# HYDRODYNAMIC LIMIT WITH GEOMETRIC CORRECTION OF STATIONARY BOLTZMANN EQUATION

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Abstract. We consider the hydrodynamic limit of a stationary Boltzmann equation in a unit plate with in-flow boundary. The classical theory claims that the solution can be approximated by the sum of interior solution which satisfies steady incompressible Navier-Stokes-Fourier system, and boundary layer derived from Milne problem. In this paper, we construct counterexamples to disprove such formulation in $L^\infty$ both for its proof and result. Also, we show the hydrodynamic limit with a different boundary layer expansion with geometric correction.
**Keywords:** normal singularity, boundary layer, geometric correction, Boussinesq relation.

Contents



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## 1. Introduction

1.1. **Problem Formulation.** We consider stationary Boltzmann equation for distribution density $F^{\epsilon}(\vec{x},\vec{v})$ in a two-dimensional unit plate $\Omega=\{\vec{x}=(x_1,x_2): \ |\vec{x}|\leq 1\}$ with velocity $\Sigma=\{\vec{v}=(v_1,v_2)\in\mathbb{R}^2\}$ as

$$\left\{\begin{array}{rcl}\epsilon\vec{v}\cdot\nabla_xF^{\epsilon}&=&Q[F^{\epsilon},F^{\epsilon}]\ \text{ in }\ \Omega\times\mathbb{R}^2,\\ F^{\epsilon}(\vec{x}_0,\vec{v})&=&B^{\epsilon}(\vec{x}_0,\vec{v})\ \text{ for }\ \vec{x}_0\in\partial\Omega\ \text{ and }\ \vec{n}(\vec{x}_0)\cdot\vec{v}<0,\end{array}\right. \tag{1.1}$$

where $\vec{n}(\vec{x}_0)$ is the outward normal vector at $\vec{x}_0$ and the Knudsen number $\epsilon$ satisfies $0<\epsilon<<1$. Here we have

$$Q[F,G]=\int_{\mathbb{R}^2}\int_{\mathcal{S}^1}q(\vec{\omega},|\vec{\mathfrak{u}}-\vec{v}|)\bigg(F(\vec{\mathfrak{u}}_{*})G(\vec{v}_{*})-F(\vec{\mathfrak{u}})G(\vec{v})\bigg)\mathrm{d}\vec{\omega}\mathrm{d}\vec{\mathfrak{u}}, \tag{1.2}$$

with

$$\vec{\mathfrak{u}}_{*}=\vec{\mathfrak{u}}+\vec{\omega}\bigg((\vec{v}-\vec{\mathfrak{u}})\cdot\vec{\omega}\bigg),\qquad \vec{v}_{*}=\vec{v}-\vec{\omega}\bigg((\vec{v}-\vec{\mathfrak{u}})\cdot\vec{\omega}\bigg), \tag{1.3}$$

and the hard-sphere collision kernel

$$q(\vec{\omega},|\vec{\mathfrak{u}}-\vec{v}|)=q_0\,|\vec{\mathfrak{u}}-\vec{v}|\,|\cos\phi|\,, \tag{1.4}$$

for positive constant $q_0$, $\vec{\omega}\cdot(\vec{v}-\vec{\mathfrak{u}})=|\vec{v}-\vec{\mathfrak{u}}|\cos\phi$ and $0\leq\phi\leq\pi/2$. We assume that the boundary data can be expanded as

$$B^{\epsilon}(\vec{x}_0,\vec{v})=\mu+\sqrt{\mu}b^{\epsilon}(\vec{x}_0,\vec{v})=\mu+\sqrt{\mu}\bigg(\sum_{k=1}^{\infty}\epsilon^kb_k(\vec{x}_0,\vec{v})\bigg), \tag{1.5}$$

where the standard Maxwellian is defined as

$$\mu(\vec{v})=\frac{1}{2\pi}\exp\left(-\frac{|\vec{v}|^2}{2}\right), \tag{1.6}$$

and $b_k(\vec{x}_0,\vec{v})$ is independent of $\epsilon$. We further assume

$$\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}\frac{B^{\epsilon}-\mu}{\sqrt{\mu}}\right|_{L^{\infty}}\leq C_0\epsilon, \tag{1.7}$$

for any $0\leq\zeta\leq 1/4$ and $\vartheta>2$, where $C_0>0$ is sufficiently small. We intend to study the behavior of $F^{\epsilon}$ as $\epsilon\rightarrow 0$.

The solution $F^{\epsilon}$ can be rewritten as a perturbation of the standard Maxwellian

$$F^{\epsilon}(\vec{x},\vec{v})=\mu+\sqrt{\mu}f^{\epsilon}(\vec{x},\vec{v}). \tag{1.8}$$

Then $f^{\epsilon}$ satisfies the equation

$$\left\{\begin{array}{rcl}\epsilon\vec{v}\cdot\nabla_xf^{\epsilon}+\mathcal{L}[f^{\epsilon}]&=&\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(\vec{x}_0,\vec{v})&=&b^{\epsilon}(\vec{x}_0,\vec{v})\ \text{ for }\ \vec{n}\cdot\vec{v}<0\ \text{ and }\ \vec{x}_0\in\partial\Omega,\end{array}\right. \tag{1.9}$$

where

$$\Gamma[f^{\epsilon},f^{\epsilon}]=\frac{1}{\sqrt{\mu}}Q[\sqrt{\mu}f^{\epsilon},\sqrt{\mu}f^{\epsilon}], \tag{1.10}$$

$$\mathcal{L}[f^{\epsilon}]=-\frac{2}{\sqrt{\mu}}Q[\mu,\sqrt{\mu}f^{\epsilon}]=\nu(\vec{v})f^{\epsilon}-K[f^{\epsilon}], \tag{1.11}$$

$$\nu(\vec{v})=\int_{\mathbb{R}^2}\int_{\mathcal{S}^1}q(\vec{v}-\vec{\mathfrak{u}},\vec{\omega})\mu(\vec{\mathfrak{u}})\mathrm{d}\vec{\omega}\mathrm{d}\vec{\mathfrak{u}}, \tag{1.12}$$

$$K[f^\epsilon](\vec{v}) = K_2[f^\epsilon](\vec{v}) - K_1[f^\epsilon](\vec{v}) \tag{1.13}$$
$$= \int_{\mathbb{R}^2} k(\vec{\mathfrak{u}},\vec{v})f^\epsilon(\vec{\mathfrak{u}})\mathrm{d}\vec{\mathfrak{u}} = \int_{\mathbb{R}^2} k_2(\vec{\mathfrak{u}},\vec{v})f^\epsilon(\vec{\mathfrak{u}})\mathrm{d}\vec{\mathfrak{u}} - \int_{\mathbb{R}^2} k_1(\vec{\mathfrak{u}},\vec{v})f^\epsilon(\vec{\mathfrak{u}})\mathrm{d}\vec{\mathfrak{u}},$$

$$K_1[f^\epsilon](\vec{v}) = \sqrt{\mu(\vec{v})}\int_{\mathbb{R}^2}\int_{\mathcal{S}^1} q(\vec{v}-\vec{\mathfrak{u}},\vec{\omega})\sqrt{\mu(\vec{\mathfrak{u}})}f^\epsilon(\vec{\mathfrak{u}})\mathrm{d}\vec{\omega}\mathrm{d}\vec{\mathfrak{u}}, \tag{1.14}$$

$$K_2[f^\epsilon](\vec{v}) = \int_{\mathbb{R}^2}\int_{\mathcal{S}^1} q(\vec{v}-\vec{\mathfrak{u}},\vec{\omega})\sqrt{\mu(\vec{\mathfrak{u}})}\bigg(\sqrt{\mu(\vec{v}_{*})}f^\epsilon(\vec{\mathfrak{u}}_{*}) + \sqrt{\mu(\vec{\mathfrak{u}}_{*})}f^\epsilon(\vec{v}_{*})\bigg)\mathrm{d}\vec{\omega}\mathrm{d}\vec{\mathfrak{u}}. \tag{1.15}$$

Hence,we only need to study the behavior of $f^\epsilon$ as $\epsilon \to 0$.

1.2. **Background.** As one of the key steps to tackle Hilbert's Six Problem, the mathematical analysis of hydrodynamic limit has a long history. The first result dates back to 1912 by Hilbert himself, using the so-called Hilbert expansion, i.e. expanding the distribution function and data in a power series of the Knudsen number. Since then, a lot of works on Boltzmann equation in $\mathbb{R}^n$ have been presented, including [?], [?], [?], [?], [?], [?], for either smooth solution or renormalized solution.

The general theory of initial-boundary-value problem was developed first in 1963 by Grad [?], and then extended by Darrozes in [?], Sone and Aoki in [?], [?], [?], [?], for both the evolutionary equation and stationary equation. In the classical books [?] and [?], Sone summarized previous results and provided a complete analysis of such approaches.

For stationary Boltzmann equation where the state of gas is close to a uniform state at rest, the leading order expansion of the perturbation $f^\epsilon$ consists of two parts: the interior solution, based on linearized Boltzmann equation, which satisfy a steady Navier-Stokes-Fourier system, and the boundary layer in the rescaled normal variable, which decays rapidly when it is away from the boundary. In [?] and [?], Sone proposed a formal expansion of boundary layer based on Milne problem. Although a rigorous proof of such expansions has not been presented, it is widely believed that the motivation of this approach is natural and the difficulties are purely technical. Besides the fact that this idea is an intuitive application of Hilbert expansion, it is strongly supported by [?] which states the well-posedness and decay in Milne problem.

However, in this paper, we show the hydrodynamic limit with a different expansion and construct a counterexample to disprove above formulation in the sense of $L^\infty$, which means Sone's approach produces approximation with $O(1)$ error. Here, due to the rescaled variables, the boundary layer is of order $\epsilon^{1/p}$ in $L^p$ norm for $1 \le p < \infty$, which implies we have to use $L^\infty$ norm to capture the leading order error of $O(1)$.

The formulation with ideas in [?] and [?] have two sections:

- **Boundary layer.**
  Expand the boundary layer $\mathscr{F}$ as $\mathscr{F} \sim \sum_{k=1}^{\infty}\epsilon^k\mathscr{F}_k$ which satisfies
  $$(\vec{v}\cdot\vec{n})\frac{\partial\mathscr{F}_k}{\partial\eta} + \mathcal{L}[\mathscr{F}_k] = -(\vec{v}\cdot\vec{\tau})\frac{1}{1-\epsilon\eta}\frac{\partial\mathscr{F}_{k-1}}{\partial\theta} + \text{nonlinear term}, \tag{1.16}$$
  where $\vec{n}$ and $\vec{\tau}$ are the normal and tangential vectors on the boundary, $\theta$ and $\eta$ are tangential and rescaled normal variable, and $\mathcal{L}$ is the linearized collision operator. Then it is intended to show $\mathscr{F}_k \to \mathscr{F}_{k,\infty} \in \ker(\mathcal{L})$ when the distance to boundary goes to infinity.
- **Interior solution.**
  Expand the interior $\mathcal{F}$ as $\mathcal{F} \sim \sum_{k=1}^{\infty}\epsilon^k\mathcal{F}_k$ which satisfies
  $$\mathcal{L}[\mathcal{F}_k] = -\vec{v}\cdot\nabla_x\mathcal{F}_{k-1} + \text{nonlinear term}, \tag{1.17}$$
  which leads to
  $$\mathcal{F}_k = \sqrt{\mu}\bigg(\rho_k + \vec{u}_k\cdot\vec{v} + \theta_k\frac{|\vec{v}|^2-2}{2}\bigg) + \text{lower order terms}, \tag{1.18}$$
  where the macroscopic variables, density $\rho$, flow velocity $\vec{u}$ and temperature $\theta$, satisfy the fluid equations, with Dirichlet-type boundary conditions $\mathcal{F}_k = \mathscr{F}_{k,\infty}$.

In [?] and [?], it is intended to construct $\mathscr{F}_k$ and $\mathcal{F}_k$ to arbitrarily higher order and approximate $f^\epsilon$ by $\mathscr{F} + \mathcal{F} - \mathscr{F}_\infty$. However, we observe that this formulation constitute two main difficulties:

- **Difficulty 1: singularity in boundary layer.**
  In [?] and [?], the singularity of solution in Milne problem is neglected. As [?] and [?] stated, the

well-posedness of $\mathscr{F}_k$ requires bounded source term $(\vec{v}\cdot\vec{\tau})\dfrac{\partial\mathscr{F}_{k-1}}{\partial\theta}$ in $L^\infty$, which further transfers this requirement to normal derivative $(\vec{v}\cdot\vec{n})\dfrac{\partial\mathscr{F}_{k-1}}{\partial\eta}$. Nevertheless, by the counterexamples in Theorem 6.1, there exists certain smooth boundary data such that the normal derivative is not a.e. bounded and this construction breaks down. Furthermore, Theorem 6.2 implies the approximation of $f^\epsilon$ by $\mathscr{F}+\mathcal{F}-\mathscr{F}_\infty$ leads to error of the same order as $f^\epsilon$ in $L^\infty$, so this approach actually provides incorrect expansion, due to this intrinsic singularity of normal derivative. Therefore, any results in kinetic equations involving boundary layer effects directly using this idea should be reexamined.

- **Difficulty 2: determination of macroscopic variables.**
  The determination of macroscopic variables is much more delicate. In particular, the leading order variables satisfies the steady Navier-Stokes-Fourier system including the well-known Boussinesq relation. In [**?**] and [**?**], the Boussinesq constant $C=\rho+\theta$ is determined, simply by total-mass condition, which is not clear from the context. It is noticeable that the boundary data of the macroscopic variables are determined by $\mathscr{F}_{k,\infty}$, which does not necessarily satisfy the Boussinesq relation. Hence, this procedure can lead to ill-posed fluid system.

In this paper, we utilize new approaches to handle these two difficulties and prove the hydrodynamic limit of stationary Boltzmann equation. The key ideas include:

- **$\epsilon$-Milne problem with geometric correction.**
  In order to avoid the estimates of normal derivative, we introduce velocity substitution $v_\eta=\vec{v}\cdot\vec{n}$ and $v_\phi=\vec{v}\cdot\vec{\tau}$ to further transformed the tangential derivative as

$$(\vec{v}\cdot\vec{\tau})\frac{\partial}{\partial\theta}\rightarrow v_\phi\frac{\partial}{\partial\theta}+v_\phi^2\frac{\partial}{\partial v_\eta}-v_\eta v_\phi\frac{\partial}{\partial v_\phi} \tag{1.19}$$

  This decomposition helps to break the transferring of $L^\infty$ singularity, since this new $\dfrac{\partial}{\partial\theta}$ does not refer to normal derivative $\dfrac{\partial}{\partial\eta}$ directly. The extra terms $v_\phi^2\dfrac{\partial}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial}{\partial v_\phi}$ are added to boundary layer as $\epsilon$-Milne problem with geometric correction for $k^{th}$ order boundary layer $\mathscr{F}_k^\epsilon$ as

$$v_\eta\frac{\partial\mathscr{F}_k^\epsilon}{\partial\eta}+\frac{1}{1-\epsilon\eta}\left(v_\phi^2\frac{\partial\mathscr{F}_k^\epsilon}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{F}_k^\epsilon}{\partial v_\phi}\right)+\mathcal{L}[\mathscr{F}_k^\epsilon]=-v_\phi\frac{1}{1-\epsilon\eta}\frac{\partial\mathscr{F}_{k-1}^\epsilon}{\partial\theta}. \tag{1.20}$$

  This idea was first used to analyze neutron transport equation in [**?**], but here it fails without modifications and the proof is much more delicate and complicated. We use several new techniques:
  – proving well-posedness and decay in $L^2$ estimate simultaneously without decoupling them;
  – using mass-flux to show entropy estimate and uniqueness.
  – redesigning the cut-off function and using linear transformation to match the limit function with boundary data.
- **Mass-flux adjustment.**
  To satisfy Boussinesq relation, we have to resort to the boundary layer. The key observation is we have an extra freedom in choosing mass-flux in defining $\epsilon$-Milne problem. We use this freedom twice to construct well-posed fluid-type equation:
  – for existence, using the bijective between mass-flux and $C=\rho^\epsilon+\theta^\epsilon$ to guarantee the Boussinesq relation is satisfied for any given $C$ on the boundary and further in the domain;
  – for uniqueness, using vanishing mass-flux condition

$$\int_{\partial\Omega}\int_{\mathbb{R}^2}f^\epsilon(\vec{x},\vec{v})(\vec{v}\cdot\vec{n})\mathrm{d}\vec{v}\mathrm{d}\vec{x}=0. \tag{1.21}$$

  to determine the Boussinesq constant $C$, instead of the ambiguous total-mass condition.
  This finalizes the construction and all the macroscopic variables are uniquely determined.
- **Counterexample by $L^2$-$L^\infty$ framework.**
  To disprove the formulation in [**?**] and [**?**], we construct counterexamples for both its proof and result. The main idea is to utilize the difference between local operator $\nu I$ and non-local operator $K$. In [**?**], maximum principle of neutron transport equation is used to capture this difference. However, in our case, the linearized Boltzmann equation does not satisfy the maximum principle, so we focus

more on the operators themselves. Roughly speaking, the local operator has order of $L^\infty$ norm, and non-local operator has order of $L^2$ norm. Then a peak-shaped function can present this contrast. The main techniques we use here is as follows:

– in Milne problem, rewriting the normal derivative as

$$\frac{\partial}{\partial\eta}=\frac{K-\nu I}{\vec{v}\cdot\vec{n}}, \tag{1.22}$$

in the bulk and using trace theorem for kinetic operator to further restrict our consideration to the boundary $\eta=0$.

– using the relation between $L^2L^2$ norm, $L^\infty L^2$ norm and $L^\infty L^\infty$ norm to show $\nu I-K$ can be strictly positive when approaching grazing set where $\vec{v}\cdot\vec{n}=0$.

– proving the $O(1)$ error by the comparison of straight-line characteristics in Milne problem and curved characteristics in $\epsilon$-Milne problem with geometric correction.

1.3. **Main Theorem.** We first present the well-posedness and hydrodynamic limit.

**Theorem 1.1.** *For given $B^\epsilon>0$ satisfying (1.5) and (1.7) and $0<\epsilon<<1$, there exists a unique positive solution $F^\epsilon=\mu+\sqrt{\mu}f^\epsilon$ to the stationary Boltzmann equation (1.1), where*

$$f^\epsilon=\epsilon^3R_N+\bigg(\sum_{k=1}^{N}\epsilon^k\mathcal{F}_k^\epsilon\bigg)+\bigg(\sum_{k=1}^{N}\epsilon^k\mathscr{F}_k^\epsilon\bigg), \tag{1.23}$$

*for $N\geq 3$, $R_N$ satisfies (5.2), $\mathcal{F}_k^\epsilon$ and $\mathscr{F}_k^\epsilon$ satisfy (3.60) and (3.53). Also, there exists a $C>0$ such that $f^\epsilon$ satisfies*

$$\left\|\langle\vec{v}\rangle^\vartheta\mathrm{e}^{\zeta|\vec{v}|^2}f^\epsilon\right\|_{L^\infty}\leq C\epsilon, \tag{1.24}$$

*for any $\vartheta>2$, $0\leq\zeta\leq 1/4$.*

Then we have a counterexample to show the result of classical approach is problematic.

**Theorem 1.2.** *For given $B^\epsilon>0$ satisfying (1.5) and (1.7) with*

$$\frac{b_1}{\sqrt{\mu}}=\left(v_\phi\mathrm{e}^{-(v_\phi^2-1)-Mv_\eta^2}\right)=h(v_\eta,v_\phi), \tag{1.25}$$

*where $v_\eta$ and $v_\phi$ are defined as in (3.34) and $M$ is sufficiently large such that*

$$h(0,1) = 1, \tag{1.26}$$

$$\left|h\right|_{L^2_-} << 1, \tag{1.27}$$

*there exists $C>0$ such that*

$$\left\|f^\epsilon-(\mathcal{F}_1+\mathscr{F}_1)\right\|_{L^\infty}\geq C\epsilon, \tag{1.28}$$

*where the interior solution $\mathcal{F}_1$ is defined in (6.44) and boundary layer $\mathscr{F}_1$ is defined in (6.38).*

1.4. **Notation and Structure of This Paper.** Throughout this paper, $C>0$ denotes a constant that only depends on the parameter $\Omega$, but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity $z$. We write $a\lesssim b$ to denote $a\leq Cb$.

Our paper is organized as follows: in Section 2, we establish the $L^\infty$ well-posedness of the linearized Boltzmann equation; in Section 3, we present the asymptotic analysis of the equation (1.9); in Section 4, we prove the well-posedness and decay of the $\epsilon$-Milne problem with geometric correction; in Section 5, we prove Theorem 1.1; finally, in Section 6, we show the classical approach and prove Theorem 1.2.

## 2. Linearized Stationary Boltzmann Equation

We consider the linearized stationary Boltzmann equation

$$\left\{\begin{array}{rcl}\epsilon\vec{v}\cdot\nabla_x f+\mathcal{L}[f] &=& S(\vec{x},\vec{v})\ \text{ in }\ \Omega,\\ f(\vec{x}_0,\vec{v}) &=& h(\vec{x}_0,\vec{v})\ \text{ for }\ \vec{x}_0\in\partial\Omega\ \text{ and }\ \vec{v}\cdot\vec{n}<0.\end{array}\right. \tag{2.1}$$

Based on the flow direction, we can divide the boundary $\gamma=\{(\vec{x}_0,\vec{v}):\ \vec{x}_0\in\partial\Omega\}$ into the in-flow boundary $\gamma_-$, the out-flow boundary $\gamma_+$, and the grazing set $\gamma_0$ as

$$\gamma_-=\{(\vec{x}_0,\vec{v}):\ \vec{x}_0\in\partial\Omega,\ \vec{v}\cdot\vec{n}(\vec{x}_0)<0\}, \tag{2.2}$$

$$\gamma_+=\{(\vec{x}_0,\vec{v}):\ \vec{x}_0\in\partial\Omega,\ \vec{v}\cdot\vec{n}(\vec{x}_0)>0\}, \tag{2.3}$$

$$\gamma_0=\{(\vec{x}_0,\vec{v}):\ \vec{x}_0\in\partial\Omega,\ \vec{v}\cdot\vec{n}(\vec{x}_0)=0\}. \tag{2.4}$$

It is easy to see $\gamma=\gamma_+\cup\gamma_-\cup\gamma_0$. Hence, the boundary condition is only given on $\gamma_-$. Let $\langle\cdot,\cdot\rangle$ be the standard $L^2$ inner product in $\Omega\times\mathbb{R}^2$. We define the $L^p$ and $L^\infty$ norms in $\Omega\times\mathbb{R}^2$ as usual:

$$\|f\|_{L^p}=\left(\int_\Omega\int_{\mathbb{R}^2}|f(\vec{x},\vec{v})|^p\,\mathrm{d}\vec{v}\mathrm{d}\vec{x}\right)^{1/p}, \tag{2.5}$$

$$\|f\|_{L^\infty}=\sup_{(\vec{x},\vec{v})\in\Omega\times\mathbb{R}^2}|f(\vec{x},\vec{v})|. \tag{2.6}$$

Define $\mathrm{d}\gamma=|\vec{v}\cdot\vec{n}|\,\mathrm{d}\varpi\mathrm{d}\vec{v}$ on the boundary $\partial\Omega\times\mathbb{R}^2$ for $\varpi$ as the surface measure. Define the $L^p$ and $L^\infty$ norms on the boundary as follows:

$$|f|_{L^p}=\left(\iint_\gamma|f(\vec{x},\vec{v})|^p\,\mathrm{d}\gamma\right)^{1/p}, \tag{2.7}$$

$$|f|_{L^p_\pm}=\left(\iint_{\gamma_\pm}|f(\vec{x},\vec{v})|^p\,\mathrm{d}\gamma\right)^{1/p}, \tag{2.8}$$

$$|f|_{L^\infty}=\sup_{(\vec{x},\vec{v})\in\gamma}|f(\vec{x},\vec{v})|, \tag{2.9}$$

$$|f|_{L^\infty_\pm}=\sup_{(\vec{x},\vec{v})\in\gamma_\pm}|f(\vec{x},\vec{v})|. \tag{2.10}$$

Also, we define

$$\|f\|_{L^2_\nu}=\left\|\sqrt{\nu}f\right\|_{L^2}. \tag{2.11}$$

Denote the Japanese bracket as

$$\langle\vec{v}\rangle=\sqrt{1+|\vec{v}|^2} \tag{2.12}$$

Define the kernel operator $\mathbb{P}$ as

$$\mathbb{P}[f]=\sqrt{\mu}\left(a_f(t,\vec{x})+\vec{v}\cdot\vec{b}_f(t,\vec{x})+\frac{|\vec{v}|^2-2}{2}c_f(t,\vec{x})\right), \tag{2.13}$$

and the non-kernel operator $\mathbb{I}-\mathbb{P}$ as

$$(\mathbb{I}-\mathbb{P})[f]=f-\mathbb{P}[f]. \tag{2.14}$$

with

$$\int_{\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f]\begin{pmatrix}1\\ \vec{v}\\ |\vec{v}|^2\end{pmatrix}\mathrm{d}\vec{v}=0 \tag{2.15}$$

Our analysis is based on the ideas in [**?**, **?**].

## 2.1. **Preliminaries.**

**Lemma 2.1.** *(Green's Identity) Assume* $f(\vec{x},\vec{v}),\ g(\vec{x},\vec{v})\in L^2(\Omega\times\mathbb{R}^2)$ *and* $\vec{v}\cdot\nabla_x f,\ \vec{v}\cdot\nabla_x g\in L^2(\Omega\times\mathbb{R}^2)$ *with* $f,\ g\in L^2(\gamma)$. *Then*

$$\iint_{\Omega\times\mathbb{R}^2}\bigg((\vec{v}\cdot\nabla_x f)g+(\vec{v}\cdot\nabla_x g)f\bigg)\mathrm{d}\vec{x}\mathrm{d}\vec{v}=\int_{\gamma_+}fg\mathrm{d}\gamma-\int_{\gamma_-}fg\mathrm{d}\gamma. \tag{2.16}$$

*Proof.* See the proof of [**?**, Lemma 2.2]. □

**Lemma 2.2.** *For any* $\lambda>0$, *there exists a unique solution* $f_\lambda(\vec{x},\vec{v})\in L^\infty(\Omega\times\mathbb{R}^2)$ *to the penalized transport equation*

$$\left\{\begin{array}{rcl}\lambda f_\lambda+\epsilon\vec{v}\cdot\nabla_x f_\lambda&=&S(\vec{x},\vec{v})\ \ in\ \ \Omega,\\ f_\lambda(\vec{x}_0,\vec{v})&=&h(\vec{x}_0,\vec{v})\ \ for\ \ \vec{x}_0\in\partial\Omega\ \ and\ \ \vec{v}\cdot\vec{n}<0,\end{array}\right. \tag{2.17}$$

*such that*

$$\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f_\lambda\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f_\lambda\right|_{L^\infty_+}\leq C\left(\frac{1}{\lambda}\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h\right|_{L^\infty_-}\right), \tag{2.18}$$

*for all* $\vartheta\geq 0$, $\zeta\geq 0$, *and*

$$\|f_\lambda\|_{L^2}^2+\frac{\epsilon}{\lambda}\left|f_\lambda\right|_{L^2_+}^2\leq C\left(\frac{1}{\lambda^2}\|S\|_{L^2}^2+\frac{\epsilon}{\lambda}\left|h\right|_{L^2_-}^2\right). \tag{2.19}$$

*Proof.* The characteristics $(X(s),V(s))$ of the equation (2.17) which goes through $(\vec{x},\vec{v})$ is defined by

$$\left\{\begin{array}{rcl}(X(0),V(0))&=&(\vec{x},\vec{v})\\ \dfrac{\mathrm{d}X(s)}{\mathrm{d}s}&=&\epsilon V(s),\\ \dfrac{\mathrm{d}V(s)}{\mathrm{d}s}&=&0.\end{array}\right. \tag{2.20}$$

which implies

$$\left\{\begin{array}{rcl}X(s)&=&\vec{x}+\epsilon s\vec{v}\\ V(s)&=&\vec{v}\end{array}\right. \tag{2.21}$$

Define the backward exit time $t_b(\vec{x},\vec{v})$ and backward exit position $\vec{x}_b(\vec{x},\vec{v})$ as

$$t_b(\vec{x},\vec{v})=\inf\{t>0:\vec{x}-\epsilon t\vec{v}\notin\Omega\}, \tag{2.22}$$

$$\vec{x}_b(\vec{x},\vec{v})=\vec{x}-\epsilon t_b(\vec{x},\vec{v})\vec{v}\notin\Omega. \tag{2.23}$$

Hence, we can rewrite the equation (2.17) along the characteristics as

$$\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f_\lambda(\vec{x},\vec{v})=\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h(\vec{x}_b,\vec{v})e^{-\lambda t_b}+\int_0^{t_b}\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S(\vec{x}_b+\epsilon s\vec{v},\vec{v})e^{-\lambda(t_b-s)}\mathrm{d}s. \tag{2.24}$$

Then we can naturally estimate

$$\begin{aligned}\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f_\lambda\right\|_{L^\infty}&\leq e^{-\lambda t_b}\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h\right|_{L^\infty_-}+\frac{1-e^{-\lambda t_b}}{\lambda}\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S\right\|_{L^\infty}\\ &\leq\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h\right|_{L^\infty_-}+\frac{1}{\lambda}\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S\right\|_{L^\infty},\end{aligned} \tag{2.25}$$

which further implies

$$\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f_\lambda\right|_{L^\infty_+}\ \leq\ \left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h\right|_{L^\infty_-}+\frac{1}{\lambda}\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S\right\|_{L^\infty}. \tag{2.26}$$

Since $f_\lambda$ can be explicitly traced back to the boundary data, the existence naturally follows from above estimate. The uniqueness and $L^2$ estimates follow from Green's identity and $\|f_\lambda\|_{L^2}\leq C\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f_\lambda\right\|_{L^\infty}$. □

### 2.2. $L^2$ Estimates of Linearized Stationary Boltzmann Equation.

**Lemma 2.3.** *For any $\lambda>0$, $m>0$, there exists a unique solution $f_{\lambda,m}(\vec x,\vec v)\in L^2(\Omega\times\mathbb{R}^2)$ to the equation*

$$\left\{\begin{array}{rcl}\lambda f_{\lambda,m}+\epsilon\vec v\cdot\nabla_x f_{\lambda,m}+\mathcal{L}_m[f_{\lambda,m}]&=&S(\vec x,\vec v)\ \ in\ \ \Omega,\\ f_{\lambda,m}(\vec x_0,\vec v)&=&h(\vec x_0,\vec v)\ \ for\ \ \vec x_0\in\partial\Omega\ \ and\ \vec v\cdot\vec n<0,\end{array}\right. \tag{2.27}$$

*with $\mathcal{L}_m$ the linearized Boltzmann operator corresponding to the cut-off cross section $q_m=\min\{q,m\}$. Also, the solution satisfies*

$$\epsilon\left\|f_{\lambda,m}\right\|^2_{L^2_\nu}+\left|f_{\lambda,m}\right|^2_{L^2_+}\leq C(\lambda,m)\left(\left\|S\right\|^2_{L^2}+\epsilon\left|h\right|^2_{L^2_-}\right). \tag{2.28}$$

*Proof.* We divide the proof into several steps:

Step 1: Definition of iteration.
Denote $\mathcal{L}_m=\nu_m-K_m$. We define the iteration in $l$: $f^0_{\lambda,m}=0$ and for $l\geq0$,

$$\left\{\begin{array}{rcl}\lambda f^{l+1}_{\lambda,m}+\epsilon\vec v\cdot\nabla_x f^{l+1}_{\lambda,m}+(1+M)\nu_m f^{l+1}_{\lambda,m}&=&S(\vec x,\vec v)-(K_m+M\nu_m)[f^l_{\lambda,m}],\\ f^{l+1}_{\lambda,m}(\vec x_0,\vec v)&=&h(\vec x_0,\vec v)\ \ \text{for}\ \ \vec x_0\in\partial\Omega\ \ \text{and}\ \vec v\cdot\vec n<0,\end{array}\right. \tag{2.29}$$

where $M>0$ is a fixed real number to be determined later. Since

$$\left\|(K_m+M\nu_m)[f^l_{\lambda,m}]\right\|_{L^\infty}\leq C(m,M)\left\|f^l_{\lambda,m}\right\|_{L^\infty}, \tag{2.30}$$

Lemma 2.2 implies $f^l_{\lambda,m}\in L^\infty(\Omega\times\mathbb{R}^2)$ are well-defined for $l\geq0$. However, we cannot directly obtain the existence of limit $f^l_{\lambda,m}$ as $l\rightarrow\infty$.

Step 2: The limit $l\rightarrow\infty$.
Based on Green's identity in Lemma 2.1, multiplying $f^{l+1}_{\lambda,m}$ on both sides of (2.29), we have

$$\begin{aligned}&\lambda\left\|f^{l+1}_{\lambda,m}\right\|^2_{L^2}+\frac{\epsilon}{2}\left|f^{l+1}_{\lambda,m}\right|^2_{L^2_+}+\left\langle(1+M)\nu_m f^{l+1}_{\lambda,m},f^{l+1}_{\lambda,m}\right\rangle\\ =&\left\langle(K_m+M\nu_m)[f^l_{\lambda,m}],f^{l+1}_{\lambda,m}\right\rangle+\frac{\epsilon}{2}\left|h\right|^2_{L^2_-}+\left\langle f^{l+1}_{\lambda,m},S\right\rangle.\end{aligned} \tag{2.31}$$

Since $\mathcal{L}_m=\nu_m-K_m$ is a non-negative symmetric operator, we can always find $M$ sufficiently large such that $K_m+M\nu_m$ is also a non-negative operator. Then we deduce

$$\begin{aligned}&\left\langle(K_m+M\nu_m)[f^l_{\lambda,m}],f^{l+1}_{\lambda,m}\right\rangle\\ \leq&\sqrt{\left\langle(K_m+M\nu_m)[f^l_{\lambda,m}],f^l_{\lambda,m}\right\rangle}\sqrt{\left\langle(K_m+M\nu_m)[f^{l+1}_{\lambda,m+1}],f^{l+1}_{\lambda,m}\right\rangle}\\ \leq&\frac{1}{2}\bigg(\left\langle(K_m+M\nu_m)[f^l_{\lambda,m}],f^l_{\lambda,m}\right\rangle+\left\langle(K_m+M\nu_m)[f^{l+1}_{\lambda,m+1}],f^{l+1}_{\lambda,m}\right\rangle\bigg)\\ \leq&\frac{1}{2}\bigg(\left\langle(1+M)\nu_m[f^l_{\lambda,m}],f^l_{\lambda,m}\right\rangle+\left\langle(1+M)\nu_m[f^{l+1}_{\lambda,m}],f^{l+1}_{\lambda,m}\right\rangle\bigg).\end{aligned} \tag{2.32}$$

Considering the fact

$$\left\langle(1+M)\nu_m[f^l_{\lambda,m}],f^l_{\lambda,m}\right\rangle\leq(1+M)m\left\|f^l_{\lambda,m}\right\|_{L^2}, \tag{2.33}$$

we obtain

$$\begin{aligned}&\left(\frac{\lambda}{(1+M)m}+1\right)\left\langle(1+M)\nu_m[f^{l+1}_{\lambda,m}],f^{l+1}_{\lambda,m}\right\rangle+\frac{\epsilon}{2}\left|f^{l+1}_{\lambda,m}\right|^2_{L^2_+}\\ \leq&\frac{1}{2}\bigg(\left\langle(1+M)\nu_m[f^l_{\lambda,m}],f^l_{\lambda,m}\right\rangle+\left\langle(1+M)\nu_m[f^l_{\lambda,m}],f^{l+1}_{\lambda,m}\right\rangle\bigg)\\ &+\frac{\epsilon}{2}\left|h\right|^2_{L^2_-}+\frac{\lambda}{2(1+M)m}\left\langle(1+M)\nu_m[f^{l+1}_{\lambda,m}],f^{l+1}_{\lambda,m}\right\rangle+\frac{(1+M)m}{2\lambda}\left\|S\right\|^2_{L^2}.\end{aligned} \tag{2.34}$$

Since

$$\frac{\lambda}{(1+M)m}+1-\frac{1}{2}-\frac{\lambda}{2(1+M)m}>\frac{1}{2}, \tag{2.35}$$

by iteration over $l$, for

$$C_1(\lambda,m)=\frac{1}{1+\frac{\lambda}{(1+M)m}}<1, \tag{2.36}$$

$$C_2(\lambda,m)=\frac{1}{1+\frac{\lambda}{(1+M)m}}\frac{(1+M)m}{\lambda}>0, \tag{2.37}$$

we have

$$\begin{aligned}&\epsilon\left|f_{\lambda,m}^{l+1}\right|_{L^2_+}^2+\left(1+\frac{\lambda}{(1+M)m}\right)\Big\langle(1+M)\nu_m[f_{\lambda,m}^{l+1}],f_{\lambda,m}^{l+1}\Big\rangle \\ \leq{}& C_1(\lambda,m)\left(\epsilon\left|f_{\lambda,m}^{l}\right|_{L^2_+}^2+\left(1+\frac{\lambda}{(1+M)m}\right)\Big\langle(1+M)\nu_m[f_{\lambda,m}^{l}],f_{\lambda,m}^{l}\Big\rangle\right) \\ &+C_2(\lambda,m)\bigg(\left\|S\right\|_{L^2}^2+\epsilon\left|h\right|_{L^2_-}^2\bigg).\end{aligned} \tag{2.38}$$

Taking the difference of $f_{\lambda,m}^{l+1}-f_{\lambda,m}^{l}$, we conclude that $f_{\lambda,m}^{l}$ is a Cauchy sequence. We take $l\rightarrow\infty$ to obtain $f_{\lambda,m}$ as a solution to the equation (2.27) satisfying

$$\epsilon\left|f_{\lambda,m}\right|_{L^2_+}^2+\left(1+\frac{\lambda}{(1+M)m}\right)\langle(1+M)\nu_m[f_{\lambda,m}],f_{\lambda,m}\rangle\leq\frac{C_2(\lambda,m)}{1-C_1(\lambda,m)}\bigg(\left\|S\right\|_{L^2}^2+\epsilon\left|h\right|_{L^2_-}^2\bigg). \tag{2.39}$$

Then our results naturally follows. □

Note that the estimate is not uniform in $\lambda$ as $\lambda\rightarrow 0$. We need to find a stronger estimate of $f_{\lambda,m}$.

