A SHARP LOWER BOUND ON THE POLYGONAL ISOPERIMETRIC DEFICIT

EMANUEL INDREI †

Abstract. It is shown that the isoperimetric deficit of a convex polygon P admits a lower bound in terms of the variance of the radii of P, the area of P, and the variance of the barycentric angles of P. The proof involves circulant matrix theory and a Taylor expansion of the deficit on a compact manifold.

1. Introduction

The polygonal isoperimetric inequality states that if $n \geq 3$ and P is an n-gon with area $|P|$ and perimeter $L(P)$, then the deficit is nonnegative,

$$\delta(P) := L^2(P) - 4n \tan\frac{\pi}{n}|P| \geq 0,$$

and uniquely minimized when P is convex and regular. A sharp stability result for this classical inequality has recently been obtained in [IN15] via a novel approach involving a functional minimization problem on a compact manifold and the spectral theory for circulant matrices. The heart of the matter is a quantitative polygonal isoperimetric inequality for convex polygons which states that

$$\sigma_s^2(P) + \sigma_r^2(P) \lesssim \delta(P), \tag{1.1}$$

where $\sigma_s^2(P)$ is the variance of the side lengths of P and $\sigma_r^2(P)$ is the variance of its radii (i.e. the distances between the vertices and their barycenter).

The starting point of the proof is the following inequality [FRS85, pg. 35] which holds for any n-gon:

$$8n^2 \sin^2\frac{\pi}{n}\sigma_r^2(P) \leq nS(P) - 4n\tan\frac{\pi}{n}|P|, \tag{1.2}$$

where $S(P)$ is the sum of the squares of the side lengths of P. Since $n^2\sigma_s^2(P) = nS(P) - L^2(P)$, it follows that (1.2) is equivalent to

$$8n^2 \sin^2\frac{\pi}{n}\ \sigma_r^2(P) \leq \delta(P) + n^2\sigma_s^2(P). \tag{1.3}$$

In order to establish (1.1), it is shown in [IN15] that

$$\sigma_s^2(P) \lesssim \delta(P) \tag{1.4}$$

† PIRE Postdoctoral Fellow.

whenever P is a convex n-gon; thereafter, a more general stability result (e.g. valid for simple n-gons) is deduced via a version of the Erdős-Nagy theorem which states that a polygon may be convexified in a finite number of "flips" while keeping the perimeter invariant. The method of proof of (1.2) given in [FRS85] is based on a polygonal Fourier decomposition, whereas the technique in [IN15] is based on a Taylor expansion of the deficit (in a suitable sense). It is natural to wonder whether one can directly deduce (1.1) via the method in [IN15] without relying on [FRS85]. A positive answer is given in this paper. In fact, a new inequality is established which combined with (1.4) improves (1.1).

Let $\sigma_a^2(P)$ denote the variance of the barycentric angles of P (i.e. the angles generated by the vertices and barycenter of the set of vertices of P, see §2). Then the following is true.

Theorem 1.1. *Let $n \geq 3$ and P be a convex n-gon. There exists $c_n > 0$ such that*

$$c_n \, \delta(P) \geq \sigma_r^2(P) + |P|\sigma_a^2(P),$$

and the exponent on the deficit is sharp.

This result directly combines with (1.4) and yields:

Corollary 1.2. *Let $n \geq 3$ and P be a convex n-gon. There exists $c_n > 0$ such that*

$$c_n \, \delta(P) \geq \sigma_s^2(P) + \sigma_r^2(P) + |P|\sigma_a^2(P).$$

Remark 1.3. The theorem holds for a more general class of polygons. The only requirement in the proof is that the barycentric angles of P sum to 2π.

Remark 1.4. An inequality of the form

$$\sigma_a^2(P) \leq c_n \delta(P)$$

cannot hold in general. One can see this by a simple scaling consideration: let P be a convex polygon and P_α be the convex polygon obtained by dilating the radii of P by $\alpha > 0$. Then $\delta(P_\alpha) = \alpha^2 \delta(P)$, but $\sigma_a^2(P_\alpha) = \sigma_a^2(P)$.

