Second-Order Γ-limit for the Cahn–Hilliard Functional

Giovanni Leoni
Carnegie Mellon University
Pittsburgh, PA, USA

Ryan Murray
Carnegie Mellon University
Pittsburgh, PA, USA

March 24, 2015

Abstract

The goal of this paper is to solve a long standing open problem, namely, the asymptotic development of order 2 by Γ-convergence of the mass-constrained Cahn–Hilliard functional. This is achieved by introducing a novel rearrangement technique, which works without Dirichlet boundary conditions.

Keywords: Second-order Γ-convergence, Rearrangement, Cahn–Hilliard functional.
AMS Mathematics Subject Classification: 49J45.

1 Introduction

The goal of this paper is to study the asymptotic development by Γ-convergence of order 2 of the Modica–Mortola or Cahn–Hilliard functional (see [35, 47, 57])

$$F_\varepsilon(u) := \int_\Omega W(u) + \varepsilon^2 |\nabla u|^2 \, dx, \qquad u \in H^1(\Omega), \tag{1.1}$$

subject to the mass constraint

$$\int_\Omega u \, dx = m. \tag{1.2}$$

Here $\Omega \subset \mathbb{R}^n$ is an open, bounded set and W is a double-well potential.

The notion of asymptotic development by Γ-convergence was introduced by Anzellotti and Baldo [7]. To be precise, given a metric space X and a family of functions $\mathcal{F}_\varepsilon : X \to \overline{\mathbb{R}}$, $\varepsilon > 0$, we say that an *asymptotic development* of *order k*

$$\mathcal{F}_\varepsilon = \mathcal{F}^{(0)} + \varepsilon \mathcal{F}^{(1)} + \cdots + \varepsilon^k \mathcal{F}^{(k)} + o(\varepsilon^k)$$

holds if there exist functions $\mathcal{F}^{(i)} : X \to \overline{\mathbb{R}}$, $i = 0, 1, \ldots, k$, such that the functions

$$\mathcal{F}_\varepsilon^{(i)} := \frac{\mathcal{F}_\varepsilon^{(i-1)} - \inf_X \mathcal{F}^{(i-1)}}{\varepsilon} \tag{1.3}$$

are well-defined and

$$\mathcal{F}_\varepsilon^{(i)} \xrightarrow{\Gamma} \mathcal{F}^{(i)}, \tag{1.4}$$

where $\mathcal{F}_\varepsilon^{(0)} := \mathcal{F}_\varepsilon$ and $\overline{\mathbb{R}}$ is the extended real line. Let

$$\mathcal{U}_i := \{\text{minimizers of } \mathcal{F}^{(i)}\}.$$

It can be shown that
$$\mathcal{F}^{(i)} \equiv \infty \text{ in } X \backslash \mathcal{U}_{i-1}, \tag{1.5}$$

and that
$$\{\text{limits of minimizers of } \mathcal{F}_{\varepsilon_m}\} \subset \mathcal{U}_k \subset \cdots \subset \mathcal{U}_0, \tag{1.6}$$

with
$$\inf \mathcal{F}_{\varepsilon_m} = \inf \mathcal{F}^{(0)} + \varepsilon_m \inf \mathcal{F}^{(1)} + \cdots + \varepsilon_m^k \inf \mathcal{F}^{(k)} + o(\varepsilon_m^k)$$

for every sequence $\varepsilon_m \to 0^+$, provided $\inf \mathcal{F}^{(i)} < \infty$ for all $i = 0, \ldots, k$.

Simple examples show that each of the inclusions in (1.6) may be strict (see [7]). Thus asymptotic development by Γ-convergence provides a selection criteria for minimizers of $\mathcal{F}^{(0)}$. Some other works that describe asymptotic development via Γ-convergence include [12], [27].

The first example of asymptotic development by Γ-convergence of order 2 for functionals of the type (1.1) was studied by Anzellotti and Baldo in [7], who considered the case in which $n = 1$, the wells of W are not points but non-degenerate intervals and the mass constraint (1.2) is replaced by a Dirichlet condition. Subsequently Anzellotti, Baldo and Orlandi [8] studied (1.1) in arbitrary dimension, in the case in which W has only one well ($W(s) = s^2$) and again with Dirichlet boundary conditions in place of (1.2).

The problem of the asymptotic development of order 2 for the Cahn–Hilliard functional (1.1) with W a double-well potential has remained an open problem, except when $n = 1$. Indeed, in the one-dimensional case and for sufficiently smooth W, one can show that $\mathcal{F}^{(2)} = 0$. This can be deduced from the work of Carr, Gurtin and Slemrod [17], Theorem 8.1, and from the recent paper [10] of Bellettini, Nayam and Novaga, who gave a very precise higher-order asymptotic estimate for $\mathcal{F}_\varepsilon(u_\varepsilon)$, where $\{u_\varepsilon\} \subset H^1(\mathbb{T})$ is any sequence converging to $u \in BV(\mathbb{T}; \{-1, 1\})$ in $L^1(\mathbb{T})$, where $\mathbb{T}$ is the one-dimensional torus and $-1, 1$ are the wells of W.

To our knowledge, the only result related to the second-order asymptotic development of (1.1) in the case $n \geq 2$ for (1.1), (1.2) has recently been obtained by the first author in collaboration with Dal Maso and Fonseca in [25]. For a double-well potential satisfying
$$W(s) = W(-s)$$

for all $s \in \mathbb{R}$ and
$$W(s) = C|1 - s|^{1+q} \tag{1.7}$$

near $s = 1$, for some $q \in (0, 1)$, and *under the additional assumption* that
$$u = 1 \text{ on } \partial\Omega, \tag{1.8}$$

in addition to (1.2), it was shown that $\mathcal{F}^{(2)} = 0$. More generally, this was proved in the case in which $\varepsilon^2 \int_\Omega |\nabla u|^2 \, dx$ is replaced by $\varepsilon^2 \int_\Omega \Phi^2(\nabla u) \, dx$, for some convex, even, positively 1-homogenous function $\Phi : \mathbb{R}^n \to [0, \infty)$. The Dirichlet condition (1.8) played a crucial role in the proof in [25] since it permitted the use of classical symmetrization techniques (see [37], [38]) in $H_0^1(\Omega)$ to reduce the problem to the radial case. Moreover, the behavior of W near the wells (see (1.7)) did not allow for C^2 potentials W. The work of [25] left open several important questions, namely the characterization of $\mathcal{F}^{(2)}$ when

- the Dirichlet condition (1.8) is not imposed,

- W is of class C^2,

- W is not even.

In this paper we address all of these questions. In particular, we show that in general $\mathcal{F}^{(2)} \neq 0$ if W is even and of class C^2, or if W is not even.

Here we take $X := L^1(\Omega)$ and define

$$\mathcal{F}_\varepsilon(u) := \begin{cases} F_\varepsilon(u) & \text{if } u \in H^1(\Omega) \text{ and if holds,} \\ \infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

The Γ-limit $\mathcal{F}^{(1)}$ of order 1 (see (1.3) and (1.4)) has been established by Carr, Gurtin and Slemrod [17] for $n = 1$ and by Modica [47] and Sternberg [57] for $n \geq 2$ (see also [34], [48]), and is known to be, under appropriate assumptions on Ω and W,

$$\mathcal{F}^{(1)}(u) := \begin{cases} 2c_W \, \mathrm{P}(\{u = a\}; \Omega) & \text{if } u \in BV(\Omega; \{a, b\}) \text{ and } (1.2) \text{ holds,} \\ \infty & \text{otherwise in } L^1(\Omega), \end{cases} \tag{1.9}$$

where $\mathrm{P}(\cdot; \Omega)$ is the perimeter in Ω (see [6, 26, 61]), a, b are the wells of W and the constant c_W is given by

$$c_W := \int_a^b W^{1/2}(s) \, ds. \tag{1.10}$$

In view of (1.5), in order to characterize the Γ-limit of order 2, $\mathcal{F}^{(2)}$ (see (1.3) (1.4)), it is important to understand the family $\mathcal{U}_1$ of minimizers of the functional $\mathcal{F}^{(1)}$ defined in (1.9). Observe that u belongs to $\mathcal{U}_1$ if and only if $u \in BV(\Omega; \{a, b\})$ and the set $\{u = a\}$ is a solution of the classical *partition problem*, namely, if it solves

$$\min\{\mathrm{P}(E; \Omega) : E \subset \Omega \text{ Borel}, \, \mathcal{L}^n(E) = r_m\}, \tag{1.11}$$

where

$$r_m := \frac{b\mathcal{L}^n(\Omega) - m}{b - a}. \tag{1.12}$$

The properties of minimizers of (1.11) have been studied by Grüter [33] (see also [31, 45, 59]), who showed that when Ω is bounded and of class C^2, minimizers E of (1.11) exist, have constant generalized mean curvature κ_E, intersect the boundary of Ω orthogonally, and their singular set is empty if $n \leq 7$, and has dimension of at most $n - 8$ if $n \geq 8$.

A crucial hypothesis in our results is that the *isoperimetric function* or *isoperimetric profile* ([55]), given by

$$\mathcal{I}_\Omega(r) := \inf\{\mathrm{P}(E; \Omega) : E \subset \Omega \text{ Borel}, \, \mathcal{L}^n(E) = r\}, \quad r \in [0, \mathcal{L}^n(\Omega)], \tag{1.13}$$

admits a Taylor expansion of order 2 at the value r_m in (1.12). In particular the differentiability of $\mathcal{I}_\Omega$ at r_m implies that (see [45])

$$\mathcal{I}'_\Omega(r_m) = (n - 1)\kappa_E \tag{1.14}$$

for every minimizer E of (1.13) at $r = r_m$. We observe that, as $\mathcal{I}_\Omega$ is semi-concave [9], a Taylor expansion of order 2 holds for $\mathcal{L}^1$ a.e. r, or equivalently for $\mathcal{L}^1$ a.e. mass m in (1.2). Under this assumption on $\mathcal{I}_\Omega$ and under other technical hypotheses on Ω, m and W (see Section 2) we will show that if W is quadratic near the wells a, b then the following theorem holds.

Theorem 1.1. *Assume that Ω, m, W satisfy hypotheses (2.1)-(2.7) with $q = 1$. Then*

$$\mathcal{F}^{(2)}(u) = \frac{2c_W^2(n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 + 2(c_{\mathrm{sym}} + c_W \tau_u)(n-1)\kappa_u \, \mathrm{P}(\{u = a\}; \Omega) \tag{1.15}$$

if $u \in \mathcal{U}_1$ and $\mathcal{F}^{(2)}(u) = \infty$ otherwise in $L^1(\Omega)$.

Here κ_u is the constant mean curvature of the set $\{u = a\}$,

$$c_{\text{sym}} := \int_{\mathbb{R}} W(z(t))t \, dt, \tag{1.16}$$

where z is the solution to the Cauchy problem

$$\begin{cases} z'(t) = \sqrt{W(z(t))} & \text{for } t \in \mathbb{R}, \\ z(0) = c, & z(t) \in [a, b], \end{cases} \tag{1.17}$$

and $\tau_u \in \mathbb{R}$ is a constant such that

$$\mathrm{P}(\{u = a\}; \Omega) \int_{\mathbb{R}} z(t - \tau_u) - \mathrm{sgn}_{a,b}(t) \, dt = \frac{2c_W(n-1)}{W''(a)(b-a)} \kappa_u, \tag{1.18}$$

where

$$\mathrm{sgn}_{a,b}(t) := \begin{cases} a & \text{if } t \leq 0, \\ b & \text{if } t > 0. \end{cases} \tag{1.19}$$

We observe that, in view of (1.14), the quantities $\mathrm{P}(\{u = a\}; \Omega)$, κ_u and τ_u are *uniquely determined* by r_m, $\mathcal{I}_\Omega(r_m)$ and $\mathcal{I}'_\Omega(r_m)$ for $u \in \mathcal{U}_1$.

Without assuming the differentiability of the isoperimetric function $\mathcal{I}_\Omega$ at r_m one can only conclude that $(n-1)\kappa_u \in [(\mathcal{I}_\Omega)'_-(r_m), (\mathcal{I}_\Omega)'_+(r_m)]$, where $(\mathcal{I}_\Omega)'_-$, $(\mathcal{I}_\Omega)'_+$ are the left and right derivatives of $\mathcal{I}_\Omega$, which must exist as $\mathcal{I}_\Omega$ is semi-concave [9]. We conjecture that (1.15) continues to hold even in this case, but we have not been able to prove it. One potential avenue of investigation involves studying L^1 isolated families of perimeter minimizers where the mean curvature is unique. While this could potentially remove the issue of differentiability it does not remove the technical necessity of a higher-order Taylor expansion of $\mathcal{I}_\Omega$ at r_m.

If such a conjecture holds then (1.15) would provide an additional selection criteria among minimizers of $\mathcal{U}_1$, namely that surfaces with larger magnitude mean curvature are energetically favored.

We can offer a heuristic explanation for the terms in (1.15). Critical points u_ε of (1.1) subject to (1.2) satisfy the Neumann problem

$$\begin{cases} 2\varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon} W'(u_\varepsilon) + \Lambda_\varepsilon & \text{in } \Omega, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν is the outward unit normal to $\partial\Omega$ and Λ_ε is a Lagrange multiplier that accounts for the constraint (1.2). In [44], Luckhaus and Modica proved that if $0 < a < b$ and $\{u_\varepsilon\}$ is a sequence of non-negative minimizers of (1.1), (1.2), uniformly bounded in $L^\infty(\Omega)$ and converging in $L^1(\Omega)$ to a minimizer of $\mathcal{F}^{(1)}$, then

$$\Lambda_\varepsilon \to \Lambda_u := \frac{2c_W(n-1)}{b-a} \kappa_u. \tag{1.20}$$

Thus the first term in equation (1.15) can be written as $\frac{\Lambda_u^2}{2W''(a)}$. This term and the τ_u terms are linked: they represent a shift outward along curved surfaces and the corresponding far-field mass correction. In one dimension this is analogous to shifting an increasing function the right some distance, and then shifting it up so that the total translation preserves mass. This term disappears if the wells have order less than two and the potential is symmetric. In the symmetric case this term is not dependent on the sign of the curvature, indicating that transition layers will generally skew in the "outward" direction of the surface ∂E.

The term involving c_{sym} can be thought of as a penalty for directional asymmetry. If the profiles are symmetric this term disappears entirely. This term is of order ε for any q that we consider.

On the other hand, if W has subquadratic growth near the wells then the following theorem holds:

Theorem 1.2. *Assume that Ω, m, W satisfy hypotheses (2.1)-(2.7) with $q \in (0,1)$. Then*

$$\mathcal{F}^{(2)}(u) = \begin{cases} 2(c_{\mathrm{sym}} + c_W \tau_u)(n-1)\kappa_u \, \mathrm{P}(\{u=a\}; \Omega) & \textit{if } u \in \mathcal{U}_1, \\ \infty & \textit{otherwise in } L^1(\Omega). \end{cases} \tag{1.21}$$

Here now τ_u is a constant such that

$$\int_{\mathbb{R}} z(t - \tau_u) - \mathrm{sgn}_{a,b}(t) \, dt = 0. \tag{1.22}$$

Note that (1.21) and (1.22) correspond to the case $W''(a) = \infty$ in (1.15) and (1.18) respectively.

To prove (1.15) and (1.21) we follow the approach of [25], namely we use rearrangement to reduce the problem to a one-dimensional one. However, since we are not imposing boundary conditions (1.8) we cannot use standard symmetrization techniques in $H_0^1(\Omega)$ (see, e.g., [25, 37, 42]). Thus we introduce *a new type of rearrangement* which makes direct use of the *isoperimetric function* (1.13). We suspect that this rearrangement may also prove useful in analyzing other types of geometric variational problems.

In particular, if W is symmetric about $(b+a)/2$, then the function z in (1.17) is symmetric, and so the constants c_{sym} and τ_u simplify to give the following:

Corollary 1.3. *Suppose that, additionally, W is symmetric about $(b+a)/2$. Then for $u \in \mathcal{U}_1$ we have that*

$$\mathcal{F}^{(2)}(u) = \begin{cases} -\frac{2c_W^2 (n-1)^2}{W''(a)(b-a)^2}\kappa_u^2 & \textit{if } q = 1, \\ 0 & \textit{if } q < 1. \end{cases}$$

Thus in the case $q \in (0,1)$, and with isotropic energy, we recover the result of [25] *without the additional Dirichlet boundary condition* (1.8). We conjecture that in this case to obtain the second nonzero term in the asymptotic development one should replace the functionals $\mathcal{F}_\varepsilon^{(2)}$ in (1.3) with the family of functionals

$$\mathcal{F}_{\varepsilon,q}^{(2)} := \frac{\mathcal{F}_\varepsilon^{(1)} - \min \mathcal{F}^{(1)}}{\varepsilon^{1/q}}.$$

We have not been able to characterize the Γ-limit of $\mathcal{F}_{\varepsilon,q}^{(2)}$.

Remark 1.4. *A straightforward calculation shows that in the case of the Cahn–Hilliard potential $W(s) = \frac{1}{2}(1 - s^2)^2$ the second-order Γ-limit takes the form*

$$\mathcal{F}^{(2)}(u) = -\frac{(n-1)^2}{9}\kappa_u^2.$$

Besides their intrinsic interest, Theorems 1 and 2 have important applications in the study of the speed of motion of the associated gradient flow in dimension $n \geq 2$. Indeed the asymptotic development of (1.1) in one dimension has been utilized by many authors to establish the slow motion of solutions of the gradient flows associated with (1.1) in different function spaces. We recall that the gradient flow associated with F_ε in $L^2(\Omega)$ without the mass constraint (1.2) is the *Allen–Cahn equation*

$$\partial_t u = \varepsilon^2 \Delta u - W'(u), \tag{1.23}$$

while mass-constrained gradient flows in $L^2(\Omega)$ and $H^{-1}(\Omega)$ of F_ε are, respectively, the *non-local Allen–Cahn equation*

$$\partial_t u = \varepsilon^2 \Delta u - W'(u) + \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega W'(u) \, dx \tag{1.24}$$

and the *Cahn–Hilliard equation*

$$\partial_t u = -\Delta(\varepsilon^2 \Delta u - W'(u)). \tag{1.25}$$

For background on these equations and their applications see, e.g. [30]. The phenomenon of slow motion of solutions to (1.23) was analyzed via variational methods first by Bronsard and Kohn [13] for $n = 1$. They demonstrate that if $u_\varepsilon(0)$ converges in $L^1(J)$ to v, with $J \subset \mathbb{R}$ an open bounded interval and v a local minimizer of $F^{(1)}$, and if $F_\varepsilon^{(1)}(u_\varepsilon(0)) \leq F^{(1)}(v) + C\varepsilon^k$ for some k, then for any $l > 0$ we have the following *slow-motion inequality*:

$$\lim_{\varepsilon \to 0^+} \sup_{0 \leq t \leq l\varepsilon^{-k}} \int_J |u_\varepsilon(x, t) - v(x)| \, dx = 0. \tag{1.26}$$

A crucial estimate in their analysis is the following higher-order asymptotic estimate: that if $\|v_\varepsilon - v\|_{L^1} < \delta$ then

$$F_\varepsilon(v_\varepsilon) \geq F^{(1)}(v) - C\varepsilon^k \tag{1.27}$$

for appropriately chosen C.

Later similar results were established for the non-local Allen–Cahn equation (1.24), as well as the Cahn–Hilliard equation (1.25) (see [14, 15, 32]). The strength of these results is that they prove this slow motion using transparent variational methods, for initial data that are generic in the sense that they only need have small initial energy. These methods require a precise asymptotic development of order 2 (or higher) of the form (1.27).

More recently, a tight, higher-order asymptotic expansion of the family F_ε was given by Bellettini, Nayam and Novaga [10]. In that paper they use their result to prove a type of slow motion bound. Their results match the well-known results of Carr and Pego [18], which state that phase boundaries of the Allen–Cahn equation should move at speed $e^{-C\varepsilon^{-1}}$.

In the case $n = 1$ these slow dynamics are generally understood to be related to the existence of *slow manifolds*, and many works focus on the existence of data that approximately moves along a slow manifold. Some critical first works in this direction include [18, 29, 54], while a more recent perspective can be found in [53] and the references therein. It can also be shown [19] that the time it takes to approach the slow manifold from arbitrary initial data is generally very short.

The slow motion of phase boundaries in higher dimension has been studied by many authors (see, e.g., [2, 3, 4, 5]). These works generally focus on the existence of solutions that move very slowly, often along slow manifolds. Generally these results require some ansatz on the initial data, such as radial data or data parametrized by the distance from a manifold. The requirement of such an ansatz in higher dimensions is, in our opinion, due to the lack of higher-order asymptotics of the functional (1.1) in dimension greater than one.