**Lemma 2.4.** *The solution $f_{\lambda,m}$ to the equation (2.27) satisfies the estimate*

$$\epsilon\left\|\mathbb{P}[f_{\lambda,m}]\right\|_{L^2}\leq C\bigg(\epsilon\left|f_{\lambda,m}\right|_{L^2_+}+\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+\left\|S\right\|_{L^2}+\epsilon\left|h\right|_{L^2_-}\bigg), \tag{2.40}$$

*for $0\leq\lambda\leq\epsilon<<1$.*

*Proof.* Applying Green's identity in Lemma 2.1 to the equation (2.27), for any $\psi\in L^2(\Omega\times\mathbb{R}^2)$ satisfying $\vec v\cdot\nabla_x\psi\in L^2(\Omega\times\mathbb{R}^2)$ and $\psi\in L^2(\gamma)$, we have

$$\begin{aligned}&\lambda\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\psi+\epsilon\int_{\gamma_+}f_{\lambda,m}\psi\mathrm{d}\gamma-\epsilon\int_{\gamma_-}f_{\lambda,m}\psi\mathrm{d}\gamma-\epsilon\iint_{\Omega\times\mathbb{R}^2}(\vec v\cdot\nabla_x\psi)f_{\lambda,m} \\ ={}&-\iint_{\Omega\times\mathbb{R}^2}\psi(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]+\iint_{\Omega\times\mathbb{R}^2}S\psi.\end{aligned} \tag{2.41}$$

Since

$$\mathbb{P}[f]=\sqrt{\mu}\bigg(a+\vec v\cdot\vec b+\frac{\left|\vec v\right|^2-2}{2}c\bigg), \tag{2.42}$$

our goal is to choose a particular test function $\psi$ to estimate $a$, $\vec b$ and $c$.

Step 1: Estimates of $c$.
We choose the test function

$$\psi=\psi_c=\sqrt{\mu(\vec v)}\left(\left|\vec v\right|^2-\beta_c\right)\left(\vec v\cdot\nabla_x\phi_c(\vec x)\right), \tag{2.43}$$

where

$$\left\{\begin{array}{rcl}-\Delta_x\phi_c(\vec x)&=&c(\vec x)\ \text{ in }\ \Omega,\\ \phi_c&=&0\ \text{ on }\ \partial\Omega,\end{array}\right. \tag{2.44}$$

and $\beta_c$ is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\left\|\phi_c\right\|_{H^2}\leq C\left\|c\right\|_{L^2}. \tag{2.45}$$

With the choice of (2.43), the right-hand side(RHS) of (2.41) is bounded by

$$\text{RHS} \leq C \left\|c\right\|_{L^2} \left( \left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2} + \left\|S\right\|_{L^2} \right). \tag{2.46}$$

We have

$$\vec{v}\cdot\nabla_x\psi_c = \sqrt{\mu(\vec{v})} \sum_{i,j=1}^{2} \left(\left|\vec{v}\right|^2 - \beta_c\right) v_i v_j \partial_{ij}\phi_c(\vec{x}), \tag{2.47}$$

so the left-hand side(LHS) of (2.41) takes the form

$$\begin{aligned} \text{LHS} = {} & \lambda \iint_{\Omega\times\mathbb{R}^2} f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i=1}^{2} v_i\partial_i\phi_c\right) \\ & + \epsilon \int_{\partial\Omega\times\mathbb{R}^2} f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i=1}^{2} v_i\partial_i\phi_c\right)(\vec{n}\cdot\vec{v}) \\ & - \epsilon \iint_{\Omega\times\mathbb{R}^2} f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i,j=1}^{2} v_i v_j\partial_{ij}\phi_c\right). \end{aligned} \tag{2.48}$$

We decompose

$$(f_{\lambda,m})_\gamma = \mathbf{1}_{\gamma_+} f_{\lambda,m} + \mathbf{1}_{\gamma_-} h \quad \text{on} \quad \gamma, \tag{2.49}$$

$$f_{\lambda,m} = \sqrt{\mu}\left(a + \vec{v}\cdot\vec{b} + \frac{\left|\vec{v}\right|^2-2}{2}c\right) + (\mathbb{I}-\mathbb{P})[f_{\lambda,m}] \quad \text{in} \quad \Omega\times\mathbb{R}^2. \tag{2.50}$$

Note that the operator $\mathbb{P}$ and $\mathbb{I}-\mathbb{P}$ are defined independent of cut-off parameter $m$. We will choose $\beta_c$ such that

$$\int_{\mathbb{R}^2} \sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right) v_i^2 \mathrm{d}\vec{v} = 0 \quad \text{for} \quad i=1,2. \tag{2.51}$$

Since $\mu(\vec{v})$ takes the form

$$\mu(\vec{v}) = C\exp\left(-\frac{\left|\vec{v}\right|^2}{2}\right), \tag{2.52}$$

this $\beta_c$ can always be achieved. Now substitute (2.49) and (2.50) into (2.48). Then based on this choice of $\beta_c$ and oddness in $\vec{v}$, there is no $\mathbb{P}[f_{\lambda,m}]$ contribution in the first term and no $a$ contribution in the third term of (2.48). Since $\vec{b}$ contribution and the off-diagonal $c$ contribution in the third term of (2.48) also vanish due to the oddness in $\vec{v}$, we can simplify (2.48) into

$$\begin{aligned} \text{LHS} = {} & \lambda \iint_{\Omega\times\mathbb{R}^2} (\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i=1}^{2} v_i\partial_i\phi_c\right) \\ & + \epsilon \int_{\partial\Omega\times\mathbb{R}^2} \mathbf{1}_{\gamma_+} f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i=1}^{2} v_i\partial_i\phi_c\right)(\vec{n}\cdot\vec{v}) \\ & + \epsilon \int_{\partial\Omega\times\mathbb{R}^2} \mathbf{1}_{\gamma_-} h\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i=1}^{2} v_i\partial_i\phi_c\right)(\vec{n}\cdot\vec{v}) \\ & - \epsilon \sum_{i=1}^{2} \int_{\mathbb{R}^2} \mu(\vec{v})\left|v_i\right|^2\left(\left|\vec{v}\right|^2-\beta_c\right)\frac{\left|\vec{v}\right|^2-2}{2}\mathrm{d}\vec{v}\int_{\Omega} c(\vec{x})\partial_{ii}\phi_c(\vec{x})\mathrm{d}\vec{x} \\ & - \epsilon \iint_{\Omega\times\mathbb{R}^2} (\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_c\right)\left(\sum_{i,j=1}^{2} v_i v_j\partial_{ij}\phi_c\right). \end{aligned} \tag{2.53}$$

Since

$$\int_{\mathbb{R}^2}\mu(\vec{v})\,|v_i|^2\left(|\vec{v}|^2-\beta_c\right)\frac{|\vec{v}|^2-2}{2}\mathrm{d}\vec{v}=C, \tag{2.54}$$

we have

$$\epsilon\left|\int_\Omega\Delta_x\phi_c(\vec{x})c(\vec{x})\mathrm{d}\vec{x}\right|\leq C\,\|c\|_{L^2}\left(\epsilon\,|f_{\lambda,m}|_{L^2_+}+(1+\epsilon+\lambda)\,\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}+\|S\|_{L^2}+\epsilon\,|h|_{L^2_-}\right), \tag{2.55}$$

where we have used the elliptic estimates and the trace estimate: $|\nabla_x\phi_c|_{L^2}\leq C\,\|\phi_c\|_{H^2}\leq C\,\|c\|_{L^2}$. Since $-\Delta_x\phi_c=c$, we know

$$\epsilon\,\|c\|_{L^2}^2\leq C\,\|c\|_{L^2}\left(\epsilon\,|f_{\lambda,m}|_{L^2_+}+(1+\epsilon+\lambda)\,\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}+\|S\|_{L^2}+\epsilon\,|h|_{L^2_-}\right), \tag{2.56}$$

which further implies

$$\epsilon\,\|c\|_{L^2}\leq C\epsilon\,|f_{\lambda,m}|_{L^2_+}+(1+\epsilon+\lambda)\,\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}+\|S\|_{L^2}+\epsilon\,|h|_{L^2_-}. \tag{2.57}$$

Step 2: Estimates of $\vec{b}$.

Step 2 - Phase 1: Estimates of $(\partial_{ij}\Delta_x^{-1}b_j)b_i$ for $i,j=1,2$.
We choose the test function

$$\psi=\psi_b^{i,j}=\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j, \tag{2.58}$$

where

$$\begin{cases}-\Delta_x\phi_b^j(\vec{x}) &= b_j(\vec{x}) \ \text{ in } \ \Omega,\\ \phi_b^j &= 0 \ \text{ on } \ \partial\Omega,\end{cases} \tag{2.59}$$

and $\beta_b$ is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\left\|\phi_b^j\right\|_{H^2}\leq C\left\|\vec{b}\right\|_{L^2}. \tag{2.60}$$

With the choice of (2.58), the right-hand side(RHS) of (2.41) is bounded by

$$\text{RHS}\leq C\left\|\vec{b}\right\|_{L^2}\bigg(\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}+\|S\|_{L^2}\bigg). \tag{2.61}$$

Hence, the left-hand side(LHS) of (2.41) takes the form

$$\begin{aligned}\text{LHS}=\lambda&\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j\\ &+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j(\vec{n}\cdot\vec{v})\\ &-\epsilon\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\left(\sum_{i=1}^2v_i\partial_{ij}\phi_b^j\right).\end{aligned} \tag{2.62}$$

Now substitute (2.49) and (2.50) into (2.62). Then based on the oddness in $\vec{v}$, there is no $\vec{b}$ contribution in the first term and no $a$ and $c$ contribution in the third term of (2.62). We can simplify (2.62) into

$$\begin{aligned}\mathrm{LHS} =& \lambda\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j \\ &+\lambda\iint_{\Omega\times\mathbb{R}^2}a(\vec{x})\mu(\vec{v})\left(v_i^2-\beta_b\right)\partial_j\phi_b^j \\ &+\lambda\iint_{\Omega\times\mathbb{R}^2}c(\vec{x})\mu(\vec{v})\frac{\left|\vec{v}\right|^2-2}{2}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j \\ &+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_+}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j(\vec{n}\cdot\vec{v}) \\ &+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_-}h\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j(\vec{n}\cdot\vec{v}) \\ &-\epsilon\sum_{l=1}^{2}\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_l^2\left(v_i^2-\beta_b\right)\partial_{lj}\phi_b^j(\vec{x})b_l \\ &-\epsilon\sum_{l=1}^{2}\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left(v_i^2-\beta_b\right)v_l\partial_{lj}\phi_b^j.\end{aligned} \tag{2.63}$$

We will choose $\beta_b$ such that

$$\int_{\mathbb{R}^2}\mu(\vec{v})\left(\left|v_i\right|^2-\beta_b\right)\mathrm{d}\vec{v}=0 \quad \text{for} \quad i=1,2. \tag{2.64}$$

Since $\mu(\vec{v})$ takes the form

$$\mu(\vec{v})=C\exp\left(-\frac{\left|\vec{v}\right|^2}{2}\right), \tag{2.65}$$

this $\beta_b$ can always be achieved. Based on this choice of $\beta_b$,we have

$$\lambda\iint_{\Omega\times\mathbb{R}^2}a_{f_{\lambda,m}}\mu(\vec{v})\left(v_i^2-\beta_b\right)\partial_j\phi_b^j=0 \tag{2.66}$$

For such $\beta_b$ and any $i\neq l$, we can directly compute

$$\int_{\mathbb{R}^2}\mu(\vec{v})\left(\left|v_i\right|^2-\beta_b\right)v_l^2\mathrm{d}\vec{v}=0, \tag{2.67}$$

$$\int_{\mathbb{R}^2}\mu(\vec{v})\left(\left|v_i\right|^2-\beta_b\right)v_i^2\mathrm{d}\vec{v}=C\neq 0. \tag{2.68}$$

Then we deduce

$$\begin{aligned}&-\epsilon\sum_{l=1}^{2}\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_l^2\left(v_i^2-\beta_b\right)\partial_{lj}\phi_b^j(\vec{x})b_i \\ =& -\epsilon\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_i^2\left(v_i^2-\beta_b\right)\partial_{ij}\phi_b^j(\vec{x})b_l-\epsilon\sum_{l\neq i}\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})v_l^2\left(v_i^2-\beta_b\right)\partial_{lj}\phi_b^j(\vec{x})b_l \\ =& C\int_{\Omega}(\partial_{ij}\Delta_x^{-1}b_j)b_i,\end{aligned} \tag{2.69}$$

and

$$\lambda\iint_{\Omega\times\mathbb{R}^2}c(\vec{x})\mu(\vec{v})\frac{\left|\vec{v}\right|^2-2}{2}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j=\lambda\iint_{\Omega\times\mathbb{R}^2}c(\vec{x})\mu(\vec{v})\frac{v_i^2-2}{2}\left(v_i^2-\beta_b\right)\partial_j\phi_b^j. \tag{2.70}$$

Hence, by (2.57), we may estimate

$$\begin{aligned}
&\epsilon\left|\int_{\Omega}(\partial_{ij}\Delta_x^{-1}b_j)b_i\right| \qquad (2.71)\\
\leq\ & C\left\|\vec{b}\right\|_{L^2}\bigg(\epsilon\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+\left\|S\right\|_{L^2}+\epsilon\left|h\right|_{L^2_-}+\lambda\left\|c\right\|_{L^2}\bigg)\\
\leq\ & C\left\|\vec{b}\right\|_{L^2}\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg).
\end{aligned}$$

Step 2 - Phase 2: Estimates of $(\partial_{jj}\Delta_x^{-1}b_i)b_i$ for $i\neq j$.
We choose the test function

$$\psi=\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\partial_j\phi_b^i\quad i\neq j. \tag{2.72}$$

The right-hand side(RHS) of (2.41) is still bounded by

$$\text{RHS}\leq C\left\|\vec{b}\right\|_{L^2}\bigg(\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+\left\|S\right\|_{L^2}\bigg). \tag{2.73}$$

Hence, the left-hand side(LHS) of (2.41) takes the form

$$\begin{aligned}
\text{LHS}=\ &\lambda\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\partial_j\phi_b^i \qquad (2.74)\\
&+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\partial_j\phi_b^i(\vec{n}\cdot\vec{v})\\
&-\epsilon\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\left(\sum_{l=1}^2v_i\partial_{lj}\phi_b^i\right).
\end{aligned}$$

Now substitute (2.49) and (2.50) into (2.74). Then based on the oddness in $\vec{v}$, there is no $\mathbb{P}[f_{\lambda,m}]$ contribution in the first term and no $a$ and $c$ contribution in the third term of (2.74). We can simplify (2.74) into

$$\begin{aligned}
\text{LHS}=\ &\lambda\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\partial_j\phi_b^i \qquad (2.75)\\
&+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_+}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\partial_j\phi_b^i(\vec{n}\cdot\vec{v})\\
&+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_-}h\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_j\partial_j\phi_b^i(\vec{n}\cdot\vec{v})\\
&-\epsilon\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})\left|\vec{v}\right|^2v_i^2v_j^2(\partial_{ij}\phi_b^i(\vec{x})b_j+\partial_{jj}\phi_b^i(\vec{x})b_i)\\
&-\epsilon\sum_{l=1}^2\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left|\vec{v}\right|^2v_iv_jv_l\partial_{lj}\phi_b^i.
\end{aligned}$$

Then we deduce

$$-\epsilon\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})\left|\vec{v}\right|^2v_i^2v_j^2(\partial_{ij}\phi_b^i(\vec{x})b_j+\partial_{jj}\phi_b^i(\vec{x})b_i)=C\bigg(\int_{\Omega}(\partial_{ij}\Delta_x^{-1}b_i)b_j+\int_{\Omega}(\partial_{jj}\Delta_x^{-1}b_i)b_i\bigg). \tag{2.76}$$

Hence, we may estimate for $i\neq j$,

$$\begin{aligned}
\epsilon\left|\int_{\Omega}(\partial_{jj}\Delta_x^{-1}b_i)b_i\right|\leq\ & C\left\|\vec{b}\right\|_{L^2}\bigg(\epsilon\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2} \qquad (2.77)\\
&+\left\|S\right\|_{L^2}+\epsilon\left|h\right|_{L^2_-}\bigg)+C\epsilon\left|\int_{\Omega}(\partial_{ij}\Delta_x^{-1}b_i)b_j\right|.
\end{aligned}$$

Moreover, by (2.71), for $i=j=1,2$,

$$\epsilon\left|\int_{\Omega}(\partial_{jj}\Delta_x^{-1}b_j)b_j\right| \leq C\left\|\vec{b}\right\|_{L^2}\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2} \\ +(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg). \tag{2.78}$$

Step 2 - Phase 3: Synthesis.
Summarizing (2.77) and (2.78), we may sum up over $j=1,2$ to obtain, for any $i=1,2$,

$$\epsilon\left\|b_i\right\|_{L^2}^2 \leq C\left\|\vec{b}\right\|_{L^2}\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2} \\ +(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg), \tag{2.79}$$

which further implies

$$\epsilon\left\|\vec{b}\right\|_{L^2} \leq C\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg). \tag{2.80}$$

Step 3: Estimates of $a$.
We choose the test function

$$\psi=\psi_a=\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\vec{v}\cdot\nabla_x\phi_a(\vec{x})\right), \tag{2.81}$$

where

$$\begin{cases} -\Delta_x\phi_a(\vec{x}) &=& a(\vec{x}) \ \text{ in } \ \Omega,\\ \phi_a &=& 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{2.82}$$

and $\beta_a$ is a real number to be determined later. Based on the standard elliptic estimates, we have

$$\left\|\phi_a\right\|_{H^2}\leq C\left\|a\right\|_{L^2}. \tag{2.83}$$

With the choice of (2.81), the right-hand side(RHS) of (2.41) is bounded by

$$\text{RHS}\leq C\left\|a\right\|_{L^2}\bigg(\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+\left\|S\right\|_{L^2}\bigg). \tag{2.84}$$

We have

$$\vec{v}\cdot\nabla_x\psi_a=\sqrt{\mu(\vec{v})}\sum_{i,j=1}^{2}\left(\left|\vec{v}\right|^2-\beta_a\right)v_iv_j\partial_{ij}\phi_a(\vec{x}), \tag{2.85}$$

so the left-hand side(LHS) of (2.41) takes the form

$$\text{LHS}=\lambda\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i=1}^{2}v_i\partial_i\phi_a\right) \\ +\epsilon\int_{\partial\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i=1}^{2}v_i\partial_i\phi_a\right)(\vec{n}\cdot\vec{v}) \\ -\epsilon\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i,j=1}^{2}v_iv_j\partial_{ij}\phi_a\right). \tag{2.86}$$

We will choose $\beta_a$ such that

$$\int_{\mathbb{R}^2}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\frac{\left|\vec{v}\right|^2-2}{2}v_i^2\mathrm{d}\vec{v}=0 \ \text{ for } \ i=1,2. \tag{2.87}$$

Since

$$\int_{\mathbb{R}^2}\sqrt{\mu(\vec{v})}\frac{\left|\vec{v}\right|^2-2}{2}v_i^2\mathrm{d}\vec{v}\neq 0, \tag{2.88}$$

this $\beta_a$ can always be achieved. Now substitute (2.49) and (2.50) into (2.86). Then based on this choice of $\beta_a$ and oddness in $\vec{v}$, there is no $a$ and $c$ contribution in the first term, and no $\vec{b}$ and $c$ contribution in the third term of (2.86). Since $\vec{b}$ contribution and the off-diagonal $c$ contribution in the third term of (2.86) also vanish due to the oddness in $\vec{v}$, we can simplify (2.86) into

$$\begin{aligned}\mathrm{LHS}=&\lambda\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i=1}^2 v_i\partial_i\phi_a\right)\\&+\lambda\iint_{\Omega\times\mathbb{R}^2}\mu(\vec{v})\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i=1}^2 b_iv_i^2\partial_i\phi_a\right)\\&+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_+}f_{\lambda,m}\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i=1}^2 v_i\partial_i\phi_a\right)(\vec{n}\cdot\vec{v})\\&+\epsilon\int_{\partial\Omega\times\mathbb{R}^2}\mathbf{1}_{\gamma_-}h\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i=1}^2 v_i\partial_i\phi_a\right)(\vec{n}\cdot\vec{v})\\&-\sum_{i=1}^2\epsilon\int_{\mathbb{R}^2}\mu(\vec{v})\left|v_i\right|^2\left(\left|\vec{v}\right|^2-\beta_a\right)\mathrm{d}\vec{v}\int_\Omega a(\vec{x})\partial_{ii}\phi_a(\vec{x})\mathrm{d}\vec{x}\\&-\epsilon\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\sqrt{\mu(\vec{v})}\left(\left|\vec{v}\right|^2-\beta_a\right)\left(\sum_{i,j=1}^2 v_iv_j\partial_{ij}\phi_a\right).\end{aligned} \tag{2.89}$$

Since

$$\int_{\mathbb{R}^2}\sqrt{\mu(\vec{v})}\left|v_i\right|^2\left(\left|\vec{v}\right|^2-\beta_a\right)\mathrm{d}\vec{v}=C, \tag{2.90}$$

we have

$$\begin{aligned}&-\epsilon\int_\Omega\Delta_x\phi_a(\vec{x})a(\vec{x})\mathrm{d}\vec{x}\\&\leq C\left\|a\right\|_{L^2}\left(\epsilon\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+\left\|S\right\|_{L^2}+\epsilon\left|h\right|_{L^2_-}+\lambda\left\|\vec{b}\right\|_{L^2}\right).\end{aligned} \tag{2.91}$$

Since $-\Delta_x\phi_a=a$, by (2.80), we know

$$\begin{aligned}\epsilon\left\|a\right\|_{L^2}^2\leq& C\left\|a\right\|_{L^2}\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}\\&+(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg),\end{aligned} \tag{2.92}$$

which further implies

$$\epsilon\left\|a\right\|_{L^2}\leq C\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg). \tag{2.93}$$

Step 4: Synthesis.
Collecting (2.57), (2.80) and (2.93), we deduce

$$\begin{aligned}&\epsilon\left\|\mathbb{P}[f_{\lambda,m}]\right\|_{L^2}\\&\leq C\bigg((\epsilon+\lambda)\left|f_{\lambda,m}\right|_{L^2_+}+(1+\epsilon+\lambda)\left\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\right\|_{L^2}+(1+\lambda)\left\|S\right\|_{L^2}+(\epsilon+\lambda)\left|h\right|_{L^2_-}\bigg).\end{aligned} \tag{2.94}$$

This completes our proof. □

**Theorem 2.5.** *There exists a unique solution $f \in L^2(\Omega\times\mathbb{R}^2)$ to the equation (2.1) that satisfies the estimate*

$$\|f\|_{L^2}+\frac{1}{\epsilon^{1/2}}\left|f\right|_{L^2_+}\leq C\bigg(\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[S]\|_{L^2}+\frac{1}{\epsilon^2}\|\mathbb{P}[S]\|_{L^2}+\frac{1}{\epsilon^{1/2}}\left|h\right|_{L^2_-}\bigg). \tag{2.95}$$

*Proof.* We square on both sides of (2.40) to obtain

$$\epsilon^2\|\mathbb{P}[f_{\lambda,m}]\|_{L^2}\leq C\bigg(\epsilon^2\left|f_{\lambda,m}\right|_{L^2_+}+\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}+\|S\|_{L^2}+\epsilon^2\left|h\right|_{L^2_-}\bigg). \tag{2.96}$$

On the other hand, by Green's identity, multiplying $f_{l,m}$ on both sides of (2.27) implies

$$\lambda\|f_{\lambda,m}\|_{L^2}^2+\langle\mathcal{L}_m f_{\lambda,m},f_{\lambda,m}\rangle+\frac{1}{2}\epsilon\left|f_{\lambda,m}\right|_{L^2_+}^2=\frac{1}{2}\epsilon\left|h\right|_{L^2_-}^2+\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}S. \tag{2.97}$$

We deduce from the spectral gap of $\mathcal{L}_m$,

$$\lambda\|f_{\lambda,m}\|_{L^2}^2+\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}+\frac{\epsilon}{2}\left|f_{\lambda,m}\right|_{L^2_+}^2\leq\epsilon\left|h\right|_{L^2_-}^2+\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}S. \tag{2.98}$$

Multiplying a small constant on both sides of (2.96) and adding to (2.98), we obtain

$$\epsilon^2\|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2+\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}^2+\epsilon\left|f_{\lambda,m}\right|_{L^2_+}^2\leq C\bigg(\epsilon\left|h\right|_{L^2_-}^2+\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}S+\|S\|_{L^2}\bigg). \tag{2.99}$$

Since

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}S&=\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}\mathbb{P}[S]+\iint_{\Omega\times\mathbb{R}^2}f_{\lambda,m}(\mathbb{I}-\mathbb{P})[S] \qquad (2.100)\\&=\iint_{\Omega\times\mathbb{R}^2}\mathbb{P}[f_{\lambda,m}]\mathbb{P}[S]+\iint_{\Omega\times\mathbb{R}^2}(\mathbb{I}-\mathbb{P})[f_{\lambda,m}](\mathbb{I}-\mathbb{P})[S]\\&\leq\bigg(C\epsilon^2\|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2+\frac{1}{4C\epsilon^2}\|\mathbb{P}[S]\|_{L^2}^2\bigg)+\bigg(C\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}^2+\frac{1}{4C}\|(\mathbb{I}-\mathbb{P})[S]\|_{L^2}^2\bigg),\end{aligned}$$

for $C$ sufficiently small, we have

$$\epsilon^2\|\mathbb{P}[f_{\lambda,m}]\|_{L^2}^2+\|(\mathbb{I}-\mathbb{P})[f_{\lambda,m}]\|_{L^2}^2+\epsilon\left|f_{\lambda,m}\right|_{L^2_+}^2\leq C\bigg(\|(\mathbb{I}-\mathbb{P})[S]\|_{L^2}^2+\frac{1}{\epsilon^2}\|\mathbb{P}[S]\|_{L^2}^2+\epsilon\left|h\right|_{L^2_-}^2\bigg). \tag{2.101}$$

Hence, we deduce

$$\|f_{\lambda,m}\|_{L^2}\leq C\bigg(\frac{1}{\epsilon}\|(\mathbb{I}-\mathbb{P})[S]\|_{L^2}+\frac{1}{\epsilon^2}\|\mathbb{P}[S]\|_{L^2}+\frac{1}{\epsilon^{1/2}}\left|h\right|_{L^2_-}\bigg). \tag{2.102}$$

Then based on (2.99), we have

$$\left|f_{\lambda,m}\right|_{L^2_+}\leq C\bigg(\frac{1}{\epsilon^{1/2}}\|(\mathbb{I}-\mathbb{P})[S]\|_{L^2}+\frac{1}{\epsilon^{3/2}}\|\mathbb{P}[S]\|_{L^2}+\left|h\right|_{L^2_-}\bigg). \tag{2.103}$$

This is a uniform estimate in $\lambda$, so we can obtain a weak solution $f_{\lambda,m}\rightarrow f_m$ with the same estimate (2.102) and (2.103). Moreover, we have

$$\left\{\begin{array}{rcl}\lambda(f_{\lambda,m}-f_m)+\epsilon\vec{v}\cdot\nabla_x(f_{\lambda,m}-f_m)+\mathcal{L}_m[f_{\lambda,m}-f_m]&=&\lambda f_m,\\(f_{\lambda,m}-f_m)(\vec{x}_0,\vec{v})&=&0.\end{array}\right. \tag{2.104}$$

Then we have the estimate

$$\|f_{\lambda,m}-f_m\|_{L^2}\leq C\bigg(\frac{\lambda}{\epsilon}\|(\mathbb{I}-\mathbb{P})[f_m]\|_{L^2}+\frac{\lambda}{\epsilon^2}\|\mathbb{P}[f_m]\|_{L^2}\bigg). \tag{2.105}$$

Hence, $f_{\lambda,m}\rightarrow f_m$ strongly in $L^2(\Omega\times\mathbb{R}^2)$ as $\lambda\rightarrow 0$. Then we can take the limit $f_m\rightarrow f$ as $m\rightarrow\infty$. By a diagonal process, there exists a unique weak solution such that $f_m\rightarrow f$ weakly in $L^2(\Omega\times\mathbb{R}^2)$. Then the weak lower semi-continuity implies $f$ satisfies the same estimate (2.102). ☐

2.3. $L^\infty$ **Estimates of Linearized Stationary Boltzmann Equation.** The characteristics $(X(s),V(s))$ of the equation (2.1) which goes through $(\vec{x},\vec{v})$ is defined by

$$\left\{\begin{array}{rcl}(X(0),V(0)) &=& (\vec{x},\vec{v})\\ \dfrac{\mathrm{d}X(s)}{\mathrm{d}s} &=& \epsilon V(s),\\ \dfrac{\mathrm{d}V(s)}{\mathrm{d}s} &=& 0.\end{array}\right. \tag{2.106}$$

which implies

$$\left\{\begin{array}{rcl}X(s) &=& \vec{x}+\epsilon s\vec{v}\\ V(s) &=& \vec{v}\end{array}\right. \tag{2.107}$$

Define the backward exit time and exit position as

$$t_b(\vec{x},\vec{v}) = \inf\{t>0 : \vec{x}-\epsilon t\vec{v}\notin\Omega\}, \tag{2.108}$$

$$x_b(\vec{x},\vec{v}) = \vec{x}-\epsilon t_b(\vec{x},\vec{v})\vec{v}\notin\Omega. \tag{2.109}$$

We define a weight function

$$w(\vec{v}) = w_{\vartheta,\zeta}(\vec{v}) = \langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}, \tag{2.110}$$

and

$$\tilde{w}(\vec{v}) = \frac{1}{\sqrt{\mu(\vec{v})}w(\vec{v})} = \sqrt{2\pi}\frac{\mathrm{e}^{\left(\frac{1}{4}-\zeta\right)|\vec{v}|^2}}{\langle\vec{v}\rangle^{\vartheta}}. \tag{2.111}$$

**Lemma 2.6.** *We have*

$$|k(\vec{v},\vec{v}')| \leq C\bigg(|\vec{v}-\vec{v}'|+\left|\frac{1}{\vec{v}-\vec{v}'}\right|\bigg)\exp\bigg(-\frac{1}{8}|\vec{v}-\vec{v}'|^2-\frac{1}{8}\frac{\left||\vec{v}|^2-|\vec{v}'|^2\right|^2}{|\vec{v}-\vec{v}'|^2}\bigg). \tag{2.112}$$

*Let $0\leq\zeta\leq 1/4$. Then there exists $0\leq C_1(\zeta)<1$ and $C_2(\zeta)>0$ such that for $0\leq\delta\leq C_1(\zeta)$,*

$$\int_{\mathbb{R}^2}\bigg(|\vec{v}-\vec{v}'|+\left|\frac{1}{\vec{v}-\vec{v}'}\right|\bigg)\exp\bigg(-\frac{1-\delta}{8}|\vec{v}-\vec{v}'|^2-\frac{1-\delta}{8}\frac{\left||\vec{v}|^2-|\vec{v}'|^2\right|^2}{|\vec{v}-\vec{v}'|^2}\bigg)\frac{\mathrm{e}^{\zeta|\vec{v}|^2}}{\mathrm{e}^{\zeta|\vec{v}'|^2}}\mathrm{d}\vec{v}' \leq \frac{C_2(\zeta)}{1+|\vec{v}|}. \tag{2.113}$$

*Proof.* See [?, Lemma 3]. □

**Theorem 2.7.** *There exists a unique solution $f\in L^\infty(\Omega\times\mathbb{R}^2)$ to the equation (2.1) that satisfies the estimate for $\vartheta>2$ and $0\leq\zeta\leq 1/4$,*

$$\begin{aligned}&\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}f\right|_{L^\infty_+} \\ &\leq C\bigg(\frac{1}{\epsilon}\|f\|_{L^2}+\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h\right|_{L^\infty_-}\bigg)\\ &\leq C\bigg(\frac{1}{\epsilon^2}\|(\mathbb{I}-\mathbb{P})[S]\|_{L^2}+\frac{1}{\epsilon^3}\|\mathbb{P}[S]\|_{L^2}+\frac{1}{\epsilon^{3/2}}|h|_{L^2_-}+\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h\right|_{L^\infty_-}\bigg).\end{aligned} \tag{2.114}$$

*Proof.* The existence and uniqueness follow from Theorem 2.5, so we focus on the estimate. We divide the proof into several steps:

Step 1: Mild formulation.
Denote

$$g = wf, \tag{2.115}$$

$$K_w[g] = wK\left[\frac{g}{w}\right] = \int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}')g(\vec{v}')\mathrm{d}\vec{v}'. \tag{2.116}$$

Since $\mathcal{L}=\nu-K$, we can rewrite the equation (2.1) along the characteristics by Duhamel's principle as

$$g(\vec{x},\vec{v})=w(\vec{v})h(\vec{x}-\epsilon t_b\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})t_1}+\int_0^{t_b}wS(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s \tag{2.117}$$
$$+\int_0^{t_b}K_w[g(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v})]\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s$$
$$=w(\vec{v})h(\vec{x}-\epsilon t_b\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})t_b}+\int_0^{t_b}wS(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s$$
$$+\int_0^{t_b}\left(\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)g(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v}_t)\mathrm{d}\vec{v}_t\right)\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s,$$

where $\vec{v}_t\in\mathbb{R}^2$ is a dummy variable. We may apply Duhamel's principle again to $g(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v}_t)$ to obtain

$$\tag{2.118}$$

$$g(\vec{x},\vec{v})$$
$$=w(\vec{v})h(\vec{x}-\epsilon t_b\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})t_b}+\int_0^{t_b}wS(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s$$
$$+\int_0^{t_b}\left(\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)w(\vec{v}_t)h(\vec{x}-\epsilon t_b\vec{v}-\epsilon s_b\vec{v}_t,\vec{v}_t)\mathrm{e}^{-\nu(\vec{v}_t)s_b}\mathrm{d}\vec{v}_t\right)\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s$$
$$+\int_0^{t_b}\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)\bigg(\int_0^{s_b}wS(\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon(s_b-r)\vec{v}_t,\vec{v}_t)\mathrm{e}^{-\nu(\vec{v}_t)(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{v}_t\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s$$
$$+\int_0^{t_b}\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)\bigg(\int_0^{s_b}\int_{\mathbb{R}^2}k_w(\vec{v}_t,\vec{v}_s)g(\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon(s_b-r)\vec{v}_t,\vec{v}_s)\mathrm{d}\vec{v}_s\mathrm{e}^{-\nu(\vec{v}_t)(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{v}_t\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s,$$

where

$$s_b(\vec{x},\vec{v},\vec{v}_t,s)=\inf\{r>0:\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon r\vec{v}_t\notin\Omega\}, \tag{2.119}$$

and $\vec{v}_s\in\mathbb{R}^2$ is another dummy variable. We need to estimate each term in (2.118).

Step 2: Estimates of source terms and boundary terms.
We can directly obtain

$$\left|w(\vec{v})h(\vec{x}-\epsilon t_b\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})t_b}\right|\leq C\left|wh\right|_{L^\infty_-}, \tag{2.120}$$

$$\left|\int_0^{t_b}wS(\vec{x}-\epsilon(t_b-s)\vec{v},\vec{v})\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s\right|\leq C\left\|wS\right\|_{L^\infty}, \tag{2.121}$$

$$\left|\int_0^{t_b}\left(\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)w(\vec{v}_t)h(\vec{x}-\epsilon t_b\vec{v}-\epsilon s_b\vec{v}_t,\vec{v}_t)\mathrm{e}^{-\nu(\vec{v}_t)s_b}\mathrm{d}\vec{v}_t\right)\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s\right| \tag{2.122}$$
$$\leq\left|wh\right|_{L^\infty_-}\left|\int_0^{t_b}\left(\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)\mathrm{e}^{-\nu(\vec{v}_t)s_b}\mathrm{d}\vec{v}_t\right)\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s\right|\leq C\left|wh\right|_{L^\infty_-},$$

$$\tag{2.123}$$
$$\left|\int_0^{t_b}\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)\bigg(\int_0^{s_b}wS(\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon(s_b-r)\vec{v}_t,\vec{v}_t)\mathrm{e}^{-\nu(\vec{v}_t)(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{v}_t\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s\right|$$
$$\leq\left\|wS\right\|_{L^\infty}\left|\int_0^{t_b}\int_{\mathbb{R}^2}k_w(\vec{v},\vec{v}_t)\bigg(\int_0^{s_b}\mathrm{e}^{-\nu(\vec{v}_t)(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{v}_t\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s\right|\leq C\left\|wS\right\|_{L^\infty}.$$

Step 3: Estimates of $K_w$ terms.