Quantitative polygonal isoperimetric inequalities turn out to be useful tools in geometric problems. For instance (1.1) was recently utilized in [CM14] to improve a result of Hales which showed up in his proof of the honeycomb conjecture [Hal01]. This was achieved by showing that the notion of asymmetry in (1.1) directly controls the Hausdorff distance between P and a specific regular polygon. Moreover, [IN15] has also been employed in [CN15] to prove a quantitative version of a Faber-Krahn inequality for the Cheeger constant of n-gons obtained in [BF15]. Related stability results for the isotropic, anisotropic, and relative isoperimetric inequalities have been obtained in [FMP08, FMP10, FI13], respectively.

Acknowledgements
The author is pleased to acknowledge support from NSF Grants OISE-0967140 (PIRE), DMS-0405343, and DMS-0635983 administered by the Center for Nonlinear Analysis at Carnegie Mellon University. Moreover, the excellent research environment provided by the Hausdorff Research Institute for Mathematics and the Rheinische Friedrich-Wilhelms-Universität Bonn is kindly acknowledged. Lastly, the author wishes to thank an anonymous referee for providing useful feedback on a preliminary version of this paper.

2. Preliminaries

Let $n \geq 3$ and $P \subset \mathbb{R}^2$ be an n-gon generated by the set of vertices $\{A_1, A_2, \dots, A_n\} \subset \mathbb{R}^2$ whose center of mass O is taken to be the origin. For $i \in \{1, 2, \dots, n\}$, the i-th side length of P, denoted by $l_i := A_iA_{i+1}$, is the length of the vector $\overrightarrow{A_iA_{i+1}}$ which connects A_i to A_{i+1}, where $A_i = A_j$ if and only if $i = j \pmod n$; with this notation in mind, $\{r_i := OA_i\}_{i=1}^n$ is the set of radii. Furthermore, x_i is the angle between the vectors $\overrightarrow{OA_i}$ and $\overrightarrow{OA_{i+1}}$ and the set $\{x_i\}_{i=1}^n$ comprises the *barycentric angles* of P.

The circulant matrix method introduced in [IN15] is based on the idea that a large class of polygons can be viewed as points in $\mathbb{R}^{2n}$ satisfying some constraints. More precisely, consider

$$\mathcal{M} := \Big\{(x; r) \in \mathbb{R}^{2n} : \ x_i, r_i \geq 0, \ (2.1), \ (2.2), \ (2.3) \text{ hold}\Big\},$$

where

$$\sum_{i=1}^{n} x_i = 2\pi, \tag{2.1}$$

$$\sum_{i=1}^{n} r_i = n. \tag{2.2}$$

$$\begin{cases} \sum_{i=1}^{n} r_i \cos\left(\sum_{k=1}^{i-1} x_k\right) = 0, \\ \sum_{i=1}^{n} r_i \sin\left(\sum_{k=1}^{i-1} x_k\right) = 0. \end{cases} \tag{2.3}$$

Note that $\mathcal{M}$ is a compact $2n-4$ dimensional manifold and each point $(x; r) \in \mathcal{M}$ represents a polygon centered at the origin with barycentric angles x and radii r; therefore, it is appropriate to name such objects *polygonal manifolds*. Indeed, a point O is the barycenter of the set of vertices of P if and only if

$$\sum_{i=1}^{n} \overrightarrow{OA_i} = 0,$$

which is equivalent to saying that the projections of $\sum_{i=1}^{n} \overrightarrow{OA_i}$ onto $\overrightarrow{OA_1}$ and $\overrightarrow{OA_1}^{\perp}$ vanish; in other words, $(x;r)$ satisfies (2.3). Furthermore, (2.1) is satisfied by all convex polygons (also many nonconvex ones) and (2.2) is a convenient technical assumption which derives from scaling considerations. Note that the convex regular n-gon corresponds to the point $(x_*;r_*) = \big(\frac{2\pi}{n},\ldots,\frac{2\pi}{n};1,\ldots,1\big)$. With this in mind, the variance of the interior angles and radii of P are represented, respectively, by the quantities

$$\sigma_a^2(P) = \sigma_a^2(x;r) := \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i\right)^2,$$

$$\sigma_r^2(P) = \sigma_r^2(x;r) := \frac{1}{n}\sum_{i=1}^{n} r_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n} r_i\right)^2.$$

