Non-transversal intersection of free and fixed boundary for fully nonlinear elliptic operators in two dimensions

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Abstract

In the study of classical obstacle problems, it is well known that in many configurations the free boundary intersects the fixed boundary tangentially. The arguments involved in producing results of this type rely on the linear structure of the operator. In this paper we employ a different approach and prove tangential touch of free and fixed boundary in two dimensions for fully nonlinear elliptic operators. Along the way, several n-dimensional results of independent interest are obtained such as BMO-estimates, $C^{1,1}$ regularity up to the fixed boundary, and a description of the behavior of blow-up solutions.

1 Introduction

Optimal interior regularity results have recently been obtained for solutions to fully nonlinear free boundary problems [FS14, IM] via methods inspired by [ALS13]. Under further thickness assumptions, these results imply C^1 regularity of the free boundary. However, a description of the dynamics on how the free boundaries intersect the fixed boundary has remained an open problem for at least a decade in the fully nonlinear setting (although partial results have been obtained in [MM04] under strong density and growth assumptions involving the solutions and a homogeneity assumption on the operator). On the other hand, extensive work has been carried out to investigate this question for the classical problem

$$\begin{cases} \Delta u = \chi_{u>0} & \text{in } B_1 \cap \{x_n > 0\}, \\ u = 0 & \text{on } \{x_n = 0\}, \end{cases}$$

and its variations [AU95, SU03, Mat05, AMM06, And07]. The conclusions have varied as a function of the boundary data, but in the homogeneous case it has been shown that the free boundary touches the fixed boundary tangentially. Dynamics of this type have also been the object of study in the classical dam problem [CG80, AG82] which is a mathematical model describing the filtration of water through a porous medium split into a wet and dry part via a free boundary.

The methods utilized in establishing the above-mentioned results strongly rely on the linear structure of the operator, e.g. in arguments involving Green's functions and monotonicity formulas. In particular, the Alt-Caffarelli-Friedman and Weiss monotonicity formulas are frequently applied: tools only available in the setting of linear operators in divergence form, see [PSU12, Chapter 8]. Thus the tangential touch problem for fully nonlinear operators requires a different approach.

In this article we prove non-transversal intersection of free and fixed boundary in two dimensions for a broad class of fully nonlinear elliptic free boundary problems. More precisely, consider the following problem

$$\begin{cases} F(D^2u) = 1 & \text{a.e. in } B_1^+ \cap \Omega, \\ |D^2u| \leq K & \text{a.e. in } B_1^+ \backslash \Omega, \\ u = 0 & \text{on } B_1', \end{cases} \tag{1}$$

where $\Omega \subset B_1^+$ is open, $K > 0$, F is C^1, and satisfies standard structural assumptions (see §3). We assume solutions u to be in $W^{2,p}(B_1^+)$ for any $1 < p < \infty$. A heuristic description of our strategy is as follows: we consider

$$M := \limsup_{|x|\to 0} \frac{1}{x_n} \sup_{e \in \mathbb{S}^{n-2} \cap e_n^\perp} \partial_e u(x).$$

By extending interior $C^{1,1}$ results (see §3), it follows that M is finite and we extract information on the nature of blow-up solutions by considering possible values for M. In particular, we show that either all blow-ups are of the form bx_n^2 if $M = 0$, or there is a sequence producing a blow-up having the form $ax_1x_n + bx_n^2$ if $M \neq 0$ (Theorem 2.1).

We then show that in $\mathbb{R}^2_+$, if $ax_1x_n + bx_n^2$ is a blow-up solution, then $\partial(\text{Int}\{u = 0\})$ stays away from the origin (Lemma 2.2) and this enables us to prove that blow-ups at the origin are unique (Theorem 2.4). Thereafter, a standard argument readily yields non-transversal intersection of the free and fixed boundary at contact points (Theorem 2.5).

The rest of the paper is organized as follows: in §1.1 we set up the problem and discuss relevant notation; §2 is the core of the paper where we rigorously develop the heuristics described above; §3 is devoted to the $C^{1,1}$ regularity up to the boundary of solutions, which follows as in [IM] once a suitable BMO result is established. The results of §3 are used in §2. We have chosen to reverse the logical ordering of these sections in order to make the tangential touch section more accessible.

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1.1 Setup and Notation

We study fully nonlinear elliptic partial differential equations of the form

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1^+ \cap \Omega, \\ |D^2u| \leq K & \text{a.e. in } B_1^+ \backslash \Omega, \\ u = 0 & \text{on } B_1', \end{cases} \tag{2}$$

where $\Omega \subseteq B_1^+$ is an open set, $B_1(x) = \{x \in \mathbb{R}^n : |x| < 1\}$, $B_r^+(x) = B_r(x) \cap \{x_n > 0\}$, $B_r'(x) = B_r(x) \cap \{x_n = 0\}$, and $B_r = B_r(0)$. Furthermore, F is assumed to satisfy the following structural conditions.

(H1) $F(0, x) \equiv 0$.

(H2) F is uniformly elliptic with ellipticity constants λ_0, $\lambda_1 > 0$ such that

$$\mathcal{P}^-(M - N) \leq F(M, x) - F(N, x) \leq \mathcal{P}^+(M - N), \qquad \forall x \in B_1^+$$

where M and N are symmetric matrices and $\mathcal{P}^\pm$ are the Pucci operators

$$\mathcal{P}^-(M) := \inf_{\lambda_0 \operatorname{Id} \leq N \leq \lambda_1 \operatorname{Id}} \operatorname{Tr} NM, \qquad \mathcal{P}^+(M) := \sup_{\lambda_0 \operatorname{Id} \leq N \leq \lambda_1 \operatorname{Id}} \operatorname{Tr} NM.$$

(H3) $F(\cdot, x)$ is concave or convex for all $x \in B_1^+$.

