}(x_0)||\nabla v_n(x)|\,\mathrm{d}x+\overline{C}\int_{B(x_0,\varsigma_k)\backslash A_{n,k}}|v_n(x)-\tilde{v}(x_0)||\nabla v_n(x)|\,\mathrm{d}x$$

$$+\int_{B(x_0,\varsigma_k)\backslash A_{n,k}}|\nabla v_n(x)-\nabla\pi_{y_{n,k}}(\tilde{v}(x_0))\nabla v_n(x)|\,\mathrm{d}x.\qquad(4.137)$$

Moreover (see (4.129)),

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{|D^c w|(B(x_0,\varsigma_k))}\int_{B(x_0,\varsigma_k)}|v_n(x)-\tilde{v}(x_0)||\nabla v_n(x)|\,\mathrm{d}x=0,$$

and so, in view of (4.137), to prove (4.135) it suffices to show that

$$\lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{|D^c w|(B(x_0,\varsigma_k))}\int_{B(x_0,\varsigma_k)}|\nabla v_n(x)-\nabla\pi_{y_{n,k}}(\tilde{v}(x_0))\nabla v_n(x)|\,\mathrm{d}x=0.\qquad(4.138)$$

In the remaining part of the proof, $\mathbb{I}_{4\times4}$ denotes the 4×4 identity matrix and $B_{n,k}$ denotes the 4×4 matrix whose last three rows and columns are those of $\nabla\pi_{y_{n,k}}(\tilde{v}(x_0))$ and the first row and column are those of the identity matrix. We observe that $|\mathbb{I}_{4\times4}-B_{n,k}|=|\mathbb{I}_{3\times3}-\nabla\pi_{y_{n,k}}(\tilde{v}(x_0))|\leqslant\sqrt{3}+2\bar{C}$. Moreover,

$$|\nabla v_n(\cdot)-\nabla\pi_{y_{n,k}}(\tilde{v}(x_0))\nabla v_n(\cdot)|=|(\mathbb{I}_{4\times4}-B_{n,k})\nabla w_n(\cdot)|=|\nabla(\vartheta_{n,k}*\rho_n)(\cdot)|,$$

46

where

$$\vartheta_{n,k}(\cdot) := (\mathbb{I}_{4\times 4} - B_{n,k})w.$$

Since

$$D\vartheta_{n,k} = (\mathbb{I}_{4\times 4} - B_{n,k})\nabla w \mathcal{L}^2_{\llcorner\Omega} + (\mathbb{I}_{4\times 4} - B_{n,k})(w^+ - w^-) \otimes \nu_w \mathcal{H}^1_{\llcorner J_w} + (\mathbb{I}_{4\times 4} - B_{n,k})W^c|D^c w|,$$

using Lemma 2.7-*ii)* with $\mathfrak{h} \equiv 1$ and observing that $1 * \rho_n \equiv 1$, we have

$$\int_{B(x_0,\varsigma_k)} |\nabla v_n(x) - \nabla \pi_{y_{n,k}}(\tilde{v}(x_0))\nabla v_n(x)| \, \mathrm{d}x \leqslant |D\vartheta_{n,k}|\Big(B\Big(x_0,\varsigma_k + \frac{1}{n}\Big)\Big)$$

$$\leqslant (\sqrt{3} + 2\bar{C})\nu\Big(B\Big(x_0,\varsigma_k + \frac{1}{n}\Big)\Big) + \int_{B(x_0,\varsigma_k+\frac{1}{n})} |(\mathbb{I}_{4\times 4} - B_{n,k})W^c(x)| \, \mathrm{d}|D^c w|(x). \tag{4.139}$$

By Lemma 4.13, $v(x) \in S^2$ for all x in Ω except possibly for x belonging to the $\mathcal{H}^1$-negligible set $S_w\backslash J_w$. Therefore, redefining v on $S_w\backslash J_w$ so that $v(x) \in S^2$ for all $x \in \Omega$ if necessary, in view of (4.23) and (4.118), we conclude that

$$\nabla \pi_{y_{n,k}}(\tilde{v}(x))W^c_v(x) = W^c_v(x) \quad \text{for } |D^c w|\text{-a.e. } x \in \Omega.$$

Hence, using once more (4.25) and recalling that $|W^c(x)| = 1$ for $|D^c w|$-a.e. $x \in \Omega$, we deduce that

$$\int_{B(x_0,\varsigma_k+\frac{1}{n})} |(\mathbb{I}_{4\times 4} - B_{n,k})W^c(x)| \, \mathrm{d}|D^c w|(x)$$

$$= \int_{B(x_0,\varsigma_k+\frac{1}{n})} |\nabla \pi_{y_{n,k}}(\tilde{v}(x))W^c_v(x) - \nabla \pi_{y_{n,k}}(\tilde{v}(x_0))W^c_v(x)| \, \mathrm{d}|D^c w|(x)$$

$$\leqslant \overline{C} \int_{B(x_0,\varsigma_k+\frac{1}{n})} |\tilde{v}(x) - \tilde{v}(x_0)| \, \mathrm{d}|D^c w|(x) \leqslant \overline{C} \int_{B(x_0,\varsigma_k+\frac{1}{n})} |\tilde{w}(x) - \tilde{w}(x_0)| \, \mathrm{d}|D^c w|(x). \tag{4.140}$$

Since $D^c w(\partial B(x_0,\varsigma_k)) = \nu(\partial B(x_0,\varsigma_k)) = 0$, from (4.139), (4.140), (4.120), and (4.121), we infer (4.138), which concludes Step 3.

Finally, (4.117) follows from (4.126), (4.128), and (4.130). $\qquad\qquad\square$

Lemma 4.17. *If $(u,v) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2)$, then for all $A \in \mathcal{A}(\Omega)$,*

$$\mathcal{F}(u,v;A \cap S_{(u,v)}) \leqslant \int_{A\cap S_{(u,v)}} K\big((u,v)^+(x),(u,v)^-(x),\nu_{(u,v)}(x)\big) \, \mathrm{d}\mathcal{H}^1(x). \tag{4.141}$$

Proof. Let $A \in \mathcal{A}(\Omega)$, and set $w := (u,v)$. We will proceed in three steps, and we closely follow the argument in [26, Step 3 in Sect. 5.2] (see also [2, Lem. 6.5]).