Moreover, in $(x;r)$ coordinates, the deficit is given by the formula

$$\delta(P) = \delta(x;r) := \left(\sum_{i=1}^{n}\left(r_{i+1}^2 + r_i^2 - 2r_{i+1}r_i\cos x_i\right)^{1/2}\right)^2 - 2n\tan\frac{\pi}{n}\sum_{i=1}^{n} r_i r_{i+1}\sin x_i.$$

3. Proof of Theorem 1.1

By a simple reduction argument, it suffices to prove the inequality on $\mathcal{M}$: let P be a convex n-gon and note that it is represented by $(x;r)\in\mathbb{R}^{2n}$, where $x\in\mathbb{R}^n$ denotes its interior angles and $r\in\mathbb{R}^n$ its radii. Convexity implies (2.1), and (2.3) follows from the definition of barycenter. If

$$\sum_{i=1}^{n} r_i = s \neq n,$$

consider (by a slight abuse of notation) the polygon $P_s = (x;\frac{n}{s}r)$ obtained by scaling the radii of P. Evidently $\sigma_a^2(P_s) = \sigma_a^2(P)$, $|P_s| = (n/s)^2|P|$, $\sigma_r^2(P_s) = (n/s)^2\sigma_r^2(P)$, $\delta(P_s) = (n/s)^2\delta(P)$. Hence if the inequality stated in the theorem holds for $P_s\in\mathcal{M}$, then it also holds for P. Now let

$$\begin{aligned}\phi(x;r) :&= n^2(|P|\sigma_a^2 + \sigma_r^2)\\ &= \frac{1}{2}\Big(\sum_{i=1}^{n} r_i r_{i+1}\sin x_i\Big)\Big(n\sum_{i=1}^{n} x_i^2 - \big(\sum_{i=1}^{n} x_i\big)^2\Big) + n\sum_{i=1}^{n} r_i^2 - \big(\sum_{i=1}^{n} r_i\big)^2,\end{aligned}$$

and note that it suffices to show

$$\phi(x;r) \le c\,\delta(x;r) \tag{3.1}$$

for all $(x;r)\in\mathcal{M}$. The polygonal isoperimetric inequality implies $\delta(x;r)\ge 0$ for every $(x;r)\in\mathcal{M}$ with $\delta(x;r)=0$ if and only if $(x;r) = z_* := (x_*;r_*)$. Since $\mathcal{M}$ is compact and δ is continuous it follows that for $\mu > 0$,

$$\inf_{\mathcal{M}\setminus B_\mu(z_*)} \delta > 0,$$

and so (3.1) follows easily on $\mathcal{M} \setminus B_\mu(z_*)$. Thus it suffices to prove (3.1) for some neighborhood B_μ of the point z_*. Direct calculations imply (recall that the notation is periodic mod n)

$$D\phi(z_*) := (D_x\phi(z_*), D_r\phi(z_*)) = 0, \tag{3.2}$$

$$D_{x_k x_l}\phi(z_*) = \begin{cases} n(n-1)\sin\frac{2\pi}{n}, & k = l, \\ -n\sin\frac{2\pi}{n}, & k \neq l, \end{cases}$$

$$D_{r_k r_l}\phi(z_*) = \begin{cases} 2(n-1), & k = l, \\ -2, & k \neq l, \end{cases}$$

and $D_{r_k x_l}\phi(z_*) = 0$. Thus by letting $\Phi := D^2\phi(z_*)$ it follows that

$$\Phi = \begin{pmatrix} n\sin\frac{2\pi}{n}\mathcal{C} & 0_{n\times n} \\ 0_{n\times n} & 2\mathcal{C} \end{pmatrix},$$

where $0_{n\times n}$ is the $n \times n$ zero matrix and

$$\mathcal{C} = \begin{pmatrix} n-1 & -1 & & \cdots & & -1 \\ -1 & n-1 & -1 & & \cdots & \\ & -1 & n-1 & -1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ & \vdots & \ddots & -1 & n-1 & -1 \\ -1 & & \cdots & & -1 & n-1 \end{pmatrix}_{n\times n}.$$