We consider the last and most complicated term $I$ in (2.118) defined as

$$\int_0^{t_b}\int_{\mathbb{R}^2} k_w(\vec{v},\vec{v}_t)\bigg(\int_0^{s_b}\int_{\mathbb{R}^2} k_w(\vec{v}_t,\vec{v}_s)g(\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon(s_b-r)\vec{v}_t,\vec{v}_s)\mathrm{d}\vec{v}_s\mathrm{e}^{-\nu(\vec{v}_t)(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{v}_t\mathrm{e}^{-\nu(\vec{v})(t_b-s)}\mathrm{d}s. \tag{2.124}$$

We can divide it into four cases:

$$I = I_1 + I_2 + I_3 + I_4. \tag{2.125}$$

Case I: $|\vec{v}| \geq N$.
Based on Lemma 2.6 with $\delta = 0$, we have

$$\left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} k_{w(\vec{v})}(\vec{v},\vec{v}_t)k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s)\mathrm{d}\vec{v}_s\mathrm{d}\vec{v}_t\right| \leq \frac{C}{1+|\vec{v}|} \leq \frac{C}{N}. \tag{2.126}$$

Hence, by Fubini's Theorem, we get

$$I_1 \leq \frac{C}{N}\left\|g\right\|_{L^\infty}. \tag{2.127}$$

Case II: $|\vec{v}| \leq N$, $|\vec{v}_t| \geq 2N$, or $|\vec{v}_t| \leq 2N$, $|\vec{v}_s| \geq 3N$.
Notice this implies either $|\vec{v}_t - \vec{v}| \geq N$ or $|\vec{v}_t - \vec{v}_s| \geq N$. Hence, either of the following is valid correspondingly:

$$\left|k_{w(\vec{v})}(\vec{v},\vec{v}_t)\right| \leq \mathrm{e}^{-\frac{\delta}{8}N^2}\left|k_{w(\vec{v})}(\vec{v},\vec{v}_t)\mathrm{e}^{\frac{\delta}{8}|\vec{v}-\vec{v}_t|^2}\right|, \tag{2.128}$$

$$\left|k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s)\right| \leq \mathrm{e}^{-\frac{\delta}{8}N^2}\left|k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s)\mathrm{e}^{\frac{\delta}{8}|\vec{v}_t-\vec{v}_s|^2}\right|. \tag{2.129}$$

Then based on Lemma 2.6,

$$\int_{\mathbb{R}^2}\left|k_{w(\vec{v})}(\vec{v},\vec{v}_t)\mathrm{e}^{\frac{\delta}{8}|\vec{v}-\vec{v}_t|^2}\right|\mathrm{d}\vec{v}_t < \infty, \tag{2.130}$$

$$\int_{\mathbb{R}^2}\left|k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s)\mathrm{e}^{\frac{\delta}{8}|\vec{v}_t-\vec{v}_s|^2}\right|\mathrm{d}\vec{v}_s < \infty. \tag{2.131}$$

Hence, we have

$$I_2 \leq C\mathrm{e}^{-\frac{\delta}{8}N^2}\left\|g\right\|_{L^\infty}. \tag{2.132}$$

Case III: $s - r \leq \delta$ and $|\vec{v}| \leq N$, $|\vec{v}_t| \leq 2N$, $|\vec{v}_s| \leq 3N$.
In this case, when $0 < \delta << 1$ is sufficiently small, since the integral in $r$ is restricted to this short interval, we have

$$I_3 \leq C\delta\left\|g\right\|_{L^\infty}. \tag{2.133}$$

Case IV: $s - r \geq \delta$ and $|\vec{v}| \leq N$, $|\vec{v}_t| \leq 2N$, $|\vec{v}_s| \leq 3N$.
Since $k_{w(\vec{v})}(\vec{v},\vec{v}_t)$ has possible integrable singularity of $1/|\vec{v}-\vec{v}_t|$, we can introduce $k_N(\vec{v},\vec{v}_t)$ smooth with compact support such that

$$\sup_{|p|\leq 3N}\int_{|\vec{v}_t|\leq 3N}\left|k_N(p,\vec{v}_t) - k_{w(p)}(p,\vec{v}_t)\right|\mathrm{d}\vec{v}_t \leq \frac{1}{N}. \tag{2.134}$$

Then we can split

$$\begin{aligned} k_{w(\vec{v})}(\vec{v},\vec{v}_t)k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s) =& k_N(\vec{v},\vec{v}_t)k_N(\vec{v}_t,\vec{v}_s) \\ &+ \bigg(k_{w(\vec{v})}(\vec{v},\vec{v}_t) - k_N(\vec{v},\vec{v}_t)\bigg)k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s) \\ &+ \bigg(k_{w(\vec{v}_t)}(\vec{v}_t,\vec{v}_s) - k_N(\vec{v}_t,\vec{v}_s)\bigg)k_N(\vec{v},\vec{v}_t). \end{aligned} \tag{2.135}$$

This means we further split $I_4$ into

$$I_4 = I_{4,1} + I_{4,2} + I_{4,3}. \tag{2.136}$$

Based on the estimate (2.134), we have

$$I_{4,2} \leq \frac{C}{N} \left\| g \right\|_{L^\infty}, \tag{2.137}$$

$$I_{4,3} \leq \frac{C}{N} \left\| g \right\|_{L^\infty}. \tag{2.138}$$

Therefore, the only remaining term is $I_{4,1}$. Note we always have $\vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s-r)\vec{v}_t \in \Omega$. Hence, we define the change of variable $\vec{y} = (y_1, y_2) = \vec{x} - \epsilon(t_b - s)\vec{v} - \epsilon(s-r)\vec{v}_t$ such that

$$\left| \frac{\mathrm{d}\vec{y}}{\mathrm{d}\vec{v}_t} \right| = \left\| \begin{matrix} \epsilon(s-r) & 0 \\ 0 & \epsilon(s-r) \end{matrix} \right\| = \epsilon^2(s-r) \geq \epsilon^2\delta^2. \tag{2.139}$$

Since $k_N$ is bounded and $|\vec{v}_s| \leq 3N$, we estimate

$$\begin{aligned} I_{4,1} &\leq C \left| \int_{|\vec{v}_t|\leq 2N} \int_{|\vec{v}_s|\leq 3N} \int_0^s \mathbf{1}_{\{\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon(s-r)\vec{v}_t\in\Omega\}} g(\vec{x}-\epsilon(t_b-s)\vec{v}-\epsilon(s-r)\vec{v}_t, \vec{v}_s)\mathrm{d}r\mathrm{d}\vec{v}_t\mathrm{d}\vec{v}_s \right| \\ &\leq \frac{C}{\epsilon\delta} \left| \int_0^s \int_{|\vec{v}_s|\leq 3N} \int_\Omega \mathbf{1}_{\{\vec{y}\in\Omega\}} g(\vec{y},\vec{v}_s)\mathrm{d}\vec{y}\mathrm{d}\vec{v}_s\mathrm{d}r \right| \\ &\leq \frac{C\left(1+N^2\right)^{\frac{\vartheta}{2}}\mathrm{e}^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g(\vec{y},\vec{v}_s)}{w(\vec{v}_s)} \right\|_{L^2} \\ &= \frac{C\left(1+N^2\right)^{\frac{\vartheta}{2}}\mathrm{e}^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2}. \end{aligned} \tag{2.140}$$

Summarize all above in Case IV, we obtain

$$I_4 \leq \frac{C}{N}\left\| g \right\|_{L^\infty} + \frac{C\left(1+N^2\right)^{\frac{\vartheta}{2}}\mathrm{e}^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2}. \tag{2.141}$$

Therefore, we already prove

$$I \leq \left( C\mathrm{e}^{-\frac{\delta}{8}N^2} + \frac{C}{N} + C\delta \right)\left\| g \right\|_{L^\infty} + \frac{C\left(1+N^2\right)^{\frac{\vartheta}{2}}\mathrm{e}^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2} \tag{2.142}$$

Step 4: Synthesis.
Collecting all above in Step 2 and Step 3, we have shown

$$\left\| g \right\|_{L^\infty} \leq \left( C\mathrm{e}^{-\frac{\delta}{8}N^2} + \frac{C}{N} + C\delta \right)\left\| g \right\|_{L^\infty} + \frac{C\left(1+N^2\right)^{\frac{\vartheta}{2}}\mathrm{e}^{\zeta N^2}}{\epsilon\delta} \left\| \frac{g}{w} \right\|_{L^2} + C\left\| wS \right\|_{L^\infty} + C\left| wh \right|_{L^\infty_-} \tag{2.143}$$

Choosing $\delta$ sufficiently small and then taking $N$ sufficiently large such that $C\mathrm{e}^{-\frac{\delta}{8}N^2} + \frac{C}{N} + C\delta \leq \frac{1}{2}$, we have

$$\left\| g \right\|_{L^\infty} \leq \frac{C}{\epsilon} \left\| \frac{g}{w} \right\|_{L^2} + C\left\| wS \right\|_{L^\infty} + C\left| wh \right|_{L^\infty_-}. \tag{2.144}$$

Based on Theorem 2.5, we obtain

$$\begin{aligned} \left\| g \right\|_{L^\infty} &\leq C\bigg( \frac{1}{\epsilon}\left\| \frac{g}{w} \right\|_{L^2} + \left\| wS \right\|_{L^\infty} + \left| wh \right|_{L^\infty_-} \bigg) \\ &= C\bigg( \frac{1}{\epsilon}\left\| f \right\|_{L^2} + \left\| wS \right\|_{L^\infty} + \left| wh \right|_{L^\infty_-} \bigg) \\ &\leq C\bigg( \frac{1}{\epsilon^2}\left\| (\mathbb{I}-\mathbb{P})[S] \right\|_{L^2} + \frac{1}{\epsilon^3}\left\| \mathbb{P}[S] \right\|_{L^2} + \frac{1}{\epsilon^{3/2}}\left| h \right|_{L^2_-} + \left\| wS \right\|_{L^\infty} + \left| wh \right|_{L^\infty_-} \bigg). \end{aligned} \tag{2.145}$$

This completes the proof. □

## 3. Asymptotic Analysis

In this section, we construct the asymptotic expansion of the equation (1.9).

### 3.1. Interior Expansion.

We define the interior expansion

$$\mathcal{F}^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\epsilon(\vec{x},\vec{v}). \tag{3.1}$$

Plugging it into the equation (1.9) and comparing the order of $\epsilon$, we obtain

$$\mathcal{L}[\mathcal{F}_1^\epsilon] = 0, \tag{3.2}$$

$$\mathcal{L}[\mathcal{F}_2^\epsilon] = -\vec{v}\cdot\nabla_x\mathcal{F}_1^\epsilon + \Gamma[\mathcal{F}_1^\epsilon,\mathcal{F}_1^\epsilon], \tag{3.3}$$

$$\mathcal{L}[\mathcal{F}_3^\epsilon] = -\vec{v}\cdot\nabla_x\mathcal{F}_2^\epsilon + \Gamma[\mathcal{F}_1^\epsilon,\mathcal{F}_2^\epsilon] + \Gamma[\mathcal{F}_2^\epsilon,\mathcal{F}_1^\epsilon], \tag{3.4}$$

$$\ldots$$

$$\mathcal{L}[\mathcal{F}_k^\epsilon] = -\vec{v}\cdot\nabla_x\mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_{k-i}^\epsilon]. \tag{3.5}$$

The solvability of

$$\mathcal{L}[\mathcal{F}_k^\epsilon] = -\vec{v}\cdot\nabla_x\mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_{k-i}^\epsilon] \tag{3.6}$$

requires

$$\int_{\mathbb{R}^2}\bigg(-\vec{v}\cdot\nabla_x\mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_{k-i}^\epsilon]\bigg)\psi(\vec{v})\mathrm{d}\vec{v} = 0 \tag{3.7}$$

for any $\psi$ satisfying $\mathcal{L}[\psi]=0$. Based on the analysis in [**?**, **?**], each $\mathcal{F}_k$ consists of three parts:

$$\mathcal{F}_k^\epsilon(\vec{x},\vec{v}) = A_k^\epsilon(\vec{x},\vec{v}) + B_k^\epsilon(\vec{x},\vec{v}) + C_k^\epsilon(\vec{x},\vec{v}), \tag{3.8}$$

where

$$A_k^\epsilon(\vec{x},\vec{v}) = \sqrt{\mu}\left(A_{k,0}^\epsilon(\vec{x}) + A_{k,1}^\epsilon(\vec{x})v_1 + A_{k,2}^\epsilon(\vec{x})v_2 + A_{k,3}^\epsilon(\vec{x})\left(\frac{\lvert\vec{v}\rvert^2-2}{2}\right)\right), \tag{3.9}$$

is the fluid part,

$$B_k^\epsilon(\vec{x},\vec{v}) = \sqrt{\mu}\left(B_{k,0}^\epsilon(\vec{x}) + B_{k,1}^\epsilon(\vec{x})v_1 + B_{k,2}^\epsilon(\vec{x})v_2 + B_{k,3}^\epsilon(\vec{x})\left(\frac{\lvert\vec{v}\rvert^2-2}{2}\right)\right), \tag{3.10}$$

with $B_k^\epsilon$ depending on $A_{s,i}^\epsilon$ for $1\leq s\leq k-1$ and $i=0,1,2,3$ as

$$B_{k,0}^\epsilon = 0, \tag{3.11}$$

$$B_{k,1}^\epsilon = \sum_{i=1}^{k-1} A_{i,0}^\epsilon A_{k-i,1}^\epsilon, \tag{3.12}$$

$$B_{k,2}^\epsilon = \sum_{i=1}^{k-1} A_{i,0}^\epsilon A_{k-i,2}^\epsilon, \tag{3.13}$$

$$B_{k,3}^\epsilon = \sum_{i=1}^{k-1}\left(A_{i,0}^\epsilon A_{k-i,3}^\epsilon + A_{i,1}^\epsilon A_{k-i,1}^\epsilon + A_{i,2}^\epsilon A_{k-i,2}^\epsilon + \sum_{j=1}^{k-1-i} A_{i,0}^\epsilon\big(A_{j,1}^\epsilon A_{k-i-j,1}^\epsilon + A_{j,2}^\epsilon A_{k-i-j,2}^\epsilon\big)\right), \tag{3.14}$$

and $C_k(\vec{x},\vec{v})$ satisfies

$$\int_{\mathbb{R}^2}\sqrt{\mu}C_k^\epsilon(\vec{x},\vec{v})\begin{pmatrix}1\\ \vec{v}\\ \lvert\vec{v}\rvert^2\end{pmatrix}\mathrm{d}\vec{v} = 0, \tag{3.15}$$

with

$$\mathcal{L}[C_k^\epsilon] = -\vec{v}\cdot\nabla_x\mathcal{F}_{k-1}^\epsilon + \sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_{k-i}^\epsilon], \tag{3.16}$$

which can be solved explicitly at any fixed $\vec{x}$. Hence, we only need to determine the relations satisfied by $A_k^\epsilon$. For convenience, we define

$$A_k^\epsilon = \sqrt{\mu}\left(\rho_k^\epsilon + u_{k,1}^\epsilon v_1 + u_{k,2}^\epsilon v_2 + \theta_k^\epsilon\left(\frac{|\vec{v}|^2-2}{2}\right)\right), \tag{3.17}$$

Then the analysis in [?, ?] shows that $A_k^\epsilon$ satisfies the equations as follows:

$0^{th}$ order equations:

$$P_1^\epsilon - (\rho_1^\epsilon + \theta_1^\epsilon) = 0, \tag{3.18}$$
$$\nabla_x P_1^\epsilon = 0, \tag{3.19}$$

$1^{st}$ order equations:

$$P_2^\epsilon - (\rho_2^\epsilon + \theta_2^\epsilon + \rho_1^\epsilon\theta_1^\epsilon) = 0, \tag{3.20}$$
$$\vec{u}_1^\epsilon\cdot\nabla_x\vec{u}_1^\epsilon - \gamma_1\Delta_x\vec{u}_1^\epsilon + \nabla_x P_2^\epsilon = 0, \tag{3.21}$$
$$\nabla_x\cdot\vec{u}_1^\epsilon = 0, \tag{3.22}$$
$$\vec{u}_1^\epsilon\cdot\nabla_x\theta_1^\epsilon - \gamma_2\Delta_x\theta_1^\epsilon = 0, \tag{3.23}$$

$k^{th}$ order equations:

$$P_{k+1}^\epsilon - \left(\rho_{k+1}^\epsilon + \theta_{k+1}^\epsilon + \sum_{i=1}^{k+1-i}\rho_i^\epsilon\theta_{k+1-i}^\epsilon\right) = 0, \tag{3.24}$$
$$\sum_{i=1}^{k}\vec{u}_i^\epsilon\cdot\nabla_x\vec{u}_{k+1-i}^\epsilon - \gamma_1\Delta_x\vec{u}_k^\epsilon + \nabla_x P_{k+1}^\epsilon = H_{k,1}^\epsilon, \tag{3.25}$$
$$\nabla_x\cdot\vec{u}_k^\epsilon = H_{k,2}^\epsilon, \tag{3.26}$$
$$\sum_{i=1}^{k}\vec{u}_i^\epsilon\cdot\nabla_x\theta_{k+1-i}^\epsilon - \gamma_2\Delta_x\theta_k^\epsilon = H_{k,3}^\epsilon, \tag{3.27}$$

where

$$H_{k,j}^\epsilon = G_{k,j}^\epsilon[\vec{x},\vec{v};\rho_1^\epsilon,\ldots,\rho_{k-1}^\epsilon;\theta_1^\epsilon,\ldots,\theta_{k-1}^\epsilon;\vec{u}_1,\ldots,\vec{u}_{k-1}^\epsilon], \tag{3.28}$$

is explicit functions depending on lower order terms, and $\gamma_1$ and $\gamma_2$ are two positive constants. Since in most cases, we are only interested in the leading order terms, so we omit the detailed description of $G_{k,j}^\epsilon$. In order to determine the boundary condition for $\vec{u}_k^\epsilon$, $\theta_k^\epsilon$ and $\rho_k^\epsilon$, we have to define the boundary layer expansion.

3.2. **Boundary Layer Expansion with Geometric Correction.** In order to define the boundary layer, we need several substitutions:

Substitution 1:
Define the substitution into polar coordinate $f^\epsilon(x_1,x_2,\vec{v})\to f^\epsilon(r,\phi,\vec{v})$ with $(r,\phi,\vec{v})\in[0,1)\times[-\pi,\pi)\times\mathbb{R}^2$ as

$$\left\{\begin{array}{rcl} x_1 &=& r\cos\phi,\\ x_2 &=& r\sin\phi,\\ \vec{v} &=& \vec{v}. \end{array}\right. \tag{3.29}$$

The equation (1.9) can be rewritten as

$$\left\{\begin{array}{l}\epsilon(\vec{v}\cdot\vec{n})\dfrac{\partial f^{\epsilon}}{\partial r}+\dfrac{\epsilon}{r}(\vec{v}\cdot\vec{\tau})\dfrac{\partial f^{\epsilon}}{\partial\phi}+\mathcal{L}[f^{\epsilon}]=\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(1,\phi,\vec{v})=b^{\epsilon}(1,\phi,\vec{v})\ \text{ for }\ \vec{v}\cdot\vec{n}<0,\end{array}\right. \tag{3.30}$$

for $\vec{n}$ the outer normal vector and $\vec{\tau}$ the counterclockwise tangential vector on $\partial\Omega$.

Substitution 2:
We further perform the scaling substitution $f^{\epsilon}(r,\phi,\vec{v})\rightarrow f^{\epsilon}(\eta,\phi,\vec{v})$ with $(\eta,\phi,\vec{v})\in[0,1/\epsilon)\times[-\pi,\pi)\times\mathbb{R}^2$ as

$$\left\{\begin{array}{rcl}\eta&=&(1-r)/\epsilon,\\ \phi&=&\phi,\\ \vec{v}&=&\vec{v},\end{array}\right. \tag{3.31}$$

which implies

$$\frac{\partial f^{\epsilon}}{\partial r}=-\frac{1}{\epsilon}\frac{\partial f^{\epsilon}}{\partial\eta}, \tag{3.32}$$

Then the equation (1.9) in $(\eta,\phi,\vec{v})$ becomes

$$\left\{\begin{array}{l}-(\vec{v}\cdot\vec{n})\dfrac{\partial f^{\epsilon}}{\partial\eta}+\dfrac{\epsilon}{1-\epsilon\eta}(\vec{v}\cdot\vec{\tau})\dfrac{\partial f^{\epsilon}}{\partial\phi}+\mathcal{L}[f^{\epsilon}]=\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(0,\phi,\vec{v})=b^{\epsilon}(0,\phi,\vec{v})\ \text{ for }\ \vec{v}\cdot\vec{n}<0.\end{array}\right. \tag{3.33}$$

Substitution 3:
We define the velocity substitution $f^{\epsilon}(r,\phi,\vec{v})\rightarrow f^{\epsilon}(\eta,\phi,\vec{v}_r)$ with $(\eta,\phi,\vec{v}_r)\in[0,1/\epsilon)\times[-\pi,\pi)\times\mathbb{R}^2$ as

$$\left\{\begin{array}{rcl}\eta&=&\eta,\\ \phi&=&\phi,\\ \vec{v}_r&=&-\vec{v}.\end{array}\right. \tag{3.34}$$

This substitution is for the convenience of Milne problem and specifying the in-flow boundary. Then the equation (1.9) in $(\eta,\phi,\vec{v})$ becomes

$$\left\{\begin{array}{l}(\vec{v}_r\cdot\vec{n})\dfrac{\partial f^{\epsilon}}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}(\vec{v}_r\cdot\vec{\tau})\dfrac{\partial f^{\epsilon}}{\partial\phi}+\mathcal{L}[f^{\epsilon}]=\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(0,\phi,\vec{v}_r)=b^{\epsilon}(0,\phi,\vec{v}_r)\ \text{ for }\ \vec{v}_r\cdot\vec{n}>0.\end{array}\right. \tag{3.35}$$

Substitution 4:
We further define the velocity decomposition with respect to the normal and tangential directions at boundary as $f^{\epsilon}(\eta,\phi,v_{r,1},v_{r,2})\rightarrow f^{\epsilon}(\eta,\phi,v_{\eta},v_{\phi})$ with $(\eta,\phi,v_{\eta},v_{\phi})\in[0,1/\epsilon)\times[-\pi,\pi)\times\mathbb{R}^2$

$$\left\{\begin{array}{rcl}\eta&=&\eta,\\ \phi&=&\phi,\\ v_{\eta}&=&v_{r,1}\cos\phi+v_{r,2}\sin\phi,\\ v_{\phi}&=&-v_{r,1}\sin\phi+v_{r,2}\cos\phi.\end{array}\right. \tag{3.36}$$

Denote $\vec{\mathfrak{v}}=(v_{\eta},v_{\phi})$. Then the equation (1.9) can be rewritten as

$$\left\{\begin{array}{l}v_{\eta}\dfrac{\partial f^{\epsilon}}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\left(v_{\phi}\dfrac{\partial f^{\epsilon}}{\partial\phi}+v_{\phi}^2\dfrac{\partial f^{\epsilon}}{\partial v_{\eta}}-v_{\eta}v_{\phi}\dfrac{\partial f^{\epsilon}}{\partial v_{\phi}}\right)+\mathcal{L}[f^{\epsilon}]=\Gamma[f^{\epsilon},f^{\epsilon}],\\ f^{\epsilon}(0,\phi,\vec{\mathfrak{v}})=b^{\epsilon}(\phi,\vec{\mathfrak{v}})\ \text{ for }\ v_{\eta}>0.\end{array}\right. \tag{3.37}$$

We define the boundary layer expansion

$$\mathscr{F}^{\epsilon}\sim\sum_{k=1}^{\infty}\epsilon^k\mathscr{F}_k^{\epsilon}(\eta,\phi,\vec{\mathfrak{v}}), \tag{3.38}$$

where $\mathscr{F}_k^\epsilon$ can be determined by plugging it into the equation (3.37) and comparing the order of $\epsilon$. In a neighborhood of the boundary, we have

$$v_\eta \frac{\partial \mathscr{F}_1^\epsilon}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_1^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_1^\epsilon}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_1^\epsilon] = 0, \tag{3.39}$$

$$v_\eta \frac{\partial \mathscr{F}_2^\epsilon}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_2^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_2^\epsilon}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_2^\epsilon] = \frac{1}{1-\epsilon\eta} v_\phi \frac{\partial \mathscr{F}_1^\epsilon}{\partial \phi} + \Gamma[\mathscr{F}_1^\epsilon, \mathscr{F}_1^\epsilon] + 2\Gamma[\mathcal{F}_1^\epsilon, \mathscr{F}_1^\epsilon], \tag{3.40}$$

$$\ldots$$

$$v_\eta \frac{\partial \mathscr{F}_k^\epsilon}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta}\left(v_\phi^2 \frac{\partial \mathscr{F}_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \mathscr{F}_k^\epsilon}{\partial v_\phi}\right) + \mathcal{L}[\mathscr{F}_k^\epsilon] = \frac{1}{1-\epsilon\eta} v_\phi \frac{\partial \mathscr{F}_{k-1}^\epsilon}{\partial \phi} + \sum_{i=1}^{k-1}\Gamma[\mathscr{F}_i^\epsilon, \mathscr{F}_{k-i}^\epsilon] + 2\sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i^\epsilon, \mathscr{F}_{k-i}^\epsilon]. \tag{3.41}$$

3.3. **Construction of Asymptotic Expansion.** The bridge between interior solution and boundary layer is the boundary condition

$$f^\epsilon(\vec{x}_0, \vec{v}) = b^\epsilon(\vec{x}_0, \vec{v}). \tag{3.42}$$

Plugging the combined expansion

$$f^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k (\mathcal{F}_k^\epsilon + \mathscr{F}_k^\epsilon), \tag{3.43}$$

into the boundary condition and comparing the order of $\epsilon$, we obtain

$$\mathcal{F}_1^\epsilon + \mathscr{F}_1^\epsilon = b_1, \tag{3.44}$$
$$\mathcal{F}_2^\epsilon + \mathscr{F}_2^\epsilon = b_2, \tag{3.45}$$
$$\ldots$$
$$\mathcal{F}_k^\epsilon + \mathscr{F}_k^\epsilon = b_k. \tag{3.46}$$

This is the boundary conditions $\mathcal{F}_k^\epsilon$ and $\mathscr{F}_k^\epsilon$ need to satisfy. We divide the construction of asymptotic expansion into several steps for each $k \geq 1$:

Step 1: $\epsilon$-Milne Problem.
We solve the $\epsilon$-Milne problem

$$\left\{\begin{array}{rcl} v_\eta \dfrac{\partial g_k^\epsilon}{\partial \eta} + G(\epsilon;\eta)\left(v_\phi^2 \dfrac{\partial g_k^\epsilon}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g_k^\epsilon}{\partial v_\phi}\right) + \mathcal{L}[g_k^\epsilon] &=& S_k^\epsilon(\eta, \phi, \vec{\mathfrak{v}}), \\ g_k^\epsilon(0, \phi, \vec{\mathfrak{v}}) &=& h_k^\epsilon(\phi, \vec{\mathfrak{v}}) \ \text{ for } \ v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_k^\epsilon(0, \phi, \vec{\mathfrak{v}}) \mathrm{d}\vec{\mathfrak{v}} &=& m_f[g_k^\epsilon](\phi), \\ \displaystyle\lim_{\eta\to\infty} g_k^\epsilon(\eta, \phi, \vec{\mathfrak{v}}) &=& g_k^\epsilon(\infty, \phi, \vec{\mathfrak{v}}), \end{array}\right. \tag{3.47}$$

for $g_k^\epsilon(\eta, \phi, \vec{\mathfrak{v}})$ with the in-flow boundary data

$$h_k^\epsilon = b_k - (B_k^\epsilon + C_k^\epsilon) \tag{3.48}$$

and source term

$$S_k^\epsilon = \frac{\Upsilon(\epsilon^{1/2}\eta)}{1-\epsilon\eta} v_\phi \frac{\partial \mathscr{F}_{k-1}^\epsilon}{\partial \phi} + \sum_{i=1}^{k-1}\Gamma[\mathscr{F}_i^\epsilon, \mathscr{F}_{k-i}^\epsilon] + 2\sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i^\epsilon, \mathscr{F}_{k-i}^\epsilon], \tag{3.49}$$

where

$$G(\epsilon;\eta) = -\frac{\epsilon\Upsilon(\epsilon^{1/2}\eta)}{1-\epsilon\eta}, \tag{3.50}$$

$$\Upsilon(z)=\left\{\begin{array}{ll}1 & 0\leq z\leq 1/2,\\ 0 & 3/4\leq z\leq\infty.\end{array}\right. \tag{3.51}$$

Here the mass-flux $m_f[g_k^\epsilon](\phi)$ will be determined later. Based on Theorem 4.15, there exist

$$\tilde{h}_k^\epsilon=\sqrt{\mu}\bigg(\tilde{D}_{k,0}^\epsilon+\tilde{D}_{k,1}^\epsilon v_\eta+\tilde{D}_{k,2}^\epsilon v_\phi+\tilde{D}_{k,3}^\epsilon\bigg(\frac{\left|\vec{\mathfrak{v}}\right|^2-2}{2}\bigg)\bigg), \tag{3.52}$$

such that the problem

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial\mathcal{G}_k^\epsilon}{\partial\eta}+G(\epsilon;\eta)\bigg(v_\phi^2\dfrac{\partial\mathcal{G}_k^\epsilon}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathcal{G}_k^\epsilon}{\partial v_\phi}\bigg)+\mathcal{L}[\mathcal{G}_k^\epsilon]&=&S_k^\epsilon(\eta,\phi,\vec{\mathfrak{v}}),\\ \mathcal{G}_k^\epsilon(0,\phi,\vec{\mathfrak{v}})&=&h_k^\epsilon(\phi,\vec{\mathfrak{v}})-\tilde{h}_k^\epsilon(\phi,\vec{\mathfrak{v}})\ \ \text{for}\ \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\mathcal{G}_k^\epsilon(0,\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&m_f[\mathcal{G}_k^\epsilon](\phi),\\ \displaystyle\lim_{\eta\rightarrow\infty}\mathcal{G}_k^\epsilon(\eta,\phi,\vec{\mathfrak{v}})&=&0,\end{array}\right. \tag{3.53}$$

is well-posed, where we need to specify

$$m_f[\mathcal{G}_k^\epsilon](\phi)=m_f[g_k^\epsilon](\phi)-\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\tilde{h}_k^\epsilon(\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}. \tag{3.54}$$

Step 2: Definition of Interior Solution and Boundary Layer with Geometric Correction.
Define

$$\mathscr{F}_k^\epsilon=\mathcal{G}_k^\epsilon\cdot\Upsilon_0(\epsilon^{1/2}\eta) \tag{3.55}$$

where $\mathcal{G}_k^\epsilon$ the solution of $\epsilon$-Milne problem (3.53) and

$$\Upsilon_0(z)=\left\{\begin{array}{ll}1 & 0\leq z\leq 1/4,\\ 0 & 1/2\leq z\leq\infty.\end{array}\right. \tag{3.56}$$

Naturally, we have

$$\lim_{\eta\rightarrow 0}\mathscr{F}_k^\epsilon(\eta,\phi,\vec{\mathfrak{v}})=0. \tag{3.57}$$

The interior solution

$$\mathcal{F}_k^\epsilon(\vec{x},\vec{v})=A_k^\epsilon(\vec{x},\vec{v})+B_k^\epsilon(\vec{x},\vec{v})+C_k^\epsilon(\vec{x},\vec{v}), \tag{3.58}$$

where $B_k^\epsilon$ and $C_k^\epsilon$ are defined in (3.11) and (3.16), and $A_k^\epsilon$ satisfies

$$A_k^\epsilon=\sqrt{\mu}\left(\rho_k^\epsilon+u_{k,1}^\epsilon v_1+u_{k,2}^\epsilon v_2+\theta_k^\epsilon\left(\frac{\left|\vec{v}\right|^2-2}{2}\right)\right), \tag{3.59}$$

and

$$P_{k+1}^\epsilon-\left(\rho_{k+1}^\epsilon+\theta_{k+1}^\epsilon+\sum_{i=1}^{k+1-i}\rho_i^\epsilon\theta_{k+1-i}^\epsilon\right)=0, \tag{3.60}$$

$$\sum_{i=1}^{k}\vec{u}_i^\epsilon\cdot\nabla_x\vec{u}_{k+1-i}^\epsilon-\gamma_1\Delta_x\vec{u}_k^\epsilon+\nabla_xP_{k+1}^\epsilon=H_{k,1}^\epsilon, \tag{3.61}$$

$$\nabla_x\cdot\vec{u}_k^\epsilon=H_{k,2}^\epsilon, \tag{3.62}$$

$$\sum_{i=1}^{k}\vec{u}_i^\epsilon\cdot\nabla_x\theta_{k+1-i}^\epsilon-\gamma_2\Delta_x\theta_k^\epsilon=H_{k,3}^\epsilon, \tag{3.63}$$

with boundary condition

$$A^{\epsilon}_{k,0} = \tilde{D}^{\epsilon}_{k,0}, \tag{3.64}$$

$$A^{\epsilon}_{k,1} = -\tilde{D}^{\epsilon}_{k,1}\cos\phi + \tilde{D}^{\epsilon}_{k,2}\sin\phi, \tag{3.65}$$

$$A^{\epsilon}_{k,2} = -\tilde{D}^{\epsilon}_{k,1}\sin\phi - \tilde{D}^{\epsilon}_{k,2}\cos\phi, \tag{3.66}$$

$$A^{\epsilon}_{k,3} = \tilde{D}^{\epsilon}_{k,3}. \tag{3.67}$$

where $\tilde{D}^{\epsilon}_{k,i}$ comes from the boundary data of $\epsilon$-Milne problem $\tilde{h}^{\epsilon}_{k}$. This determines $A^{\epsilon}_{k,0}$, $A^{\epsilon}_{k,1}$, $A^{\epsilon}_{k,2}$ and $A^{\epsilon}_{k,3}$. Now it is easy to verify the boundary data are satisfied as

$$\mathcal{F}^{\epsilon}_{k} + \mathscr{F}^{\epsilon}_{k} = b_k. \tag{3.68}$$

Step 3: Boussinesq relation and Vanishing Mass-Flux.
Note that the fluid-type equations satisfied by $A^{\epsilon}_{k}$ imply Boussinesq relation. In detail,

$$P^{\epsilon}_{k+1} - \left(\rho^{\epsilon}_{k+1} + \theta^{\epsilon}_{k+1} + \sum_{i=1}^{k+1-i}\rho^{\epsilon}_{i}\theta^{\epsilon}_{k+1-i}\right) = 0 \tag{3.69}$$

yields

$$\rho^{\epsilon}_{k} + \theta^{\epsilon}_{k} + \sum_{i=1}^{k-i}\rho^{\epsilon}_{i}\theta^{\epsilon}_{k-i} = E_k \tag{3.70}$$

which is actually

$$\rho^{\epsilon}_{k} + \theta^{\epsilon}_{k} = E_k - \sum_{i=1}^{k-i}\rho^{\epsilon}_{i}\theta^{\epsilon}_{k-i} \tag{3.71}$$

for some constant $E_k$ which is free to choose. To enforce this relation, we need to adjust the mass-flux in the $\epsilon$-Milne problem (3.53). Note that the Boussinesq relation (3.71) leads to a given $\tilde{D}^{\epsilon}_{k,0}(\phi)+\tilde{D}^{\epsilon}_{k,3}(\phi)$ for any $\phi$ up to a constant in the $\epsilon$-Milne problem. Theorem 4.8 implies we can always adjust the mass-flux $m_f[\mathcal{G}^{\epsilon}_{k}](\phi)$ to guarantee the Boussinesq relation. Based on the proof of Theorem 4.8, we know this can determine the mass-flux $m_f[\mathcal{G}^{\epsilon}_{k}](\phi)$ up to a constant.

On the other hand, $f^{\epsilon}(\vec{x},\vec{v})$ satisfies the vanishing mass-flux

$$\int_{\partial\Omega}\int_{\mathbb{R}^2} f^{\epsilon}(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma = 0. \tag{3.72}$$

Then we need

$$\int_{\partial\Omega}\int_{\mathbb{R}^2} (\mathcal{F}^{\epsilon}_{k} + \mathscr{F}^{\epsilon}_{k})(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma = 0. \tag{3.73}$$

for any $k \geq 1$. The definition of $\mathscr{F}^{\epsilon}_{k}$ implies

$$\int_{\partial\Omega}\int_{\mathbb{R}^2} \mathscr{F}^{\epsilon}_{k}(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma = 0. \tag{3.74}$$

Then the remaining relation

$$\int_{\partial\Omega}\int_{\mathbb{R}^2} \mathcal{F}^{\epsilon}_{k}(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma = 0. \tag{3.75}$$

finally determines the free constant in mass-flux $m_f[\mathcal{G}^{\epsilon}_{k}](\phi)$.