<u>Step 1.</u> We prove that (4.141) holds whenever w is of the form

$$w(x) = a\chi_E(x) + b\chi_{E^c}(x),$$

where $a, b \in [\alpha,\beta] \times S^2$ and $E \subset \Omega$ is a set of finite perimeter in Ω.

Substep 1.1. We start by considering the case in which $A = \kappa + \lambda Q_\nu$ and

$$w(x) = \begin{cases} b & \text{if } x \cdot \nu > \sigma \text{ and } x \in A, \\ a & \text{if } x \cdot \nu < \sigma \text{ and } x \in A, \end{cases} \tag{4.142}$$

for some $\kappa \in \mathbb{R}^2$, $\lambda \in \mathbb{R}^+$, $\nu \in S^1$, and $\sigma \in \mathbb{R}$.

Without loss of generality, we may assume that $A \cap \{x \in \mathbb{R}^2 \colon x \cdot \nu = \sigma\} \neq \emptyset$. Fix $\varepsilon > 0$, and let $\vartheta = (\varphi,\psi) \in \mathcal{P}(a,b,\nu)$, depending on ε, be such that

$$K(a,b,\nu) + \varepsilon > \int_{Q_\nu} f^\infty(\vartheta(y),\nabla\vartheta(y)) \, \mathrm{d}y. \tag{4.143}$$

Substep 1.1.1. Assume that $\nu = e_2$.

Set $Q := Q_{e_2}$, and for $n \in \mathbb{N}$, define $w_n \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2; [\alpha, \beta] \times S^2)$ as

$$w_n(x) = (u_n(x), v_n(x)) := \begin{cases} b & \text{if } x_2 > \dfrac{\lambda}{2(2n+1)} + \sigma, \\ \vartheta\left((2n+1)\dfrac{x - (\kappa_1, \sigma)}{\lambda}\right) & \text{if } |x_2 - \sigma| \leqslant \dfrac{\lambda}{2(2n+1)}, \\ a & \text{if } x_2 < -\dfrac{\lambda}{2(2n+1)} + \sigma. \end{cases} \tag{4.144}$$

For all $n \in \mathbb{N}$ large enough, we have that $A \cap \left\{x \in \mathbb{R}^2 : x_2 = \frac{\lambda}{2(2n+1)} + \sigma\right\} \neq \emptyset$ and $A \cap \left\{x \in \mathbb{R}^2 : x_2 = -\frac{\lambda}{2(2n+1)} + \sigma\right\} \neq \emptyset$. For all such $n \in \mathbb{N}$, a change of variables yields

$$\int_A |w_n(x) - w(x)| \, dx = \int_\sigma^{\frac{\lambda}{2(2n+1)} + \sigma} \int_{\kappa_1 - \frac{\lambda}{2}}^{\kappa_1 + \frac{\lambda}{2}} \left|\vartheta\left((2n+1)\frac{x - (\kappa_1, \sigma)}{\lambda}\right) - b\right| dx_1 \, dx_2$$

$$+ \int_{-\frac{\lambda}{2(2n+1)} + \sigma}^{\sigma} \int_{\kappa_1 - \frac{\lambda}{2}}^{\kappa_1 + \frac{\lambda}{2}} \left|\vartheta\left((2n+1)\frac{x - (\kappa_1, \sigma)}{\lambda}\right) - a\right| dx_1 \, dx_2$$

$$= \frac{\lambda^2}{2n+1}\left(\int_0^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\vartheta((2n+1)y_1, y_2) - b| \, dy_1 \, dy_2\right.$$

$$\left. + \int_{-\frac{1}{2}}^0 \int_{-\frac{1}{2}}^{\frac{1}{2}} |\vartheta((2n+1)y_1, y_2) - a| \, dy_1 \, dy_2\right). \tag{4.145}$$

By the Riemann–Lebesgue Lemma, by the 1-periodicity of ϑ in the e_1 direction, and by the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n \to \infty} \left(\int_0^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\vartheta((2n+1)y_1, y_2) - b| \, dy_1 \, dy_2 + \int_{\frac{1}{2}}^0 \int_{-\frac{1}{2}}^{\frac{1}{2}} |\vartheta((2n+1)y_1, y_2) - a| \, dy_1 \, dy_2\right)$$

$$= \left(\int_0^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\vartheta(z_1, y_2) - b| \, dz_1 \, dy_2 + \int_{\frac{1}{2}}^0 \int_{-\frac{1}{2}}^{\frac{1}{2}} |\vartheta(z_1, y_2) - a| \, dz_1 \, dy_2\right).$$

Hence, passing (4.145) to the limit as $n \to \infty$, we conclude that

$$\lim_{n \to \infty} \|w_n - w\|_{L^1(A; \mathbb{R} \times \mathbb{R}^3)} = 0. \tag{4.146}$$

Consequently,

$$\mathcal{F}(w; A) \leqslant \liminf_{n \to \infty} \int_A f(w_n(x), \nabla w_n(x)) \, dx$$

$$= \liminf_{n \to \infty} \int_{-\frac{\lambda}{2(2n+1)} + \sigma}^{\frac{\lambda}{2(2n+1)} + \sigma} \int_{\kappa_1 - \frac{\lambda}{2}}^{\kappa_1 + \frac{\lambda}{2}} f\left(\vartheta\left((2n+1)\frac{x - (\kappa_1, \sigma)}{\lambda}\right), \frac{2n+1}{\lambda} \nabla\vartheta\left((2n+1)\frac{x - (\kappa_1, \sigma)}{\lambda}\right)\right) dx_1 \, dx_2$$

$$= \liminf_{n \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{2n+1}{2}}^{\frac{2n+1}{2}} \frac{\lambda^2}{(2n+1)^2} f\left(\vartheta(y), \frac{2n+1}{\lambda} \nabla\vartheta(y)\right) dy_1 \, dy_2$$

$$= \liminf_{n \to \infty} \lambda \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\lambda}{2n+1} f\left(\vartheta(y), \frac{2n+1}{\lambda} \nabla\vartheta(y)\right) dy_1 \, dy_2,$$

where we used $f(\cdot, \cdot, 0, 0) = 0$ and in the last equality we invoked the 1-periodicity of ϑ in the e_1 direction. Hence, Fatou's Lemma, together with (1.14), and (4.143) yield

$$\mathcal{F}(w; A) \leqslant \lambda \int_Q f^\infty(\vartheta(y), \nabla\vartheta(y)) \, dy < \lambda K(a, b, e_2) + \lambda\varepsilon = K(a, b, e_2)\mathcal{H}^1(A \cap S_w) + \lambda\varepsilon,$$

from which we obtain (4.141) by letting $\varepsilon \to 0^+$.