Moreover, $D\delta(z_*)$ is given by

$$\begin{cases} D_{x_k}\delta(z_*) = 2n\tan\frac{\pi}{n}, \\ D_{r_k}\delta(z_*) = 0; \end{cases}$$

hence, (2.1) implies

$$\begin{aligned} \Big\langle D\delta(z_*), (x - x_*; r - r_*)\Big\rangle &= \Big\langle D_x\delta(z_*), x - x_*\Big\rangle + \Big\langle D_r\delta(z_*), r - r_*\Big\rangle \\ &= 2n\tan\frac{\pi}{n}\sum_{i=1}^{n}(x_i - (x_*)_i) = 0. \end{aligned} \tag{3.3}$$

Since $\phi(z_*) = \delta(z_*) = 0$, by utilizing (3.2), (3.3), and performing a Taylor expansion, it follows that for z close enough to z_*,

$$\begin{aligned}\phi(z) =&\frac{1}{2}\langle D^2\phi(z_*)(z-z_*),(z-z_*)\rangle\\&+\frac{1}{6}\sum_{i,j,k=1}^{2n} D_{ijk}\phi((1-\theta_z)z_*+\theta_z z)(z-z_*)_i(z-z_*)_j(z-z_*)_k,\end{aligned}$$

and

$$\begin{aligned}\delta(z) =&\frac{1}{2}\langle D^2\delta(z_*)(z-z_*),(z-z_*)\rangle\\&+\frac{1}{6}\sum_{i,j,k=1}^{2n} D_{ijk}\delta((1-\tau_z)z_*+\tau_z z)(z-z_*)_i(z-z_*)_j(z-z_*)_k,\end{aligned}$$

for some $\theta_z, \tau_z \in (0,1)$. Furthermore, since in a neighborhood of z_*, ϕ and δ are C^3 and $\mathcal{M}$ is compact, there exists $C>0$ such that

$$\frac{1}{6}\left|\sum_{i,j,k=1}^{2n} D_{ijk}\phi((1-\theta_z)z_*+\theta_z z)(z-z_*)_i(z-z_*)_j(z-z_*)_k\right| \leq C|z-z_*|^3,$$

$$\frac{1}{6}\left|\sum_{i,j,k=1}^{2n} D_{ijk}\delta((1-\theta_z)z_*+\theta_z z)(z-z_*)_i(z-z_*)_j(z-z_*)_k\right| \leq C|z-z_*|^3,$$

for $z\in\mathcal{M}$ sufficiently close to z_*. Thus there exists $C>0$ for which

$$\left|\phi(z)-\frac{1}{2}\langle D^2\phi(z_*)(z-z_*),(z-z_*)\rangle\right| \leq C|z-z_*|^3, \tag{3.4}$$

$$\left|\delta(z)-\frac{1}{2}\langle D^2\delta(z_*)(z-z_*),(z-z_*)\rangle\right| \leq C|z-z_*|^3, \tag{3.5}$$

in a neighborhood of z_*. In particular, there exists $\eta=\eta(n)>0$ such that

$$\phi(z) \leq \frac{1}{2}||\Phi||_2|z-z_*|^2 + C|z-z_*|^3 \tag{3.6}$$

for all $z\in B_\eta(z_*)$. By the results of [IN15, §3.6], it follows that

$$\inf_{w\in S_{\mathcal{H}}}\langle D^2\delta(z_*)w,w\rangle =: \sigma > 0,$$ [1]

[1] In fact, something stronger is proved: namely that $\inf_{w\in S_{\mathcal{H}}}\langle D^2 f(z_*)w,w\rangle =: \sigma>0$ where f is an explicit function for which $D^2 f \leq D^2\delta$. This is achieved via the spectral theory for circulant matrices and an analysis involving the tangent space of $\mathcal{M}$ at z_* and the identification of a suitable coordinate system in which calculations can be performed efficiently. The barycentric condition (2.3) built into the definition of $\mathcal{M}$ comes up in this analysis.