In summary, the free mass-flux $m_f[\mathcal{G}^{\epsilon}_{k}](\phi)$ can help to enforce two relations: the Boussinesq relation

$$\rho^{\epsilon}_{k} + \theta^{\epsilon}_{k} = E_k - \sum_{i=1}^{k-i}\rho^{\epsilon}_{i}\theta^{\epsilon}_{k-i}, \tag{3.76}$$

and vanishing mass-flux relation

$$\int_{\partial\Omega}\int_{\mathbb{R}^2} \mathcal{F}^{\epsilon}_{k}(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma = 0. \tag{3.77}$$

Therefore, $m_f[\mathcal{G}_k^\epsilon](\phi)$ is completed determined and so are $\mathcal{F}_k^\epsilon$ and $\mathscr{F}_k^\epsilon$.

In particular, when $k=1$, $\mathscr{F}_1^\epsilon$ satisfies

$$\left\{\begin{array}{rcl}\mathscr{F}_1^\epsilon(\eta,\phi,\vec{\mathfrak{v}})&=&\mathcal{G}_1^\epsilon(\eta,\phi,\vec{\mathfrak{v}})\cdot\Upsilon_0(\sqrt{\epsilon\eta})\\ v_\eta\dfrac{\partial\mathcal{G}_1^\epsilon}{\partial\eta}+G(\epsilon;\eta)\bigg(v_\phi^2\dfrac{\partial\mathcal{G}_1^\epsilon}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathcal{G}_1^\epsilon}{\partial v_\phi}\bigg)+\mathcal{L}[\mathcal{G}_1^\epsilon]&=&0,\\ \mathcal{G}_1^\epsilon(0,\phi,\vec{\mathfrak{v}})&=&b_1(\phi,\vec{\mathfrak{v}})-\tilde{h}^\epsilon(\phi,\vec{\mathfrak{v}})\ \ \text{for}\ \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\mathcal{G}_1^\epsilon(0,\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&m_f[\mathcal{G}_1^\epsilon],\\ \displaystyle\lim_{\eta\to\infty}\mathcal{G}_1^\epsilon(\eta,\phi,\vec{\mathfrak{v}})&=&0,\end{array}\right. \tag{3.78}$$

where

$$\tilde{h}_1^\epsilon=\sqrt{\mu(\vec{v})}\bigg(\tilde{D}_{1,0}^\epsilon+\tilde{D}_{1,1}^\epsilon v_\eta+\tilde{D}_{1,2}^\epsilon v_\phi+\tilde{D}_{1,3}^\epsilon\bigg(\frac{\left|\vec{v}\right|^2-2}{2}\bigg)\bigg), \tag{3.79}$$

and $\mathcal{F}_1^\epsilon$ satisfies

$$\mathcal{F}_1^\epsilon=\sqrt{\mu}\bigg(\rho_1^\epsilon+u_{1,1}^\epsilon v_1+u_{1,2}^\epsilon v_2+\theta_1^\epsilon\bigg(\frac{\left|\vec{v}\right|^2-2}{2}\bigg)\bigg), \tag{3.80}$$

with

$$\left\{\begin{array}{rcl}\nabla_x(\rho_1^\epsilon+\theta_1^\epsilon)&=&0,\\ \vec{u}_1^\epsilon\cdot\nabla_x\vec{u}_1^\epsilon-\gamma_1\Delta_x\vec{u}_1^\epsilon+\nabla_xP_2^\epsilon&=&0,\\ \nabla_x\cdot\vec{u}_1^\epsilon&=&0,\\ \vec{u}_1^\epsilon\cdot\nabla_x\theta_1^\epsilon-\gamma_2\Delta_x\theta_1^\epsilon&=&0,\\ \rho_1^\epsilon(\vec{x}_0)&=&\tilde{D}_{1,0}^\epsilon(\vec{x}_0),\\ u_{1,1}^\epsilon(\vec{x}_0)&=&-\tilde{D}_{1,1}^\epsilon(\vec{x}_0)\cos\phi+\tilde{D}_{1,2}^\epsilon(\vec{x}_0)\sin\phi,\\ u_{1,2}^\epsilon(\vec{x}_0)&=&-\tilde{D}_{1,1}^\epsilon(\vec{x}_0)\sin\phi-\tilde{D}_{1,2}^\epsilon(\vec{x}_0)\cos\phi,\\ \theta_1^\epsilon(\vec{x}_0)&=&\tilde{D}_{1,3}^\epsilon(\vec{x}_0),\end{array}\right. \tag{3.81}$$

where the free mass-flux $m_f[\mathcal{G}_1^\epsilon](\phi)$ is chosen to enforce the Boussinesq relation

$$\rho_1^\epsilon+\theta_1^\epsilon=E_1, \tag{3.82}$$

and vanishing mass-flux relation

$$\int_{\partial\Omega}\int_{\mathbb{R}^2}\mathcal{F}_1^\epsilon(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma=0. \tag{3.83}$$

Similarly, we can define any $\mathcal{F}_k^\epsilon$ and $\mathscr{F}_k^\epsilon$ for $k\geq1$.

## 4. $\epsilon$-Milne Problem with Geometric Correction

We consider the $\epsilon$-Milne problem for $g^\epsilon(\eta,\phi,\vec{\mathfrak{v}})$ in the domain $(\eta,\phi,\vec{\mathfrak{v}})\in[0,\infty)\times[-\pi,\pi)\times\mathbb{R}^2$ as

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial g^\epsilon}{\partial\eta}+G(\epsilon;\eta)\bigg(v_\phi^2\dfrac{\partial g^\epsilon}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial g^\epsilon}{\partial v_\phi}\bigg)+\mathcal{L}[g^\epsilon]&=&S^\epsilon(\eta,\phi,\vec{\mathfrak{v}}),\\ g^\epsilon(0,\phi,\vec{\mathfrak{v}})&=&h^\epsilon(\phi,\vec{\mathfrak{v}})\ \ \text{for}\ \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}g^\epsilon(0,\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&m_f[g^\epsilon](\phi),\\ \displaystyle\lim_{\eta\to\infty}g^\epsilon(\eta,\phi,\vec{\mathfrak{v}})&=&g_\infty^\epsilon(\phi,\vec{\mathfrak{v}}),\end{array}\right. \tag{4.1}$$

where the velocity variables

$$\vec{\mathfrak{v}}=(v_\eta,v_\phi), \tag{4.2}$$

the standard Maxwellian

$$\mu(\vec{\mathfrak{v}})=\frac{1}{2\pi}\exp\bigg(-\frac{\left|\vec{\mathfrak{v}}\right|^2}{2}\bigg), \tag{4.3}$$

$m_f[g^\epsilon]$ is the mass-flux given a priori, the limit function

$$g^\epsilon_\infty(\phi,\vec{\mathfrak{v}})=\sqrt{\mu}\bigg(D^\epsilon_0(\phi)+D^\epsilon_1(\phi)v_\eta+D^\epsilon_2(\phi)v_\phi+D^\epsilon_3(\phi)\frac{\left|\vec{\mathfrak{v}}\right|^2-2}{2}\bigg), \tag{4.4}$$

the forcing term

$$G(\epsilon;\eta)=\;-\frac{\epsilon\Upsilon(\epsilon^{1/2}\eta)}{1-\epsilon\eta}, \tag{4.5}$$

and the cut-off function $\Upsilon(z)\in C^\infty$ is defined as

$$\Upsilon(z)=\left\{\begin{array}{ll}1 & 0\leq z\leq 1/2,\\ 0 & 3/4\leq z\leq\infty.\end{array}\right. \tag{4.6}$$

We assume the boundary data and source term satisfy for $0\leq\zeta\leq 1/4$ and $\vartheta\geq 0$,

$$\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h^\epsilon(\phi,\vec{\mathfrak{v}})\right|\leq M, \tag{4.7}$$

and

$$\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S^\epsilon(\eta,\phi,\vec{\mathfrak{v}})\right|\leq M\mathrm{e}^{-K\eta}, \tag{4.8}$$

for $M(\zeta,\vartheta)\geq 0$ and $K(\zeta,\vartheta)>0$ uniform in $\epsilon$ and $\phi$, where the Japanese bracket is defined as

$$\langle\vec{\mathfrak{v}}\rangle=(1+\left|\vec{\mathfrak{v}}\right|^2)^{1/2}. \tag{4.9}$$

We define a potential function $W(\epsilon;\eta)$ as $G(\epsilon;\eta)=-\partial_\eta W(\epsilon;\eta)$ with $W(\epsilon;0)=0$. Our main goal is to find

$$\tilde{h}^\epsilon(\phi,\vec{\mathfrak{v}})=\sqrt{\mu}\bigg(\tilde{D}^\epsilon_0(\phi)+\tilde{D}^\epsilon_1(\phi)v_\eta+\tilde{D}^\epsilon_2(\phi)v_\phi+\tilde{D}^\epsilon_3(\phi)\frac{\left|\vec{\mathfrak{v}}\right|^2-2}{2}\bigg), \tag{4.10}$$

such that the $\epsilon$-Milne problem for $\mathcal{G}^\epsilon(\eta,\phi,\vec{\mathfrak{v}})$ in the domain $(\eta,\phi,\vec{\mathfrak{v}})\in[0,\infty)\times[-\pi,\pi)\times\mathbb{R}^2$ as

$$\left\{\begin{array}{rcl}
v_\eta\dfrac{\partial\mathcal{G}^\epsilon}{\partial\eta}+G(\epsilon;\eta)\bigg(v_\phi^2\dfrac{\partial\mathcal{G}^\epsilon}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathcal{G}^\epsilon}{\partial v_\phi}\bigg)+\mathcal{L}[\mathcal{G}^\epsilon] &=& S^\epsilon(\eta,\phi,\vec{\mathfrak{v}}),\\
\mathcal{G}^\epsilon(0,\phi,\vec{\mathfrak{v}}) &=& h^\epsilon(\phi,\vec{\mathfrak{v}})-\tilde{h}^\epsilon(\phi,\vec{\mathfrak{v}})\ \ \text{for}\ \ v_\eta>0,\\
\displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\mathcal{G}^\epsilon(0,\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}} &=& m_f[\mathcal{G}^\epsilon](\phi),\\
\displaystyle\lim_{\eta\rightarrow\infty}\mathcal{G}^\epsilon(\eta,\phi,\vec{\mathfrak{v}}) &=& 0,
\end{array}\right. \tag{4.11}$$

is well-posed, where

$$m_f[\mathcal{G}^\epsilon](\phi)=m_f[g^\epsilon](\phi)-\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\tilde{h}^\epsilon(\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}. \tag{4.12}$$

For notational simplicity, we omit superscript $\epsilon$ and $\phi$ dependence in $g^\epsilon$ and $\mathcal{G}^\epsilon$ in this section. The same convention also applies to $G(\epsilon;\eta)$, $W(\epsilon;\eta)$, $S^\epsilon(\eta,\phi,\vec{\mathfrak{v}})$ and $h^\epsilon(\phi,\vec{\mathfrak{v}})$. It is easy to see the estimates are uniform in $\epsilon$ and $\phi$. Our analysis is based on the ideas in [**?**, **?**, **?**, **?**].

In this section, we introduce some special notations to describe the norms in the space $(\eta,\vec{\mathfrak{v}})\in[0,\infty)\times\mathbb{R}^2$. Define the $L^2$ norm as follows:

$$\|f(\eta)\|_{L^2}\;=\bigg(\int_{\mathbb{R}^2}\left|f(\eta,\vec{\mathfrak{v}})\right|^2\mathrm{d}\vec{\mathfrak{v}}\bigg)^{1/2}, \tag{4.13}$$

$$|||f|||_{L^2L^2}=\bigg(\int_0^\infty\int_{\mathbb{R}^2}\left|f(\eta,\vec{\mathfrak{v}})\right|^2\mathrm{d}\vec{\mathfrak{v}}\mathrm{d}\eta\bigg)^{1/2}. \tag{4.14}$$

Define the inner product in $\vec{\mathfrak{v}}$ space

$$\langle f,g\rangle(\eta)=\int_{\mathbb{R}^2}f(\eta,\vec{\mathfrak{v}})g(\eta,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}. \tag{4.15}$$

Define the weighted $L^{\infty}$ norm as follows:

$$\|f(\eta)\|_{L^{\infty}_{\vartheta,\zeta}} = \sup_{\vec{\mathfrak{v}}\in\mathbb{R}^2}\bigg(\left\langle\vec{\mathfrak{v}}\right\rangle^{\vartheta}\mathrm{e}^{\zeta\left|\vec{\mathfrak{v}}\right|^2}\left|f(\eta,\vec{\mathfrak{v}})\right|\bigg), \tag{4.16}$$

$$|||f|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}} = \sup_{(\eta,\vec{\mathfrak{v}})\in[0,\infty)\times\mathbb{R}^2}\bigg(\left\langle\vec{\mathfrak{v}}\right\rangle^{\vartheta}\mathrm{e}^{\zeta\left|\vec{\mathfrak{v}}\right|^2}\left|f(\eta,\vec{\mathfrak{v}})\right|\bigg), \tag{4.17}$$

$$\|f\|_{L^{\infty}L^2_{\zeta}} = \sup_{\eta\in[0,\infty)}\bigg(\int_{\mathbb{R}^2}\left|e^{2\zeta\left|\vec{\mathfrak{v}}\right|^2}f(\eta,\vec{\mathfrak{v}})\right|^2\mathrm{d}\vec{\mathfrak{v}}\bigg)^{1/2}. \tag{4.18}$$

Since the boundary data $h(\vec{\mathfrak{v}})$ is only defined on $v_\eta > 0$, we naturally extend above definitions on this half-domain as follows:

$$\|h\|_{L^2} = \bigg(\int_{v_\eta>0}\left|h(\vec{\mathfrak{v}})\right|^2\mathrm{d}\vec{\mathfrak{v}}\bigg)^{1/2}, \tag{4.19}$$

$$\|h\|_{L^{\infty}_{\vartheta,\zeta}} = \sup_{v_\eta>0}\bigg(\left\langle\vec{\mathfrak{v}}\right\rangle^{\vartheta}\mathrm{e}^{\zeta\left|\vec{\mathfrak{v}}\right|^2}\left|h(\vec{\mathfrak{v}})\right|\bigg). \tag{4.20}$$

Since the null of operator $\mathcal{L}$ is $\mathcal{N} = \sqrt{\mu}\bigg\{1, v_\eta, v_\phi, \dfrac{\left|\vec{\mathfrak{v}}\right|^2-2}{2}\bigg\} = \{\psi_0,\psi_1,\psi_2,\psi_3\}$, we can decompose the solution as

$$g = w_g + q_g, \tag{4.21}$$

where

$$q_g = \sqrt{\mu}\bigg(q_{g0} + q_{g1}v_\eta + q_{g2}v_\phi + q_{g3}\frac{\left|\vec{\mathfrak{v}}\right|^2-2}{2}\bigg) = q_{g0}\psi_0 + q_{g1}\psi_1 + q_{g2}\psi_2 + q_{g3}\psi_3 \in \mathcal{N}, \tag{4.22}$$

and

$$w_g \in \mathcal{N}^{\perp}. \tag{4.23}$$

When there is no confusion, we will simply write $g = w + q$.

**Lemma 4.1.** *When $\epsilon \leq 1/2$, the force $G(\eta)$ and potential $W(\eta)$ satisfy the following properties:*

*1. $W(\eta)$ is an increasing function satisfying for any $\eta > 0$,*

$$0 \leq W(\eta) \leq 1, \tag{4.24}$$

*2.*

$$\lim_{\epsilon\to 0} W(\infty) = 0. \tag{4.25}$$

*3.*

$$\int_0^{\infty}\bigg(\mathrm{e}^{-W(\eta)} - \mathrm{e}^{-W(\infty)}\bigg)^2\mathrm{d}\eta \leq C. \tag{4.26}$$

*4.*

$$\int_0^{\infty} G^2(\eta)\mathrm{d}\eta \leq C. \tag{4.27}$$

*5.*

$$\int_0^{\infty}\int_{\eta}^{\infty} G^2(y)\mathrm{d}y\mathrm{d}\eta \leq C. \tag{4.28}$$

*Proof.* Since $G(\eta)$ is always zero or negative, we know $W(\eta)$ is increasing. Then we directly compute

$$\begin{aligned}\int_0^{\infty}\left|G(\epsilon;\eta)\right|\mathrm{d}\eta = \int_0^{\infty}\frac{\epsilon\Upsilon(\epsilon^{1/2}\eta)}{1-\epsilon\eta}\mathrm{d}\eta \leq \int_0^{\frac{3}{4\sqrt{\epsilon}}}\frac{\epsilon}{1-\epsilon\eta}\mathrm{d}\eta \\ = -\ln(1-\epsilon\eta)\bigg|_0^{\frac{3}{4\sqrt{\epsilon}}} = -\ln\bigg(1-\frac{3}{4}\epsilon^{1/2}\bigg).\end{aligned} \tag{4.29}$$

This naturally implies (4.24) and (4.25). Also, we have

$$\int_0^{\infty}\Big(\mathrm{e}^{-W(\eta)}-\mathrm{e}^{-W(\infty)}\Big)^2\mathrm{d}\eta=\mathrm{e}^{-2W(\infty)}\int_0^{\frac{3}{4\sqrt{\epsilon}}}\Big(\mathrm{e}^{W(\infty)-W(\eta)}-1\Big)^2\mathrm{d}\eta. \tag{4.30}$$

Then we obtain

$$\int_0^{\frac{3}{4\sqrt{\epsilon}}}\mathrm{e}^{W(\infty)-W(\eta)}\mathrm{d}\eta=\int_0^{\frac{3}{4\sqrt{\epsilon}}}\exp\bigg(-\int_\eta^{\frac{3}{4\sqrt{\epsilon}}}G(y)\mathrm{d}y\bigg)\mathrm{d}\eta=\int_0^{\frac{3}{4\sqrt{\epsilon}}}\exp\bigg(\int_\eta^{\frac{3}{4\sqrt{\epsilon}}}\frac{\epsilon\Upsilon(\epsilon^{1/2}y)}{1-\epsilon y}\mathrm{d}y\bigg)\mathrm{d}\eta.$$

Note that

$$1\leq\exp\bigg(\int_\eta^{\frac{3}{4\sqrt{\epsilon}}}\frac{\epsilon\Upsilon(\epsilon^{1/2}y)}{1-\epsilon y}\mathrm{d}y\bigg)\leq\exp\bigg(\int_\eta^{\frac{3}{4\sqrt{\epsilon}}}\frac{\epsilon}{1-\epsilon y}\mathrm{d}y\bigg)=\exp\bigg(-\ln(1-\epsilon y)\bigg|_\eta^{\frac{3}{4\sqrt{\epsilon}}}\bigg)=\frac{1-\epsilon\eta}{1-\frac{3\sqrt{\epsilon}}{4}} \tag{4.31}$$

Hence, we have

$$\Big(\mathrm{e}^{W(\infty)-W(\eta)}-1\Big)^2\leq\max\bigg\{\bigg(\frac{1-\epsilon\eta}{1-\frac{3\sqrt{\epsilon}}{4}}\bigg)^2-1,2\bigg(\frac{1-\epsilon\eta}{1-\frac{3\sqrt{\epsilon}}{4}}-1\bigg)\bigg\}\leq C\epsilon\eta+C\sqrt{\epsilon}, \tag{4.32}$$

for some constant $C$ independent of $\epsilon$. Then we have

$$\int_0^{\frac{3}{4\sqrt{\epsilon}}}\Big(\mathrm{e}^{W(\infty)-W(\eta)}-1\Big)^2\mathrm{d}\eta\leq C\int_0^{\frac{3}{4\sqrt{\epsilon}}}\Big(\epsilon\eta+\sqrt{\epsilon}\Big)\mathrm{d}\eta\leq C. \tag{4.33}$$

This proves (4.26). Furthermore,

$$\int_0^{\infty}G^2(\eta)\mathrm{d}\eta=\int_0^{\infty}\bigg(\frac{\epsilon\Upsilon(\epsilon^{1/2}\eta)}{1-\epsilon\eta}\bigg)^2\mathrm{d}\eta\leq\int_0^{\frac{3}{4\sqrt{\epsilon}}}\bigg(\frac{\epsilon}{1-\epsilon\eta}\bigg)^2\mathrm{d}\eta=-\epsilon\ln(1-\epsilon\eta)\bigg|_0^{\frac{3}{4\sqrt{\epsilon}}}=-\epsilon\ln\bigg(1-\frac{3\sqrt{\epsilon}}{4}\bigg). \tag{4.34}$$

Therefore, (4.27) is obvious. Moreover, we compute

$$\begin{aligned}\int_0^{\infty}\int_\eta^{\infty}G^2(y)\mathrm{d}y\mathrm{d}\eta&=\int_0^{\infty}\int_\eta^{\infty}\bigg(\frac{\epsilon\Upsilon(\epsilon^{1/2}y)}{1-\epsilon y}\bigg)^2\mathrm{d}y\mathrm{d}\eta=\int_0^{\frac{3}{4\sqrt{\epsilon}}}\int_\eta^{\frac{3}{4\sqrt{\epsilon}}}\bigg(\frac{\epsilon\Upsilon(\epsilon^{1/2}y)}{1-\epsilon y}\bigg)^2\mathrm{d}y\mathrm{d}\eta \\ &\leq\int_0^{\frac{3}{4\sqrt{\epsilon}}}\int_\eta^{\frac{3}{4\sqrt{\epsilon}}}\bigg(\frac{\epsilon}{1-\epsilon y}\bigg)^2\mathrm{d}y\mathrm{d}\eta=-\int_0^{\frac{3}{4\sqrt{\epsilon}}}\bigg(\frac{\epsilon}{1-\epsilon y}\bigg|_\eta^{\frac{3}{4\sqrt{\epsilon}}}\bigg)\mathrm{d}\eta \\ &=\int_0^{\frac{3}{4\sqrt{\epsilon}}}\bigg(\frac{\epsilon}{1-\epsilon\eta}-\frac{\epsilon}{1-\frac{3\sqrt{\epsilon}}{4}}\bigg)\mathrm{d}\eta \\ &=-\ln(1-\epsilon\eta)\bigg|_0^{\frac{3}{4\sqrt{\epsilon}}}+\frac{\frac{3\sqrt{\epsilon}}{4}}{1-\frac{3\sqrt{\epsilon}}{4}}\leq C.\end{aligned} \tag{4.35}$$

□

**Lemma 4.2.** *For the operator $\mathcal{L}=\nu-K$, we have the estimates*

$$\nu_0(1+|\vec{\mathfrak{v}}|)\leq\nu(\vec{\mathfrak{v}})\leq\nu_1(1+|\vec{\mathfrak{v}}|), \tag{4.36}$$

$$\langle g,\mathcal{L}[g]\rangle(\eta)=\langle w,\mathcal{L}[w]\rangle(\eta)\geq C\left\|\sqrt{\nu}w(\eta)\right\|_{L^2}^2, \tag{4.37}$$

*for $\nu_0$, $\nu_1$ and $C$ positive constants.*

*Proof.* See [**?**, Chapter 3]. □

4.1. $L^2$ **Estimates.**

4.1.1. $L^2$ *Estimates in a finite slab.* We first consider the case with zero source term for $g^L(\eta,\vec{\mathfrak{v}})$ in a finite slab $[0,L]\times\mathbb{R}^2$ as

$$\left\{\begin{array}{rcl} v_{\eta}\dfrac{\partial g^L}{\partial\eta}+G(\eta)\bigg(v_{\phi}^2\dfrac{\partial g^L}{\partial v_{\eta}}-v_{\eta}v_{\phi}\dfrac{\partial g^L}{\partial v_{\phi}}\bigg)+\mathcal{L}[g^L]&=&0,\\ g^L(0,\vec{\mathfrak{v}})&=&h(\vec{\mathfrak{v}})\ \text{ for }\ v_{\eta}>0,\\ g^L(L,R[\vec{\mathfrak{v}}])&=&g^L(L,\vec{\mathfrak{v}}), \end{array}\right. \tag{4.38}$$

where

$$R[v_{\eta},v_{\phi}]=(-v_{\eta},v_{\phi}). \tag{4.39}$$

Similarly, we can decompose $g^L$ as

$$g^L=w^L+q^L. \tag{4.40}$$

**Lemma 4.3.** *There exists a solution of the equation (4.38) satisfying the estimates*

$$\int_0^L\left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2}^2\mathrm{d}\eta\leq C, \tag{4.41}$$

$$\left\|q^L(\eta)\right\|_{L^2}^2\leq C\bigg(1+\eta+\left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2}^2\bigg), \tag{4.42}$$

*where $C$ is a constant independent of $L$. Also, the solution satisfies the orthogonal relation*

$$\left\langle v_{\eta}\psi_i,w^L\right\rangle(\eta)=0,\quad\textit{for}\ \ i=0,2,3. \tag{4.43}$$

*Proof.* The existence follows from a standard argument by adding penalty term $\lambda g^L$ on the left-hand side of the equation for $0<\lambda<<1$ and estimate along the characteristics. Hence, we concentrate on the estimates. We divide the proof into several steps:

Step 1: Estimate of $w^L$.
Multiplying $g^L$ on both sides of (4.38) and integrating over $\vec{\mathfrak{v}}\in\mathbb{R}^2$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_{\eta}g^L,g^L\right\rangle+G(\eta)\left\langle v_{\phi}^2\frac{\partial g^L}{\partial v_{\eta}}-v_{\eta}v_{\phi}\frac{\partial g^L}{\partial v_{\phi}},g^L\right\rangle=-(g^L,\mathcal{L}[g^L]). \tag{4.44}$$

An integration by parts implies

$$\left\langle v_{\phi}^2\frac{\partial g^L}{\partial v_{\eta}}-v_{\eta}v_{\phi}\frac{\partial g^L}{\partial v_{\phi}},g^L\right\rangle=\frac{1}{2}\left\langle v_{\phi}^2,\frac{\partial(g^L)^2}{\partial v_{\eta}}\right\rangle-\frac{1}{2}\left\langle v_{\eta}v_{\phi},\frac{\partial(g^L)^2}{\partial v_{\phi}}\right\rangle=\frac{1}{2}\left\langle v_{\eta}g^L,g^L\right\rangle. \tag{4.45}$$

Therefore, using Lemma 4.2, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_{\eta}g^L,g^L\right\rangle+\frac{1}{2}G(\eta)\left\langle v_{\eta}g^L,g^L\right\rangle=-(w^L,\mathcal{L}[w^L]). \tag{4.46}$$

Define

$$\alpha(\eta)=\frac{1}{2}\left\langle v_{\eta}g^L,g^L\right\rangle(\eta), \tag{4.47}$$

which implies

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\eta}+G(\eta)\alpha=-(w^L,\mathcal{L}[w^L]). \tag{4.48}$$

Then we have

$$\alpha(\eta)=\alpha(L)\exp\bigg(\int_{\eta}^L G(y)\mathrm{d}y\bigg)+\int_{\eta}^L\exp\bigg(-\int_{\eta}^y G(z)\mathrm{d}z\bigg)\Big(\left\langle w^L,\mathcal{L}[w^L]\right\rangle(y)\Big)\mathrm{d}y, \tag{4.49}$$

$$\alpha(\eta)=\alpha(0)\exp\bigg(-\int_0^{\eta}G(y)\mathrm{d}y\bigg)+\int_0^{\eta}\exp\bigg(\int_y^{\eta}G(z)\mathrm{d}z\bigg)\Big(-\left\langle w^L,\mathcal{L}[w^L]\right\rangle(y)\Big)\mathrm{d}y. \tag{4.50}$$

Since $\alpha(L)=0$ due to reflexive boundary, (4.49) implies

$$\alpha(\eta)\geq 0. \tag{4.51}$$

By

$$\alpha(0)=\frac{1}{2}\int_{v_\eta>0}v_\eta(g^L)^2(0)\mathrm{d}\vec{\mathfrak{v}}+\frac{1}{2}\int_{v_\eta<0}v_\eta(g^L)^2(0)\mathrm{d}\vec{\mathfrak{v}}\leq\frac{1}{2}\int_{v_\eta>0}v_\eta h^2\mathrm{d}\vec{\mathfrak{v}}\leq C, \tag{4.52}$$

and (4.50), we obtain

$$\alpha(\eta)\leq C. \tag{4.53}$$

Hence, (4.49) and (4.53) lead to

$$\int_0^L \exp\bigg(-\int_0^y G(z)\mathrm{d}z\bigg)\bigg(\left\langle w^L,\mathcal{L}[w^L]\right\rangle(y)\bigg)\mathrm{d}y\leq C, \tag{4.54}$$

which, by Lemma 4.1 and Lemma 4.2, further yields

$$\int_0^L \left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2}^2\mathrm{d}\eta\leq C \tag{4.55}$$

Step 2: Estimate of $q^L$.
Multiplying $v_\eta\psi_j$ with $j\neq 1$ on both sides of (4.38) and integrating over $\vec{\mathfrak{v}}\in\mathbb{R}^2$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta^2\psi_j,g^L\right\rangle+G(\eta)\left\langle v_\eta\psi_j,v_\phi^2\frac{\partial g^L}{\partial v_\eta}-v_\eta v_\phi\frac{\partial g^L}{\partial v_\phi}\right\rangle=-\left\langle v_\eta\psi_j,\mathcal{L}[w^L]\right\rangle. \tag{4.56}$$

Define $\tilde{q}^L=q^L-q_1^L\psi_1$ and

$$\beta_j(\eta)=\left\langle v_\eta^2\psi_j,\tilde{q}^L\right\rangle(\eta), \tag{4.57}$$

$$\beta(\eta)=\Big(\beta_0(\eta),\beta_1(\eta),\beta_2(\eta),\beta_3(\eta)\Big)^T \tag{4.58}$$

$$\tilde{\beta}(\eta)=\Big(\beta_0(\eta),\beta_2(\eta),\beta_3(\eta)\Big)^T. \tag{4.59}$$

By definition, $\beta_1=0$. For $j\neq 1$, using integration by parts, we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta^2\psi_j,g^L\right\rangle=G(\eta)\left\langle\frac{\partial}{\partial v_\eta}(v_\eta v_\phi^2\psi_j)-\frac{\partial}{\partial v_\phi}(v_\eta^2v_\phi\psi_j),g^L\right\rangle-\left\langle v_\eta\psi_j,\mathcal{L}[w^L]\right\rangle, \tag{4.60}$$

which further implies

$$\frac{\mathrm{d}\beta_j}{\mathrm{d}\eta}=G(\eta)\left\langle\frac{\partial}{\partial v_\eta}(v_\eta v_\phi^2\psi_j)-\frac{\partial}{\partial v_\phi}(v_\eta^2v_\phi\psi_j),\tilde{q}^L+q_1^L\psi_1+w^L\right\rangle-\left\langle v_\eta\psi_j,\mathcal{L}[w^L]\right\rangle-\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta^2\psi_j,w^L\right\rangle. \tag{4.61}$$

Put $\tilde{q}_i^L=q_i^L-\delta_{i1}q_1^L$. Then we can write

$$\left\langle\frac{\partial}{\partial v_\eta}(v_\eta v_\phi^2\psi_j)-\frac{\partial}{\partial v_\phi}(v_\eta^2v_\phi\psi_j),\tilde{q}^L\right\rangle(\eta)=\sum_i B_{ji}\tilde{q}_i^L(\eta), \tag{4.62}$$

for $i,j=0,2,3$, where

$$B_{ji}=\left\langle\frac{\partial}{\partial v_\eta}(v_\eta v_\phi^2\psi_j)-\frac{\partial}{\partial v_\phi}(v_\eta^2v_\phi\psi_j),\psi_i\right\rangle. \tag{4.63}$$

Moreover,

$$\beta_j(\eta)=\sum_k A_{jk}\tilde{q}_k^L(\eta), \tag{4.64}$$

where

$$A_{jk}=\left\langle v_\eta^2\psi_j,\psi_k\right\rangle, \tag{4.65}$$

is a non-singular matrix such that we can express back

$$\tilde{q}_j^L(\eta)=\sum_k A_{jk}^{-1}\beta_k(\eta). \tag{4.66}$$

Hence, (4.61) can be rewritten as

$$\frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}\eta} = G(BA^{-1})\tilde{\beta} + D, \tag{4.67}$$

where

$$D_j = G(\eta)\left\langle \frac{\partial}{\partial v_\eta}(v_\eta v_\phi^2\psi_j) - \frac{\partial}{\partial v_\phi}(v_\eta^2 v_\phi\psi_j), q_1^L\psi_1 + w^L\right\rangle - \left\langle v_\eta\psi_j, \mathcal{L}[w^L]\right\rangle - \frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta^2\psi_j, w^L\right\rangle. \tag{4.68}$$

We can solve for $\beta$ as

$$\tilde{\beta}(\eta) = \exp\bigg(-W(\eta)BA^{-1}\bigg)\theta - \zeta(\eta) + \int_0^\eta \exp\bigg((W(\eta)-W(y))BA^{-1}\bigg)Z(y)\mathrm{d}y, \tag{4.69}$$

where

$$\theta_j = \left\langle v_\eta^2\psi_j, g^L\right\rangle(0), \quad j \neq 1, \tag{4.70}$$

$$\zeta_j(\eta) = \left\langle v_\eta^2\psi_j, w^L\right\rangle(\eta), \tag{4.71}$$

and

$$Z = D + \frac{\mathrm{d}\zeta}{\mathrm{d}\eta} + G(BA^{-1})\zeta, \tag{4.72}$$

where we use the fact

$$\int_0^\eta \exp\bigg((W(\eta)-W(y))BA^{-1}\bigg)\frac{\mathrm{d}\zeta}{\mathrm{d}y}\mathrm{d}y = \zeta(\eta) - \exp\bigg(-W(\eta)BA^{-1}\bigg)\zeta(0) - \int_0^\eta G(BA^{-1})\zeta(y)\mathrm{d}y. \tag{4.73}$$

Hence, using the boundedness of $W(\eta)$ and $BA^{-1}$ due to Lemma 4.1, we can directly estimate (4.69) to get

$$\left|\beta_j(\eta)\right| \leq C\left|\theta_j\right| + \left|\zeta_j(\eta)\right| + C\int_0^\eta \left|Z_j(y)\right|\mathrm{d}y \quad \text{for} \quad i = 0, 2, 3. \tag{4.74}$$

By Cauchy's inequality, we obtain

$$\left|\zeta_j(\eta)\right| \leq \left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2} \tag{4.75}$$

$$\left|Z_j(\eta)\right| \leq C\bigg(\left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2} + \left\|q_1^L(\eta)\psi_1\right\|_{L^2}\bigg). \tag{4.76}$$

Multiplying $\sqrt{\mu}$ on both sides of (4.38) and integrating over $\vec{\mathfrak{v}} \in \mathbb{R}^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle\sqrt{\mu}v_\eta, g^L\right\rangle = G(\eta)\left\langle\frac{\partial}{\partial v_\eta}(v_\phi^2) - \frac{\partial}{\partial v_\phi}(v_\eta v_\phi), \sqrt{\mu}g^L\right\rangle = -G(\eta)\left\langle\sqrt{\mu}v_\eta, g^L\right\rangle, \tag{4.77}$$

which is actually

$$\frac{\mathrm{d}q_1^L}{\mathrm{d}\eta} = -G(\eta)q_1^L. \tag{4.78}$$

Since $q_1^L(L) = 0$, we have for any $\eta \in [0, L]$,

$$q_1^L(\eta) = 0. \tag{4.79}$$

Also,

$$\left\langle v_\eta^2\psi_j, g^L\right\rangle(0) \leq C\left\langle\left|v_\eta\right|g^L(0), g^L(0)\right\rangle^{1/2}\left\langle\left|v_\eta\right|^3, \psi_j^2\right\rangle^{1/2} \leq C\left\langle\left|v_\eta\right|g^L(0), g^L(0)\right\rangle^{1/2}, \tag{4.80}$$

$$\left\langle\left|v_\eta\right|g^L(0), g^L(0)\right\rangle = \int_{v_\eta>0}\mu v_\eta h^2 - \int_{v_\eta<0}\mu v_\eta(g^L(0))^2. \tag{4.81}$$

Since

$$\int_{v_\eta>0}\mu v_\eta h^2 + \int_{v_\eta<0}\mu v_\eta(g^L(0))^2 = 2\alpha(0) \geq 0, \tag{4.82}$$

we have

$$\theta_j = \left\langle v_\eta^2\psi_j, g^L\right\rangle(0) \leq 2C\int_{v_\eta>0}\mu v_\eta h^2 \leq C. \tag{4.83}$$

In conclusion, collecting (4.74), (4.75), (4.76), (4.79), and (4.83), we have

$$|\beta_j(\eta)| \leq C\bigg(1+\left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2}+\int_0^\eta \left\|\sqrt{\nu}w^L(y)\right\|_{L^2}\mathrm{d}y\bigg) \ \text{ for } \ j=0,2,3, \tag{4.84}$$

which further implies

$$\left|q_j^L(\eta)\right| \leq C\bigg(1+\left\|\sqrt{\nu}w^L(\eta)\right\|_{L^2}+\int_0^\eta \left\|\sqrt{\nu}w^L(y)\right\|_{L^2}\mathrm{d}y\bigg) \ \text{ for } \ j=0,2,3, \tag{4.85}$$

and $q_1^L(\eta)=0$. An application of Cauchy's inequality leads to our desired result.