Substep 1.1.2. We complete Substep 1.1.

Let $R \in SO(2)$ be such that $Re_2 = \nu$, and define

$$\bar{w}(x) := w(Rx), \ \ x \in R^T A = R^T \kappa + \lambda Q_{e_2}, \quad \bar{\vartheta}(y) := \vartheta(Ry), \ \ y \in Q.$$

Let $\{\bar{w}_n\}_{n \in \mathbb{N}}$ be the sequence in (4.144) with ϑ replaced by $\bar{\vartheta}$. Then, (4.146) reads as $\bar{w}_n \to \bar{w}$ in $L^1(R^T A; \mathbb{R} \times \mathbb{R}^3)$, which in turn implies that $w_n \to w$ in $L^1(A; \mathbb{R} \times \mathbb{R}^3)$, where

$$w_n(x) := \bar{w}_n(R^T x), \quad x \in A, \ n \in \mathbb{N}.$$

Finally, arguing as in Substep 1.1.1, we obtain

$$\mathcal{F}(w; A) \leqslant \liminf_{n \to \infty} \int_A f(w_n(x), \nabla w_n(x)) \, \mathrm{d}x = \liminf_{n \to \infty} \int_{R^T A} f(\bar{w}_n(x), \nabla \bar{w}_n(x) R^T) \, \mathrm{d}x$$

$$\leqslant \lambda \int_{Q_{e_2}} f^\infty(\bar{\vartheta}(y), \nabla \bar{\vartheta}(y) R^T) \, \mathrm{d}y = \lambda \int_{Q_\nu} f^\infty(\vartheta(y), \nabla \vartheta(y)) \, \mathrm{d}y,$$

which, together with (4.143), concludes Substep 1.1.

Substep 1.2. We prove that if $\kappa \in \mathbb{R}^2$, $\lambda \in \mathbb{R}^+$, $\nu \in S^1$, $\sigma \in \mathbb{R}$, and $\delta > 0$ are such that for $x \in \kappa + (\lambda + \delta)Q_\nu$, we have

$$w(x) = \begin{cases} b & \text{if } x \cdot \nu > \sigma, \\ a & \text{if } x \cdot \nu < \sigma, \end{cases}$$

then

$$\mathcal{F}(w; \kappa + \lambda \overline{Q}_\nu) \leqslant K(a, b, \nu) \, \mathcal{H}^1((\kappa + \lambda \overline{Q}_\nu) \cap S_w). \tag{4.147}$$

Let $\{\lambda_n\}_{n \in \mathbb{N}} \subset (\lambda, \lambda + \delta)$ be a strictly decreasing sequence converging to λ. By Lemma 4.12 and Substep 1.1, we have

$$\mathcal{F}(w; \kappa + \lambda \overline{Q}_\nu) = \lim_{n \to \infty} \mathcal{F}(w; \kappa + \lambda_n Q_\nu) \leqslant \lim_{n \to \infty} K(a, b, \nu) \mathcal{H}^1((\kappa + \lambda_n Q_\nu) \cap S_w) = K(a, b, \nu) \, \mathcal{H}^1((\kappa + \lambda \overline{Q}_\nu) \cap S_w),$$

which proves (4.147).

Substep 1.3. We treat the case in which $A \in \mathcal{A}(\Omega)$ is arbitrary and w is of the form (4.142).

Let $\nu_1 \in S^1$ be a fixed vector such that $\{\nu_1, \nu\}$ is an orthonormal basis of $\mathbb{R}^2$, and let $\{c_{ij} \in \mathbb{R}^2 : i, j \in \mathbb{Z}\}$ be the collection of nodes of a grid in $\mathbb{R}^2$ of size 1 with respect to the basis $\{\nu_1, \nu\}$ such that

$$\left\{ \frac{c_{ij}}{2^{n-1}} \in \mathbb{R}^2 : i, j \in \mathbb{Z}, \, n \in \mathbb{N} \right\} \cap \{x \in \mathbb{R}^2 : x \cdot \nu = \sigma\} = \emptyset. \tag{4.148}$$

Next, we write A as a union of convenient closed squares for which the previous substep applies. For $i, j \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $Q_{ij}^{(n)}$ represent the open square of size $\frac{1}{2^{n-1}}$ whose left inferior vertex is $\frac{c_{ij}}{2^{n-1}}$, and let $\kappa_{ij}^{(n)} \in \mathbb{R}^2$ be such that $Q_{ij}^{(n)} = \kappa_{ij}^{(n)} + \frac{1}{2^{n-1}} Q_\nu$. Set $B^{(0)} := \emptyset$, and define recursively the sets

$$B^{(n)} := \left\{ Q_{ij}^{(n)} = \kappa_{ij}^{(n)} + \frac{1}{2^{n-1}} Q_\nu : i, j \in \mathbb{Z}, \, \overline{Q_{ij}^{(n)}} \subset A, \, Q_{ij}^{(n)} \cap \cup_{l=0}^{n-1} B^{(l)} = \emptyset \right\}, \ \ n \in \mathbb{N}.$$

Without loss of generality, we may assume that $B^{(n)} \neq \emptyset$ for all $n \in \mathbb{N}$, otherwise we simply consider the subsequence of the sequence $\{B^{(n)}\}_{n \in \mathbb{N}}$ obtained by removing all its empty sets. Let $I^{(n)} := \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : Q_{ij}^{(n)} \in B^{(n)}\} \in \mathbb{N}$ and $A_k := \cup_{n=1}^k \tilde{A}_n$, where $\tilde{A}_n := \cup_{(i,j) \in I^{(n)}} \left(\kappa_{ij}^{(n)} + \frac{1}{2^{n-1}} \overline{Q}_\nu \right)$. Then,

$$A = \bigcup_{k=1}^\infty A_k, \quad A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots,$$

and, by construction and (4.148), $\{\kappa_{ij}^{(n)} + \frac{1}{2^{n-1}}Q_\nu : n \in \mathbb{N}, (i,j) \in I^{(n)}\}$ is a family of mutually disjoint sets such that

$$\mathcal{H}^1\Big(\partial\Big(\kappa_{ij}^{(n)} + \frac{1}{2^{n-1}}Q_\nu\Big) \cap \{x \in \mathbb{R}^2 : x \cdot \nu = \sigma\}\Big) = 0.$$

Hence, using Lemma 4.12 and (4.147), we conclude that

$$\mathcal{F}(w; A) = \lim_{k\to\infty} \mathcal{F}(w; A_k) \leqslant \liminf_{k\to\infty} \sum_{n=1}^{k} \sum_{(i,j)\in I^{(n)}} \mathcal{F}\Big(w; \kappa_{ij}^{(n)} + \frac{1}{2^{n-1}}\overline{Q}_\nu\Big)$$

$$\leqslant \liminf_{j\to\infty} \sum_{n=1}^{k} \sum_{(i,j)\in I^{(n)}} K(a,b,\nu)\,\mathcal{H}^1\Big(\Big(\kappa_{ij}^{(n)} + \frac{1}{2^{n-1}}\overline{Q}_\nu\Big) \cap S_w\Big) = K(a,b,\nu)\,\mathcal{H}^1(A \cap S_w).$$

This concludes Substep 1.2.