where $\mathcal{H}$ is the tangent space of $\mathcal{M}$ at z_* and $S_{\mathcal{H}}$ is the unit sphere in $\mathcal{H}$ with center z_*. Moreover by continuity there exists a neighborhood $U \subset \mathbb{R}^{2n}$ of $S_{\mathcal{H}}$ such that

$$\langle D^2\delta(z_*)w, w\rangle \geq \frac{\sigma}{2},$$

for all $w \in U$. Note that $\frac{z-z_*}{|z-z_*|} \in U$ for $z \in \mathcal{M}$ sufficiently close to z_*. Hence there exists $\mu = \mu(\eta, \sigma) \in (0, \eta]$ such that

$$\langle D^2\delta(z_*)(z - z_*), (z - z_*)\rangle \geq \frac{\sigma}{2}|z - z_*|^2$$

for $z \in B_\mu(z_*)$. In particular, for $\tilde{\mu} := \min\{\mu, \frac{\sigma}{8C}\}$ and $z \in B_{\tilde{\mu}}(z_*)$,

$$\delta(z) \geq \frac{1}{4}\langle D^2\delta(z_*)(z - z_*), (z - z_*)\rangle;$$

thus, recalling (3.6),

$$\phi(z) \leq \Big(\frac{1}{\sigma}||\Phi||_2 + \frac{2C}{\sigma}|z - z_*|\Big)\langle D^2\delta(z_*)(z - z_*), (z - z_*)\rangle \leq c_n\delta(z),$$

where $c_n := \frac{4}{\sigma}||\Phi||_2 + \frac{8C}{\sigma}\tilde{\mu}$. To achieve the second part of the theorem, it suffices to prove the existence of $c > 0$ such that

$$\langle \Phi\,(x; r), (x; r)\rangle \geq c|(x; r)|^2, \tag{3.7}$$

for

$$(x; r) \in \mathcal{Z} := \left\{(x; r) :\ \sum_{i=1}^{n} x_i = 0,\ \sum_{i=1}^{n} r_i = 0\right\}.$$

Indeed, if (3.7) holds, let $\omega : [0, \infty] \to [0, \infty]$ be any modulus of continuity (i.e. $\omega(0+) = 0$) such that

$$\phi(z) \leq c_n\omega(\delta(z)).$$

Then for $z \in \mathcal{M}$ close to z_*, (3.5) implies

$$\delta(z) \leq c_0|z - z_*|^2,$$

for some $c_0 > 0$. Moreover, $z - z_* \in \mathcal{Z}$ since $z \in \mathcal{M}$, and by combining (3.4) with (3.7) it follows that

$$\delta(z) \leq c_0|z - z_*|^2 \leq c_1\langle \Phi(z - z_*), (z - z_*)\rangle \leq c_2\phi(z) \leq \tilde{c}\omega(\delta(z)), \tag{3.8}$$

for some $\tilde{c} > 0$ provided z is close to z_*; however, since $\delta(z) \to 0$ as $z \to z_*$ and $\delta(z) > 0$ for $z \neq z_*$, (3.8) leads to a contradiction if

$$\liminf_{t\to 0^+} \frac{\omega(t)}{t} = 0.$$

Thus the lim inf is strictly greater than zero and this implies ω is at most linear at zero. To verify (3.7), note first that $\mathcal{C}$ is a real, symmetric, circulant matrix generated by the vector $(n-1,-1,\dots,-1)$. A calculation shows that the eigenvalues of $\mathcal{C}$, say λ_k, are given by

$$\lambda_0 = 0 \qquad \text{and} \qquad \lambda_k = n \qquad \text{for } k=1,\dots,n-1. \tag{3.9}$$

Moreover, let $v_0 := (1,\dots,1)$, and for $l \in \{1,\dots,\lfloor \frac{n}{2} \rfloor\}$ define

$$\begin{aligned} v_{2l-1} &:= \left(1, \cos\frac{2\pi l}{n}, \cos\frac{4\pi l}{n}, \dots, \cos\frac{2\pi l(n-1)}{n}\right), \\ v_{2l} &:= \left(0, \sin\frac{2\pi l}{n}, \sin\frac{4\pi l}{n}, \dots, \sin\frac{2\pi l(n-1)}{n}\right). \end{aligned}$$

