DIFFUSIVE LIMIT WITH GEOMETRIC CORRECTION OF UNSTEADY NEUTRON TRANSPORT EQUATION

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Abstract. We consider the diffusive limit of an unsteady neutron transport equation in a two-dimensional plate with one-speed velocity. We show the solution can be approximated by the sum of interior solution, initial layer, and boundary layer with geometric correction. Also, we construct a counterexample to the classical theory in [1] which states the behavior of solution near boundary can be described by the Knudsen layer derived from the Milne problem.
Keywords: compatibility condition, ϵ-Milne problem, Knudsen layer, geometric correction.

1. Introduction and Notation

1.1. **Problem Formulation.** We consider a homogeneous isotropic unsteady neutron transport equation in a two-dimensional unit plate $\Omega = \{\vec{x} = (x_1, x_2) : \ |\vec{x}| \leq 1\}$ with one-speed velocity $\Sigma = \{\vec{w} = (w_1, w_2) : \vec{w} \in \mathcal{S}^1\}$ as

$$\left\{\begin{array}{rcl}\epsilon^2\partial_t u^\epsilon + \epsilon\vec{w}\cdot\nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon &=& 0 \quad \text{in} \quad [0,\infty)\times\Omega,\\ u^\epsilon(0,\vec{x},\vec{w}) &=& h(\vec{x},\vec{w}) \quad \text{in} \quad \Omega\\ u^\epsilon(t,\vec{x}_0,\vec{w}) &=& g(t,\vec{x}_0,\vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{n}<0 \quad \text{and} \quad \vec{x}_0\in\partial\Omega,\end{array}\right. \tag{1.1}$$

where

$$\bar{u}^\epsilon(t,\vec{x}) = \frac{1}{2\pi}\int_{\mathcal{S}^1} u^\epsilon(t,\vec{x},\vec{w})\mathrm{d}\vec{w}. \tag{1.2}$$

and $\vec{n}$ is the outward normal vector on $\partial\Omega$, with the Knudsen number $0 < \epsilon << 1$. The initial and boundary data satisfy the compatibility condition

$$h(\vec{x}_0,\vec{w}) = g(0,\vec{x}_0,\vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{n}<0 \quad \text{and} \quad \vec{x}_0\in\partial\Omega. \tag{1.3}$$

We intend to study the diffusive limit of u^ϵ as $\epsilon \to 0$.

Based on the flow direction, we can divide the boundary $\Gamma = \{(\vec{x},\vec{w}) : \ \vec{x} \in \partial\Omega\}$ into the in-flow boundary Γ^-, the out-flow boundary Γ^+, and the grazing set Γ^0 as

$$\Gamma^- = \{(\vec{x},\vec{w}) : \ \vec{x}\in\partial\Omega, \ \vec{w}\cdot\vec{n}<0\}, \tag{1.4}$$

$$\Gamma^+ = \{(\vec{x},\vec{w}) : \ \vec{x}\in\partial\Omega, \ \vec{w}\cdot\vec{n}>0\}, \tag{1.5}$$

$$\Gamma^0 = \{(\vec{x},\vec{w}) : \ \vec{x}\in\partial\Omega, \ \vec{w}\cdot\vec{n}=0\}. \tag{1.6}$$

It is easy to see $\Gamma = \Gamma^+ \cup \Gamma^- \cup \Gamma^0$.

The study of neutron transport equation dates back to 1950s. The main methods include the explicit formula and spectral analysis of the transport operators(see [5], [4], [6], [7], [8], [9], [10], [11], [12]). In the classical paper [1], a systematic construction of boundary layer was provided via Milne problem. However, this construction was proved to be problematic for steady equation in [13] and a new boundary layer construction based on ϵ-Milne problem with geometric correction was presented. In this paper, we extend this result to unsteady equation and consider a more complicated case with initial layer involved.

1.2. **Main Results.** We first present the well-posedness of the equation (1.1).

Theorem 1.1. *Assume $g(t,x_0,\vec{w}) \in L^\infty([0,\infty)\times\Gamma^-)$ and $h(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathcal{S}^1)$. Then for the unsteady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(t,\vec{x},\vec{w}) \in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying*

$$\|u^\epsilon\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} \leq C(\Omega)\bigg(\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|g\|_{L^\infty([0,\infty)\times\Gamma^-)}\bigg). \tag{1.7}$$

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Then we can show the diffusive limit of the equation (1.1).

Theorem 1.2. *Assume $g(t,\vec x_0,\vec w)\in C^2([0,\infty)\times\Gamma^-)$ and $h(\vec x,\vec w)\in C^2(\Omega\times\mathcal{S}^1)$. Then for the unsteady neutron transport equation (1.1), the unique solution $u^\epsilon(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfies*

$$\left\|u^\epsilon-U_0^\epsilon-\mathscr{U}_{I,0}^\epsilon-\mathscr{U}_{B,0}^\epsilon\right\|_{L^\infty}=o(1), \tag{1.8}$$

where the interior solution U_0^ϵ is defined in (3.56), the initial layer $\mathscr{U}_{I,0}^\epsilon$ is defined in (3.55), and the boundary layer $\mathscr{U}_{B,0}^\epsilon$ is defined in (3.54). Moreover, if $g(t,\theta,\phi)=t^2\mathrm{e}^{-t}\cos\phi$ and $h(\vec x,\vec w)=0$, then there exists a $C>0$ such that

$$\left\|u^\epsilon-U_0-\mathscr{U}_{I,0}-\mathscr{U}_{B,0}\right\|_{L^\infty}\geq C>0 \tag{1.9}$$

when ϵ is sufficiently small, where the interior solution U_0 is defined in (6.19), the initial layer $\mathscr{U}_{I,0}$ is defined in (6.18), and the boundary layer $\mathscr{U}_{B,0}^\epsilon$ is defined in (6.17).

Remark 1.3. *θ and ϕ are defined in (3.34) and (3.39).*

It is easy to see, by a similar argument, the results in Theorem 1.1 and Theorem 1.2 also hold for the one-dimensional unsteady neutron transport equation, where the temporal domain is $[0,\infty)$, spacial domain is $[0,L]$ for fixed $L>0$, and velocity domain is $[-1/2,1/2]$.

1.3. **Notation and Structure of This Paper.** Throughout this paper, $C>0$ denotes a constant that only depends on the parameter Ω, but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a\lesssim b$ to denote $a\leq Cb$.

Our paper is organized as follows: in Section 2, we establish the L^∞ well-posedness of the equation (1.1) and prove Theorem 1.1; in Section 3, we present the asymptotic analysis of the equation (1.1); in Section 4, we give the main results of the ϵ-Milne problem with geometric correction; in Section 5, we prove the first part of Theorem 1.2; finally, in Section 6, we prove the second part of Theorem 1.2.

2. Well-posedness of Unsteady Neutron Transport Equation

In this section, we consider the well-posedness of the unsteady neutron transport equation

$$\left\{\begin{array}{rcl}\epsilon^2\partial_t u+\epsilon\vec w\cdot\nabla_x u+u-\bar u&=&f(t,\vec x,\vec w)\ \text{ in }\ [0,\infty)\times\Omega,\\ u(0,\vec x,\vec w)&=&h(\vec x,\vec w)\ \text{ in }\ \Omega\\ u(t,\vec x_0,\vec w)&=&g(t,\vec x_0,\vec w)\ \text{ for }\ \vec w\cdot\vec n<0\ \text{ and }\ \vec x_0\in\partial\Omega,\end{array}\right. \tag{2.1}$$

where the initial and boundary data satisfy the compatibility condition

$$h(\vec x_0,\vec w)=g(0,\vec x_0,\vec w)\ \text{ for }\ \vec w\cdot\vec n<0\ \text{ and }\ \vec x_0\in\partial\Omega. \tag{2.2}$$

We define the L^2 and L^∞ norms in $\Omega\times\mathcal{S}^1$ as usual:

$$\|f\|_{L^2(\Omega\times\mathcal{S}^1)}=\left(\int_\Omega\int_{\mathcal{S}^1}|f(\vec x,\vec w)|^2\,\mathrm{d}\vec w\mathrm{d}\vec x\right)^{1/2}, \tag{2.3}$$

$$\|f\|_{L^\infty(\Omega\times\mathcal{S}^1)}=\sup_{(\vec x,\vec w)\in\Omega\times\mathcal{S}^1}|f(\vec x,\vec w)|. \tag{2.4}$$

Define the L^2 and L^∞ norms on the boundary as follows:

$$\|f\|_{L^2(\Gamma)}=\left(\iint_\Gamma|f(\vec x,\vec w)|^2\,|\vec w\cdot\vec n|\,\mathrm{d}\vec w\mathrm{d}\vec x\right)^{1/2}, \tag{2.5}$$

$$\|f\|_{L^2(\Gamma^\pm)}=\left(\iint_{\Gamma^\pm}|f(\vec x,\vec w)|^2\,|\vec w\cdot\vec n|\,\mathrm{d}\vec w\mathrm{d}\vec x\right)^{1/2}, \tag{2.6}$$

$$\|f\|_{L^\infty(\Gamma)}=\sup_{(\vec x,\vec w)\in\Gamma}|f(\vec x,\vec w)|, \tag{2.7}$$

$$\|f\|_{L^\infty(\Gamma^\pm)}=\sup_{(\vec x,\vec w)\in\Gamma^\pm}|f(\vec x,\vec w)|. \tag{2.8}$$

Similar notation also applies to the space $[0,\infty)\times\Omega\times\mathcal{S}^1$, $[0,\infty)\times\Gamma$, and $[0,\infty)\times\Gamma^\pm$.

2.1. **Preliminaries.** In order to show the L^2 and L^∞ well-posedness of the equation (2.1), we start with some preparations of the penalized neutron transport equation.

Lemma 2.1. *Assume $f(t,\vec{x},\vec{w})\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$, $h(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(t,x_0,\vec{w})\in L^\infty([0,\infty)\times\Gamma^-)$. Then for the penalized transport equation*

$$
\left\{
\begin{array}{rcl}
\lambda u_\lambda+\epsilon^2\partial_t u_\lambda+\epsilon\vec{w}\cdot\nabla_x u_\lambda+u_\lambda &=& f(t,\vec{x},\vec{w})\ \ \text{in}\ \ [0,\infty)\times\Omega,\\
u_\lambda(0,\vec{x},\vec{w}) &=& h(\vec{x},\vec{w})\ \ \text{in}\ \ \Omega\\
u_\lambda(t,\vec{x}_0,\vec{w}) &=& g(t,\vec{x}_0,\vec{w})\ \ \text{for}\ \ \vec{w}\cdot\vec{n}<0\ \ \text{and}\ \ \vec{x}_0\in\partial\Omega.
\end{array}
\right. \tag{2.9}
$$

with $\lambda>0$ as a penalty parameter, there exists a solution $u_\lambda(t,\vec{x},\vec{w})\in L^\infty([0,T]\times\Omega\times\mathcal{S}^1)$ satisfying

$$
\|u_\lambda\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\|g\|_{L^\infty([0,\infty)\times\Gamma^-)}+\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}. \tag{2.10}
$$

Proof. The characteristics $(T(s),X(s),W(s))$ of the equation (2.9) which goes through $(t,\vec{x},\vec{w})$ is defined by

$$
\left\{
\begin{array}{rcl}
(T(0),X(0),W(0)) &=& (t,\vec{x},\vec{w})\\
\dfrac{\mathrm{d}T(s)}{\mathrm{d}s} &=& \epsilon^2,\\
\dfrac{\mathrm{d}X(s)}{\mathrm{d}s} &=& \epsilon W(s),\\
\dfrac{\mathrm{d}W(s)}{\mathrm{d}s} &=& 0.
\end{array}
\right. \tag{2.11}
$$

which implies

$$
\left\{
\begin{array}{rcl}
T(s) &=& t+\epsilon^2 s,\\
X(s) &=& \vec{x}+(\epsilon\vec{w})s,\\
W(s) &=& \vec{w},
\end{array}
\right. \tag{2.12}
$$

Hence, we can rewrite the equation (2.9) along the characteristics as

$$
\begin{aligned}
&u_\lambda(t,\vec{x},\vec{w}) \qquad (2.13)\\
=\ &\mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(g(t-\epsilon^2t_b,\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)t_b}+\int_0^{t_b}f(t-\epsilon^2(t_b-s),\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s\bigg)\\
&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(h(\vec{x}-(\epsilon t\vec{w})/\epsilon^2,\vec{w})\mathrm{e}^{-(1+\lambda)t/\epsilon^2}+\int_0^{t/\epsilon^2}f(\epsilon^2s,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w})\mathrm{e}^{-(1+\lambda)(t/\epsilon^2-s)}\mathrm{d}s\bigg),
\end{aligned}
$$

where the backward exit time t_b is defined as

$$
t_b(\vec{x},\vec{w})=\inf\{s\geq0:(\vec{x}-\epsilon s\vec{w},\vec{w})\in\Gamma^-\}. \tag{2.14}
$$

Then we can naturally estimate

$$
\begin{aligned}
&\|u_\lambda\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\\
\leq\ &\mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(\mathrm{e}^{-(1+\lambda)t_b}\|g\|_{L^\infty([0,\infty)\times\Gamma^-)}+\frac{1-\mathrm{e}^{(1+\lambda)t_b}}{1+\lambda}\|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\bigg)\\
&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(\mathrm{e}^{-(1+\lambda)t/\epsilon^2}\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1-\mathrm{e}^{(1+\lambda)t/\epsilon^2}}{1+\lambda}\|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\bigg)\\
\leq\ &\|g\|_{L^\infty([0,\infty)\times\Gamma^-)}+\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}.
\end{aligned} \tag{2.15}
$$

Since u_λ can be explicitly traced back to the initial or boundary data, the existence naturally follows from above estimate. $\square$

Lemma 2.2. *Assume $f(t,\vec{x},\vec{w})\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$, $h(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(t,x_0,\vec{w})\in L^\infty([0,\infty)\times\Gamma^-)$. Then for the penalized neutron transport equation*

$$
\left\{
\begin{array}{rcl}
\lambda u_\lambda+\epsilon^2\partial_t u_\lambda+\epsilon\vec{w}\cdot\nabla_x u_\lambda+u_\lambda-\bar{u}_\lambda &=& f(t,\vec{x},\vec{w})\ \ \text{in}\ \ [0,\infty)\times\Omega,\\
u_\lambda(0,\vec{x},\vec{w}) &=& h(\vec{x},\vec{w})\ \ \text{in}\ \ \Omega\\
u_\lambda(t,\vec{x}_0,\vec{w}) &=& g(t,\vec{x}_0,\vec{w})\ \ \text{for}\ \ \vec{x}_0\in\partial\Omega\ \ \text{and}\ \vec{w}\cdot\vec{n}<0.
\end{array}
\right. \tag{2.16}
$$

with $\lambda>0$*, there exists a solution* $u_{\lambda}(t,\vec x,\vec w)\in L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)$ *satisfying*

$$\|u_{\lambda}\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\frac{1+\epsilon}{\lambda}\bigg(\|g\|_{L^{\infty}([0,\infty)\times\Gamma^-)}+\|h\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\|f\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\bigg).\tag{2.17}$$

Proof. We define an approximating sequence $\{u_{\lambda}^k\}_{k=0}^{\infty}$, where $u_{\lambda}^0=0$ and

$$\left\{\begin{array}{rcl}\lambda u_{\lambda}^k+\epsilon^2\partial_tu_{\lambda}^k+\epsilon\vec w\cdot\nabla_xu_{\lambda}^k+u_{\lambda}^k-\bar u_{\lambda}^{k-1}&=&f(t,\vec x,\vec w)\ \text{ in }\ [0,\infty)\times\Omega,\\ u_{\lambda}^k(0,\vec x,\vec w)&=&h(\vec x,\vec w)\ \text{ in }\ \Omega\\ u_{\lambda}^k(t,\vec x_0,\vec w)&=&g(t,\vec x_0,\vec w)\ \text{ for }\ \vec x_0\in\partial\Omega\ \text{ and }\vec w\cdot\vec n<0.\end{array}\right.\tag{2.18}$$

By Lemma 2.1, this sequence is well-defined and $\left\|u_{\lambda}^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}<\infty$.

The characteristics and the backward exit time are defined as (2.11) and (2.14), so we rewrite equation (2.18) along the characteristics as

(2.19)

$$\begin{aligned}&u_{\lambda}^k(t,\vec x,\vec w)\\=\;&\mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(g(t-\epsilon^2t_b,\vec x-\epsilon t_b\vec w,\vec w)\mathrm{e}^{-(1+\lambda)t_b}+\int_0^{t_b}(\bar u_{\lambda}^{k-1}+f)(t-\epsilon^2(t_b-s),\vec x-\epsilon(t_b-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s\bigg)\\&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(h(\vec x-(\epsilon t\vec w)/\epsilon^2,\vec w)\mathrm{e}^{-(1+\lambda)t/\epsilon^2}+\int_0^{t/\epsilon^2}(\bar u_{\lambda}^{k-1}+f)(\epsilon^2s,\vec x-\epsilon(t/\epsilon^2-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t/\epsilon^2-s)}\mathrm{d}s\bigg),\end{aligned}$$

We define the difference $v^k=u_{\lambda}^k-u_{\lambda}^{k-1}$ for $k\geq1$. Then v^k satisfies

$$\begin{aligned}v^{k+1}(\vec x,\vec w)\;=\;&\mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(\int_0^{t_b}\bar v_{\lambda}^{k-1}(t-\epsilon^2(t_b-s),\vec x-\epsilon(t_b-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s\bigg)\\&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(\int_0^{t/\epsilon^2}\bar v_{\lambda}^{k-1}(\epsilon^2s,\vec x-\epsilon(t/\epsilon^2-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t/\epsilon^2-s)}\mathrm{d}s\bigg),\end{aligned}\tag{2.20}$$

Since $\left\|\bar v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\left\|v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}$, we can directly estimate

$$\begin{aligned}\left\|v^{k+1}\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}&\leq\left\|v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\int_0^{\max\{t/\epsilon^2,t_b\}}\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s\\&\leq\frac{1-\mathrm{e}^{-(1+\lambda)\max\{t/\epsilon^2,t_b\}}}{1+\lambda}\left\|v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}.\end{aligned}\tag{2.21}$$

Hence, we naturally have

$$\left\|v^{k+1}\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\frac{1}{1+\lambda}\left\|v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}.\tag{2.22}$$

Thus, this is a contraction sequence for $\lambda>0$. Considering $v^1=u_{\lambda}^1$, we have

$$\left\|v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\left(\frac{1}{1+\lambda}\right)^{k-1}\left\|u_{\lambda}^1\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)},\tag{2.23}$$

for $k\geq1$. Therefore, u_{λ}^k converges strongly in L^{∞} to a limit solution u_{λ} satisfying

$$\|u_{\lambda}\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\sum_{k=1}^{\infty}\left\|v^k\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\frac{1+\lambda}{\lambda}\left\|u_{\lambda}^1\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}.\tag{2.24}$$

Since u_{λ}^1 can be rewritten along the characteristics as

(2.25) $u_{\lambda}^1(\vec x,\vec w)$

$$\begin{aligned}=\;&\mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(g(t-\epsilon^2t_b,\vec x-\epsilon t_b\vec w,\vec w)\mathrm{e}^{-(1+\lambda)t_b}+\int_0^{t_b}f(t-\epsilon^2(t_b-s),\vec x-\epsilon(t_b-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s\bigg)\\&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(h(\vec x-(\epsilon t\vec w)/\epsilon^2,\vec w)\mathrm{e}^{-(1+\lambda)t/\epsilon^2}+\int_0^{t/\epsilon^2}f(\epsilon^2s,\vec x-\epsilon(t/\epsilon^2-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t/\epsilon^2-s)}\mathrm{d}s\bigg),\end{aligned}$$

based on Lemma 2.1, we can directly estimate

$$\left\|u_\lambda^1\right\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} \leq \|g\|_{L^\infty([0,\infty)\times\Gamma^-)} + \|h\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}. \tag{2.26}$$

Combining (2.24) and (2.26), we can easily deduce the lemma. □

2.2. L^2 **Estimate.** It is easy to see when $\lambda\to 0$, the estimate in Lemma 2.2 blows up. Hence, we need to show a uniform estimate of the solution to the penalized neutron transport equation (2.16).