Substep 1.4. We now treat the case in which $A \in \mathcal{A}(\Omega)$ is arbitrary and w has a polygonal interface; that is,

$$w(x) = a\chi_E(x) + b\chi_{E^c}(x),$$

where E is polyhedral open set with $\partial E = \cup_{i=1}^{M} H_i$, H_i a closed segment of a line of the type $\{x \in \mathbb{R}^2 : x \cdot \nu_i = \sigma_i\}$ for some $\nu_i \in S^1$ and $\sigma_i \in \mathbb{R}$, $i \in \{1, ..., M\}$.

Let $I := \{i \in \{1, ..., M\} : \mathcal{H}^1(A \cap H_i) > 0\}$. Note that since A is open and H_i is a closed segment, $\mathcal{H}^1(A \cap H_i) = 0$ is equivalent to saying that $A \cap H_i = \emptyset$. As in Substep 1.1, the only nontrivial case is the case in which card $I > 0$.

Assume that card $I = 1$, and let $i \in \{1, ..., M\}$ be such that $\mathcal{H}^1(A \cap H_i) > 0$. Define the sets

$$A_1 := A \cap \overline{E}^c, \qquad A_2 := A \cap E,$$
$$A_3 := \{x \in A \cap \overline{E}^c : x \cdot \nu_i > \sigma_i\} \cup \{x \in A \cap E : x \cdot \nu_i < \sigma_i\} \cup (A \cap H_i).$$

We have that A_1, A_2, and A_3 are open and satisfy $A = A_1 \cup A_2 \cup A_3$, $w \equiv b$ in A_1, $w \equiv a$ in A_2, and

$$w(x) = \begin{cases} b & \text{if } x \cdot \nu_i > \sigma_i \text{ and } x \in A_3, \\ a & \text{if } x \cdot \nu_i < \sigma_i \text{ and } x \in A_3. \end{cases}$$

Since $f(\cdot, \cdot, 0, 0) = 0$, we obtain $\mathcal{F}(w; A_1) = \mathcal{F}(w; A_2) = 0$, which, together with Lemma 4.12 and Substep 1.2, yields

$$\mathcal{F}(w; A) \leqslant \mathcal{F}(w, A_3) \leqslant K(a,b,\nu)\mathcal{H}^1(A_3 \cap S_w) = K(a,b,\nu)\mathcal{H}^1(A \cap S_w).$$

By induction, we assume that the statement holds true if card $I = k$ for some $k \in \{1, ..., M-1\}$ and we prove that it is also true if card $I = k + 1$. Assume that

$$A \cap \partial E = (A \cap H_1) \cup \cdots \cup (A \cap H_{k+1}),$$

and define

$$A_1 := \{x \in A : \text{dist}(x, H_1) < \text{dist}(x, H_2 \cup \cdots \cup H_M)\}, \quad A_2 := A \backslash \overline{A}_1.$$

We have that A_1 and A_2 are open sets such that $A_1 \cap H_1 \neq \emptyset$, $A_1 \cap (H_2 \cup \cdots \cup H_{k+1}) = \emptyset$, $A_2 \cap H_1 = \emptyset$, and $A_2 \cap (H_2 \cup \cdots \cup H_{k+1}) \neq \emptyset$. Moreover, we observe that $\partial A_1 \cap \partial A_2 \subset S := \{x \in \mathbb{R}^2 : \text{dist}(x, H_1) = \text{dist}(x, H_2 \cup \cdots \cup H_M)\}$ and $\mathcal{H}^1(S \cap S_w) = 0$ since $\mathcal{H}^1(H_i \cap H_j) = 0$ for $i \neq j$. Fix $\delta > 0$. By the induction hypothesis applied to A_1 and A_2, there exist sequences $\{w_n^1\}_{n\in\mathbb{N}} \subset W^{1,1}(A_1; [\alpha,\beta] \times S^2)$ and $\{w_n^2\}_{n\in\mathbb{N}} \subset W^{1,1}(A_2; [\alpha,\beta] \times S^2)$ such that

$$\lim_{n\to\infty} \|w_n^1 - w\|_{L^1(A_1;\mathbb{R}\times\mathbb{R}^3)} = 0, \quad \lim_{n\to\infty} \|w_n^2 - w\|_{L^1(A_2;\mathbb{R}\times\mathbb{R}^3)} = 0,$$

$$\lim_{n\to\infty} \int_{A_1} f(w_n^1(x), \nabla w_n^1(x))\,\mathrm{d}x \leqslant \int_{A_1 \cap S_w} K(a,b,\nu_w(x))\,\mathrm{d}\mathcal{H}^1(x) + \delta, \tag{4.149}$$

$$\lim_{n\to\infty} \int_{A_2} f(w_n^2(x), \nabla w_n^2(x))\,\mathrm{d}x \leqslant \int_{A_2 \cap S_w} K(a,b,\nu_w(x))\,\mathrm{d}\mathcal{H}^1(x) + \delta.$$

Let A_1', $A_2' \in \mathcal{A}_\infty(\Omega)$ satisfy $A_1' \subset\subset A_1$, $A_2' \subset\subset A_2$, and

$$|Dw|(A_1 \backslash \overline{A_1'}) \leqslant \delta, \qquad |Dw|(A_2 \backslash \overline{A_2'}) \leqslant \delta. \tag{4.150}$$