An immediate consequence of Corollary 1.3 is that when v is a global minimizer of $\mathcal{F}^{(1)}$ we then have, for any sequence v_ε converging to v,

$$\mathcal{F}_\varepsilon^{(1)}(v_\varepsilon) \geq \mathcal{F}^{(1)}(v) - C\varepsilon.$$

Using exactly the techniques from [13] it is possible to establish generic, slow motion results similar to (1.26), for the non-local Allen–Cahn and Cahn–Hilliard equations in dimension greater than one, for data that are close to *global* perimeter minimizers [49]. We are currently investigating extensions of this type of result in the more interesting case of local perimeter minimizers.

This paper is organized as follows. In Section 2 we state our technical assumptions. In Section 3 we develop our new rearrangement result. In Section 4 we analyze a weighted, one-dimensional functional problem. In Section 5 we use the results in Sections 3 and 4 to prove Theorems 1.1 and 1.2.

2 Preliminaries and Main Assumptions

In this paper we consider the Cahn–Hilliard functional (1.1), where we assume that $\Omega \subset \mathbb{R}^n, n \leq 7$, is an open, connected, bounded set with

$$\mathcal{L}^n(\Omega) = 1 \quad \text{and} \quad \partial\Omega \text{ is of class } C^{2,\alpha}, \quad \alpha \in (0, 1]. \tag{2.1}$$

We observe that the restriction to $n \leq 7$ is necessary to guarantee regularity of minimizers of the problem (1.11) [31, 33, 45, 59], while the assumption that $\mathcal{L}^n(\Omega) = 1$ is for simplicity (the general case follows by a scaling argument). We assume that the mass m in (1.2) satisfies

$$a < m < b, \tag{2.2}$$

where a, b are the wells of W, and that the isoperimetric function $\mathcal{I}_\Omega$ defined in (1.13) satisfies the Taylor expansion

$$\mathcal{I}_\Omega(r) = \mathcal{I}_\Omega(r_m) + \mathcal{I}'_\Omega(r_m)(r - r_m) + O(|r - r_m|^{1+\beta}) \tag{2.3}$$

for all r close to $r_m = \frac{b-m}{b-a}$ (see (1.12)) and for some $\beta \in (0, 1]$. As remarked in the introduction, for domains of class C^2, $\mathcal{I}_\Omega$ is semi-concave (see [9]) and so (2.3) holds with $\beta = 1$ at $\mathcal{L}^1$ a.e. r_m in $[0, 1]$ (see [36]), or equivalently for $\mathcal{L}^1$ a.e. $m \in (a, b)$.

We also make the following assumptions on the potential $W : \mathbb{R} \to [0, \infty)$:

$$W \text{ is of class } C^2(\mathbb{R}\backslash\{a, b\}) \text{ and has precisely two zeros at } a < b, \tag{2.4}$$

$$\lim_{s \to a} \frac{W''(s)}{|s - a|^{q-1}} = \lim_{s \to b} \frac{W''(s)}{|s - b|^{q-1}} := \ell > 0, \quad q \in (0, 1], \tag{2.5}$$

$$W' \text{ has exactly 3 zeros at } a < c < b, \quad W''(c) > 0, \tag{2.6}$$

$$\liminf_{|s| \to \infty} |W'(s)| > 0. \tag{2.7}$$

Most of these assumptions are standard (see [35]). We note that in the case where $q = 1$ we have that ℓ is simply $W''(a)$. In particular we note that $q = 1$ when $W(s) = \frac{1}{2}(s^2 - 1)^2$, which is the classical Cahn–Hilliard potential (see, e.g., [16]). While it is possible to deal with different limits at a and b in (2.5), we do not handle those cases in our analysis for clarity of presentation.

Remark 2.1. *In view of* (2.4)-(2.7)*, we have that there exists an* $\hat{L} > 0$ *and* $\hat{T} > 0$ *so that*

$$W(s) \geq \hat{L}|s| \tag{2.8}$$

for all $|s| > \hat{T}$.

Remark 2.2. *In view of* (2.4) *and* (2.5) *if follows from L'Hôpital's rule that*

$$\lim_{s \to a} \frac{W(s)}{|s - a|^{1+q}} = \lim_{s \to b} \frac{W(s)}{|s - b|^{1+q}} = \frac{\ell}{q(1 + q)}, \tag{2.9}$$

$$\lim_{s \to a} \frac{W'(s)}{(s - a)|s - a|^{q-1}} = \lim_{s \to b} \frac{W'(s)}{(s - b)|s - b|^{q-1}} = \frac{\ell}{q}. \tag{2.10}$$

In turn, by (2.4), there exist $c_1, c_2 > 0$ such that $c_1^2(b - s)^{1+q} \leq W(s) \leq c_2^2(b - s)^{1+q}$ for all $s \in [\frac{a+b}{2}, b]$. It follows that the solution z of the Cauchy problem (1.17) satisfies

$$\left[(b - z(t_0))^{\frac{1-q}{2}} - \frac{(1 - q)c_2}{2}(t - t_0) \right]_+^{\frac{2}{1-q}} \leq b - z(t)$$

$$\leq \left[(b - z(t_0))^{\frac{1-q}{2}} - \frac{(1 - q)c_1}{2}(t - t_0) \right]_+^{\frac{2}{1-q}}$$

for all $t \geq t_0 \geq 0$ if $0 < q < 1$ and

$$(b - z(t_0))e^{-c_2(t-t_0)} \leq b - z(t) \leq (b - z(t_0))e^{-c_1(t-t_0)} \tag{2.11}$$

7

for all $t \geq t_0 \geq 0$ for $q = 1$. In particular, in the case $0 < q < 1$, since $z(0) = c$, there exists a constant

$$\left(\frac{b-a}{2}\right)^{\frac{1-q}{2}} \frac{2}{c_2(1-q)} \leq t_b \leq \left(\frac{b-a}{2}\right)^{\frac{1-q}{2}} \frac{2}{c_1(1-q)}$$

such that

$$z(t) \equiv b \quad \text{for all } t \geq t_b. \tag{2.12}$$

Similar estimates hold near a, so that $z(t) \equiv a$ for all $t \leq t_a < 0$ when $0 < q < 1$.

In what follows, given a set $E \subset \mathbb{R}^n$, we define d_E to be the *signed distance function* from the set E, namely

$$d_E(t) := \begin{cases} d(t, \partial E) & \text{if } t \in E^c, \\ -d(t, \partial E) & \text{if } t \in E. \end{cases} \tag{2.13}$$

3   A Pólya–Szegő Type Inequality

The classical Pólya–Szegő inequality states that in $\mathbb{R}^n$ the decreasing spherical rearrangement u^* of a positive function $u \in W^{1,p}(\mathbb{R}^n)$ will not increase the L^p norm of the gradient [37, 42]. This permits complicated problems in arbitrary dimensions to be reduced to radial, one-dimensional problems. For Dirichlet problems it is often possible to obtain similar inequalities for functions on a bounded domain Ω. Specifically, if $u \in H_0^1(\Omega)$ is positive then we can use the Pólya–Szegő inequality in the whole space (after extending the function u) to show that $\|\nabla u^*\|_{L^p(\Omega^*)} \leq \|\nabla u\|_{L^p(\Omega)}$. A classical example where this technique is used is in the proof of Talenti's inequality [60]. In this section we propose a new type of rearrangement that does not require extending functions to all of $\mathbb{R}^n$, and is thus better suited to analyzing certain Neumann problems.

In this section only we assume that Ω is bounded, connected, has measure $\mathcal{L}^n(\Omega) = 1$ and has Lipschitz boundary. Then the isoperimetric function (see (1.13)) satisfies

$$\mathcal{I}_\Omega(r) \geq C_1 \min\{r, 1-r\}^{\frac{n-1}{n}} \quad \text{for all } r \in [0,1]. \tag{3.1}$$

Indeed, this bound follows from Corollary 3 in Section 5.2.1 of [46] (see also [1] and [22]). By considering sets and their complements it is clear that $\mathcal{I}_\Omega(1/2 + r) = \mathcal{I}_\Omega(1/2 - r)$. We now prove an elementary proposition.

Proposition 3.1. *Suppose that $\mathcal{I}_\Omega$ satisfies (2.3) and (3.1). Then there exists a function $\mathcal{I}_\Omega^* \in C_{\mathrm{loc}}^{1,\beta}(0,1)$ such that*

$$\mathcal{I}_\Omega^*(r) = \mathcal{I}_\Omega^*(1-r) \quad \text{for all } r \in (0,1), \tag{3.2}$$

$$\mathcal{I}_\Omega \geq \mathcal{I}_\Omega^* > 0 \quad \text{in } (0,1), \tag{3.3}$$

$$\mathcal{I}_\Omega(r_m) = \mathcal{I}_\Omega^*(r_m), \quad \mathcal{I}_\Omega'(r_m) = (\mathcal{I}_\Omega^*)'(r_m), \tag{3.4}$$

$$\mathcal{I}_\Omega^*(r) = C_0 r^{\frac{n-1}{n}} \quad \text{for all } r \in (0, \delta) \tag{3.5}$$

for some $C_0 > 0$ and $0 < \delta < 1$.

Proof. Assume first that $r_m \in (0, 1/2)$. By (2.3) there exist $\delta, C > 0$ so that

$$|\mathcal{I}_\Omega(r) - \mathcal{I}_\Omega(r_m) - \mathcal{I}_\Omega'(r_m)(r - r_m)| \leq C|r - r_m|^{1+\beta} \tag{3.6}$$

for all $r \in (r_m - \delta, r_m + \delta)$. By taking δ smaller we can assume that

$$\delta < \frac{\mathcal{I}_\Omega(r_m)}{2(1 + |\mathcal{I}_\Omega'(r_m)|)}. \tag{3.7}$$

Define

$$\widehat{\mathcal{I}_\Omega}(r) := \mathcal{I}_\Omega(r_m) + \mathcal{I}'_\Omega(r_m)(r - r_m) - 2C|r - r_m|^{1+\beta} \tag{3.8}$$

for $r \in (r_m - \delta, r_m + \delta) \cap (0, 1/2)$. Then by (3.6), $\mathcal{I}_\Omega \geq \widehat{\mathcal{I}_\Omega}$ in $(r_m - \delta, r_m + \delta) \cap (0, 1/2)$ with strict inequality for $r \neq r_m$. Moreover by (3.7),

$$\widehat{\mathcal{I}_\Omega}(r) \geq \mathcal{I}_\Omega(r_m)/4 > 0$$

for $r \in (r_m - \delta, r_m + \delta) \cap (0, 1/2)$, provided we take δ even smaller. Since $\mathcal{I}_\Omega(r) > \mathcal{I}^*_\Omega(r)$ for all $r \in (r_m - \delta, r_m + \delta) \cap (0, 1/2) \setminus \{r_m\}$, and since $\mathcal{I}_\Omega(r) \geq C_1 r^{n-1}$ for $r \in (0, 1/2)$ and $\mathcal{I}_\Omega$ is continuous and positive in $[0, 1]$, and is symmetric, we can extend $\widehat{\mathcal{I}_\Omega}$ to a function $\mathcal{I}^*_\Omega$ that satisfies (3.2)-(3.5).

If $r_m = 1/2$ then $\mathcal{I}'_\Omega(r_m) = 0$, by the symmetry of $\mathcal{I}_\Omega$ with respect to $1/2$. In turn the function $\widehat{\mathcal{I}_\Omega}$ in (3.8) is also symmetric with respect to $1/2$ and so we can define $\widehat{\mathcal{I}_\Omega}$ as in (3.8) for all $r \in (1/2 - \delta, 1/2 + \delta)$ and continue as before. The case $r_m \in (1/2, 1)$ follows by symmetry. $\qquad\square$

In subsequent sections it is more convenient to work with $\mathcal{I}^*_\Omega$ instead of $\mathcal{I}_\Omega$. The results of this section, however, hold when using $\mathcal{I}_\Omega$ instead of $\mathcal{I}^*_\Omega$.

Next we define a function V_Ω as the solution to the following Cauchy problem:

$$\frac{d}{dt}V_\Omega(t) = \mathcal{I}^*_\Omega(V_\Omega(t)), \quad V_\Omega(0) = 1/2. \tag{3.9}$$

We can extend $\mathcal{I}^*_\Omega$ to be zero outside of $[0, 1]$. Since $\mathcal{I}^*_\Omega$ is bounded and continuous (see Proposition 3.1), the Cauchy problem (3.9) admits a global solution $V_\Omega : \mathbb{R} \to [0, 1]$. It follows from inequality (3.1) that there is a $T > 0$ so that $0 < V_\Omega(t)$ for $-T < t < 0$ and $V_\Omega(-T) = 0$. Moreover by equation (3.2) we have that $V_\Omega(T) = 1$ and $V_\Omega(t) < 1$ for all $0 < t < T$. Define

$$I := (-T, T). \tag{3.10}$$

Observe that V_Ω is uniquely defined because $\mathcal{I}^*_\Omega$ is locally Lipschitz on any compact subset of $(0, 1)$, and $\mathcal{I}^*_\Omega = 0$ outside of $(0, 1)$.

In what follows for $y \in \mathbb{R}^n$ we use the notation $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Next we define a set $\Omega^* \subset \mathbb{R}^n$, which will be a type of rearrangement of Ω. This set is defined by:

$$\Omega^* := \{y : \ y_n \in I, \ y' \in B_{n-1}(0, r(y_n))\}, \tag{3.11}$$

where for $t \in I$,

$$r(t) := \left(\frac{\mathcal{I}^*_\Omega(V_\Omega(t))}{\alpha_{n-1}}\right)^{1/(n-1)} \quad \text{and} \quad \alpha_{n-1} := \mathcal{L}^{n-1}(B_{n-1}(0, 1)).$$

Note that the definition of $r(t)$ implies that

$$\mathcal{L}^{n-1}(B_{n-1}(0, r(t))) = \mathcal{I}^*_\Omega(V_\Omega(t)) \tag{3.12}$$

for all $t \in \bar{I}$.

The following lemma motivates our choice of the Cauchy problem (3.9).

Lemma 3.2. *For any $t \in \bar{I}$ the following equalities hold:*

$$V_\Omega(t) = \mathcal{L}^n(\Omega^* \cap \{y_n < t\}), \tag{3.13}$$
$$\mathcal{I}^*_\Omega(V_\Omega(t)) = \mathrm{P}(\{y_n < t\}; \Omega^*). \tag{3.14}$$

Proof. We prove equation (3.13) by using Fubini's theorem, equation (3.12), the Cauchy problem (3.9), the fundamental theorem of calculus, and the fact that $V_\Omega(-T) = 0$, in that order:

$$\mathcal{L}^n(\Omega^* \cap \{y_n < t\}) = \int_{-T}^t \mathcal{H}^{n-1}(\Omega^* \cap \{y_n = s\})\,ds$$

$$= \int_{-T}^t \mathcal{I}_\Omega^*(V_\Omega(s))\,ds$$

$$= V_\Omega(t) - V_\Omega(-T) = V_\Omega(t).$$

Equality (3.14) follows immediately from equation (3.12) and Definition 3.11. $\qquad\square$

Now given any measurable function $u : \Omega \to \mathbb{R}$, we define the distribution function $\varrho_u(s) := \mathcal{L}^n(\{u > s\})$ and the following function:

$$g_u(t) := \sup\{s \in \mathbb{R} : \varrho_u(s) > V_\Omega(t)\}. \tag{3.15}$$

We then define a function $u^* : \Omega^* \to \mathbb{R}$ as follows:

$$u^*(y', y_n) := g_u(y_n). \tag{3.16}$$

The first important property of our rearranged function can be summarized by the following lemma:

Lemma 3.3. *Let $u : \Omega \to \mathbb{R}$ be a measurable function. Then the functions u^* and u are equimeasurable, meaning that $\varrho_u = \varrho_{u^*}$. This implies that for any Borel function $\psi : \mathbb{R} \to \mathbb{R}$,*

$$\int_\Omega \psi(u)\,dx = \int_{\Omega^*} \psi(u^*)\,dy = \int_I \psi(g_u)\mathcal{I}_\Omega^*(V_\Omega)\,dt,$$

assuming that the previous integrals are well-defined. In particular the L^p norms of u and u^* are preserved, as well as the integral of $W \circ u$.

Proof. First we note that, by standard arguments, ϱ_u is decreasing and right continuous and that g_u is decreasing and left continuous (see, e.g., [42], p. 478).

Let $h(t) := \sup\{s : g_u(s) > t\}$. Since g_u is decreasing we have that

$$\varrho_{u^*}(t) = \mathcal{L}^n(\{y \in \Omega^* : g_u(y_n) > t\})$$
$$= \mathcal{L}^n(\{y \in \Omega^* : y_n < h(t)\}) = V_\Omega(h(t)),$$

where in the last equality we have used Lemma 3.2.

We then claim that $V_\Omega(h(t)) = \varrho_u(t)$. To see this observe that since $\mathcal{I}_\Omega^* > 0$ in $(0,1)$, by (3.9) we have that V_Ω is strictly increasing and of class C^1 in I. Hence:

$$V_\Omega(h(t)) = V_\Omega(\sup\{s : g_u(s) > t\}) = \sup\{V_\Omega(s) : g_u(s) > t\}$$
$$= \sup\{V_\Omega(s) : \sup\{\tau : \varrho_u(\tau) > V_\Omega(s)\} > t\}$$
$$= \sup\{\rho : \sup\{\tau : \varrho_u(\tau) > \rho\} > t\}. \tag{3.17}$$

For every ρ such that $\sup\{\tau : \varrho_u(\tau) > \rho\} > t$, there exists $\tau > t$ such that $\varrho_u(\tau) > \rho$. But since ϱ_u is decreasing we have that $\varrho_u(t) \geq \varrho_u(\tau) > \rho$, which then shows that

$$V_\Omega(h(t)) \leq \varrho_u(t).$$

Now if $V_\Omega(h(t)) < \varrho_u(t)$, then $V_\Omega(h(t)) < \varrho_u(t) - \epsilon$ for some $\epsilon > 0$. By equation (3.17) this implies that

$$\sup_s\{s : \varrho_u(s) > \varrho_u(t) - \varepsilon\} \leq t.$$

10

By the right continuity of ϱ_u for some $\delta > 0$ we have that $\varrho_u(t + \delta) > \varrho_u(t) - \epsilon$, which violates the previous inequality. This then implies that $\varrho_u(t) = V_\Omega(h(t)) = \varrho_{u^*}(t)$ for all t, which is the desired conclusion.

To see the integral equality stated, we note that (see, e.g., Theorem B.61 in [42]):

$$\int_\Omega \psi(u(x)) \, dx = \int_\mathbb{R} \psi(s) \, d\varrho_u(s) = \int_\mathbb{R} \psi(s) \, d\varrho_{u^*}(s) = \int_{\Omega^*} \psi(u^*(y)) \, dy.$$

This concludes the proof.

$\square$

Next we prove that the operation of rearrangement is a contraction in L^1. It should be possible to prove a more general version of this proposition, but this suffices for our purposes.

Proposition 3.4. *Suppose that $u_1, u_2 \in L^1(\Omega)$. Then*

$$\|u_1^* - u_2^*\|_{L^1(\Omega^*)} \le \|u_1 - u_2\|_{L^1(\Omega)}. \tag{3.18}$$

Proof. By the definition of u^* (see (3.15) and (3.16)) we have that if $u_1 \le u_2$ then $u_1^* \le u_2^*$. Applying Proposition 1 in [23], in light of Lemma 3.3, then implies (3.18). $\square$

Next we prove a lemma stating that truncation and rearrangement commute. This will later allow us to establish that the rearrangement of a Sobolev function is still a Sobolev function.

Lemma 3.5. *Let $u : \Omega \to \mathbb{R}$ be measurable. Given $s_1 < s_2$, let $\mathrm{Tr}_{s_1, s_2}(s) := (s \vee s_1) \wedge s_2, s \in \mathbb{R}$. Then the following equality holds:*

$$\mathrm{Tr}_{s_1, s_2}(u^*) = (\mathrm{Tr}_{s_1, s_2}(u))^*.$$

Proof. Set $v := \mathrm{Tr}_{s_1, s_2}(u)$. It suffices to show that $\mathrm{Tr}_{s_1, s_2} g_u = g_v$. Let $t_1 := \sup\{t : g_u(t) \ge s_2\}$ and $t_2 := \inf\{t : g_u(t) \le s_1\}$.

Step 1: By Lemma 3.3 and the definition of truncation we can deduce the following:

$$\mathcal{L}^n(\{\mathrm{Tr}_{s_1, s_2}(u^*) = s_2\}) = \mathcal{L}^n(\{u^* \ge s_2\}) = \mathcal{L}^n(\{u \ge s_2\})$$
$$= \mathcal{L}^n(\{v = s_2\}) = \mathcal{L}^n(\{v^* = s_2\}).$$

As g_u is decreasing we find that for $t < t_1$, $\mathrm{Tr}_{s_1, s_2} g_u(t) = s_2$. Since g_v is decreasing and is bounded above by s_2 the previous chain of equalities implies that for $t < t_1$ we have that $g_v(t) = s_2$, which then implies that $g_v(t) = \mathrm{Tr}_{s_1, s_2} g_u(t)$ for such t.