Lemma 2.3. *(Green's Identity) Assume $f(t,\vec x,\vec w),\ g(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ and $\partial_t f+\vec w\cdot\nabla_x f,\ \partial_t g+\vec w\cdot\nabla_x g\in L^2([0,\infty)\times\Omega\times\mathcal{S}^1)$ with $f,\ g\in L^2([0,\infty)\times\Gamma)$. Then for almost all $s,t\in[0,\infty)$,*

$$\begin{aligned}\int_s^t\iint_{\Omega\times\mathcal{S}^1}\bigg((\partial_t f+\vec w\cdot\nabla_x f)g+(\partial_t g+\vec w\cdot\nabla_x g)f\bigg)\mathrm{d}\vec x\mathrm{d}\vec w\mathrm{d}r \\ = \int_s^t\int_\Gamma fg\mathrm{d}\gamma\mathrm{d}r+\iint_{\Omega\times\mathcal{S}^1}f(t)g(t)\mathrm{d}\vec x\mathrm{d}\vec w-\iint_{\Omega\times\mathcal{S}^1}f(s)g(s)\mathrm{d}\vec x\mathrm{d}\vec w,\end{aligned}\tag{2.27}$$

where $\mathrm{d}\gamma=(\vec w\cdot\vec n)\mathrm{d}s$ on the boundary.

Proof. See [2, Chapter 9] and [3]. □

Lemma 2.4. *The solution u_λ to the equation (2.16) satisfies the uniform estimate in time interval $[s,t]$,*

$$\begin{aligned}\epsilon\left\|\bar u_\lambda\right\|_{L^2([s,t]\times\Omega\times\mathcal{S}^1)} \leq{}& C(\Omega)\bigg(\left\|u_\lambda-\bar u_\lambda\right\|_{L^2([s,t]\times\Omega\times\mathcal{S}^1)}+\|f\|_{L^2([s,t]\times\Omega\times\mathcal{S}^1)}+\epsilon\left\|u_\lambda\right\|_{L^2([s,t]\times\Gamma^+)} \\ &+\epsilon\|g\|_{L^2([s,t]\times\Gamma^-)}\bigg)+\epsilon^2G(t)-\epsilon^2G(s),\end{aligned}\tag{2.28}$$

where $G(t)$ is a function satisfying

$$G(t)\leq C(\Omega)\left\|u_\lambda(t)\right\|_{L^2(\Omega\times\mathcal{S}^1)},\tag{2.29}$$

for $0\leq\lambda<<1$ and $0<\epsilon<<1$.

Proof. We divide the proof into several steps:

Step 1:
Applying Lemma 2.3 to the solution of the equation (2.16). Then for any $\phi\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying $\epsilon\partial_t\phi+\vec w\cdot\nabla_x\phi\in L^2([0,\infty)\times\Omega\times\mathcal{S}^1)$ and $\phi\in L^2([0,\infty)\times\Gamma)$, we have

$$\begin{aligned}\lambda\int_s^t\iint_{\Omega\times\mathcal{S}^1}u_\lambda\phi-\epsilon^2\int_s^t\iint_{\Omega\times\mathcal{S}^1}\partial_t\phi u_\lambda-\epsilon\int_s^t\iint_{\Omega\times\mathcal{S}^1}(\vec w\cdot\nabla_x\phi)u_\lambda+\int_s^t\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar u_\lambda)\phi \\ = -\epsilon\int_s^t\int_\Gamma u_\lambda\phi\mathrm{d}\gamma-\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t)\phi(t)+\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(s)\phi(s)+\int_s^t\iint_{\Omega\times\mathcal{S}^1}f\phi.\end{aligned}\tag{2.30}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function $\zeta(t)$. Since $u_\lambda(t)\in L^\infty(\Omega\times\mathcal{S}^1)$, it naturally implies $\bar u_\lambda(t)\in L^\infty(\Omega)$ which further leads to $\bar u_\lambda(t)\in L^2(\Omega)$. We define $\zeta(t,\vec x)$ on Ω satisfying

$$\left\{\begin{array}{rcl}\Delta\zeta(t)&=&\bar u_\lambda(t)\ \text{ in }\ \Omega,\\ \zeta(t)&=&0\ \text{ on }\ \partial\Omega.\end{array}\right.\tag{2.31}$$

In the bounded domain Ω, based on the standard elliptic estimate, we have

$$\|\zeta(t)\|_{H^2(\Omega)}\leq C(\Omega)\left\|\bar u_\lambda(t)\right\|_{L^2(\Omega)}.\tag{2.32}$$

Step 2:
Without loss of generality, we only prove the case with $s=0$. We plug the test function

$$\phi(t)=-\vec w\cdot\nabla_x\zeta(t)\tag{2.33}$$

into the weak formulation (2.30) and estimate each term there. Naturally, we have

$$\|\phi(t)\|_{L^2(\Omega)}\leq C\|\zeta(t)\|_{H^1(\Omega)}\leq C(\Omega)\left\|\bar u_\lambda(t)\right\|_{L^2(\Omega)}.\tag{2.34}$$

Easily we can decompose

$$-\epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u_\lambda = -\epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)\bar{u}_\lambda-\epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)(u_\lambda-\bar{u}_\lambda). \tag{2.35}$$

We estimate the two term on the right-hand side of (2.35) separately. By (2.31) and (2.33), we have

$$\begin{aligned}-\epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)\bar{u}_\lambda &= \epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}\bar{u}_\lambda\bigg(w_1(w_1\partial_{11}\zeta+w_2\partial_{12}\zeta)+w_2(w_1\partial_{12}\zeta+w_2\partial_{22}\zeta)\bigg)\\ &= \epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}\bar{u}_\lambda\bigg(w_1^2\partial_{11}\zeta+w_2^2\partial_{22}\zeta\bigg)\\ &= \epsilon\pi\int_0^t\int_\Omega\bar{u}_\lambda(\partial_{11}\zeta+\partial_{22}\zeta)\\ &= \epsilon\pi\left\|\bar{u}_\lambda\right\|^2_{L^2([0,t]\times\Omega)}\\ &= \frac{1}{2}\epsilon\left\|\bar{u}_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{2.36}$$

In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathcal{S}^1$. On the other hand, for the second term in (2.35), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned}-\epsilon\int_0^t\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)(u_\lambda-\bar{u}_\lambda) &\leq C(\Omega)\epsilon\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\bigg(\int_0^t\left\|\zeta\right\|^2_{H^2(\Omega)}\bigg)^{1/2}\\ &\leq C(\Omega)\epsilon\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\left\|\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{2.37}$$

Based on (2.32), (2.34), the boundary condition of the penalized neutron transport equation (2.16), the trace theorem, Hölder's inequality and the elliptic estimate, we have

$$\begin{aligned}\epsilon\int_0^t\int_\Gamma u_\lambda\phi\,\mathrm{d}\gamma &= \epsilon\int_0^t\int_{\Gamma^+}u_\lambda\phi\,\mathrm{d}\gamma+\epsilon\int_0^t\int_{\Gamma^-}u_\lambda\phi\,\mathrm{d}\gamma\\ &\leq C(\Omega)\bigg(\epsilon\left\|u_\lambda\right\|_{L^2([0,t]\times\Gamma^+)}\left\|\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\epsilon\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\left\|\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\bigg),\end{aligned} \tag{2.38}$$

$$\begin{aligned}\lambda\int_0^t\iint_{\Omega\times\mathcal{S}^1}u_\lambda\phi &= \lambda\int_0^t\iint_{\Omega\times\mathcal{S}^1}\bar{u}_\lambda\phi+\lambda\int_0^t\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi=\lambda\int_0^t\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi\\ &\leq C(\Omega)\lambda\left\|\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)},\end{aligned} \tag{2.39}$$

$$\int_0^t\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi\leq C(\Omega)\left\|\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}, \tag{2.40}$$

$$\int_0^t\iint_{\Omega\times\mathcal{S}^1}f\phi\leq C(\Omega)\left\|\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\left\|f\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}. \tag{2.41}$$

Note that we will take

$$-\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t)\phi(t)+\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(0)\phi(0)=\epsilon^2\bigg(G(t)-G(0)\bigg), \tag{2.42}$$

where $G(t)=-\displaystyle\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t)\phi(t)$. Then the only remaining term is

$$\begin{aligned}-\epsilon^2\int_0^t\iint_{\Omega\times\mathcal{S}^1}\partial_t\phi u_\lambda &= -\epsilon^2\int_s^t\iint_{\Omega\times\mathcal{S}^1}\partial_t\phi(u_\lambda-\bar{u}_\lambda)\\ &\leq \left\|\partial_t\nabla\zeta\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\left\|u_\lambda-\bar{u}_\lambda\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{2.43}$$

Now we have to tackle $\left\|\partial_t\nabla\zeta\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}$.

Step 3:
For test function $\phi(\vec{x},\vec{w})$ which is independent of time t, in time interval $[t-\delta,t]$ the weak formulation in (2.30) can be simplified as

$$\begin{aligned}
&\lambda\int_{t-\delta}^{t}\iint_{\Omega\times\mathcal{S}^1}u_\lambda\phi-\epsilon\int_{t-\delta}^{t}\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u_\lambda+\int_{t-\delta}^{t}\iint_{\Omega\times\mathcal{S}^1}(u_\lambda-\bar{u}_\lambda)\phi \\
=\;&-\epsilon\int_{t-\delta}^{t}\int_{\Gamma}u_\lambda\phi\mathrm{d}\gamma-\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t)\phi+\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t-\delta)\phi+\int_{t-\delta}^{t}\iint_{\Omega\times\mathcal{S}^1}f\phi.
\end{aligned}\tag{2.44}$$

Taking difference quotient as $\delta\to0$, we know

$$\frac{\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t)\phi-\epsilon^2\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t-\delta)\phi}{\delta}\to\epsilon^2\iint_{\Omega\times\mathcal{S}^1}\partial_t u_\lambda(t)\phi.\tag{2.45}$$

Then (2.44) can be simplified into

$$\begin{aligned}
&\epsilon^2\iint_{\Omega\times\mathcal{S}^1}\partial_t u_\lambda(t)\phi \\
=\;&-\lambda\iint_{\Omega\times\mathcal{S}^1}u_\lambda(t)\phi+\epsilon\iint_{\Omega\times\mathcal{S}^1}(\vec{w}\cdot\nabla_x\phi)u_\lambda(t)-\iint_{\Omega\times\mathcal{S}^1}(u_\lambda(t)-\bar{u}_\lambda(t))\phi \\
&-\epsilon\int_{\Gamma}u_\lambda(t)\phi\mathrm{d}\gamma+\iint_{\Omega\times\mathcal{S}^1}f(t)\phi.
\end{aligned}\tag{2.46}$$

For fixed t, taking $\phi=\Phi(\vec{x})$ which satisfies

$$\left\{\begin{array}{rcl}\Delta\Phi&=&\partial_t\bar{u}_\lambda(t)\ \text{ in }\ \Omega,\\ \Phi&=&0\ \text{ on }\ \partial\Omega,\end{array}\right.\tag{2.47}$$

which further implies $\Phi=\partial_t\zeta$. Then the left-hand side of (2.46) is actually

$$\begin{aligned}
LHS&=\epsilon^2\iint_{\Omega\times\mathcal{S}^1}\Phi\partial_t u_\lambda(t)=\epsilon^2\iint_{\Omega\times\mathcal{S}^1}\Phi\partial_t\bar{u}_\lambda \\
&=\epsilon^2\iint_{\Omega\times\mathcal{S}^1}\Phi\Delta\Phi=\epsilon^2\iint_{\Omega\times\mathcal{S}^1}|\nabla\Phi|^2 \\
&=\|\partial_t\nabla\zeta(t)\|^2_{L^2(\Omega\times\mathcal{S}^1)}.
\end{aligned}\tag{2.48}$$

By a similar argument as in Step 2 and the Poincaré inequality, the right-hand side of (2.46) can be bounded as

$$RHS\lesssim\|\partial_t\nabla\zeta(t)\|_{L^2(\Omega\times\mathcal{S}^1)}\bigg(\|u_\lambda(t)-\bar{u}_\lambda(t)\|_{L^2(\Omega\times\mathcal{S}^1)}+\lambda\|\bar{u}_\lambda(t)\|_{L^2(\Omega\times\mathcal{S}^1)}+\|f(t)\|_{L^2(\Omega\times\mathcal{S}^1)}\bigg).\tag{2.49}$$

Therefore, we have

$$\|\partial_t\nabla\zeta(t)\|_{L^2(\Omega\times\mathcal{S}^1)}\lesssim\|u_\lambda(t)-\bar{u}_\lambda(t)\|_{L^2(\Omega\times\mathcal{S}^1)}+\lambda\|\bar{u}_\lambda(t)\|_{L^2(\Omega\times\mathcal{S}^1)}+\|f(t)\|_{L^2(\Omega\times\mathcal{S}^1)}.\tag{2.50}$$

For all t, we can further integrate over $[0,t]$ to obtain

$$\begin{aligned}
&\|\partial_t\nabla\zeta(t)\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)} \\
\lesssim\;&\|u_\lambda(t)-\bar{u}_\lambda(t)\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\lambda\|\bar{u}_\lambda(t)\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\|f(t)\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}.
\end{aligned}\tag{2.51}$$

Step 4:
Collecting terms in (2.36), (2.37), (2.38), (2.39), (2.40), (2.41), (2.42), (2.43), and (2.51), we obtain

$$\begin{aligned}
&\epsilon\|\bar{u}_\lambda\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)} \\
\leq\;&C(\Omega)\bigg((1+\epsilon+\lambda)\|u_\lambda-\bar{u}_\lambda\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\epsilon\|u_\lambda\|_{L^2([0,t]\times\Gamma^+)}+\|f\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\epsilon\|g\|_{L^2([0,t]\times\Gamma^-)}\bigg) \\
&+\epsilon^2G(t)-\epsilon^2G(0).
\end{aligned}\tag{2.52}$$

When $0\leq\lambda<1$ and $0<\epsilon<1$, we get the desired uniform estimate with respect to λ. $\square$

Theorem 2.5. *Assume $\mathrm{e}^{\lambda_0 t}f(t,\vec x,\vec w)\in L^2([0,\infty)\times\Omega\times\mathcal{S}^1)$, $h(\vec x,\vec w)\in L^2(\Omega\times\mathcal{S}^1)$ and $\mathrm{e}^{\lambda_0 t}g(t,x_0,\vec w)\in L^2([0,\infty)\times\Gamma^-)$ for some $\lambda_0>0$. Then for the unsteady neutron transport equation (2.1), there exists λ_0^* satisfying $0<\lambda_0^*\leq\lambda_0$ and a unique solution $u(t,\vec x,\vec w)\in L^2([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying*

$$
\begin{aligned}
&\frac{1}{\epsilon^{1/2}}\left\|\mathrm{e}^{\lambda t}u\right\|_{L^2([0,\infty)\times\Gamma^+)}+\left\|\mathrm{e}^{\lambda t}u(t)\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\left\|\mathrm{e}^{\lambda t}u\right\|_{L^2([0,\infty)\times\Omega\times\mathcal{S}^1)}\\
\leq\;& C(\Omega)\bigg(\frac{1}{\epsilon^2}\left\|\mathrm{e}^{\lambda t}f\right\|_{L^2([0,\infty)\times\Omega\times\mathcal{S}^1)}+\left\|h\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|\mathrm{e}^{\lambda t}g\right\|_{L^2([0,\infty)\times\Gamma^-)}\bigg),
\end{aligned}\tag{2.53}
$$

for any $0\leq\lambda\leq\lambda_0^*$. When $\lambda_0=0$, we have $\lambda_0^*=0$.

Proof. We divide the proof into several steps:

Step 1: Weak formulation.
In the weak formulation (2.30), we may take the test function $\phi=u_\lambda$ to get the energy estimate

$$
\begin{aligned}
&\lambda\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{2}\epsilon\int_0^t\int_\Gamma\left|u_\lambda\right|^2\mathrm{d}\gamma\\
&+\frac{1}{2}\epsilon^2\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}-\frac{1}{2}\epsilon^2\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|u_\lambda-\bar u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\\
=\;&\int_0^t\iint_{\Omega\times\mathcal{S}^1}fu_\lambda.
\end{aligned}\tag{2.54}
$$

Hence, this naturally implies

$$
\begin{aligned}
&\frac{1}{2}\epsilon\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Gamma^+)}+\frac{1}{2}\epsilon^2\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|u_\lambda-\bar u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\\
\leq\;&\int_0^t\iint_{\Omega\times\mathcal{S}^1}fu_\lambda+\frac{1}{2}\epsilon^2\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{2}\epsilon\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}.
\end{aligned}\tag{2.55}
$$

On the other hand, we can square on both sides of (2.28) to obtain

$$
\begin{aligned}
&\epsilon^2\left\|\bar u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\\
\leq\;& C(\Omega)\bigg(\left\|u_\lambda-\bar u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\left\|f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\epsilon^2\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Gamma^+)}+\epsilon^2\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\\
&+\epsilon^4\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon^4\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}\bigg).
\end{aligned}\tag{2.56}
$$

Multiplying a sufficiently small constant on both sides of (2.56) and adding it to (2.55) to absorb $\left\|u_\lambda\right\|^2_{L^2(\Gamma^+)}$, $\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$ and $\left\|u_\lambda-\bar u_\lambda\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}$, we deduce

$$
\begin{aligned}
&\epsilon\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Gamma^+)}+\epsilon^2\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon^2\left\|\bar u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\left\|u_\lambda-\bar u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\\
&\leq C(\Omega)\bigg(\left\|f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\int_0^t\iint_{\Omega\times\mathcal{S}^1}fu_\lambda+\epsilon^2\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}\tag{2.57}
$$

Hence, we have

$$
\begin{aligned}
&\epsilon\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Gamma^+)}+\epsilon^2\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon^2\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\\
\leq\;& C(\Omega)\bigg(\left\|f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\int_0^t\iint_{\Omega\times\mathcal{S}^1}fu_\lambda+\epsilon^2\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\epsilon\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}\tag{2.58}
$$

A simple application of Cauchy's inequality leads to

$$
\int_0^t\iint_{\Omega\times\mathcal{S}^1}fu_\lambda\leq\frac{1}{4C\epsilon^2}\left\|f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+C\epsilon^2\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}.\tag{2.59}
$$

Taking C sufficiently small, we can divide (2.58) by ϵ^2 to obtain

$$
\begin{aligned}
&\frac{1}{\epsilon}\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Gamma^+)}+\left\|u_\lambda(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)} \\
\leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^4}\left\|f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}\tag{2.60}
$$

Step 2: Convergence.
Since above estimate does not depend on λ, it gives a uniform estimate for the penalized neutron transport equation (2.16). Thus, we can extract a weakly convergent subsequence $u_\lambda\rightarrow u$ as $\lambda\rightarrow0$. The weak lower semi-continuity of norms $\left\|\cdot\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}$ and $\left\|\cdot\right\|_{L^2([0,t]\times\Gamma^+)}$ implies u also satisfies the estimate (2.60). Hence, in the weak formulation (2.30), we can take $\lambda\rightarrow0$ to deduce that u satisfies equation (2.1). Also $u_\lambda-u$ satisfies the equation

$$
\left\{
\begin{array}{rcl}
\epsilon^2\partial_t(u_\lambda-u)+\epsilon\vec w\cdot\nabla_x(u_\lambda-u)+(u_\lambda-u)-(\bar u_\lambda-\bar u)&=&-\lambda u_\lambda\ \text{ in }\ \Omega,\\
(u_\lambda-u)(0,\vec x,\vec w)&=&0\ \text{ in }\ \Omega,\\
(u_\lambda-u)(\vec x_0,\vec w)&=&0\ \text{ for }\ \vec x_0\in\partial\Omega\ \text{ and }\ \vec w\cdot\vec n<0.
\end{array}
\right.\tag{2.61}
$$

By a similar argument as above, we can achieve

$$
\left\|u_\lambda-u\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\leq C(\Omega)\bigg(\frac{\lambda}{\epsilon^4}\left\|u_\lambda\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\bigg).\tag{2.62}
$$

When $\lambda\rightarrow0$, the right-hand side approaches zero, which implies the convergence is actually in the strong sense. The uniqueness easily follows from the energy estimates.

Step 3: L^2 Decay.
Let $v=\mathrm{e}^{\lambda t}u$. Then v satisfies the equation

$$
\left\{
\begin{array}{rcl}
\epsilon^2\partial_t v+\epsilon\vec w\cdot\nabla_x v+v-\bar v&=&f+\lambda\epsilon^2v\ \text{ in }\ \Omega,\\
v(0,\vec x,\vec w)&=&h(\vec x,\vec w)\ \text{ in }\ \Omega,\\
v(\vec x_0,\vec w)&=&\mathrm{e}^{\lambda t}g(t,\vec x_0,\vec w)\ \text{ for }\ \vec x_0\in\partial\Omega\ \text{ and }\ \vec w\cdot\vec n<0.
\end{array}
\right.\tag{2.63}
$$

Similar to the argument in Step 1, we can obtain

$$
\begin{aligned}
&\frac{1}{\epsilon}\left\|v\right\|^2_{L^2([0,t]\times\Gamma^+)}+\left\|v(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|v\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)} \\
\leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^4}\left\|\mathrm{e}^{\lambda t}f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^4}\left\|\lambda^2\epsilon^4v\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|\mathrm{e}^{\lambda t}g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg).
\end{aligned}\tag{2.64}
$$

Then when λ is sufficiently small, we have

$$
\begin{aligned}
&\frac{1}{\epsilon}\left\|v\right\|^2_{L^2([0,t]\times\Gamma^+)}+\left\|v(t)\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\left\|v\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)} \\
\leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^4}\left\|\mathrm{e}^{\lambda t}f\right\|^2_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\left\|h\right\|^2_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|\mathrm{e}^{\lambda t}g\right\|^2_{L^2([0,t]\times\Gamma^-)}\bigg),
\end{aligned}\tag{2.65}
$$

which implies exponential decay of u.