(H4)

$$|F(M, x) - F(M, y)| \leq \overline{C}(|M| + 1)|x - y|^{\bar{\alpha}}$$

for some $\bar{\alpha} \in (0, 1]$ and x, $y \in B_1^+$.

Moreover, let

$$\beta(x, x^0) := \sup_{M \in \mathcal{S}} \frac{|F(M, x) - F(M, x^0)|}{|M| + 1}$$

where $\mathcal{S}$ is the space of $n \times n$ symmetric real valued matrices.

Notation Points in $\mathbb{R}^n$ are generally denoted by x, x^0, y etc. while subscripts are used for components, i.e. $x = (x_1, \ldots, x_n)$, scalar sequences, and functions. The notation x' is used for $(n-1)$-dimensional vectors. Similarly, ∇ and ∇' will be used, respectively, for the gradient and the gradient with respect to the first $n-1$ variables.

$\mathbb{R}^n_+$	is the upper half space $\{x \in \mathbb{R}^n \,:\, x_n > 0\}$;
Ω	is an open set in $\mathbb{R}^n_+$;
Γ	is the set $\mathbb{R}^n_+ \cap \partial\Omega$;
Γ_i	is the set $\mathbb{R}^n_+ \cap \partial\, \mathrm{Int}\{u = 0\}$;
$B_r(x^0)$	is the open ball $\{x \in \mathbb{R}^n \,:\, \lvert x - x^0\rvert < r\}$;
$B^+_r(x^0)$	is the truncated open ball $\{x \in \mathbb{R}^n \,:\, \lvert x - x^0\rvert < r,\, x_n > 0\}$;
B'_r	is the ball $\{x' \in \mathbb{R}^{n-1} \,:\, \lvert x'\rvert < r\}$;
$\mathbb{S}^{n-1}$	is the $(n-1)$-sphere $\{x \in \mathbb{R}^n \,:\, \lvert x\rvert = 1\}$;
$e^\perp$	is the vector space orthogonal to $e \in \mathbb{S}^{n-1}$;
$C^{k,\alpha}(\Omega)$	denotes the usual Hölder space;
$C^{k,\alpha}_{\mathrm{loc}}(\Omega)$	denotes the local Hölder space;
$W^{k,p}(\Omega)$	denotes the usual Sobolev space.

The term "blow-ups of u at x^0" will be used for limits of the form $\lim\limits_{j\to\infty} \dfrac{u(x^0 + r_j x)}{r_j^2}$, where r_j is a sequence such that $r_j \to 0^+$ as $j \to \infty$; $\mathrm{Int}\{u = 0\} = \{u = 0\}^\circ$ means the interior of the set $\{u = 0\} := \{x \in \mathbb{R}^n_+ \,:\, u(x) = 0\}$. Finally, $S(\varphi)$ denotes the space of viscosity solutions corresponding to φ and the ellipticity constants λ_0 and λ_1 in (H2), see [CC95].

2 Main Result

Our first result gives a natural dichotomy of blow-ups of solutions to (1) in any dimension.

Theorem 2.1 (Blow-up Alternative)**.** *Let u be a solution to* (1) *and suppose $\{\nabla u \neq 0\} \cap \{x_n > 0\} \subset \Omega$, $0 \in \overline{\{u \neq 0\}}$, and $\nabla u(0) = 0$. Then exactly one of the following holds:*

(i) *All blow-ups of u at the origin are of the form $u_0(x) = b x_n^2$ for some $b > 0$;*

(ii) *There exists a blow-up of u at the origin of the form*

$$u_0(x) = a x_1 x_n + b x_n^2,$$

for $a \neq 0$, $b \in \mathbb{R}$.

Proof. Firstly, since $u(x', 0) = 0$, it follows that $\partial_{x_i} u(x', 0) = 0$ for all $i \in \{1, \ldots, n-1\}$. By $C^{1,1}$ regularity (Theorem 3.1), there is a constant $C > 0$ such that

$$\left| \frac{1}{x_n} \partial_{x_i} u(x', x_n) \right| = \left| \frac{1}{x_n} \big(\partial_{x_i} u(x', x_n) - \partial_{x_i} u(x', 0) \big) \right| \leq C, \qquad x_n > 0.$$

Define

$$M := \limsup_{\substack{|x|\to 0\\ x_n>0}} \frac{1}{x_n} \sup_{e\in\mathbb{S}^{n-2}\cap e_n^\perp} \partial_e u(x).$$

In particular, $0 \le M \le C < \infty$ and there exists a sequence $x^j \to 0$ with $x_n^j > 0$ and directions $e_{x^j} \in \mathbb{S}^{n-2}$ such that

$$\lim_{j\to\infty} \frac{1}{x_n^j}\partial_{e_{x^j}} u(x^j) = M.$$

Moreover, there exists $e \in \mathbb{S}^{n-2}$ such that (up to a subsequence) $e_{x^j} \to e$. Next note

$$\begin{aligned}\left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot e - M\right| &\le \left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot (e - e_{x^j})\right| + \left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot e_{x^j} - M\right| \\ &\le C|e-e_{x^j}| + \left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot e_{x^j} - M\right| \to 0,\end{aligned}$$

as $j\to\infty$. Thus, up to a rotation,

$$\lim_{j\to\infty} \frac{1}{x_n^j}\partial_{x_1} u(x^j) = M.$$

Now consider a sequence $\{s_j\}$ such that $s_j \to 0^+$ and the corresponding blow-up procedure so that

$$u_j(x) := \frac{u(s_j x)}{s_j^2} \to u_0(x)$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_+)$ for any $\alpha \in [0,1)$, and u_0 satisfies

$$\begin{cases} F(D^2u_0) = 1 & \text{a.e. in } \mathbb{R}^n_+ \cap \Omega_0, \\ |\nabla u_0| = 0 & \text{in } \mathbb{R}^n_+ \setminus \Omega_0, \\ u = 0 & \text{on } \mathbb{R}^{n-1}_+, \end{cases} \tag{3}$$