Using an identical argument we find that for $t > t_2$ we have that $\mathrm{Tr}_{s_1, s_2} g_u(t) = g_v(t)$.

Step 2: Next we consider $t \in (t_1, t_2)$. First, for $s \in (s_1, s_2)$ we have that $\varrho_u(s) = \varrho_v(s)$. Next, we note that for $t \in (t_1, t_2)$ we have that $g_u(t) \in (s_1, s_2)$ and that $g_v(t) \in (s_1, s_2)$. Thus we can write the following for $t \in (t_1, t_2)$:

$$\mathrm{Tr}_{s_1, s_2}(g_u(t)) = g_u(t) = \sup\{s \in \mathbb{R} : \varrho_u(s) > V_\Omega(t)\}$$
$$= \sup\{s \in (s_1, s_2) : \varrho_u(s) > V_\Omega(t)\}$$
$$= \sup\{s \in (s_1, s_2) : \varrho_v(s) > V_\Omega(t)\}$$
$$= \sup\{s \in \mathbb{R} : \varrho_v(s) > V_\Omega(t)\} = g_v(t).$$

Step 3: As $\mathrm{Tr}_{s_1, s_2}(g_u)$ and g_v are both left continuous, we then have that $\mathrm{Tr}_{s_1, s_2} g_u = g_v$ everywhere, as desired. $\square$

Next we state a simple identity related to level sets of functions. This is well-known (see [20]), but we include the proof for completeness.

Lemma 3.6. *For $u \in W^{1,1}(\Omega)$ there exists a representative of u such that the following equality holds for all $s_1 < s_2$:*

$$\int_{s_1}^{s_2} \int_{u^{-1}(s)} |\nabla u(x)|^{-1} d\mathcal{H}^{n-1} ds = \mathcal{L}^n(\{x \in \Omega : u(x) \in (s_1, s_2), \nabla u(x) \neq 0\}). \tag{3.19}$$

Proof. Let $H_\varepsilon := (\varepsilon + |\nabla u|)^{-1}$. By the coarea formula (see [26]) we find that:

$$\int_{\{s_1 < u < s_2, \; \nabla u \neq 0\}} H_\varepsilon |\nabla u| \, dx = \int_{\{s_1 < u < s_2\}} H_\varepsilon |\nabla u| \, dx$$
$$= \int_{s_1}^{s_2} \int_{u^{-1}(s)} H_\varepsilon \, d\mathcal{H}^{n-1} ds.$$

By noting that $H_\varepsilon \to |\nabla u|^{-1}$ monotonically in the set $\{\nabla u \neq 0\}$, we find that (3.19) holds. $\square$

Next we state and prove a simple lemma, which is essentially an isoperimetric inequality.

Lemma 3.7. *Given $u \in BV(\Omega)$, for any t the following must hold:*

$$\mathrm{P}(\{u^* > t\}; \Omega^*) \leq \mathrm{P}(\{u > t\}; \Omega).$$

Proof. As g_u is a decreasing function (see (3.15)), we note that the set $\{u^* > t\}$ is actually a set of the form $\{y_n < s\}$. By Lemma 3.2 we have that $V_\Omega(s) = \mathcal{L}^n(\Omega^* \cap \{y_n < s\}) = \varrho_{u^*}(t)$. By then recalling that u and u^* are equimeasurable (see Lemma 3.3) and by Lemma 3.2 we have the following:

$$\mathrm{P}(\{u^* > t\}; \Omega^*) = \mathcal{I}_\Omega^*(\varrho_{u^*}(t)) = \mathcal{I}_\Omega^*(\varrho_u(t)) \leq \mathcal{I}_\Omega(\varrho_u(t))$$
$$\leq \mathrm{P}(\{u > t\}; \Omega),$$

where we have used the fact that $\mathcal{I}_\Omega^* \leq \mathcal{I}_\Omega$ and (1.13). This concludes the proof. $\square$

Next we prove two lemmas that are preliminary to establishing our Pólya–Szegő type result.

Lemma 3.8. *Given $u \in BV(\Omega)$, we have that $u^* \in BV(\Omega^*)$ and that the following inequality holds:*

$$\int_I \mathcal{I}_\Omega^*(V_\Omega(s)) d|Dg_u|(s) = |Du^*|(\Omega^*) \leq |Du|(\Omega).$$

Proof. By Lemma 3.3 we have that $u^* \in L^1(\Omega^*)$. By (3.15) and by the fact that g_u is decreasing, it follows that $g_u \in BV_{\mathrm{loc}}(I)$ (see, e.g., Theorem 7.2 in [42]).

Moreover by our definition of u^* (see (3.11), (3.12), (3.15), and Lemma 3.3) we can write the following:

$$|Du^*|(\Omega^*) = \sup\left\{ \int_{\Omega^*} \phi(y', y_n) d(Dg_u)(y_n) : \phi \in C_0(\Omega^*), \|\phi\|_{C_0} \leq 1 \right\}$$

$$= \sup\left\{ \int_I \left(\int_{B_{n-1}(0, r(y_n))} \phi(y', y_n) \, dy' \right) d(Dg_u)(y_n) : \phi \in C_0(\Omega^*), \|\phi\|_{C_0} \leq 1 \right\}$$

$$= \sup\left\{ \int_I \mathcal{I}_\Omega^*(V_\Omega(y_n)) \psi(y_n) \, d(Dg_u)(y_n) : \psi \in C_0(-T, T), \|\psi\|_{C_0} \leq 1 \right\}$$

$$= \int_I \mathcal{I}_\Omega^*(V_\Omega(y_n)) \, d|Dg_u|(y_n).$$

Next we utilize the coarea formula and Lemma 3.7 as follows:

$$|Du^*|(\Omega^*) = \int_{\mathbb{R}} \mathrm{P}(\{u^* > t\}; \Omega^*) \, dt \leq \int_{\mathbb{R}} \mathrm{P}(\{u > t\}; \Omega) \, dt = |Du|(\Omega).$$

This proves the desired lemma. $\square$

Lemma 3.9. *Given $u \in W^{1,1}(\Omega)$, it follows that $u^* \in W^{1,1}(\Omega^*)$.*

Proof. By (3.16) it suffices to show that g_u is absolutely continuous on any sub-interval $[t_0, t_1]$ compactly contained in I. Fix $\epsilon > 0$, and let δ be small enough such that for any measurable $E \subset \Omega$ with $\mathcal{L}^n(E) < \delta$ the following holds (see (3.3)):

$$\int_E |\nabla u| \, dx \leq \epsilon \min_{t \in [t_0, t_1]} \mathcal{I}_\Omega^*(V_\Omega(t)). \tag{3.20}$$

Now consider any finite collection of non-overlapping subintervals (a_k, b_k) of $[t_0, t_1]$, satisfying

$$\sum_{k=1}^N (b_k - a_k) \leq \frac{\delta}{\max_{t \in [t_0, t_1]} \mathcal{I}_\Omega^*(V_\Omega(t))}. \tag{3.21}$$

The following estimate holds by (3.9), (3.13), (3.15), (3.16), Lemma 3.3 and (3.21):

$$\mathcal{L}^n \left(\bigcup_{k=1}^N \{x \in \Omega : g_u(b_k) < u(x) < g_u(a_k)\} \right)$$

$$= \sum_{k=1}^N \mathcal{L}^n(\{y \in \Omega^* : g_u(b_k) < u^*(y) < g_u(a_k)\})$$

$$\leq \sum_{k=1}^N (V_\Omega(b_k) - V_\Omega(a_k)) \leq \max_{t \in [t_0, t_1]} \mathcal{I}_\Omega^*(V_\Omega(t)) \sum_{k=1}^N (b_k - a_k) \leq \delta. \tag{3.22}$$

Next, set $s_1 := g_u(b_k)$ and $s_2 := g_u(a_k)$ and let $v := \mathrm{Tr}_{s_1, s_2} u$. By applying Lemma 3.5, Lemma 3.8 above and Theorem 7.2 in [42] we obtain

$$\min_{t \in [t_0, t_1]} \mathcal{I}_\Omega^*(V_\Omega(t)) |g_u(a_k) - g_u(b_k)|$$

$$\leq \int_{a_k}^{b_k} \mathcal{I}_\Omega^*(V_\Omega(t)) \, d|Dg_u|(t) = \int_I \mathcal{I}_\Omega^*(V_\Omega(t)) \, d|D(\mathrm{Tr}_{s_1, s_2} g_u)|(t)$$

$$= |Dv^*|(\Omega^*) \leq |Dv|(\Omega) = \int_{\{g_u(b_k) < u < g_u(a_k)\}} |\nabla u| \, dx.$$

We then find the following:

$$\min_{t \in [t_0, t_1]} \mathcal{I}_\Omega^*(V_\Omega(t)) \sum |g_u(a_k) - g_u(b_k)| \leq \int_{\bigcup_k \{g_u(b_k) < u < g_u(a_k)\}} |\nabla u| \, dx$$

$$\leq \min_{t \in [t_0, t_1]} (\mathcal{I}_\Omega^*(V_\Omega(t))) \epsilon,$$

where we have used (3.20) and (3.22). This implies that g_u is absolutely continuous on $[t_0, t_1]$, as claimed. $\qquad \square$

Now we prove the main result of this section.

Theorem 3.10. *If $u \in W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$ then $u^* \in W^{1,p}(\Omega^*)$ and furthermore:*

$$\int_I |g_u'|^p \mathcal{I}_\Omega^*(V_\Omega) \, ds = \int_{\Omega^*} |\nabla u^*|^p \, dy \leq \int_\Omega |\nabla u|^p \, dx.$$

13

Proof. Lemmas 3.8 and 3.9 immediately give this inequality if $p = 1$. For $p > 1$ we can still apply the previous lemmas to show that $u^* \in W^{1,1}(\Omega^*)$, because Ω has finite measure.

Next we note that the following equality holds (by using the coarea formula):

$$\varrho_u(t) = \mathcal{L}^n(\{u > t\} \cap \{\nabla u = 0\}) + \int_t^\infty \int_{\{u=s, \nabla u \neq 0\}} |\nabla u|^{-1} \, d\mathcal{H}^{n-1} \, ds =: f_1^u(t) + f_2^u(t). \quad (3.23)$$

Clearly f_2^u is absolutely continuous, and f_1^u is decreasing. Thus ϱ_u is differentiable for a.e. t, with:

$$\varrho_u'(t) \leq - \int_{\{u=t, \nabla u \neq 0\}} |\nabla u|^{-1} \, d\mathcal{H}^{n-1}. \quad (3.24)$$

Next we claim that (following [20]) for a.e. t:

$$\frac{d}{dt} f_1^{u^*}(t) = \frac{d}{dt} \mathcal{L}^n(\{u^* > t\} \cap \{\nabla u^* = 0\}) = 0. \quad (3.25)$$

To establish this claim, we first note that for any open interval J we have the following:

$$\mathcal{L}^1(g_u(J)) \leq \int_J |g_u'| \, ds.$$

By approximating measurable sets with disjoint open intervals we can then establish that

$$\mathcal{L}^1(g_u(\{g_u' = 0\})) \leq \int_{\{g_u'=0\}} |g_u'| \, ds = 0.$$

Following [21] we then find that:

$$\mathcal{L}^1(u^*(\{\nabla u^* = 0\})) = \mathcal{L}^1(g_u(\{g_u' = 0\})) = 0.$$

Thus there exists a Borel set F_0 in $\mathbb{R}$ so that $\mathcal{L}^1(F_0) = 0$ and so that $u^*(\{\nabla u^* = 0\}) \subset F_0$.

We then claim that for any Borel set B in $\mathbb{R}$ we have that

$$|Df_1^{u^*}|(B) = \mathcal{L}^n((u^*)^{-1}(B) \cap \{\nabla u^* = 0\}).$$

To see this, we first note that $f_1^{u^*}$ is right continuous and decreasing. We then have that

$$|Df_1^{u^*}|((t_1, t_2)) = f_1^{u^*}(t_1) - \lim_{t \to t_2^-} f_1^{u^*}(t_2)$$
$$= \mathcal{L}^n(\{u^* > t_1\} \cap \{\nabla u^* = 0\}) - \lim_{t \to t_2^-} \mathcal{L}^n(\{u^* > t\} \cap \{\nabla u^* = 0\})$$
$$= \mathcal{L}^n(\{u^* > t_1\} \cap \{\nabla u^* = 0\}) - \mathcal{L}^n(\{u^* \geq t_2\} \cap \{\nabla u^* = 0\})$$
$$= \mathcal{L}^n((u^*)^{-1}((t_1, t_2)) \cap \{\nabla u^* = 0\}).$$

As both $|Df_1^{u^*}|$ and $\mathcal{L}^n((u^*)^{-1}(\cdot) \cap \{\nabla u^* = 0\})$ are Borel measures, and as they are equal on open intervals, they must be equal on all Borel sets. This and the fact that $u^*(\{\nabla u^* = 0\}) \subset F_0$ immediately give that

$$|Df_1^{u^*}|(\mathbb{R} \backslash F_0) = \mathcal{L}^n((u^*)^{-1}(\mathbb{R} \backslash F_0) \cap \{\nabla u^* = 0\}) = \mathcal{L}^n(\emptyset) = 0,$$

which proves (3.25). Utilizing (3.23) this then immediately implies that for a.e. t,

$$\varrho_{u^*}'(t) = - \int_{\{u^*=t, \nabla u^* \neq 0\}} |\nabla u^*|^{-1} \, d\mathcal{H}^{n-1}. \quad (3.26)$$

14

By the coarea formula we can write the following:

$$\int_{\Omega^*} |\nabla u^*|^p \, dy = \int_{\Omega^* \cap \{\nabla u^* \neq 0\}} |\nabla u^*|^p \, dy$$

$$= \int_{\mathbb{R}} \int_{\{u^* = t\} \cap \{\nabla u^* \neq 0\}} |\nabla u^*|^{p-1} \, d\mathcal{H}^{n-1} \, dt.$$

By (3.16) we know that $\nabla u^*(y) = (0, g_u'(y_n)) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Since g_u is decreasing we have that the set $\{u^* = t\}$ is a set of the form $\{y : y_n \in [t_1, t_2], \, y' \in B_{n-1}(0, r(y_n))\}$, for some $t_1 \leq t_2$ with possibly $t_1 = t_2$. If $t_1 = t_2$ then clearly ∇u^* is constant on the set $\{u^* = t\}$. If $t_1 \neq t_2$ then g_u' is zero on the set (t_1, t_2), and is either zero at t_1, t_2 or is undefined. Since ∇u^* is constant on level sets of u^* (where it's defined) we can then write

$$\int_{\Omega^*} |\nabla u^*|^p \, dy = \int_{\mathbb{R}} \frac{\left(\mathcal{H}^{n-1}(\{u^* = t\} \cap \{\nabla u^* \neq 0\}) \right)^p}{\left(\int_{\{u^* = t\} \cap \{\nabla u^* \neq 0\}} |\nabla u^*|^{-1} \, d\mathcal{H}^{n-1} \right)^{p-1}} \, dt.$$

By (3.26) we have that

$$\int_{\Omega^*} |\nabla u^*|^p \, dy = \int_{\mathbb{R}} \frac{P(\{u^* > t\}; \Omega^*)^p}{(-\varrho_{u^*}'(t))^{p-1}} \, dt.$$

Next we utilize Lemma 3.3 and Lemma 3.7 to find that

$$\int_{\Omega^*} |\nabla u^*|^p \, dy \leq \int_{\mathbb{R}} \frac{P(\{u > t\}; \Omega)^p}{(-\varrho_u'(t))^{p-1}} \, dt.$$

Next (3.24) gives

$$\int_{\Omega^*} |\nabla u^*|^p \, dy \leq \int_{\mathbb{R}} \frac{P(\{u > t\}; \Omega)^p}{(\int_{\{u = t\}} |\nabla u|^{-1} \, d\mathcal{H}^{n-1})^{p-1}} \, dt.$$

Jensen's inequality on $f(s) = s^{-(p-1)}$ then implies that

$$\int_{\Omega^*} |\nabla u^*|^p \, dy \leq \int_{\mathbb{R}} \int_{\{u = t\}} |\nabla u|^{p-1} \, d\mathcal{H}^{n-1} \, dt,$$

which after applying the coarea formula gives the desired result. $\qquad \square$

Remark 3.11. *In this section we have considered a rearrangement of the function u, via the decreasing function $g_u : I \to \mathbb{R}$. However, all of the arguments would hold for an increasing rearrangement. Indeed, by utilizing (3.2) and by Lemma 3.3 for any Borel function $\psi : \mathbb{R} \to [0, \infty)$, the function $f_u(t) := g_u(-t)$ satisfies the following*

$$\int_I \psi(f_u(t)) \mathcal{I}_\Omega^*(V_\Omega(t)) \, dt = \int_I \psi(g_u(t)) \mathcal{I}_\Omega^*(V_\Omega(t)) \, dt = \int_\Omega \psi(u) \, dx,$$

$$\int_I |f_u'(t)|^p \mathcal{I}_\Omega^*(V_\Omega(t)) \, dt = \int_I |g_u'(t)|^p \mathcal{I}_\Omega^*(V_\Omega(t)) \, dt \leq \int_\Omega |\nabla u|^p \, dx.$$

We chose to work with the decreasing rearrangement in this section because that is the standard convention chosen in the literature involving rearrangement. However, we will work with the increasing rearrangement f_u of u in subsequent sections because we prefer to think of phase transitions as increasing functions.

The following corollary is the motivation for our development of the rearrangement in this section, and is a simple application of Lemma 3.3 and Theorem 3.10.

Corollary 3.12. *Let $u \in H^1(\Omega)$. Then the following inequality holds:*

$$\int_\Omega W(u) + \varepsilon^2 |\nabla u|^2 \, dx \geq \int_I (W(f_u) + \varepsilon^2 (f_u')^2) \mathcal{I}_\Omega^*(V_\Omega) \, dt. \tag{3.27}$$

Moreover

$$\int_\Omega u \, dx = \int_I f_u \mathcal{I}_\Omega^*(V_\Omega) \, dt.$$

4 A 1D Functional Problem

In light of Corollary 3.12, one possible avenue for studying the Γ-$\liminf$ of $\mathcal{F}_\varepsilon^{(1)}$ is to study the weighted, one-dimensional functional in (3.27). This was precisely the approach in [25], where the radial case was studied. Because we do not have a specific form for $\mathcal{I}_\Omega^*(V_\Omega)$, it is necessary for us to consider a much more general class of weights. The general weighted case, to our knowledge, has only been studied in [41]. They studied monotonicity properties of minimizers along curved strips in $\mathbb{R}^2$. Our work focuses on an entirely different question and applies to a wider class of weights.

We recall (3.10), namely

$$I := (-T, T).$$

In this section only T may be any positive number. We will assume that the weight η satisfies the following:

$$\eta \in C^{1,\beta}(I), \quad \eta > 0 \text{ in } I, \tag{4.1}$$

$$d_1(t+T)^{n_1-1} \leq \eta(t) \leq d_2(t+T)^{n_1-1} \text{ for } t \in (-T, -T+t^*], \tag{4.2}$$

$$d_3(T-t)^{n_2-1} \leq \eta(t) \leq d_4(T-t)^{n_2-1} \text{ for } t \in [T-t^*, T), \tag{4.3}$$

$$|\eta'(t)| \leq \frac{d_5 \eta(t)}{\min\{T-t, t+T\}} \quad \text{for } t \in I, \tag{4.4}$$

for some constants $\beta \in (0,1]$, $d_1, \ldots, d_5 > 0$, $n_1, n_2 \in \mathbb{N}$ and $t^* > 0$. These assumptions are naturally satisfied if $\eta = \mathcal{I}_\Omega^*(V_\Omega)$.

Remark 4.1. *Two important weights are covered in our analysis. The unweighted case $\eta \equiv 1$ can be recovered by taking $n_1 = n_2 = 1$ and $d_i = 1$ for $i = 1, \ldots, 4$, while the radial weight $\eta(t) = (T+t)^{n-1}$ can be obtained by taking $n_1 = n$, $n_2 = 1$, $d_1 = d_2 = 1$ and appropriate d_3 and d_4. Both of these cases have been previously studied by various authors (see, e.g., [10, 51]).*

By way of notation, we will write L_η^p to be the space $L^p(I; \mathbb{R}, \eta)$, where $p \geq 1$. We will also write BV_η to be the space $BV(I; \mathbb{R}, \eta)$ with weight η, meaning that

$$\|v\|_{BV_\eta} := \int_I |v(t)| \eta(t) \, dt + \int_I \eta(t) \, d|Dv|(t).$$

For $v \in BV_\eta$ we will also write the weighted total variation of the derivative in the following manner

$$|Dv|_\eta(E) = \int_E \eta(t) \, d|Dv|(t).$$

We will write H_η^1 to be the analogous weighted version of H^1. We conduct our analysis in the weighted spaces because it is the natural setting for this variational problem.