□

2.3. L^∞ Estimate.

Theorem 2.6. *Assume* $\mathrm{e}^{\lambda_0t}f(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$*,* $h(\vec x,\vec w)\in L^\infty(\Omega\times\mathcal{S}^1)$ *and* $\mathrm{e}^{\lambda_0t}g(t,x_0,\vec w)\in L^\infty([0,\infty)\times\Gamma^-)$ *for some* $\lambda_0>0$*. Then for the unsteady neutron transport equation (2.1), there exists* λ_0^* *satisfying* $0<\lambda_0^*\leq\lambda_0$ *and a unique solution* $u(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ *satisfying*

$$
\begin{aligned}
&\left\|\mathrm{e}^{\lambda t}u\right\|_{L^\infty([0,\infty)\times\Gamma^+)}+\left\|\mathrm{e}^{\lambda t}u(t)\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\left\|\mathrm{e}^{\lambda t}u\right\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} \\
\leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\left\|\mathrm{e}^{\lambda t}f\right\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|\mathrm{e}^{\lambda t}g\right\|_{L^\infty([0,\infty)\times\Gamma^-)}\bigg),
\end{aligned}\tag{2.66}
$$

for any $0\leq\lambda\leq\lambda_0^*$*. When* $\lambda_0=0$*, we have* $\lambda_0^*=0$*.*

Proof. We divide the proof into several steps to bootstrap an L^2 solution to an L^∞ solution:

Step 1: Double Duhamel iterations.
The characteristics of the equation (2.1) is given by (2.11). Hence, we can rewrite the equation (2.1) along the characteristics as

$$
\begin{aligned}
(2.67)\qquad & u(t,\vec{x},\vec{w})\\
=\ & \mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(g(t-\epsilon^2t_b,\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}+\int_0^{t_b}(\bar{u}+f)(t-\epsilon^2(t_b-s),\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w})\mathrm{e}^{-(t_b-s)}\mathrm{d}s\bigg)\\
&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(h(\vec{x}-(\epsilon t\vec{w})/\epsilon^2,\vec{w})\mathrm{e}^{-t/\epsilon^2}+\int_0^{t/\epsilon^2}(\bar{u}+f)(\epsilon^2s,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w})\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\bigg),
\end{aligned}
$$

where the backward exit time t_b is defined as (2.14). For the convenience of analysis, we transform it into a simpler form

$$
\begin{aligned}
(2.68)\qquad & u(t,\vec{x},\vec{w})\\
=\ & \mathbf{1}_{\{t\geq\epsilon^2t_b\}}g(t-\epsilon^2t_b,\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}h(\vec{x}-(\epsilon t\vec{w})/\epsilon^2,\vec{w})\mathrm{e}^{-t/\epsilon^2}\\
&+\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}f(\epsilon^2s,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w})\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\
&+\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}u(\epsilon^2s,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w}_t)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s.
\end{aligned}
$$

Here $a\wedge b$ denotes $\min\{a,b\}$. Note we have replaced $\bar{u}$ by the integral of u over the dummy velocity variable $\vec{w}_t$. For the last term in this formulation, we apply the Duhamel's principle again to $u(\epsilon^2s,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w}_t)$ and obtain

$$
\begin{aligned}
(2.69)\ \ & u(t,\vec{x},\vec{w})\\
=\ & \mathbf{1}_{\{t\geq\epsilon^2t_b\}}g(t-\epsilon^2t_b,\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}h(\vec{x}-(\epsilon t\vec{w})/\epsilon^2,\vec{w})\mathrm{e}^{-t/\epsilon^2}\\
&+\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}f(\epsilon^2s,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w})\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\
&+\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}\mathbf{1}_{\{s\geq s_b\}}g(\epsilon^2(s-s_b),\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w},\vec{w}_t)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\
&+\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}\mathbf{1}_{\{s\geq s_b\}}h(\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w}-\epsilon s\vec{w}_t,\vec{w}_t)\mathrm{e}^{-s}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\
&+\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}\int_{s-s_b\wedge s}^{s}f(\epsilon^2r,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w}-\epsilon(s-r)\vec{w}_t,\vec{w}_t)\mathrm{e}^{-(s-r)}\mathrm{d}r\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\
&+\frac{1}{(2\pi)^2}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\\
&\bigg(\int_{-\pi}^{\pi}\int_{s-s_b\wedge s}^{s}\bigg(\int_{-\pi}^{\pi}u(\epsilon^2r,\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w}-\epsilon(s-r)\vec{w}_t,\vec{w}_s)\mathrm{e}^{-(s-r)}\mathrm{d}\vec{w}_s\bigg)\mathrm{d}r\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s.
\end{aligned}
$$

where we introduce another dummy velocity variable $\vec{w}_s$ and

$$
(2.70)\qquad s_b(\vec{x},\vec{w},s,\vec{w}_t)=\inf\{r\geq0:(\vec{x}-\epsilon(t/\epsilon^2-s)\vec{w}-\epsilon r\vec{w}_t,\vec{w}_t)\in\Gamma^-\}.
$$

Step 2: Estimates of all but the last term in (2.69).
We can directly estimate as follows:

$$
(2.71)\qquad \left|\mathbf{1}_{\{t\geq\epsilon^2t_b\}}g(t-\epsilon^2t_b,\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}\right|\ \leq\ \left\lVert g\right\rVert_{L^\infty([0,t]\times\Gamma^-)},
$$

$$
(2.72)\qquad \left|\mathbf{1}_{\{t\leq\epsilon^2t_b\}}h(\vec{x}-(\epsilon t\vec{w})/\epsilon^2,\vec{w})\mathrm{e}^{-t/\epsilon^2}\right|\leq\left\lVert h\right\rVert_{L^\infty(\Omega\times\mathcal{S}^1)},
$$

$$\left|\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2} f(\epsilon^2 s,\vec x-\epsilon(t/\epsilon^2-s)\vec w,\vec w)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\right|\leq\|f\|_{L^\infty([0,t]\times\Omega\times\mathcal{S}^1)}, \tag{2.73}$$

$$\begin{aligned}&\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}\mathbf{1}_{\{s\geq s_b\}}g(\epsilon^2(s-s_b),\vec x-\epsilon(t/\epsilon^2-s)\vec w,\vec w_t)\mathrm{e}^{-s_b}\mathrm{d}\vec w_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\ \leq\ &\|g\|_{L^\infty([0,t]\times\Gamma^-)},\end{aligned} \tag{2.74}$$

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}\mathbf{1}_{\{s\geq s_b\}}h(\vec x-\epsilon(t/\epsilon^2-s)\vec w-\epsilon s\vec w_t,\vec w_t)\mathrm{e}^{-s}\mathrm{d}\vec w_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\right|\\ \leq\ &\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)},\end{aligned} \tag{2.75}$$

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\bigg(\int_{-\pi}^{\pi}\int_{s-s_b\wedge s}^{s}f(\epsilon^2 r,\vec x-\epsilon(t/\epsilon^2-s)\vec w-\epsilon(s-r)\vec w_t,\vec w_t)\mathrm{e}^{-(s-r)}\mathrm{d}r\mathrm{d}\vec w_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\right|\\ \leq\ &\|f\|_{L^\infty([0,t]\times\Omega\times\mathcal{S}^1)}.\end{aligned} \tag{2.76}$$

Step 3: Estimates of the last term in (2.69).
We can first transform the last term I in (2.69) into

$$\begin{aligned}|I|\ \leq\ &\frac{1}{(2\pi)^2}\int_{t/\epsilon^2-t_b\wedge(t/\epsilon^2)}^{t/\epsilon^2}\\ &\bigg(\int_{-\pi}^{\pi}\int_{s-s_b\wedge s}^{s}\bigg(\int_{-\pi}^{\pi}\big|u(\epsilon^2 r,\vec x-\epsilon(t/\epsilon^2-s)\vec w-\epsilon(s-r)\vec w_t,\vec w_s)\big|\,\mathrm{e}^{-(s-r)}\mathrm{d}\vec w_s\bigg)\mathrm{d}r\mathrm{d}\vec w_t\bigg)\mathrm{e}^{-(t/\epsilon^2-s)}\mathrm{d}s\\ \leq\ &\frac{1}{(2\pi)^2}\int_0^{t_b}\bigg(\int_{-\pi}^{\pi}\int_0^{s_b}\bigg(\int_{-\pi}^{\pi}\\ &\big|u(\epsilon^2(r^*+s^*+t/\epsilon^2-t_b-s_b),\vec x-\epsilon(t_b-s^*)\vec w-\epsilon(s_b-r^*)\vec w_t,\vec w_s)\big|\,\mathrm{e}^{-(s_b-r^*)}\mathrm{d}\vec w_s\bigg)\mathrm{d}r^*\mathrm{d}\vec w_t\bigg)\mathrm{e}^{-(t_b-s^*)}\mathrm{d}s^*,\end{aligned} \tag{2.77}$$

by substitution $s\rightarrow s^*=(s-t/\epsilon^2+t_b)$ and $r\rightarrow r^*=(r-s+s_b)$. Now we decompose the right-hand side in (2.77) as

$$\int_0^{t_b}\int_{\mathcal{S}^1}\int_0^{s_b}\int_{\mathcal{S}^1}=\int_0^{t_b}\int_{\mathcal{S}^1}\int_{s_b-r^*\leq\delta}\int_{\mathcal{S}^1}+\int_0^{t_b}\int_{\mathcal{S}^1}\int_{s_b-r^*\geq\delta}\int_{\mathcal{S}^1}=I_1+I_2, \tag{2.78}$$

for some $\delta>0$. We can estimate I_1 directly as

$$I_1\ \leq\ \int_0^{t_b}\mathrm{e}^{-(t_b-r^*)}\bigg(\int_{\max\{0,s_b-\delta\}}^{s_b}\|u\|_{L^\infty(\Omega\times\mathcal{S}^1)}\mathrm{d}r^*\bigg)\mathrm{d}s^*\leq\delta\|u\|_{L^\infty([0,t]\times\Omega\times\mathcal{S}^1)}. \tag{2.79}$$

Then we can bound I_2 as

$$\begin{aligned}I_2\ \leq\ &C\int_0^{t_b}\int_{\mathcal{S}^1}\int_0^{\max\{0,s_b-\delta\}}\int_{\mathcal{S}^1}\\ &\big|u(\epsilon^2(r^*+s^*+t/\epsilon^2-t_b-s_b),\vec x-\epsilon(t_b-s^*)\vec w-\epsilon(s_b-r^*)\vec w_t,\vec w_s)\big|\,\mathrm{e}^{-(t_b-r^*)}\mathrm{d}\vec w_s\mathrm{d}r^*\mathrm{d}\vec w_t\mathrm{d}s^*.\end{aligned} \tag{2.80}$$

Define new variables

$$s'\ =\ \epsilon(t_b-s^*), \tag{2.81}$$

$$r'\ =\ \epsilon(s_b-r^*), \tag{2.82}$$

which implies

$$\frac{\mathrm{d}s'}{\mathrm{d}s^*} = -\epsilon, \tag{2.83}$$

$$\frac{\mathrm{d}r'}{\mathrm{d}r^*} = -\epsilon. \tag{2.84}$$

By the definition of t_b and s_b, we always have $\vec{x}-\epsilon(t_b-s^*)\vec{w}-\epsilon(s_b-r^*)\vec{w}_t=\vec{x}-s'\vec{w}-r'\vec{w}_t\in\bar{\Omega}$. Hence, we may interchange the order of integration and apply Hölder's inequality to obtain

$$\begin{aligned} I_2 &\leq \frac{C}{\epsilon^2}\int_0^{\epsilon t_b}\int_{\epsilon\min\{\delta,s_b\}}^{\epsilon s_b}\int_{\mathcal{S}^1}\int_{\mathcal{S}^1}\mathbf{1}_{\Omega}(\vec{x}-s'\vec{w}-r'\vec{w}_t) \\ &\quad |u(\vec{x}-s'\vec{w}-r'\vec{w}_t|\,\mathrm{e}^{-s'/\epsilon}\mathrm{e}^{-r'/\epsilon}\mathrm{d}\vec{w}_t\mathrm{d}\vec{w}_s\mathrm{d}r'\mathrm{d}s' \\ &\leq \frac{C}{\epsilon^2}\int_0^{\epsilon t_b}\int_{\mathcal{S}^1}\left(\int_{\epsilon\min\{\delta,s_b\}}^{\epsilon s_b}\int_{\mathcal{S}^1}\mathbf{1}_{\Omega}(\vec{x}-s'\vec{w}-r'\vec{w}_t)\mathrm{e}^{-2r'/\epsilon}\mathrm{d}\vec{w}_t\mathrm{d}r'\right)^{1/2} \\ &\quad \left(\int_0^{\epsilon s_b}\int_{\mathcal{S}^1}|u(\vec{x}-s'\vec{w}-r'\vec{w}_t|^2\,\mathrm{d}\vec{w}_t\mathrm{d}r'\right)^{1/2}\mathrm{e}^{-s'/\epsilon}\mathrm{d}\vec{w}_s\mathrm{d}s'. \end{aligned} \tag{2.85}$$

Note $\vec{w}_t\in\mathcal{S}^1$, which is essentially a one-dimensional variable. Thus, we may write it in a new variable ψ as $\vec{w}_t=(\cos\psi,\sin\psi)$. Then we define the change of variable $[-\pi,\pi)\times\mathbb{R}\to\Omega:(\psi,r)\to(y_1,y_2)=\vec{y}=\vec{x}-s'\vec{w}-r'\vec{w}_t$, i.e.

$$\left\{\begin{array}{rcl} y_1 &=& x_1-s'w_1-r'\cos\psi, \\ y_2 &=& x_2-s'w_2-r'\sin\psi. \end{array}\right. \tag{2.86}$$

Therefore, for $s_b-r\geq\delta$, we can directly compute the Jacobian

$$\left|\frac{\partial(y_1,y_2)}{\partial(\psi,r)}\right| = \left|\left|\begin{array}{cc} -r'\sin\psi & \cos\psi \\ r'\cos\psi & \sin\psi \end{array}\right|\right| = r'\geq\delta. \tag{2.87}$$

Hence, we may simplify (2.85) as

$$\begin{aligned} I_2 &\leq \frac{C}{\epsilon^2}\int_0^{\epsilon t_b}\int_{\mathcal{S}^1}\left(\int_{\epsilon\min\{\delta,s_b\}}^{\epsilon s_b}\int_{\mathcal{S}^1}\mathbf{1}_{\Omega}(\vec{x}-s'\vec{w}-r'\vec{w}_t)\mathrm{e}^{-2r'/\epsilon}\mathrm{d}\vec{w}_t\mathrm{d}r'\right)^{1/2} \\ &\quad \left(\int_{\Omega}|u(\vec{y})|^2\,\mathrm{d}\vec{y}\right)^{1/2}\mathrm{e}^{-s'/\epsilon}\mathrm{d}\vec{w}_s\mathrm{d}s'. \end{aligned}$$

Then we may further utilize Cauchy's inequality and the L^2 estimate of u in Theorem 2.5 to obtain

$$\begin{aligned} I_2 &\leq \frac{C}{\epsilon^2\sqrt{\delta}}\left\|u\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}\int_0^{\epsilon t_b}\int_{\mathcal{S}^1}\left(\int_{\epsilon\min\{\delta,s_b\}}^{\epsilon s_b}\int_{\mathcal{S}^1}\mathbf{1}_{\Omega}(\vec{x}-s'\vec{w}-r'\vec{w}_t)\mathrm{e}^{-2r'/\epsilon}\mathrm{d}\vec{w}_t\mathrm{d}r'\right)^{1/2}\mathrm{e}^{-s'/\epsilon}\mathrm{d}\vec{w}_s\mathrm{d}s' \\ &\leq \frac{C}{\sqrt{\epsilon\delta}}\left\|u\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)} \\ &\leq \frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^2([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^2(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^2([0,t]\times\Gamma^-)}\right) \\ &\leq \frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^\infty([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\right). \end{aligned} \tag{2.88}$$

Step 4: L^∞ estimate.
In summary, collecting (2.71), (2.72), (2.73), (2.74), (2.75), (2.76), (2.79) and (2.88), for fixed $0<\delta<1$, we have

$$\begin{aligned} &|u(t,\vec{x},\vec{w})| \\ &\quad\leq \delta\left\|u\right\|_{L^\infty([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{C(\Omega)}{\sqrt{\delta}}\left(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^\infty([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^\infty([0,t]\times\Gamma^-)}\right). \end{aligned} \tag{2.89}$$

Then we may take $0<\delta\leq 1/2$ to obtain

$$\begin{aligned}(2.90)\quad &\left|u(t,\vec x,\vec w)\right|\\ \leq\ & \frac{1}{2}\left\|u\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{C(\Omega)}{\sqrt{\delta}}\bigg(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^{\infty}([0,t]\times\Gamma^-)}\bigg).\end{aligned}$$

Taking supremum of u over all $(t,\vec x,\vec w)$, we have

$$\begin{aligned}(2.91)\quad &\left\|u\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}\\ \leq\ & \frac{1}{2}\left\|u\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{C(\Omega)}{\sqrt{\delta}}\bigg(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^{\infty}([0,t]\times\Gamma^-)}\bigg).\end{aligned}$$

Finally, absorbing $\left\|u\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}$, for fixed $0<\delta\leq 1/2$, we get

(2.92)
$$\left\|u\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}\leq C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^{\infty}([0,t]\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^{\infty}([0,t]\times\Gamma^-)}\bigg).$$

Step 5: L^∞ Decay.
Let $v=\mathrm{e}^{\lambda t}u$. Then v satisfies the equation

$$(2.93)\quad \left\{\begin{array}{rcl}\epsilon^2\partial_t v+\epsilon\vec w\cdot\nabla_x v+(1-\lambda\epsilon^2)v-\bar v&=&f\ \text{ in }\ \Omega,\\ v(0,\vec x,\vec w)&=&h(\vec x,\vec w)\ \text{ in }\ \Omega,\\ v(\vec x_0,\vec w)&=&\mathrm{e}^{\lambda t}g(t,\vec x_0,\vec w)\ \text{ for }\ \vec x_0\in\partial\Omega\ \text{ and }\ \vec w\cdot\vec n<0.\end{array}\right.$$

By a similar argument as in Step 3 and Step 4, combined with the L^2 decay, we can finally show the desired estimate. □

Based on the proof of Theorem 2.6, we actually have a more delicate estimate for the equation (2.1).