One can readily check that v_k is an eigenvector of $\mathcal{C}$ corresponding to the eigenvalue $\lambda_{\lceil \frac{k}{2} \rceil}$, and that the set $\{v_0, v_1, \dots, v_{n-1}\}$ forms a real orthogonal basis of $\mathbb{R}^n$ (see e.g. Proposition 2.1 in [IN15]). For $k = 1, 2, \dots, n$, define $b_k := (v_{k-1}; 0, \dots, 0) \in \mathbb{R}^{2n}$ and $b_k := (0, \dots, 0; v_{k-n-1}) \in \mathbb{R}^{2n}$ for $k = n+1, \dots, 2n$. Since the set $\{b_k\}_{k=1}^{2n}$ forms a real orthogonal basis of $\mathbb{R}^{2n}$, given $(x;r) \in \mathbb{R}^{2n}$ there exist unique coefficients $\alpha_k \in \mathbb{R}$ such that

$$(x;r) = \sum_{k=1}^{2n} \alpha_k b_k.$$

Thus, by utilizing (3.9) it follows that

$$\begin{aligned} \langle \Phi(x;r), (x;r) \rangle &= \sum_{k,k'=1}^{2n} \alpha_k \alpha_{k'} \langle \Phi b_k, b_{k'} \rangle \\ &= n \sin\frac{2\pi}{n} \sum_{k=1}^{n} \alpha_k^2 \lambda_{\lceil \frac{k-1}{2} \rceil} |b_k|^2 + 2 \sum_{k=n+1}^{2n} \alpha_k^2 \lambda_{\lceil \frac{k-n-1}{2} \rceil} |b_k|^2 \\ &= n^2 \sin\frac{2\pi}{n} \sum_{k=2}^{n} \alpha_k^2 |b_k|^2 + 2n \sum_{k=n+2}^{2n} \alpha_k^2 |b_k|^2. \end{aligned}$$

Furthermore, if $(x;r) \in \mathcal{Z}$,

$$\alpha_1 = \frac{\langle (x;r), b_1 \rangle}{|b_1|^2} = \sum_{i=1}^{n} x_i = 0,$$

$$\alpha_{n+1} = \frac{\langle (x;r), b_{n+1} \rangle}{|b_1|^2} = \sum_{i=1}^{n} r_i = 0;$$

hence,

$$
\begin{aligned}
\langle \Phi\,(x;r),(x;r)\rangle &= n^2 \sin\frac{2\pi}{n}\sum_{k=1}^{n}\alpha_k^2|b_k|^2 + 2n\sum_{k=n+1}^{2n}\alpha_k^2|b_k|^2 \\
&\geq 2n\sum_{k=1}^{2n}\alpha_k^2|b_k|^2,
\end{aligned}
$$

and this concludes the proof.

References

[BF15] M. Bucur and I. Fragala. A Faber-Krahn inequality for the Cheeger constant of N-gons. J. Geom. Anal., DOI 10.1007/s12220-014-9539-5 (2015).

[CM14] M. Caroccia and F. Maggi. A sharp quantitative version of Hales' isoperimetric honeycomb theorem. arXiv:1410.6128 (2014).

[CN15] M. Caroccia and R. Neumayer. A note on the stability of the Cheeger constant of N-gons. J. Convex Anal., to appear (2015).

[FI13] A. Figalli and E. Indrei. A sharp stability result for the relative isoperimetric inequality inside convex cones. J. Geom. Anal., 23(2):938–969, 2013.

[FMP08] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. Ann. of Math. (2), 168(3):941–980, 2008.

[FMP10] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. Invent. Math., 182(1):167–211, 2010.

[FRS85] J. Chris Fisher, D. Ruoff, and J. Shilleto. Perpendicular polygons. Amer. Math. Monthly, 92(1):23–37, 1985.

[Hal01] T. C. Hales. The honeycomb conjecture. Discrete Comput. Geom., 25(1):1–22, 2001.

[IN15] E. Indrei and L. Nurbekyan. On the stability of the polygonal isoperimetric inequality. Advances in Mathematics, 276:62 – 86, 2015.

Emanuel Indrei

Center for Nonlinear Analysis
Carnegie Mellon University
Pittsburgh, PA 15213, USA
email: egi@cmu.edu