By Lemma 4.10, there exist sequences $\{\tilde{w}_n^1\}_{n \in \mathbb{N}} \subset W^{1,1}(A_1'; [\alpha,\beta] \times S^2)$ and $\{\tilde{w}_n^2\}_{n \in \mathbb{N}} \subset W^{1,1}(A_2'; [\alpha,\beta] \times S^2)$ satisfying

$$\lim_{n \to \infty} \|\tilde{w}_n^1 - w\|_{L^1(A_1'; \mathbb{R} \times \mathbb{R}^3)} = 0, \qquad \lim_{n \to \infty} \|\tilde{w}_n^2 - w\|_{L^1(A_2'; \mathbb{R} \times \mathbb{R}^3)} = 0,$$
$$\tilde{w}_n^1 = w \text{ on } \partial A_1', \qquad \tilde{w}_n^2 = w \text{ on } \partial A_2',$$
$$\limsup_{n \to \infty} \int_{A_1'} f(\tilde{w}_n^1(x), \nabla \tilde{w}_n^1(x)) \, dx \leqslant \liminf_{n \to \infty} \int_{A_1'} f(w_n^1(x), \nabla w_n^1(x)) \, dx, \tag{4.151}$$
$$\limsup_{n \to \infty} \int_{A_2'} f(\tilde{w}_n^2(x), \nabla \tilde{w}_n^2(x)) \, dx \leqslant \liminf_{n \to \infty} \int_{A_2'} f(w_n^2(x), \nabla w_n^2(x)) \, dx.$$

Moreover, by Lemma 4.7 and Remark 4.8, together with the fact that $\text{dist}(\overline{A_1'}, \overline{A_2'}) > 0$, there exist a positive constant $\widetilde{C}$ and a sequence $\{\tilde{w}_n^3\}_{n \in \mathbb{N}} \subset W^{1,1}(A \backslash (\overline{A_1'} \cup \overline{A_2'}); [\alpha,\beta] \times S^2)$ such that

$$\lim_{n \to \infty} \|\tilde{w}_n^3 - w\|_{L^1(A \backslash (\overline{A_1'} \cup \overline{A_2'}); \mathbb{R} \times \mathbb{R}^3)} = 0, \qquad \tilde{w}_n^3 = w \text{ on } \partial A_1' \cup \partial A_2',$$
$$\limsup_{n \to \infty} \int_{A \backslash (\overline{A_1'} \cup \overline{A_2'})} |\nabla \tilde{w}_n^3(x)| \, dx \leqslant \widetilde{C} |Dw|\big(A \backslash (\overline{A_1'} \cup \overline{A_2'})\big). \tag{4.152}$$

Define for $n \in \mathbb{N}$,

$$w_n := \begin{cases} \tilde{w}_n^1 & \text{in } A_1', \\ \tilde{w}_n^2 & \text{in } A_2', \\ \tilde{w}_n^3 & \text{in } A \backslash (\overline{A_1'} \cup \overline{A_2'}). \end{cases}$$

In view of (4.151) and (4.152), we have that $\{w_n\}_{n \in \mathbb{N}} \subset W^{1,1}(A; [\alpha,\beta] \times S^2)$ and $\lim_{n \to \infty} \|w_n - w\|_{L^1(A; \mathbb{R} \times \mathbb{R}^3)} = 0$. Consequently, by definition of $\mathcal{F}(w; A)$, and by (4.149), (1.14), and (4.150), we obtain

$$\mathcal{F}(w; A) \leqslant \liminf_{n \to \infty} \left(\int_{A_1'} f(\tilde{w}_n^1(x), \nabla \tilde{w}_n^1(x)) \, dx + \int_{A_2'} f(\tilde{w}_n^2(x), \nabla \tilde{w}_n^2(x)) \, dx \right.$$
$$\left. + \int_{A \backslash (\overline{A_1'} \cup \overline{A_2'})} f(\tilde{w}_n^3(x), \nabla \tilde{w}_n^3(x)) \, dx \right)$$
$$\leqslant \int_{A_1 \cap S_w} K(a, b, \nu_w(x)) \, d\mathcal{H}^1(x) + \int_{A_2 \cap S_w} K(a, b, \nu_w(x)) \, d\mathcal{H}^1(x) + 2\delta$$
$$+ (3 + \beta) \limsup_{n \to \infty} \int_{A \backslash (\overline{A_1'} \cup \overline{A_2'})} |\nabla \tilde{w}_n^3(x)| \, dx$$
$$\leqslant \int_{A \cap S_w} K(a, b, \nu_w(x)) \, d\mathcal{H}^1(x) + 2\delta[1 + \widetilde{C}(3 + \beta)],$$

and we deduce Substep 1.3 by letting $\delta \to 0^+$.

Substep 1.5. We conclude Step 1.

Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of polyhedral open sets such that (see Remark 2.20)

$$\lim_{n \to \infty} \mathcal{L}^2(E_n \Delta E) = 0, \qquad \lim_{n \to \infty} \text{Per}_\Omega(E_n) = \text{Per}_\Omega(E),$$

and define

$$w_n(x) := a \chi_{E_n}(x) + b \chi_{E_n^c}(x).$$

Then

$$\lim_{n \to \infty} \|w_n - w\|_{L^1(\Omega; \mathbb{R} \times \mathbb{R}^3)} = 0, \qquad \lim_{n \to \infty} |Dw_n|(\Omega) = |Dw|(\Omega).$$

We now consider the homogeneous of degree-one extension $\widetilde{K}(a,b,\cdot)$ of $K(a,b,\cdot)$ to the whole $\mathbb{R}^2$ defined for $v \in \mathbb{R}^2$ by

$$\widetilde{K}(a,b,v) := \begin{cases} 0 & \text{if } v = 0, \\ |v|K(a,b,v/|v|) & \text{if } v \neq 0. \end{cases}$$

In view of Lemma 4.5, $\widetilde{K}(a,b,\cdot)$ is an upper semicontinuous function in $\mathbb{R}^2$ satisfying $\widetilde{K}(a,b,v) \leqslant C|v|$ for all $v \in \mathbb{R}^2$ and for some positive constant C. Therefore, we can find a decreasing sequence $\{h_m\}_{m \in \mathbb{N}}$ of continuous functions satisfying for all $v \in \mathbb{R}^2$,

$$\widetilde{K}(a,b,v) \leqslant h_m(v) \leqslant 2C|v|, \qquad \widetilde{K}(a,b,v) = \inf_{m \in \mathbb{N}} h_m(v).$$

Using the lower semicontinuity of $\mathcal{F}(\cdot; A)$ with respect to the L^1-convergence (of sequences taking values on $[\alpha, \beta] \times S^2$), Substep 1.3, and Reshetnyak's Continuity Theorem, for every $m \in \mathbb{N}$, we obtain

$$\mathcal{F}(w;A) \leqslant \liminf_{n \to \infty} \mathcal{F}(w_n; A) \leqslant \liminf_{n \to \infty} \int_{S_{w_n} \cap A} \widetilde{K}(a,b,\nu(x)) \, \mathrm{d}\mathcal{H}^1(x)$$

$$\leqslant \liminf_{n \to \infty} \int_{S_{w_n} \cap A} h_m(\nu(x)) \, \mathrm{d}\mathcal{H}^1(x) = \int_{S_w \cap A} h_m(\nu(x)) \, \mathrm{d}\mathcal{H}^1(x).$$

We conclude Step 1 by letting $m \to \infty$ and using Lebesgue's Monotone Convergence Theorem.