where $\Omega_0 = \{\nabla u_0 \ne 0\} \cap \{x_n > 0\}$ (via non-degeneracy). The definition of M implies

$$M \ge \lim_j \left|\frac{\partial_{x_i} u(s_j x)}{s_j x_n}\right| = \lim_j \left|\frac{\partial_{x_i} u_j(x)}{x_n}\right| = \left|\frac{\partial_{x_i} u_0(x)}{x_n}\right|, \tag{4}$$

for all $i \in \{1,\ldots,n-1\}$. In particular, let $v = \partial_{x_1} u_0$ so that in $\mathbb{R}^n_+$,

$$|v(x)| \le M x_n. \tag{5}$$

If $M = 0$, then (4) implies $\partial_{x_i} u_0 = 0$ for all $i \in \{1,\ldots,n-1\}$ so that $u_0(x) = u_0(x_n)$. However, since $u_0(0) = |\nabla u_0(0)| = 0$, $0 \in \overline{\{u_0 \ne 0\}}$ and u_0 satisfies (3), the uniform ellipticity of F readily implies

$$u_0(x) = bx_n^2,$$

for some $b > 0$. This shows that if $M = 0$, then any blow-up at the origin is of the form stated in (i).

Now suppose $M > 0$. In order to prove (ii), we cook up a specific blow-up: let $r_j := |x^j|$ (recall that $\{x^j\}$ is the sequence achieving the lim sup in the definition of M) so that as before

$$u_j(x) := \frac{u(r_j x)}{r_j^2} \to u_0(x)$$

in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_+)$ for any $\alpha \in [0, 1)$, and u_0 satisfies (3), (4), and (5). Set $y^j = \frac{x^j}{r_j} \in \mathbb{S}^{n-1} \cap \{x_n > 0\}$ and note that along a subsequence, $y^j \to y \in \mathbb{S}^{n-1} \cap \{x_n \geq 0\}$. Moreover, by the choice of the sequence $\{x^j\}$ and the $C^{1,\alpha}$ convergence of u_j to u_0, if $y_n > 0$, then

$$\lim_j \frac{v(y^j)}{y_n^j} = \lim_j \frac{\partial_{x_1} u_j(y^j)}{y_n^j} = \lim_j \frac{\partial_{x_1} u(x^j)}{x_n^j} = M,$$

so that

$$v(y) = M y_n, \tag{6}$$

and note that (6) also holds if $y_n = 0$. We consider several possibilities keeping in mind that $M > 0$.

Case 1: If $y \in \Omega_0$, then by differentiating (3) we get the elliptic equation

$$a_{ij} \partial_{ij}(v(x) - M x_n) = 0$$

for some measurable a_{ij}, and by (5), (6), and the maximum principle, it follows that $v(x) = M x_n$ in the connected component of Ω_0 containing y, say Ω_0^y. If there exists $x \in \partial\Omega_0^y \cap \{x_n > 0\}$, then $M x_n = v(x) = 0$ so we must have $M = 0$, a contradiction. Thus, $v(x) = M x_n$ in $\mathbb{R}^n_+$ and by integrating,

$$u_0(x) = M x_1 x_n + h(x_2, \dots, x_n).$$

Now, Krylov/Safonov's up to the boundary $C^{2,\alpha}$ estimate (see e.g. Theorem 3.3) applied to $u_0(Rx)/R^2$ yields

$$\frac{|D^2 u_0(x) - D^2 u_0(y)|}{|x - y|^\alpha} \leq \frac{C}{R^\alpha}, \qquad x \neq y \in B_R^+,$$

and taking $R \to \infty$ implies that $D^2 u_0$ is a constant matrix and thus h is a second order polynomial. Since u_0 vanishes on $\{x_n = 0\}$, it follows that

$$h(x_2, \dots, x_n) = x_n \sum_{i \neq n} \alpha_i x_i + b x_n^2,$$

and so up to a rotation,

$$u_0(x) = a x_1 x_n + b x_n^2,$$

with a or $b \neq 0$.

Case 2: If $y \in \partial\Omega_0 \cap \{x_n > 0\}$, then $My_n = v(y) = 0$, a contradiction.

Case 3: If $y \in \overline{\Omega}_0^c$, then for all but finitely many j, $y^j \in \Omega_0^c$ and since $\{\nabla u_0 \neq 0\} \subset \Omega_0$, it follows that $v(y^j) = 0$ if $j \geq N$ for some $N \in \mathbb{N}$. However, $y_n^j > 0$ and so $0 = \lim_j \frac{v(y^j)}{y_n^j} = M$, a contradiction.

Case 4: If $y \in \partial\Omega_0 \cap \{x_n = 0\}$, by differentiating (3) in Ω_0, it can be seen that for $r > 0$ (to be picked later), v satisfies

$$Lv = 0 \text{ in } B_r(y)^+ \cap \Omega_0,$$

where $L = F_{ij}(D^2 u_0)\partial_{ij}$ is elliptic. Since $u_0 \in C^{1,1}(B_r^+(y))$, it follows that the $F_{ij}(D^2 u_0)$ are bounded and measurable on $B_r^+(y)$.