In this section we study the functional

$$G_\varepsilon(v) := \int_I (W(v) + \varepsilon^2 (v')^2) \eta \, dt, \quad v \in H_\eta^1,$$

subject to the constraint that

$$\int_I v\eta \, dt = m \in \left(a \int_I \eta \, dt, b \int_I \eta \, dt \right). \tag{4.5}$$

We extend G_ε to L_η^1 by setting $G_\varepsilon(v) := \infty$ if $v \in L_\eta^1 \backslash H_\eta^1$ or if (4.5) fails.

4.1 Zero and First-Order Γ-limit of G_ε

We begin by establishing the zeroth-order Γ-limit of the functional G_ϵ.

Theorem 4.2. *Assume that W satisfies hypotheses (2.4)-(2.7) and that η satisfies hypotheses (4.1)-(4.4). Then the family $\{G_\varepsilon\}$ Γ-converges to $G^{(0)}$ in L_η^1, where*

$$G^{(0)}(v) := \begin{cases} \int_I W(v)\eta \, dt & \text{if } v \in L_\eta^1 \text{ and } \int_I v\eta \, dt = m, \\ \infty & \text{otherwise in } L_\eta^1. \end{cases}$$

Proof. For the $\liminf$ inequality assume that $v_\varepsilon \to v$ in L_η^1. By utilizing Fatou's lemma along with (2.4) we have that

$$\liminf_{\varepsilon \to 0^+} G_\varepsilon(v_\varepsilon) \geq \liminf_{\varepsilon \to 0^+} \int_I W(v_\varepsilon)\eta \, dt \geq \int_I W(v)\eta \, dt.$$

For the $\limsup$ inequality, we begin by assuming that v is bounded and satisfies (4.5) (the case where v does not satisfy (4.5) is trivial). Let ϕ_δ be the standard mollifier, let $\tilde{v}$ be v extended to all of $\mathbb{R}$ by zero and consider $\tilde{v}_\varepsilon := \phi_{\delta_\varepsilon} * \tilde{v}$, where we select δ_ε so that $\|v - \tilde{v}_\varepsilon\|_{L_\eta^1} = o(1)$ and so that

$$\int_I (\tilde{v}_\varepsilon')^2 \eta \, dt \leq C\varepsilon^{-1}.$$

We then select $d_\varepsilon \in \mathbb{R}$ so that $v_\varepsilon := \tilde{v}_\varepsilon + d_\varepsilon$ satisfies (4.5). It is evident that $d_\varepsilon = o(1)$. Finally, by the Lebesgue dominated convergence theorem we have that

$$\lim_{\varepsilon \to 0} \int_I W(v_\varepsilon)\eta \, dt = \int_I W(v)\eta \, dt,$$

which gives the desired result for v bounded. We recall that the Γ-$\limsup$ is upper semicontinuous (see, e.g., [24]), and that bounded functions are dense in L_η^1. This completes the proof. $\square$

Clearly we have that $\inf G^{(0)} = 0$, and thus

$$G_\varepsilon^{(1)}(v) = \varepsilon^{-1} G_\varepsilon(v) = \int_I \left(\frac{W(v)}{\varepsilon} + \varepsilon|v'|^2 \right) \eta \, dt \tag{4.6}$$

for all $v \in H_\eta^1$ satisfying (4.5), and $G_\varepsilon^{(1)}(v) = \infty$ otherwise in L_η^1. We now state a compactness result, which utilizes arguments from [28].

Proposition 4.3. *Let $v_\varepsilon \in H_\eta^1$ be such that $\sup_\varepsilon G_\varepsilon^{(1)}(v_\varepsilon) < \infty$. Then up to a subsequence $v_\varepsilon \to v \in \mathcal{C}$ in L_η^1, where*

$$\mathcal{C} := \{v \in BV_\eta(I; \{a,b\}) : v \text{ satisfies (4.5)}\}. \tag{4.7}$$

Proof. We first show that $\{v_\varepsilon\}$ is uniformly bounded in L_η^1 and equi-integrable. This is since, by applying (2.8),

$$\int_{|v_\varepsilon|>\hat{T}} |v_\varepsilon|\eta \, ds \leq \hat{L}^{-1} \int_I W(v_\varepsilon)\eta \, ds \leq C\varepsilon G_\varepsilon^{(1)}(v_\varepsilon) \leq C\varepsilon,$$

17

which, as $\int_I \eta \, ds < \infty$ then implies that the sequence $\{v_\varepsilon\}$ is bounded in L^1_η and equi-integrable.

Next, define $W_1(s) := \min\{W(s), K\}$, where $K = \max_{s \in [a,b]} W(s)$. Then define $\Phi_1(t) := \int_a^t W_1^{1/2}(s) \, ds$. Using Young's inequality, and the fact that $0 \leq W_1 \leq W$ we have that

$$2 \int_I W_1^{1/2}(v_\varepsilon)|v'_\varepsilon|\eta \, ds \leq G_\varepsilon^{(1)}(v_\varepsilon) \leq C.$$

Utilizing the chain rule, we find that

$$\int_I |(\Phi_1 \circ v_\varepsilon)'|\eta \leq C.$$

Furthermore, as Φ_1 is Lipshitz, and $\Phi_1(a) = 0$, we have that $\Phi_1 \circ v_\varepsilon$ is uniformly bounded in L^1_η. This then implies, by BV compactness, that, up to a subsequence, not relabeled,

$$\Phi_1 \circ v_\varepsilon \to \tilde{v} \quad \text{in } L^1_\eta$$

for some function $\tilde{v} \in BV_\eta$. It is easy to show, using (2.6), that Φ_1 has a continuous inverse. This implies that, up to a subsequence, v_ε must converge pointwise to $v := \Phi_1^{-1}(\tilde{v})$. Thus, up to a subsequence, the v_ε converge in L^1_η to v. Using (1.5), it is evident that $v \in L^1_\eta(I; \{a, b\})$. As $\tilde{v} \in BV_\eta$, this implies that $v \in BV_\eta(I; \{a, b\})$. The L^1_η convergence of the v_ε then implies that v satisfies (4.5). This concludes the proof.

$\square$

We now state the first main theorem of this section, which characterizes the first-order Γ-limit of G_ε.

Theorem 4.4. *Assume that W satisfies (2.4)-(2.7) and that η satisfies (4.1)-(4.4). Then the family $\{G_\varepsilon^{(1)}\}$ Γ-converges to the functional*

$$G^{(1)}(v) = \begin{cases} \frac{2c_W}{b-a}|Dv|_\eta(I) & \text{if } v \in \mathcal{C}, \\ \infty & \text{otherwise in } L^1_\eta, \end{cases} \tag{4.8}$$

where c_W is the constant given in (1.10) and $\mathcal{C}$ defined in (4.7).

We note that here

$$|Dv|_\eta = (b-a) \sum \eta(t_i),$$

where t_i are the locations of jumps of the function v. We also note that Proposition 4.3 and Theorem 4.4 are completely analogous to classical results (e.g. [47, 57]) in the unweighted, higher-dimensional case.

Proof. We first characterize the Γ-lim sup. Specifically, given a $v \in \mathcal{C}$, we construct a family of functions v_ε that converge in L^1_η to v satisfying

$$\limsup_{\varepsilon \to 0} G_\varepsilon^{(1)}(v_\varepsilon) \leq G^{(1)}(v). \tag{4.9}$$

To begin with, we assume that v is of the form

$$v(t) = \begin{cases} a & \text{if } t \in [t_{2k}, t_{2k+1}), \\ b & \text{otherwise,} \end{cases}$$

18

where $-T = t_0 < t_1 < \cdots < t_{2N} = T$. Define

$$
f(t) := \begin{cases}
t - t_1 & \text{if } t \in [t_0, t_1), \\
-\min\{t - t_{2k}, t_{2k+1} - t\} & \text{if } t \in [t_{2k}, t_{2k+1}), \text{ and } k \geq 1, \\
\min\{t - t_{2k+1}, t_{2k+2} - t\} & \text{if } t \in (t_{2k+1}, t_{2k+2}], \text{ and } k < N - 1, \\
t - t_{2N-1} & \text{if } t \in [t_{2N-1}, t_{2N}).
\end{cases}
$$

Observe that f is the signed distance function (see (2.13)) of the set $E := \{t \in I : v(t) = a\}$, where we naturally are considering ∂E relative to I, not $\mathbb{R}$. We note that $v(t) = \mathrm{sgn}_{a,b}(f(t))$, where $\mathrm{sgn}_{a,b}$ is the function given in (1.19). Thus our goal is to construct smooth approximations of the function $\mathrm{sgn}_{a,b}$ that make the energy $G_\varepsilon^{(1)}$ small.

We will follow the construction of [47]. Although the argument is almost identical, we include it for completeness. Consider the function

$$
\Psi_\varepsilon(s) := \int_a^s \left(\frac{\varepsilon^2}{\varepsilon + W(r)} \right)^{1/2} dr, \tag{4.10}
$$

and define the constant

$$
\xi_\varepsilon := \Psi_\varepsilon(b).
$$

We note that since $W \geq 0$, equation (4.10) gives

$$
0 \leq \xi_\varepsilon \leq (b - a)\varepsilon^{1/2}.
$$

Note that Ψ_ε is strictly increasing and differentiable. Now define $\phi_\varepsilon : [0, \xi_\varepsilon] \to [a, b]$ to be the inverse of Ψ_ε on the interval $[a, b]$. By the fundamental theorem of calculus and the inverse function theorem ϕ_ε will satisfy the equation

$$
\varepsilon \phi_\varepsilon'(t) = (\varepsilon + W(\phi_\varepsilon(t)))^{1/2}.
$$

Next, extend ϕ_ε to be equal to a for $t < 0$ and b for $t > \xi_\varepsilon$. Note that for all $t \in \mathbb{R}$ we have that $\phi_\varepsilon(t) \leq \mathrm{sgn}_{a,b}(t)$ and that $\phi_\varepsilon(t + \xi_\varepsilon) \geq \mathrm{sgn}_{a,b}(t)$. Thus as $v \in \mathcal{C}$ we can find a $\tau_\varepsilon \in (0, \xi_\varepsilon)$ that gives

$$
\int_I \phi_\varepsilon(f(t) + \tau_\varepsilon)\eta(t)\, dt = m.
$$

Define $v_\varepsilon(t) := \phi_\varepsilon(f(t) + \tau_\varepsilon)$. As $\{v_\varepsilon\}$ converges to v pointwise and $|v_\varepsilon| < C$ we have that $v_\varepsilon \to v$ in L_η^1. We then examine the energy associated with v_ε, when ε is sufficiently small that transition layers do not overlap or leave $\overline{I}$:

$$
\begin{aligned}
G_\varepsilon^{(1)}(v_\varepsilon) &= \sum_{k=1}^{2N-1} \int_0^{\xi_\varepsilon} \left(\varepsilon(\phi_\varepsilon'(t))^2 + \varepsilon^{-1}W(\phi_\varepsilon(t)) \right) \eta(t_k + (t - \tau_\varepsilon)(-1)^{k+1})\, dt \\
&\leq \sum_{k=1}^{2N-1} \int_0^{\xi_\varepsilon} 2(\varepsilon + W(\phi_\varepsilon(t)))^{1/2} \phi_\varepsilon'(t) \eta(t_k + (t - \tau_\varepsilon)(-1)^{k+1})\, dt \\
&\leq \sum_{k=1}^{2N-1} \sup\{\eta(t_k + (s - \tau_\varepsilon)(-1)^{k+1}) : s \in (0, \xi_\varepsilon)\} \int_0^{\xi_\varepsilon} 2(\varepsilon + W(\phi_\varepsilon(t)))^{1/2} \phi_\varepsilon'(t)\, dt \\
&= \sum_{k=1}^{2N-1} \sup\{\eta(t_k + (s - \tau_\varepsilon)(-1)^{k+1}) : s \in (0, \xi_\varepsilon)\} \int_a^b 2(\varepsilon + W(s))^{1/2}\, ds.
\end{aligned}
$$

Thus taking the limit as $\varepsilon \to 0^+$ we find that

$$\limsup_{\varepsilon \to 0} G_\varepsilon^{(1)}(v_\varepsilon) \leq 2c_W \sum_{k=1}^{2N-1} \eta(t_k) = G^{(1)}(v).$$

The cases where v has a finite number of jump points, but starting or ending at different values than we assumed are analogous. By noting that functions with a finite number of jumps are dense in $\mathcal{C}$, and as the Γ-lim sup is upper semicontinuous, we then have (4.9).

Next we will establish our Γ-lim inf. Assume that $v_\varepsilon \to v$ in L_η^1. By Lemma 4.3 if $v \notin \mathcal{C}$ then $\liminf_{\varepsilon \to 0} G_\varepsilon^{(1)} = \infty$, and there is nothing to prove. We claim that for any sequence $\{v_\varepsilon\}$ that converges in L_η^1 to some $v \in \mathcal{C}$ the following inequality holds:

$$\liminf_{\varepsilon \to 0} G_\varepsilon^{(1)}(v_\varepsilon) \geq G^{(1)}(v). \tag{4.11}$$

To establish this inequality we use Young's inequality, the chain rule and lower semicontinuity of $\|\cdot\|_{BV_\eta}$ (see, e.g., [56]) as follows:

$$\liminf_{\varepsilon \to 0} G_\varepsilon^{(1)}(v_\varepsilon) \geq \liminf_{\varepsilon \to 0} \int_I (\varepsilon^{-1} W_1(v_\varepsilon) + \varepsilon(v_\varepsilon')^2) \eta \, dt$$

$$\geq \liminf_{\varepsilon \to 0} 2 \int_I |(\Phi_1 \circ v_\varepsilon)'| \eta \, dt \geq 2 \int_I \eta \, d|D\Phi_1(v)|$$

$$= 2 \int_I \eta \, d|D\Phi(v)| = \frac{2c_W}{b-a} \int_I \eta \, d|Dv| = G^{(1)}(v_0),$$

Here we have used the fact that $\Phi_1 \circ v_\varepsilon$ converges to $\Phi_1 \circ v$ in L_η^1 (because Φ_1 is Lipschitz), and the fact that $\Phi_1 \circ v = \Phi \circ v$, where $\Phi := \int_a^t W^{1/2}(s) \, ds$. This proves the claim. $\qquad \square$

Properties of Γ-limits [24] along with Lemma 4.3 then establishes the following corollary.

Corollary 4.5. *Under the hypotheses of Theorem 4.4 if v_ε are minimizers of $G_\varepsilon^{(1)}$ then, up to a subsequence, they converge in L_η^1 to v which is a minimizer of $G^{(1)}$. Furthermore the v_ε will satisfy the following*

$$\lim_{\varepsilon \to 0} G_\varepsilon^{(1)}(v_\varepsilon) = G^{(1)}(v).$$

To conclude this subsection we prove two theorems that will be important later in our analysis. We select t_0 so that

$$v_0(t) := \operatorname{sgn}_{a,b}(t - t_0)$$

satisfies (4.5). By (4.1) it is clear that t_0 is uniquely determined. First we prove that v_0 is an isolated local minimizer of $G^{(1)}$ in L_η^1.

Theorem 4.6. *Assume that W satisfies (2.4)-(2.7) and that η satisfies (4.1)-(4.4). Then there exists $\delta > 0$ such that v_0 is an isolated δ-local minimizer of $G^{(1)}$ in L_η^1, that is, there is no $v_1 \in \mathcal{C}$ (see (4.7)), with $0 < \|v_1 - v_0\|_{L_\eta^1} \leq \delta$ such that*

$$G^{(1)}(v_1) \leq G^{(1)}(v_0).$$

Proof. Assume by contradiction that such v_1 exists. By continuity of η, for every $\epsilon > 0$ there is $r_\epsilon > 0$ such that

$$|\eta(t) - \eta(t_0)| \leq \epsilon \tag{4.12}$$

for all $t \in [t_0 - r_\epsilon, t_0 + r_\epsilon]$. Let $M_0 := \max|\eta'| + 1$ and fix

$$0 < r_0 < \min\left\{\frac{1}{2}t^*, T - t_0, T + t_0, \frac{d_1 n_1 \eta(t_0)}{2d_2 M_0}, \frac{d_3 n_2 \eta(t_0)}{2d_4 M_0}\right\}, \qquad (4.13)$$

where t^*, n_1, n_2 and the constants $d_i, i = 1\ldots 4$ are given in (4.2) and (4.3). Then define

$$I_0 := [-T + r_0, T - r_0],$$

and fix $0 < \epsilon_1 < \min\{\min_{I_0} \eta, \eta(t_0)/2\}$ in (4.12) and let r_{ϵ_1} be the corresponding r_ϵ.
Step 1: We claim that v_1 has a jump at some $t_1 \in B(t_0, r_{\epsilon_1})$. If not, then either $v_1 \equiv a$ in $B(t_0, r_{\epsilon_1})$ or $v_1 \equiv b$ in $B(t_0, r_{\epsilon_1})$. Assume that $v_1 \equiv a$ in $B(t_0, r_{\epsilon_1})$. Then by (4.12),

$$\delta \geq \int_{B(t_0, r_{\epsilon_1})} |v_1 - v_0| \eta \, dt \geq (b - a)\frac{\eta(t_0)}{2} r_{\epsilon_1},$$

where we used the fact that $0 < \epsilon_1 < \eta(t_0)/2$. Since the case $v_1 \equiv b$ gives an identical estimate, the claim follows provided

$$0 < \delta < (b - a)\frac{\eta(t_0)}{2} r_{\epsilon_1}.$$

Step 2: We claim that v_1 has no jump other than t_1 in I_0. Indeed, assume that there is a second jump $t_2 \neq t_1$ in I_0. Then by (4.12) and Step 1,

$$G^{(1)}(v_1) \geq 2c_W(\eta(t_1) + \eta(t_2))$$
$$\geq 2c_W(\eta(t_0) - \epsilon_1 + \min_{I_0} \eta) > 2c_W \eta(t_0) = G^{(1)}(v_0),$$

where in the last inequality we used the fact that $0 < \epsilon_1 < \min_{I_0} \eta$. This is impossible since we are assuming that $G^{(1)}(v_1) \leq G^{(1)}(v_0)$.
Step 3: We claim that v_1 jumps from a to b at t_1. Suppose not, and suppose that $t_1 \leq t_0$. Then

$$\delta \geq \int_{B(t_0, r_{\epsilon_1})} |v_1 - v_0| \eta \, dt \geq (b - a)\frac{\eta(t_0)}{2} r_{\epsilon_1},$$

which again leads to a contradiction if δ is chosen small enough. The case $t_1 > t_0$ is analogous.
Step 4: We claim that $t_1 = t_0$. Indeed, if $t_1 > t_0$, then

$$0 = \int_I (v_1 - v_0)\eta \, dt = \int_{-T}^{-T+r_0} (v_1 - a)\eta \, dt + \int_{t_0}^{t_1} (a - b)\eta \, dt + \int_{T-r_0}^{T} (v_1 - b)\eta \, dt,$$

which implies, as the last two terms are negative, that there must be a jump t_3 that belongs to $(-T, -T + r_0)$, with

$$0 < \frac{\eta(t_0)}{2}(b - a)(t_1 - t_0) \leq \int_{t_0}^{t_1} (b - a)\eta \, dt \leq (b - a)\int_{-T}^{t_3} \eta \, dt \leq d_2(b - a)\frac{(T + t_3)^{n_1}}{n_1}, \qquad (4.14)$$

where in the last equality we used (4.2), in conjunction with (4.13). By the mean value theorem and inequality (4.14), for some $\theta \in (t_0, t_1)$,

$$\eta(t_1) = \eta(t_0) + \eta'(\theta)(t_1 - t_0) \geq \eta(t_0) - M_0|t_1 - t_0|$$
$$\geq \eta(t_0) - \frac{2M_0 d_2}{n_1 \eta(t_0)}(T + t_3)^{n_1}.$$

21

Hence by (4.13),

$$G^{(1)}(v_1) \geq 2c_W(\eta(t_1) + \eta(t_3))$$

$$\geq 2c_W\eta(t_0) - 2c_W\frac{2M_0 d_2}{n_1\eta(t_0)}(T+t_3)^{n_1} + 2c_W d_1(T+t_3)^{n_1-1}$$

$$> 2c_W\eta(t_0) = G^{(1)}(v_0),$$

which violates our assumption. The case $t_1 < t_0$ is analogous. This proves that $t_1 = t_0$, and so $G^{(1)}(v_1) \geq 2c_W\eta(t_0) = G^{(1)}(v_0)$, which implies that $G^{(1)}(v_1) = G^{(1)}(v_0)$. In particular, v_1 has no jumps in $I \setminus I_0$. But then $v_1 = v_0$, which is a contradiction. This completes the proof. $\square$

We have seen in Theorem 4.6 that v_0 is a local minimizer for $G^{(1)}$. In general v_0 will not be a global minimizer without further assumptions on η (e.g., $\eta \equiv$ constant). However, for the applications in the n-dimensional case later on it will be important to study a type of second-order asymptotic development of G_ε where in the definition of $G_\varepsilon^{(2)}$ (see (1.3)) in place of $\inf G^{(1)}$ we take $G^{(1)}(v_0)$. We will see that this corresponds to studying the second-order asymptotic development of the localized functional

$$J_\varepsilon(v) := \begin{cases} G_\varepsilon(v) & \text{if } \|v - v_0\|_{L_\eta^1} \leq \delta, \\ \infty & \text{otherwise.} \end{cases} \tag{4.15}$$

We conclude this subsection by proving a Γ-lim sup inequality. In the next subsection we will prove that this corresponds to the Γ-lim sup inequality for J_ε, (see (1.3)). For this theorem we will state and use weaker assumptions on η than we have in the rest of this section. This is because we will use Theorem 4.7 later to establish a Γ-lim sup estimate in higher dimensions by using η equal to the perimeter of level sets of the distance function, which may not satisfy (4.1).