<u>Step 2.</u> We prove that (4.141) holds whenever w is of the form

$$w(x) = \sum_{i=1}^{k} a_i \chi_{E_i}(x), \tag{4.153}$$

where $k \in \mathbb{N}$, $a_i \in [\alpha, \beta] \times S^2$, $i \in \{1, ..., k\}$, and $\{E_i\}_{i=1}^{k}$ is a family of mutually disjoint sets of finite perimeter in Ω, which covers Ω.

By Theorem 2.19 (see also (4.71)), we have that for all $i,j \in \{1, ..., k\}$,

$$(w^+(x), w^-(x), \nu_w(x)) \sim (a_i, a_j, \nu_{E_i}(x)) \text{ for all } x \in \mathcal{F}^* E_i \cap \mathcal{F}^* E_j,$$
$$\bigcup_{i<j} (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j) \subset S_w \subset B \cup \bigcup_{i<j} (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j)$$

where B is a suitable Borel set satisfying $\mathcal{H}^1(B) = 0$ and $(\mathcal{F}^* E_i \cap \mathcal{F}^* E_j) \cap (\mathcal{F}^* E_l \cap \mathcal{F}^* E_m) = \emptyset$ for all $i,j,l,m \in \{1, ..., k\}$ such that $i \neq j$, $l \neq m$, and $\{i,j\} \neq \{l,m\}$. Moreover,

$$Dw = \sum_{i=1}^{k} a_i \otimes \nu_{E_i} \mathcal{H}^1 \lfloor_{\mathcal{F}^* E_i} = \sum_{i<j} (a_i - a_j) \otimes \nu_{E_i} \mathcal{H}^1 \lfloor_{(\mathcal{F}^* E_i \cap \mathcal{F}^* E_j)}.$$

Therefore, having in mind (4.48) and the identification observed at the beginning of Subsection 4.4, we conclude that $\mathcal{F}(w; \cdot) \ll |Dw| \ll \mathcal{H}^1 \lfloor_{S_w}$ and

$$\mathcal{F}(w;A) = \mathcal{F}(w; A \cap S_w) = \sum_{i<j} \mathcal{F}(w; A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j))$$

$$= \sum_{i<j} \mathcal{F}(a_i \chi_{E_i} + a_j \chi_{E_i^c}; A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j)).$$

On the other hand, by Step 1 together with Theorem 2.19, we obtain for $i,j \in \{1, ..., k\}$ with $i < j$,

$$\mathcal{F}(a_i \chi_{E_i} + a_j \chi_{E_i^c}; A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j))$$
$$= \inf \left\{ \mathcal{F}(a_i \chi_{E_i} + a_j \chi_{E_i^c}; A') \colon A' \in \mathcal{A}(\Omega),\, A' \supset A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j) \right\}$$
$$\leqslant \inf \left\{ \int_{A' \cap \mathcal{F}^* E_i} K(a_i, a_j, \nu_{E_i}(x)) \, \mathrm{d}\mathcal{H}^1(x) \colon A' \in \mathcal{A}(\Omega),\, A' \supset A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j) \right\}$$

$$= \int_{A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j)} K(a_i, a_j, \nu_{E_i}(x)) \, \mathrm{d}\mathcal{H}^1(x).$$

Consequently,

$$\mathcal{F}(w; A) \leqslant \sum_{i<j} \int_{A \cap (\mathcal{F}^* E_i \cap \mathcal{F}^* E_j)} K(a_i, a_j, \nu_{E_i}(x)) \, \mathrm{d}\mathcal{H}^1(x) = \int_{A \cap S_w} K(w^+(x), w^-(x), \nu_w(x)) \, \mathrm{d}\mathcal{H}^1(x),$$

which concludes Step 2.

Step 3. We establish Lemma 4.17.

Let $\phi \in C_c^\infty(\mathbb{R}^3; [0, 1])$ be a smooth cut-off function such that $\phi(z) = 0$ if $|z| \leqslant \frac{1}{4}$, and $\phi(z) = 1$ if $|z| \geqslant \frac{3}{4}$. Let $\bar{\phi} : \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) \to [\alpha, \beta] \times S^2$ be the function defined by $\bar{\phi}(r, s) := (\tilde{r}, \tilde{s})$, where $\tilde{r}$ and $\tilde{s}$ are given by (4.3). Note that for all $\delta > 0$, $\bar{\phi}$ is a Lipschitz function in $\mathbb{R} \times (\mathbb{R}^3 \setminus B(0, \delta))$. Set $\delta = \frac{1}{8}$, and let $L_{\bar{\phi}} := Lip(\bar{\phi}_{|\mathbb{R} \times (\mathbb{R}^3 \setminus B(0, \frac{1}{8}))})$. Consider the extension $\overline{K} : (\mathbb{R} \times \mathbb{R}^3) \times (\mathbb{R} \times \mathbb{R}^3) \times S^1 \to [0, +\infty)$ of K defined for $a = (r_1, s_1)$, $b = (r_2, s_2) \in \mathbb{R} \times \mathbb{R}^3$ and $\nu \in S^1$, by

$$\overline{K}(a, b, \nu) := \begin{cases} 0 & \text{if } s_1 = 0 \text{ or } s_2 = 0, \\ \phi(s_1)\phi(s_2)K(\bar{\phi}(a), \bar{\phi}(b), \nu) & \text{if } s_1 \neq 0 \text{ and } s_2 \neq 0. \end{cases}$$