Corollary 2.7. *Assume $f(t,\vec x,\vec w)\in L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)$, $h(\vec x,\vec w)\in L^{\infty}(\Omega\times\mathcal{S}^1)$ and $g(t,x_0,\vec w)\in L^{\infty}([0,\infty)\times\Gamma^-)$. Then for the unsteady neutron transport equation (2.1), there exists a unique solution $u(t,\vec x,\vec w)\in L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying*

$$\begin{aligned}(2.94)\qquad &\left\|u\right\|_{L^{\infty}([0,\infty)\times\Gamma^+)}+\left\|u(t)\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\left\|u\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}\\ \leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\left\|f\right\|_{L^{2}([0,\infty)\times\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon^{1/2}}\left\|h\right\|_{L^{2}(\Omega\times\mathcal{S}^1)}+\frac{1}{\epsilon}\left\|g\right\|_{L^{2}([0,\infty)\times\Gamma^-)}\bigg)\\ &+C(\Omega)\bigg(\left\|f\right\|_{L^{\infty}([0,\infty)\times\Omega\times\mathcal{S}^1)}+\left\|h\right\|_{L^{\infty}(\Omega\times\mathcal{S}^1)}+\left\|g\right\|_{L^{\infty}([0,\infty)\times\Gamma^-)}\bigg).\end{aligned}$$

2.4. **Maximum Principle.**

Theorem 2.8. *When $f=0$, the solution $u(t,\vec x,\vec w)$ to the unsteady neutron transport equation (2.1) satisfies the maximum principle, i.e.*

$$(2.95)\qquad \inf\{g(t,\vec x_0,\vec w),h(\vec x,\vec w)\}\leq u(t,\vec x,\vec w)\leq\sup\{g(t,\vec x_0,\vec w),h(\vec x,\vec w)\}.$$

Proof. We claim that it suffices to show $u(t,\vec x,\vec w)\leq 0$ whenever $g(t,\vec x_0,\vec w)\leq 0$ and $h(\vec x,\vec w)\leq 0$. Suppose the claim is justified. Then define

$$(2.96)\qquad m=\inf\{g(t,\vec x_0,\vec w),h(\vec x,\vec w)\},$$
$$(2.97)\qquad M=\sup\{g(t,\vec x_0,\vec w),h(\vec x,\vec w)\}.$$

We have $u_1=u-M$ satisfies the equation

$$(2.98)\quad \left\{\begin{array}{rcl}\epsilon^2\partial_t u_1+\epsilon\vec w\cdot\nabla_x u_1+u_1-\bar u_1&=&0\ \text{ in }\ [0,\infty)\times\Omega,\\ u_1(0,\vec x,\vec w)&=&h(\vec x,\vec w)-M\ \text{ in }\ \Omega\\ u_1(t,\vec x_0,\vec w)&=&g(t,\vec x_0,\vec w)-M\ \text{ for }\ \vec w\cdot\vec n<0\ \text{ and }\ \vec x_0\in\partial\Omega,\end{array}\right.$$

Hence, $h-M\leq 0$ and $g-M\leq 0$ implies $u_1\leq 0$, which is actually $u\leq M$. Similarly, we have $u_2=m-u$ satisfies the equation

$$
\left\{
\begin{array}{rcl}
\epsilon^2\partial_t u_2+\epsilon\vec w\cdot\nabla_x u_2+u_2-\bar u_2 &=& 0 \ \text{ in } \ [0,\infty)\times\Omega,\\
u_2(0,\vec x,\vec w) &=& m-h(\vec x,\vec w) \ \text{ in } \ \Omega\\
u_2(t,\vec x_0,\vec w) &=& m-g(t,\vec x_0,\vec w) \ \text{ for } \ \vec w\cdot\vec n<0 \ \text{ and } \ \vec x_0\in\partial\Omega,
\end{array}
\right.
\tag{2.99}
$$

Hence, $m-h\leq 0$ and $m-g\leq 0$ implies $u_2\leq 0$, which is actually $u\geq m$. Therefore, the maximum principle is established.

We now prove the claim that if $g(t,\vec x_0,\vec w)\leq 0$ and $h(\vec x,\vec w)\leq 0$, we have $u(t,\vec x,\vec w)\leq 0$. We first consider the penalized neutron transport equation

$$
\left\{
\begin{array}{rcl}
\lambda u_\lambda+\epsilon^2\partial_t u_\lambda+\epsilon\vec w\cdot\nabla_x u_\lambda+u_\lambda-\bar u_\lambda &=& 0 \ \text{ in } \ [0,\infty)\times\Omega,\\
u_\lambda(0,\vec x,\vec w) &=& h(\vec x,\vec w) \ \text{ in } \ \Omega\\
u_\lambda(t,\vec x_0,\vec w) &=& g(t,\vec x_0,\vec w) \ \text{ for } \ \vec x_0\in\partial\Omega \ \text{ and } \vec w\cdot\vec n<0.
\end{array}
\right.
\tag{2.100}
$$

with $\lambda>0$. Based on Lemma 2.2, there exists a solution $u_\lambda(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$. We use the notation in the proof of Lemma 2.2. Define an approximating sequence $\{u_\lambda^k\}_{k=0}^\infty$, where $u_\lambda^0=0$ and

$$
\left\{
\begin{array}{rcl}
\lambda u_\lambda^k+\epsilon^2\partial_t u_\lambda^k+\epsilon\vec w\cdot\nabla_x u_\lambda^k+u_\lambda^k-\bar u_\lambda^{k-1} &=& 0 \ \text{ in } \ [0,\infty)\times\Omega,\\
u_\lambda^k(0,\vec x,\vec w) &=& h(\vec x,\vec w) \ \text{ in } \ \Omega\\
u_\lambda^k(t,\vec x_0,\vec w) &=& g(t,\vec x_0,\vec w) \ \text{ for } \ \vec x_0\in\partial\Omega \ \text{ and } \vec w\cdot\vec n<0.
\end{array}
\right.
\tag{2.101}
$$

By Lemma 2.1, this sequence is well-defined and $\left\|u_\lambda^k\right\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}<\infty$. Then we rewrite equation (2.18) along the characteristics as

$$
\begin{aligned}
&u_\lambda^k(t,\vec x,\vec w)\\
=\;& \mathbf{1}_{\{t\geq\epsilon^2t_b\}}\bigg(g(t-\epsilon^2t_b,\vec x-\epsilon t_b\vec w,\vec w)\mathrm{e}^{-(1+\lambda)t_b}+\int_0^{t_b}\bar u_\lambda^{k-1}(t-\epsilon^2(t_b-s),\vec x-\epsilon(t_b-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t_b-s)}\mathrm{d}s\bigg)\\
&+\mathbf{1}_{\{t\leq\epsilon^2t_b\}}\bigg(h(\vec x-(\epsilon t\vec w)/\epsilon^2,\vec w)\mathrm{e}^{-(1+\lambda)t/\epsilon^2}+\int_0^{t/\epsilon^2}\bar u_\lambda^{k-1}(\epsilon^2s,\vec x-\epsilon(t/\epsilon^2-s)\vec w,\vec w)\mathrm{e}^{-(1+\lambda)(t/\epsilon^2-s)}\mathrm{d}s\bigg),
\end{aligned}
\tag{2.102}
$$

where

$$
t_b(\vec x,\vec w)=\inf\{s\geq 0:(\vec x-\epsilon s\vec w,\vec w)\in\Gamma^-\}.
\tag{2.103}
$$

Since $u_\lambda^k(t,\vec x,\vec w)\leq 0$ naturally implies $\bar u_\lambda^k(t,\vec x)\leq 0$, we naturally have $u_\lambda^k(t,\vec x,\vec w)\leq 0$ when $g(t,\vec x_0,\vec w)\leq 0$ and $h(\vec x,\vec w)\leq 0$. In the proof of Lemma 2.2, we have shown $u_\lambda^k\to u_\lambda$ in L^∞ as $k\to\infty$. Therefore, we have $u_\lambda(t,\vec x,\vec w)\leq 0$. Based on the proof of Lemma 2.5, we know $u_\lambda\to u$ in L^2 as $\lambda\to 0$, where u is the solution of the equation (2.1). Then we naturally obtain $u\leq 0$. Also, this is the unique solution to the equation (2.1). This justifies the claim and completes the proof. $\square$

Theorem 2.8 naturally leads to the L^∞ estimate of the equation (2.1).

Corollary 2.9. *Assume $h(\vec x,\vec w)\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(t,x_0,\vec w)\in L^\infty([0,\infty)\times\Gamma^-)$. Then for the unsteady neutron transport equation (2.1) with $f=0$, there exists a unique solution $u(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying*

$$
\|u\|_{L^\infty([0,\infty)\times\Gamma^+)}+\|u(t)\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|u\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\leq\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|g\|_{L^\infty([0,\infty)\times\Gamma^-)}\,.
\tag{2.104}
$$

2.5. **Well-posedness of Transport Equation.** Combining the results in Corollary 2.7 and Corollary 2.9, we can show an improved L^∞ estimate of the equation (2.1).

Theorem 2.10. *Assume $f(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$, $h(\vec x,\vec w)\in L^\infty(\Omega\times\mathcal{S}^1)$ and $g(t,x_0,\vec w)\in L^\infty([0,\infty)\times\Gamma^-)$. Then for the unsteady neutron transport equation (2.1), there exists a unique solution $u(t,\vec x,\vec w)\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying*

$$
\begin{aligned}
&\|u\|_{L^\infty([0,\infty)\times\Gamma^+)}+\|u(t)\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|u\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\\
\leq\;& C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\|f\|_{L^2([0,\infty)\times\Omega\times\mathcal{S}^1)}+\|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\bigg)+\|h\|_{L^\infty(\Omega\times\mathcal{S}^1)}+\|g\|_{L^\infty([0,\infty)\times\Gamma^-)}\,.
\end{aligned}
\tag{2.105}
$$

Proof. Since the equation (2.1) is a linear equation, then we can utilize the superposition property, i.e. we can separate the solution $u = u_1 + u_2$ where u_1 satisfies the equation

$$\left\{\begin{array}{rcl} \epsilon^2\partial_t u_1 + \epsilon\vec{w}\cdot\nabla_x u_1 + u_1 - \bar{u}_1 &=& 0 \ \text{ in } \ [0,\infty)\times\Omega,\\ u_1(0,\vec{x},\vec{w}) &=& h(\vec{x},\vec{w}) \ \text{ in } \ \Omega\\ u_1(t,\vec{x}_0,\vec{w}) &=& g(t,\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{n}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{array}\right. \tag{2.106}$$

and u_2 satisfies the equation

$$\left\{\begin{array}{rcl} \epsilon^2\partial_t u_2 + \epsilon\vec{w}\cdot\nabla_x u_2 + u_2 - \bar{u}_2 &=& f(t,\vec{x},\vec{w}) \ \text{ in } \ [0,\infty)\times\Omega,\\ u_2(0,\vec{x},\vec{w}) &=& 0 \ \text{ in } \ \Omega\\ u_2(t,\vec{x}_0,\vec{w}) &=& 0 \ \text{ for } \ \vec{w}\cdot\vec{n}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{array}\right. \tag{2.107}$$

Note that the data in (2.106) and (2.107) satisfy the compatibility condition (1.3). Therefore, we can apply the previous results in this section. Corollary 2.9 yields

$$\|u_1\|_{L^\infty([0,\infty)\times\Gamma^+)} + \|u_1(t)\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|u_1\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} \leq \|h\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|g\|_{L^\infty([0,\infty)\times\Gamma^-)}. \tag{2.108}$$

Also, Corollary 2.7 leads to

$$\begin{aligned} &\|u_2\|_{L^\infty([0,\infty)\times\Gamma^+)} + \|u_2(t)\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|u_2\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\\ \leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\|f\|_{L^2([0,\infty)\times\Omega\times\mathcal{S}^1)} + \|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\bigg). \end{aligned} \tag{2.109}$$

Combining (2.108) and (2.109), we have the desired result. ∎

Finally, we can apply Theorem 2.10 to the equation (1.1) and obtain Theorem 1.1.

Theorem 2.11. *Assume $g(t,x_0,\vec{w})\in L^\infty([0,\infty)\times\Gamma^-)$ and $h(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathcal{S}^1)$. Then for the unsteady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(t,\vec{x},\vec{w})\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfying*

$$\|u^\epsilon\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} \leq \|h\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|g\|_{L^\infty([0,\infty)\times\Gamma^-)}. \tag{2.110}$$

3. Asymptotic Analysis

In this section, we construct the asymptotic expansion of the equation (1.1).

3.1. **Discussion of Compatibility Condition.**

The initial and boundary data satisfy the compatibility condition

$$h(\vec{x}_0,\vec{w}) = g(0,\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{n}<0. \tag{3.1}$$

Then in the half-space $\vec{w}\cdot\vec{n}<0$ at $(0,\vec{x}_0,\vec{w})$, the equation

$$\epsilon^2\partial_t u^\epsilon + \epsilon\vec{w}\cdot\nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon = 0, \tag{3.2}$$

is valid, which implies

$$\epsilon^2\partial_t g(0,\vec{x}_0,\vec{w}) + \epsilon\vec{w}\cdot\nabla_x h(\vec{x}_0,\vec{w}) + h(\vec{x}_0,\vec{w}) - \bar{h}(\vec{x}_0) = 0. \tag{3.3}$$

In order to show the diffusive limit, the condition (3.3) holds for arbitrary ϵ. Since g and h are all independent of ϵ, we must have for $\vec{w}\cdot\vec{n}<0$,

$$\partial_t g(0,\vec{x}_0,\vec{w}) = 0, \tag{3.4}$$

$$\vec{w}\cdot\nabla_x h(\vec{x}_0,\vec{w}) = 0, \tag{3.5}$$

$$h(\vec{x}_0,\vec{w}) - \bar{h}(\vec{x}_0) = 0. \tag{3.6}$$

The relation (3.6) implies the improved compatibility condition

$$h(\vec{x}_0,\vec{w}) = g(0,\vec{x}_0,\vec{w}) = C_0 \ \text{ for } \ \vec{w}\cdot\vec{n}<0, \tag{3.7}$$

for some constant C_0. This fact is of great importance in the following analysis.

3.2. **Interior Expansion.** We define the interior expansion as follows:

$$U^{\epsilon}(t,\vec{x},\vec{w}) \sim \sum_{k=0}^{\infty}\epsilon^k U_k^{\epsilon}(t,\vec{x},\vec{w}), \tag{3.8}$$

where U_k^{ϵ} can be defined by comparing the order of ϵ via plugging (3.8) into the equation (1.1). Thus, we have

$$U_0^{\epsilon}-\bar{U}_0^{\epsilon} = 0, \tag{3.9}$$
$$U_1^{\epsilon}-\bar{U}_1^{\epsilon} = -\vec{w}\cdot\nabla_x U_0^{\epsilon}, \tag{3.10}$$
$$U_2^{\epsilon}-\bar{U}_2^{\epsilon} = -\vec{w}\cdot\nabla_x U_1^{\epsilon}-\partial_t U_0^{\epsilon}, \tag{3.11}$$
$$\ldots$$
$$U_k^{\epsilon}-\bar{U}_k^{\epsilon} = -\vec{w}\cdot\nabla_x U_{k-1}^{\epsilon}-\partial_t U_{k-2}^{\epsilon}. \tag{3.12}$$

The following analysis reveals the equation satisfied by U_k^{ϵ}:
Plugging (3.9) into (3.10), we obtain

$$U_1^{\epsilon}=\bar{U}_1^{\epsilon}-\vec{w}\cdot\nabla_x\bar{U}_0^{\epsilon}. \tag{3.13}$$

Plugging (3.13) into (3.11), we get

$$U_2^{\epsilon}-\bar{U}_2^{\epsilon}+\partial_t U_0^{\epsilon}=-\vec{w}\cdot\nabla_x(\bar{U}_1^{\epsilon}-\vec{w}\cdot\nabla_x\bar{U}_0^{\epsilon})=-\vec{w}\cdot\nabla_x\bar{U}_1^{\epsilon}+\left|\vec{w}\right|^2\Delta_x\bar{U}_0^{\epsilon}+2w_1w_2\partial_{x_1x_2}\bar{U}_0^{\epsilon}. \tag{3.14}$$

Integrating (3.14) over $\vec{w}\in\mathcal{S}^1$, we achieve the final form

$$\partial_t\bar{U}_0^{\epsilon}-\Delta_x\bar{U}_0^{\epsilon}=0, \tag{3.15}$$

which further implies $U_0^{\epsilon}(t,\vec{x},\vec{w})$ satisfies the equation

$$\left\{\begin{array}{rcl} U_0^{\epsilon} &=& \bar{U}_0^{\epsilon},\\ \partial_t\bar{U}_0^{\epsilon}-\Delta_x\bar{U}_0^{\epsilon} &=& 0. \end{array}\right. \tag{3.16}$$

Similarly, we can derive $U_1^{\epsilon}(t,\vec{x},\vec{w})$ satisfies

$$\left\{\begin{array}{rcl} U_1^{\epsilon} &=& \bar{U}_1^{\epsilon}-\vec{w}\cdot\nabla_x U_0^{\epsilon},\\ \partial_t\bar{U}_1^{\epsilon}-\Delta_x\bar{U}_1^{\epsilon} &=& 0, \end{array}\right. \tag{3.17}$$

and $U_k^{\epsilon}(t,\vec{x},\vec{w})$ for $k\geq 2$ satisfies

$$\left\{\begin{array}{rcl} U_k^{\epsilon} &=& \bar{U}_k^{\epsilon}-\vec{w}\cdot\nabla_x U_{k-1}^{\epsilon}-\partial_t U_{k-2}^{\epsilon},\\ \partial_t\bar{U}_k^{\epsilon}-\Delta_x\bar{U}_k^{\epsilon} &=& 0. \end{array}\right. \tag{3.18}$$

Note that in order to determine U_k^{ϵ}, we need to define the initial data and boundary data.

3.3. **Initial Layer Expansion.** In order to determine the initial condition for U_k^{ϵ}, we need to define the initial layer expansion. Hence, we need a substitution:

Temporal Substitution:
We define the stretched variable τ by making the scaling transform for $u^{\epsilon}(t)\rightarrow u^{\epsilon}(\tau)$ with $\tau\in[0,\infty)$ as

$$\tau = \frac{t}{\epsilon^2}, \tag{3.19}$$

which implies

$$\frac{\partial u^{\epsilon}}{\partial t}=\frac{1}{\epsilon^2}\frac{\partial u^{\epsilon}}{\partial\tau}. \tag{3.20}$$

In this new variable, equation (1.1) can be rewritten as

$$\left\{\begin{array}{l} \partial_\tau u^{\epsilon}+\epsilon\vec{w}\cdot\nabla_x u^{\epsilon}+u^{\epsilon}-\bar{u}^{\epsilon}=0,\\ u^{\epsilon}(0,\vec{x},\vec{w})=h(\vec{x},\vec{w}),\\ u^{\epsilon}(\tau,\vec{x}_0,\vec{w})=g(\tau,\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{n}<0. \end{array}\right. \tag{3.21}$$

We define the initial layer expansion as follows:

$$\mathscr{U}_I^\epsilon(\tau,\vec{x},\vec{w}) \sim \sum_{k=0}^{\infty} \epsilon^k \mathscr{U}_{I,k}^\epsilon(\tau,\vec{x},\vec{w}), \tag{3.22}$$

where $\mathscr{U}_{I,k}^\epsilon$ can be determined by comparing the order of ϵ via plugging (3.22) into the equation (3.21). Thus, we have

$$\partial_\tau \mathscr{U}_{I,0}^\epsilon + \mathscr{U}_{I,0}^\epsilon - \bar{\mathscr{U}}_{I,0}^\epsilon = 0, \tag{3.23}$$

$$\partial_\tau \mathscr{U}_{I,1}^\epsilon + \mathscr{U}_{I,1}^\epsilon - \bar{\mathscr{U}}_{I,1}^\epsilon = -\vec{w}\cdot\nabla_x \mathscr{U}_{I,0}^\epsilon, \tag{3.24}$$

$$\partial_\tau \mathscr{U}_{I,2}^\epsilon + \mathscr{U}_{I,2}^\epsilon - \bar{\mathscr{U}}_{I,2}^\epsilon = -\vec{w}\cdot\nabla_x \mathscr{U}_{I,1}^\epsilon, \tag{3.25}$$

$$\ldots$$

$$\partial_\tau \mathscr{U}_{I,k}^\epsilon + \mathscr{U}_{I,k}^\epsilon - \bar{\mathscr{U}}_{I,k}^\epsilon = -\vec{w}\cdot\nabla_x \mathscr{U}_{I,k-1}^\epsilon. \tag{3.26}$$

The following analysis reveals the equation satisfied by $\mathscr{U}_{I,k}^\epsilon$:
Integrate (3.23) over $\vec{w}\in\mathcal{S}^1$, we have

$$\partial_\tau \bar{\mathscr{U}}_{I,0}^\epsilon = 0. \tag{3.27}$$

which further implies

$$\bar{\mathscr{U}}_{I,0}^\epsilon(\tau,\vec{x}) = \bar{\mathscr{U}}_{I,0}^\epsilon(0,\vec{x}). \tag{3.28}$$

Therefore, from (3.23), we can deduce

$$\begin{aligned} \mathscr{U}_{I,0}^\epsilon(\tau,\vec{x},\vec{w}) &= \mathrm{e}^{-\tau}\mathscr{U}_{I,0}^\epsilon(0,\vec{x},\vec{w}) + \int_0^\tau \bar{\mathscr{U}}_{I,0}^\epsilon(s,\vec{x})\mathrm{e}^{s-\tau}\mathrm{d}s \\ &= \mathrm{e}^{-\tau}\mathscr{U}_{I,0}^\epsilon(0,\vec{x},\vec{w}) + (1-\mathrm{e}^{-\tau})\bar{\mathscr{U}}_{I,0}^\epsilon(0,\vec{x}). \end{aligned} \tag{3.29}$$

This means we have

$$\begin{cases} \partial_\tau \bar{\mathscr{U}}_{I,0}^\epsilon = 0, \\ \mathscr{U}_{I,0}^\epsilon(\tau,\vec{x},\vec{w}) = \mathrm{e}^{-\tau}\mathscr{U}_{I,0}^\epsilon(0,\vec{x},\vec{w}) + (1-\mathrm{e}^{-\tau})\bar{\mathscr{U}}_{I,0}^\epsilon(0,\vec{x}). \end{cases} \tag{3.30}$$