Theorem 4.7. *Assume that W satisfies (2.4)-(2.7), and that $\eta : I \to [0, \infty)$ is measurable, bounded, differentiable at t_0, $\eta(t_0) > 0$ and*

$$|\eta(t) - \eta(t_0) - \eta'(t_0)(t - t_0)| \leq C|t - t_0|^{1+\beta} \tag{4.16}$$

for some constant $C > 0$ and for all t in a neighborhood of t_0. Then there exists a sequence $\{v_\varepsilon\}$ converging to v_0 in L_η^1 so that

$$\limsup_{\varepsilon \to 0} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \leq 2\eta'(t_0)(\tau_0 c_W + c_{\text{sym}})$$

$$+ \begin{cases} \frac{\lambda_0^2}{2W''(a)}\int_I \eta\, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases} \tag{4.17}$$

where c_W and c_{sym} are given by (1.10), (1.16), τ_0 is determined by the equation

$$\eta(t_0)\int_\mathbb{R}(z(s - \tau_0) - \text{sgn}_{a,b})\, ds = \begin{cases} \frac{\lambda_0}{W''(a)}\int_I \eta\, dt & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases} \tag{4.18}$$

and λ_0 is defined by

$$\lambda_0 := \frac{2\eta'(t_0)c_W}{(b-a)\eta(t_0)}. \tag{4.19}$$

Proof. **Step 1:** Assume $q = 1$. Define $z_\varepsilon(t) := z(\frac{t-t_0}{\varepsilon})$ and then define

$$v_\varepsilon(t) := z_\varepsilon(t - \varepsilon\tau_\varepsilon) - \frac{\lambda_0\varepsilon}{W''(a)}, \tag{4.20}$$

22

where τ_ε is selected so that (4.5) is satisfied. We first claim that $\tau_\varepsilon = \tau_0 + o(1)$. To this end, we can write, via (4.5),

$$\int_I v_\varepsilon \eta \, dt = \int_I v_0 \eta \, dt = m.$$

In turn this implies that

$$\int_I (z_\varepsilon(t - \varepsilon\tau_\varepsilon) - z_\varepsilon(t - \varepsilon\tau_0))\eta \, dt = \int_I (\mathrm{sgn}_{a,b}(t - t_0) - z_\varepsilon(t - \varepsilon\tau_0))\eta \, dt \tag{4.21}$$
$$+ \frac{\varepsilon\lambda_0}{W''(a)} \int_I \eta \, dt.$$

After the change of variables $s = \frac{t - t_0}{\varepsilon}$ we can write the right-hand side as

$$\varepsilon \int_{\frac{-T - t_0}{\varepsilon}}^{\frac{T - t_0}{\varepsilon}} (\mathrm{sgn}_{a,b}(s) - z(s - \tau_0))\eta(\varepsilon s + t_0) \, ds + \frac{\varepsilon\lambda_0}{W''(a)} \int_I \eta \, dt. \tag{4.22}$$

By our choice of τ_0 (via (4.18)) and (1.19) this is equal to

$$\varepsilon \int_{\frac{-T - t_0}{\varepsilon}}^{\frac{T - t_0}{\varepsilon}} (\mathrm{sgn}_{a,b}(s) - z(s - \tau_0))(\eta(\varepsilon s + t_0) - \eta(t_0)) \, ds$$
$$- \varepsilon\eta(t_0) \int_{-\infty}^{\frac{-T - t_0}{\varepsilon}} (a - z(s - \tau_0)) \, ds - \varepsilon\eta(t_0) \int_{\frac{T - t_0}{\varepsilon}}^{\infty} (b - z(s - \tau_0)). \tag{4.23}$$

By (4.16) there exists a $R_0 > 0$ such that $|\eta(t) - \eta(t_0)| \leq (|\eta'(t_0)| + 1)|t - t_0|$ for all $t \in B(t_0, R_0)$. Since η is bounded by assumption, we thus have for all $t \in I \setminus B(t_0, R_0)$ that $|\eta(t) - \eta(t_0)| \leq 2\frac{\|\eta\|_\infty}{R_0}|t - t_0|$. Hence, using (2.11) the first term in (4.23) can be bounded by

$$2(b - a)\varepsilon \int_{\frac{-T - t_0}{\varepsilon}}^{\frac{T - t_0}{\varepsilon}} e^{-c_1|s|}|\eta(\varepsilon s + t_0) - \eta(t_0)| \, ds \leq 2(b - a)C_\eta \varepsilon^2 \int_\mathbb{R} e^{-\hat{d}|s|}|s| \, ds,$$

where $C_\eta > 0$. By (2.11) we know that the last two terms of (4.23) are bounded from above by $\frac{(b - a)}{c_1}\|\eta\|_\infty \varepsilon^2 e^{-\frac{c_1 T_1}{\varepsilon}}$, where $T_1 := \min(T - t_0, T + t_0) > 0$. Hence, the right-hand side of (4.21) is bounded from above by $C\varepsilon^2$ for all $\varepsilon > 0$ sufficiently small.

Now assume that the τ_ε do not converge to τ_0. Assume without loss of generality, that for some subsequence (not relabeled) the $\tau_\varepsilon \leq \tau_0 - C_0$ for some $C_0 > 0$ (the case where $\tau_\varepsilon \geq \tau_0 + C_0$ is similar). Since z is increasing (see (1.17)), by (4.21) and what we just proved,

$$C\varepsilon^2 \geq \int_I (z_\varepsilon(t - \varepsilon\tau_\varepsilon) - z_\varepsilon(t - \varepsilon\tau_0))\eta(t) \, dt \geq \inf_{B(t_0 + \varepsilon\tau_0, k_1\varepsilon)} \eta \int_{B(t_0 + \varepsilon\tau_0, k_1\varepsilon)} \int_{t - \varepsilon\tau_0}^{t - \varepsilon\tau_\varepsilon} z'_\varepsilon(s) \, ds \, dt$$

$$\geq \inf_{B(t_0 + \varepsilon\tau_0, k_1\varepsilon)} \eta \int_{B(t_0 + \varepsilon\tau_0, k_1\varepsilon)} \int_{t - \varepsilon\tau_0}^{t - \varepsilon(\tau_0 - k_2)} \varepsilon^{-1}\sqrt{W(z(\varepsilon^{-1}(s - t_0)))} \, ds \, dt$$

$$\geq 2k_1 k_2 \varepsilon \inf_{t \in B(0, k_1 + k_2)} \sqrt{W(z(t))} \inf_{B(t_0 + \varepsilon\tau_0, k_1\varepsilon)} \eta,$$

where $0 < k_1 < 1$ and $0 < k_2 < C_0$ and where we have used the facts that η is continuous at t_0 and that $\eta(t_0) > 0$. Since $z(0) = c$, by taking k_1 and k_2 sufficiently small we can assume that $z(t) \in B(c, \min\{\frac{c - a}{2}, \frac{b - c}{2}\})$ for all $t \in B(0, k_1 + k_2)$. In turn the right-hand side of the previous inequality is bounded from below by $C_1\varepsilon$ for some $C_1 > 0$. This is a contradiction, which proves our claim.

23

Next we prove (4.17). We will write $R_\varepsilon := C_k \varepsilon |\log \varepsilon|$, with $C_k > 0$ to be chosen later. We then write

$$
\begin{aligned}
\frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W \eta(t_0)}{\varepsilon} &= \varepsilon^{-1} \left(\int_{B(t_0, R_\varepsilon)} (\varepsilon^{-1} W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2) \eta \, dt - 2c_W \eta(t_0) \right) \\
&\quad + \int_{I \setminus B(t_0, R_\varepsilon)} (\varepsilon^{-2} W(v_\varepsilon) + (v_\varepsilon')^2) \eta \, dt.
\end{aligned}
\tag{4.24}
$$

First we examine the second term, namely the tail integral. We first note that by (2.11) and the fact that the $\tau_\varepsilon \to \tau_0$ we then have that

$$
b - z_\varepsilon(t - \varepsilon \tau_\varepsilon) \leq \frac{b-a}{2} e^{c_1(1 + |\tau_0|)} \varepsilon^{c_1 C_k} \leq \varepsilon^k
$$

for $t \in [t_0 + R_\varepsilon, T]$ and for ε small, provided $C_k \geq 2 \frac{k}{c_1}$. Similarly, $z_\varepsilon(t - \varepsilon \tau_\varepsilon) - a < \varepsilon^k$ for $t \in [-T, t_0 - R_\varepsilon]$. Thus for $t \in I \setminus B(t_0, R_\varepsilon)$ we have that

$$
|z_\varepsilon(t - \varepsilon \tau_\varepsilon) - v_0(t)| \leq \varepsilon^k
$$

which in turn implies, after recalling (4.20), that, for k large,

$$
(v_\varepsilon(t) - v_0)^2 \leq \frac{\lambda_0^2 \varepsilon^2}{W''(a)^2} + C \varepsilon^{k+1}
\tag{4.25}
$$

for all $t \in I \setminus B(t_0, R_\varepsilon)$ and for some fixed $C > 0$.

We then fix $\gamma > 0$. By (2.9) there exists s_γ such that

$$
W(s) \leq \left(\frac{W''(a)}{2} + \gamma \right) (s - a)^2
\tag{4.26}
$$

for all s with $|s - a| \leq s_\gamma$, and

$$
W(s) \leq \left(\frac{W''(a)}{2} + \gamma \right) (s - b)^2
\tag{4.27}
$$

for all s with $|s - b| \leq s_\gamma$. By (4.25), (4.26) and (4.27) we then have for ε sufficiently small that

$$
\int_{I \setminus B(t_0, R_\varepsilon)} W(v_\varepsilon) \eta \, dt \leq \left(\frac{W''(a)}{2} + \gamma \right) \varepsilon^2 \lambda_0^2 W''(a)^{-2} \int_I \eta \, dt + O(\varepsilon^{k+1}).
$$

On the other hand, using (1.17), (4.24), (4.26), and (4.27),

$$
(v_\varepsilon'(t))^2 = W(z_\varepsilon(t + \varepsilon \tau_\varepsilon)) \leq C(z_\varepsilon(t + \varepsilon \tau_\varepsilon) - v_0(t))^2 \leq C \varepsilon^{2k}
$$

for $t \in I \setminus B(t_0, R_\varepsilon)$. After taking limits (first as $\varepsilon \to 0^+$ and then as $\gamma \to 0^+$) we thus find that

$$
\limsup_{\varepsilon \to 0^+} \int_{I \setminus B(t_0, R_\varepsilon)} (\varepsilon^{-2} W(v_\varepsilon) + (v_\varepsilon')^2) \eta \, dt \leq \frac{\lambda_0^2}{2 W''(a)} \int_I \eta \, dt.
\tag{4.28}
$$

Next we estimate the energy in the region $B(t_0, R_\varepsilon)$. We will define $s_1^\varepsilon := v_\varepsilon(t_0 - R_\varepsilon)$ and $s_2^\varepsilon := v_\varepsilon(t_0 - R_\varepsilon)$. Note that by (4.25), $s_1^\varepsilon = a + O(\varepsilon)$ and $s_2^\varepsilon = b + O(\varepsilon)$. Thus recalling the definition of c_W, (1.10), and (2.9), we find that

$$
c_W = \int_{s_1^\varepsilon}^{s_2^\varepsilon} W^{1/2}(s) \, ds + O(\varepsilon^2) = \int_{B(t_0, R_\varepsilon)} W^{1/2}(v_\varepsilon) v_\varepsilon' \, dt + O(\varepsilon^2),
$$

24

where we have used the change of variables $s = v_\varepsilon(t)$. Thus we have that

$$
\int_{B(t_0,R_\varepsilon)} (\varepsilon^{-1}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2)\eta \, dt - 2c_W\eta(t_0)
$$
$$
= \int_{B(t_0,R_\varepsilon)} (\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}v'_\varepsilon)^2\eta + W^{1/2}(v_\varepsilon)v'_\varepsilon(2\eta - 2\eta(t_0)) \, dt + O(\varepsilon^2).
$$
(4.29)

Recalling (1.17), the fact that $|W^{1/2}(s_1) - W^{1/2}(s_2)| \le C|s_1 - s_2|$ for all $s_1, s_2 \in [a-1, b+1]$ (see (2.4) and (2.5)), it follows from (1.17), (4.20), and the boundedness of η, that

$$
\int_{B(t_0,R_\varepsilon)} (\varepsilon^{-1/2}W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}v'_\varepsilon)^2\eta \, dt \le \varepsilon^{-1} \int_{B(t_0,R_\varepsilon)} (W^{1/2}(v_\varepsilon(t)) - W^{1/2}(z_\varepsilon(t - \varepsilon\tau_\varepsilon)))^2\eta(t) \, dt
$$
$$
\le C\varepsilon^{-1} \int_{B(t_0,R_\varepsilon)} \left(\frac{\varepsilon\lambda_0}{W''(a)}\right)^2 \eta \, dt \le C\varepsilon^2|\log\varepsilon|. \qquad (4.30)
$$

Next we will use (4.16) to obtain:

$$
2\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon)v'_\varepsilon(\eta - \eta(t_0)) \, dt
$$
$$
= 2\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon(t))v'_\varepsilon(t)(\eta'(t_0)(t - t_0) + O(|t - t_0|^{1+\beta})) \, dt
$$
$$
= 2\eta'(t_0)\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon(t))v'_\varepsilon(t)(t - t_0) \, dt + O(\varepsilon^{2+\beta}|\log\varepsilon|^{2+\beta}).
$$

Changing variables to $s = \frac{t - t_0 - \varepsilon\tau_\varepsilon}{\varepsilon}$ we can then write

$$
2\int_{B(t_0,R_\varepsilon)} W^{1/2}(v_\varepsilon)v'_\varepsilon(\eta - \eta(t_0)) \, dt
$$
$$
= 2\eta'(t_0)\varepsilon \int_{B(\tau_\varepsilon, C_k|\log\varepsilon|)} W^{1/2}(z(s) - \lambda_0 W''(a)^{-1}\varepsilon)z'(s)(s + \tau_\varepsilon) \, ds + O(\varepsilon^{2+\beta}|\log\varepsilon|^{2+\beta}). (4.31)
$$

By then combining estimates (4.24), (4.28), (4.29), (4.30), (4.31), and using (1.10), (1.16) and the fact that the $\tau_\varepsilon \to \tau_0$ we find that

$$
\limsup_{\varepsilon \to 0} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \le 2\eta'(t_0)(\tau_0 c_W + c_{\mathrm{sym}}) + \frac{\lambda_0^2}{2W''(a)}\int_I \eta \, dt,
$$

which is the desired conclusion.

Step 2: The case $q < 1$ is simpler since by (2.12) the function z in (1.17) satisfies $z(t) \equiv b$ for $t \ge t_b$ and $z(t) \equiv a$ for $t \le t_a$. We define $v_\varepsilon(t) := z_\varepsilon(t - \varepsilon\tau_\varepsilon)$. Then the second term in the right hand side of (4.21) should be replaced by 0, while (4.22) becomes

$$
\int_{t_a+\tau_0}^{t_b+\tau_0} (\mathrm{sgn}_{a,b}(s) - z(s - \tau_0))\eta(\varepsilon s + t_0) \, ds.
$$

In turn, in (4.23) the first integral is over $[t_a + \tau_0, t_b + \tau_0]$, while the other two integrals vanish. Using the regularity of η near t_0 we can bound in integral in the new (4.23) by $2(b - a)C_\eta\varepsilon^2(t_b - t_a)$. We can continue as before to conclude that $\tau_\varepsilon \to \tau_0$.

By (1.10), (1.17) and (2.4), in place of (4.24) we now have

$$
\frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} = \varepsilon^{-1}\int_{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_a}^{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_b} W^{1/2}(v_\varepsilon(t))v'_\varepsilon(t)(\eta(t) - \eta(t_0)) \, dt.
$$

25

Using (4.16) and the fact that $\tau_\varepsilon \to \tau_0$, the right-hand side can be bounded from above by

$$\leq 2\varepsilon^{-1}\eta'(t_0) \int_{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_a}^{t_0+\varepsilon\tau_\varepsilon+\varepsilon t_b} W^{1/2}(v_\varepsilon(t))v_\varepsilon'(t)(t-t_0)\, dt + O(\varepsilon^\beta)$$

$$= 2\eta'(t_0) \int_{t_a}^{t_b} W^{1/2}(z(s))z'(s)(s+\tau_\varepsilon)\, ds + O(\varepsilon^\beta),$$

where we have used a change of variables $s = \frac{t-t_0-\varepsilon\tau_\varepsilon}{\varepsilon}$. It now suffices to let $\varepsilon \to 0^+$.

$\square$

4.2 Local Minimizers of G_ε

In this subsection we prove the existence of certain types of local minimizers of G_ε and study their qualitative properties. In the next subsection these properties will enable us to characterize the second-order asymptotic development of the family J_ε defined in (4.15). We begin with the following proposition, which is based on an argument from [39] (see also [11]). We include the proof for completeness.

Proposition 4.8. *Assume that W satisfies (2.4)-(2.7) and that η satisfies (4.1)-(4.4). Then for all $\varepsilon > 0$ there exists a global minimizer v_ε of the functional J_ε. Furthermore, the functions v_ε must converge to v_0 in L_η^1, and thus for ε small enough v_ε is a local minimizer of G_ε. Additionally, the following equality holds:*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon^{(1)}(v_\varepsilon) = G^{(1)}(v_0). \tag{4.32}$$

Proof. First we prove the existence of a global minimizer. Fix $\varepsilon > 0$ and suppose that $\{f_k\}$ is a minimizing sequence in the sense that

$$\lim_{k\to\infty} J_\varepsilon(f_k) = \inf_v J_\varepsilon(v) < \infty.$$

In particular, $\|f_k - v_0\|_{L_\eta^1} \leq \delta$ for all k sufficiently large. By (4.6) and (4.15) it follows that $\{f_k'\}$ is bounded in L_η^2. Since $\{f_k\}$ is bounded in L_η^1, by (4.1) and a diagonal argument, we may find a function $v_\varepsilon \in H_{\eta,\text{loc}}^1$ such that $f_k' \rightharpoonup v_\varepsilon'$ in L_η^2 and $f_k \to v_\varepsilon$ in $L_{\eta,\text{loc}}^1$, and pointwise a.e.. By Fatou's Lemma, and the weak lower semi-continuity of the L_η^2 norm, we then have that

$$G_\varepsilon(v_\varepsilon) \leq \liminf_k G_\varepsilon(f_k) = \inf_v J_\varepsilon(v)$$

and that $\|v_\varepsilon - v_0\|_{L_\eta^1} \leq \delta$. It remains to show that $v_\varepsilon \in L_\eta^2$. Since v_ε is locally absolutely continuous, by Hölder's inequality, for $-T < t < -T + t^*$ we have

$$v_\varepsilon^2(t)\eta(t) = \eta(t)\left(v_\varepsilon(-T+t^*) - \int_t^{-T+t^*} v_\varepsilon'(s)\, ds\right)^2$$

$$\leq 2\eta(t)v_\varepsilon^2(-T+t^*) + 2\eta(t)\left(\int_t^{-T+t^*} v_\varepsilon'(s)\frac{\eta^{1/2}(s)}{\eta^{1/2}(s)}\, ds\right)^2$$

$$\leq 2\eta(t)v_\varepsilon^2(-T+t^*) + 2\eta(t)\int_t^{-T+t^*} \frac{1}{\eta(s)}\, ds \int_t^{-T+t^*} |v_\varepsilon'(s)|^2\eta(s)\, ds$$

$$\leq 2\eta(t)v_\varepsilon^2(-T+t^*) + 2\frac{d_2}{d_1}t^* \int_I |v_\varepsilon'(s)|^2\eta(s)\, ds,$$

where we have used the fact that if $t < s < -T + t^*$ then $\eta(s) \geq \frac{d_1}{d_2}\eta(t)$ (see (4.2)). By integrating in t over $(-T, -T+t^*)$ we observe that $v_\varepsilon \in L_\eta^2((-T, -T+t^*))$. A similar estimate can be obtained

26

on the interval $(T - t^*, T)$. On the other hand, by (4.1), we have that $\eta \geq \eta_0 > 0$ in $[-T+t^*, T-t^*]$, and thus again by the fundamental theorem of calculus $v_\varepsilon \in H^1((-T+t^*, T-t^*))$, which then implies that $v_\varepsilon \in H^1_\eta$, as desired. This establishes the existence of a global minimizer, v_ε.