Then, the properties stated in Lemma 4.5 hold in $(\mathbb{R} \times \mathbb{R}^3) \times (\mathbb{R} \times \mathbb{R}^3) \times S^1$ for $\overline{K}$, where the corresponding constant depends on the constant in Lemma 4.5, on $L_{\bar{\phi}}$, and on $\|\phi\|_{1,\infty}$. Because w takes values on $[\alpha, \beta] \times S^2$, arguing as in [5, Step 2 of Prop. 4.8] we can construct a sequence $\{w_n\}_{n \in \mathbb{N}} \subset BV(\Omega; \mathbb{R} \times \mathbb{R}^3)$ where each w_n is of the type (4.153) (but whose coefficients do not necessarily belong to $[\alpha, \beta] \times S^2$) and such that

$$\lim_{n \to \infty} \|w_n - w\|_{L^\infty(\Omega; \mathbb{R} \times \mathbb{R}^3)} = 0, \tag{4.154}$$

$$\liminf_{n \to \infty} \int_{A \cap S_{w_n}} \overline{K}(w_n^+(x), w_n^-(x), \nu_{w_n}(x)) \, \mathrm{d}\mathcal{H}^1(x) \tag{4.155}$$

$$\leqslant C|Dw|(A \setminus S_w) + \int_{A \cap S_w} \overline{K}(w^+(x), w^-(x), \nu_w(x)) \, \mathrm{d}\mathcal{H}^1(x)$$

$$= C|Dw|(A \setminus S_w) + \int_{A \cap S_w} K(w^+(x), w^-(x), \nu_w(x)) \, \mathrm{d}\mathcal{H}^1(x), \tag{4.156}$$

where C is a positive constant depending only on the constants in Lemma 4.5 for $\overline{K}$, and where in the last equality we used Lemma 4.13 and Theorem 2.9.

In view of (4.154) and since w takes values on $[\alpha, \beta] \times S^2$, w_n takes values in $\mathbb{R} \times (\mathbb{R}^3 \setminus B(0, 3/4))$ for all $n \in \mathbb{N}$ sufficiently large. Then, also $w_n^\pm(x) \in \mathbb{R} \times (\mathbb{R}^3 \setminus B(0, 3/4))$ for $\mathcal{H}^1$-a.e. $x \in S_{w_n}$ and for all $n \in \mathbb{N}$ sufficiently large. For all such $n \in \mathbb{N}$, the function

$$\bar{w}_n := \bar{\phi}(w_n)$$

belongs to $BV(\Omega; \mathbb{R} \times \mathbb{R}^3)$, takes values on $[\alpha, \beta] \times S^2$, and is of the type (4.153). Moreover, by the Lipschitz continuity of $\bar{\phi}$, the equality $\bar{\phi}(w) = w$, and (4.154), we also have $\lim_{n \to \infty} \|\bar{w}_n - w\|_{L^1(\Omega; \mathbb{R} \times \mathbb{R}^3)} = 0$. Furthermore, using Proposition 2.6 (a)-iii), (b)-iii) and Theorem 2.9 (b), we have $S_{\bar{w}_n} \subset S_{w_n}$, $\mathcal{H}^1(S_{w_n} \setminus (J_{w_n} \cap J_{\bar{w}_n})) = 0$, and $(\bar{w}_n^+(x), \bar{w}_n^-(x), \nu_{\bar{w}_n}(x)) = (\bar{\phi}(w_n^+(x)), \bar{\phi}(w_n^-(x)), \nu_{w_n}(x))$ for all $x \in J_{w_n} \cap J_{\bar{w}_n}$. Thus,

$$\int_{A \cap S_{\bar{w}_n}} K(\bar{w}_n^+(x), \bar{w}_n^-(x), \nu_{\bar{w}_n}(x)) \, \mathrm{d}\mathcal{H}^1(x) \leqslant \int_{A \cap S_{w_n}} \overline{K}(w_n^+(x), w_n^-(x), \nu_{w_n}(x)) \, \mathrm{d}\mathcal{H}^1(x). \tag{4.157}$$

Hence, using the lower semicontinuity of $\mathcal{F}(\cdot, A)$ with respect to the L^1-convergence (of sequences taking values in $[\alpha, \beta] \times S^2$), Step 2, (4.156), and (4.157), yields

$$\mathcal{F}(w, A) \leqslant \liminf_{n \to \infty} \mathcal{F}(\bar{w}_n, A) \leqslant \liminf_{n \to \infty} \int_{A \cap S_{\bar{w}_n}} K(\bar{w}_n^+(x), \bar{w}_n^-(x), \nu_{\bar{w}_n}(x)) \, \mathrm{d}\mathcal{H}^1(x)$$

$$\leqslant C|Dw|(A \setminus S_w) + \int_{A \cap S_w} K(w^+(x), w^-(x), \nu_w(x)) \, \mathrm{d}\mathcal{H}^1(x).$$

53

Finally,

$$\mathcal{F}(w, A \cap S_w) = \inf \left\{ \mathcal{F}(w; A') \colon A' \in \mathcal{A}(\Omega), \ A' \supset A \cap S_w \right\}$$

$$\leqslant \inf \left\{ C|Dw|(A' \backslash S_w) + \int_{A' \cap S_w} K(w^+(x), w^-(x), \nu_w(x)) \, \mathrm{d}\mathcal{H}^1(x) \colon A' \in \mathcal{A}(\Omega), \ A' \supset A \cap S_w \right\}$$

$$= \int_{A \cap S_w} K(w^+(x), w^-(x), \nu_w(x)) \, \mathrm{d}\mathcal{H}^1(x),$$

which concludes the proof of Lemma 4.17. $\qquad\square$

Acknowledgements

References

[1] Giovanni Alberti. Rank one property for derivatives of functions with bounded variation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(2):239–274, 1993.

[2] Roberto Alicandro, Antonio Corbo Esposito, and Chiara Leone. Relaxation in BV of integral functionals defined on Sobolev functions with values in the unit sphere. *J. Convex Anal.*, 14(1):69–98, 2007.

[3] Luigi Ambrosio and Gianni Dal Maso. On the relaxation in BV($\Omega; \mathbf{R}^m$) of quasi-convex integrals. *J. Funct. Anal.*, 109(1):76–97, 1992.

[4] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[5] Luigi Ambrosio, Stefano Mortola, and Vincenzo M. Tortorelli. Functionals with linear growth defined on vector valued BV functions. *J. Math. Pures Appl. (9)*, 70(3):269–323, 1991.