Similarly, we can derive $\mathscr{U}_{I,k}^\epsilon(\tau,\vec{x},\vec{w})$ for $k\geq 1$ satisfies

$$\begin{cases} \partial_\tau \bar{\mathscr{U}}_{I,k}^\epsilon = -\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x \mathscr{U}_{I,k-1}^\epsilon\bigg)\mathrm{d}\vec{w}, \\ \mathscr{U}_{I,k}^\epsilon(\tau,\vec{x},\vec{w}) = \mathrm{e}^{-\tau}\mathscr{U}_{I,k}^\epsilon(0,\vec{x},\vec{w}) + \displaystyle\int_0^\tau \bigg(\bar{\mathscr{U}}_{I,k}^\epsilon - \vec{w}\cdot\nabla_x \mathscr{U}_{I,k-1}^\epsilon\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s. \end{cases} \tag{3.31}$$

3.4. **Boundary Layer Expansion with Geometric Correction.** In order to determine the boundary condition for U_k^ϵ, we need to define the boundary layer expansion. Hence, we need several substitutions:

Spacial Substitution 1:
We consider the substitution into quasi-polar coordinates $u^\epsilon(x_1,x_2)\to u^\epsilon(\mu,\theta)$ with $(\mu,\theta)\in[0,1)\times[-\pi,\pi)$ defined as

$$\begin{cases} x_1 = (1-\mu)\cos\theta, \\ x_2 = (1-\mu)\sin\theta. \end{cases} \tag{3.32}$$

Here μ denotes the distance to the boundary $\partial\Omega$ and θ is the space angular variable. In these new variables, equation (1.1) can be rewritten as

(3.33)

$$\begin{cases} \epsilon^2\dfrac{\partial u^\epsilon}{\partial t} - \epsilon\bigg(w_1\cos\theta + w_2\sin\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\mu} - \dfrac{\epsilon}{1-\mu}\bigg(w_1\sin\theta - w_2\cos\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\theta} + u^\epsilon - \dfrac{1}{2\pi}\displaystyle\int_{\mathcal{S}^1} u^\epsilon\mathrm{d}\vec{w} = 0, \\ u^\epsilon(0,\mu,\theta,w_1,w_2) = h(\mu,\theta,w_1,w_2), \\ u^\epsilon(t,0,\theta,w_1,w_2) = g(t,\theta,w_1,w_2) \quad \text{for} \quad w_1\cos\theta + w_2\sin\theta < 0. \end{cases}$$

Spacial Substitution 2:

We further define the stretched variable η by making the scaling transform for $u^\epsilon(\mu,\theta)\to u^\epsilon(\eta,\theta)$ with $(\eta,\theta)\in[0,1/\epsilon)\times[-\pi,\pi)$ as

$$\left\{\begin{array}{rcl}\eta&=&\dfrac{\mu}{\epsilon},\\ \theta&=&\theta,\end{array}\right. \tag{3.34}$$

which implies

$$\frac{\partial u^\epsilon}{\partial\mu}=\frac{1}{\epsilon}\frac{\partial u^\epsilon}{\partial\eta}. \tag{3.35}$$

Then equation (1.1) is transformed into

(3.36)
$$\left\{\begin{array}{l}\epsilon^2\dfrac{\partial u^\epsilon}{\partial t}-\bigg(w_1\cos\theta+w_2\sin\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\bigg(w_1\sin\theta-w_2\cos\theta\bigg)\dfrac{\partial u^\epsilon}{\partial\theta}+u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{\mathcal{S}^1}u^\epsilon\mathrm{d}\vec{w}=0,\\ u^\epsilon(0,\eta,\theta,\vec{w})=h(\eta,\theta,w_1,w_2),\\ u^\epsilon(t,0,\theta,w_1,w_2)=g(t,\theta,w_1,w_2)\quad\text{for}\quad w_1\cos\theta+w_2\sin\theta<0.\end{array}\right.$$

Spacial Substitution 3:
Define the velocity substitution for $u^\epsilon(w_1,w_2)\to u^\epsilon(\xi)$ with $\xi\in[-\pi,\pi)$ as

$$\left\{\begin{array}{rcl}w_1&=&-\sin\xi,\\ w_2&=&-\cos\xi.\end{array}\right. \tag{3.37}$$

Here ξ denotes the velocity angular variable. We have the succinct form for (1.1) as

$$\left\{\begin{array}{l}\epsilon^2\dfrac{\partial u^\epsilon}{\partial t}+\sin(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\cos(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\theta}+u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}u^\epsilon\mathrm{d}\xi=0,\\ u^\epsilon(0,\eta,\theta,\xi)=h(\eta,\theta,\xi),\\ u^\epsilon(t,0,\theta,\xi)=g(t,\theta,\xi)\quad\text{for}\quad\sin(\theta+\xi)>0.\end{array}\right. \tag{3.38}$$

Spacial Substitution 4:
We make the rotation substitution for $u^\epsilon(\xi)\to u^\epsilon(\phi)$ with $\phi\in[-\pi,\pi)$ as

$$\phi\;=\;\theta+\xi, \tag{3.39}$$

and transform the equation (1.1) into

$$\left\{\begin{array}{l}\epsilon^2\dfrac{\partial u^\epsilon}{\partial t}+\sin\phi\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\dfrac{\partial u^\epsilon}{\partial\phi}+\dfrac{\partial u^\epsilon}{\partial\theta}\bigg)+u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}u^\epsilon\mathrm{d}\phi=0,\\ u^\epsilon(0,\eta,\theta,\phi)=h(\eta,\theta,\phi),\\ u^\epsilon(t,0,\theta,\phi)=g(t,\theta,\phi)\quad\text{for}\quad\sin\phi>0.\end{array}\right. \tag{3.40}$$

We define the boundary layer expansion with geometric correction as follows:

$$\mathscr{U}_B^\epsilon(t,\eta,\theta,\phi)\sim\sum_{k=0}^\infty\epsilon^k\mathscr{U}_{B,k}^\epsilon(t,\eta,\theta,\phi), \tag{3.41}$$

where $\mathscr{U}_{B,k}^\epsilon$ can be determined by comparing the order of ϵ via plugging (3.41) into the equation (3.40). Following the idea in [13], in a neighborhood of the boundary, we require

$$\sin\phi\frac{\partial\mathscr{U}_{B,0}^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,0}^\epsilon}{\partial\phi}+\mathscr{U}_{B,0}^\epsilon-\bar{\mathscr{U}}_{B,0}^\epsilon\;=\;0, \tag{3.42}$$

$$\sin\phi\frac{\partial\mathscr{U}_{B,1}^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,1}^\epsilon}{\partial\phi}+\mathscr{U}_{B,1}^\epsilon-\bar{\mathscr{U}}_{B,1}^\epsilon\;=\;\frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,0}^\epsilon}{\partial\theta}, \tag{3.43}$$

$$\sin\phi\frac{\partial\mathscr{U}_{B,2}^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,2}^\epsilon}{\partial\phi}+\mathscr{U}_{B,2}^\epsilon-\bar{\mathscr{U}}_{B,2}^\epsilon\;=\;\frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,1}^\epsilon}{\partial\theta}-\frac{\partial\mathscr{U}_{B,0}^\epsilon}{\partial t}, \tag{3.44}$$

$$\cdots$$

$$\sin\phi\frac{\partial\mathscr{U}_{B,k}^\epsilon}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,k}^\epsilon}{\partial\phi}+\mathscr{U}_{B,k}^\epsilon-\bar{\mathscr{U}}_{B,k}^\epsilon\;=\;\frac{1}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_{B,k-1}^\epsilon}{\partial\theta}-\frac{\partial\mathscr{U}_{B,k-2}^\epsilon}{\partial t}. \tag{3.45}$$

where

$$\bar{\mathscr{U}}^{\epsilon}_{B,k}(t,\eta,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{U}^{\epsilon}_{B,k}(t,\eta,\theta,\phi)\mathrm{d}\phi. \tag{3.46}$$

It is important to note the solution $\mathscr{U}^{\epsilon}_{B,k}$ depends on ϵ and this is the reason why we add the superscript ϵ to U^{ϵ}_{k}, $\mathscr{U}^{\epsilon}_{I,k}$ and $\mathscr{U}^{\epsilon}_{B,k}$.

3.5. **Initial-Boundary Layer Expansion.** Above construction of initial layer and boundary layer yields an interesting fact that at the corner point $(t,\vec{x})=(0,\vec{x}_0)$ for $\vec{x}_0\in\partial\Omega$, the initial layer starting from this point has a contribution on the boundary data, and the boundary layer starting from this point has a contribution on the initial data. Therefore, we have to find some additional functions to compensate these noises. The classical theory of asymptotic analysis requires the so-called initial-boundary layer, where both the temporal scaling and spacial scaling should be used simultaneously. Fortunately, based on our analysis, the improved compatibility condition (3.7) implies the value at this corner point is a constant for $\vec{w}\cdot\vec{n}<0$. Then These contribution must be zero at the zeroth order, i.e.

$$\mathscr{U}^{\epsilon}_{I,0}(\tau,\vec{x}_0,\vec{w}) = 0, \tag{3.47}$$

$$\mathscr{U}^{\epsilon}_{B,0}(0,\eta,\theta,\phi) = 0, \tag{3.48}$$

Therefore, the zeroth order initial-boundary layer is absent.

3.6. **Construction of Asymptotic Expansion.** The bridge between the interior solution, the initial layer, and the boundary layer is the initial and boundary condition of (1.1). To avoid the introduction of higher order initial-boundary layer, we only require the zeroth order expansion of initial and boundary data be satisfied, i.e. we have

$$U^{\epsilon}_0(0,\vec{x},\vec{w})+\mathscr{U}^{\epsilon}_{I,0}(0,\vec{x},\vec{w})+\mathscr{U}^{\epsilon}_{B,0}(0,\vec{x},\vec{w}) = h(\vec{x},\vec{w}), \tag{3.49}$$

$$U^{\epsilon}_0(t,\vec{x}_0,\vec{w})+\mathscr{U}^{\epsilon}_{I,0}(t,\vec{x}_0,\vec{w})+\mathscr{U}^{\epsilon}_{B,0}(t,\vec{x}_0,\vec{w}) = g(t,\vec{x}_0,\vec{w}). \tag{3.50}$$

The construction of U^{ϵ}_{k}. $\mathscr{U}^{\epsilon}_{I,k}$ and $\mathscr{U}^{\epsilon}_{B,k}$ are as follows:

Assume the cut-off function ψ and ψ_0 are defined as

$$\psi(\mu)=\left\{\begin{array}{ll}1 & 0\leq\mu\leq 1/2,\\ 0 & 3/4\leq\mu\leq\infty.\end{array}\right. \tag{3.51}$$

$$\psi_0(\mu)=\left\{\begin{array}{ll}1 & 0\leq\mu\leq 1/4,\\ 0 & 3/8\leq\mu\leq\infty.\end{array}\right. \tag{3.52}$$

and define the force as

$$F(\epsilon;\eta)=-\frac{\epsilon\psi(\epsilon\eta)}{1-\epsilon\eta}, \tag{3.53}$$

Step 1: Construction of zeroth order terms.
The zeroth order boundary layer solution is defined as

$$\left\{\begin{array}{rcl}\mathscr{U}^{\epsilon}_{B,0}(t,\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f^{\epsilon}_0(t,\eta,\theta,\phi)-f^{\epsilon}_0(t,\infty,\theta)\bigg),\\ \sin\phi\dfrac{\partial f^{\epsilon}_0}{\partial\eta}-F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}_0}{\partial\phi}+f^{\epsilon}_0-\bar{f}^{\epsilon}_0 &=& 0,\\ f^{\epsilon}_0(t,0,\theta,\phi) &=& g(t,\theta,\phi)\ \text{ for }\ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f^{\epsilon}_0(t,\eta,\theta,\phi) &=& f^{\epsilon}_0(t,\infty,\theta).\end{array}\right. \tag{3.54}$$

Assuming $g\in L^{\infty}$, by Theorem 4.1, we can show there exists a unique solution $f^{\epsilon}_0(t,\eta,\theta,\phi)\in L^{\infty}$. Hence, $\mathscr{U}^{\epsilon}_{B,0}$ is well-defined.

The zeroth order initial layer is defined as

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{I,0}(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_0(\tau,\vec{x},\vec{w})-\mathcal{F}^{\epsilon}_0(\infty,\vec{x})\\
\partial_{\tau}\bar{\mathcal{F}}^{\epsilon}_0 &=& 0,\\
\mathcal{F}^{\epsilon}_0(\tau,\vec{x},\vec{w}) &=& \mathrm{e}^{-\tau}\mathcal{F}^{\epsilon}_0(0,\vec{x},\vec{w})+(1-\mathrm{e}^{-\tau})\bar{\mathcal{F}}^{\epsilon}_0(0,\vec{x}),\\
\mathcal{F}^{\epsilon}_0(0,\vec{x},\vec{w}) &=& h(\vec{x},\vec{w}),\\
\lim_{\tau\rightarrow\infty}\mathcal{F}^{\epsilon}_0(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_0(\infty,\vec{x}).
\end{array}
\right.
\tag{3.55}
$$

Assuming $h\in L^{\infty}$. Then we can show there exists a unique solution $\mathcal{F}^{\epsilon}_0(\tau,\vec{x},\vec{w})\in L^{\infty}$. Hence, $\mathscr{U}^{\epsilon}_{I,0}$ is well-defined.
Then we can define the zeroth order interior solution as

$$
\left\{
\begin{array}{rcl}
U^{\epsilon}_0 &=& \bar{U}^{\epsilon}_0,\\
\partial_t\bar{U}^{\epsilon}_0-\Delta_x\bar{U}^{\epsilon}_0 &=& 0,\\
\bar{U}^{\epsilon}_0(0,\vec{x}) &=& \mathcal{F}^{\epsilon}_0(\infty,\vec{x}) \ \ \text{in} \ \ \Omega,\\
\bar{U}^{\epsilon}_0(t,\vec{x}_0) &=& f^{\epsilon}_0(t,\infty,\theta) \ \ \text{on} \ \ \partial\Omega,
\end{array}
\right.
\tag{3.56}
$$

where $(t,\vec{x},\vec{w})$ is the same point as (τ,η,θ,ϕ). Note that due to the improved compatibility condition (3.7), we have $\mathscr{U}^{\epsilon}_{B,0}(0,\eta,\theta,\phi)=\mathscr{U}^{\epsilon}_{I,0}(\tau,\vec{x}_0,\vec{w})=0$.

Step 2: Construction of first order terms.
Define the first order boundary layer solution as

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{B,1}(t,\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f^{\epsilon}_1(t,\eta,\theta,\phi)-f^{\epsilon}_1(t,\infty,\theta)\bigg),\\
\sin\phi\dfrac{\partial f^{\epsilon}_1}{\partial\eta}-F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}_1}{\partial\phi}+f^{\epsilon}_1-\bar{f}^{\epsilon}_1 &=& \cos\phi\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}^{\epsilon}_{B,0}}{\partial\theta},\\
f^{\epsilon}_1(t,0,\theta,\phi) &=& \vec{w}\cdot\nabla_xU^{\epsilon}_0(t,\vec{x}_0,\vec{w}) \ \ \text{for} \ \ \sin\phi>0,\\
\lim_{\eta\rightarrow\infty}f^{\epsilon}_1(t,\eta,\theta,\phi) &=& f^{\epsilon}_1(t,\infty,\theta).
\end{array}
\right.
\tag{3.57}
$$

Define the first order initial layer as

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{I,1}(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_1(\tau,\vec{x},\vec{w})-\mathcal{F}^{\epsilon}_1(\infty,\vec{x})\\
\partial_{\tau}\bar{\mathcal{F}}^{\epsilon}_1 &=& -\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}^{\epsilon}_{I,0}\bigg)\mathrm{d}\vec{w},\\
\mathcal{F}^{\epsilon}_1(\tau,\vec{x},\vec{w}) &=& \mathrm{e}^{-\tau}\mathcal{F}^{\epsilon}_1(0,\vec{x},\vec{w})+\displaystyle\int_0^{\tau}\bigg(\bar{\mathcal{F}}^{\epsilon}_1-\vec{w}\cdot\nabla_x\mathscr{U}^{\epsilon}_{I,0}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\
\mathcal{F}^{\epsilon}_1(0,\vec{x},\vec{w}) &=& \vec{w}\cdot\nabla_xU^{\epsilon}_0(0,\vec{x},\vec{w}),\\
\lim_{\tau\rightarrow\infty}\mathcal{F}^{\epsilon}_1(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_1(\infty,\vec{x}).
\end{array}
\right.
\tag{3.58}
$$

Define the first order interior solution as

$$
\left\{
\begin{array}{rcl}
U^{\epsilon}_1 &=& \bar{U}^{\epsilon}_1-\vec{w}\cdot\nabla_xU^{\epsilon}_0,\\
\partial_t\bar{U}^{\epsilon}_1-\Delta_x\bar{U}^{\epsilon}_1 &=& 0,\\
\bar{U}^{\epsilon}_1(0,\vec{x}) &=& \mathcal{F}^{\epsilon}_1(\infty,\vec{x}) \ \ \text{in} \ \ \Omega,\\
\bar{U}^{\epsilon}_1(t,\vec{x}) &=& f^{\epsilon}_1(t,\infty,\theta) \ \ \text{on} \ \ \partial\Omega.
\end{array}
\right.
\tag{3.59}
$$

Step 3: Construction of $\mathscr{U}^{\epsilon}_2$ and U^{ϵ}_2.
Define the second order boundary layer solution as

(3.60)

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{B,2}(t,\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f^{\epsilon}_2(t,\eta,\theta,\phi)-f^{\epsilon}_2(t,\infty,\theta)\bigg),\\
\sin\phi\dfrac{\partial f^{\epsilon}_2}{\partial\eta}-F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}_2}{\partial\phi}+f^{\epsilon}_2-\bar{f}^{\epsilon}_2 &=& \cos\phi\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}^{\epsilon}_{B,1}}{\partial\theta}-\dfrac{\partial\mathscr{U}^{\epsilon}_{B,0}}{\partial t},\\
f^{\epsilon}_2(t,0,\theta,\phi) &=& \vec{w}\cdot\nabla_xU^{\epsilon}_1(t,\vec{x}_0,\vec{w})+\partial_tU^{\epsilon}_0(t,\vec{x}_0,\vec{w}) \ \ \text{for} \ \ \sin\phi>0,\\
\lim_{\eta\rightarrow\infty}f^{\epsilon}_2(t,\eta,\theta,\phi) &=& f^{\epsilon}_2(t,\infty,\theta).
\end{array}
\right.
$$

Define the second order initial layer as

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{I,2}(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_2(\tau,\vec{x},\vec{w})-\mathcal{F}^{\epsilon}_2(\infty,\vec{x})\\
\partial_{\tau}\bar{\mathcal{F}}^{\epsilon}_2 &=& -\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}^{\epsilon}_{I,1}\bigg)\mathrm{d}\vec{w},\\
\mathcal{F}^{\epsilon}_2(\tau,\vec{x},\vec{w}) &=& \mathrm{e}^{-\tau}\mathcal{F}^{\epsilon}_2(0,\vec{x},\vec{w})+\displaystyle\int_0^{\tau}\bigg(\bar{\mathcal{F}}^{\epsilon}_2-\vec{w}\cdot\nabla_x\mathscr{U}^{\epsilon}_{I,1}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\
\mathcal{F}^{\epsilon}_2(0,\vec{x},\vec{w}) &=& \vec{w}\cdot\nabla_x U^{\epsilon}_1(0,\vec{x},\vec{w})+\partial_t U^{\epsilon}_0(0,\vec{x},\vec{w}),\\
\lim_{\tau\rightarrow\infty}\mathcal{F}^{\epsilon}_2(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_2(\infty,\vec{x}).
\end{array}
\right.
\tag{3.61}
$$

Define the first order interior solution as

$$
\left\{
\begin{array}{rcl}
U^{\epsilon}_2 &=& \bar{U}^{\epsilon}_2-\vec{w}\cdot\nabla_x U^{\epsilon}_1-\partial_t U^{\epsilon}_0,\\
\partial_t\bar{U}^{\epsilon}_2-\Delta_x\bar{U}^{\epsilon}_2 &=& 0,\\
\bar{U}^{\epsilon}_2(0,\vec{x}) &=& \mathcal{F}^{\epsilon}_2(\infty,\vec{x})\ \ \text{in}\ \ \Omega,\\
\bar{U}^{\epsilon}_2(t,\vec{x}) &=& f^{\epsilon}_2(t,\infty,\theta)\ \ \text{on}\ \ \partial\Omega.
\end{array}
\right.
\tag{3.62}
$$

Step 4: Generalization to arbitrary k.
Similar to above procedure, we can define the k^{th} order boundary layer solution as
(3.63)