By Theorem 4.4 we know that there exists a sequence $\{\tilde{v}_\varepsilon\}$ converging to v_0 in L^1_η with $G^{(1)}_\varepsilon(\tilde{v}_\varepsilon) \to G^{(1)}(v_0)$. In particular $\|\tilde{v}_\varepsilon - v_0\|_{L^1_\eta} \leq \delta$ for ε sufficiently small. Since v_ε is a global minimizer of J_ε we then know that $G_\varepsilon(v_\varepsilon) \leq G_\varepsilon(\tilde{v}_\varepsilon)$ for ε small. Thus

$$\limsup_{\varepsilon \to 0^+} G^{(1)}_\varepsilon(v_\varepsilon) \leq \limsup_{\varepsilon \to 0^+} G^{(1)}_\varepsilon(\tilde{v}_\varepsilon) \leq G^{(1)}(v_0).$$

By Lemma 4.3 we then have that (up to a subsequence, not relabeled), $v_\varepsilon \to \tilde{v}$ in L^1_η, with $\tilde{v} \in \mathcal{C}$ and with $\|\tilde{v} - v_0\|_{L^1_\eta} \leq \delta$. By again applying Theorem 4.4 we find that

$$G^{(1)}(\tilde{v}) \leq \liminf_{\varepsilon \to 0^+} G^{(1)}_\varepsilon(v_\varepsilon) \leq \limsup_{\varepsilon \to 0^+} G^{(1)}_\varepsilon(v_\varepsilon) \leq G^{(1)}(v_0). \tag{4.33}$$

Theorem 4.6 then implies that $\tilde{v} = v_0$, which along with (4.33) implies (4.32). As $v_\varepsilon \to v_0$ in L^1_η we then have that the v_ε must be local minimizers of G_ε, for ε sufficiently small. This completes the proof. $\qquad \square$

In light of the fact that the global minimizers of J_ε are local minimizers of G_ε for ε sufficiently small we can then establish the Euler–Lagrange equations.

Theorem 4.9. *Under the hypotheses of Proposition 4.8 the sequence $\{v_\varepsilon\}$ of global minimizers of the functionals J_ε will satisfy the following Euler–Lagrange equations (for ε sufficiently small):*

$$2\varepsilon^2(v'_\varepsilon(t)\eta(t))' - W'(v_\varepsilon(t))\eta(t) = \varepsilon\lambda_\varepsilon\eta(t), \tag{4.34}$$

where $\lambda_\varepsilon \in \mathbb{R}$. Moreover the Lagrange multipliers λ_ε satisfy

$$\lim_{\varepsilon \to 0^+} \lambda_\varepsilon = \lambda_0, \tag{4.35}$$

where λ_0 is the number given in (4.19).

Proof. Reasoning somewhat as in the proof of step 4 in [25] we have that $v_\varepsilon \in C^2(I)$ and satisfies (4.34). Next, we will prove (4.35), namely the limit of the Lagrange multipliers λ_ε. The argument here follows [44], with the necessary adaptations to the weighted setting.

To prove (4.35), fix some $\psi \in C^\infty_c(I)$. We multiply the Euler–Lagrange equations (4.34) by $\psi v'_\varepsilon$ and integrate to obtain

$$\varepsilon\lambda_\varepsilon \int_I \psi v'_\varepsilon \eta \, dt = \int_I (2\varepsilon^2(v''_\varepsilon\eta + v'_\varepsilon\eta') - W'(v_\varepsilon)\eta)\psi v'_\varepsilon \, dt.$$

Integrating by parts, we find that

$$\varepsilon\lambda_\varepsilon \int_I \psi v'_\varepsilon \eta \, dt = \int_I (W(v_\varepsilon) - \varepsilon^2 v'^2_\varepsilon)(\eta\psi)' + 2\varepsilon^2(v'_\varepsilon)^2 \eta' \psi \, dt. \tag{4.36}$$

By Theorem 4.4 and Proposition 4.8 we know that

$$\lim_{\varepsilon \to 0} \int_I (\varepsilon^{-1}W(v_\varepsilon) + \varepsilon(v'_\varepsilon)^2)\eta \, dt = 2c_W\eta(t_0).$$

Furthermore, as in the proof of (4.11), by lower semicontinuity

$$\liminf_{\varepsilon \to 0} 2\int_I W^{1/2}(v_\varepsilon)|v'_\varepsilon|\eta \, dt = \liminf_{\varepsilon \to 0} 2\int_I |(\Phi(v_\varepsilon))'|\eta \, dt \geq 2c_W\eta(t_0), \tag{4.37}$$

27

where we recall that $\Phi(t) := \int_a^t W^{1/2}(s)\,ds$. These together give the following:

$$0 \le \limsup_{\varepsilon \to 0} \int_I (\varepsilon^{-1/2} W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}(v_\varepsilon'))^2 \eta\,dt$$

$$= \limsup_{\varepsilon \to 0} \int_I (\varepsilon^{-1} W(v_\varepsilon) + \varepsilon(v_\varepsilon')^2 - 2W^{1/2}(v_\varepsilon)|v_\varepsilon'|)\eta\,dt \le 0.$$

We thus have that $\varepsilon^{-1/2} W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'|$ goes to zero in L_η^2. Moreover, the liminf in (4.37) is actually a limit and equality holds, so that

$$\lim_{\varepsilon \to 0} \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\,dt = c_W \eta(t_0). \tag{4.38}$$

Additionally, we can write the following:

$$\lim_{\varepsilon \to 0} \int_I |\varepsilon^{-1} W(v_\varepsilon) - \varepsilon(v_\varepsilon')^2|\eta\,dt$$

$$= \lim_{\varepsilon \to 0} \int_I \left| \varepsilon^{-1/2} W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'| \right| \left| \varepsilon^{-1/2} W^{1/2}(v_\varepsilon) + \varepsilon^{1/2}|v_\varepsilon'| \right| \eta\,dt$$

$$\le \lim_{\varepsilon \to 0} \left(\int_I \left(\varepsilon^{-1/2} W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'| \right)^2 \eta\,dt \right)^{1/2}$$

$$\times \left(\int_I \left(\varepsilon^{-1/2} W^{1/2}(v_\varepsilon) + \varepsilon^{1/2}|v_\varepsilon'| \right)^2 \eta\,dt \right)^{1/2}$$

$$\le \lim_{\varepsilon \to 0} C \left(\int_I \left(\varepsilon^{-1/2} W^{1/2}(v_\varepsilon) - \varepsilon^{1/2}|v_\varepsilon'| \right)^2 \eta\,dt \right)^{1/2} = 0,$$

where we have used Hölder's inequality, Minkowski's inequality and the boundedness of $G_\varepsilon^{(1)}(v_\varepsilon)$. By (4.1) we can deduce that $\varepsilon^{-1} W(v_\varepsilon) - \varepsilon(v_\varepsilon')^2$ goes to zero in $L^1_{\mathrm{loc}}(I)$. Thus by dividing (4.36) by ε, and recalling that ψ is compactly supported in I, we obtain

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon \int_I \psi v_\varepsilon' \eta\,dt = \lim_{\varepsilon \to 0} 2 \int_I \varepsilon(v_\varepsilon')^2 \eta' \psi\,dt.$$

We then use the L^2 convergence shown above to estimate the following

$$\lim_{\varepsilon \to 0} \left| \int_I (\varepsilon(v_\varepsilon')^2 - W^{1/2}(v_\varepsilon)|v_\varepsilon'|)\eta'\psi\,dt \right|$$

$$= \lim_{\varepsilon \to 0} \left| \int_I \varepsilon^{1/2}|v_\varepsilon|(\varepsilon^{1/2}|v_\varepsilon'| - \varepsilon^{-1/2} W^{1/2}(v_\varepsilon))\eta'\psi\,dt \right|$$

$$\le \lim_{\varepsilon \to 0} \left(\int_I \varepsilon(v_\varepsilon')^2 \left(\frac{\eta'\psi}{\eta} \right)^2 \eta\,dt \right)^{1/2} \left(\int_I (\varepsilon^{1/2}|v_\varepsilon'| - \varepsilon^{-1/2} W^{1/2}(v_\varepsilon))^2 \eta\,dt \right)^{1/2} = 0,$$

where we have used the fact that $\frac{\psi\eta'}{\eta}$ is uniformly bounded, since ψ has compact support in I.

Thus we can write the following:

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon \int_I \psi v_\varepsilon' \eta\,dt = \lim_{\varepsilon \to 0} 2 \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta'\psi\,dt. \tag{4.39}$$

We know that $v_\varepsilon' \mathcal{L}^1 \lfloor I \overset{*}{\rightharpoonup} Dv_0 = (b-a)\delta_{t_0}$ and $W^{1/2}(v_\varepsilon)v_\varepsilon'\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} D(\Phi \circ v_0) = c_W \delta_{t_0}$, both in $(C_0(\bar{I}))'$. In turn, $W^{1/2}(v_\varepsilon)v_\varepsilon'\eta\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} c_W\eta(t_0)\delta_{t_0}$. In view of (4.38), it follows from Proposition 4.30 in [45] that $W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta\mathcal{L}^1\lfloor I \overset{*}{\rightharpoonup} c_W\eta(t_0)\delta_{t_0}$. Hence,

$$\lim_{\varepsilon \to 0} \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta'\psi\,dt = \lim_{\varepsilon \to 0} \int_I W^{1/2}(v_\varepsilon)|v_\varepsilon'|\eta \frac{\eta'}{\eta}\psi\,dt = c_W\eta(t_0)\frac{\eta'(t_0)}{\eta(t_0)}\psi(t_0).$$

28

We thus take limits in (4.39) to find that

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon (b - a)\psi(t_0)\eta(t_0) = 2\eta'(t_0)c_W \psi(t_0).$$

This then gives the desired conclusion, namely that (4.35) holds.

$\square$

Next we establish tight bounds on the functions v_ε, as well as a Neumann condition.

Theorem 4.10. *Under the hypotheses of Proposition 4.8, for all $\varepsilon > 0$ sufficiently small the minimizers v_ε of J_ε satisfy*

$$a_\varepsilon \leq v_\varepsilon(t) \leq b_\varepsilon, \quad t \in I, \tag{4.40}$$
$$v'_\varepsilon(-T) = v'_\varepsilon(T) = 0, \tag{4.41}$$

where $a_\varepsilon < c_\varepsilon < b_\varepsilon$ are the only zeros of $W' + \lambda_\varepsilon \varepsilon$. Moreover

$$a_\varepsilon = a - \lambda_\varepsilon |\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q} + o(\varepsilon^{1/q}), \tag{4.42}$$
$$c_\varepsilon = c - \lambda_\varepsilon W''(c)^{-1}\varepsilon + o(\varepsilon), \tag{4.43}$$
$$b_\varepsilon = b - \lambda_\varepsilon |\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q} + o(\varepsilon^{1/q}). \tag{4.44}$$

Proof. By hypothesis (2.7), $|W'(s)| \geq w_0 > 0$ for all $|s| \geq C$. Since W' has only three zeros at a, b, c and is strictly monotonic in a ball centered at each of these points with radius $\zeta_0 > 0$ (see (2.5) and (2.6)), by taking w_0 smaller we can assume that $|W'(s)| \geq w_0$ for all $s \in \mathbb{R} \setminus (B(a, \zeta_0) \cup B(c, \zeta_0) \cup B(b, \zeta_0))$. By (4.35), $|\varepsilon\lambda_\varepsilon| \leq w_0/2$ for all $\varepsilon > 0$ small. Hence $W' + \varepsilon\lambda_\varepsilon$ has only three zeros

$$a_\varepsilon < b_\varepsilon < c_\varepsilon, \tag{4.45}$$

for all $\varepsilon > 0$ small. Furthermore by (2.6) and (2.10) we can derive the explicit forms forms in (4.42)-(4.44).

Next, consider the open set $U_\varepsilon := \{t \in I : v_\varepsilon(t) < a_\varepsilon\}$. We claim that U_ε is empty. Indeed, if not, let I_ε be a maximal subinterval of U_ε, and since $W'(v_\varepsilon) + \varepsilon\lambda_\varepsilon < 0$ for all $t \in I_\varepsilon$ by (4.34) we have that $(v'_\varepsilon(t)\eta(t)))' < 0$ for all $t \in I_\varepsilon$. Since $\eta > 0$ on I by (4.1), this implies that v'_ε has at most one zero in $\overline{I}_\varepsilon$. Hence there exist $\lim_{t \to t_\varepsilon^+} v_\varepsilon(t) = \ell_\varepsilon$ and $\lim_{t \to T_\varepsilon^-} v_\varepsilon(t) = L_\varepsilon$, where $t_\varepsilon, T_\varepsilon$ are the left and right endpoints of I_ε, respectively. Note that $\ell_\varepsilon, L_\varepsilon$ could be infinite if one of the endpoints is $-T$ or T. Consider $\inf_{I_\varepsilon} v_\varepsilon$. If there exists $s_\varepsilon \in I_\varepsilon^\circ$ such that $v_\varepsilon(s_\varepsilon) = \inf_{I_\varepsilon} v_\varepsilon$, then $v'_\varepsilon(s_\varepsilon) = 0$ and $v''_\varepsilon(s_\varepsilon) \geq 0$. This is impossible, as $(v'_\varepsilon\eta)' < 0$ on I_ε. Thus it follows that $\inf_{I_\varepsilon} v_\varepsilon$ is either ℓ_ε or L_ε. Assume first that $\inf_{I_\varepsilon} v_\varepsilon = \ell_\varepsilon$. By the definition of I_ε it cannot be that $\ell_\varepsilon = a_\varepsilon$, but then, by the maximality of I_ε, necessarily $t_\varepsilon = -T$. By (4.34) for all $t_1, t_2 \in I_\varepsilon$, with $t_1 < t_2$:

$$2\varepsilon^2 v'_\varepsilon(t_2)\eta(t_2) - 2\varepsilon^2 v'_\varepsilon(t_1)\eta(t_1) = \int_{t_1}^{t_2} (W'(v_\varepsilon(s)) + \varepsilon\lambda_\varepsilon)\eta(s)\, ds. \tag{4.46}$$

Since $W'(v_\varepsilon(t)) + \varepsilon\lambda_\varepsilon < 0$ for all $t \in I_\varepsilon$, the integral $\int_{-T}^{t_2}(W'(v_\varepsilon(s)) + \varepsilon\lambda_\varepsilon)\eta(s)\, ds$ is well-defined in $\mathbb{R} \cup \{-\infty\}$. Hence, letting $t_1 \to -T^+$ in (4.46), it follows that there exists

$$\lim_{t \to -T^+} v'_\varepsilon(t)\eta(t) = M_\varepsilon \in \overline{\mathbb{R}}. \tag{4.47}$$

Assume, for the sake of contradiction, that $M_\varepsilon \neq 0$. Then also by (4.2), $|v'_\varepsilon(t)| \geq C_0(T + t)^{-n_1-1}$ for all $t \in (-T, -T + \delta_\varepsilon)$, for some $\delta_\varepsilon > 0$. It would then follow that

$$\int_{-T}^{-T+\delta_\varepsilon} |v'_\varepsilon|^2 \eta\, dt \geq d_1 \int_{-T}^{-T+\delta_\varepsilon} C_0^2(T + t)^{-n_1+1}\, dt = \infty$$

29

if $n_1 \geq 2$. On the other hand, if $n_1 = 1$ then $v'_\varepsilon(-T) = 0$, since v_ε is a minimizer. Thus in both cases we must have that $M_\varepsilon = 0$. In turn, letting $t_1 \to -T^+$ in (4.46) it follows that $v'_\varepsilon(t) < 0$ for all $t \in I_\varepsilon$, which contradicts the fact that $\ell_\varepsilon = \inf_{I_\varepsilon} v_\varepsilon$. Using a similar argument we can exclude the case that $L_\varepsilon = \inf_{I_\varepsilon} v_\varepsilon$. This proves that I_ε, and in turn U_ε, is empty. Thus $v_\varepsilon \geq a_\varepsilon$ in I. Similarly, we can show that $v_\varepsilon \leq b_\varepsilon$ in I.

It remains to prove the Neumann boundary condition (4.41) when $n_i \geq 2$. Since v_ε is bounded by what we just proved, it follows that the integral on the right-hand side of (4.46) is bounded for all $t \in I$. Hence as in the first part of the proof we can conclude that the limit M_ε in (4.47) exists and must be zero. Hence letting $t_1 \to -T^+$ in (4.46) we obtain

$$2\varepsilon^2 v'_\varepsilon(t)\eta(t) = \int_{-T}^{t} (W'(v_\varepsilon) + \lambda_\varepsilon \varepsilon)\eta(s)\,ds.$$

Using again the fact that v_ε is bounded, along with (2.4) and (4.2), we have that

$$0 \leq 2\varepsilon^2 |v'_\varepsilon(t)| \leq \frac{C}{d_1(T+t)^{n_1-1}} \int_{-T}^{t} d_2(T+s)^{n_1-1}\,ds = \frac{Cd_2}{d_1 n_1}(T+t) \to 0$$

as $t \to -T^+$. A similar estimate holds near T. This completes the proof. $\qquad \square$

In the following theorem we specify the qualitative behavior of v_ε, which are global minimizers of J_ε. Despite the fact that $v_\varepsilon \to v_0 \in L^1_\eta$ by Proposition 4.8, v_ε need not be increasing. Indeed in the radial case $\eta(t) \equiv (t+T)^{n-1}$, Ni [50] has shown that all positive solutions of (4.34) approach c_ε as $t \to \infty$ in an oscillatory way. The presence of possible oscillations makes the analysis significantly more involved. However, the overall idea of the proof is the same as the proof of Theorem 4.6.

Fix

$$\theta_i \in \left(\frac{1}{n_i}, \frac{1}{n_i - 1}\right), \quad i = 1, 2, \tag{4.48}$$

where n_i are the exponents given in (4.2) and (4.3). Let $k \in \mathbb{N}$ and define

$$O_\varepsilon := \{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : a_\varepsilon + \varepsilon^k \leq v_\varepsilon(t) \leq b_\varepsilon - \varepsilon^k\}, \tag{4.49}$$

with $c(n_i) := 0$ if $n_i = 1$ and 1 otherwise.

Theorem 4.11. *Assume that W satisfies (2.4)-(2.7), and that η satisfies (4.1)-(4.4). Let v_ε be a minimizer of J_ε. Write $I_0 := [-T + r_0, T - r_0]$, with $r_0 > 0$ a constant to be defined. Then for all $\varepsilon > 0$ sufficiently small the following properties hold:*

i) *$\Gamma_\varepsilon := O_\varepsilon \cap I_0$ has exactly one component $[T_1^\varepsilon, T_2^\varepsilon]$, with $v_\varepsilon(T_1^\varepsilon) = a_\varepsilon + \varepsilon^k$ and $v_\varepsilon(T_2^\varepsilon) = b_\varepsilon - \varepsilon^k$. Moreover, there exists $0 < r_1 < r_0$ so that $\Gamma_\varepsilon \subset B(t_0, r_1)$.*

ii) *The points in Γ_ε where $v_\varepsilon = c_\varepsilon$ are at most distance $O(\varepsilon)$ apart.*

iii) *For $t \in (-T, T_1^\varepsilon)$ we have that $v_\varepsilon(t) \in [a_\varepsilon, a_\varepsilon + \varepsilon^k)$ except on a set of $\eta\mathcal{L}^1$ measure $o(\varepsilon)$. Similarly for $t \in (T_2^\varepsilon, T)$ we have that $v_\varepsilon(t) \in (b_\varepsilon - \varepsilon^k, b_\varepsilon]$ except on a set of $\eta\mathcal{L}^1$ measure $o(\varepsilon)$.*

We delay the proof of this theorem until after we establish some preliminary results. Let $r_0 > 0$ be chosen as in (4.13). As $v_\varepsilon \to v_0$ in L^1_η, by selecting a subsequence, we can assume that $v_\varepsilon(t) \to v_0(t)$ for $\mathcal{L}^1$ a.e. $t \in I$. Hence, given

$$0 < \rho < \frac{1}{2}\min\{c - a, b - c\}, \tag{4.50}$$

there exists $\varepsilon_\rho > 0$ such that

$$|v_\varepsilon(T_1) - a| < \rho, \quad |v_\varepsilon(T_2) - a| < \rho, \quad |v_\varepsilon(T_3) - b| < \rho, \quad |v_\varepsilon(T_4) - b| < \rho \tag{4.51}$$

for all $0 < \varepsilon \leq \varepsilon_\rho$ sufficiently small and some $T_1 \in (-T, -T + r_0)$, $T_2 \in (-T + 2r_0, t_0 - r_0)$, $T_3 \in (t_0 + r_0, T - 2r_0)$ and $T_4 \in (T - r_0, T)$. Fix $\varepsilon > 0$ sufficiently small so that (4.51) holds.