Step 3: Orthogonal Properties.
In the equation (4.38), multiplying $\sqrt{\mu}$ on both sides and integrating over $\vec{\mathfrak{v}}\in\mathbb{R}^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle\sqrt{\mu}v_\eta,g^L\right\rangle = G(\eta)\left\langle\frac{\partial}{\partial v_\eta}(v_\phi^2)-\frac{\partial}{\partial v_\phi}(v_\eta v_\phi),\sqrt{\mu}g^L\right\rangle = -G\left\langle\sqrt{\mu}v_\eta,g^L\right\rangle. \tag{4.86}$$

Since $\left\langle\sqrt{\mu}v_\eta,g^L\right\rangle(L)=0$, we have

$$\left\langle\sqrt{\mu}v_\eta,g^L\right\rangle(\eta)=0. \tag{4.87}$$

It is easy to check that

$$\left\langle v_\eta\psi_i,q^L\right\rangle = 0 \ \ i\neq 1, \tag{4.88}$$

$$\left\langle v_\eta q^L,q^L\right\rangle = 0. \tag{4.89}$$

Multiplying $\psi_i$ for $i=2,3$ on both sides of (4.38) and integrating over $\vec{\mathfrak{v}}\in\mathbb{R}^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta\psi_i,w^L\right\rangle = = -CG\left\langle v_\eta\psi_i,w^L\right\rangle. \tag{4.90}$$

Since $\left\langle v_\eta\psi_i,w^L(L)\right\rangle=0$, then we have

$$\left\langle v_\eta\psi_i,w^L(\eta)\right\rangle=0. \tag{4.91}$$

In summary, we have

$$\left\langle v_\eta\psi_i,w^L(\eta)\right\rangle=0 \ \text{ for } \ i=0,2,3. \tag{4.92}$$

□

4.1.2. $L^2$ *Estimates in an infinite slab.* We consider the case with zero source term and zero mass flux in an infinite slab

$$\left\{\begin{aligned} v_\eta\frac{\partial g}{\partial\eta}+G(\eta)\bigg(v_\phi^2\frac{\partial g}{\partial v_\eta}-v_\eta v_\phi\frac{\partial g}{\partial v_\phi}\bigg)+\mathcal{L}[g] &= 0,\\ g(0,\vec{\mathfrak{v}}) &= h(\vec{\mathfrak{v}}) \ \text{ for } \ v_\eta>0,\\ \int_{\mathbb{R}^2}v_\eta\sqrt{\mu}g(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}} &= 0\\ \lim_{\eta\to\infty}g(\eta,\vec{\mathfrak{v}}) &= g_\infty(\vec{\mathfrak{v}}).\end{aligned}\right. \tag{4.93}$$

**Lemma 4.4.** *There exists a unique solution of the equation (4.93) satisfying the estimate*

$$\left|\left|\left|\sqrt{\nu}w\right|\right|\right|_{L^2L^2} \leq C, \tag{4.94}$$

$$|q_{i,\infty}| \leq C, \tag{4.95}$$

$$|||q-q_\infty|||_{L^2L^2} \leq C, \tag{4.96}$$

*where* $q_\infty=\sum_{i=0}^3 q_{i,\infty}\psi_i$ *and the orthogonal properties:*

$$\langle v_\eta\psi_i,w\rangle=0 \ \text{ for } \ i=0,2,3. \tag{4.97}$$

*Proof.* We divide the proof into several steps:

Step 1: Weak convergence.
We can extend the solution $g^L$ by passing $L \to \infty$. Hence, we can always take weakly convergent subsequence

$$q_i^L(\eta) \to q_i(\eta) \quad \text{in} \quad L^2_{\text{loc}}([0,\infty)), \tag{4.98}$$

$$w^L \to w \quad \text{in} \quad L^2_{\text{loc}}([0,\infty), L^2(\mathbb{R}^2)). \tag{4.99}$$

Therefore,

$$g = \sum_{i=0}^{3} q_i\psi_i + w, \tag{4.100}$$

is a weak solution of the equation (4.93). Also, by the weak lower semi-continuity, the estimate (4.94) of $w$ is obvious. Also, we can show the orthogonal properties (4.97) when $L \to \infty$.

Step 2: Estimate of $q_\infty$.
It is easy to see

$$q_1(\eta) = m_f[g] = 0, \tag{4.101}$$

so we do not need to bother with it. Since $\mathcal{L} : L^2(\mathbb{R}^2) \to \mathcal{N}^\perp$ with null space $\mathcal{N}$ and image $\mathcal{N}^\perp$, we have $\tilde{\mathcal{L}} : L^2/\mathcal{N} \to \mathcal{N}^\perp$ is bijective, where $L^2/\mathcal{N} = \mathcal{N}^\perp$ is the quotient space. Then we can define its inverse, i.e. the pseudo-inverse of $\mathcal{L}$ as $\mathcal{L}^{-1} : \mathcal{N}^\perp \to \mathcal{N}^\perp$ satisfying $\mathcal{L}\mathcal{L}^{-1}[f] = f$ for any $f \in \mathcal{N}^\perp$.

We intend to multiply $\mathcal{L}^{-1}[v_\eta\psi_i]$ for $i = 2,3$ on both sides of (4.93) and integrating over $\vec{\mathfrak{v}}$. Notice that $v_\eta\psi_2 \in \mathcal{N}^\perp$, but $v_\eta\psi_3 \notin \mathcal{N}^\perp$. Actually, it is easy to verify $v_\eta(\psi_3 - \psi_0) \in \mathcal{N}^\perp$. To avoid introducing new notation, we still use $\psi_3$ to denote $\psi_3 - \psi_0$ in the following proof and it is easy to see there is no confusion. Then we get

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle \mathcal{L}^{-1}[\psi_i v_\eta], v_\eta g\right\rangle + G(\eta)\left\langle \mathcal{L}^{-1}[\psi_i v_\eta], \left(v_\phi^2\frac{\partial g}{\partial v_\eta} - v_\eta v_\phi\frac{\partial g}{\partial v_\phi}\right)\right\rangle = -\left\langle \mathcal{L}^{-1}[\psi_i v_\eta], \mathcal{L}[w]\right\rangle. \tag{4.102}$$

Since $\mathcal{L}$ is self-adjoint, combining with the orthogonal properties, we have

$$\left\langle \mathcal{L}^{-1}[\psi_i v_\eta], \mathcal{L}[w]\right\rangle = \left\langle \mathcal{L}\Big[\mathcal{L}^{-1}[\psi_i v_\eta]\Big], w\right\rangle = \langle \psi_i v_\eta, w\rangle = 0. \tag{4.103}$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta\mathcal{L}^{-1}[\psi_i v_\eta], g\right\rangle + G(\eta)\left\langle \mathcal{L}^{-1}[\psi_i v_\eta], \left(v_\phi^2\frac{\partial g}{\partial v_\eta} - v_\eta v_\phi\frac{\partial g}{\partial v_\phi}\right)\right\rangle = 0. \tag{4.104}$$

Since $\psi_1 \in \mathcal{N}$ and $\mathcal{L}^{-1}[\psi_i v_\eta] \in \mathcal{N}^\perp$, we have

$$\left\langle \psi_1, \mathcal{L}^{-1}[\psi_i v_\eta]\right\rangle = 0. \tag{4.105}$$

For $i, k = 2, 3$, put

$$N_{i,k} = \left\langle v_\eta\mathcal{L}^{-1}[\psi_i v_\eta], \psi_k\right\rangle, \tag{4.106}$$

$$P_{i,k} = \left\langle \left(v_\phi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\phi\frac{\partial}{\partial v_\phi} + v_\eta\right)\mathcal{L}^{-1}[\psi_i v_\eta], \psi_k\right\rangle. \tag{4.107}$$

Thus,

$$\Omega_i = \left\langle v_\eta\mathcal{L}^{-1}[\psi_i v_\eta], q\right\rangle = \sum_{k=2}^{3} N_{i,k}q_k(\eta), \tag{4.108}$$

and

$$\left\langle \left(v_\phi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\phi\frac{\partial}{\partial v_\phi} + v_\eta\right)\mathcal{L}^{-1}[\psi_i v_\eta], q\right\rangle = \sum_{k=2}^{3} P_{i,k}q_k(\eta). \tag{4.109}$$

Since matrix $N$ is invertible (see [?]), from (4.104) and integration by parts, we have for $i=2,3$,

$$\begin{aligned}\frac{\mathrm{d}\Omega_i}{\mathrm{d}\eta} =& -\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle v_\eta\mathcal{L}^{-1}[\psi_i v_\eta], w\right\rangle \\ &+\sum_{k=2}^{3} G(\eta)(PN^{-1})_{ik}\Omega_k + G(\eta)\left\langle\left(v_\phi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\phi\frac{\partial}{\partial v_\phi} + v_\eta\right)\mathcal{L}^{-1}[\hat{\psi}_i v_\eta], w\right\rangle.\end{aligned} \tag{4.110}$$

Denote

$$\Omega' = \exp\Big(W(\eta)PN^{-1}\Big)\Omega. \tag{4.111}$$

Let $\hat{\psi} = (\psi_2, \psi_3)^T$, we can solve

$$\begin{aligned}\Omega'(\eta) =& \left\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta], g\right\rangle(0) - \exp\Big(W(\eta)PN^{-1}\Big)\left\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta], w\right\rangle(\eta) \\ &+\int_0^\eta \exp\Big(W(y)PN^{-1}\Big)G(y)\Bigg(\left\langle\left(v_\phi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\phi\frac{\partial}{\partial v_\phi} + v_\eta\right)\mathcal{L}^{-1}[\hat{\psi}v_\eta], w\right\rangle(y) \\ &+\sum_{k=2}^{3} PN^{-1}\left\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta], w\right\rangle(y)\Bigg)\mathrm{d}y.\end{aligned} \tag{4.112}$$

By a similar method as in the proof of Lemma 4.3 to bound $\theta_i(0)$, we can show

$$\left\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta], g\right\rangle(0) < \infty. \tag{4.113}$$

Since $w \in L^2([0,\infty)\times\mathbb{R}^2)$, considering $W(\eta)$ and $PN^{-1}$ are bounded, and $G(\eta) \in L^\infty$, we define

$$\begin{aligned}\Omega'_\infty =& \left\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta], g\right\rangle(0) + \int_0^\infty \exp\Big(W(y)PN^{-1}\Big)G(y)\Bigg(\left\langle\left(v_\phi^2\frac{\partial}{\partial v_\eta} - v_\eta v_\phi\frac{\partial}{\partial v_\phi} + v_\eta\right)\mathcal{L}^{-1}[\hat{\psi}v_\eta], w\right\rangle(y) \\ &+\sum_{k=2}^{3} PN^{-1}\left\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta], w\right\rangle(y)\Bigg)\mathrm{d}y.\end{aligned} \tag{4.114}$$

Let $\hat{q} = (q_2, q_3)^T$. Then we can define

$$\hat{q}_\infty = N^{-1}\exp\Big(-W(\infty)PN^{-1}\Big)\Omega'_\infty. \tag{4.115}$$

Finally, we consider $q_0$. Multiplying $\psi_1$ on both sides of (4.93) and integrating over $\vec{\mathfrak{v}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle\psi_1 g, v_\eta\right\rangle = -G(\eta)\left\langle\psi_1, \left(v_\phi^2\frac{\partial g}{\partial v_\eta} - v_\eta v_\phi\frac{\partial g}{\partial v_\phi}\right)\right\rangle = G(\eta)\left\langle g, (v_\phi^2 - v_\eta^2)\sqrt{\mu}\right\rangle. \tag{4.116}$$

Then integrating over $[0,\eta]$, we obtain

$$\begin{aligned}\left\langle\psi_1 g, v_\eta\right\rangle(\eta) =& \left\langle\psi_1 g, v_\eta\right\rangle(0) + \int_0^\eta G(y)\left\langle g, (v_\phi^2 - v_\eta^2)\sqrt{\mu}\right\rangle\mathrm{d}y \\ =& \left\langle\psi_1 g, v_\eta\right\rangle(0) + \int_0^\eta G(y)\left\langle w, (v_\phi^2 - v_\eta^2)\sqrt{\mu}\right\rangle(y)\mathrm{d}y.\end{aligned} \tag{4.117}$$

Since $w \in L^2([0,\infty)\times\mathbb{R}^2)$ and we can also bound $\left\langle v_\eta g, v_\eta\right\rangle(0)$, we have

$$\lim_{\eta\to\infty}\left\langle\psi_1 g, v_\eta\right\rangle(\eta) = \left\langle\psi_1 g, v_\eta\right\rangle(0) + \int_0^\infty G(y)\left\langle w, (v_\phi^2 - v_\eta^2)\sqrt{\mu}\right\rangle(y)\mathrm{d}y \tag{4.118}$$

exists. Note that

$$\lim_{\eta\to\infty}\left\langle\psi_1 q_{1,\infty}, v_\eta\right\rangle(\eta) = \lim_{\eta\to\infty}\left\langle\psi_1 q_{2,\infty}, v_\eta\right\rangle(\eta) = 0. \tag{4.119}$$

Then we define

$$q_{0,\infty} = \frac{\lim_{\eta\to\infty} \langle \psi_1 g, v_\eta\rangle (\eta) - q_{3,\infty} \langle \psi_1\psi_3, v_\eta\rangle}{\langle v_\eta\psi_0, v_\eta\rangle}. \tag{4.120}$$

Then to summarize all above, we have defined $q_\infty$ which satisfies $|q_{i,\infty}| \leq C$ for $i = 0, 1, 2, 3$.

Step 3: $L^2$ Decay of $w$.
The orthogonal property and zero mass-flux imply

$$\langle v_\eta q, w\rangle (\eta) = \sum_{k=0}^{3} \langle v_\eta \psi_k, w\rangle (\eta) = 0. \tag{4.121}$$

The oddness and zero mass-flux imply

$$\langle v_\eta q, q\rangle (\eta) = 0. \tag{4.122}$$

Therefore, we deduce that

$$\langle v_\eta g, g\rangle (\eta) = \langle v_\eta w, w\rangle (\eta). \tag{4.123}$$

Multiplying $\mathrm{e}^{2K_0\eta} g$ on both sides of (4.93) and integrating over $\vec{\mathfrak{v}}$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\bigg(\mathrm{e}^{2K_0\eta+W(\eta)} \langle v_\eta w, w\rangle\bigg) - \mathrm{e}^{2K_0\eta+W(\eta)}\bigg(K_0 \langle v_\eta w, w\rangle - \langle w, \mathcal{L}w\rangle\bigg) = 0. \tag{4.124}$$

Since

$$\langle \mathcal{L}[w], w\rangle \geq \nu_0 \left\langle (1 + \left|\vec{\mathfrak{v}}\right|)w, w\right\rangle, \tag{4.125}$$

and $W(\eta)$ is bounded, for $K_0$ sufficiently small, we have

$$\langle \mathcal{L}[w], w\rangle - \langle K_0 v_\eta w, w\rangle \geq C \langle w, w\rangle. \tag{4.126}$$

Then by a similar argument as in Lemma 4.3, we can show

$$\int_0^\infty \mathrm{e}^{2K_0\eta} \langle \nu w, w\rangle (\eta)\mathrm{d}\eta \leq C. \tag{4.127}$$

Step 4: Estimate of $q - q_\infty$.
We first consider $\hat{q} = (q_2, q_3)^T$, which satisfies

$$\hat{q}(\eta) = N^{-1} \exp\bigg( - W(\eta)PN^{-1}\bigg)\Omega'(\eta), \tag{4.128}$$

Let

$$\delta = \Big\langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], g\Big\rangle (0) \tag{4.129}$$

$$\Delta = G\Bigg( \bigg\langle \bigg( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} + v_\eta \bigg)\mathcal{L}^{-1}[\hat{\psi} v_\eta], w\bigg\rangle + \sum_{k=2}^{3} PN^{-1} \Big\langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], w\Big\rangle \Bigg) \tag{4.130}$$

Then we have

$$\begin{aligned}\hat{q}(\eta) = {} & N^{-1} \exp\bigg( - W(\eta)PN^{-1}\bigg)\delta - N^{-1} \Big\langle v_\eta \mathcal{L}^{-1}[\hat{\psi} v_\eta], w\Big\rangle (\eta) \\ & + \int_0^\eta \exp\bigg( (W(y) - W(\eta))PN^{-1}\bigg)\Delta(y)\mathrm{d}y.\end{aligned} \tag{4.131}$$

Also, we know

$$\hat{q}_\infty = N^{-1} \exp\bigg( - W(\infty)PN^{-1}\bigg)\delta + \int_0^\infty \exp\bigg( (W(y) - W(\infty))PN^{-1}\bigg)\Delta(y)\mathrm{d}y. \tag{4.132}$$

Then we have

$$\hat{q}(\eta)-\hat{q}_\infty = N^{-1}\bigg(\exp\Big(-W(\eta)PN^{-1}\Big)-\exp\Big(-W(\infty)PN^{-1}\Big)\bigg)\delta - N^{-1}\Big\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta],w\Big\rangle(\eta) \tag{4.133}$$
$$+N^{-1}\bigg(\exp\Big(-W(\eta)PN^{-1}\Big)-\exp\Big(-W(\infty)PN^{-1}\Big)\bigg)\int_0^\infty\exp\Big(W(y)PN^{-1}\Big)\Delta(y)\mathrm{d}y$$
$$+N^{-1}\int_\eta^\infty\exp\Big((W(y)-W(\eta))PN^{-1}\Big)\Delta(y)\mathrm{d}y.$$

Then we have

$$|||\hat{q}-\hat{q}_\infty|||_{L^2L^2} \tag{4.134}$$
$$\leq\bigg|\bigg|\bigg|N^{-1}\bigg(\exp\Big(-W(\eta)PN^{-1}\Big)-\exp\Big(-W(\infty)PN^{-1}\Big)\bigg)\delta\bigg|\bigg|\bigg|_{L^2L^2}+\bigg|\bigg|\bigg|N^{-1}\Big\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta],w\Big\rangle\bigg|\bigg|\bigg|_{L^2L^2}$$
$$+\bigg|\bigg|\bigg|N^{-1}\bigg(\exp\Big(-W(\eta)PN^{-1}\Big)-\exp\Big(-W(\infty)PN^{-1}\Big)\bigg)\int_0^\infty\exp\Big(W(y)PN^{-1}\Big)\Delta(y)\mathrm{d}y\bigg|\bigg|\bigg|_{L^2L^2}$$
$$+\bigg|\bigg|\bigg|N^{-1}\int_\eta^\infty\exp\Big((W(y)-W(\eta))PN^{-1}\Big)\Delta(y)\mathrm{d}y\bigg|\bigg|\bigg|_{L^2L^2}.$$

We need to estimate each term on the right-hand side of (4.134). By Lemma 4.1, we have

$$\bigg|\bigg|\bigg|N^{-1}\bigg(\exp\Big(-W(\eta)PN^{-1}\Big)-\exp\Big(-W(\infty)PN^{-1}\Big)\bigg)\delta\bigg|\bigg|\bigg|^2_{L^2L^2} \tag{4.135}$$
$$\leq C\delta\Big|\Big|\Big|\mathrm{e}^{-W(\eta)}-\mathrm{e}^{-W(\infty)}\Big|\Big|\Big|_{L^2L^2}\leq C. \tag{4.136}$$

Since $w\in L^2([0,\infty)\times\mathbb{R}^2)$, we have

$$\bigg|\bigg|\bigg|N^{-1}\Big\langle v_\eta\mathcal{L}^{-1}[\hat{\psi}v_\eta],w\Big\rangle\bigg|\bigg|\bigg|_{L^2L^2}\leq C|||w|||_{L^2L^2}\leq C. \tag{4.137}$$

Similarly, we can show

$$\bigg|\bigg|\bigg|N^{-1}\bigg(\exp\Big(-W(\eta)PN^{-1}\Big)-\exp\Big(-W(\infty)PN^{-1}\Big)\bigg)\int_0^\infty\exp\Big(W(y)PN^{-1}\Big)\Delta(y)\mathrm{d}y\bigg|\bigg|\bigg|_{L^2L^2} \tag{4.138}$$
$$\leq C\Big|\Big|\Big|\mathrm{e}^{-W(\eta)}-\mathrm{e}^{-W(\infty)}\Big|\Big|\Big|_{L^2L^2}|||r|||_{L^2L^2}\leq C.$$

For the last term, we have to resort to exponential decay of $w$. We estimate

$$\bigg|\bigg|\bigg|N^{-1}\int_\eta^\infty\exp\Big((W(y)-W(\eta))PN^{-1}\Big)\Delta(y)\mathrm{d}y\bigg|\bigg|\bigg|_{L^2L^2} \tag{4.139}$$
$$\leq C\int_0^\infty\bigg(\int_\eta^\infty\Delta(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta\leq\int_0^\infty\bigg(\int_\eta^\infty\mathrm{e}^{-2K_0y}\mathrm{d}y\bigg)\bigg(\int_\eta^\infty w^2(y)\mathrm{e}^{2K_0y}\mathrm{d}y\bigg)\mathrm{d}\eta$$
$$\leq\int_0^\infty C\mathrm{e}^{-2K_0\eta}\mathrm{d}\eta\leq C.$$

Collecting all above, we have

$$|||\hat{q}-\hat{q}_\infty|||_{L^2L^2}\leq C. \tag{4.140}$$

Then we turn to $q_0$. We have

$$q_0(\eta)=\frac{\langle\psi_1g,v_\eta\rangle(\eta)-q_3(\eta)\langle\psi_1\psi_3,v_\eta\rangle}{\langle v_\eta\psi_0,v_\eta\rangle}, \tag{4.141}$$

where

$$\langle\psi_1g,v_\eta\rangle(\eta)=\langle\psi_1g,v_\eta\rangle(0)+\int_0^\eta G(y)\left\langle w,(v_\phi^2-v_\eta^2)\sqrt{\mu}\right\rangle(y)\mathrm{d}y. \tag{4.142}$$

Also, we have

$$q_{0,\infty}(\eta) = \frac{\lim_{\eta\to\infty} \langle \psi_1 g, v_\eta \rangle (\eta) - q_{3,\infty} \langle \psi_1\psi_3, v_\eta \rangle}{\langle v_\eta \psi_0, v_\eta \rangle}, \tag{4.143}$$

where

$$\lim_{\eta\to\infty} \langle \psi_1 g, v_\eta \rangle (\eta) = \langle \psi_1 g, v_\eta \rangle (0) + \int_0^\infty G(y) \left\langle w, (v_\phi^2 - v_\eta^2)\sqrt{\mu} \right\rangle (y)\mathrm{d}y. \tag{4.144}$$

Therefore, we have

$$q_0(\eta) - q_{0,\infty} = \frac{\displaystyle\int_\eta^\infty G(y) \left\langle w, (v_\phi^2 - v_\eta^2)\sqrt{\mu} \right\rangle (y)\mathrm{d}y - (q_3(\eta) - q_{3,\infty}) \langle \psi_1\psi_3, v_\eta \rangle}{\langle v_\eta \psi_0, v_\eta \rangle} \tag{4.145}$$

Then we can naturally estimate

$$|||q_0 - q_{0,\infty}|||_{L^2L^2} \leq C\left|\left|\left|\int_\eta^\infty G(y) \left\langle w, (v_\phi^2 - v_\eta^2)\sqrt{\mu} \right\rangle (y)\mathrm{d}y\right|\right|\right|_{L^2L^2} + C|||q_3(\eta) - q_{3,\infty}|||_{L^2L^2} \tag{4.146}$$

$|||q_3(\eta) - q_{3,\infty}|||_{L^2L^2}$ is bounded due to the estimate of $|||\hat{q}(\eta) - \hat{q}_\infty|||_{L^2L^2}$. Then by Cauchy's inequality and Lemma 4.1, we obtain

$$\left|\left|\left|\int_\eta^\infty G(y) \left\langle w, (v_\phi^2 - v_\eta^2)\sqrt{\mu} \right\rangle (y)\mathrm{d}y\right|\right|\right|_{L^2L^2} \leq \int_0^\infty \left(\int_\eta^\infty G(y) \|r(y)\|_{L^2} \mathrm{d}y\right)^2 \mathrm{d}\eta \tag{4.147}$$

$$\leq |||r|||_{L^2L^2} \int_0^\infty \int_\eta^\infty G^2(y)\mathrm{d}y\mathrm{d}\eta \leq C. \tag{4.148}$$

Therefore, we have shown

$$|||q_0(\eta) - q_{0,\infty}|||_{L^2L^2} \leq C. \tag{4.149}$$

In summary, we prove that

$$|||q - q_\infty|||_{L^2L^2} \leq C. \tag{4.150}$$

Step 5: Uniqueness.
If $g_1$ and $g_2$ are two solutions of (4.93), define $g' = g_1 - g_2$. Then $g'$ satisfies the equation

$$\begin{cases} v_\eta \dfrac{\partial g'}{\partial \eta} + G(\eta)\left(v_\phi^2 \dfrac{\partial g'}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g'}{\partial v_\phi}\right) + \mathcal{L}[g'] &= 0, \\ g'(0, \vec{\mathfrak{v}}) &= 0 \ \text{ for } \ v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g'(0, \vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}} &= 0, \\ \lim_{\eta\to\infty} g'(\eta, \vec{\mathfrak{v}}) &= g'_\infty(\vec{\mathfrak{v}}), \end{cases} \tag{4.151}$$

Similarly, we can define $g' = w' + q'$. Define the linearized entropy as

$$H[g'](\eta) = \langle v_\eta g', g' \rangle (\eta). \tag{4.152}$$

Multiplying $g'$ on both sides of (4.151) and integrating over $\vec{\mathfrak{v}}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta} \langle v_\eta g', g' \rangle = \langle w', \mathcal{L}[w'] \rangle - \frac{1}{2}G(\eta) \langle v_\eta g', g' \rangle. \tag{4.153}$$

Hence, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\mathrm{e}^W \langle v_\eta g', g' \rangle\right) = -\mathrm{e}^W \langle w', \mathcal{L}[w'] \rangle, \tag{4.154}$$

which implies $\mathrm{e}^W H[g']$ is decreasing. Furthermore, we have

$$\mathrm{e}^{W(\eta)} H[g'](\eta) = H[g'](0) - \int_0^\eta \mathrm{e}^{W(y)} \langle w', \mathcal{L}[w'] \rangle (y)\mathrm{d}y < \infty. \tag{4.155}$$

Hence, we can take a subsequence such that $\|\sqrt{\nu}w'(\eta_n)\|_{L^2}$ goes to zero. Then we can always assume $q'(\eta_n)$ goes to $q'_\infty$. Therefore, we have

$$\mathrm{e}^{W(\eta_n)}H[g'](\eta_n)\rightarrow\left\langle v_\eta q'_\infty,q'_\infty\right\rangle. \tag{4.156}$$

Since $m_f[g']=0$, we naturally obtain

$$\mathrm{e}^{W(\eta_n)}H[g'](\eta_n)\rightarrow 0\ \text{ as }\ \eta_n\rightarrow\infty. \tag{4.157}$$

Hence, we have

$$\mathrm{e}^{W(\eta)}H[g'](\eta)\geq 0, \tag{4.158}$$

and

$$\mathrm{e}^{W(\eta)}H[g'](\eta)\rightarrow 0\ \text{ as }\ \eta\rightarrow\infty. \tag{4.159}$$

In (4.154), integrating over $[0,\infty)$, we achieve

$$-\int_{v_\eta<0}v_\eta(g')^2(0)\mathrm{d}\vec{\mathfrak{v}}+\int_0^\infty\mathrm{e}^{W(\eta)}\left\langle w',\mathcal{L}[w']\right\rangle(\eta)\mathrm{d}\eta=\int_{v_\eta>0}v_\eta(g')^2(0)\mathrm{d}\vec{\mathfrak{v}}=0. \tag{4.160}$$

Hence, we have

$$\int_{v_\eta<0}v_\eta(g')^2(0)\mathrm{d}\vec{\mathfrak{v}}=\int_0^\infty\left\langle w',\mathcal{L}[w']\right\rangle(\eta)\mathrm{d}\eta=0, \tag{4.161}$$

which implies $g'(0)=0$ and $w'=0$. Hence, $g'=q'$ and satisfies

$$v_\eta\frac{\partial g'}{\partial\eta}+G(\eta)\bigg(v_\phi^2\frac{\partial g'}{\partial v_\eta}-v_\eta v_\phi\frac{\partial g'}{\partial v_\phi}\bigg)=0. \tag{4.162}$$

$m_f[g']=0$ implies $q_1'=0$. Therefore, multiplying $v_\eta\psi_i$ for $i\neq 1$ on both sides of (4.162) and integrating over $\vec{\mathfrak{v}}$, we obtain a linear system on $q_k'$ for $k=1,2,3$ with initial data zero, which possesses a unique solution zero. This means $g'=0$. Hence, the solution is unique. □

4.1.3. *$L^2$ Estimates with general source term and non-vanishing mass-flux.* We consider the Milne problem with general source term and non-vanishing mass-flux.

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial g}{\partial\eta}+G(\eta)\bigg(v_\phi^2\dfrac{\partial g}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial g}{\partial v_\phi}\bigg)+\mathcal{L}[g]&=&S,\\ g(0,\vec{\mathfrak{v}})&=&h(\vec{\mathfrak{v}})\ \text{ for }\ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}g(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&m_f[g]\\ \lim_{\eta\rightarrow\infty}g(\eta,\vec{\mathfrak{v}})&=&g_\infty(\vec{\mathfrak{v}}).\end{array}\right. \tag{4.163}$$

**Lemma 4.5.** *There exists a unique solution of the equation (4.163) satisfying the estimate*

$$\left\|\left|\sqrt{\nu}w\right|\right\|_{L^2L^2}\leq C, \tag{4.164}$$

$$\left|q_{i,\infty}\right|\leq C, \tag{4.165}$$

$$\left\|\left|q-q_\infty\right|\right\|_{L^2L^2}\leq C, \tag{4.166}$$

*where* $q_\infty=\sum_{i=0}^3 q_{i,\infty}\psi_i$.

*Proof.* For the non-vanishing mass flux problem, we can see $\hat{g}=g-m_f[g]\sqrt{\mu}\mathrm{e}^{-\eta}v_\eta$ satisfies the $\epsilon$-Milne problem with zero mass flux with the source term

$$\hat{S}=S+m_f[g]\sqrt{\mu}\bigg(v_\eta^2-G(\eta)v_\phi^2\bigg)\mathrm{e}^{-\eta}, \tag{4.167}$$

and the boundary data

$$\hat{h}=h-m_f[g]\sqrt{\mu}v_\eta. \tag{4.168}$$

Therefore, we only need to consider the case with general source term and zero mass-flux. However, if $\displaystyle\int_{\mathbb{R}^2}\sqrt{\mu}S(\eta,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}\neq 0$, the mass-flux is not conserved when $\eta$ changes. The construction of solutions can be divided into several steps:

Step 1: Decomposition of the source term.
We decompose the source term as

$$S = S_Q + S_W, \tag{4.169}$$

where $S_Q \in \mathcal{N}$ is the kernel part and $S_W = S - S_Q \in \mathcal{N}^{\perp}$.

Step 2: Construction of $g_1$.
We first solve the problem with source term $S_W$ as

$$\begin{cases} v_\eta \dfrac{\partial g_1}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \dfrac{\partial g_1}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g_1}{\partial v_\phi}\bigg) + \mathcal{L}[g_1] &= S_W, \\ g_1(0, \vec{\mathfrak{v}}) &= h(\vec{\mathfrak{v}}) \quad \text{for} \quad v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_1(0, \vec{\mathfrak{v}}) \mathrm{d}\vec{\mathfrak{v}} &= 0 \\ \lim_{\eta\to\infty} g_1(\eta, \vec{\mathfrak{v}}) &= g_{1,\infty}(\vec{\mathfrak{v}}). \end{cases} \tag{4.170}$$

In this case, we apply similar techniques as in the analysis of $S = 0$ case. All the results can be generalized in a natural way. Hence, we know $g_1$ is well-posed.

Step 3: Construction of $g_2$.
We try to find a function $g_2$ such that

$$\mathcal{L}\left[v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi}\bigg) + S_Q\right] = 0. \tag{4.171}$$

which further means

$$\int_{\mathbb{R}^2} \psi_i \left(v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi}\bigg) + S_Q\right) \mathrm{d}\vec{\mathfrak{v}} = 0. \tag{4.172}$$

for $i = 0, 1, 2, 3$. Consider

$$S_Q = \sqrt{\mu}\bigg(a(\eta) + \vec{b}(\eta) \cdot \vec{\mathfrak{v}} + c(\eta) \left|\vec{\mathfrak{v}}\right|^2\bigg). \tag{4.173}$$

We make an ansatz that

$$g_2 = \sqrt{\mu}\bigg(A(\eta)v_\eta + B_1(\eta) + B_2(\eta)v_\eta v_\phi + C(\eta)v_\eta \left|\vec{\mathfrak{v}}\right|^2\bigg). \tag{4.174}$$

Plugging this ansatz into the equation (4.172), we obtain a system of linear ordinary differential equations which is well-posed. Hence, we can naturally obtain $g_2$. Furthermore, $g_2$ decays exponentially with respect to $\eta$ as long as the boundary data are taken properly.

Step 4: Construction of $g_3$.
We may directly verify

$$\mathcal{L}\left[v_\eta \frac{\partial g_2}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \frac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi}\bigg) + \mathcal{L}[g_2] + S_Q\right] = 0 \tag{4.175}$$

Then we may define $g_3$ as the solution of the equation

$$\begin{cases} v_\eta \dfrac{\partial g_3}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \dfrac{\partial g_3}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g_3}{\partial v_\phi}\bigg) + \mathcal{L}[g_3] &= v_\eta \dfrac{\partial g_2}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \dfrac{\partial g_2}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g_2}{\partial v_\phi}\bigg) + \mathcal{L}[g_2] + S_Q, \\ g_3(0, \vec{\mathfrak{v}}) &= -g_2(0, \vec{\mathfrak{v}}) \quad \text{for} \quad v_\eta > 0, \\ \displaystyle\int_{\mathbb{R}^2} v_\eta \sqrt{\mu} g_3(0, \vec{\mathfrak{v}}) \mathrm{d}\vec{\mathfrak{v}} &= 0 \\ \lim_{\eta\to\infty} g_3(\eta, \vec{\mathfrak{v}}) &= g_{3,\infty}(\vec{\mathfrak{v}}). \end{cases} \tag{4.176}$$

We can obtain $g_3$ is well-posed.

Step 5: Construction of $g_4$.
We may directly verify $g_4=g_2+g_3$ satisfies the equation

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial g_4}{\partial\eta}+G(\eta)\bigg(v_\phi^2\dfrac{\partial g_4}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial g_4}{\partial v_\phi}\bigg)+\mathcal{L}[g_4]&=&S_Q,\\ g_4(0,\vec{\mathfrak{v}})&=&h(\vec{\mathfrak{v}})\ \text{ for }\ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}g_4(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&0\\ \lim_{\eta\to\infty}g_4(\eta,\vec{\mathfrak{v}})&=&g_{4,\infty}(\vec{\mathfrak{v}}).\end{array}\right.\tag{4.177}$$

In summary, we know $g=g_1+g_4$ satisfies the equation (4.163) with zero mass-flux and is well-posed. □

**Lemma 4.6.** *Assume (4.7) and (4.8) hold. There exists a unique solution $g(\eta,\vec{\mathfrak{v}})$ to the $\epsilon$-Milne problem (4.1) satisfying*

$$\lVert\lvert g-g_\infty\rvert\rVert_{L^2L^2}\leq C.\tag{4.178}$$

*Proof.* Taking $g_\infty=q_\infty$, we can naturally obtain the desired result. □

Then we turn to the construction of $\tilde{h}$ and the well-posedness of the equation (4.11).