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{B,k}(t,\eta,\theta,\phi) &=& \psi_0(\epsilon\eta)\bigg(f^{\epsilon}_k(t,\eta,\theta,\phi)-f^{\epsilon}_k(t,\infty,\theta)\bigg),\\
\sin\phi\dfrac{\partial f^{\epsilon}_k}{\partial\eta}-F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}_k}{\partial\phi}+f^{\epsilon}_k-\bar{f}^{\epsilon}_k &=& \cos\phi\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}^{\epsilon}_{B,k-1}}{\partial\theta}-\dfrac{\partial\mathscr{U}^{\epsilon}_{B,k-2}}{\partial t},\\
f^{\epsilon}_k(t,0,\theta,\phi) &=& \vec{w}\cdot\nabla_x U^{\epsilon}_{k-1}(t,\vec{x}_0,\vec{w})+\partial_t U^{\epsilon}_{k-2}(t,\vec{x}_0,\vec{w})\ \ \text{for}\ \ \sin\phi>0,\\
\lim_{\eta\rightarrow\infty}f^{\epsilon}_k(t,\eta,\theta,\phi) &=& f^{\epsilon}_k(t,\infty,\theta).
\end{array}
\right.
$$

Define the k^{th} order initial layer as

$$
\left\{
\begin{array}{rcl}
\mathscr{U}^{\epsilon}_{I,k}(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_k(\tau,\vec{x},\vec{w})-\mathcal{F}^{\epsilon}_k(\infty,\vec{x})\\
\partial_{\tau}\bar{\mathcal{F}}^{\epsilon}_k &=& -\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}^{\epsilon}_{I,k-1}\bigg)\mathrm{d}\vec{w},\\
\mathcal{F}^{\epsilon}_k(\tau,\vec{x},\vec{w}) &=& \mathrm{e}^{-\tau}\mathcal{F}^{\epsilon}_k(0,\vec{x},\vec{w})+\displaystyle\int_0^{\tau}\bigg(\bar{\mathcal{F}}^{\epsilon}_k-\vec{w}\cdot\nabla_x\mathscr{U}^{\epsilon}_{I,k-1}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\
\mathcal{F}^{\epsilon}_k(0,\vec{x},\vec{w}) &=& \vec{w}\cdot\nabla_x U^{\epsilon}_{k-1}(0,\vec{x},\vec{w})+\partial_t U^{\epsilon}_{k-2}(0,\vec{x},\vec{w}),\\
\lim_{\tau\rightarrow\infty}\mathcal{F}^{\epsilon}_k(\tau,\vec{x},\vec{w}) &=& \mathcal{F}^{\epsilon}_k(\infty,\vec{x}).
\end{array}
\right.
\tag{3.64}
$$

Define the k^{th} order interior solution as

$$
\left\{
\begin{array}{rcl}
U^{\epsilon}_k &=& \bar{U}^{\epsilon}_k-\vec{w}\cdot\nabla_x U^{\epsilon}_{k-1}-\partial_t U^{\epsilon}_{k-2},\\
\partial_t\bar{U}^{\epsilon}_k-\Delta_x\bar{U}^{\epsilon}_k &=& 0,\\
\bar{U}^{\epsilon}_k(0,\vec{x}) &=& \mathcal{F}^{\epsilon}_k(\infty,\vec{x})\ \ \text{in}\ \ \Omega,\\
\bar{U}^{\epsilon}_k &=& f^{\epsilon}_k(t,\infty,\theta)\ \ \text{on}\ \ \partial\Omega.
\end{array}
\right.
\tag{3.65}
$$

When g and h are sufficiently smooth, then all the functions defined above are well-posed. The key point here is in the boundary layer, the source term including $\partial_{\theta}\mathscr{U}^{\epsilon}_{B,k}$ is in L^{∞} due to the substitution (3.39).

4. ϵ-Milne Problem

In this section, we study the ϵ-Milne problem for $f^{\epsilon}(\eta,\theta,\phi)$ in the domain $(\eta,\theta,\phi)\in[0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)$

$$
\left\{
\begin{array}{rcl}
\sin\phi\dfrac{\partial f^{\epsilon}}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial f^{\epsilon}}{\partial\phi}+f^{\epsilon}-\bar{f}^{\epsilon} &=& S^{\epsilon}(\eta,\theta,\phi),\\
f^{\epsilon}(0,\theta,\phi) &=& H^{\epsilon}(\theta,\phi)\ \ \text{for}\ \ \sin\phi>0,\\
\lim_{\eta\rightarrow\infty}f^{\epsilon}(\eta,\theta,\phi) &=& f^{\epsilon}_{\infty}(\theta),
\end{array}
\right.
\tag{4.1}
$$

where

$$
\bar{f}^{\epsilon}(\eta,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{\epsilon}(\eta,\theta,\phi)\mathrm{d}\phi,
\tag{4.2}
$$

$$F(\epsilon;\eta)=-\frac{\epsilon\psi(\epsilon\eta)}{1-\epsilon\eta}, \tag{4.3}$$

$$\psi(\mu)=\begin{cases}1 & 0\leq\mu\leq 1/2,\\ 0 & 3/4\leq\mu\leq\infty,\end{cases} \tag{4.4}$$

$$\left|H^{\epsilon}(\theta,\phi)\right|\leq M, \tag{4.5}$$

and

$$\left|S^{\epsilon}(\eta,\theta,\phi)\right|\leq Me^{-K\eta}, \tag{4.6}$$

for $M>0$ and $K>0$ uniform in ϵ and θ. In this section, since the key variables here are η and ϕ, we temporarily ignore the dependence on ϵ and θ. We define the norms in the space $(\eta,\phi)\in[0,\infty)\times[-\pi,\pi)$ as follows:

$$\left\|f\right\|_{L^2L^2} = \bigg(\int_0^{\infty}\int_{-\pi}^{\pi}\left|f(\eta,\phi)\right|^2\mathrm{d}\phi\mathrm{d}\eta\bigg)^{1/2}, \tag{4.7}$$

$$\left\|f\right\|_{L^{\infty}L^{\infty}} = \sup_{(\eta,\phi)\in[0,\infty)\times[-\pi,\pi)}\left|f(\eta,\phi)\right|. \tag{4.8}$$

In [13, Section 4], the authors proved the following results:

Theorem 4.1. *There exists a unique solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.1) satisfying*

$$\left\|f-f_{\infty}\right\|_{L^2L^2}\leq C\bigg(1+M+\frac{M}{K}\bigg). \tag{4.9}$$

Theorem 4.2. *There exists a unique solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.1) satisfying*

$$\left\|f-f_{\infty}\right\|_{L^{\infty}L^{\infty}}\leq C\bigg(1+M+\frac{M}{K}\bigg). \tag{4.10}$$

Theorem 4.3. *For $K_0>0$ sufficiently small, the solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.1) satisfies*

$$\left\|e^{K_0\eta}(f-f_{\infty})\right\|_{L^2L^2}\leq C\bigg(1+M+\frac{M}{K}\bigg), \tag{4.11}$$

Theorem 4.4. *For $K_0>0$ sufficiently small, the solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.1) satisfies*

$$\left\|e^{K_0\eta}(f-f_{\infty})\right\|_{L^{\infty}L^{\infty}}\leq C\bigg(1+M+\frac{M}{K}\bigg), \tag{4.12}$$

Theorem 4.5. *The solution $f(\eta,\phi)$ to the ϵ-Milne problem (4.1) with $S=0$ satisfies the maximum principle, i.e.*

$$\min_{\sin\phi>0}h(\phi)\leq f(\eta,\phi)\leq\max_{\sin\phi>0}h(\phi). \tag{4.13}$$

Remark 4.6. *Note that when $F=0$, Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4, and Theorem 4.5 still hold. Hence, we can deduce the well-posedness, decay and maximum principle of the classical Milne problem*

$$\left\{\begin{array}{rcl}\sin\phi\dfrac{\partial f}{\partial\eta}+f-\bar{f} &=& S(\eta,\phi),\\ f(0,\phi) &=& h(\phi)\ \ for\ \ \sin\phi>0,\\ \lim_{\eta\rightarrow\infty}f(\eta,\phi) &=& f_{\infty}.\end{array}\right. \tag{4.14}$$

5. Diffusive Limit

In this section, we prove the first part of Theorem 1.2.

Theorem 5.1. *Assume $g(t,\vec{x}_0,\vec{w}) \in C^2([0,\infty)\times\Gamma^-)$ and $h(\vec{x},\vec{w}) \in C^2(\Omega\times\mathcal{S}^1)$. Then for the unsteady neutron transport equation (1.1), the unique solution $u^\epsilon(t,\vec{x},\vec{w}) \in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfies*

$$\left\|u^\epsilon - U_0^\epsilon - \mathscr{U}_{I,0}^\epsilon - \mathscr{U}_{B,0}^\epsilon\right\|_{L^\infty} = o(1), \tag{5.1}$$

where the interior solution U_0^ϵ is defined in (3.56), the initial layer $\mathscr{U}_{I,0}^\epsilon$ is defined in (3.55), and the boundary layer $\mathscr{U}_{B,0}^\epsilon$ is defined in (3.54).

Proof. We divide the proof into several steps:

Step 1: Remainder definitions.
We may rewrite the asymptotic expansion as follows:

$$u^\epsilon \quad\sim\quad \sum_{k=0}^{\infty}\epsilon^k U_k^\epsilon + \sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_{I,k}^\epsilon + \sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_{B,k}^\epsilon. \tag{5.2}$$

The remainder can be defined as

$$R_N \quad=\quad u^\epsilon - \sum_{k=0}^{N}\epsilon^k U_k^\epsilon - \sum_{k=0}^{N}\epsilon^k\mathscr{U}_{I,k}^\epsilon - \sum_{k=0}^{N}\epsilon^k\mathscr{U}_{B,k}^\epsilon = u^\epsilon - Q_N - \mathscr{Q}_{I,N} - \mathscr{Q}_{B,N}, \tag{5.3}$$

where

$$Q_N \quad=\quad \sum_{k=0}^{N}\epsilon^k U_k^\epsilon, \tag{5.4}$$

$$\mathscr{Q}_{I,N} \quad=\quad \sum_{k=0}^{N}\epsilon^k \mathscr{U}_{I,k}^\epsilon, \tag{5.5}$$

$$\mathscr{Q}_{B,N} \quad=\quad \sum_{k=0}^{N}\epsilon^k \mathscr{U}_{B,k}^\epsilon. \tag{5.6}$$

Noting the equation is equivalent to the equations (3.21) and (3.40), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\mathcal{L}u \quad=\quad \epsilon^2\partial_t u + \epsilon\vec{w}\cdot\nabla_x u + u - \bar{u} \tag{5.7}$$

$$\quad=\quad \partial_\tau u + \epsilon\vec{w}\cdot\nabla_x u + u - \bar{u} \tag{5.8}$$

$$\quad=\quad \epsilon^2\frac{\partial u}{\partial t} + \sin\phi\frac{\partial u}{\partial\eta} - \frac{\epsilon}{1-\epsilon\eta}\cos\phi\left(\frac{\partial u}{\partial\phi} + \frac{\partial u}{\partial\theta}\right) + u - \bar{u}.$$

Step 2: Estimates of $\mathcal{L}Q_N$.
The interior contribution can be estimated as

$$\begin{aligned}\mathcal{L}Q_0 \quad&=\quad \epsilon^2\partial_t Q_0 + \epsilon\vec{w}\cdot\nabla_x Q_0 + Q_0 - \bar{Q}_0\\ &=\quad \epsilon^2\partial_t U_0^\epsilon + \epsilon\vec{w}\cdot\nabla_x U_0^\epsilon + (U_0^\epsilon - \bar{U}_0^\epsilon) = \epsilon^2\partial_t U_0^\epsilon + \epsilon\vec{w}\cdot\nabla_x U_0^\epsilon.\end{aligned} \tag{5.9}$$

We have

$$\left|\epsilon^2\partial_t U_0^\epsilon\right| \quad\leq\quad C\epsilon^2\left|\partial_t U_0^\epsilon\right| \leq C\epsilon^2, \tag{5.10}$$

$$\left|\epsilon\vec{w}\cdot\nabla_x U_0^\epsilon\right| \quad\leq\quad C\epsilon\left|\nabla_x U_0^\epsilon\right| \leq C\epsilon. \tag{5.11}$$

This implies

$$\left|\mathcal{L}Q_0\right| \leq C\epsilon. \tag{5.12}$$

Similarly, for higher order term, we can estimate

$$\mathcal{L}Q_N = \epsilon^2\partial_t Q_N + \epsilon\vec{w}\cdot\nabla_x Q_N + Q_N - \bar{Q}_N \quad=\quad \epsilon^{N+2}\partial_t U_N^\epsilon + \epsilon^{N+1}\vec{w}\cdot\nabla_x U_N^\epsilon. \tag{5.13}$$

We have

$$\left|\epsilon^{N+1}\partial_t U_N^\epsilon\right| \leq C\epsilon^{N+2}\left|\partial_t U_N^\epsilon\right| \leq C\epsilon^{N+2}, \tag{5.14}$$

$$\left|\epsilon^{N+1}\vec{w}\cdot\nabla_x U_N^\epsilon\right| \leq C\epsilon^{N+1}\left|\nabla_x U_N^\epsilon\right| \leq C\epsilon^{N+1}. \tag{5.15}$$

This implies

$$\left|\mathcal{L}Q_N\right| \leq C\epsilon^{N+1}. \tag{5.16}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}_{I,N}$.
The initial layer contribution can be estimated as

$$\begin{aligned}\mathcal{L}\mathscr{Q}_{I,0} &= \partial_\tau\mathscr{Q}_{I,0}+\epsilon\vec{w}\cdot\nabla_x\mathscr{Q}_{I,0}+\mathscr{Q}_{I,0}-\bar{\mathscr{Q}}_{I,0}\\ &= \partial_\tau\mathscr{U}_{I,0}^\epsilon+\epsilon\vec{w}\cdot\nabla_x\mathscr{U}_{I,0}^\epsilon+\mathscr{U}_{I,0}^\epsilon-\bar{\mathscr{U}}^\epsilon{}_{I,0}=\epsilon\nabla_x\mathscr{U}_{I,0}^\epsilon.\end{aligned} \tag{5.17}$$

Based on the smoothness of $\mathscr{U}_{I,0}^\epsilon$, we have

$$\left|\mathcal{L}\mathscr{Q}_{I,0}\right| = \left|\epsilon\nabla_x\mathscr{U}_{I,0}^\epsilon\right| \leq C\epsilon. \tag{5.18}$$

Similarly, we have

$$\mathcal{L}\mathscr{Q}_{I,N} = \partial_\tau\mathscr{Q}_{I,N}+\epsilon\vec{w}\cdot\nabla_x\mathscr{Q}_{I,N}+\mathscr{Q}_{I,N}-\bar{\mathscr{Q}}_{I,N}=\epsilon^{N+1}\nabla_x\mathscr{U}_{I,N}^\epsilon. \tag{5.19}$$

Therefore, we have

$$\left|\mathcal{L}\mathscr{Q}_{I,N}\right| = \left|\epsilon^{N+1}\nabla_x\mathscr{U}_{I,N}^\epsilon\right| \leq C\epsilon^{N+1}. \tag{5.20}$$

Step 4: Estimates of $\mathcal{L}\mathscr{Q}_{B,N}$.
The boundary layer solution is $\mathscr{U}_k^\epsilon = (f_k^\epsilon - f_k^\epsilon(\infty))\cdot\psi_0 = \mathscr{V}_k\psi_0$ where $f_k^\epsilon(\eta,\theta,\phi)$ solves the ϵ-Milne problem and $\mathscr{V}_k = f_k^\epsilon - f_k^\epsilon(\infty)$. Notice $\psi_0\psi=\psi_0$, so the boundary layer contribution can be estimated as

(5.21)

$$\begin{aligned}\mathcal{L}\mathscr{Q}_{B,0} &= \epsilon^2\frac{\partial\mathscr{Q}_{B,0}}{\partial t}+\sin\phi\frac{\partial\mathscr{Q}_{B,0}}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\frac{\partial\mathscr{Q}_{B,0}}{\partial\phi}+\frac{\partial\mathscr{Q}_{B,0}}{\partial\theta}\bigg)+\mathscr{Q}_{B,0}-\bar{\mathscr{Q}}_{B,0}\\
&= \epsilon^2\frac{\partial\mathscr{V}_0}{\partial t}+\sin\phi\bigg(\psi_0\frac{\partial\mathscr{V}_0}{\partial\eta}+\mathscr{V}_0\frac{\partial\psi_0}{\partial\eta}\bigg)-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\frac{\partial\mathscr{V}_0}{\partial\phi}+\frac{\partial\mathscr{V}_0}{\partial\theta}\bigg)+\psi_0\mathscr{V}_0-\psi_0\bar{\mathscr{V}}_0\\
&= \epsilon^2\frac{\partial\mathscr{V}_0}{\partial t}+\sin\phi\bigg(\psi_0\frac{\partial\mathscr{V}_0}{\partial\eta}+\mathscr{V}_0\frac{\partial\psi_0}{\partial\eta}\bigg)-\frac{\psi_0\psi\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\frac{\partial\mathscr{V}_0}{\partial\phi}+\frac{\partial\mathscr{V}_0}{\partial\theta}\bigg)+\psi_0\mathscr{V}_0-\psi_0\bar{\mathscr{V}}_0\\
&= \epsilon^2\frac{\partial\mathscr{V}_0}{\partial t}+\psi_0\bigg(\sin\phi\frac{\partial\mathscr{V}_0}{\partial\eta}-\frac{\epsilon\psi}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\phi}+\mathscr{V}_0-\bar{\mathscr{V}}_0\bigg)+\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_0-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\theta}\\
&= \epsilon^2\frac{\partial\mathscr{V}_0}{\partial t}+\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_0-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\theta}.\end{aligned}$$

It is easy to see

$$\left|\epsilon^2\frac{\partial\mathscr{V}_0}{\partial t}\right| \leq \epsilon^2\left|\frac{\partial\mathscr{V}_0}{\partial t}\right| \leq C\epsilon^2. \tag{5.22}$$

Since $\psi_0=1$ when $\eta\leq 1/(4\epsilon)$, the effective region of $\partial_\eta\psi_0$ is $\eta\geq 1/(4\epsilon)$ which is further and further from the origin as $\epsilon\to 0$. By Theorem 4.2, the first term in (5.21) can be controlled as

$$\left|\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_0\right| \leq C\mathrm{e}^{-\frac{K_0}{\epsilon}} \leq C\epsilon. \tag{5.23}$$

For the second term in (5.21), we have

$$\left|-\frac{\psi_0\epsilon}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_0}{\partial\theta}\right| \leq C\epsilon\left|\frac{\partial\mathscr{V}_0}{\partial\theta}\right| \leq C\epsilon. \tag{5.24}$$

This implies

$$\left|\mathcal{L}\mathscr{Q}_{B,0}\right| \leq C\epsilon. \tag{5.25}$$

Similarly, for higher order term, we can estimate

$$\begin{aligned}
\mathcal{L}\mathscr{Q}_{B,N} &= \epsilon^2\frac{\partial\mathscr{Q}_{B,N}}{\partial t}+\sin\phi\frac{\partial\mathscr{Q}_{B,N}}{\partial\eta}-\frac{\epsilon}{1-\epsilon\eta}\cos\phi\bigg(\frac{\partial\mathscr{Q}_{B,N}}{\partial\phi}+\frac{\partial\mathscr{Q}_{B,N}}{\partial\theta}\bigg)+\mathscr{Q}_{B,N}-\bar{\mathscr{Q}}_{B,N}\\
&= \epsilon^{N+2}\frac{\partial\mathscr{V}_N}{\partial t}+\sum_{i=0}^{k}\epsilon^i\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_i-\frac{\psi_0\epsilon^{k+1}}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_k}{\partial\theta}.
\end{aligned}\tag{5.26}$$

It is obvious that

$$\left|\epsilon^{N+2}\frac{\partial\mathscr{V}_N}{\partial t}\right|\leq\epsilon^{N+2}\left|\frac{\partial\mathscr{V}_N}{\partial t}\right|\leq C\epsilon^{N+2}.\tag{5.27}$$

Away from the origin, the first term in (5.26) can be controlled as

$$\left|\sum_{i=0}^{k}\epsilon^i\sin\phi\frac{\partial\psi_0}{\partial\eta}\mathscr{V}_i\right| \leq C\mathrm{e}^{-\frac{K_0}{\epsilon}}\leq C\epsilon^{k+1}.\tag{5.28}$$

For the second term in (5.26), we have

$$\left|-\frac{\psi_0\epsilon^{k+1}}{1-\epsilon\eta}\cos\phi\frac{\partial\mathscr{V}_k}{\partial\theta}\right| \leq C\epsilon^{k+1}\left|\frac{\partial\mathscr{V}_k}{\partial\theta}\right|\leq C\epsilon^{k+1}.\tag{5.29}$$