First, we prove adaptations of two lemmas from [58].

Lemma 4.12. *Let $s_0, s_1 > 0$ be such that $a_\varepsilon + s_0 < c_\varepsilon < b_\varepsilon - s_1$ for all $\varepsilon > 0$ sufficiently small. Fix any such ε. Let $I_\varepsilon \subseteq I$ be a non-empty maximal interval such that $a_\varepsilon + s_0 < v_\varepsilon(t) < b_\varepsilon - s_1$ for all $t \in I_\varepsilon$. Then there exists $t_0 \in \overline{I_\varepsilon}$ such that $v_\varepsilon(t_0) = c_\varepsilon$.*

Proof. If not, then either $a_\varepsilon + s_0 \leq v_\varepsilon(t) < c_\varepsilon$ for all $t \in \overline{I_\varepsilon}$ or $c_\varepsilon < v_\varepsilon(t) \leq b_\varepsilon - s_1$ for all $t \in \overline{I_\varepsilon}$. Consider the second case. Then $W'(v_\varepsilon(t)) + \varepsilon\lambda_\varepsilon < 0$ for all $t \in I_\varepsilon$, and so by (4.34) we have that $(v'_\varepsilon(t)\eta(t))' < 0$ for all $t \in I_\varepsilon$. Let $\tilde{t} \in \overline{I_\varepsilon}$ be the point of minimum of v_ε in $\overline{I_\varepsilon}$. Reasoning as in the proof of (4.40), we have that $\tilde{t}$ cannot belong to I_ε, and so $\tilde{t} \in \partial I_\varepsilon$. If $\tilde{t} \in I$, then necessarily, $v_\varepsilon(\tilde{t}) = c_\varepsilon$, which contradicts the fact that $c_\varepsilon < v_\varepsilon(t) < b_\varepsilon - s_1$ for all $t \in \overline{I_\varepsilon}$. it follows that $\tilde{t} \in \{-T, T\}$. We can now continue as in the proof of (4.40) to exclude this possibility. $\square$

Lemma 4.13. *Let ρ be as in (4.50) and suppose that I_ε is a maximal subinterval of the set $\{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : v_\varepsilon(t) \geq c + \rho\}$. Then there exists a $\mu > 0$ such that we have the following estimate for all $t \in I_\varepsilon$:*

$$b_\varepsilon - v_\varepsilon(t) \leq 2(b_\varepsilon - c - \rho)e^{-\mu d(t, I_\varepsilon^c)\varepsilon^{-1}}.$$

In addition an analogous bound holds for the set $\{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : v_\varepsilon(t) \leq c - \rho\}$.

Proof. First, we claim that there exists a μ such that for any $s \in [c + \rho, b_\varepsilon]$ the following inequality holds

$$-(W'(s) + \varepsilon\lambda_\varepsilon) \geq 2\mu^2(b_\varepsilon - s). \tag{4.52}$$

If $q = 1$ in (2.5), then also by (2.4) we have that $W \in C^2(\mathbb{R})$. Since $W''(b) > 0$ by continuity we have that $W''(s) \geq 2\mu^2 > 0$ for all $s \in B(b, R_1)$, for some $\mu \neq 0$, and $R_1 > 0$. It follows from (4.45) that

$$W'(s) + \varepsilon\lambda_\varepsilon = -\int_s^{b_\varepsilon} W''(r)\,dr \leq -2\mu^2(b_\varepsilon - s)$$

for all $s \in B(b, R_1)$, with $s < b_\varepsilon$. Using the fact that $W' + \varepsilon\lambda_\varepsilon < 0$ in $(c_\varepsilon, b_\varepsilon)$ (see Theorem 4.10), and by taking μ smaller, if necessary, we can assume that

$$W'(s) + \varepsilon\lambda_\varepsilon \leq -2\mu^2(b_\varepsilon - s)$$

for all $s \in [c + \rho, b_\varepsilon]$. Note that μ depends upon ρ but not on ε. On the other hand, if $0 < q < 1$ then since $\lim_{s \to b} W''(s) = \infty$ by (2.5), we can still assume that $W''(s) \geq \mu^2 > 0$ near b. Hence we can continue as before to conclude that (4.52) holds even in this case. This proves the claim.

Write $I_\varepsilon = [t_1, t_2]$ and define

$$\phi(t) := (b_\varepsilon - v_\varepsilon(t_1))e^{-\mu(t-t_1)\varepsilon^{-1}} + (b_\varepsilon - v_\varepsilon(t_2))e^{-\mu(t_2-t)\varepsilon^{-1}} \tag{4.53}$$

with μ fixed by (4.52). We note that ϕ satisfies the following differential inequality:

$$(\phi'\eta)' = \frac{\mu^2}{\varepsilon^2}\phi\eta + \frac{\mu}{\varepsilon}\eta'\left(-(b_\varepsilon - v_\varepsilon(t_1))e^{-\mu(t-t_1)\varepsilon^{-1}} + (b_\varepsilon - v_\varepsilon(t_2))e^{-\mu(t_2-t)\varepsilon^{-1}}\right)$$

$$\leq \frac{1}{\varepsilon^2}\left(\mu^2 + \varepsilon\frac{|\eta'|}{\eta}\mu\right)\phi\eta.$$

If $n_1 > 1$ in (4.2), then $c(n_1) = 1$ in (4.49) and so by (4.4),

$$\varepsilon\frac{|\eta'(t)|}{\eta(t)} \leq \frac{\varepsilon d_5}{t + T} \leq d_5\varepsilon^{1-\theta_1} \leq \mu$$

for all $t \in [-T + \varepsilon^{\theta_1}, 0]$ and all ε sufficiently small. On the other hand, if $n_1 = 1$ in (4.2), then $c(n_1) = 0$ in (4.49) and so by (4.1) and (4.3), $\eta(t) \geq \eta_0 > 0$ for all $t \in [-T, 0]$. Thus,

$$\varepsilon\frac{|\eta'(t)|}{\eta(t)} \leq \varepsilon\frac{\max|\eta'|}{\eta_0} \leq \mu$$

31

for all $t \in [-T, 0]$ and all ε sufficiently small. Similar inequalities hold in $[0, T - c(n_2)\varepsilon^{\theta_2}]$. Thus in I_ε,

$$(\phi'\eta)' \le 2\varepsilon^{-2}\mu^2\phi\eta. \tag{4.54}$$

We then set $g(t) := b_\varepsilon - v_\varepsilon(t)$ and using (4.52) we have that

$$(g'\eta)' = -\varepsilon^{-2}(W'(v_\varepsilon) + \varepsilon\lambda_\varepsilon)\eta \ge 2\varepsilon^{-2}\mu^2 g\eta. \tag{4.55}$$

We define $\Psi := g - \phi$. By (4.53), (4.54) and (4.55), for ε small we have the following:

$$(\Psi'\eta)' \ge 2\varepsilon^{-2}\mu^2\Psi\eta,$$
$$\Psi(t_1) \le 0, \quad \Psi(t_2) \le 0.$$

The maximum principle implies that $\Psi \le 0$ for all $t \in I_\varepsilon$. Thus

$$b_\varepsilon - v(t) \le \phi(t) \le 2(b_\varepsilon - c - \rho))e^{-\mu\varepsilon^{-1}d(t, I_\varepsilon^c)},$$

which is the desired result. $\qquad\square$

Corollary 4.14. *Let ρ be as in* (4.50) *and let*

$$A_\varepsilon := \{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : a_\varepsilon + \varepsilon^k \le v_\varepsilon(t) \le c - \rho\},$$
$$B_\varepsilon := \{t \in [-T + c(n_1)\varepsilon^{\theta_1}, T - c(n_2)\varepsilon^{\theta_2}] : c + \rho \le v_\varepsilon(t) \le b_\varepsilon - \varepsilon^k\}.$$

Then for any maximal interval I_ε contained in $A_\varepsilon \cup B_\varepsilon$,

$$\operatorname{diam} I_\varepsilon \le C\varepsilon|\log\varepsilon|$$

for all $\varepsilon > 0$ sufficiently small and for some constant $C > 0$ depending only on W, k, μ, ρ.

Proof. Assume $(t_1, t_2) = I_\varepsilon^\circ \subset B_\varepsilon$. By Lemma 4.13 we have that for $t = \frac{t_1 + t_2}{2}$:

$$\varepsilon^k \le b_\varepsilon - v_\varepsilon(t) \le 2(b_\varepsilon - c - \rho)e^{-\mu 2^{-1}(t_2 - t_1)\varepsilon^{-1}},$$

which implies that $-\frac{\mu}{2}(t_2 - t_1)\varepsilon^{-1} \ge k\log\varepsilon - \log 2(b_\varepsilon - c - \rho)$, that is,

$$0 \le t_2 - t_1 \le 2\mu^{-1}k\varepsilon|\log\varepsilon| + 2\mu^{-1}\varepsilon\log 2(b_\varepsilon - c - \rho).$$

This shows that $\operatorname{diam} I_\varepsilon \le C\varepsilon|\log\varepsilon|$. The proof for the case $I_\varepsilon \subset A_\varepsilon$ is similar and we omit it. $\qquad\square$

Next we state a lemma from [58], which allows us to estimate the size of certain sets. In what follows given a set E and $s > 0$ we define the set

$$E^s := \{x \in \mathbb{R}^n : d(x, E) \le s\} \tag{4.56}$$

Lemma 4.15. *Given a measurable set $A \subset \mathbb{R}^n$, and for all numbers $0 < s < t$ we have that*

$$\frac{\mathcal{L}^n(A^t)}{\mathcal{L}^n(A^s)} \le C_n \left(\frac{t}{s}\right)^n,$$

where we are using the notation (4.56).

Next we establish an estimate on the derivative of v_ε.

Lemma 4.16. *There exists a constant $C > 0$ such that*

$$|v_\varepsilon'(t)| \le C\varepsilon^{-1}$$

for all $t \in I$.

Proof. By (4.34) and the fact that $v'_\varepsilon(-T) = 0$,

$$2\varepsilon^2 v'_\varepsilon(t)\eta(t) = \int_{-T}^{t} (W'(v_\varepsilon(s)) + \varepsilon\lambda_\varepsilon)\eta(s)\ ds$$

for every $t \in \bar{I}$. In light of (4.1)-(4.2) we know that that there exist constants $c_1, c_2 > 0$ so that $c_1(T+t)^{n_1-1} \le \eta(t) \le c_2(T+t)^{n_1-1}$ for all $t \in [-T, T-t^*]$. Since v_ε is bounded by (4.40), this implies that

$$2\varepsilon^2 |v'_\varepsilon(t)| \le \frac{C}{\eta(t)} \int_{-T}^{t} \eta(s)\ ds \le \frac{C}{c_1(T+t)^{n_1-1}} \int_{-T}^{t} c_2(T+s)^{n_1-1}\ ds$$
$$= \frac{Cc_2}{c_1 n_1}(T+t)$$

for all $t \in (-T, T-t^*)$. Using a similar argument in $(-T+t^*, T)$, we conclude that

$$\varepsilon^2 |v'_\varepsilon(t)| \le C\min\{T+t, T-t\}$$

for all $t \in I$. By (4.34), v_ε satisfies

$$2\varepsilon^2 v''_\varepsilon(t) + 2\varepsilon^2 \frac{\eta'(t)}{\eta(t)} v'_\varepsilon(t) = W'(v_\varepsilon(t)) + \varepsilon\lambda_\varepsilon.$$

Using by (4.4), (4.40) and the previous inequality we get

$$2\varepsilon^2 |v''_\varepsilon(t)| \le \left| \frac{\eta'(t)}{\eta(t)} \right| 2\varepsilon^2 |v'_\varepsilon(t)| + C \le C.$$

Next we use a classical interpolation result. Let $t \in I$ and consider $t_1 \in I$ with $|t - t_1| = \varepsilon$. By the mean value theorem $v_\varepsilon(t) - v_\varepsilon(t_1) = v'_\varepsilon(\theta)(t - t_1)$ and so by the fundamental theorem of calculus

$$v'_\varepsilon(t) = v'_\varepsilon(\theta) + \int_{\theta}^{t} v''_\varepsilon(s)\ ds = \frac{v_\varepsilon(t) - v_\varepsilon(t_1)}{t - t_1} + \int_{\theta}^{t} v''_\varepsilon(s)\ ds.$$

Again by (4.40) it follows that

$$|v'_\varepsilon(t)| \le \frac{C}{\varepsilon} + \sup |v''_\varepsilon||t - \theta| \le \frac{C}{\varepsilon} + \frac{C}{\varepsilon^2}\varepsilon.$$

This concludes the proof. $\qquad \square$

We are now prepared to prove Theorem 4.11. By way of notation, for every measurable subset $E \subset I$ and for every $v \in H^1_\eta$ satisfying $\|v - v_0\|_{L^1_\eta} \le \delta$ and (4.5) we define the localized energy

$$J_\varepsilon^{(1)}(v; E) := \int_E \left(\frac{1}{\varepsilon}W(v) + \varepsilon(v')^2 \right) \eta\ dt. \tag{4.57}$$

Proof of Theorem 4.11. By Lemma 4.7 there exists $\tilde{v}_\varepsilon$ converging to v_0 in L^1_η such that

$$G_\varepsilon^{(1)}(v_\varepsilon) = J_\varepsilon^{(1)}(v_\varepsilon) \le J_\varepsilon^{(1)}(\tilde{v}_\varepsilon) \le G_\varepsilon^{(1)}(\tilde{v}_\varepsilon) \le G^{(1)}(v_0) + C\varepsilon = 2c_W \eta(t_0) + C\varepsilon, \tag{4.58}$$

where we have used the fact that v_ε is a minimizer of J_ε. We will take $r_1 := r_{\epsilon_1}$, where r_{ϵ_1} is given as in the proof of Theorem 4.6 and ϵ_1 will be taken even smaller. Define

$$c_- := \int_a^c W^{1/2}(s)\ ds, \quad c_+ := \int_c^b W^{1/2}(s)\ ds. \tag{4.59}$$

Step 1: We claim that $\mathcal{L}^1(O_\varepsilon) = o(1)$ (see (4.49)). Define the set

$$D_\varepsilon := O_\varepsilon \cap v_\varepsilon^{-1}([c - \rho, c + \rho]\}).$$

By Lemma 4.16, $|v_\varepsilon'| \leq C_0\varepsilon^{-1}$, and so, using the notation in (4.56), $(D_\varepsilon)^{l\varepsilon} \subset v_\varepsilon^{-1}([c - 2\rho, c + 2\rho])$, provided $0 < l \leq \rho C_0^{-1}$. In turn

$$\mathcal{L}^1((D_\varepsilon)^{l\varepsilon}) \leq \mathcal{L}^1(v_\varepsilon^{-1}([c - 2\rho, c + 2\rho]))$$

$$\leq \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + \left(\min_{[c-2\rho,c+2\rho]} W\right)^{-1} \int_{-T+\varepsilon^{\theta_1}}^{T-\varepsilon^{\theta_2}} W(v_\varepsilon)\, dt \qquad (4.60)$$

$$\leq \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\left(\varepsilon^{-\theta_1(n_1-1)} + \varepsilon^{-\theta_2(n_2-1)}\right) \int_{-T+\varepsilon^{\theta_1}}^{T-\varepsilon^{\theta_2}} W(v_\varepsilon)\eta\, dt$$

$$\leq \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\left(\varepsilon^{1-\theta_1(n_1-1)} + \varepsilon^{1-\theta_2(n_2-1)}\right),$$

where we have used (2.4), (4.1)-(4.3), (4.50) and (4.58).

Next we claim that

$$O_\varepsilon \subset (D_\varepsilon)^{C\varepsilon|\log \varepsilon|} \cup [-T, -T + c(n_1)\varepsilon^{\theta_1} + C\varepsilon|\log \varepsilon|] \cup [T - c(n_2)\varepsilon^{\theta_2} - C\varepsilon|\log \varepsilon|, T]. \qquad (4.61)$$

Indeed, as $O_\varepsilon = A_\varepsilon \cup B_\varepsilon \cup D_\varepsilon$, it suffices to consider $\tilde{t} \in A_\varepsilon$, as the case $\tilde{t} \in B_\varepsilon$ is analagous. Let I_ε be the maximal subinterval of A_ε containing $\tilde{t}$. By Corollary 4.14, $\operatorname{diam} I_\varepsilon \leq C\varepsilon|\log \varepsilon|$. If I_ε intersects D_ε, then $d(\tilde{t}, D_\varepsilon) \leq \operatorname{diam} I_\varepsilon \leq C\varepsilon|\log \varepsilon|$. Otherwise, since reasoning as in Lemma 4.12 it cannot happen that v_ε takes the value $b_\varepsilon - \varepsilon^k$ at both endpoints of I_ε, it follows that one of the endpoints of I_ε is $-T + c(n_1)\varepsilon^{\theta_1}$ or $T - c(n_2)\varepsilon^{\theta_2}$, say, $-T + c(n_1)\varepsilon^{\theta_1}$. Thus

$$d(\tilde{t}, [-T, -T + c(n_1)\varepsilon^{\theta_1}]) \leq C\varepsilon|\log \varepsilon|.$$

This proves (4.61).

By Lemma 4.15 and (4.60) we then have that

$$\mathcal{L}^1(O_\varepsilon) \leq \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\varepsilon|\log \varepsilon| + \mathcal{L}^1((D_\varepsilon)^{l\varepsilon}) \leq \varepsilon^{\theta_1} + \varepsilon^{\theta_2} + C\varepsilon|\log \varepsilon| + C|\log \varepsilon|\mathcal{L}^1(D^{l\varepsilon}) = o(1).$$

Step 2: We claim if I_ε is a maximal subinterval of the set O_ε (see (4.49)) that intersects the interval $J_0 := [-T + 2r_0, T - 2r_0]$, then I_ε is contained in I_0 for all $\varepsilon > 0$ sufficiently small, with

$$\mathcal{L}^1(I_\varepsilon) \leq C\varepsilon|\log \varepsilon|. \qquad (4.62)$$

The first part of the claim, namely, that $I_\varepsilon \subset I_0$, follows immediately from Step 1. Lemma 4.12 then implies that $I_\varepsilon \cap D_\varepsilon \neq \emptyset$. Reasoning as in the proof of (4.60) but using the fact that $\eta \geq \eta_0 > 0$ in I_0 we find that $\mathcal{L}^1((I_\varepsilon \cap D_\varepsilon)^{C\varepsilon}) < C\varepsilon$. Again due to the fact that $I_\varepsilon \subset I_0$, reasoning as in the proof of (4.61) we can show that $I_\varepsilon \subset (I_\varepsilon \cap D_\varepsilon)^{C\varepsilon|\log \varepsilon|}$. Using Lemma 4.15 once more gives (4.62).

Step 3: We claim that there exist t_1^ε, $t_2^\varepsilon \in \overline{B(t_0, r_{\epsilon_1}/2)}$ such that

$$v_\varepsilon(t_1^\varepsilon) \leq c - \rho, \quad v_\varepsilon(t_2^\varepsilon) \geq c + \rho$$

provided $\varepsilon > 0$ is sufficiently small. Indeed, if t_1^ε does not exist, then $c - \rho < v_\varepsilon$ in $\overline{B(t_0, r_{\epsilon_1}/2)}$, and so by (4.12),

$$\delta \geq \int_{B(t_0, r_{\epsilon_1}/2)} |v_\varepsilon - v_0|\eta\, dt \geq (c - a - \rho)\frac{\eta(t_0)}{2}r_{\epsilon_1},$$

where we used the fact that $0 < \epsilon_1 < \eta(t_0)/2$ (see (4.12)). This gives a contradiction provided

$$0 < \delta < (c - a - \rho)\frac{\eta(t_0)}{2}r_{\epsilon_1}.$$

34

Hence, t_1^ε exists, and with a similar argument we can prove the existence of t_2^ε.

Since v_ε is continuous, by the intermediate value theorem it will take all values between $c - \rho$ and $c + \rho$ in $\overline{B(t_0, r_{\epsilon_1}/2)}$. Let Γ_ε^- be a maximal subinterval of O_ε intersecting $\overline{B(t_0, r_{\epsilon_1}/2)}$ such that $v_\varepsilon(\Gamma_\varepsilon^-) \supset [c - \rho, c]$ and let Γ_ε^- be a maximal subinterval of O_ε intersecting $\overline{B(t_0, r_{\epsilon_1}/2)}$ such that $v_\varepsilon(\Gamma_\varepsilon^+) \supset [c, c + \rho]$. By (4.62) both intervals are contained in $I_1 := [t_0 - r_{\epsilon_1}, t_0 + r_{\epsilon_1}]$.