**Theorem 4.7.** *Assume (4.7) and (4.8) hold. There exists $\tilde{h}$ satisfying the condition (4.10) such that there exists a unique solution $\mathcal{G}(\eta,\vec{\mathfrak{v}})$ to the $\epsilon$-Milne problem (4.11) satisfying*

$$\lVert\lvert\mathcal{G}\rvert\rVert_{L^2L^2}\leq C.\tag{4.179}$$

*Proof.* The key part is the construction of $\tilde{h}$. Our main idea is to find $\tilde{h}\in\mathcal{N}$ such that the equation

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial\tilde{g}}{\partial\eta}+G(\eta)\bigg(v_\phi^2\dfrac{\partial\tilde{g}}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\tilde{g}}{\partial v_\phi}\bigg)+\mathcal{L}[\tilde{g}]&=&0,\\ \tilde{g}(0,\vec{\mathfrak{v}})&=&\tilde{h}(\vec{\mathfrak{v}})\ \text{ for }\ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\tilde{g}(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&\displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\tilde{h}(\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}},\\ \displaystyle\lim_{\eta\to\infty}\tilde{g}(\eta,\vec{\mathfrak{v}})&=&\tilde{g}_\infty(\vec{\mathfrak{v}}),\end{array}\right.\tag{4.180}$$

for $\tilde{g}(\eta,\vec{\mathfrak{v}})$ is well-posed, where

$$\tilde{g}_\infty(\vec{\mathfrak{v}})=g_\infty(\vec{\mathfrak{v}})=q_{0,\infty}\psi_0+q_{1,\infty}\psi_1+q_{2,\infty}\psi_2+q_{3,\infty}\psi_3,\tag{4.181}$$

is given by the equation of $g$. Note that

$$\tilde{h}^\epsilon(\vec{\mathfrak{v}})=\tilde{D}_0^\epsilon\psi_0+\tilde{D}_1^\epsilon\psi_1+\tilde{D}_2^\epsilon\psi_2+\tilde{D}_3^\epsilon\psi_3.\tag{4.182}$$

We consider the endomorphism $T$ in $\mathcal{N}$ defined as $T:\tilde{h}\to T[\tilde{h}]=\tilde{g}_\infty$. Therefore, we only need to study the matrix of $T$ at the basis $\{\psi_0,\psi_1,\psi_2,\psi_3\}$. It is easy to check when $\tilde{h}=\psi_0$ and $\tilde{h}=\psi_3$, $T$ is an identity mapping, i.e.

$$T[\psi_0]=\psi_0\tag{4.183}$$

$$T[\psi_3]=\psi_3\tag{4.184}$$

Multiplying $\sqrt{\mu}$ on both sides of (4.180) and integrating over $\vec{\mathfrak{v}}\in\mathbb{R}^2$ imply conserved mass-flux, which further leads to

$$T[\psi_1]=\psi_1\tag{4.185}$$

The main obstacle is when $\tilde{h}=\psi_2$. In this case, define $\tilde{g}'=\tilde{g}-\psi_2$. Then $\tilde{g}'$ satisfies the equation

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial\tilde{g}'}{\partial\eta}+G(\eta)\bigg(v_\phi^2\dfrac{\partial\tilde{g}'}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\tilde{g}'}{\partial v_\phi}\bigg)+\mathcal{L}[\tilde{g}']&=&G(\eta)\sqrt{\mu}v_\eta v_\phi,\\ \tilde{g}'(0,\vec{\mathfrak{v}})&=&0\ \text{ for }\ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\tilde{g}'(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&0,\\ \displaystyle\lim_{\eta\to\infty}\tilde{g}'(\eta,\vec{\mathfrak{v}})&=&\tilde{g}'_\infty(\vec{\mathfrak{v}}).\end{array}\right.\tag{4.186}$$

Although $G(\eta)\sqrt{\mu}v_\eta v_\phi$ does not decay exponentially, based on Lemma 4.1, $L^1$ and $L^2$ norm of $G$ can be sufficiently small as $\epsilon\rightarrow 0$ and $G(\eta)\sqrt{\mu}v_\eta v_\phi\in\mathcal{N}$. Using a natural extension of Lemma 4.4 for $\mathcal{L}[S]=0$, we know $|\tilde{q}'_{\infty}|$ is also sufficiently small, where $\tilde{q}'$ is the projection of $\tilde{g}'$ on $\mathcal{N}$. Note that we do not need exponential decay of source term in order to show the bound of $\tilde{q}_{\infty}$. This means

$$T[\psi_0,\psi_1,\psi_2,\psi_3]=[\psi_0,\psi_1,\psi_2,\psi_3]\left(\begin{array}{cccc}1&0&\tilde{q}'_{0,\infty}&0\\ 0&1&\tilde{q}'_{1,\infty}&0\\ 0&0&1+\tilde{q}'_{2,\infty}&0\\ 0&0&\tilde{q}'_{3,\infty}&1\end{array}\right) \tag{4.187}$$

For $\epsilon$ sufficiently small, this matrix is invertible, which means $T$ is bijective. Therefore, we can always find $\tilde{h}$ such that $\tilde{g}_{\infty}=g_{\infty}$, which is desired. Then by Lemma 4.6 and superposition property, when define $\mathcal{G}^{\epsilon}=g^{\epsilon}-\tilde{g}$, the theorem naturally follows. □

In the $\epsilon$-Milne problem (4.11), even if the boundary data and source term are determined, we still have the freedom to choose mass-flux $m_f[g]$. Next theorem shows that we can adjust the mass-flux $m_f[\mathcal{G}]$ to obtain desired properties of $\tilde{h}$.

**Theorem 4.8.** *Assume (4.7) and (4.8) hold. In the $\epsilon$-Milne problem (4.11), for any constant $C_0$, there exists an $m_f[g]$ such that $\tilde{\gamma}=\tilde{D}_0^{\epsilon}+\tilde{D}_3^{\epsilon}=C_0$.*

*Proof.* The key point is to study the equation for $\bar{g}$ as

$$\left\{\begin{array}{rcl}\displaystyle v_\eta\frac{\partial\bar{g}}{\partial\eta}+G(\epsilon;\eta)\bigg(v_\phi^2\frac{\partial\bar{g}}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\bar{g}}{\partial v_\phi}\bigg)+\mathcal{L}[\bar{g}]&=&0,\\ \bar{g}(0,\vec{\mathfrak{v}})&=&0\ \ \text{for}\ \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\bar{g}(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&m_f[\bar{g}],\\ \displaystyle\lim_{\eta\rightarrow\infty}\bar{g}(\eta,\vec{\mathfrak{v}})&=&\bar{g}_{\infty}(\vec{\mathfrak{v}}),\end{array}\right. \tag{4.188}$$

where

$$\bar{g}_{\infty}(\vec{\mathfrak{v}})=E_0\psi_0+E_1\psi_1+E_2\psi_2+E_3\psi_3, \tag{4.189}$$

with zero boundary data and source term but non-vanishing mass-flux, i.e. $m_f[\bar{g}]\neq0$. We claim $E_1+E_3\neq 0$. If this claim is true, then by superposition property, in the equation (4.1), we can obtain the desired $\gamma=g_{0,\infty}+g_{3,\infty}$ by adding a multiple of the equation (4.188). Then as in the proof of Theorem 4.7, we know the endomorphism $T$ leads to $g_{0,\infty}+g_{3,\infty}=\tilde{D}_0^{\epsilon}+\tilde{D}_3^{\epsilon}$. Then our work is done.

Next, we prove this claim by contradiction. Let us assume the claim is not true, i.e $E_1+E_3=0$ for some $m_f[\bar{g}]\neq0$. We decompose $\bar{g}=\bar{w}+\bar{q}$. By the construction in the proof of Lemma 4.5, we know

$$0<\lvert\lvert\lvert\bar{w}\rvert\rvert\rvert_{L^2L^2}\leq C. \tag{4.190}$$

Note there the first inequality is valid since we can directly verify $\bar{g}\in\mathcal{N}$ cannot be a solution. Define the linearized entropy as

$$H[\bar{g}](\eta)=\langle v_\eta\bar{g},\bar{g}\rangle(\eta). \tag{4.191}$$

Multiplying $\bar{g}$ on both sides of (4.188) and integrating over $\vec{\mathfrak{v}}$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle v_\eta\bar{g},\bar{g}\rangle=\langle\bar{w},\mathcal{L}[\bar{w}]\rangle-\frac{1}{2}G(\eta)\langle v_\eta\bar{g},\bar{g}\rangle. \tag{4.192}$$

Hence, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\bigg(\mathrm{e}^{W}\langle v_\eta\bar{g},\bar{g}\rangle\bigg)=-\mathrm{e}^{W}\langle\bar{w},\mathcal{L}[\bar{w}]\rangle, \tag{4.193}$$

which implies $\mathrm{e}^{W}H[\bar{g}]$ is decreasing. Furthermore, we have

$$\mathrm{e}^{W(\eta)}H[\bar{g}](\eta)=H[\bar{g}](0)-\int_0^{\eta}\mathrm{e}^{W(y)}\langle\bar{w},\mathcal{L}[\bar{w}]\rangle(y)\mathrm{d}y<\infty. \tag{4.194}$$

Hence, we can take a subsequence such that $\|\sqrt{\nu}\bar{w}(\eta_n)\|_{L^2}$ goes to zero. Then we can always assume $\bar{q}(\eta_n)$ goes to $\bar{q}_\infty$. Therefore, we have

$$\mathrm{e}^{W(\eta_n)}H[\bar{g}](\eta_n)\to\langle v_\eta\bar{q}_\infty,\bar{q}_\infty\rangle=2E_1(E_0+E_3)=0. \tag{4.195}$$

Then we naturally obtain

$$\mathrm{e}^{W(\eta_n)}H[\bar{g}](\eta_n)\to 0\ \text{ as }\ \eta_n\to\infty. \tag{4.196}$$

Hence, we have

$$\mathrm{e}^{W(\eta)}H[\bar{g}](\eta)\geq 0, \tag{4.197}$$

and

$$\mathrm{e}^{W(\eta)}H[\bar{g}](\eta)\to 0\ \text{ as }\ \eta\to\infty. \tag{4.198}$$

In (4.193), integrating over $[0,\infty)$, we achieve

$$-\int_{v_\eta<0}v_\eta\bar{g}^2(0)\mathrm{d}\vec{\mathfrak{v}}+\int_0^\infty\mathrm{e}^{W(\eta)}\left\langle\bar{w},\mathcal{L}[\bar{w}]\right\rangle(\eta)\mathrm{d}\eta=\int_{v_\eta>0}v_\eta\bar{g}^2(0)\mathrm{d}\vec{\mathfrak{v}}=0. \tag{4.199}$$

Hence, we have

$$\int_{v_\eta<0}v_\eta\bar{g}^2(0)\mathrm{d}\vec{\mathfrak{v}}=\int_0^\infty\left\langle\bar{w},\mathcal{L}[\bar{w}]\right\rangle(\eta)\mathrm{d}\eta=0, \tag{4.200}$$

which implies $\bar{g}(0)=0$ and $\bar{w}=0$. This contradicts (4.190). Therefore, the claim is valid. □

### 4.2. $L^\infty$ Estimates.

4.2.1. *Mild formulation in a finite slab.* Consider the $\epsilon$-transport problem for $g^L(\eta,\vec{\mathfrak{v}})$ in a finite slab

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial g^L}{\partial\eta}+G(\eta)\bigg(v_\phi^2\dfrac{\partial g^L}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial g^L}{\partial v_\phi}\bigg)+\nu g^L&=&Q(\eta,\vec{\mathfrak{v}}),\\ g^L(0,\vec{\mathfrak{v}})&=&h(\vec{\mathfrak{v}})\ \text{ for }\ v_\eta>0,\\ g^L(L,R[\vec{\mathfrak{v}}])&=&g^L(\vec{\mathfrak{v}}),\end{array}\right. \tag{4.201}$$

We define the characteristics starting from $(\eta(0),v_\eta(0),v_\phi(0))$ as $(\eta(s),v_\eta(s),v_\phi(s))$ defined by

$$\frac{\mathrm{d}\eta}{\mathrm{d}s}=v_\eta \tag{4.202}$$

$$\frac{\mathrm{d}v_\eta}{\mathrm{d}s}=G(\eta)v_\phi^2 \tag{4.203}$$

$$\frac{\mathrm{d}v_\phi}{\mathrm{d}s}=-G(\eta)v_\eta v_\phi \tag{4.204}$$

which leads to

$$v_\eta^2(s)+v_\phi^2(s)=C_1, \tag{4.205}$$

$$v_\phi(s)\mathrm{e}^{-W(s)}=C_2, \tag{4.206}$$

where $C_1$ and $C_2$ are two constants depending on the starting point. Along the characteristics, the equation (4.201) can be rewritten as

$$v_\eta\frac{\partial g}{\partial\eta}+\nu g=Q. \tag{4.207}$$

Define the energy

$$E(\eta,\vec{\mathfrak{v}})=v_\eta^2(\eta)+v_\phi^2(\eta). \tag{4.208}$$

and

$$v_\phi'(\eta,\vec{\mathfrak{v}};\eta')=v_\phi e^{W(\eta')-W(\eta)}. \tag{4.209}$$

For $E \geq v_{\phi}'^2$, define

$$v_{\eta}'(\eta, \vec{\mathfrak{v}}; \eta') = \sqrt{E - v_{\phi}'^2(\eta, \vec{\mathfrak{v}}; \eta')}, \tag{4.210}$$

$$\vec{\mathfrak{v}}'(\eta, \eta') = (v_{\eta}'(\eta, \vec{\mathfrak{v}}; \eta'), v_{\phi}'(\eta, \vec{\mathfrak{v}}; \eta')), \tag{4.211}$$

$$R[\vec{\mathfrak{v}}'(\eta, \eta')] = (-v_{\eta}'(\eta, \vec{\mathfrak{v}}; \eta'), v_{\phi}'(\eta, \vec{\mathfrak{v}}; \eta')). \tag{4.212}$$

Basically, this means $(\eta, v_\eta, v_\phi)$ and $(\eta', v_\eta', v_\phi')$ are on the same characteristics. Moreover, define an implicit function $\eta^+(\eta, \vec{\mathfrak{v}})$ by the equation

$$E(\eta, \vec{\mathfrak{v}}) = v_{\phi}'^2(\eta, \vec{\mathfrak{v}}; \eta^+). \tag{4.213}$$

We know $(\eta^+, \vec{\mathfrak{v}})$ at the axis $v_\eta = 0$ is on the same characteristics as $(\eta, \vec{\mathfrak{v}})$. Finally put

$$G_{\eta,\eta'} = \int_{\eta'}^{\eta} \frac{\nu(\vec{\mathfrak{v}}'(\eta, y))}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, y)} \mathrm{d}y, \tag{4.214}$$

$$R[G_{\eta,\eta'}] = \int_{\eta'}^{\eta} \frac{\nu(R[\vec{\mathfrak{v}}'(\eta, y)])}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, y)} \mathrm{d}y. \tag{4.215}$$

We can rewrite the solution to the equation (4.201) along the characteristics as follows:

Case I:
For $v_\eta > 0$,

$$g^L(\eta, \vec{\mathfrak{v}}) = h(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) \exp(-G_{\eta,0}) + \int_0^{\eta} \frac{Q(\eta', \vec{\mathfrak{v}}(\eta, \vec{\mathfrak{v}}; \eta'))}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, \eta')} \exp(-G_{\eta,\eta'}) \mathrm{d}\eta'. \tag{4.216}$$

Case II:
For $v_\eta < 0$ and $\big|E(\eta, \vec{\mathfrak{v}})\big| \geq v_{\phi}'(\eta, \vec{\mathfrak{v}}; L)$,

$$\begin{aligned} g^L(\eta, \vec{\mathfrak{v}}) = {}& h(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) \exp(-G_{L,0} - R[G_{L,\eta}]) \\ & + \bigg( \int_0^L \frac{Q(\eta', \vec{\mathfrak{v}}(\eta, \vec{\mathfrak{v}}; \eta'))}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, \eta')} \exp(-G_{L,\eta'} - R[G_{L,\eta}]) \mathrm{d}\eta' \\ & + \int_{\eta}^{L} \frac{Q(\eta', R[\vec{\mathfrak{v}}(\eta, \vec{\mathfrak{v}}; \eta')])}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, \eta')} \exp(R[G_{\eta,\eta'}]) \mathrm{d}\eta' \bigg). \end{aligned} \tag{4.217}$$

Case III:
For $v_\eta < 0$ and $\big|E(\eta, \vec{\mathfrak{v}})\big| \leq v_{\phi}'(\eta, \vec{\mathfrak{v}}; L)$,

$$\begin{aligned} g^L(\eta, \vec{\mathfrak{v}}) = {}& h(\vec{\mathfrak{v}}'(\eta, \vec{\mathfrak{v}}; 0)) \exp(-G_{\eta^+,0} - R[G_{\eta^+,\eta}]) \\ & + \bigg( \int_0^{\eta^+} \frac{Q(\eta', \vec{\mathfrak{v}}(\eta, \vec{\mathfrak{v}}; \eta'))}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, \eta')} \exp(-G_{\eta^+,\eta'} - R[G_{\eta^+,\eta}]) \mathrm{d}\eta' \\ & + \int_{\eta}^{\eta^+} \frac{Q(\eta', R[\vec{\mathfrak{v}}(\eta, \vec{\mathfrak{v}}; \eta')])}{v_{\eta}'(\eta, \vec{\mathfrak{v}}, \eta')} \exp(R[G_{\eta,\eta'}]) \mathrm{d}\eta' \bigg). \end{aligned} \tag{4.218}$$

4.2.2. *Mild formulation in an infinite slab.* Consider the $\epsilon$-transport problem for $g(\eta, \vec{\mathfrak{v}})$ in an infinite slab

$$\left\{ \begin{array}{rcl} v_\eta \dfrac{\partial g}{\partial \eta} + G(\eta)\bigg(v_\phi^2 \dfrac{\partial g}{\partial v_\eta} - v_\eta v_\phi \dfrac{\partial g}{\partial v_\phi}\bigg) + \nu g &=& Q(\eta, \vec{\mathfrak{v}}), \\ g(0, \vec{\mathfrak{v}}) &=& h(\vec{\mathfrak{v}}) \ \text{ for } \ v_\eta > 0, \\ \lim_{\eta \to \infty} g(\eta, \vec{\mathfrak{v}}) &=& g_\infty(\vec{\mathfrak{v}}), \end{array} \right. \tag{4.219}$$

We can define the solution via taking limit $L \to \infty$ in (4.216), (4.217) and (4.218) as follows:

$$g(\eta, \vec{\mathfrak{v}}) = \mathcal{A}[h(\vec{\mathfrak{v}})] + \mathcal{T}[Q(\eta, \vec{\mathfrak{v}})], \tag{4.220}$$

where

Case I:
For $v_\eta > 0$,

$$\mathcal{A}[h(\vec{\mathfrak{v}})] = h(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};0))\exp(-G_{\eta,0}), \tag{4.221}$$

$$\mathcal{T}[Q(\eta,\vec{\mathfrak{v}})] = \int_0^\eta \frac{Q(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.222}$$

Case II:
For $v_\eta < 0$ and $\left|E(\eta,\vec{\mathfrak{v}})\right| \geq v_\phi'(\eta,\vec{\mathfrak{v}};\infty)$,

$$\mathcal{A}[h(\vec{\mathfrak{v}})] = 0, \tag{4.223}$$

$$\mathcal{T}[Q(\eta,\vec{\mathfrak{v}})] = \int_\eta^\infty \frac{Q(\eta',R[\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(R[G_{\eta,\eta'}])\mathrm{d}\eta'. \tag{4.224}$$

Case III:
For $v_\eta < 0$ and $\left|E(\eta,\vec{\mathfrak{v}})\right| \leq v_\phi'(\eta,\vec{\mathfrak{v}};\infty)$,

$$\mathcal{A}[h(\vec{\mathfrak{v}})] = h(\vec{\mathfrak{v}}'(\eta,\vec{\mathfrak{v}};0))\exp(-G_{\eta^+,0} - R[G_{\eta^+,\eta}]), \tag{4.225}$$

$$\begin{aligned}\mathcal{T}[Q(\eta,\vec{\mathfrak{v}})] = \bigg(&\int_0^{\eta^+} \frac{Q(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(-G_{\eta^+,\eta'} - R[G_{\eta^+,\eta}])\mathrm{d}\eta' \\ &+ \int_\eta^{\eta^+} \frac{Q(\eta',R[\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')])}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(R[G_{\eta,\eta'}])\mathrm{d}\eta'\bigg).\end{aligned} \tag{4.226}$$

Notice that

$$\lim_{L\to\infty}\exp(-G_{L,\eta}) = 0, \tag{4.227}$$

for $v_\eta < 0$ and $\left|E(\eta,\vec{\mathfrak{v}})\right| \leq v_\phi'(\eta,\vec{\mathfrak{v}};\infty)$,. Hence, above derivation is valid. In order to achieve the estimate of $g$, we need to control $\mathcal{A}[h]$ and $\mathcal{T}[Q]$.

4.2.3. *Preliminaries.*

**Lemma 4.9.** *There is a positive $0 < \beta < \nu_0$ such that for any $\vartheta \geq 0$ and $0 \leq \zeta \leq 1/4$,*

$$\left\|\mathrm{e}^{\beta\eta}\mathcal{A}[h]\right\|_{L^\infty_{\vartheta,\zeta}} \leq C\left\|h\right\|_{L^\infty_{\vartheta,\zeta}}. \tag{4.228}$$

*Proof.* Based on Lemma 4.2, we know

$$\frac{\nu(\vec{\mathfrak{v}}'(\eta,y))}{v_\eta'(\eta,\vec{\mathfrak{v}},y)} \geq \nu_0 \tag{4.229}$$

$$\frac{\nu(R[\vec{\mathfrak{v}}'(\eta,y)])}{v_\eta'(\eta,\vec{\mathfrak{v}},y)} \geq \nu_0 \tag{4.230}$$

It follows that

$$\exp(-G_{\eta,0}) \leq \mathrm{e}^{-\beta\eta} \tag{4.231}$$

$$\exp(-G_{\eta^+,0} - R[G_{\eta^+,\eta}]) \leq \mathrm{e}^{-\beta\eta} \tag{4.232}$$

Then our results are obvious. □

**Lemma 4.10.** *For any integer $\vartheta \geq 0$, $0 \leq \zeta \leq 1/4$ and $\beta \leq \nu_0/2$, there is a constant $C$ such that*

$$\left|\left|\left|\mathcal{T}[Q]\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C\left|\left|\left|\frac{Q}{\nu}\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}}. \tag{4.233}$$

*Moreover, we have*

$$\left|\left|\left|\mathrm{e}^{\beta\eta}\mathcal{T}[Q]\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C\left|\left|\left|\frac{\mathrm{e}^{\beta\eta}Q}{\nu}\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}}. \tag{4.234}$$

*Proof.* The first inequality is a special case of the second one, so we only need to prove the second inequality. For $v_\eta > 0$ case, we have

$$\beta(\eta-\eta') - G_{\eta,\eta'} \leq \beta(\eta-\eta') - \frac{\nu_0(\eta-\eta')}{2} - \frac{G_{\eta,\eta'}}{2} \leq -\frac{G_{\eta,\eta'}}{2}. \tag{4.235}$$

It is natural that

$$\int_0^\eta \frac{\nu(\vec{\mathfrak{v}}'(\eta,\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')} \exp(\beta(\eta-\eta') - G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_0^\infty \exp\left(-\frac{z}{2}\right)\mathrm{d}z = 2. \tag{4.236}$$

Then we estimate

$$\begin{aligned}\left|\left\langle\vec{\mathfrak{v}}\right\rangle^\vartheta \mathrm{e}^{\zeta|\vec{\mathfrak{v}}|^2}\mathrm{e}^{\beta\eta}\mathcal{T}[Q]\right| &\leq \mathrm{e}^{\beta\eta}\int_0^\eta \left\langle\vec{\mathfrak{v}}\right\rangle^\vartheta \mathrm{e}^{\zeta|\vec{\mathfrak{v}}|^2}\frac{\left|Q(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))\right|}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ &\leq \left|\!\left|\!\left|\frac{\mathrm{e}^{\beta\eta}Q}{\nu}\right|\!\right|\!\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \int_0^\eta \frac{\nu(\vec{\mathfrak{v}}'(\eta,\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(\beta(\eta-\eta')-G_{\eta,\eta'})\mathrm{d}\eta' \\ &\leq C\left|\!\left|\!\left|\frac{\mathrm{e}^{\beta\eta}Q}{\nu}\right|\!\right|\!\right|_{L^\infty L^\infty_{\vartheta,\zeta}}.\end{aligned} \tag{4.237}$$

The $v_\eta < 0$ case can be proved in a similar fashion, so we omit it here. ☐

**Lemma 4.11.** *For any $\delta > 0$, $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$, there is a constant $C(\delta)$ such that*

$$\left\|\mathcal{T}[Q]\right\|_{L^\infty L^2_\zeta} \leq C(\delta)\left|\!\left|\!\left|\nu^{-1/2}Q\right|\!\right|\!\right|_{L^2L^2} + \delta|\!|\!|Q|\!|\!|_{L^\infty L^\infty_{\vartheta,\zeta}}. \tag{4.238}$$

*Proof.* We divide the proof into several cases:

Case I: For $v_\eta > 0$,

$$\mathcal{T}[Q(\eta,\vec{\mathfrak{v}})] = \int_0^\eta \frac{Q(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.239}$$

We need to estimate

$$\int_{\mathbb{R}^2} \mathrm{e}^{2\zeta|\vec{\mathfrak{v}}|^2}\left(\int_0^\eta \frac{Q(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\right)^2\mathrm{d}\vec{\mathfrak{v}}. \tag{4.240}$$

Assume $m > 0$ is sufficiently small, $M > 0$ is sufficiently large and $\sigma > 0$ is sufficiently small which will be determined in the following. We can split the integral into the following parts

$$I = I_1 + I_2 + I_3 + I_4. \tag{4.241}$$

Case I - Type I: $\chi_1$: $M \leq v_\eta'(\eta,\vec{\mathfrak{v}},\eta')$ or $M \leq v_\phi'(\eta,\vec{\mathfrak{v}},\eta')$.
By Lemma 4.2, we have

$$\left|\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')\right| + 1 \leq C\nu(\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')). \tag{4.242}$$

Then for $\vartheta > 2$, since $\left|\vec{\mathfrak{v}}\right|$ is conserved along the characteristics, we have

$$\begin{aligned}I_1 &\leq C|\!|\!|Q|\!|\!|^2_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\chi_1\left(\int_0^\eta \frac{1}{\left\langle\vec{\mathfrak{v}}'\right\rangle^\vartheta}\frac{\exp(-G_{\eta,\eta'})}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\right)^2\mathrm{d}\vec{\mathfrak{v}} \\ &\leq \frac{C}{M^\vartheta}|\!|\!|Q|\!|\!|^2_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\frac{1}{\left\langle\vec{\mathfrak{v}}\right\rangle^\vartheta}\left(\int_0^\eta \frac{\exp(-G_{\eta,\eta'})}{v_\eta'(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\right)^2\mathrm{d}\vec{\mathfrak{v}} \\ &\leq \frac{C}{M^\vartheta}|\!|\!|Q|\!|\!|^2_{L^\infty L^\infty_{\vartheta,\zeta}}.\end{aligned} \tag{4.243}$$

since

$$\left|\int_0^\eta \frac{\exp(-G_{\eta,\eta'})}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\right| \leq \left|\int_0^\eta \frac{-\nu(\vec{\mathfrak{v}}'(\eta,\eta'))\exp(G_{\eta,\eta'})}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\right| \leq \int_0^\infty \mathrm{e}^{-y}\mathrm{d}y = 1. \tag{4.244}$$

Case I - Type II: $\chi_2$: $m \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta') \leq M$ and $v'_\phi(\eta,\vec{\mathfrak{v}},\eta') \leq M$.
Since along the characteristics, $\left|\vec{\mathfrak{v}}\right|^2$ can be bounded by $2M^2$ and the integral domain for $\vec{\mathfrak{v}}$ is finite. Then by Cauchy's inequality, we have

$$\begin{aligned} I_2 &\leq C\frac{\mathrm{e}^{4\zeta M^2}}{m}\int_0^\eta \frac{Q^2}{\nu}(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))\mathrm{d}\eta'\int_0^\eta \frac{\nu(\vec{\mathfrak{v}}'(\eta,\eta'))\exp(-2G_{\eta,\eta'})}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta' \\ &\leq C\frac{\mathrm{e}^{4\zeta M^2}}{m}\left\|\left|\nu^{-1/2}Q\right|\right\|^2_{L^2L^2}, \end{aligned} \tag{4.245}$$

where

$$\int_0^\eta \frac{\nu(\vec{\mathfrak{v}}'(\eta,\eta'))}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\exp(-2G_{\eta,\eta'})\mathrm{d}\eta'\mathrm{d}\vec{\mathfrak{v}} \leq \int_0^\infty \mathrm{e}^{-2y}\mathrm{d}y = \frac{1}{2}. \tag{4.246}$$

Case I - Type III: $\chi_3$: $0 \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta') \leq m$, $v'_\phi(\eta,\vec{\mathfrak{v}},\eta') \leq M$ and $\eta-\eta' \geq \sigma$.
In this case, we know

$$G_{\eta,\eta'} \geq \frac{\sigma}{m}. \tag{4.247}$$

Then after substitution, the integral is not from zero, but from $-\sigma/m$. Hence, we have

$$\begin{aligned} I_3 &\leq C\|\!|Q|\!\|^2_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\chi_3\bigg(\int_0^\eta \frac{1}{\left\langle\vec{\mathfrak{v}}'\right\rangle^\vartheta}\frac{\exp(-G_{\eta,\eta'})}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\bigg)^2\mathrm{d}\vec{\mathfrak{v}} \\ &\leq C\|\!|Q|\!\|^2_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\frac{\chi_3}{\left\langle\vec{\mathfrak{v}}\right\rangle^{2\vartheta}}\bigg(\int_0^\eta \frac{\nu(\vec{\mathfrak{v}}'(\eta,\eta'))\exp(-G_{\eta,\eta'})}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\bigg)^2\mathrm{d}\vec{\mathfrak{v}} \\ &\leq C\|\!|Q|\!\|^2_{L^\infty L^\infty_{\vartheta,\zeta}}\bigg(\int_{-\sigma/m}^\infty \mathrm{e}^{-y}\mathrm{d}y\bigg)^2 \\ &\leq C\mathrm{e}^{-\frac{\sigma}{m}}\|\!|Q|\!\|^2_{L^\infty L^\infty_{\vartheta,\zeta}}. \end{aligned} \tag{4.248}$$

Case I - Type IV: $\chi_4$: $0 \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta') \leq m$, $v'_\phi(\eta,\vec{\mathfrak{v}},\eta') \leq M$ and $\eta-\eta' \leq \sigma$.
For $\eta' \leq \eta$ and $\eta-\eta' \leq \sigma$, we have

$$v_\eta \leq Cv'_\eta(\eta,\vec{\mathfrak{v}},\eta') \leq C(m+\sigma). \tag{4.249}$$

Therefore, the integral domain for $v_\eta$ is very small. We have the estimate

$$\begin{aligned} I_4 &\leq C\|\!|Q|\!\|_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\chi_4\bigg(\int_0^\eta \frac{1}{\left\langle\vec{\mathfrak{v}}'\right\rangle^\vartheta}\frac{\exp(-G_{\eta,\eta'})}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\bigg)^2\mathrm{d}\vec{\mathfrak{v}} \\ &\leq C\|\!|Q|\!\|_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\frac{\chi_4}{\left\langle\vec{\mathfrak{v}}\right\rangle^\vartheta}\mathrm{d}\vec{\mathfrak{v}} \\ &\leq C(m+\sigma)\|\!|Q|\!\|_{L^\infty L^\infty_{\vartheta,\zeta}}. \end{aligned} \tag{4.250}$$

Collecting all four types, we have

$$I \leq C\frac{\mathrm{e}^{4\zeta M^2}}{m}\left\|\left|\nu^{-1/2}Q\right|\right\|_{L^2L^2} + C\bigg(\frac{1}{M^\vartheta}+m+\sigma+\mathrm{e}^{-\frac{\sigma}{m}}\bigg)\|\!|Q|\!\|_{L^\infty L^\infty_{\vartheta,\zeta}}. \tag{4.251}$$

Taking $M$ sufficiently large, $\sigma$ sufficiently small and $m << \sigma$, this is the desired result.

Case II:
For $v_\eta < 0$ and $\left|E(\eta,\vec{\mathfrak{v}})\right| \geq v'_\phi(\eta,\vec{\mathfrak{v}};\infty)$,

$$\mathcal{T}[Q(\eta,\vec{\mathfrak{v}})] = \int_\eta^\infty \frac{Q(\eta', R[\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')])}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\exp(R[G_{\eta,\eta'}])\mathrm{d}\eta'. \tag{4.252}$$

We need to estimate

$$\int_{\mathbb{R}^2} \mathrm{e}^{2\zeta|\vec{\mathfrak{v}}|^2}\bigg(\int_\eta^\infty \frac{Q(\eta', R[\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')])}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\exp(R[G_{\eta,\eta'}])\mathrm{d}\eta'\bigg)^2\mathrm{d}\vec{\mathfrak{v}}. \tag{4.253}$$

We can split the integral into the following types:

$$II = II_1 + II_2 + II_3. \tag{4.254}$$

Case II - Type I: $\chi_1$: $M \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta')$ or $M \leq v'_\phi(\eta,\vec{\mathfrak{v}},\eta')$.
Similar to Case I - Type I, we have

$$\begin{aligned} II_1 &\leq C|||Q|||_{L^\infty L^\infty_{\vartheta,\zeta}}\int_{\mathbb{R}^2}\chi_1\bigg(\int_\eta^\infty \frac{1}{\left\langle\vec{\mathfrak{v}}'\right\rangle^\vartheta}\frac{\exp(R[G_{\eta,\eta'}])}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta'\bigg)^2\mathrm{d}\vec{\mathfrak{v}} \\ &\leq C\frac{1}{M^\vartheta}|||Q|||_{L^\infty L^\infty_{\vartheta,\zeta}}. \end{aligned} \tag{4.255}$$

Case II - Type II: $\chi_2$: $m \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta') \leq M$ and $v'_\phi(\eta,\vec{\mathfrak{v}},\eta') \leq M$.
Similar to Case I - Type II, by Cauchy's inequality, we have

$$\begin{aligned} II_2 &\leq C\frac{\mathrm{e}^{4\zeta M^2}}{m}\int_\eta^\infty \frac{Q^2}{\nu}(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))\mathrm{d}\eta'\int_\eta^\infty \frac{\exp(2R[G_{\eta,\eta'}])}{v'_\eta(\eta,\vec{\mathfrak{v}},\eta')}\mathrm{d}\eta' \\ &\leq C\frac{\mathrm{e}^{4\zeta M^2}}{m}\left|\left|\left|\nu^{-1/2}Q\right|\right|\right|_{L^2L^2}. \end{aligned} \tag{4.256}$$

Case I - Type III: $\chi_3$: $0 \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta') \leq m$ and $v'_\phi(\eta,\vec{\mathfrak{v}},\eta') \leq M$.
In this case, we can directly verify the fact

$$v_\eta \leq v'_\eta(\eta,\vec{\mathfrak{v}},\eta') \tag{4.257}$$

for $\eta \leq \eta'$. Then we know the integral of $v_\eta$ is always in a small domain. Similar to Case I - Type IV, we have the estimate

$$II_3 \leq Cm|||Q|||_{L^\infty L^\infty_{\vartheta,\zeta}}. \tag{4.258}$$

Hence, collecting all three types, we obtain

$$II \leq C\frac{\mathrm{e}^{4\zeta M^2}}{m}\left|\left|\left|\nu^{-1/2}Q\right|\right|\right|_{L^2L^2} + C\bigg(\frac{1}{M^\vartheta}+m\bigg)|||Q|||_{L^\infty L^\infty_{\vartheta,\zeta}}. \tag{4.259}$$

Taking $M$ sufficiently large and $m$ sufficiently small, this is the desired result.