We know that $Mx_n - v(x) \geq 0$ in $\mathbb{R}^n_+$, and if equality holds everywhere, $u_0(x) = ax_1x_n + bx_n^2$ just as in Case 1. If there is a point z where strict inequality holds, $Mz_n - v(z) > 0$, we can choose a ball $B_r^+(y)$ so that, by continuity of v, $v(x) < Mx_n$ in a neighborhood $B_s(z)$, where z is a boundary point of $B_r^+(y)$. Note that this strict inequality necessarily occurs on $\partial B_r^+(y) \cap \{x_n > 0\}$ since both v and Mx_n are zero on the hyperplane $\{x_n = 0\}$. Now choose a smooth non-negative (but not identically zero) function ϕ supported on $B_s(z)$ small enough such that $Mx_n - \phi(x) \geq v(x)$ in $\mathbb{R}^n_+$ and $Mx_n - \phi(x) > 0$ (this can be done since $B_s(z)$ is some distance away from the hyperplane $\{x_n = 0\}$). Then if

$$\begin{cases} Lw = 0 & \text{in } B_r^+(y), \\ w = Mx_n - \phi & \text{on } \partial B_r^+(y), \end{cases}$$

we have that $w > 0$ in $B_r^+(y)$ by the strong maximum principle since $Mx_n - \phi(x) > 0$. In particular, $w > v = 0$ on $\partial\Omega$, and since $v \leq w$ on $\partial B_r^+(y)$, the strong maximum principle again gives $w > v$ in $B_r^+(y) \cap \Omega$. Note also by linearity that $w = Mx_n - h$ where h solves

$$\begin{cases} Lh = 0 & \text{in } B_r^+(y), \\ h = \phi & \text{on } \partial B_r^+(y), \end{cases}$$

Once more, the strong maximum principle shows that $h > 0$ in $B_r^+(y)$, so the boundary Harnack comparison principle implies that $cx_n \leq h(x)$ in $B_{r/2}^+(y)$, where $c > 0$ depends on ellipticity constants and ϕ. With this,

$$M = \lim_{j\to\infty} \frac{v(y^j)}{y_n^j} \leq \limsup_{\substack{x_n \to 0^+ \\ x \in B_{r/4}^+(y)}} \frac{w(x)}{x_n} \leq \lim_{\substack{x_n \to 0^+ \\ x \in B_{r/4}^+(y)}} \frac{Mx_n - cx_n}{x_n} = M - c,$$

a contradiction.

□

The next lemma shows that in two dimensions, if (ii) in Theorem 2.1 occurs, then $\Gamma_i = \mathbb{R}^n_+ \cap \partial\,\mathrm{Int}\{u = 0\}$ stays away from the origin.

Lemma 2.2. *Let u be a solution to* (1) *with* $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_2 > 0\} \subset \mathbb{R}^2_+$*. If there exists* $\{r_j\} \subset \mathbb{R}^+$ *such that* $r_j \to 0$ *as* $j \to \infty$ *and*

$$u_j(x) := \frac{u(r_j x)}{r_j^2} \to u_0(x) = a x_1 x_2 + b x_2^2$$

in $C^{1,\alpha}_{loc}(\mathbb{R}^n_+)$ *as* $j \to \infty$*, for* $a \neq 0$*,* $b \in \mathbb{R}$*, then there exists* $\delta \in (0,1)$ *such that* $B^+_\delta \cap \Gamma_i = \emptyset$*.*

Proof. We may assume $a > 0$. Set $v_j := \partial_1 u_j$ and let $R > 2$, $\mu \in (0, \frac{1}{4})$, and $\delta \in (0, \frac{1}{4})$. Then select $j_0 = j_0(R, \mu, \delta) > 0$ such that for all $j \geq j_0$

$$|\nabla u_j(x)| > 0, \quad x \in B^+_R \setminus B^+_\delta; \tag{7}$$

$$v_j(x) > 0, \quad x \in B^+_R \cap \{x_2 \geq \mu\}. \tag{8}$$

(the two-dimensional setting is crucial for (7)). Consider $z \in \partial B_1 \cap \{x_2 = 0\}$ and note that

$$B^+_{\frac{3}{4}}(z) \subset B^+_R \setminus B^+_\delta.$$

Thanks to (7), u_j satisfies $F(D^2 u_j) = 1$ in $B_{\frac{3}{4}}(z)^+$ for all $j \geq j_0$. $C^{2,\alpha}$ estimates up to the boundary (see Theorem 3.3) implies

$$\sup_j \|u_j\|_{C^{2,\alpha}\left(B^+_{\frac{3}{4}}(z)\right)} < \infty.$$

Thus, along a subsequence, $v_j \to a x_2$ in $C^{0,1}$ ($C^{2,\alpha}$ is compactly contained in $C^{1,1}$) and so

$$c_j := \sup_{\substack{x,y \in B^+_{3/4}(z) \\ x \neq y}} \frac{|(v_j(x) - v_j(y)) - (v(x) - v(y))|}{|x - y|} \to 0.$$

In particular, since $v_j(x_1, 0) = v(x_1, 0) = 0$, it follows that

$$\frac{|v_j(x) - a x_2|}{x_2} \leq c_j$$

and so

$$v_j(x) \geq (a - c_j) x_2.$$

Now we select j large such that $v_j(x) \geq 0$ on ∂B_1. Note that $L v_j = 0$ in $B^+_1 \cap \Omega(u_j)$ where L is an elliptic second order operator obtained by differentiating (1). Indeed, u_j satisfies

$$\begin{cases} F(D^2 u_j) = 1 & \text{a.e. in } B^+_{1/r_j} \cap \Omega(u_j), \\ |D^2 u| \leq K & \text{a.e. in } B^+_{1/r_j} \setminus \Omega(u_j), \\ u_j = 0 & \text{on } B'_{1/r_j}, \end{cases}$$