[6] Gilles Aubert and Jean-François Aujol. Modeling very oscillating signals. Application to image processing. *Appl. Math. Optim.*, 51(2):163–182, 2005.

[7] Gilles Aubert and Pierre Kornprobst. *Mathematical problems in image processing. Partial differential equations and the calculus of variations. Foreword by Olivier Faugeras. 2nd ed.* New York, NY: Springer, 2006.

[8] Jean-François Aujol, Gilles Aubert, Laure Blanc-Féraud, and Antonin Chambolle. Image decomposition into a bounded variation component and an oscillating component. *J. Math. Imaging Vision*, 22(1):71–88, 2005.

[9] Jean-François Babadjian and Vincent Millot. Homogenization of variational problems in manifold valued BV-spaces. *Calc. Var. Partial Differential Equations*, 36(1):7–47, 2009.

[10] Jean-François Babadjian and Vincent Millot. Homogenization of variational problems in manifold valued Sobolev spaces. *ESAIM Control Optim. Calc. Var.*, 16(4):833–855, 2010.

[11] Fabrice Bethuel. The approximation problem for Sobolev maps between two manifolds. *Acta Math.*, 167(3-4):153–206, 1991.

[12] Fabrice Bethuel and Xiao Min Zheng. Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.*, 80(1):60–75, 1988.

[13] Haïm Brézis, Jean-Michel Coron, and Elliott H. Lieb. Estimations d'énergie pour des applications de $\mathbf{R}^3$ à valeurs dans S^2. *C. R. Acad. Sci. Paris Sér. I Math.*, 303(5):207–210, 1986.

[14] Graça Carita, Irene Fonseca, and Giovanni Leoni. Relaxation in $\mathrm{SBV}_p(\Omega; S^{d-1})$. *Calc. Var. Partial Differ. Equ.*, 42(1-2):211–255, 2011.

[15] Tony Chan, Antonio Marquina, and Pep Mulet. High-order total variation-based image restoration. *SIAM J. Sci. Comput.*, 22(2):503–516, 2000.

[16] Tony F. Chan, Selim Esedoglu, Frederick Park, and Andy M. Yip. Total variation image restoration: overview and recent developments. In *Handbook of Mathematical Models in Computer Vision*, pages 17–31. Springer, New York, 2006.

[17] Tony F. Chan, Sung Ha Kang, and Jianhong Shen. Total variation denoising and enhancement of color images based on the CB and HSV color models. *Journal of Visual Communication and Image Representation*, 12(4):422–435, DEC 2001.

[18] Bernard Dacorogna, Irene Fonseca, Jan Malý, and Konstantina Trivisa. Manifold constrained variational problems. *Calc. Var. Partial Differential Equations*, 9(3):185–206, 1999.

[19] G. Dal Maso, I. Fonseca, G. Leoni, and M. Morini. A higher order model for image restoration: the one-dimensional case. *SIAM J. Math. Anal.*, 40(6):2351–2391, 2009.

[20] Gianni Dal Maso. *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston Inc., Boston, MA, 1993.

[21] Ennio De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.

[22] Vincent Duval, Jean-François Aujol, and Luminita A. Vese. Mathematical modeling of textures: application to color image decomposition with a projected gradient algorithm. *J. Math. Imaging Vision*, 37(3):232–248, 2010.

[23] Jerald L. Ericksen. Liquid crystals with variable degree of orientation. *Arch. Rational Mech. Anal.*, 113(2):97–120, 1990.

[24] Irene Fonseca, Giovanni Leoni, and Stefan Müller. $\mathcal{A}$-quasiconvexity: weak-star convergence and the gap. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(2):209–236, 2004.

[25] Irene Fonseca and Jan Malý. Relaxation of multiple integrals below the growth exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(3):309–338, 1997.

[26] Irene Fonseca and Stefan Müller. Relaxation of quasiconvex functionals in $\mathrm{BV}(\Omega, \mathbf{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Rational Mech. Anal.*, 123(1):1–49, 1993.

[27] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[28] Victor Guillemin and Alan Pollack. *Differential topology.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[29] Fengbo Hang and Fanghua Lin. Topology of Sobolev mappings. II. *Acta Math.*, 191(1):55–107, 2003.

[30] Robert Hardt, David Kinderlehrer, and Fang-Hua Lin. Existence and partial regularity of static liquid crystal configurations. *Comm. Math. Phys.*, 105(4):547–570, 1986.

[31] Robert Hardt and Fang-Hua Lin. Mappings minimizing the L^p norm of the gradient. *Comm. Pure Appl. Math.*, 40(5):555–588, 1987.

[32] Sung Ha Kang and Riccardo March. Variational models for image colorization via chromaticity and brightness decomposition. *IEEE Transactions on Image Processing*, 16(9):2251–2261, SEP 2007.

[33] Yves Meyer. *Oscillating patterns in image processing and nonlinear evolution equations*, volume 22 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001. The fifteenth Dean Jacqueline B. Lewis memorial lectures.

[34] Giuseppe Mingione and Domenico Mucci. Integral functionals and the gap problem: Sharp bounds for relaxation and energy concentration. *SIAM J. Math. Anal.*, 36(5):1540–1579, 2005.

[35] Domenico Mucci. Relaxation of isotropic functionals with linear growth defined on manifold constrained Sobolev mappings. *ESAIM Control Optim. Calc. Var.*, 15(2):295–321, 2009.

[36] Stanley Osher, Andrés Solé, and Luminita Vese. Image decomposition and restoration using total variation minimization and the H^{-1} norm. *Multiscale Model. Simul.*, 1(3):349–370 (electronic), 2003.

[37] Leonid I. Rudin, Stanley Osher, and Emad Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60(1-4):259–268, 1992.

[38] Bei Tang, Guillermo Sapiro, and Vicent Caselles. Color image enhancement via chromaticity diffusion. *IEEE Trans. Image Process.*, 10(5):701–707, 2001.

[39] Andrey N. Tikhonov and Vasiliy Y. Arsenin. *Solutions of ill-posed problems. Translation editor Frity John.* 1977.

[40] Luminita A. Vese and Stanley J. Osher. Modeling textures with total variation minimization and oscillating patterns in image processing. *J. Sci. Comput.*, 19(1-3):553–572, 2003. Special issue in honor of the sixtieth birthday of Stanley Osher.

[41] Epifanio G. Virga. *Variational theories for liquid crystals*, volume 8 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1994.