This implies

$$|\mathcal{L}\mathscr{Q}_{B,N}|\leq C\epsilon^{k+1}.\tag{5.30}$$

Step 5: Synthesis.
In summary, since $\mathcal{L}u^\epsilon=0$, collecting (5.3), (5.16), (5.20), and (5.30), we can prove

$$|\mathcal{L}R_N|\leq C\epsilon^{N+1}.\tag{5.31}$$

Consider the asymptotic expansion to $N=2$, then the remainder R_2 satisfies the equation

(5.32)
$$\left\{\begin{aligned}
\epsilon\partial_tR_2+\epsilon\vec{w}\cdot\nabla_xR_2+R_2-\bar{R}_2 &= \mathcal{L}R_2,\\
R_2(0,\vec{x},\vec{w}) &= \epsilon\mathscr{U}^\epsilon_{B,1}(0,\vec{x},\vec{w})+\epsilon^2\mathscr{U}^\epsilon_{B,2}(0,\vec{x},\vec{w}),\\
R_2(t,\vec{x}_0,\vec{w}) &= \epsilon\mathscr{U}^\epsilon_{I,1}(t,\vec{x}_0,\vec{w})+\epsilon^2\mathscr{U}^\epsilon_{I,2}(t,\vec{x}_0,\vec{w})\ \text{ for }\ \vec{w}\cdot\vec{n}<0\ \text{ and }\ \vec{x}_0\in\partial\Omega.
\end{aligned}\right.$$

Note that the initial data and boundary data are nonzero due the contribution of initial layer and boundary data at the point $(t,\vec{x})=(0,\vec{x}_0)$. By Theorem 2.10, we have

$$\begin{aligned}
\|R_2\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} &\leq C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\|\mathcal{L}R_2\|_{L^2([0,\infty)\times\Omega\times\mathcal{S}^1)}+\|\mathcal{L}R_2\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}\bigg)\\
&\quad+\left\|\epsilon\mathscr{U}^\epsilon_{B,1}(0,\vec{x},\vec{w})+\epsilon^2\mathscr{U}^\epsilon_{B,2}(0,\vec{x},\vec{w})\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}\\
&\quad+\left\|\epsilon\mathscr{U}^\epsilon_{I,1}(\infty,\vec{x}_0,\vec{w})+\epsilon^2\mathscr{U}^\epsilon_{I,2}(t,\vec{x}_0,\vec{w})\right\|_{L^\infty([0,t]\times\Gamma^-)}\\
&\leq C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}(C\epsilon^3)+(C\epsilon^3)\bigg)+C\epsilon+C\epsilon=C(\Omega)\epsilon^{1/2}.
\end{aligned}\tag{5.33}$$

Hence, we have

$$\left\|u^\epsilon-\sum_{k=0}^{2}\epsilon^kU^\epsilon_k-\sum_{k=0}^{2}\epsilon^k\mathscr{U}^\epsilon_{I,k}-\sum_{k=0}^{2}\epsilon^k\mathscr{U}^\epsilon_{B,k}\right\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)}=o(1).\tag{5.34}$$

Since it is easy to see

$$\left\|\sum_{k=1}^{2}\epsilon^kU^\epsilon_k+\sum_{k=1}^{2}\epsilon^k\mathscr{U}^\epsilon_{I,k}+\sum_{k=1}^{2}\epsilon^k\mathscr{U}^\epsilon_{B,k}\right\|_{L^\infty(\Omega\times\mathcal{S}^1)}=O(\epsilon),\tag{5.35}$$

our result naturally follows. □

6. Counterexample for Classical Approach

In this section, we present the classical approach in [1] to construct asymptotic expansion, especially the boundary layer expansion, and give a counterexample to show this method is problematic in unsteady equation.

6.1. Discussion on Expansions except Boundary Layer.

Basically, the expansions for interior solution and initial layer are identical to our method, so omit the details and only present the notation. We define the interior expansion as follows:

$$U(t,\vec{x},\vec{w})\sim\sum_{k=0}^{\infty}\epsilon^kU_k(t,\vec{x},\vec{w}), \tag{6.1}$$

$U_0(t,\vec{x},\vec{w})$ satisfies the equation

$$\left\{\begin{array}{rcl} U_0&=&\bar{U}_0,\\ \partial_t\bar{U}_0-\Delta_x\bar{U}_0&=&0.\end{array}\right. \tag{6.2}$$

$U_1(t,\vec{x},\vec{w})$ satisfies

$$\left\{\begin{array}{rcl} U_1&=&\bar{U}_1-\vec{w}\cdot\nabla_xU_0,\\ \partial_t\bar{U}_1-\Delta_x\bar{U}_1&=&0,\end{array}\right. \tag{6.3}$$

and $U_k(t,\vec{x},\vec{w})$ for $k\geq 2$ satisfies

$$\left\{\begin{array}{rcl} U_k&=&\bar{U}_k-\vec{w}\cdot\nabla_xU_{k-1}-\partial_tU_{k-2},\\ \partial_t\bar{U}_k-\Delta_x\bar{U}_k&=&0.\end{array}\right. \tag{6.4}$$

With the substitution (3.19), we define the initial layer expansion as follows:

$$\mathscr{U}_I(\tau,\vec{x},\vec{w})\sim\sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_{I,k}(\tau,\vec{x},\vec{w}), \tag{6.5}$$

where $\mathscr{U}_{I,0}$ satisfies

$$\left\{\begin{array}{rcl} \partial_\tau\bar{\mathscr{U}}_{I,0}&=&0,\\ \mathscr{U}_{I,0}(\tau,\vec{x},\vec{w})&=&\mathrm{e}^{-\tau}\mathscr{U}_{I,0}(0,\vec{x},\vec{w})+(1-\mathrm{e}^{-\tau})\bar{\mathscr{U}}_{I,0}(0,\vec{x}).\end{array}\right. \tag{6.6}$$

and $\mathscr{U}_{I,k}(\tau,\vec{x},\vec{w})$ for $k\geq 1$ satisfies

$$\left\{\begin{array}{rcl} \partial_\tau\bar{\mathscr{U}}_{I,k}&=&-\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}_{I,k-1}\bigg)\mathrm{d}\vec{w},\\ \mathscr{U}_{I,k}(\tau,\vec{x},\vec{w})&=&\mathrm{e}^{-\tau}\mathscr{U}_{I,k}(0,\vec{x},\vec{w})+\displaystyle\int_0^\tau\bigg(\bar{\mathscr{U}}_{I,k}-\vec{w}\cdot\nabla_x\mathscr{U}_{I,k-1}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s.\end{array}\right. \tag{6.7}$$

6.2. Boundary Layer Expansion.

By the idea in [1], the boundary layer expansion can be defined by introducing substitutions (3.32), (3.34), and (3.37). Note that we terminate here and do not further use substitution (3.39). Hence, we have the transformed equation for (1.1) as

$$\left\{\begin{array}{l} \epsilon^2\dfrac{\partial u^\epsilon}{\partial t}+\sin(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\eta}-\dfrac{\epsilon}{1-\epsilon\eta}\cos(\theta+\xi)\dfrac{\partial u^\epsilon}{\partial\theta}+u^\epsilon-\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}u^\epsilon\mathrm{d}\xi=0,\\ u^\epsilon(0,\eta,\theta,\xi)=h(\eta,\theta),\\ u^\epsilon(0,\theta,\xi)=g(\theta,\xi)\quad\text{for}\quad\sin(\theta+\xi)>0.\end{array}\right. \tag{6.8}$$

We now define the Milne expansion of boundary layer as follows:

$$\mathscr{U}(t,\eta,\theta,\phi)\sim\sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_k(t,\eta,\theta,\phi), \tag{6.9}$$

where $\mathscr{U}_k$ can be determined by comparing the order of ϵ via plugging (6.9) into the equation (6.8). Thus, in a neighborhood of the boundary, we have

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_0}{\partial\eta}+\mathscr{U}_0-\bar{\mathscr{U}}_0 = 0, \tag{6.10}$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_1}{\partial\eta}+\mathscr{U}_1-\bar{\mathscr{U}}_1 = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_0}{\partial\theta}, \tag{6.11}$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_2}{\partial\eta}+\mathscr{U}_2-\bar{\mathscr{U}}_2 = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_1}{\partial\theta}-\frac{\partial \mathscr{U}_0}{\partial t}, \tag{6.12}$$

$$\ldots$$

$$\sin(\theta+\xi)\frac{\partial \mathscr{U}_k}{\partial\eta}+\mathscr{U}_k-\bar{\mathscr{U}}_k = \frac{1}{1-\epsilon\eta}\cos(\theta+\xi)\frac{\partial \mathscr{U}_{k-1}}{\partial\theta}-\frac{\partial \mathscr{U}_{k-2}}{\partial t}, \tag{6.13}$$

where

$$\bar{\mathscr{U}}_k(t,\eta,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{U}_k(t,\eta,\theta,\xi)\mathrm{d}\xi. \tag{6.14}$$

6.3. **Classical Approach to Construct Asymptotic Expansion.** Similarly, we require the zeroth order expansion of initial and boundary data be satisfied, i.e. we have

$$U_0(0,\vec{x},\vec{w})+\mathscr{U}_{I,0}(0,\vec{x},\vec{w})+\mathscr{U}_{B,0}(0,\vec{x},\vec{w}) = h, \tag{6.15}$$

$$U_0(t,\vec{x}_0,\vec{w})+\mathscr{U}_{I,0}(t,\vec{x}_0,\vec{w})+\mathscr{U}_{B,0}(t,\vec{x}_0,\vec{w}) = g. \tag{6.16}$$

The construction of U_k, $\mathscr{U}_{I,k}$, and $\mathscr{U}_{B,k}$ by the idea in [1] can be summarized as follows:

Assume the cut-off function ψ and ψ_0 are defined as (3.51) and (3.52).

Step 1: Construction of zeroth order terms.
The zeroth order boundary layer solution is defined as

$$\begin{cases} \mathscr{U}_0(t,\eta,\theta,\xi) = \psi_0(\epsilon\eta)\bigg(f_0(t,\eta,\theta,\xi)-f_0(t,\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_0}{\partial\eta}+f_0-\bar{f}_0 = 0,\\ f_0(t,0,\theta,\xi) = g(t,\theta,\xi)\ \text{ for }\ \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty}f_0(t,\eta,\theta,\xi) = f_0(t,\infty,\theta). \end{cases} \tag{6.17}$$

The zeroth order initial layer is defined as

$$\begin{cases} \mathscr{U}_{I,0}(\tau,\vec{x},\vec{w}) = \mathscr{F}_0(\tau,\vec{x},\vec{w})-\mathscr{F}_0(\infty,\vec{x})\\ \partial_\tau\bar{\mathscr{F}}_0 = 0,\\ \mathscr{F}_0(\tau,\vec{x},\vec{w}) = \mathrm{e}^{-\tau}\mathscr{F}_0(0,\vec{x},\vec{w})+(1-\mathrm{e}^{-\tau})\bar{\mathscr{F}}_0(0,\vec{x}),\\ \mathscr{F}_0(0,\vec{x},\vec{w}) = h(\vec{x},\vec{w}),\\ \lim_{\tau\to\infty}\mathscr{F}_0(\tau,\vec{x},\vec{w}) = \mathscr{F}_0(\infty,\vec{x}). \end{cases} \tag{6.18}$$

Then we can define the zeroth order interior solution as

$$\begin{cases} U_0 = \bar{U}_0,\\ \partial_t\bar{U}_0-\Delta_x\bar{U}_0 = 0,\\ \bar{U}_0(0,\vec{x}) = \mathscr{F}_0(\infty,\vec{x})\ \text{ in }\ \Omega,\\ \bar{U}_0(t,\vec{x}_0) = f_0(t,\infty,\theta)\ \text{ on }\ \partial\Omega, \end{cases} \tag{6.19}$$

where $(t,\vec{x},\vec{w})$ is the same point as (τ,η,θ,ξ).

Step 2: Construction of first order terms.

Define the first order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_1(t,\eta,\theta,\xi) &=& \psi_0(\epsilon\eta)\bigg(f_1(t,\eta,\theta,\xi)-f_1(t,\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_1}{\partial\eta}+f_1-\bar{f}_1 &=& \cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_0}{\partial\theta},\\ f_1(t,0,\theta,\xi) &=& \vec{w}\cdot\nabla_xU_0(t,\vec{x}_0,\vec{w}) \quad\text{for}\quad \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty}f_1(t,\eta,\theta,\xi) &=& f_1(t,\infty,\theta).\end{array}\right. \tag{6.20}$$

Define the first order initial layer as

$$\left\{\begin{array}{rcl}\mathscr{U}_{I,1}(\tau,\vec{x},\vec{w}) &=& \mathcal{F}_1(\tau,\vec{x},\vec{w})-\mathcal{F}_1(\infty,\vec{x})\\ \partial_\tau\bar{\mathcal{F}}_1 &=& -\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}_{I,0}\bigg)\mathrm{d}\vec{w},\\ \mathcal{F}_1(\tau,\vec{x},\vec{w}) &=& \mathrm{e}^{-\tau}\mathcal{F}_1(0,\vec{x},\vec{w})+\displaystyle\int_0^\tau\bigg(\bar{\mathcal{F}}_1-\vec{w}\cdot\nabla_x\mathscr{U}_{I,0}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\ \mathcal{F}_1(0,\vec{x},\vec{w}) &=& \vec{w}\cdot\nabla_xU_0^\epsilon(0,\vec{x},\vec{w}),\\ \lim_{\tau\to\infty}\mathcal{F}_1(\tau,\vec{x},\vec{w}) &=& \mathcal{F}_1(\infty,\vec{x}).\end{array}\right. \tag{6.21}$$

Define the first order interior solution as

$$\left\{\begin{array}{rcl}U_1 &=& \bar{U}_1-\vec{w}\cdot\nabla_xU_0,\\ \partial_t\bar{U}_1-\Delta_x\bar{U}_1 &=& 0,\\ \bar{U}_1(0,\vec{x}) &=& \mathcal{F}_1(\infty,\vec{x}) \quad\text{in}\quad\Omega,\\ \bar{U}_1(t,\vec{x}) &=& f_1(t,\infty,\theta) \quad\text{on}\quad\partial\Omega.\end{array}\right. \tag{6.22}$$

Step 3: Construction of second order terms.
Define the second order boundary layer solution as

$$\left\{\begin{array}{rcl}\mathscr{U}_2(t,\eta,\theta,\xi) &=& \psi_0(\epsilon\eta)\bigg(f_2(t,\eta,\theta,\xi)-f_2(t,\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_2}{\partial\eta}+f_2-\bar{f}_2 &=& \cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_1}{\partial\theta}-\dfrac{\partial\mathscr{U}_0}{\partial t},\\ f_2(t,0,\theta,\xi) &=& \vec{w}\cdot\nabla_xU_1(t,\vec{x}_0,\vec{w})+\partial_tU_0(t,\vec{x}_0,\vec{w}) \quad\text{for}\quad \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty}f_2(t,\eta,\theta,\xi) &=& f_2(t,\infty,\theta).\end{array}\right. \tag{6.23}$$

Define the second order initial layer as

$$\left\{\begin{array}{rcl}\mathscr{U}_{I,2}(\tau,\vec{x},\vec{w}) &=& \mathcal{F}_2(\tau,\vec{x},\vec{w})-\mathcal{F}_2(\infty,\vec{x})\\ \partial_\tau\bar{\mathcal{F}}_2 &=& -\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}_{I,1}\bigg)\mathrm{d}\vec{w},\\ \mathcal{F}_2(\tau,\vec{x},\vec{w}) &=& \mathrm{e}^{-\tau}\mathcal{F}_2(0,\vec{x},\vec{w})+\displaystyle\int_0^\tau\bigg(\bar{\mathcal{F}}_2-\vec{w}\cdot\nabla_x\mathscr{U}_{I,1}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\ \mathcal{F}_2(0,\vec{x},\vec{w}) &=& \vec{w}\cdot\nabla_xU_1^\epsilon(0,\vec{x},\vec{w})+\partial_tU_0^\epsilon(0,\vec{x},\vec{w}),\\ \lim_{\tau\to\infty}\mathcal{F}_2(\tau,\vec{x},\vec{w}) &=& \mathcal{F}_2(\infty,\vec{x}).\end{array}\right. \tag{6.24}$$

Define the first order interior solution as

$$\left\{\begin{array}{rcl}U_2 &=& \bar{U}_2-\vec{w}\cdot\nabla_xU_1-\partial_tU_0,\\ \partial_t\bar{U}_2-\Delta_x\bar{U}_2 &=& 0,\\ \bar{U}_2(0,\vec{x}) &=& \mathcal{F}_2(\infty,\vec{x}) \quad\text{in}\quad\Omega,\\ \bar{U}_2(t,\vec{x}) &=& f_2(t,\infty,\theta) \quad\text{on}\quad\partial\Omega.\end{array}\right. \tag{6.25}$$

Step 4: Generalization to arbitrary k.

Similar to above procedure, we can define the k^{th} order boundary layer solution as

(6.26)
$$\left\{\begin{array}{rcl}\mathscr{U}_k(t,\eta,\theta,\xi)&=&\psi_0(\epsilon\eta)\bigg(f_k(t,\eta,\theta,\xi)-f_k(t,\infty,\theta)\bigg),\\ \sin(\theta+\xi)\dfrac{\partial f_k}{\partial\eta}+f_k-\bar{f}_k&=&\cos(\theta+\xi)\dfrac{\psi(\epsilon\eta)}{1-\epsilon\eta}\dfrac{\partial\mathscr{U}_{k-1}}{\partial\theta}-\dfrac{\partial\mathscr{U}_{k-2}}{\partial t},\\ f_k(t,0,\theta,\xi)&=&\vec{w}\cdot\nabla_xU_{k-1}(t,\vec{x}_0,\vec{w})+\partial_tU_{k-2}(t,\vec{x}_0,\vec{w})\ \text{ for }\ \sin(\theta+\xi)>0,\\ \lim_{\eta\rightarrow\infty}f_k(t,\eta,\theta,\xi)&=&f_k(t,\infty,\theta).\end{array}\right.$$

Define the k^{th} order initial layer as

$$\left\{\begin{array}{rcl}\mathscr{U}_{I,k}(\tau,\vec{x},\vec{w})&=&\mathcal{F}_k(\tau,\vec{x},\vec{w})-\mathcal{F}_k(\infty,\vec{x})\\ \partial_\tau\bar{\mathcal{F}}_k&=&-\displaystyle\int_{\mathcal{S}^1}\bigg(\vec{w}\cdot\nabla_x\mathscr{U}_{I,k-1}\bigg)\mathrm{d}\vec{w},\\ \mathcal{F}_k(\tau,\vec{x},\vec{w})&=&\mathrm{e}^{-\tau}\mathcal{F}_k(0,\vec{x},\vec{w})+\displaystyle\int_0^\tau\bigg(\bar{\mathcal{F}}_k-\vec{w}\cdot\nabla_x\mathscr{U}_{I,k-1}\bigg)(s,\vec{x},\vec{w})\mathrm{e}^{s-\tau}\mathrm{d}s,\\ \mathcal{F}_k(0,\vec{x},\vec{w})&=&\vec{w}\cdot\nabla_xU^\epsilon_{k-1}(0,\vec{x},\vec{w})+\partial_tU^\epsilon_{k-2}(0,\vec{x},\vec{w}),\\ \lim_{\tau\rightarrow\infty}\mathcal{F}_k(\tau,\vec{x},\vec{w})&=&\mathcal{F}_k(\infty,\vec{x}).\end{array}\right. \tag{6.27}$$

Define the k^{th} order interior solution as

$$\left\{\begin{array}{rcl}U_k&=&\bar{U}_k-\vec{w}\cdot\nabla_xU_{k-1}-\partial_tU_{k-2},\\ \partial_t\bar{U}_k-\Delta_x\bar{U}_k&=&0,\\ \bar{U}_k(0,\vec{x})&=&\mathcal{F}_k(\infty,\vec{x})\ \text{ in }\ \Omega,\\ \bar{U}_k&=&f_k(t,\infty,\theta)\ \text{ on }\ \partial\Omega.\end{array}\right. \tag{6.28}$$

By the idea in [1], we should be able to prove the following result:

Theorem 6.1. *Assume $g(t,\vec{x}_0,\vec{w})$ and $h(\vec{x},\vec{w})$ are sufficiently smooth. Then for the unsteady neutron transport equation (1.1), the unique solution $u^\epsilon(t,\vec{x},\vec{w})\in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ satisfies*

$$\|u^\epsilon-U_0-\mathscr{U}_{I,0}-\mathscr{U}_{B,0}\|_{L^\infty}=O(\epsilon). \tag{6.29}$$

Similar to the analysis in [13, Section 2.2], considering a crucial observation that based on Remark 4.6, we know that the existence of solution f_1 requires

$$\frac{\partial}{\partial\theta}\bigg(f_0(t,\eta,\theta,\xi)-f_0(t,\infty,\theta)\bigg)\in L^\infty([0,\infty)^2\times[-\pi,\pi)\times[-\pi,\pi)). \tag{6.30}$$

This in turn requires

$$\frac{\partial f_0}{\partial\eta}\in L^\infty([0,\infty)^2\times[-\pi,\pi)\times[-\pi,\pi)). \tag{6.31}$$

On the other hand, as shown by the Appendix of [13], we can show for specific g, it holds that $\partial_\eta f_0\notin L^\infty([0,\infty)^2\times[-\pi,\pi)\times[-\pi,\pi))$. Due to intrinsic singularity for (6.17), this construction breaks down.