where $\Omega(u_j)$ is the dilated set Ω/r_j, and without loss we may assume $r_j < \frac{1}{2}$. Since v_j vanishes on $\partial\Omega(u_j)$ and is non-negative on ∂B_1^+, the maximum principle implies $v_j > 0$ in $B_1^+ \cap \Omega(u_j)$ (note that v_j is not identically zero by (8)). If $\Gamma_i(u_j) \cap B_\delta^+ \neq \emptyset$, consider a ball N in the interior of $\{u_j = 0\} \cap B_\delta^+$. For $t \in \mathbb{R}$, let $N_t = N + te_1$. Note that by taking t negative we can move N_t to the left so that eventually $N_t \subset B_1^+ \backslash B_\delta^+$. Consider the strip $S = \cup_{t\in\mathbb{R}} N_t$. The next claim is that there exists a ball in the set $(S \cap B_1^+) \setminus B_\delta^+$ such that $u_j \neq 0$ in this ball: if not, then for each point $z \in (S \cap B_1^+) \setminus B_\delta^+$ there exists a sequence $\{z_k\} \subset \{u_j = 0\}$ such that $z_k \to z$; by continuity, $u_j(z) = 0$, so $u_j = 0$ in $(S \cap B_1^+) \setminus B_\delta^+$ and therefore the gradient also vanishes there, a contradiction to (7). Denote the ball by $N_{\tilde{t}} \subset \Omega(u_j)$ and note that $u_j < 0$ on $N_{\tilde{t}}$ since for each $z \in N_{\tilde{t}}$, there exists $t_z > 0$ such that $z + e_1 t_z \in \{u_j = 0\}$ and $v_j > 0$ in $B_1^+ \cap \Omega(u_j)$. Thus, $N_{\tilde{t}} \subset \Omega(u_j) \cap \{u_j < 0\}$. Now move $N_{\tilde{t}}$ to the right until the first time it touches $\{u_j = 0\}$, and let y be the contact point. If $\nabla u_j(y) = 0$, we immediately obtain a contradiction via Hopf's lemma. Thus we may assume $\nabla u_j(y) \neq 0$ which implies $y \in \Omega(u_j)$; whence $v_j(y) > 0$ (recall that $v_j > 0$ in $\Omega(u_j)$). By continuity $v_j > 0$ in $B_r(y)$ for some $r > 0$ so in particular $v_j(y+te_1) > 0$ for all $t > 0$ small. Since $\{y+te_1 : t \in (0,r)\} \subset \Omega(u_j)$, $t_* := \sup\{t > 0 : y + te_1 \in \Omega(u_j)\}$ is positive. Note that $y + te_1$ will eventually enter N as t gets larger. However,

$$u_j(y + t_*e_1) - u_j(y) = \int_0^{t_*} v_j(y + se_1)ds > 0,$$

and this implies $0 = u_j(y + t_*e_1) > u_j(y) = 0$, a contradiction. Thus $\Gamma_i(u_j) \cap B_\delta^+ = \emptyset$ and the result follows. □

Before proving uniqueness of blowups and tangential touch, we require one more lemma.

Lemma 2.3. *Let u be a solution to* (1) *with $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_n > 0\}$. If $s \in (0,1]$ and $(B_s^+ \setminus \Omega)^\circ = \emptyset$, then $|B_s^+ \setminus \Omega| = 0$.*

Proof. Since $u \in W^{2,n}(B_1^+)$, it follows that $D^2u = 0$ a.e. on $B_s^+ \setminus \Omega$. Let $Z := \{D^2u = 0\} \cap (B_s^+ \setminus \Omega)$ and note that $|Z| = |B_s^+ \setminus \Omega|$. Thus if $Z \subset (B_s^+ \setminus \Omega)^\circ$, then the result follows. Let $x^0 \in Z$ and suppose $x^0 \notin (B_s^+ \setminus \Omega)^\circ$. Then consider a sequence of points $x^j \to x^0$ such that $u(x^j) \neq 0$ and let $r_j := |x^0 - x^j|$. Non-degeneracy (see e.g. Lemma 3.1 in [IM]) implies that for j large,

$$\sup_{\partial B_{r_j}(x^0)} \frac{u}{r_j^2} \geq c > 0,$$

or in other words

$$\sup_{\partial B_1(0)} \frac{u(x^0 + r_j x)}{r_j^2} \geq c > 0.$$

Now for each j large enough, let $y^j \in \partial B_1(0)$ be the element achieving the supremum in the previous expression; note that since $u(x^0) = |\nabla u(x^0)| = |D^2u(x^0)| = 0$, we have

$$u(x^0 + r_j y^j) = o(r_j^2),$$

a contradiction. □

Theorem 2.1, Lemma 2.2, and Lemma 2.3 imply uniqueness of blow-ups in two dimensions.

Theorem 2.4 (Uniqueness of Blow-ups)**.** *Let u be a solution to* (1) *with* $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_2 > 0\} \subset \mathbb{R}^2_+$*. If* $0 \in \overline{\{u \neq 0\}}$ *and* $\nabla u(0) = 0$*, then all blow-up limits* u_0 *of* u *at the origin are of the form*

$$u_0(x) = ax_1x_2 + bx_2^2$$

where $a, b \in \mathbb{R}$ *with at least one of them non-zero.*

Proof. We divide the proof into two cases.

Case 1, $\mathbf{0} \in \overline{\mathbf{\Gamma}}_\mathbf{i}$: Lemma 2.2 implies the non-existence of a blow-up u_0 of u of the form

$$ax_1x_2 + bx_2^2,$$

$a \neq 0$, $b \in \mathbb{R}$ from which it follows that (i) holds in Theorem 2.1.