Case III:
For $v_\eta < 0$ and $\left|E(\eta,\vec{\mathfrak{v}})\right| \leq v'_\phi(\eta,\vec{\mathfrak{v}};\infty)$,

$$\begin{aligned} \mathcal{T}[Q(\eta,\vec{\mathfrak{v}})] = \bigg(&\int_0^{\eta^+}\frac{Q(\eta',\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}{\nu(\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}\exp(-G_{\eta^+,\eta'} - R[G_{\eta^+,\eta}])\mathrm{d}\eta' \\ &+ \int_\eta^{\eta^+}\frac{Q(\eta', R[\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta')])}{\nu(\vec{\mathfrak{v}}(\eta,\vec{\mathfrak{v}};\eta'))}\exp(R[G_{\eta,\eta'}])\mathrm{d}\eta'\bigg). \end{aligned} \tag{4.260}$$

This is a combination of Case I and Case II, so it naturally holds. □

4.2.4. *Estimates of $\epsilon$-Milne problem.*

**Lemma 4.12.** *Assume (4.7) and (4.8) hold. The solution $g(\eta,\vec{\mathfrak{v}})$ to the $\epsilon$-Milne problem (4.1) satisfies for $\vartheta>2$ and $0\leq\zeta\leq 1/4$,*

$$|||g-g_{\infty}|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}\leq C+C|||g-g_{\infty}|||_{L^2L^2}. \tag{4.261}$$

*Proof.* Define $u=g-g_{\infty}$. Then $u$ satisfies the equation

$$\left\{\begin{array}{rcl}v_{\eta}\dfrac{\partial u}{\partial\eta}+G(\eta)\bigg(v_{\phi}^2\dfrac{\partial u}{\partial v_{\eta}}-v_{\eta}v_{\phi}\dfrac{\partial u}{\partial v_{\phi}}\bigg)+\mathcal{L}[u]&=&S(\eta,\vec{\mathfrak{v}})+g_{2,\infty}G(\eta)\sqrt{\mu}v_{\eta}v_{\phi}=\tilde{S},\\ u(0,\vec{\mathfrak{v}})&=&(h-g_{\infty})(\vec{\mathfrak{v}})=p(\vec{\mathfrak{v}})\ \ \text{for}\ \ v_{\eta}>0,\\ \displaystyle\int_{\mathbb{R}^2}v_{\eta}\sqrt{\mu}u(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&m_f[g]-g_{1,\infty},\\ \lim_{\eta\rightarrow\infty}u(\eta,\vec{\mathfrak{v}})&=&0,\end{array}\right. \tag{4.262}$$

Since $u=\mathcal{A}[p]+\mathcal{T}[K[u]+\tilde{S}]$, based on Lemma 4.11, we have

$$\begin{aligned}\|u-\mathcal{A}[p]\|_{L^{\infty}L^2_{\zeta}}&=\Big\|\mathcal{T}[K[u]+\tilde{S}]\Big\|_{L^{\infty}L^2}\\ &\leq C(\delta)\bigg(\Big|\Big|\Big|\nu^{-1/2}K[u]\Big|\Big|\Big|_{L^2L^2}+\Big|\Big|\Big|\nu^{-1/2}\tilde{S}\Big|\Big|\Big|_{L^2L^2}\bigg)+\delta\bigg(|||K[u]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\Big|\Big|\Big|\tilde{S}\Big|\Big|\Big|_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\leq C(\delta)\bigg(|||u|||_{L^2L^2}+\Big|\Big|\Big|\tilde{S}\Big|\Big|\Big|_{L^2L^2}\bigg)+\delta\bigg(|||K[u]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\Big|\Big|\Big|\tilde{S}\Big|\Big|\Big|_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}\bigg),\end{aligned} \tag{4.263}$$

where we can directly verify

$$\Big|\Big|\Big|\nu^{-1/2}K[u]\Big|\Big|\Big|_{L^2L^2}\leq|||u|||_{L^2L^2}, \tag{4.264}$$

$$\Big|\Big|\Big|\nu^{-1/2}\tilde{S}\Big|\Big|\Big|_{L^2L^2}\leq\Big|\Big|\Big|\tilde{S}\Big|\Big|\Big|_{L^2L^2}. \tag{4.265}$$

In [**?**, Lemma 3.3.1], it is shown that

$$|||K[u]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}\leq|||u|||_{L^{\infty}L^{\infty}_{\vartheta-1,\zeta}}, \tag{4.266}$$

$$|||K[u]|||_{L^{\infty}L^{\infty}_{0,\zeta}}\leq\|u\|_{L^{\infty}L^2_{\zeta}}. \tag{4.267}$$

Since $u=\mathcal{A}[p]+\mathcal{T}[K[u]+\tilde{S}]$, for $\epsilon$ and $\delta$ sufficiently small, we can estimate

$$\begin{aligned}|||u|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}&\leq C\bigg(|||\mathcal{T}[K[u]]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+|||\mathcal{T}[S]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\|\mathcal{A}[p]\|_{L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\leq C\bigg(|||K[u]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+|||S|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\|\mathcal{A}[p]\|_{L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\leq C\bigg(|||u|||_{L^{\infty}L^{\infty}_{\vartheta-1,\zeta}}+|||S|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\|\mathcal{A}p\|_{L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\leq\ \ldots\\ &\leq C\bigg(|||K[u]|||_{L^{\infty}L^{\infty}_{0,\zeta}}+|||S|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\|\mathcal{A}[p]\|_{L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\leq C\bigg(\|u\|_{L^{\infty}L^2_{\zeta}}+|||S|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\|\mathcal{A}[p]\|_{L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\leq C(\delta)\bigg(\Big|\Big|\Big|\nu^{-1/2}K[u]\Big|\Big|\Big|_{L^2L^2}+\Big|\Big|\Big|\nu^{-1/2}\tilde{S}\Big|\Big|\Big|_{L^2L^2}\bigg)+\delta\bigg(|||K[u]|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\Big|\Big|\Big|\tilde{S}\Big|\Big|\Big|_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}\bigg)\\ &\quad+C\bigg(|||S|||_{L^{\infty}L^{\infty}_{\vartheta,\zeta}}+\|\mathcal{A}[p]\|_{L^{\infty}_{\vartheta,\zeta}}\bigg).\end{aligned} \tag{4.268}$$

Therefore, absorbing $\delta\left|\left|\left|K[u]\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}}$ into the right-hand side of the second inequality implies

$$\left|\left|\left|K[u]\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C\bigg(\left|\left|\left|\nu^{-1/2}K[u]\right|\right|\right|_{L^2L^2} + \left|\left|\left|S\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} + \left\|p\right\|_{L^\infty_{\vartheta,\zeta}}\bigg) \tag{4.269}$$

Therefore, we have

$$\begin{aligned}\left|\left|\left|u\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} &\leq C\bigg(\left|\left|\left|K[u]\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} + \left|\left|\left|S\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} + \left\|\mathcal{A}[p]\right\|_{L^\infty_{\vartheta,\zeta}}\bigg)\\ &\leq C\bigg(\left|\left|\left|u\right|\right|\right|_{L^2L^2} + \left|\left|\left|S\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} + \left\|p\right\|_{L^\infty_{\vartheta,\zeta}}\bigg).\end{aligned}$$

Then our result naturally follows. □

**Lemma 4.13.** *Assume (4.7) and (4.8) hold. There exists a unique solution $g(\eta,\vec{\mathfrak{v}})$ to the $\epsilon$-Milne problem (4.1) satisfying for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\left|\left|\left|g - g_\infty\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C. \tag{4.270}$$

*Proof.* Based on Lemma 4.6 and Lemma 4.12, this is obvious. □

**Theorem 4.14.** *Assume (4.7) and (4.8) hold. There exists a unique solution $\mathcal{G}(\eta,\vec{\mathfrak{v}})$ to the $\epsilon$-Milne problem (4.11) satisfying for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\left|\left|\left|\mathcal{G}\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C. \tag{4.271}$$

*Proof.* Based on Theorem 4.7 and Lemma 4.13, this is obvious. □

### 4.3. **Exponential Decay.**

**Theorem 4.15.** *Assume (4.7) and (4.8) hold. For sufficiently small $K_0$, there exists a unique solution $\mathcal{G}(\eta,\vec{\mathfrak{v}})$ to the $\epsilon$-Milne problem (4.11) satisfying for $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$,*

$$\left|\left|\left|\mathrm{e}^{K_0\eta}\mathcal{G}\right|\right|\right|_{L^\infty L^\infty_{\vartheta,\zeta}} \leq C. \tag{4.272}$$

*Proof.* Define $U = \mathrm{e}^{K_0\eta}\mathcal{G}$. Then $U$ satisfies the equation

$$\left\{\begin{array}{rcl} v_\eta\dfrac{\partial U}{\partial\eta} + G(\eta)\bigg(v_\phi^2\dfrac{\partial U}{\partial v_\eta} - v_\eta v_\phi\dfrac{\partial U}{\partial v_\phi}\bigg) + \mathcal{L}[U] &=& \mathrm{e}^{K_0\eta}S(\eta,\vec{\mathfrak{v}}) + K_0v_\eta U,\\ U(0,\vec{\mathfrak{v}}) &=& \mathrm{e}^{K_0\eta}(h-\tilde{h})(\vec{\mathfrak{v}}) \ \text{ for } \ v_\eta > 0,\\ \displaystyle\int_{\mathbb{R}^2} v_\eta\sqrt{\mu}U(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}} &=& m_f[g] - \displaystyle\int_{\mathbb{R}^2}\sqrt{\mu}\tilde{h}(\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}},\\ \lim_{\eta\to\infty} U(\eta,\vec{\mathfrak{v}}) &=& 0,\end{array}\right. \tag{4.273}$$

We divide the proof into several steps:

Step 1: $L^2$ Estimates for $S = 0$ and $m_f[U] = 0$.
In the proof of Lemma 4.4, we already show

$$\int_0^\infty \mathrm{e}^{2K_0\eta}\left\langle w_{\mathcal{G}}, w_{\mathcal{G}}\right\rangle(\eta)\mathrm{d}\eta \leq C. \tag{4.274}$$

We can decompose $\mathcal{G} = w_{\mathcal{G}} + q_{\mathcal{G}}$. Since $\lim_{\eta\to\infty}\mathcal{G}(\eta,\vec{\mathfrak{v}}) = 0$, we naturally have $q_{\mathcal{G}} = 0$. Then using the orthogonal relation and zero mass-flux, we have

$$\int_0^\infty \mathrm{e}^{2K_0\eta}\int_{\mathbb{R}^2}\mathcal{G}^2(\eta,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}\mathrm{d}\eta = \int_0^\infty \mathrm{e}^{2K_0\eta}\left\langle q_{\mathcal{G}}, q_{\mathcal{G}}\right\rangle(\eta)\mathrm{d}\eta + \int_0^\infty \mathrm{e}^{2K_0\eta}\left\langle w_{\mathcal{G}}, w_{\mathcal{G}}\right\rangle(\eta)\mathrm{d}\eta. \tag{4.275}$$

Similar to Step 4 in the proof of Lemma 4.4, using the exponential decay of $w_{\mathcal{G}}$, we have

$$\int_0^\infty \mathrm{e}^{2K_0\eta}\left\langle q_{\mathcal{G}}, q_{\mathcal{G}}\right\rangle(\eta)\mathrm{d}\eta \leq C\int_0^\infty \mathrm{e}^{2K_0\eta}\left\langle w_{\mathcal{G}}, w_{\mathcal{G}}\right\rangle(\eta)\mathrm{d}\eta \tag{4.276}$$

This shows

$$\left|\left|\left|U\right|\right|\right|_{L^2L^2} < C. \tag{4.277}$$

Step 2: $L^2$ Estimates for general source term and mass flux.
We follow the idea in the proof of Lemma 4.5. Note that all the auxiliary functions we construct decays exponentially. Hence, the result naturally follows.

Step 3: $L^\infty$ Estimates.
By a similar argument of Lemma 4.12, Since $u = \mathcal{A}[p] + \mathcal{T}[K[u] + \tilde{S}]$, similar to the proof of Lemma 4.12, we have

$$\left\|\left|\mathrm{e}^{K_0\eta}\mathcal{G}\right|\right\|_{L^\infty L^\infty_{\vartheta,\varsigma}} \leq C\bigg(\left\|\left|\mathrm{e}^{K_0\eta}\mathcal{G}\right|\right\|_{L^2L^2} + \left\|\left|\mathrm{e}^{K_0\eta}S\right|\right\|_{L^\infty L^\infty_{\vartheta,\varsigma}} + \|p\|_{L^\infty_{\vartheta,\varsigma}}\bigg) \tag{4.278}$$

Then we naturally obtain the result. □

Our results can also be applied to the Milne problem without geometric correction.

**Remark 4.16.** *Taking $G = 0$, we consider the Milne problem*

$$\left\{\begin{array}{rcl} v_\eta\dfrac{\partial g}{\partial\eta} + \mathcal{L}[g] &=& S(\eta,\phi,\vec{\mathfrak{v}}),\\ g(0,\phi,\vec{\mathfrak{v}}) &=& h(\phi,\vec{\mathfrak{v}}) \ \ for \ \ v_\eta > 0,\\ \displaystyle\int_{\mathbb{R}^2} v_\eta\sqrt{\mu}g(0,\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}} &=& m_f[g],\\ \displaystyle\lim_{\eta\rightarrow\infty} g(\eta,\phi,\vec{\mathfrak{v}}) &=& g_\infty(\phi,\vec{\mathfrak{v}}). \end{array}\right. \tag{4.279}$$

*Then there exists*

$$\tilde{h}(\phi,\vec{\mathfrak{v}}) = \tilde{D}_0(\phi)\psi_0 + \tilde{D}_1(\phi)\psi_1 + \tilde{D}_2(\phi)\psi_2 + \tilde{D}_3(\phi)\psi_3, \tag{4.280}$$

*such that the Milne problem for $\mathcal{G}(\eta,\phi,\vec{\mathfrak{v}})$ in the domain $(\eta,\phi,\vec{\mathfrak{v}}) \in [0,\infty)\times[-\pi,\pi)\times\mathbb{R}^2$*

$$\left\{\begin{array}{rcl} v_\eta\dfrac{\partial \mathcal{G}}{\partial\eta} + \mathcal{L}[\mathcal{G}] &=& S(\eta,\phi,\vec{\mathfrak{v}}),\\ \mathcal{G}(0,\phi,\vec{\mathfrak{v}}) &=& h(\phi,\vec{\mathfrak{v}}) - \tilde{h}(\phi,\vec{\mathfrak{v}}) \ \ for \ \ v_\eta > 0,\\ \displaystyle\int_{\mathbb{R}^2} v_\eta\sqrt{\mu}\mathcal{G}(0,\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}} &=& m_f[g] - \displaystyle\int_{\mathbb{R}^2} v_\eta\sqrt{\mu}\tilde{h}(\phi,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}},\\ \displaystyle\lim_{\eta\rightarrow\infty} \mathcal{G}(\eta,\phi,\vec{\mathfrak{v}}) &=& 0. \end{array}\right. \tag{4.281}$$

*is well-posed in $L^\infty$ and decays exponentially.*

## 5. Diffusive Limit and Well-Posedness

We prove the diffusive limit and well-posedness of the Boltzmann equation (1.9).

**Theorem 5.1.** *For given $B^\epsilon > 0$ satisfying (1.5) and (1.7) and $0 < \epsilon << 1$, there exists a unique positive solution $F^\epsilon = \mu + \sqrt{\mu}f^\epsilon$ to the stationary Boltzmann equation (1.1), where*

$$f^\epsilon = \epsilon^3 R_N + \bigg(\sum_{k=1}^N \epsilon^k\mathcal{F}_k^\epsilon\bigg) + \bigg(\sum_{k=1}^N \epsilon^k\mathscr{F}_k^\epsilon\bigg), \tag{5.1}$$

*for $N \geq 3$ and $R_N$ satisfies*

$$\left\{\begin{array}{rcl} \epsilon\vec{v}\cdot\nabla_x R_N + \mathcal{L}[R_N] &=& \epsilon^3\Gamma[R_N,R_N] + 2\Gamma[R_N,\mathcal{Q}_N + \mathscr{Q}_N] + S_N \ \ in \ \ \Omega,\\ R_N(\vec{x}_0,\vec{v}) &=& h_N \ \ for \ \ \vec{v}\cdot\vec{n} < 0 \ \ and \ \ \vec{x}_0 \in \partial\Omega, \end{array}\right. \tag{5.2}$$

*with*

$$\begin{aligned} S_N = &-\sum_{1\leq i,j\leq N}^{i+j\geq N+1} \epsilon^{i+j-3}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_j^\epsilon] - \Upsilon_0\sum_{1\leq i,j\leq N}^{i+j\geq N+1}\epsilon^{i+j-3}\bigg(\Gamma[\mathscr{F}_i^\epsilon,\mathscr{F}_j^\epsilon] + 2\Gamma[\mathcal{F}_i^\epsilon,\mathscr{F}_j^\epsilon]\bigg)\\ &-\epsilon^{N-2}\vec{v}\cdot\nabla_x\mathcal{F}_N^\epsilon - \sum_{k=1}^N \epsilon^{k-3}v_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon + \epsilon^{N-2}\frac{\Upsilon_0}{1-\epsilon\eta}v_\phi\frac{\partial\mathcal{G}_N^\epsilon}{\partial\phi}, \end{aligned}$$

*and*

$$h_N = \sum_{k=N+1}^{\infty} \epsilon^{k-3} b_k,$$

*$\mathcal{F}_k^\epsilon$ and $\mathscr{F}_k^\epsilon$ satisfy (3.60) and (3.53). Also, there exists a $C > 0$ such that $f^\epsilon$ satisfies*

$$\left\| \langle \vec{v} \rangle^{\vartheta} \mathrm{e}^{\zeta |\vec{v}|^2} f^\epsilon \right\|_{L^\infty} \leq C\epsilon, \tag{5.3}$$

*for any $\vartheta > 2$ and $0 \leq \zeta \leq 1/4$*

.

*Proof.* We divide the proof into several steps:

Step 1: Remainder definitions.
We combine the interior solution and boundary layer as follows:

$$f^\epsilon \sim \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k^\epsilon + \sum_{k=1}^{\infty} \epsilon^k \mathscr{F}_k^\epsilon. \tag{5.4}$$

Define the remainder as

$$R_N = \frac{1}{\epsilon^3}\bigg( f^\epsilon - \sum_{k=1}^{N} \epsilon^k \mathcal{F}_k^\epsilon - \sum_{k=1}^{N} \epsilon^k \mathscr{F}_k^\epsilon \bigg) = \frac{1}{\epsilon^3}\bigg( f^\epsilon - \mathcal{Q}_N - \mathscr{Q}_N \bigg), \tag{5.5}$$

where

$$\mathcal{Q}_N = \sum_{k=1}^{N} \epsilon^k \mathcal{F}_k^\epsilon, \tag{5.6}$$

$$\mathscr{Q}_N = \sum_{k=1}^{N} \epsilon^k \mathscr{F}_k^\epsilon. \tag{5.7}$$

Noting the equation (3.37) is equivalent to the equation (1.9), we write $\mathscr{L}$ to denote the linearized Boltzmann operator as follows:

$$\begin{aligned} \mathscr{L}[f] &= \epsilon \vec{v} \cdot \nabla_x u + \mathcal{L}[f] \\ &= v_\eta \frac{\partial f}{\partial \eta} - \frac{\epsilon}{1-\epsilon\eta}\bigg( v_\phi \frac{\partial f}{\partial \phi} + v_\phi^2 \frac{\partial f}{\partial v_\eta} - v_\eta v_\phi \frac{\partial f}{\partial v_\phi} \bigg) + \mathcal{L}[f]. \end{aligned} \tag{5.8}$$

Step 2: Estimates of $\mathscr{L}[R_N]$.
The interior contribution can be estimated as

$$\begin{aligned} \mathscr{L}[\mathcal{Q}_N] &= \epsilon \vec{v} \cdot \nabla_x \mathcal{Q}_N + \mathcal{L}[\mathcal{Q}_N] = \sum_{k=1}^{N} \epsilon^k \bigg( \epsilon \vec{v} \cdot \nabla_x \mathcal{F}_k^\epsilon + \mathcal{L}[\mathcal{F}_k^\epsilon] \bigg) \\ &= \mathcal{L}[\mathcal{F}_1^\epsilon] + \sum_{k=2}^{N} \epsilon^k \bigg( \vec{v} \cdot \nabla_x \mathcal{F}_{k-1}^\epsilon + \mathcal{L}[\mathcal{F}_k^\epsilon] \bigg) + \epsilon^{N+1} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon \\ &= \epsilon^{N+1} \vec{v} \cdot \nabla_x \mathcal{F}_N^\epsilon + \sum_{1 \leq i,j \leq N}^{i+j \leq N} \epsilon^{i+j} \Gamma[\mathcal{F}_i^\epsilon, \mathcal{F}_j^\epsilon]. \end{aligned} \tag{5.9}$$

The boundary layer is $\mathscr{F}_k^\epsilon=\mathcal{G}_k^\epsilon\cdot\Upsilon_0$ where $\mathcal{G}_k^\epsilon$ solves the $\epsilon$-Milne problem. Notice $\Upsilon_0\Upsilon=\Upsilon_0$, so the boundary layer contribution can be estimated as

$$
\begin{aligned}
\mathscr{L}[\mathscr{Q}_N]&=v_\eta\frac{\partial\mathscr{Q}_N}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\bigg(v_\phi\frac{\partial\mathscr{Q}_N}{\partial\phi}+v_\phi^2\frac{\partial\mathscr{Q}_N}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{Q}_N}{\partial v_\phi}\bigg)+\mathcal{L}[\mathscr{Q}_N] \qquad (5.10)\\
&=\sum_{k=1}^N\epsilon^k\Bigg(v_\eta\frac{\partial\mathscr{F}_k^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\bigg(v_\phi\frac{\partial\mathscr{F}_k^\epsilon}{\partial\phi}+v_\phi^2\frac{\partial\mathscr{F}_k^\epsilon}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathscr{F}_k^\epsilon}{\partial v_\phi}\bigg)+\mathcal{L}[\mathscr{F}_k^\epsilon]\Bigg)\\
&=\sum_{k=1}^N\epsilon^k\Bigg(v_\eta\frac{\partial\mathcal{G}_k^\epsilon}{\partial\eta}\Upsilon_0-\frac{\epsilon\Upsilon_0}{1-\epsilon\eta}\bigg(v_\phi\frac{\partial\mathcal{G}_k^\epsilon}{\partial\phi}+v_\phi^2\frac{\partial\mathcal{G}_k^\epsilon}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathcal{G}_k^\epsilon}{\partial v_\phi}\bigg)+\Upsilon_0\mathcal{L}[\mathcal{G}_k^\epsilon]\Bigg)\\
&\quad+\sum_{k=1}^N\epsilon^kv_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon\\
&=\sum_{k=1}^N\epsilon^k\Bigg(v_\eta\frac{\partial\mathcal{G}_k^\epsilon}{\partial\eta}\Upsilon_0-\frac{\epsilon\Upsilon_0\Upsilon}{1-\epsilon\eta}\bigg(v_\phi\frac{\partial\mathcal{G}_k^\epsilon}{\partial\phi}+v_\phi^2\frac{\partial\mathcal{G}_k^\epsilon}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathcal{G}_k^\epsilon}{\partial v_\phi}\bigg)+\Upsilon_0\mathcal{L}[\mathcal{G}_k^\epsilon]\Bigg)\\
&\quad+\sum_{k=1}^N\epsilon^kv_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon\\
&=\Upsilon_0\sum_{k=1}^N\epsilon^k\Bigg(v_\eta\frac{\partial\mathcal{G}_k^\epsilon}{\partial\eta}-\frac{\epsilon\Upsilon}{1-\epsilon\eta}\bigg(v_\phi^2\frac{\partial\mathcal{G}_k^\epsilon}{\partial v_\eta}-v_\eta v_\phi\frac{\partial\mathcal{G}_k^\epsilon}{\partial v_\phi}\bigg)+\mathcal{L}[\mathcal{G}_k^\epsilon]\Bigg)\\
&\quad+\sum_{k=1}^N\epsilon^kv_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon-\sum_{k=1}^N\epsilon^{k+1}\frac{\Upsilon_0}{1-\epsilon\eta}v_\phi\frac{\partial\mathcal{G}_k^\epsilon}{\partial\phi}\\
&=\sum_{k=1}^N\epsilon^kv_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon-\epsilon^{N+1}\frac{\Upsilon_0}{1-\epsilon\eta}v_\phi\frac{\partial\mathcal{G}_N^\epsilon}{\partial\phi}+\Upsilon_0\sum_{1\leq i,j\leq N}^{i+j\leq N}\epsilon^{i+j}\bigg(\Gamma[\mathscr{F}_i^\epsilon,\mathscr{F}_j^\epsilon]+2\Gamma[\mathcal{F}_i^\epsilon,\mathscr{F}_j^\epsilon]\bigg).
\end{aligned}
$$

Note that for any $f,g\in L^2$,

$$
\mathbb{P}[\Gamma(f,g)]=0. \qquad (5.11)
$$

Since

$$
\mathscr{L}[f^\epsilon]=\Gamma[f^\epsilon,f^\epsilon], \qquad (5.12)
$$

then we can naturally obtain

$$
\begin{aligned}
\mathscr{L}[R_N]&=\frac{1}{\epsilon^3}\mathscr{L}[f^\epsilon-\mathcal{Q}_N-\mathscr{Q}_N]=\frac{1}{\epsilon^3}\mathscr{L}[f^\epsilon]-\frac{1}{\epsilon^3}\mathscr{L}[\mathcal{Q}_N]-\frac{1}{\epsilon^3}\mathscr{L}[\mathscr{Q}_N] \qquad (5.13)\\
&=\frac{1}{\epsilon^3}\Gamma[\mathcal{Q}_N+\mathscr{Q}_N+\epsilon^3R_N,\mathcal{Q}_N+\mathscr{Q}_N+\epsilon^3R_N]\\
&\quad-\sum_{1\leq i,j\leq N}^{i+j\leq N}\epsilon^{i+j-3}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_j^\epsilon]-\psi_0\sum_{1\leq i,j\leq N}^{i+j\leq N}\epsilon^{i+j-3}\bigg(\Gamma[\mathscr{F}_i^\epsilon,\mathscr{F}_j^\epsilon]+2\Gamma[\mathcal{F}_i^\epsilon,\mathscr{F}_j^\epsilon]\bigg)\\
&\quad-\epsilon^{N-2}\vec v\cdot\nabla_x\mathcal{F}_N^\epsilon-\sum_{k=1}^N\epsilon^{k-3}v_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon+\epsilon^{N-2}\frac{\Upsilon_0}{1-\epsilon\eta}v_\phi\frac{\partial\mathcal{G}_N^\epsilon}{\partial\phi}\\
&=\epsilon^3\Gamma[R_N,R_N]+2\Gamma[R_N,\mathcal{Q}_N+\mathscr{Q}_N]\\
&\quad-\sum_{1\leq i,j\leq N}^{i+j\geq N+1}\epsilon^{i+j-3}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_j^\epsilon]-\psi_0\sum_{1\leq i,j\leq N}^{i+j\geq N+1}\epsilon^{i+j-3}\bigg(\Gamma[\mathscr{F}_i^\epsilon,\mathscr{F}_j^\epsilon]+2\Gamma[\mathcal{F}_i^\epsilon,\mathscr{F}_j^\epsilon]\bigg)\\
&\quad-\epsilon^{N-2}\vec v\cdot\nabla_x\mathcal{F}_N^\epsilon-\sum_{k=1}^N\epsilon^{k-3}v_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon+\epsilon^{N-2}\frac{\Upsilon_0}{1-\epsilon\eta}v_\phi\frac{\partial\mathcal{G}_N^\epsilon}{\partial\phi}.
\end{aligned}
$$

Step 3: Estimates of $R_N$.

$R_N$ satisfies the equation

$$\left\{\begin{array}{rcl}\epsilon\vec{v}\cdot\nabla_x R_N+\mathcal{L}[R_N]&=&\epsilon^3\Gamma[R_N,R_N]+2\Gamma[R_N,\mathcal{Q}_N+\mathscr{Q}_N]+S_N\ \ \text{in}\ \ \Omega,\\ R_N(\vec{x}_0,\vec{v})&=&h_N\ \ \text{for}\ \ \vec{v}\cdot\vec{n}<0\ \ \text{and}\ \ \vec{x}_0\in\partial\Omega,\end{array}\right. \tag{5.14}$$

where

$$\begin{aligned}S_N=&-\sum_{1\leq i,j\leq N}^{i+j\geq N+1}\epsilon^{i+j-3}\Gamma[\mathcal{F}_i^\epsilon,\mathcal{F}_j^\epsilon]-\psi_0\sum_{1\leq i,j\leq N}^{i+j\geq N+1}\epsilon^{i+j-3}\Big(\Gamma[\mathscr{F}_i^\epsilon,\mathscr{F}_j^\epsilon]+2\Gamma[\mathcal{F}_i^\epsilon,\mathscr{F}_j^\epsilon]\Big)\\&-\epsilon^{N-2}\vec{v}\cdot\nabla_x\mathcal{F}_N^\epsilon-\sum_{k=1}^{N}\epsilon^{k-3}v_\eta\frac{\partial\Upsilon_0}{\partial\eta}\mathcal{G}_k^\epsilon+\epsilon^{N-2}\frac{\Upsilon_0}{1-\epsilon\eta}v_\phi\frac{\partial\mathcal{G}_N^\epsilon}{\partial\phi},\end{aligned}$$

and

$$h_N=\sum_{k=N+1}^{\infty}\epsilon^{k-3}b_k.$$

By the classical estimate of two-dimensional Stokes-Fourier equations, exponential decay of $\mathscr{F}_k^\epsilon$ and (5.11), we can directly verify

$$\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}S_N\right\|_{L^\infty}\leq C\epsilon^{N-2}, \tag{5.15}$$

$$\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}h_N\right|_{L^\infty_-}\leq C\epsilon^{N-2}. \tag{5.16}$$

Based on Theorem 2.5, we have

$$\left\|R_N\right\|_{L^2}+\left|R_N\right|_{L^2_+}\leq C\bigg(\frac{1}{\epsilon^2}\left\|S_N\right\|_{L^2}+\frac{1}{\epsilon}\left\|\epsilon^3\Gamma[R_N,R_N]\right\|_{L^2}+\frac{1}{\epsilon}\left\|2\Gamma[R_N,\mathcal{Q}_N+\mathscr{Q}_N]\right\|_{L^2}+\frac{1}{\epsilon^{1/2}}\left|h_N\right|_{L^2_-}\bigg). \tag{5.17}$$

Then since

$$\left|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}(B^\epsilon-\mu)\right|_{L^\infty}\leq C_0\epsilon, \tag{5.18}$$

for $C_0$ is sufficiently small, we deduce $b_1$ is sufficiently small. Hence, small boundary data naturally yields

$$\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}(\mathcal{Q}_N+\mathscr{Q}_N)\right\|_{L^\infty}\leq\delta\epsilon, \tag{5.19}$$

which further implies

$$\frac{1}{\epsilon}\left\|2\Gamma[R_N,\mathcal{Q}_N+\mathscr{Q}_N]\right\|_{L^2}\leq\frac{C}{\epsilon}\left\|R_N\right\|_{L^2}\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}(\mathcal{Q}_N+\mathscr{Q}_N)\right\|_{L^\infty}\leq\delta\left\|R_N\right\|_{L^2}, \tag{5.20}$$

for some small $\delta>0$. Hence, absorbing them into the left-hand side of (5.17) yields

$$\left\|R_N\right\|_{L^2}+\left|R_N\right|_{L^2_+}\leq C\bigg(\frac{1}{\epsilon^2}\left\|S_N\right\|_{L^2}+\epsilon^2\left\|\Gamma[R_N,R_N]\right\|_{L^2}+\frac{1}{\epsilon^{1/2}}\left|h_N\right|_{L^2_-}\bigg). \tag{5.21}$$

Based on Theorem 2.7, we have

$$
\begin{aligned}
&\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right|_{L^\infty_+} \qquad (5.22)\\
\leq{}& C\bigg(\frac{1}{\epsilon}\left\|R_N\right\|_{L^2}+\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}S_N\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}h_N\right|_{L^\infty_-}\\
&+\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\epsilon^2\Gamma[R_N,R_N]\right\|_{L^\infty}+\left\|2\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\Gamma[R_N,\mathcal{Q}_N+\mathscr{Q}_N]\right\|_{L^\infty}\bigg)\\
\leq{}& C\bigg(\epsilon\left\|\Gamma[R_N,R_N]\right\|_{L^2}+\frac{1}{\epsilon^3}\left\|S_N\right\|_{L^2}+\frac{1}{\epsilon^{3/2}}\left|h_N\right|_{L^2_-}+\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}S_N\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}h_N\right|_{L^\infty_-}\\
&+\epsilon^3\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\Gamma[R_N,R_N]\right\|_{L^\infty}+\left\|2\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\Gamma[R_N,\mathcal{Q}_N+\mathscr{Q}_N]\right\|_{L^\infty}\bigg)\\
\leq{}& C\bigg(\epsilon\left\|\Gamma[R_N,R_N]\right\|_{L^2}+\epsilon^3\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\Gamma[R_N,R_N]\right\|_{L^\infty}+\epsilon\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}+\epsilon^{N-5}\bigg).
\end{aligned}
$$

Moreover, we can directly estimate

$$\left\|\epsilon\Gamma[R_N,R_N]\right\|_{L^2}\leq C\epsilon\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}^2 \tag{5.23}$$

$$\epsilon^3\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\Gamma[R_N,R_N]\right\|_{L^\infty}\leq C\epsilon^3\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}^2. \tag{5.24}$$

Then if $N\geq 4$, we obtain

$$\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right|_{L^\infty_+}\leq C\epsilon^{N-5}+C\epsilon\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}^2, \tag{5.25}$$

which further implies

$$\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right\|_{L^\infty}+\left|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}R_N\right|_{L^\infty_+}\leq\frac{C}{\epsilon}. \tag{5.26}$$

for $\epsilon$ sufficiently small. This means we have shown

$$\frac{1}{\epsilon^3}\left\|\langle\vec{v}\rangle^{\vartheta}\mathrm{e}^{\zeta|\vec{v}|^2}\bigg(f^\epsilon-\sum_{k=1}^{N}\epsilon^k\mathcal{F}_k^\epsilon-\sum_{k=1}^{N}\epsilon^k\mathscr{F}_k^\epsilon\bigg)\right\|_{L^\infty}=O\bigg(\frac{1}{\epsilon}\bigg), \tag{5.27}$$

which naturally leads to the desired result. □

Hence, combining the estimate of steady Navier-Stokes-type equations and $\epsilon$-Milne problem, we have

$$f^\epsilon=\epsilon^3R_N+\mathcal{Q}_N+\mathscr{Q}_N, \tag{5.28}$$

exists and is well-posed. The uniqueness and positivity follows from a standard argument as in [**?**].

## 6. Counterexample for Classical Approach

In this section, we present the classical approach with the idea in [**?**, **?**] to construct asymptotic expansion, especially the boundary layer expansion, and provide counterexamples to show this method is problematic.

6.1. **Interior Expansion.** Basically, the expansion for interior solution is identical to our method, so we omit the details and only present the notation. We define the interior expansion

$$\mathcal{F}\sim\sum_{k=1}^{\infty}\epsilon^k\mathcal{F}_k(\vec{x},\vec{v}), \tag{6.1}$$

with

$$\mathcal{F}_k(\vec{x},\vec{v})=A_k(\vec{x},\vec{v})+B_k(\vec{x},\vec{v})+C_k(\vec{x},\vec{v}), \tag{6.2}$$

where

$$A_k(\vec{x},\vec{v})=\sqrt{\mu}\bigg(A_{k,0}(\vec{x})+A_{k,1}(\vec{x})v_1+A_{k,2}(\vec{x})v_2+A_{k,3}(\vec{x})\left(\frac{|\vec{v}|^2-2}{2}\right)\bigg), \tag{6.3}$$

$$B_k(\vec{x},\vec{v})=\sqrt{\mu}\bigg(B_{k,0}(\vec{x})+B_{k,1}(\vec{x})v_1+B_{k,2}(\vec{x})v_2+B_{k,3}(\vec{x})\left(\frac{\left|\vec{v}\right|^2-2}{2}\right)\bigg), \tag{6.4}$$

with $B_k$ depending on $A_{s,i}$ in $1\leq s\leq k-1$ and $i=0,1,2,3$ as

$$B_{k,0}=0, \tag{6.5}$$

$$B_{k,1}=\sum_{i=1}^{k-1}A_{i,0}A_{k-i,1}, \tag{6.6}$$

$$B_{k,2}=\sum_{i=1}^{k-1}A_{i,0}A_{k-i,2}, \tag{6.7}$$

$$\begin{aligned}B_{k,3}=\sum_{i=1}^{k-1}\bigg(&A_{i,0}A_{k-i,3}+A_{i,1}A_{k-i,1}+A_{i,2}A_{k-i,2}\\&+\sum_{j=1}^{k-1-i}A_{i,0}(A_{j,1}A_{k-i-j,1}+A_{j,2}A_{k-i-j,2})\bigg),\end{aligned} \tag{6.8}$$

and $C_k(\vec{x},\vec{v})$ satisfies

$$\int_{\mathbb{R}^2}\sqrt{\mu(\vec{v})}C_k(\vec{x},\vec{v})\left(\begin{array}{c}1\\ \vec{v}\\ \left|\vec{v}\right|^2\end{array}\right)\mathrm{d}\vec{v}=0, \tag{6.9}$$

with

$$\mathcal{L}[C_k]=\ -\vec{v}\cdot\nabla_x\mathcal{F}^{\epsilon}_{k-1}+\sum_{i=1}^{k-1}\Gamma[\mathcal{F}^{\epsilon}_i,\mathcal{F}^{\epsilon}_{k-i}], \tag{6.10}$$

which can be solved explicitly at any fixed $\vec{x}$. We define

$$A_k=\sqrt{\mu}\bigg(\rho_k+u_{k,1}v_1+u_{k,2}v_2+\theta_k\left(\frac{\left|\vec{v}\right|^2-2}{2}\right)\bigg), \tag{6.11}$$

Then $A_k$ satisfies the equations as follows:

$0^{th}$ order equations:

$$P_1-(\rho_1+\theta_1)=0, \tag{6.12}$$

$$\nabla_xP_1=0, \tag{6.13}$$

$1^{st}$ order equations:

$$P_2-(\rho_2+\theta_2+\rho_1\theta_1)=0, \tag{6.14}$$

$$\vec{u}_1\cdot\nabla_x\vec{u}_1-\gamma_1\Delta_x\vec{u}_1+\nabla_xP_2=0, \tag{6.15}$$

$$\nabla_x\cdot\vec{u}_1=0, \tag{6.16}$$

$$\vec{u}_1\cdot\nabla_x\theta_1-\gamma_2\Delta_x\theta_1=0, \tag{6.17}$$

$k^{th}$ order equations:

$$P_{k+1}-\left(\rho_{k+1}+\theta_{k+1}+\sum_{i=1}^{k+1-i}\rho_i\theta_{k+1-i}\right)=0, \tag{6.18}$$

$$\sum_{i=1}^{k}\vec{u}_i\cdot\nabla_x\vec{u}_{k+1-i}-\gamma_1\Delta_x\vec{u}_k+\nabla_xP_{k+1}=G_{k,1}, \tag{6.19}$$

$$\nabla_x\cdot\vec{u}_k=G_{k,2}, \tag{6.20}$$

$$\sum_{i=1}^{k}\vec{u}_i\cdot\nabla_x\theta_{k+1-i}-\gamma_2\Delta_x\theta_k=G_{k,3}, \tag{6.21}$$

where

$$G_{k,j}=G_{k,j}[\vec{x},\vec{v};\rho_1,\ldots,\rho_{k-1};\theta_1,\ldots,\theta_{k-1};\vec{u}_1,\ldots,\vec{u}_{k-1}], \tag{6.22}$$

is explicit functions depending on lower order terms, and $\gamma_1$ and $\gamma_2$ are two positive constants.