6.4. **Counterexample to Classical Approach.**

Theorem 6.2. *If $g(t,\theta,\phi)=t^2\mathrm{e}^{-t}\cos\phi$ and $h(\vec{x},\vec{w})=0$, then there exists a $C>0$ such that*

$$\|u^\epsilon-U_0-\mathscr{U}_{I,0}-\mathscr{U}_{B,0}\|_{L^\infty}\geq C>0 \tag{6.32}$$

when ϵ is sufficiently small, where the interior solution U_0 is defined in (6.19), the initial layer $\mathscr{U}_{I,0}$ is defined in (6.18), and the boundary layer $\mathscr{U}^\epsilon_{B,0}$ is defined in (6.17).

Proof. We divide the proof into several steps:

Step 1: Basic settings.
By (6.17), the solution f_0 satisfies the Milne problem

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial f_0}{\partial\eta}+f_0-\bar{f}_0 &= 0,\\ f_0(t,0,\theta,\xi) &= g(t,\theta,\xi) \quad \text{for} \quad \sin(\theta+\xi)>0,\\ \lim_{\eta\to\infty} f_0(t,\eta,\theta,\xi) &= f_0(t,\infty,\theta). \end{cases} \tag{6.33}$$

For convenience of comparison, we make the substitution $\phi=\theta+\xi$ to obtain

$$\begin{cases} \sin\phi\dfrac{\partial f_0}{\partial\eta}+f_0-\bar{f}_0 &= 0,\\ f_0(t,0,\theta,\phi) &= g(t,\theta,\phi) \quad \text{for} \quad \sin\phi>0,\\ \lim_{\eta\to\infty} f_0(t,\eta,\theta,\phi) &= f_0(t,\infty,\theta). \end{cases} \tag{6.34}$$

Assume the theorem is incorrect, i.e.

$$\lim_{\epsilon\to0}\left\|(U_0+\mathscr{U}_{I,0}+\mathscr{U}_{B,0})-(U_0^\epsilon+\mathscr{U}_{I,0}^\epsilon+\mathscr{U}_{B,0}^\epsilon)\right\|_{L^\infty}=0. \tag{6.35}$$

We can easily show the zeroth order initial layer $\mathscr{U}_{B,0}=\mathscr{U}_{B,0}^\epsilon=0$ due to $h(\vec{x},\vec{w})=0$. Since the boundary $g(t,\theta,\phi)=t^2\mathrm{e}^{-t}\cos\phi$ independent of θ, by (6.17) and (3.54), it is obvious the limit of zeroth order boundary layer $f_0(t,\infty,\theta)$ and $f_0^\epsilon(t,\infty,\theta)$ satisfy $f_0(t,\infty,\theta)=C_1(t)$ and $f_0^\epsilon(t,\infty,\theta)=C_2(t)$ for some constant $C_1(t)$ and $C_2(t)$ independent of θ. By (6.19), (3.56) and solution continuity of heat equation, we can derive the interior solutions are smooth and are close to constants $U_0=C_1(t)$ and $U_0^\epsilon=C_2(t)$ in a neighborhood $O(\epsilon)$ of the boundary with difference $O(\epsilon)$. Hence, we may further derive in this neighborhood,

$$\lim_{\epsilon\to0}\|(f_0(\infty)+\mathscr{U}_0)-(f_0^\epsilon(\infty)+\mathscr{U}_0^\epsilon)\|_{L^\infty}=0. \tag{6.36}$$

For $0\leq\eta\leq1/(2\epsilon)$, we have $\psi_0=1$, which means $f_0=\mathscr{U}_0+f_0(\infty)$ and $f_0^\epsilon=\mathscr{U}_0^\epsilon+f_0^\epsilon(\infty)$ in this neighborhood of the boundary. Define $u=f_0+2$, $U=f_0^\epsilon+2$ and $G=g+2=t^2\mathrm{e}^{-t}\cos\phi+2$, then $u(\eta,\phi)$ satisfies the equation

$$\begin{cases} \sin\phi\dfrac{\partial u}{\partial\eta}+u-\bar{u} &= 0,\\ u(0,\phi) &= G(\phi) \quad \text{for} \quad \sin\phi>0,\\ \lim_{\eta\to\infty} u(\eta,\phi) &= 2+f_0(\infty), \end{cases} \tag{6.37}$$

and $U(\eta,\phi)$ satisfies the equation

$$\begin{cases} \sin\phi\dfrac{\partial U}{\partial\eta}+F(\epsilon;\eta)\cos\phi\dfrac{\partial U}{\partial\phi}+U-\bar{U} &= 0,\\ U(0,\phi) &= G(\phi) \quad \text{for} \quad \sin\phi>0,\\ \lim_{\eta\to\infty} U(\eta,\phi) &= 2+f_0^\epsilon(\infty). \end{cases} \tag{6.38}$$

Based on (6.36), we have

$$\lim_{\epsilon\to0}\|U(\eta,\phi)-u(\eta,\phi)\|_{L^\infty}=0. \tag{6.39}$$

Then it naturally implies

$$\lim_{\epsilon\to0}\left\|\bar{U}(\eta)-\bar{u}(\eta)\right\|_{L^\infty}=0. \tag{6.40}$$

Step 2: Continuity of $\bar{u}$ and $\bar{U}$ at $\eta=0$.
For the problem (6.37), we have for any $r_0>0$

$$|\bar{u}(\eta)-\bar{u}(0)| \leq \frac{1}{2\pi}\bigg(\int_{\sin\phi\leq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi+\int_{\sin\phi\geq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi\bigg). \tag{6.41}$$

Since we have shown $u\in L^\infty([0,\infty)\times[-\pi,\pi))$, then for any $\delta>0$, we can take r_0 sufficiently small such that

$$\frac{1}{2\pi}\int_{\sin\phi\leq r_0}|u(\eta,\phi)-u(0,\phi)|\,\mathrm{d}\phi \leq \frac{C}{2\pi}\arcsin r_0\leq\frac{\delta}{2}. \tag{6.42}$$

For fixed r_0 satisfying above requirement, we estimate the integral on $\sin\phi \geq r_0$. By Ukai's trace theorem, $u(0,\phi)$ is well-defined in the domain $\sin\phi \geq r_0$ and is continuous. Also, by consider the relation

$$\frac{\partial u}{\partial \eta}(0,\phi) = \frac{\bar{u}(0) - u(0,\phi)}{\sin\phi}, \tag{6.43}$$

we can obtain in this domain $\partial_\eta u$ is bounded, which further implies $u(\eta,\phi)$ is uniformly continuous at $\eta = 0$. Then there exists $\delta_0 > 0$ sufficiently small, such that for any $0 \leq \eta \leq \delta_0$, we have

$$\frac{1}{2\pi}\int_{\sin\phi\geq r_0} |u(\eta,\phi) - u(0,\phi)|\,\mathrm{d}\phi \;\leq\; \frac{1}{2\pi}\int_{\sin\phi\geq r_0} \frac{\delta}{2}\mathrm{d}\phi \leq \frac{\delta}{2}. \tag{6.44}$$

In summary, we have shown for any $\delta > 0$, there exists $\delta_0 > 0$ such that for any $0 \leq \eta \leq \delta_0$,

$$|\bar{u}(\eta) - \bar{u}(0)| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \tag{6.45}$$

Hence, $\bar{u}(\eta)$ is continuous at $\eta = 0$. By a similar argument along the characteristics, we can show $\bar{U}(\eta,\phi)$ is also continuous at $\eta = 0$.

In the following, by the continuity, we assume for arbitrary $\delta > 0$, there exists a $\delta_0 > 0$ such that for any $0 \leq \eta \leq \delta_0$, we have

$$|\bar{u}(\eta) - \bar{u}(0)| \;\leq\; \delta, \tag{6.46}$$

$$\left|\bar{U}(\eta) - \bar{U}(0)\right| \;\leq\; \delta. \tag{6.47}$$

Step 3: Milne formulation.

We consider the solution at a specific point $(\eta,\phi) = (n\epsilon,\epsilon)$ for some fixed $n > 0$. The solution along the characteristics can be rewritten as follows:

$$u(n\epsilon,\epsilon) = G(\epsilon)\mathrm{e}^{-\frac{1}{\sin\epsilon}n\epsilon} + \int_0^{n\epsilon} \mathrm{e}^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\bar{u}(\kappa)\mathrm{d}\kappa, \tag{6.48}$$

$$U(n\epsilon,\epsilon) = G(\epsilon_0)\mathrm{e}^{-\int_0^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta} + \int_0^{n\epsilon} \mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\bar{U}(\kappa)\mathrm{d}\kappa, \tag{6.49}$$

where we have the conserved energy along the characteristics

$$E(\eta,\phi) = \cos\phi\,\mathrm{e}^{-V(\eta)}, \tag{6.50}$$

in which $(0,\epsilon_0)$ and $(\zeta,\phi(\zeta))$ are in the same characteristics of $(n\epsilon,\epsilon)$.

Step 4: Estimates of (6.48).

We turn to the Milne problem for u. We have the natural estimate

$$\begin{aligned}\int_0^{n\epsilon} \mathrm{e}^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\mathrm{d}\kappa &= \int_0^{n\epsilon} \mathrm{e}^{-\frac{1}{\epsilon}(n\epsilon-\kappa)}\frac{1}{\epsilon}\mathrm{d}\kappa + o(\epsilon)\\ &= \mathrm{e}^{-n}\int_0^{n\epsilon} \mathrm{e}^{\frac{\kappa}{\epsilon}}\frac{1}{\epsilon}\mathrm{d}\kappa + o(\epsilon)\\ &= \mathrm{e}^{-n}\int_0^{n} \mathrm{e}^{\zeta}\mathrm{d}\zeta + o(\epsilon)\\ &= (1-\mathrm{e}^{-n}) + o(\epsilon).\end{aligned} \tag{6.51}$$

Then for $0 < \epsilon \leq \delta_0$, we have $|\bar{u}(0) - \bar{u}(\kappa)| \leq \delta$, which implies

$$\begin{aligned}\int_0^{n\epsilon} \mathrm{e}^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\bar{u}(\kappa)\mathrm{d}\kappa &= \int_0^{n\epsilon} \mathrm{e}^{-\frac{1}{\sin\epsilon}(n\epsilon-\kappa)}\frac{1}{\sin\epsilon}\bar{u}(0)\mathrm{d}\kappa + O(\delta)\\ &= (1-\mathrm{e}^{-n})\bar{u}(0) + o(\epsilon) + O(\delta).\end{aligned} \tag{6.52}$$

For the boundary data term, it is easy to see

$$G(\epsilon)\mathrm{e}^{-\frac{1}{\sin\epsilon}n\epsilon} \;=\; \mathrm{e}^{-n}G(\epsilon) + o(\epsilon) \tag{6.53}$$

In summary, we have

$$u(n\epsilon,\epsilon) = (1-\mathrm{e}^{-n})\bar{u}(0) + \mathrm{e}^{-n}G(\epsilon) + o(\epsilon) + O(\delta). \tag{6.54}$$

Step 5: Estimates of (6.49).
We consider the ϵ-Milne problem for U. For $\epsilon << 1$ sufficiently small, $\psi(\epsilon) = 1$. Then we may estimate

$$\cos\phi(\zeta)\mathrm{e}^{-V(\zeta)} = \cos\epsilon\mathrm{e}^{-V(n\epsilon)}, \tag{6.55}$$

which implies

$$\cos\phi(\zeta) = \frac{1-n\epsilon^2}{1-\epsilon\zeta}\cos\epsilon. \tag{6.56}$$

and hence

$$\sin\phi(\zeta) = \sqrt{1-\cos^2\phi(\zeta)} = \sqrt{\frac{\epsilon(n\epsilon-\zeta)(2-\epsilon\zeta-n\epsilon^2)}{(1-\epsilon\zeta)^2}\cos^2\epsilon + \sin^2\epsilon}. \tag{6.57}$$

For $\zeta \in [0,\epsilon]$ and $n\epsilon$ sufficiently small, by Taylor's expansion, we have

$$1-\epsilon\zeta = 1+o(\epsilon), \tag{6.58}$$
$$2-\epsilon\zeta-n\epsilon^2 = 2+o(\epsilon), \tag{6.59}$$
$$\sin^2\epsilon = \epsilon^2+o(\epsilon^3), \tag{6.60}$$
$$\cos^2\epsilon = 1-\epsilon^2+o(\epsilon^3). \tag{6.61}$$

Hence, we have

$$\sin\phi(\zeta) = \sqrt{\epsilon(\epsilon+2n\epsilon-2\zeta)} + o(\epsilon^2). \tag{6.62}$$

Since $\sqrt{\epsilon(\epsilon+2n\epsilon-2\zeta)} = O(\epsilon)$, we can further estimate

$$\frac{1}{\sin\phi(\zeta)} = \frac{1}{\sqrt{\epsilon(\epsilon+2n\epsilon-2\zeta)}} + o(1) \tag{6.63}$$

$$-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta = \sqrt{\frac{\epsilon+2n\epsilon-2\zeta}{\epsilon}}\bigg|_\kappa^{n\epsilon} + o(\epsilon) = 1-\sqrt{\frac{\epsilon+2n\epsilon-2\kappa}{\epsilon}} + o(\epsilon). \tag{6.64}$$

Then we can easily derive the integral estimate

$$\begin{aligned}\int_0^{n\epsilon}\mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\mathrm{d}\kappa &= \mathrm{e}^1\int_0^{n\epsilon}\mathrm{e}^{-\sqrt{\frac{\epsilon+2n\epsilon-2\kappa}{\epsilon}}}\frac{1}{\sqrt{\epsilon(\epsilon+2n\epsilon-2\kappa)}}\mathrm{d}\kappa + o(\epsilon)\\ &= \frac{1}{2}\mathrm{e}^1\int_\epsilon^{(1+2n)\epsilon}\mathrm{e}^{-\sqrt{\frac{\sigma}{\epsilon}}}\frac{1}{\sqrt{\epsilon\sigma}}\mathrm{d}\sigma + o(\epsilon)\\ &= \frac{1}{2}\mathrm{e}^1\int_1^{1+2n}\mathrm{e}^{-\sqrt{\rho}}\frac{1}{\sqrt{\rho}}\mathrm{d}\rho + o(\epsilon)\\ &= \mathrm{e}^1\int_1^{\sqrt{1+2n}}\mathrm{e}^{-t}\mathrm{d}t + o(\epsilon)\\ &= (1-\mathrm{e}^{1-\sqrt{1+2n}}) + o(\epsilon).\end{aligned} \tag{6.65}$$

Then for $0 < \epsilon \leq \delta_0$, we have $\left|\bar{U}(0)-\bar{U}(\kappa)\right| \leq \delta$, which implies

$$\begin{aligned}\int_0^{n\epsilon}\mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\bar{U}(\kappa)\mathrm{d}\kappa &= \int_0^{n\epsilon}\mathrm{e}^{-\int_\kappa^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta}\frac{1}{\sin\phi(\kappa)}\bar{U}(0)\mathrm{d}\kappa + O(\delta)\\ &= (1-\mathrm{e}^{1-\sqrt{1+2n}})\bar{U}(0) + o(\epsilon) + O(\delta).\end{aligned} \tag{6.66}$$

For the boundary data term, since $G(\phi)$ is C^1, a similar argument shows

$$G(\epsilon_0)\mathrm{e}^{-\int_0^{n\epsilon}\frac{1}{\sin\phi(\zeta)}\mathrm{d}\zeta} = \mathrm{e}^{1-\sqrt{1+2n}}G(\sqrt{1+2n}\epsilon) + o(\epsilon). \tag{6.67}$$

Therefore, we have

$$U(n\epsilon,\epsilon) = (1-\mathrm{e}^{1-\sqrt{1+2n}})\bar{U}(0) + \mathrm{e}^{1-\sqrt{1+2n}}G(\sqrt{1+2n}\epsilon) + o(\epsilon) + O(\delta). \tag{6.68}$$

Step 6: Contradiction.
In summary, we have the estimate

$$u(n\epsilon,\epsilon) = (1-\mathrm{e}^{-n})\bar{u}(0)+\mathrm{e}^{-n}G(\epsilon)+o(\epsilon)+O(\delta), \tag{6.69}$$

$$U(n\epsilon,\epsilon) = (1-\mathrm{e}^{1-\sqrt{1+2n}})\bar{U}(0)+\mathrm{e}^{1-\sqrt{1+2n}}G(\sqrt{1+2n}\epsilon)+o(\epsilon)+O(\delta). \tag{6.70}$$

The boundary data is $G=t^2\mathrm{e}^{-t}\cos\phi+2$. Fix $t=1$. Then by the maximum principle in Theorem 4.3, we can achieve $1\leq u(0,\phi)\leq 3$ and $1\leq U(0,\phi)\leq 3$. Since

$$\begin{aligned}\bar{u}(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}u(0,\phi)\mathrm{d}\phi=\frac{1}{2\pi}\int_{\sin\phi>0}u(0,\phi)\mathrm{d}\phi+\frac{1}{2\pi}\int_{\sin\phi<0}u(0,\phi)\mathrm{d}\phi\\ &= \frac{1}{2\pi}\int_{\sin\phi>0}(2+\mathrm{e}^{-1}\cos\phi)\mathrm{d}\phi+\frac{1}{2\pi}\int_{\sin\phi<0}u(0,\phi)\mathrm{d}\phi\\ &= 2+\frac{1}{2\pi}\int_{\sin\phi>0}\mathrm{e}^{-1}\cos\phi\mathrm{d}\phi+\frac{1}{2\pi}\int_{\sin\phi<0}u(0,\phi)\mathrm{d}\phi,\end{aligned} \tag{6.71}$$

we naturally obtain

$$2-\frac{1}{2}\mathrm{e}^{-1}\leq\bar{u}(0)\leq 2+\frac{1}{2}\mathrm{e}^{-1}. \tag{6.72}$$

Similarly, we can obtain

$$2-\frac{1}{2}\mathrm{e}^{-1}\leq\bar{U}(0)\leq 2+\frac{1}{2}\mathrm{e}^{-1}. \tag{6.73}$$

Furthermore, for ϵ sufficiently small, we have

$$G(\sqrt{1+2n}\epsilon) = 2+\mathrm{e}^{-1}+o(\epsilon), \tag{6.74}$$

$$G(\epsilon) = 2+\mathrm{e}^{-1}+o(\epsilon). \tag{6.75}$$

Hence, we can obtain

$$u(n\epsilon,\epsilon) = \bar{u}(0)+\mathrm{e}^{-n}(-\bar{u}(0)+2+\mathrm{e}^{-1})+o(\epsilon)+O(\delta), \tag{6.76}$$

$$U(n\epsilon,\epsilon) = \bar{U}(0)+\mathrm{e}^{1-\sqrt{1+2n}}(-\bar{U}(0)+2+\mathrm{e}^{-1})+o(\epsilon)+O(\delta). \tag{6.77}$$

Then we can see $\lim_{\epsilon\rightarrow0}\left\|\bar{U}(0)-\bar{u}(0)\right\|_{L^{\infty}}=0$ naturally leads to $\lim_{\epsilon\rightarrow0}\left\|(-\bar{u}(0)+2+\mathrm{e}^{-1})-(-\bar{U}(0)+2+\mathrm{e}^{-1})\right\|_{L^{\infty}}=0$. Also, we have $-\bar{u}(0)+2+\mathrm{e}^{-1}=O(1)$ and $-\bar{U}(0)+2+\mathrm{e}^{-1}=O(1)$. Due to the smallness of ϵ and δ, and also $\mathrm{e}^{-n}\neq\mathrm{e}^{1-\sqrt{1+2n}}$, we can obtain

$$\left|U(n\epsilon,\epsilon)-u(n\epsilon,\epsilon)\right|=O(1). \tag{6.78}$$

However, above result contradicts our assumption that $\lim_{\epsilon\rightarrow0}\left\|U(\eta,\phi)-u(\eta,\phi)\right\|_{L^{\infty}}=0$ for any (η,ϕ). This completes the proof. □

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