Case 2, $\mathbf{0} \notin \overline{\mathbf{\Gamma}}_\mathbf{i}$: In this case there exists $\delta > 0$ such that $\Gamma_i \cap B_\delta^+ = \emptyset$. Since $0 \in \overline{\{u \neq 0\}}$ (by assumption), it follows that $B_\delta^+ \not\subset \{u = 0\}^\circ$ and as $\Gamma_i \cap B_\delta^+ = \emptyset$, we may conclude that $\{u = 0\}^\circ \cap B_\delta^+ = \emptyset$. Thus the hypotheses of Lemma 2.3 are satisfied and by applying the lemma we obtain that $F(D^2u) = 1$ a.e. in B_δ^+. Therefore $u \in C^{2,\alpha}(B_{\delta/2}^+)$ and the blow-up limit u_0 is uniquely given by

$$\begin{aligned}\lim_{r \to 0} \frac{u(rx)}{r^2} &= \lim_{r \to 0} \frac{u(0) + \nabla u(0) \cdot rx + \langle rx, D^2u(0)rx \rangle + o(r^2)}{r^2} \\ &= \langle x, D^2u(0)x \rangle = ax_1x_2 + bx_2^2.\end{aligned}$$

The last equality follows from the boundary condition. Furthermore, u_0 solves the same equation as u so $F(D^2u_0) = F(D^2u(0)) = 1$ and so a and b cannot both be zero due to (H1).

□

If blow-ups are unique and of the form given above, it is rather standard to show that the free boundary touches the fixed boundary tangentially (see e.g. Chapter 8 in [PSU12]). The proof is included for completeness.

Theorem 2.5 (Tangential Touch)**.** *Let u be a solution to* (1) *with* $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_2 > 0\} \subset \mathbb{R}^2_+$*. Then there exists a constant* $r_0 > 0$ *and a modulus of continuity* $\omega_u(r)$ *such that*

$$\Gamma(u) \cap B_{r_0}^+ \subset \{x : x_2 \leq \omega_u(|x|)|x|\},$$

if $0 \in \overline{\Gamma(u)}$*, where* $\Gamma(u) := \partial\Omega \cap \mathbb{R}^2_+$*.*

Proof. By Theorem 2.4 the blow-up of u at the origin is not identically zero and given by $u_0(x) = ax_1x_2 + bx_2^2$. In particular, $\Gamma(u_0) = \emptyset$. It suffices to show that for any $\epsilon > 0$ there exists $\rho_\epsilon = \rho_\epsilon(u) > 0$ such that

$$\Gamma(u) \cap B^+_{\rho_\epsilon} \subset B^+_{\rho_\epsilon} \setminus \mathcal{C}_\epsilon,$$

where $\mathcal{C}_\epsilon := \{x_2 > \epsilon|x_1|\}$. Suppose not, then there exists a solution u to (1) satisfying the hypotheses of the theorem and $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists

$$x^k \in \Gamma(u) \cap B^+_{\frac{1}{k}} \cap \mathcal{C}_\epsilon.$$

Let $r_k := |x^k|$ and $y^k := \frac{x^k}{r_k} \in \partial B_1 \cap \mathcal{C}_\epsilon$. Note that along a subsequence

$$y^k \to y \in \partial B_1 \cap \mathcal{C}_\epsilon.$$

Define

$$u_k(x) := \frac{u(r_k x)}{r_k^2}$$

so that $u_k \to u_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_+)$ (along a subsequence). In particular $y \in \Gamma(u_0)$ which contradicts that $\Gamma(u_0) = \emptyset$. □

3 $C^{1,1}$ Regularity up to the Boundary

In this section we show BMO-estimates as well as $C^{1,1}$ regularity up to the fixed boundary of solutions to (2).

Theorem 3.1 ($C^{1,1}$ regularity)**.** *Let $f \in C^\alpha(B_1^+)$ be a given function and $\Omega \subseteq B_1^+$ a domain such that $u : B_1^+ \to \mathbb{R}$ is a $W^{2,n}$ solution of (2). Assume F satisfies (H1)-(H4). Then there exists a constant C depending on $\|u\|_{W^{2,n}(B_1^+)}$, $\|f\|_{C^\alpha(B_1^+)}$, and universal constants such that*

$$|D^2u| \leq C, \qquad a.e. \text{ in } B^+_{1/2}$$

There are three key tools needed to prove this theorem. The first two are $C^{2,\alpha}$ and $W^{2,p}$ estimates up to the boundary for the following classical fully nonlinear problem

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1^+, \\ u = 0 & \text{on } B_1', \end{cases} \tag{9}$$

and the last involves BMO-estimates. The $C^{2,\alpha}$ and $W^{2,p}$ estimates are well-known [Wan92, Saf94, Win09, Kry82]. We have been unable to find a reference for the BMO-estimates and thus provide a proof which is an adaptation of the interior case. For convenience, we record the following estimates, see e.g. [Win09, Theorem 4.3] and [Saf94, Theorem 7.1].

Theorem 3.2 ($W^{2,p}$ Regularity)**.** *Let u be a $W^{2,p}$ viscosity solution to* (9) *and $f \in L^p(B_1^+)$ for $n \leq p \leq \infty$. If $\beta(x^0, y) \leq \beta_0$ in $B_r^+(x^0) \cap B_1^+$ for all $x^0 \in B_1^+$ and $0 < r \leq r_0$, where β_0 and r_0 are universal constants, then $u \in W^{2,p}(B_{1/2}^+)$ and*

$$\|u\|_{W^{2,p}(B_{1/2}^+)} \leq C(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}),$$

where $C = C(n, \lambda_0, \lambda_1, \bar{\alpha}, \overline{C}, p) > 0$.

Theorem 3.3 ($C^{2,\alpha}$ Regularity)**.** *Let u be a $W^{2,n}$ viscosity solution to* (9)*. Then if $\beta(x^0, y) \leq \beta_0$ in $B_r^+(x^0) \cap B_1^+$ for all $x^0 \in B_1$ and $0 < r \leq r_0$, where β_0 and r_0 are universal constants, then $u \in C^{2,\alpha}(B_{1/2}^+)$ and*

$$\|u\|_{C^{2,\alpha}(B_{1/2}^+)} \leq C(\|u\|_{L^\infty(B_1^+)} + \|f\|_{C^{\bar{\alpha}}(B_1^+)}),$$

where $C = C(n, \lambda_0, \lambda_1, \bar{\alpha}, \overline{C}) > 0$.