6.2. **Boundary Layer Expansion.** By the idea in [?, ?], the boundary layer expansion can be defined by introducing substitutions (3.29), (3.31) and (3.34). Note that we terminate here and do not further use substitution (3.36). Hence, we have transformed the equation (1.9) into

$$\left\{\begin{array}{l}(\vec{v}_r\cdot\vec{n})\dfrac{\partial f^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}(\vec{v}_r\cdot\vec{\tau})\dfrac{\partial f^\epsilon}{\partial\phi}+\mathcal{L}[f^\epsilon]=\Gamma[f^\epsilon,f^\epsilon],\\ f^\epsilon(0,\phi,\vec{v}_r)=b^\epsilon(0,\phi,\vec{v}_r)\ \text{ for }\ \vec{v}_r\cdot\vec{n}>0.\end{array}\right. \tag{6.23}$$

We define the boundary layer expansion

$$\mathscr{F}\sim\sum_{k=1}^{\infty}\epsilon^k\mathscr{F}_k(\eta,\phi,\vec{v}_r), \tag{6.24}$$

where $\mathscr{F}_k$ can be determined by plugging it into the equation (6.23) and comparing the order of $\epsilon$. Thus in a neighborhood of the boundary, we have

$$(\vec{v}_r\cdot\vec{n})\frac{\partial\mathscr{F}_1}{\partial\eta}+\mathcal{L}[\mathscr{F}_1]=0, \tag{6.25}$$

$$(\vec{v}_r\cdot\vec{n})\frac{\partial\mathscr{F}_2}{\partial\eta}+\mathcal{L}[\mathscr{F}_2]=\frac{1}{1-\epsilon\eta}(\vec{v}_r\cdot\vec{\tau})\frac{\partial\mathscr{F}_1}{\partial\phi}+\Gamma[\mathscr{F}_1,\mathscr{F}_1]+2\Gamma[\mathcal{F}_1,\mathscr{F}_1], \tag{6.26}$$

$$\ldots$$

$$(\vec{v}_r\cdot\vec{n})\frac{\partial\mathscr{F}_k}{\partial\eta}+\mathcal{L}[\mathscr{F}_k]=\frac{1}{1-\epsilon\eta}(\vec{v}_r\cdot\vec{\tau})\frac{\partial\mathscr{F}_{k-1}}{\partial\phi}+\sum_{i=1}^{k-1}\Gamma[\mathscr{F}_i,\mathscr{F}_{k-i}]+2\sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i,\mathscr{F}_{k-i}]. \tag{6.27}$$

The bridge between the interior solution and boundary layer is the boundary condition

$$f^\epsilon(\vec{x}_0,\vec{v})=b^\epsilon(\vec{x}_0,\vec{v}). \tag{6.28}$$

Plugging the combined expansion

$$f^\epsilon\sim\sum_{k=1}^{\infty}\epsilon^k(\mathcal{F}_k+\mathscr{F}_k), \tag{6.29}$$

into the boundary condition and comparing the order of $\epsilon$, we obtain

$$\mathcal{F}_1+\mathscr{F}_1=b_1, \tag{6.30}$$

$$\mathcal{F}_2+\mathscr{F}_2=b_2, \tag{6.31}$$

$$\ldots$$

$$\mathcal{F}_k+\mathscr{F}_k=b_k. \tag{6.32}$$

This is the boundary conditions $\mathcal{F}_k$ and $\mathscr{F}_k$ need to satisfy.

6.3. **Classical Approach to Construct Asymptotic Expansion.** We divide the construction of asymptotic expansion into several steps for each $k\geq1$:

Step 1: Milne Problem.

We solve the $\epsilon$-Milne problem

$$\left\{\begin{array}{rcl}(\vec{v}_r\cdot\vec{n})\dfrac{\partial g_k}{\partial\eta}+\mathcal{L}[g_k]&=&S_k(\eta,\phi,\vec{v}_r),\\ g_k(0,\phi,\vec{v}_r)&=&h_k(\phi,\vec{v}_r)\ \text{ for }\ \vec{v}_r\cdot\vec{n}>0,\\ \displaystyle\int_{\mathbb{R}^2}(\vec{v}_r\cdot\vec{n})\sqrt{\mu}g_k(0,\phi,\vec{v}_r)\mathrm{d}\vec{v}_r&=&m_f[g_k](\phi),\\ \displaystyle\lim_{\eta\rightarrow\infty}g_k(\eta,\phi,\vec{v}_r)&=&g_k(\infty,\phi,\vec{v}_r),\end{array}\right. \tag{6.33}$$

for $g_k(\eta,\phi,\vec{v}_r)$ with the in-flow boundary data

$$h_k=b_k-(B_k+C_k) \tag{6.34}$$

and source term

$$S_k=\frac{\Upsilon(\sqrt{\epsilon}\eta)}{1-\epsilon\eta}(\vec{v}_r\cdot\vec{\tau})\frac{\partial\mathscr{F}_{k-1}}{\partial\phi}+\sum_{i=1}^{k-1}\Gamma[\mathscr{F}_i,\mathscr{F}_{k-i}]+2\sum_{i=1}^{k-1}\Gamma[\mathcal{F}_i,\mathscr{F}_{k-i}], \tag{6.35}$$

where

$$\Upsilon(z)=\left\{\begin{array}{ll}1 & 0\leq z\leq 1/2,\\ 0 & 3/4\leq z\leq\infty.\end{array}\right. \tag{6.36}$$

Here the mass-flux $m_f[g_k](\phi)$ will be determined later. Based on Remark 4.16, there exist

$$\tilde{h}_k=\sqrt{\mu}\Bigg(\tilde{D}_{k,0}+\tilde{D}_{k,1}v_{r,1}+\tilde{D}_{k,2}v_{r,2}+\tilde{D}^{\epsilon}_{k,3}\bigg(\frac{\left|\vec{v}_r\right|^2-2}{2}\bigg)\Bigg), \tag{6.37}$$

such that the problem

$$\left\{\begin{array}{rcl}(\vec{v}_r\cdot\vec{n})\dfrac{\partial\mathcal{G}_k}{\partial\eta}+\mathcal{L}[\mathcal{G}_k]&=&S_k(\eta,\phi,\vec{v}_r),\\ \mathcal{G}_k(0,\phi,\vec{v}_r)&=&h_k(\phi,\vec{v}_r)-\tilde{h}_k(\phi,\vec{v}_r)\ \text{ for }\ \vec{v}_r\cdot\vec{n}>0,\\ \displaystyle\int_{\mathbb{R}^2}(\vec{v}_r\cdot\vec{n})\sqrt{\mu}\mathcal{G}_k(0,\phi,\vec{v}_r)\mathrm{d}\vec{v}_r&=&m_f[\mathcal{G}_k](\phi),\\ \displaystyle\lim_{\eta\rightarrow\infty}\mathcal{G}_k(\eta,\phi,\vec{v}_r)&=&0,\end{array}\right. \tag{6.38}$$

is well-posed.
Step 2: Definition of Interior Solution and Boundary Layer.
Define

$$\mathscr{F}_k=\mathcal{G}_k\cdot\Upsilon_0(\epsilon^{1/2}\eta) \tag{6.39}$$

where $\mathcal{G}_k$ the solution of $\epsilon$-Milne problem (6.38) and

$$\Upsilon_0(z)=\left\{\begin{array}{ll}1 & 0\leq z\leq 1/4,\\ 0 & 1/2\leq z\leq\infty.\end{array}\right. \tag{6.40}$$

Naturally, we have

$$\lim_{\eta\rightarrow 0}\mathscr{F}_k(\eta,\phi,\vec{v}_r)=0. \tag{6.41}$$

The interior solution

$$\mathcal{F}_k(\vec{x},\vec{v})=A_k(\vec{x},\vec{v})+B_k(\vec{x},\vec{v})+C_k(\vec{x},\vec{v}), \tag{6.42}$$

where $A_k$ satisfies

$$A_k=\sqrt{\mu}\left(\rho_k+u_{k,1}v_1+u_{k,2}v_2+\theta_k\left(\frac{\left|\vec{v}\right|^2-2}{2}\right)\right), \tag{6.43}$$

and

$$P_{k+1}-\left(\rho_{k+1}+\theta_{k+1}+\sum_{i=1}^{k+1-i}\rho_i\theta_{k+1-i}\right)=0, \tag{6.44}$$

$$\sum_{i=1}^{k}\vec{u}_i\cdot\nabla_x\vec{u}_{k+1-i}-\gamma_1\Delta_x\vec{u}_k+\nabla_xP_{k+1}=H_{k,1}, \tag{6.45}$$

$$\nabla_x\cdot\vec{u}_k=H_{k,2}, \tag{6.46}$$

$$\sum_{i=1}^{k}\vec{u}_i\cdot\nabla_x\theta_{k+1-i}-\gamma_2\Delta_x\theta_k=H_{k,3}, \tag{6.47}$$

with boundary condition

$$A_{k,0}=\tilde{D}_{k,0}, \tag{6.48}$$

$$A_{k,1}=-\tilde{D}_{k,0}, \tag{6.49}$$

$$A_{k,2}=-\tilde{D}_{k,0}, \tag{6.50}$$

$$A_{k,3}=\tilde{D}_{k,3}. \tag{6.51}$$

where $\tilde{D}_{k,i}$ comes from the boundary data of Milne problem $\tilde{h}_k$. This determines $A_{k,0}$, $A_{k,1}$, $A_{k,2}$ and $A_{k,3}$. Now it is easy to verify the boundary data are satisfied as

$$\mathcal{F}_k+\mathscr{F}_k=b_k. \tag{6.52}$$

Step 3: Boussinesq relation and Vanishing Mass-Flux.
Similarly, the free mass-flux $m_f[\mathcal{G}_k](\phi)$ can help to enforce two relations: the Boussinesq relation

$$\rho_k+\theta_k=E_k-\sum_{i=1}^{k-i}\rho_i\theta_{k-i}, \tag{6.53}$$

and vanishing mass-flux relation

$$\int_{\partial\Omega}\int_{\mathbb{R}^2}\mathcal{F}_k(\vec{x},\vec{v})\mathrm{d}\vec{v}\mathrm{d}\gamma=0. \tag{6.54}$$

Therefore, $m_f[\mathcal{G}_k](\phi)$ is completed determined and so are $\mathcal{F}_k$ and $\mathscr{F}_k$.

The analysis in [**?**, **?**] anticipates this process can be generalized to arbitrary $k$. However, In order to show the hydrodynamic limit, we at least need to expand to $k=2$ at least. Therefore, based on Remark 4.16, we require $S_2\in L^\infty$ to obtain a well-posed $\mathscr{F}_2$, i.e. we need

$$\frac{\partial\mathscr{F}_1}{\partial\phi}\in L^\infty, \tag{6.55}$$

which further requires

$$\frac{\partial\mathscr{F}_1}{\partial\eta}\in L^\infty. \tag{6.56}$$

Theorem 6.1 states that for certain boundary data $B^\epsilon$, this is invalid. Hence, this formulation breaks down.

6.4. **Singularity in Derivative of Milne Problem.** Now we present the singularity of the normal derivative in the Milne problem. For convenience, we use the notation $\vec{\mathfrak{v}}=(v_\eta,v_\phi)$.

**Theorem 6.1.** *For the Milne problem*

$$\left\{\begin{array}{rcl}v_\eta\dfrac{\partial g}{\partial\eta}+\mathcal{L}[g]&=&0,\\ g(0,\vec{\mathfrak{v}})&=&h(\vec{\mathfrak{v}})\ \ for\ \ v_\eta>0,\\ \displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}g(0,\vec{\mathfrak{v}})\mathrm{d}\vec{\mathfrak{v}}&=&0,\\ \displaystyle\lim_{\eta\rightarrow\infty}g(\eta,\vec{\mathfrak{v}})&=&g_\infty(\vec{\mathfrak{v}}).\end{array}\right. \tag{6.57}$$

*with*

$$h = v_\phi \mathrm{e}^{-(v_\phi^2-1)-Mv_\eta^2}, \tag{6.58}$$

*where $v_\eta$ and $v_\phi$ are defined as in (3.34) and $M$ is sufficiently large such that*

$$h(0,1) = 1, \tag{6.59}$$

$$|h|_{L^2_-} << 1, \tag{6.60}$$

*then we have*

$$\left\|\frac{\partial g}{\partial \eta}\right\|_{L^\infty} \notin L^\infty([0,\infty)\times\mathbb{R}^2). \tag{6.61}$$

*Proof.* We divide the proof into several steps: We first assume $\partial_\eta g \in L^\infty([0,\infty)\times\mathbb{R}^2)$ and then show it can lead to a contradiction.

Step 1: Definition of trace.
It is easy to see $\partial_\eta g$ satisfies the Milne problem

$$v_\eta \frac{\partial(\partial_\eta g)}{\partial \eta} + \mathcal{L}[\partial_\eta g] = 0. \tag{6.62}$$

Since $k(\vec{\mathfrak{u}},\vec{\mathfrak{v}}) = k_2(\vec{\mathfrak{u}},\vec{\mathfrak{v}}) - k_1(\vec{\mathfrak{u}},\vec{\mathfrak{v}})$ is in $L^1$ with respect to $\vec{\mathfrak{u}}$ uniformly in $\vec{\mathfrak{v}}$, then we have $K[\partial_\eta g] \in L^\infty([0,\infty)\times\mathbb{R}^2)$. For fixed $N > 0$, $\nu(\vec{\mathfrak{v}})$ is bounded in the domain $S = \{\left|\vec{\mathfrak{v}}\right| \leq N\}$. Hence, we have $\nu(\vec{\mathfrak{v}})\partial_\eta g \in L^\infty([0,\infty)\times S)$, which further implies $\mathcal{L}[\partial_\eta g] \in L^\infty([0,\infty)\times S)$. Therefore, by a standard cut-off argument and Ukai's trace theorem, we deduce $\partial_\eta g(0) \in L^\infty(S)$ is well-defined.

However, we can define the trace of $\partial_\eta g$ in another fashion. For any $v_\eta \neq 0$, since we have $\nu(\vec{\mathfrak{v}})g \in L^\infty([0,\infty)\times S)$ as well as $K[g] \in L^\infty[0,\infty)\times S$, by the Milne problem (6.57), it is naturally to define for $\eta > 0$

$$\partial_\eta g(\eta,\vec{\mathfrak{v}}) = \frac{K[g](\eta,\vec{\mathfrak{v}}) - \nu g(\eta,\vec{\mathfrak{v}})}{v_\eta}. \tag{6.63}$$

Since $\partial_\eta g \in L^\infty([0,\infty)\times S$, we know $g$ is continuous with respect to $\eta$ for a.e. $\vec{v}$. Taking $\eta \to 0$ defines the trace for $\partial_\eta g$ at$(0,\vec{\mathfrak{v}})$

$$\partial_\eta g(0,\vec{\mathfrak{v}}) = \frac{K[g](0,\vec{\mathfrak{v}}) - \nu g(0,\vec{\mathfrak{v}})}{v_\eta}. \tag{6.64}$$

Since the grazing set $\{\vec{\mathfrak{v}} : v_\eta = 0\}$ is zero-measured on the boundary $\eta = 0$, then we have the trace of $\partial_\eta g$ is a.e. well-defined.

By the uniqueness of trace of $\partial_\eta g$, above two types of traces must coincide with each other a.e.. Then we may combine them both and obtain $\partial_\eta g(0,\vec{\mathfrak{v}}) \in L^\infty(S)$ is a.e. well-defined and satisfies the formula

$$\partial_\eta g(0,\vec{\mathfrak{v}}) = \frac{K[g](0,\vec{\mathfrak{v}}) - \nu g(0,\vec{\mathfrak{v}})}{v_\eta}. \tag{6.65}$$

Step 2: Limiting Process.
Therefore, we may consider the limiting process

$$\lim_{\vec{\mathfrak{v}}\to(0,1)} \frac{\partial g}{\partial \eta}(0,\vec{\mathfrak{v}}) = \lim_{\vec{\mathfrak{v}}\to(0,1)} \frac{K[g](0,\vec{\mathfrak{v}}) - \nu g(0,\vec{\mathfrak{v}})}{v_\eta}. \tag{6.66}$$

Based on [?, Lemma 3.3.1], we have

$$\|K[g](0)\|_{L^\infty_{0,0}} \leq |||K[g]|||_{L^\infty L^\infty_{0,0}} \leq C\|g\|_{L^\infty L^2_0}. \tag{6.67}$$

By Lemma 4.11, we have

$$\|g\|_{L^\infty L^2_0} \leq C(\delta)|||g|||_{L^2L^2} + \delta|||g|||_{L^\infty L^\infty_{\vartheta,0}}, \tag{6.68}$$

for $\delta>0$ sufficiently small and $\vartheta>2$. Combining this with Theorem 4.7 and Theorem 4.14, we know

$$|||g|||_{L^2L^2} \leq C\left\|h\right\|_{L^2} \tag{6.69}$$

$$|||g|||_{L^\infty L^\infty_{\vartheta,0}} \leq C<\infty. \tag{6.70}$$

Taking $\delta$ sufficiently small, and then taking $M$ sufficiently large, we have

$$\left\|g\right\|_{L^\infty L^2_0} << 1. \tag{6.71}$$

On the other hand, we can see

$$\nu(0,1)g(0,0,1)\geq Ch(0,1)\geq C_0>0, \tag{6.72}$$

for some positive constant $C_0$. Therefore, we have shown

$$\mathcal{L}[g](0,\vec{\mathfrak{v}})\geq \frac{C_0}{2}>0, \tag{6.73}$$

when $\vec{\mathfrak{v}}\rightarrow(0,1)$ which implies $v_\eta\rightarrow 0$. Hence, we can solve the normal derivative as

$$\frac{\partial g}{\partial\eta}=-\frac{\mathcal{L}[g](0)}{v_\eta}\rightarrow\infty, \tag{6.74}$$

which contradicts our assumption that $\partial_\eta g(0,\vec{\mathfrak{v}})\in L^\infty(S)$. □

6.5. **Counterexample to Classical Approach.** We present a counterexample to show this classical approach can lead to wrong result.

**Theorem 6.2.** *For given $B^\epsilon>0$ satisfying (1.5) and (1.7) with*

$$\frac{b_1}{\sqrt{\mu}}=\left(v_\phi \mathrm{e}^{-(v_\phi^2-1)-Mv_\eta^2}\right)=h(v_\eta,v_\phi), \tag{6.75}$$

*where $v_\eta$ and $v_\phi$ are defined as in (3.34) and we take $M$ sufficiently large such that*

$$h(0,1) \;=\; 1, \tag{6.76}$$

$$\left|h\right|_{L^2_-} \;<<1, \tag{6.77}$$

*there exists $C>0$ such that*

$$\left\|f^\epsilon-(\mathcal{F}_1+\mathscr{F}_1)\right\|_{L^\infty}\geq C\epsilon, \tag{6.78}$$

*where the interior solution $\mathcal{F}_1$ is defined in (6.44) and boundary layer $\mathscr{F}_1$ is defined in (6.38).*

*Proof.* Define $\mathscr{W}^\epsilon=\mathcal{F}_1^\epsilon+\mathscr{F}_1^\epsilon$ and $\mathscr{W}=\mathcal{F}_1+\mathscr{F}_1$. Consider the $\epsilon$-Milne problem

$$\left\{\begin{array}{rcl}
v_\eta\dfrac{\partial\mathscr{W}^\epsilon}{\partial\eta}+G(\epsilon;\eta)\left(v_\phi^2\dfrac{\partial\mathscr{W}^\epsilon}{\partial v_\eta}-v_\eta v_\phi\dfrac{\partial\mathscr{W}^\epsilon}{\partial v_\phi}\right)+\mathcal{L}[\mathscr{W}^\epsilon]&=&0,\\
\mathscr{W}^\epsilon(0,v_\eta,v_\phi)&=&h(v_\eta,v_\phi)\ \ \text{for}\ \ v_\eta>0,\\
\displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\mathscr{W}^\epsilon(0,v_\eta,v_\phi)\mathrm{d}v_\eta\mathrm{d}v_\phi&=&m_f(\phi),\\
\displaystyle\lim_{\eta\rightarrow\infty}\mathscr{W}^\epsilon(\eta,v_\eta,v_\phi)&=&\mathscr{W}^\epsilon_\infty(v_\eta,v_\phi),
\end{array}\right. \tag{6.79}$$

and Milne problem

$$\left\{\begin{array}{rcl}
v_\eta\dfrac{\partial\mathscr{W}}{\partial\eta}+\mathcal{L}[\mathscr{W}]&=&0,\\
\mathscr{W}(0,v_\eta,v_\phi)&=&h(v_\eta,v_\phi)\ \ \text{for}\ \ v_\eta>0,\\
\displaystyle\int_{\mathbb{R}^2}v_\eta\sqrt{\mu}\mathscr{W}(0,v_\eta,v_\phi)\mathrm{d}v_\eta\mathrm{d}v_\phi&=&m_f(\phi),\\
\displaystyle\lim_{\eta\rightarrow\infty}\mathscr{W}(\eta,v_\eta,v_\phi)&=&\mathscr{W}_\infty(v_\eta,v_\phi).
\end{array}\right. \tag{6.80}$$

For convenience, we use the same velocity variables. Note that $\mathscr{W}^\epsilon$ actually satisfies an $\epsilon$-Milne problem with non-trivial source term. However, based on the proof of Theorem 4.7, this source term will add a $O(\epsilon^{1/2})$ perturbation to $\mathscr{W}^\epsilon$, so we can omit it and concentrate on above simpler form.

We divide the proof into several steps:

Step 1: Continuity of $K[\mathscr{W}^{\epsilon}]$ and $K[\mathscr{W}]$ at $\eta=0$.
For any $R_0>r_0>0$ and $\vec{\mathfrak{u}}=(\mathfrak{u}_\eta,\mathfrak{u}_\phi)$, we have

$$\begin{aligned}&\left|K[\mathscr{W}](0,\vec{\mathfrak{v}})-K[\mathscr{W}](\eta,\vec{\mathfrak{v}})\right| \qquad (6.81)\\ \leq&\int_{\mathfrak{u}_\eta\leq r_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\left|\mathscr{W}(0,\vec{\mathfrak{u}})-\mathscr{W}(\eta,\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}+\int_{\mathfrak{u}_\eta\geq R_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\left|\mathscr{W}(0,\vec{\mathfrak{u}})-\mathscr{W}(\eta,\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\\ &+\int_{r_0\leq\mathfrak{u}_\eta\leq R_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\left|\mathscr{W}(0,\vec{\mathfrak{u}})-\mathscr{W}(\eta,\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\end{aligned}$$

Since we know $\mathscr{W}\in L^\infty([0,\infty)\times\mathbb{R}^2)$, then for any $\delta>0$ we can take $r_0$ sufficiently small such that

$$\int_{\mathfrak{u}_\eta\leq r_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\left|\mathscr{W}(0,\vec{\mathfrak{u}})-\mathscr{W}(\eta,\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\leq C\int_{\mathfrak{u}_\eta\leq r_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\mathrm{d}\vec{\mathfrak{u}}\leq\frac{\delta}{3}. \tag{6.82}$$

Since we know

$$\left\|\langle\vec{v}\rangle^{\vartheta}\,\mathrm{e}^{\zeta|\vec{v}|^2}(\mathscr{W}-\mathscr{W}_\infty)\right\|_{L^\infty}\leq C<\infty, \tag{6.83}$$

then there exists a $R_0>0$, such that for $\mathfrak{u}_\eta\geq R_0$,

$$|\mathscr{W}(\eta,\vec{\mathfrak{u}})|\leq\tilde{\delta}, \tag{6.84}$$

where $\tilde{\delta}$ is sufficiently small. Therefore, we have

$$\int_{\mathfrak{u}_\eta\geq R_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\left|\mathscr{W}(0,\vec{\mathfrak{u}})-\mathscr{W}(\eta,\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\leq 2\tilde{\delta}\int_{\mathfrak{u}_\eta\geq R_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\mathrm{d}\vec{\mathfrak{u}}\leq\frac{\delta}{3}. \tag{6.85}$$

For fixed $r_0$ and $R_0$ satisfying above requirement, we estimate the integral on $r_0\leq\mathfrak{u}_\eta\leq R_0$. By Ukai's trace theorem, we have $\mathscr{W}(0,\vec{\mathfrak{v}})$ is well-defined and

$$\partial_\eta\mathscr{W}(0,\vec{\mathfrak{u}})=\frac{K[\mathscr{W}](0,\vec{\mathfrak{v}})-\nu(\vec{\mathfrak{v}})\mathscr{W}(0,\vec{\mathfrak{v}})}{v_\eta}. \tag{6.86}$$

The in $r_0\leq\mathfrak{u}_\eta\leq R_0$, $\partial_\eta\mathscr{W}$ is bounded, which implies $\mathscr{W}(\eta,\vec{\mathfrak{v}})$ is uniformly continuous at $\eta=0$. Then there exists a $\eta_0$ such that for $0\leq\eta\leq\eta_0$,

$$\int_{r_0\leq\mathfrak{u}_\eta\leq R_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\left|\mathscr{W}(0,\vec{\mathfrak{u}})-\mathscr{W}(\eta,\vec{\mathfrak{u}})\right|\mathrm{d}\vec{\mathfrak{u}}\leq C\tilde{\delta}\int_{r_0\leq\mathfrak{u}_\eta\leq R_0}\left|k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})\right|\mathrm{d}\vec{\mathfrak{u}}\leq\frac{\delta}{3}. \tag{6.87}$$

In summary, we have shown for any $\delta>0$, there exists a $\eta_0>0$ such that for any $0\leq\eta\leq\eta_0$ and fixed $\vec{\mathfrak{v}}$,

$$\left|K[\mathscr{W}](0,\vec{\mathfrak{v}})-K[\mathscr{W}](\eta,\vec{\mathfrak{v}})\right|\leq\delta. \tag{6.88}$$

Therefore, $K[\mathscr{W}]$ is continuous at $\eta=0$. A similar argument can be implemented to $\mathscr{W}^{\epsilon}$. It is easy to see above estimate is uniform in $\vec{\mathfrak{v}}$ since $L^1$ estimate of $k(\vec{\mathfrak{u}},\vec{\mathfrak{v}})$ in $\vec{\mathfrak{u}}$ is uniform with respect to $\vec{\mathfrak{v}}$. Also, it is obvious to see $K$ is continuous with respect to $\vec{\mathfrak{v}}$ at $\eta=0$.

Step 2: Milne formulation.
We consider the solution at a specific point $\eta=n\epsilon$, $v_\eta=\epsilon$ and $v_\phi=\sqrt{1-\epsilon^2}$ for some fixed $n>0$. The solution along the characteristics can be rewritten as follows:

$$\mathscr{W}(n\epsilon,\epsilon,\sqrt{1-\epsilon^2})=h(\epsilon,\sqrt{1-\epsilon^2})\mathrm{e}^{-\frac{\nu(1)}{\epsilon}n\epsilon}+\int_0^{n\epsilon}\mathrm{e}^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)}\frac{1}{\epsilon}K[\mathscr{W}](\kappa,\epsilon,\sqrt{1-\epsilon^2})\mathrm{d}\kappa, \tag{6.89}$$

$$\mathscr{W}^{\epsilon}(n\epsilon,\epsilon,\sqrt{1-\epsilon^2})=h(\epsilon_0,\sqrt{1-\epsilon_0^2})\mathrm{e}^{-\int_0^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta}+\int_0^{n\epsilon}\mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta}\frac{1}{v_\eta(\kappa)}K[\mathscr{W}^{\epsilon}](\kappa,v_\eta(\kappa),v_\phi(\kappa))\mathrm{d}\kappa, \tag{6.90}$$

where $\nu(1)$ denote the value of $\nu(\vec{\mathfrak{v}})$ at $\left|\vec{\mathfrak{v}}\right|=1$ and we have the conserved energy along the characteristics

$$E(\eta,v_\eta,v_\phi)=v_\phi e^{-W(\eta)}, \tag{6.91}$$

in which $(0,\epsilon_0,\sqrt{1-\epsilon_0^2})$ and $(\zeta,v_\eta(\zeta),\sqrt{1-v_\eta^2(\zeta)})$ are in the same characteristics of $(n\epsilon,\epsilon,\sqrt{1-\epsilon^2})$.

Step 3: Estimates of (6.89).
We turn to the Milne problem for $\mathscr{W}$. We have the natural estimate

$$\int_0^{n\epsilon} \mathrm{e}^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)}\frac{1}{\epsilon}\mathrm{d}\kappa = \mathrm{e}^{-n\nu(1)}\int_0^{n\epsilon} \mathrm{e}^{\frac{\nu(1)\kappa}{\epsilon}}\frac{1}{\epsilon}\mathrm{d}\kappa = \mathrm{e}^{-n\nu(1)}\int_0^{n} e^{\nu(1)\zeta}\mathrm{d}\zeta = \frac{1}{\nu(1)}\Big(1-\mathrm{e}^{-n\nu(1)}\Big). \tag{6.92}$$

Then for $0<\epsilon\leq\eta_0$, we have $\left|K[\mathscr{W}](0,0,1)-K[\mathscr{W}](\kappa,\epsilon,\sqrt{1-\epsilon^2})\right|\leq\delta+\epsilon$, which implies

$$\begin{aligned}\int_0^{n\epsilon} \mathrm{e}^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)}\frac{1}{\epsilon}K[\mathscr{W}](\kappa,\epsilon,\sqrt{1-\epsilon^2})\mathrm{d}\kappa &= \int_0^{n\epsilon} \mathrm{e}^{-\frac{\nu(1)}{\epsilon}(n\epsilon-\kappa)}\frac{1}{\epsilon}K[\mathscr{W}](0,0,1)\mathrm{d}\kappa + O(\delta)+O(\epsilon) \\ &= \frac{1}{\nu(1)}(1-\mathrm{e}^{-n\nu(1)})K[\mathscr{W}](0,0,1)+O(\delta)+O(\epsilon).\end{aligned} \tag{6.93}$$

For the boundary data term, it is easy to see

$$h(\epsilon,\sqrt{1-\epsilon^2})\mathrm{e}^{-\frac{\nu(1)}{\epsilon}n\epsilon} = \mathrm{e}^{-n\nu(1)}h(\epsilon,\sqrt{1-\epsilon^2}). \tag{6.94}$$

In summary, we have

$$\mathscr{W}(n\epsilon,\epsilon,\sqrt{1-\epsilon^2}) = \frac{1}{\nu(1)}(1-\mathrm{e}^{-n\nu(1)})K[\mathscr{W}](0,0,1)+\mathrm{e}^{-n\nu(1)}h(0,1)+O(\delta)+O(\epsilon). \tag{6.95}$$

Step 4: Estimates of (6.90).
We consider the $\epsilon$-Milne problem for $\mathscr{W}^\epsilon$. For $\epsilon<<1$ sufficiently small, $\psi(\epsilon)=1$. Then we may estimate

$$v_\phi(\zeta)e^{-W(\zeta)} = \sqrt{1-\epsilon^2}e^{-W(n\epsilon)}, \tag{6.96}$$

which implies

$$v_\phi(\zeta) = \frac{1-n\epsilon^2}{1-\epsilon\zeta}\sqrt{1-\epsilon^2}. \tag{6.97}$$

and hence

$$v_\eta(\zeta) = \sqrt{1-v_\phi^2(\zeta)} = \sqrt{\frac{\epsilon(n\epsilon-\zeta)(2-\epsilon\zeta-n\epsilon^2)}{(1-\epsilon\zeta)^2}(1-\epsilon^2)+\epsilon^2}. \tag{6.98}$$

For $\zeta\in[0,\epsilon]$ and $n\epsilon$ sufficiently small, by Taylor's expansion, we have

$$1-\epsilon\zeta = 1+o(\epsilon), \tag{6.99}$$

$$2-\epsilon\zeta-n\epsilon^2 = 2+o(\epsilon). \tag{6.100}$$

Hence, we have

$$v_\eta(\zeta) = \sqrt{\epsilon(\epsilon+2n\epsilon-2\zeta)}+o(\epsilon^2). \tag{6.101}$$

Since $\sqrt{\epsilon(\epsilon+2n\epsilon-2\zeta)}=O(\epsilon)$, we can further estimate

$$\frac{1}{v_\eta(\zeta)} = \frac{1}{\sqrt{\epsilon(\epsilon+2n\epsilon-2\zeta)}}+o(1) \tag{6.102}$$

$$-\int_\kappa^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta = \nu(1)\sqrt{\frac{\epsilon+2n\epsilon-2\zeta}{\epsilon}}\bigg|_\kappa^{n\epsilon}+o(\epsilon) = \nu(1)\bigg(1-\sqrt{\frac{\epsilon+2n\epsilon-2\kappa}{\epsilon}}\bigg)+o(\epsilon). \tag{6.103}$$

Then we can easily derive the integral estimate

$$\begin{aligned}\int_0^{n\epsilon} \mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta}\frac{1}{v_\eta(\kappa)}\mathrm{d}\kappa &= \mathrm{e}^{\nu(1)}\int_0^{n\epsilon}\mathrm{e}^{-\nu(1)\sqrt{\frac{\epsilon+2n\epsilon-2\kappa}{\epsilon}}}\frac{1}{\sqrt{\epsilon(\epsilon+2n\epsilon-2\kappa)}}\mathrm{d}\kappa+o(\epsilon) \\ &= \frac{1}{2}\mathrm{e}^{\nu(1)}\int_\epsilon^{(1+2n)\epsilon}\mathrm{e}^{-\nu(1)\sqrt{\frac{\sigma}{\epsilon}}}\frac{1}{\sqrt{\epsilon\sigma}}\mathrm{d}\sigma+o(\epsilon) \\ &= \frac{1}{2}\mathrm{e}^{\nu(1)}\int_1^{1+2n}\mathrm{e}^{-\nu(1)\sqrt{\rho}}\frac{1}{\sqrt{\rho}}\mathrm{d}\rho+o(\epsilon) \\ &= \mathrm{e}^{\nu(1)}\int_1^{\sqrt{1+2n}}\mathrm{e}^{-\nu(1)t}\mathrm{d}t+o(\epsilon) \\ &= \frac{1}{\nu(1)}(1-\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})})+o(\epsilon).\end{aligned} \tag{6.104}$$

Then for $0<\epsilon\leq\eta_0$, we have $|K[\mathscr{W}^\epsilon](0,0,1)-K[\mathscr{W}^\epsilon](\kappa),v_\eta(\kappa),v_\phi(\epsilon)|\leq\delta$, which implies

$$\begin{aligned}&\int_0^{n\epsilon} \mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta}\frac{1}{v_\eta(\kappa)}K[\mathscr{W}^\epsilon](\kappa,v_\eta(\kappa),v_\phi(\epsilon))\mathrm{d}\kappa \\ =& \int_0^{n\epsilon} \mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta}\frac{1}{v_\eta(\kappa)}K[\mathscr{W}^\epsilon](0,0,1)\mathrm{d}\kappa+O(\delta)+O(\epsilon) \\ =& \frac{1}{\nu(1)}(1-\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})})K[\mathscr{W}^\epsilon](0,0,1)+O(\epsilon)+O(\delta).\end{aligned} \tag{6.105}$$

For the boundary data term, since $h(v_\eta,v_\phi)$ is $C^1$, a similar argument shows

$$h(\epsilon_0,\sqrt{1-\epsilon_0^2})\mathrm{e}^{-\int_0^{n\epsilon}\frac{\nu(1)}{v_\eta(\zeta)}\mathrm{d}\zeta}=\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})}h(\sqrt{1+2n}\epsilon,\sqrt{1-(1+2n)\epsilon^2})+O(\epsilon). \tag{6.106}$$

Therefore, we have

$$\mathscr{W}^\epsilon(n\epsilon,\epsilon)=\frac{1}{\nu(1)}(1-\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})})K[\mathscr{W}^\epsilon](0,0,1)+\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})}h(0,1)+O(\epsilon)+O(\delta). \tag{6.107}$$

Step 5: Estimate of Difference.
Collecting all above, we can estimate the value at point $\eta=n\epsilon$, $v_\eta=\epsilon$ and $v_\phi=\sqrt{1-\epsilon^2}$ as

$$\mathscr{W}^\epsilon(n\epsilon,\epsilon,\sqrt{1-\epsilon^2})=\frac{1}{\nu(1)}(1-\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})})K[\mathscr{W}^\epsilon](0,0,1)+\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})}h(0,1)+O(\epsilon)+O(\delta), \tag{6.108}$$

$$\mathscr{W}(n\epsilon,\epsilon,\sqrt{1-\epsilon^2})=\frac{1}{\nu(1)}(1-\mathrm{e}^{-\nu(1)n})K[\mathscr{W}](0,0,1)+\mathrm{e}^{-\nu(1)n}h(0,1)+O(\epsilon)+O(\delta). \tag{6.109}$$

By our assumptions on $h$ and a similar argument as in the proof of Theorem 6.1, we know $|K[\mathscr{W}^\epsilon](0,0,1)|<<1$ and $|K[\mathscr{W}](0,0,1)|<<1$. However, $h(0,1)=1$. Since $n$ is arbitrary and $\mathrm{e}^{\nu(1)(1-\sqrt{1+2n})}\neq\mathrm{e}^{-\nu(1)n}$, we always have

$$\left|\mathscr{W}^\epsilon(\epsilon,n\epsilon,\sqrt{1-n^2\epsilon^2})-\mathscr{W}(\epsilon,n\epsilon,\sqrt{1-n^2\epsilon^2})\right|\geq C>0, \tag{6.110}$$

which further implies

$$|||\mathscr{W}^\epsilon-\mathscr{W}|||_{L^\infty L^\infty_{0,0}}\geq C>0. \tag{6.111}$$

□

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## References

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