The next results are technical tools utilized in the proof of the BMO-estimate (i.e. Proposition 3.6). The first is an approximation lemma, see e.g. [Wan92, Lemma 1.4].

Lemma 3.4 (Approximation)**.** *Let $\epsilon > 0$, $u \in W^{2,p}(B_1^+(x^0))$, and let v solve*

$$\begin{cases} F(D^2 v, x^0) = a & \text{in } B_{1/2}^+(x^0), \\ v = u & \text{on } \partial B_{1/2}^+(x^0). \end{cases}$$

Then there exists $\delta > 0$ and $\eta > 0$ such that if

$$\beta(x, x^0) := \sup_M \frac{|F(M, x) - F(M, x^0)|}{|M| + 1} \leq \delta$$

and $|f(x) - a| \leq \eta$ a.e. for $f(x) := F(D^2 u(x), x)$ in $B_1^+(x^0)$, then

$$|u - v| \leq \epsilon \qquad \text{in } B_{1/2}^+.$$

Lemma 3.5. *Let u be a $W^{2,n}(B_1^+)$ solution to* (2) *such that $|u| \leq 1$, $\beta(x, y)$ satisfies (H4) and $|F(D^2 u(x), x)| \leq \delta$ a.e. in B_1^+ for δ as in Lemma 3.4. Then there exists a universal constant $\rho > 0$ such that*

$$|D^2 P_{k,x^0} - D^2 P_{k-1,x^0}| \leq C_0(n, \lambda_0, \lambda_1)$$

and

$$|u(x) - P_{k,x^0}(x)| \leq \rho^{2k}, \qquad \text{inside } B_{\min(\rho^k,1)}^+(x^0),\ k \in \mathbb{N}_0,$$

where P_{k,x^0} is a second order polynomial such that $F(D^2 P_{k,x^0}, x^0) = 0$ and $x^0 \in B_{1/2}^+$.

Proof. For $k = 0$ and $k = -1$, the statement is true for $P_{k,x^0}(x) \equiv 0$ by assumption (recall (H1)). If we assume it is true up to some k, define $u_k := \frac{u(\rho^k x + x^0) - P_{k,x^0}(\rho^k x + x^0)}{\rho^{2k}}$ and

$$F_k(M, x) := F(M + D^2 P_{k,x^0}, \rho^k x + x^0), \qquad x \in B_1 \cap \{x_n > -\frac{x_n^0}{\rho^k}\}.$$

Then $|F_k(D^2 u_k, x)| = |F((D^2 u)(\rho^k x + x^0), \rho^k x + x^0)| \leq \delta$ a.e. Also,

$$\begin{aligned}
\beta_k(x, 0) &= \sup_{M \in \mathcal{S}} \frac{|F_k(M, x) - F_k(M, 0)|}{|M| + 1} \\
&= \sup_{M \in \mathcal{S}} \frac{|F(M + D^2 P_{k,x^0}, \rho^k x + x^0) - F(M + D^2 P_{k,x^0}, x^0)|}{|M| + 1} \\
&= \sup_{M \in \mathcal{S}} \frac{|F(M, \rho^k x + x^0) - F(M, x^0)|}{|M - D^2 P_{k,x^0}| + 1} \\
&= \sup_{M \in \mathcal{S}} \frac{|F(M, \rho^k x + x^0) - F(M, x^0)|}{|M| + 1} \frac{|M| + 1}{|M - D^2 P_{k,x^0}| + 1} \\
&\leq \beta(\rho^k x + x^0, x^0) \sup_{M \in \mathcal{S}} \frac{|M| + 1}{|M - D^2 P_{k,x^0}| + 1} \\
&\leq \overline{C} \rho^{\overline{\alpha} k} \sup_{M \in \mathcal{S}} \frac{|M| + 1}{||M| - |D^2 P_{k,x^0}|| + 1} \\
&\leq \overline{C} \rho^{\overline{\alpha} k} (|D^2 P_{k,x^0}| + 1),
\end{aligned}$$

where the last inequality follows from a calculation of the maximum of the function $\frac{x+1}{|x-a|+1}$, x, $a \geq 0$. However, from the induction hypothesis,

$$|D^2 P_{k,x^0}| \leq \sum_{j=1}^{k} |D^2 P_{j-1,x^0} - D^2 P_{j,x^0}| \leq C_0 k$$

so

$$\overline{C} \rho^{\overline{\alpha} k} (|D^2 P_{k,x^0}| + 1) \leq \overline{C} \rho^{\overline{\alpha} k} C_0 k \leq \eta$$

if ρ is chosen small enough (depending only on universal constants) and η as in Lemma 3.4. Thus $|v_k - u_k| \leq \epsilon$ in $B_{1/2} \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\}$ by Lemma 3.4, where v_k solves

$$\begin{cases} F_k(D^2 v_k, x^0) = 0 & \text{in } B_{1/2} \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\}, \\ v_k = u_k & \text{on } \partial(B_{1/2} \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\}). \end{cases}$$

Since

$$\|v_k\|_{L^\infty(B_{1/2} \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\})} \leq \|u_k\|_{L^\infty(B_{1/2} \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\})} \leq 1$$

by the maximum principle, Theorem 3.3 gives

$$\|v_k\|_{C^{2,\alpha}(B_{1/4} \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\})} \leq C_0. \tag{10}$$

Now define $\hat{P}_{k,x^0}$ as the second order Taylor expansion of v_k at the origin, and note that $F_k(D^2\hat{P}_{k,x^0}, x^0) = F_k(D^2 v_k(0), x^0) = 0$. Then

$$|v_k - \hat{P}_{k,x^0}| \le C_0\rho^{2+\alpha} \qquad \text{in } B_\rho \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\}$$

for $\rho < 1/4$, which gives

$$|u_k - \hat{P}_{k,x^0}| \le |u_k - v_k| + |v_k - \hat{P}_{k,x^0}| \le \epsilon + C_0\rho^{2+\alpha} \qquad \text{in } B_\rho \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\}.$$

For $\rho^\alpha \le \frac{1}{2C_0}$ and $\epsilon \le \rho^2/2$, we get

$$|u_k - \hat{P}_{k,x^0}| \le \rho^2 \qquad \text{in } B_\rho \cap \{x : x_n > -\frac{x_n^0}{\rho^k}\},$$

or, in other words,

$$|u - P_{k+1,x^0}| \le \rho^{2(k+1)} \qquad \text{in } B^+_{\rho^{k+1}}(x^0),$$

for

$$P_{k+1,x^0}(x) := P_{k,x^0}(x) + \rho^{2k}\hat{P}_{k,x^0}\left(\frac{x - x^0}{\rho^k}\right).$$

Also, since $F_k(D^2\hat{P}_k, x^0) = 0$, we have

$$F(D^2P_{k+1,x^0}, x^0) = F(D^2P_{k,x^0} + D^2\hat{P}_k, x^0) = F_k(D^2\hat{P}_k, 0) = 0,$$

and

$$|D^2P_{k+1,x^0} - D^2P_{k,x^0}| = |D^2\hat{P}_{k,x^0}| = |D^2v_k(0)| \le C_0,$$

by (10). □

Proposition 3.6 (BMO-estimate)**.** *Let u be a viscosity solution to* (2)*, and P_{k,x^0} and ρ be as in Lemma 3.5. Then*

$$\fint_{B^+_{\rho^k/2}(x^0)} |D^2u(y) - D^2P_{k,x^0}|^2 \le C, \qquad x^0 \in \overline{B}^+_{1/2}$$

if ρ is smaller than a constant which depends only on $\|u\|_{W^{2,p}(B_1)}$, f , $\overline{C}$ in (H4), and universal constants.

Proof. Let $x^0 \in \overline{B}^+_{1/2}$ and define $v(x) := u(x/R)$ and $G(M,x) := \frac{1}{R^2}F(R^2M, \frac{x}{R})$ for $R = R(\overline{C}, f, K, \delta)$ ($\overline{C}$ as in (H4)) chosen so that $|G(D^2v, x)| \le \delta$ in B^+_R for δ as in Lemma 3.4. Note also that $\beta_G(x,y) := \sup_{M\in\mathcal{S}} \frac{|G(M,x)-G(M,y)|}{|M|+1}$ satisifies (H4). Then v solves

$$\begin{cases} G(D^2v, x) = \frac{f(x/R)}{R^2} & \text{a.e. in } B^+_R \cap (R\Omega), \\ |D^2v| \le \frac{K}{R^2} & \text{a.e. in } B^+_R \setminus (R\Omega), \\ v = 0 & \text{on } B'_R, \end{cases}$$

and there is a polynomial $\tilde{P}_{k,x^0}$ for which $G(D^2\tilde{P}_{k,x^0}, Rx^0) = 0$, and a constant $\tilde{\rho}$ such that

$$|v(x) - \tilde{P}_{k,x^0}(x)| \le \tilde{\rho}^{2k}, \qquad x \in B^+_{\tilde{\rho}^k}(Rx^0),$$

i.e.

$$|u(x) - P_{k,x^0}(x)| \le R^2\rho^{2k}, \qquad x \in B^+_{\rho^k}(x^0),$$

for $P_{k,x^0}(x) := \tilde{P}_{k,x^0}(Rx)$ and $\rho^k := \tilde{\rho}^k/R$. Note also that

$$F(D^2P_{k,x^0}, x^0) = F(R^2D^2\tilde{P}_{k,x^0}, Rx^0/R) = R^2G(D^2\tilde{P}_{k,x^0}, Rx^0) = 0.$$

In particular, for $u_k(x) := \frac{u(\rho^k x + x^0) - P_{k,x^0}(\rho^k x + x^0)}{\rho^{2k}}$,

$$F_k(M, x) := F(M + D^2P_{k,x^0}, \rho^k x + x^0)$$

and β_k as in the proof of Lemma 3.5, we have $|u_k| \le R^2$, $\beta_k(x, y) \le \eta$ and $|F_k(u_k, x)| \le C$. Therefore we can apply Theorem 3.2 to deduce

$$\|u_k\|_{W^{2,p}(B_{1/2} \cap \{x_n \ge -x^0/\rho^k\})} \le C,$$

or

$$⨍_{B^+_{\rho^k/2}(x^0)} |D^2u(x) - D^2P_{k,x^0}|^p \, dx \le C.$$

□

From this it is straightforward to show that there exists a second order polynomial $P_{r,x^0}(x)$ with $F(D^2P_{r,x^0}, x^0) = f(x^0)$ such that

$$\sup_{r \in (0,1/4)} ⨍_{B^+_r(x^0)} |D^2u(y) - D^2P_{r,x^0}|^2 \, dy \le C,$$

where $x^0 \in \overline{B}^+_{1/2}(0)$. The proof of $C^{1,1}$ regularity now follows as in [IM] up to minor modifications (see also [FS14]). The idea is that $D^2P_{r,x^0}(x)$ provides a suitable approximation to $D^2u(x^0)$ and one may consider two cases: firstly, if $D^2P_{r,x^0}(x)$ stays bounded in r, then one can show that $D^2u(x^0)$ is also bounded by a constant depending only on the initial ingredients; next, if $D^2P_{r,x^0}(x)$ blows up in r, one can show that the set

$$A_r(x^0) := \frac{(B^+_r(x^0) \backslash \Omega) - x^0}{r} = B_1 \backslash ((\Omega - x^0)/r) \cap \left\{ y : y_n > -\frac{x^0_n}{r} \right\}$$

decays fast enough to ensure yet again a bound on $D^2u(x^0)$.

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