We claim that either $v_\varepsilon(\Gamma_\varepsilon^-) = [a_\varepsilon + \varepsilon^k, b_\varepsilon - \varepsilon^k]$ or $v_\varepsilon(\Gamma_\varepsilon^+) = [a_\varepsilon + \varepsilon^k, b_\varepsilon - \varepsilon^k]$. Indeed, if this is not the case, then by the maximality of Γ_ε^- and Γ_ε^+, Lemma 4.12 and the definition of O_ε (see (4.49)) $v_\varepsilon = a_\varepsilon + \varepsilon^k$ at both endpoints of Γ_ε^- and $v_\varepsilon = b_\varepsilon - \varepsilon^k$ at both endpoints of Γ_ε^+. Hence, by (4.57), and a change of variables,

$$J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon^-) \geq 4 \min_{\Gamma_\varepsilon^-} \eta \int_{\Gamma_\varepsilon^-} W^{1/2}(v_\varepsilon) |v_\varepsilon'| \, dt$$

$$\geq 4 \min_{\Gamma_\varepsilon^-} \eta \int_{a_\varepsilon + \varepsilon^k}^{c} W^{1/2}(s) \, ds \geq 4 c_- \min_{\Gamma_\varepsilon^-} \eta - C\varepsilon^{(q+3)/2q}, \tag{4.63}$$

where we have used (4.59) and the fact that

$$\int_{a}^{a_\varepsilon + \varepsilon^k} W^{1/2}(s) \, ds \leq C|a - a_\varepsilon - \varepsilon^k|^{(q+3)/2} \leq C\varepsilon^{(q+3)/2q}$$

by (2.9) and (4.40). A similar inequality holds for $J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon^+)$ with the only difference that c_- should be replaced by c_+. Hence, also by (4.12) and (4.58),

$$2c_W \eta(t_0) + C\varepsilon \geq J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon^-) + J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon^+) \geq 4c_W(\eta(t_0) - \epsilon_1) - C\varepsilon^{(q+3)/2q},$$

which gives

$$C\varepsilon \geq 2(\eta(t_0) - 2\epsilon_1)c_W.$$

This is a contradiction, provided $\epsilon_1 < \eta(t_0)/2$ and ε is sufficiently small. This proves the claim. We denote by Γ_ε a maximal subinterval of O_ε intersecting $\overline{B(t_0, r_{\epsilon_1}/2)}$ such that $v_\varepsilon(\Gamma_\varepsilon) = [a_\varepsilon + \varepsilon^k, b_\varepsilon - \varepsilon^k]$.

First we claim that v_ε takes the values $a_\varepsilon + \varepsilon^k$ and $b_\varepsilon - \varepsilon^k$ on the endpoints of Γ_ε. If not then reasoning as in (4.63) we would have

$$J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) \geq 4c_W \min_{\Gamma_\varepsilon} \eta - C\varepsilon^{(q+3)/2}$$

which is a contradiction. Next let t_3^ε and t_4^ε be the first time and last time in Γ_ε that v_ε equals c_ε. We claim that

$$t_4^\varepsilon - t_3^\varepsilon \leq C\varepsilon. \tag{4.64}$$

Indeed, if $v_\varepsilon(t) \in [c - \rho, c + \rho]$ for all $t \in [t_3^\varepsilon, t_4^\varepsilon]$, then by (4.12),

$$J_\varepsilon^{(1)}(v_\varepsilon; [t_3^\varepsilon, t_4^\varepsilon]) \geq \varepsilon^{-1} \frac{\eta(t_0)}{2} (t_4^\varepsilon - t_3^\varepsilon) \min_{[c-\rho, c+\rho]} W,$$

and so (4.64) follows from (4.58), while if there exists $t_5^\varepsilon \in [t_3^\varepsilon, t_4^\varepsilon]$ such that $|v_\varepsilon(t_5^\varepsilon) - c| \geq \rho$, say, $v_\varepsilon(t_5^\varepsilon) \geq c + \rho$, then, reasoning as in (4.63) we get

$$J_\varepsilon^{(1)}(v_\varepsilon; [t_3^\varepsilon, t_4^\varepsilon]) \geq 2 \frac{\eta(t_0)}{2} \int_c^{c+\rho} W^{1/2}(s) \, ds - C\varepsilon^{(q+3)/2q}.$$

Furthermore, by again reasoning as in (4.63), and using the fact that v_ε takes the values $a_\varepsilon + \varepsilon^k$ and $b_\varepsilon - \varepsilon^k$ on the endpoints of Γ_ε we have that

$$J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon \setminus [t_3^\varepsilon, t_4^\varepsilon]) \geq 2 \min_{\Gamma_\varepsilon} \eta \int_{a_\varepsilon + \varepsilon^k}^{b_\varepsilon - \varepsilon^k} W^{1/2}(s) \, ds \geq 2c_W \min_{\Gamma_\varepsilon} \eta - C\varepsilon^{(q+3)/2q}. \tag{4.65}$$

35

Hence, by (4.12), (4.58), and (4.65),

$$2c_W\eta(t_0) + C\varepsilon \geq J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon \setminus [t_3^\varepsilon, t_4^\varepsilon]) + J_\varepsilon^{(1)}(v_\varepsilon; [t_3^\varepsilon, t_4^\varepsilon])$$

$$\geq 2c_W(\eta(t_0) - \epsilon_1) + \eta(t_0) \int_c^{c+\rho} W^{1/2}(s)\, ds - C\varepsilon^{(q+3)/2q},$$

which gives

$$C_1\varepsilon \geq \eta(t_0) \int_c^{c+\rho} W^{1/2}(s)\, ds - 2c_W\epsilon_1,$$

which is a contradiction provided $0 < \epsilon_1 < \frac{1}{2}\eta(t_0) \int_c^{c+\rho} W^{1/2}(s)\, ds/c_W$ and ε is sufficiently small. The case where $v_\varepsilon(t_5^\varepsilon) \leq c - \rho$ is analagous.

Step 4: We claim that for all $\varepsilon > 0$ sufficiently small, Γ_ε is the only maximal subinterval of the set O_ε that intersects the interval J_0 defined in Step 2. Indeed, assume that there exists another maximal subinterval I_ε of O_ε that intersects J_0. By Step 1, $I_\varepsilon \subset I_0$ and (4.62) holds. In view of Lemma 4.12 there exists $t_\varepsilon \in I_\varepsilon$ such that $v_\varepsilon(t_\varepsilon) = c_\varepsilon$. Since I_ε is a maximal interval of O_ε at one of the endpoints it attains either the value $a_\varepsilon + \varepsilon^k$ or $b_\varepsilon - \varepsilon^k$. In the first case, reasoning as in (4.63), we get

$$J_\varepsilon^{(1)}(v_\varepsilon; I_\varepsilon) \geq 2 \min_{I_\varepsilon} \eta \int_{I_\varepsilon} W^{1/2}(v_\varepsilon)|v_\varepsilon'|\, dt \geq 2 \min_{I_\varepsilon} \eta \int_{a_\varepsilon + \varepsilon^k}^{c_\varepsilon} W^{1/2}(s)\, ds$$

$$\geq 2c_- \min_{I_\varepsilon} \eta - C|c - c_\varepsilon| - C\varepsilon^{(q+3)/2q}.$$

A similar inequality holds in the second case, with c_+ in place of c_-. Hence, by (4.12), (4.58), and by (4.65),

$$2c_W\eta(t_0) + C\varepsilon \geq J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) + J_\varepsilon^{(1)}(v_\varepsilon; I_\varepsilon)$$

$$\geq 2c_W \min_{\Gamma_\varepsilon} \eta + 2\min\{c_-, c_+\} \min_{I_\varepsilon} \eta - C_1\varepsilon$$

$$\geq 2c_W(\eta(t_0) - \epsilon_1) + 2\min\{c_-, c_+\} \min_{I_0} \eta - C_1\varepsilon,$$

which gives

$$C_2\varepsilon \geq 2\min\{c_-, c_+\} \min_{I_0} \eta - 2c_W\epsilon_1,$$

which is a contradiction provided $0 < \epsilon_1 < \frac{1}{2}\min\{c_-, c_+\} \min_{I_0} \eta/c_W$ and ε is sufficiently small.

This proves that Γ_ε is the only maximal subinterval of O_ε that intersects J_0. In view of (4.51) it follows that v_ε takes the value $a_\varepsilon + \varepsilon^k$ on its left endpoint of Γ_ε and $b_\varepsilon - \varepsilon^k$ on the right endpoint. At this point we have established parts (i) and (ii) of our theorem.

Step 5: We claim that $v_\varepsilon < c - \rho$ in $[-T + c(n_1)\varepsilon^{\theta_1}, -T + r_0]$. We first consider the case where $n_1 > 1$ in (4.2). Suppose the claim does not hold, and let t_5^ε be the last time in $[-T + \varepsilon^{\theta_1}, -T + r_0]$ such that $v_\varepsilon(t_5^\varepsilon) = c - \rho$. Note that t_5^ε exists in view of (4.51). We claim that

$$|t_3^\varepsilon - t_0| \leq C\varepsilon|\log\varepsilon| + C_1(T + t_5^\varepsilon)^{n_1}, \tag{4.66}$$

where we recall that t_3^ε and t_4^ε are the first time and last time in Γ_ε that v_ε equals c_ε. If $t_3^\varepsilon \leq t_0 \leq t_4^\varepsilon$, then this follows from (4.64). Assume next that $t_0 < t_3^\varepsilon$. Then from (4.5),

$$0 = \int_I (v_\varepsilon - v_0)\eta\, dt = \int_{-T}^{t_0} (v_\varepsilon - a)\eta\, dt + \int_{t_0}^{t_3^\varepsilon} (v_\varepsilon - b)\eta\, dt + \int_{t_3^\varepsilon}^T (v_\varepsilon - b)\eta\, dt. \tag{4.67}$$

36

By (4.12),

$$0 < \frac{\eta(t_0)}{2}(b - c_\varepsilon)(t_3^\varepsilon - t_0) \le \int_{t_0}^{t_3^\varepsilon} (b - v_\varepsilon)\eta \, dt \tag{4.68}$$

$$= \int_{-T}^{t_0} (v_\varepsilon - a)\eta \, dt + \int_{t_3^\varepsilon}^{T} (v_\varepsilon - b)\eta \, dt.$$

We now estimate the two terms on the right-hand side of (4.68). By (4.40) and (4.44),

$$\int_{t_3^\varepsilon}^{T} (v_\varepsilon - b)\eta \, dt \le |b_\varepsilon - b| 2T \max \eta \le C\varepsilon^{1/q}.$$

We decompose the interval $[-T, t_0]$ as follows

$$[-T, t_0] = [-T, t_5^\varepsilon] \cup [t_5^\varepsilon, -T + r_0] \cup [-T + r_0, t_0] \setminus \Gamma_\varepsilon) \cup ([-T + r_0, t_0] \cap \Gamma_\varepsilon),$$

and estimate the integrals over each of these subintervals. By (4.2), (4.40), and (4.44),

$$\int_{-T}^{t_5^\varepsilon} (v_\varepsilon - a)\eta \, dt \le (b_\varepsilon - a)d_2 \int_{-T}^{t_5^\varepsilon} (T + t)^{n_1 - 1} \, dt \le 2(b - a)d_2(T + t_5^\varepsilon)^{n_1}.$$

Let $Q_\varepsilon := [t_5^\varepsilon, -T + r_0] \cap O_\varepsilon$. Since $v_\varepsilon(t_5^\varepsilon) = c - \rho$, we have that $t_5^\varepsilon \in Q_\varepsilon$. Moreover, $v_\varepsilon < c - \rho$ in $(t_5^\varepsilon, -T + r_0]$. Reasoning as in Corollary 4.14, we get that

$$\mathcal{L}^1(Q_\varepsilon) \le C\varepsilon |\log \varepsilon|, \tag{4.69}$$

and so, by (4.1) and (4.40),

$$\int_{Q_\varepsilon} (v_\varepsilon - a)\eta \, dt \le C\varepsilon |\log \varepsilon|.$$

On the other hand, since $v_\varepsilon \le a_\varepsilon + \varepsilon^k$ in $[t_5^\varepsilon, -T + r_0] \setminus Q_\varepsilon$, by Lemma (4.40) and (4.42),

$$\int_{[t_5^\varepsilon, -T + r_0] \setminus Q_\varepsilon} (v_\varepsilon - a)\eta \, dt \le |a_\varepsilon + \varepsilon^k - a|d_2 \int_{-T}^{-T + r_0} (T + t)^{n_1 - 1} \, dt \le Cr_0^{n_1} \varepsilon^{1/q}.$$

Since the set O_ε intersects the interval J_0 only in Γ_ε by Step 3, and as $t_0 < t_3^\varepsilon$, we have that $v_\varepsilon \le a_\varepsilon + \varepsilon^k$ in $[-T + r_0, t_0] \setminus \Gamma_\varepsilon$. Hence, by (4.40) and (4.42),

$$\int_{[-T + r_0, t_0] \setminus \Gamma_\varepsilon} (v_\varepsilon - a)\eta \, dt \le |a_\varepsilon + \varepsilon^k - a| 2T \max \eta \le C\varepsilon^{1/q}.$$

Again by Step 3, $[-T + r_0, t_0] \cap \Gamma_\varepsilon = [t_0 - r_{\epsilon_1}, t_0] \cap \Gamma_\varepsilon$. Hence, by (4.40) and (4.62),

$$\int_{[t_0 - r_{\epsilon_1}, t_0] \cap \Gamma_\varepsilon} (v_\varepsilon - a)\eta \, dt \le C\varepsilon |\log \varepsilon|. \tag{4.70}$$

Combining the inequalities (4.68)–(4.70) gives

$$\frac{\eta(t_0)}{2}(b - c_\varepsilon)(t_3^\varepsilon - t_0) \le C\varepsilon |\log \varepsilon| + 2(b - a)d_2(T + t_5^\varepsilon)^{n_1},$$

which implies (4.66) in the case $t_0 < t_3^\varepsilon$.

It remains to prove (4.66) in the case $t_4^\varepsilon < t_0$. Then (4.67) should be replaced by

$$0 = \int_{-T}^{T} (v_\varepsilon - v_0)\eta \, dt = \int_{-T}^{t_4^\varepsilon} (v_\varepsilon - a)\eta \, dt + \int_{t_4^\varepsilon}^{t_0} (v_\varepsilon - a)\eta \, dt + \int_{t_0}^{T} (v_\varepsilon - b)\eta \, dt$$

37

and by

$$0 < \frac{\eta(t_0)}{2}(c_\varepsilon - a)(t_0 - t_4^\varepsilon) \le \int_{t_4^\varepsilon}^{t_0}(v_\varepsilon - a)\eta \, dt \le \int_{t_0}^T (b - v_\varepsilon)\eta \, dt + \int_{-T}^{t_4^\varepsilon}(a - v_\varepsilon)\eta \, dt.$$

By (4.40) and (4.42),

$$\int_{-T}^{t_4^\varepsilon}(a - v_\varepsilon)\eta \, dt \le |a - a_\varepsilon|2T \le C\varepsilon^{1/q}.$$

The integral $\int_{t_0}^T (b - v_\varepsilon)\eta \, dt$ can be estimated as in the case $t_0 < t_3^\varepsilon$. We omit the details. Hence, we have shown that (4.66) holds in all cases.

Since $t_3^\varepsilon \in \Gamma_\varepsilon$ and by (4.62), $\mathcal{L}^1(\Gamma_\varepsilon) \le C\varepsilon|\log \varepsilon|$, it follows that for any $t \in \Gamma_\varepsilon$,

$$|t - t_0| \le |t - t_3^\varepsilon| + |t_3^\varepsilon - t_0| \le C\varepsilon|\log \varepsilon| + C_1(T + t_5^\varepsilon)^{n_1}.$$

In turn, by the mean value theorem

$$\eta(t) = \eta(t_0) + \eta'(\theta)(t - t_0) \ge \eta(t_0) - M_0|t - t_0|$$
$$\ge \eta(t_0) - C\varepsilon|\log \varepsilon| - M_0 C_1(T + t_5^\varepsilon)^{n_1},$$

where we recall that $M_0 = \max|\eta'| + 1$. Hence, also by (4.65) we get

$$J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) \ge 2c_W \min_{\Gamma_\varepsilon} \eta - C\varepsilon^{(q+3)/2q} \ge 2c_W\eta(t_0) - C\varepsilon|\log \varepsilon| - C_2(T + t_5^\varepsilon)^{n_1}.$$

On the other hand, since $v_\varepsilon(t_5^\varepsilon) = c - \rho$, there exists a maximal subinterval S_ε of Q_ε that contains t_5^ε. Since $v_\varepsilon \le c - \rho$ in $[t_5^\varepsilon, -T + 2r_0]$, $v_\varepsilon(S_\varepsilon) \supset [a_\varepsilon + \varepsilon^k, c - \rho]$, and so reasoning as in (4.63), by (4.2), which can be applied since $2r_0 < t^*$ by (4.13) and (4.69) holds,

$$J_\varepsilon^{(1)}(v_\varepsilon; S_\varepsilon) \ge 2 \min_{S_\varepsilon} \eta \int_{a_\varepsilon + \varepsilon^k}^{c-\rho} W^{1/2}(s) \, ds$$

$$\ge 2d_1(T + t_5^\varepsilon)^{n_1-1} \int_{a+\rho}^{c-\rho} W^{1/2}(s) \, ds,$$

for $\varepsilon > 0$ small enough. Combining these last two estimates, it follows from (4.58) that

$$2c_W\eta(t_0) + C\varepsilon \ge J_\varepsilon^{(1)}(v_\varepsilon; \Gamma_\varepsilon) + J_\varepsilon^{(1)}(v_\varepsilon; S_\varepsilon) \ge 2c_W\eta(t_0) - C\varepsilon|\log \varepsilon| - C_2(T + t_5^\varepsilon)^{n_1}$$

$$+ 2d_1(T + t_5^\varepsilon)^{n_1-1} \int_{a+\rho}^{c-\rho} W^{1/2}(s) \, ds,$$

which gives

$$C\varepsilon|\log \varepsilon| \ge (T + t_5^\varepsilon)^{n_1-1}\left(2d_1 \int_{a+\rho}^{c-\rho} W^{1/2}(s) \, ds - C_2(T + t_5^\varepsilon)\right).$$

Since $-T + \varepsilon^{\theta_1} \le t_5^\varepsilon \le -T + r_0$, by taking

$$0 < r_0 < \frac{2d_1}{C_2} \int_{a+\rho}^{c-\rho} W^{1/2}(s) \, ds,$$

we get a contradiction, since $\theta_1(n_1 - 1) < 1$ by (4.48).

Finally we consider the case where $n_1 = 1$. In this case we can use energy estimates, as in Step 4, the fact that $\eta \ge C > 0$ on $[-T, -T + r_0]$, and Lemma 4.12 to show that $v_\varepsilon(t) < a_\varepsilon + \varepsilon^k$ on the interval $[-T, -T + r_0]$. We omit the details.

Step 6: Finally, we prove the last claim in our theorem. We write $\Gamma_\varepsilon = [T_1^\varepsilon, T_2^\varepsilon]$. In the case $n_1 = 1$ we are already done, so we only need to consider the case $n_1 > 1$. In view of Step 5 we can use the barrier method in Lemma 4.13 to show that for $t \in [-T + \varepsilon^{\theta_1}, T_1^\varepsilon]$

$$|v_\varepsilon(t) - a_\varepsilon| \le C e^{-\mu \varepsilon^{-1} d(t, \{-T + \varepsilon^{\theta_1}, T_1^\varepsilon\})}$$

This clearly implies that $v_\varepsilon(t) \in [a_\varepsilon, a_\varepsilon + \varepsilon^k)$ for all $t \in (-T + \varepsilon^{\theta_1} + 2k\mu^{-1}\varepsilon|\log\varepsilon|, T_1^\varepsilon)$. Using (4.2) we then estimate the η measure of the remaining set as follows:

$$\int_{-T}^{-T + \varepsilon^{\theta_1} + 2k\mu^{-1}\varepsilon|\log\varepsilon|} \eta \, dt \le \frac{d_2}{n_1}(\varepsilon^{\theta_1} + C\varepsilon|\log\varepsilon|)^{n_1} \le C\varepsilon^{n_1\theta_1}$$

Since $n_1\theta_1 > 1$ by (4.48), then we have the desired estimate. Thus the result holds to the left of T_1^ε. We can use the same argument to the right of T_2^ε to obtain the desired result. $\qquad\square$

4.3 Second-Order Γ-limit

In this subsection we prove the $\liminf$ counterpart of Theorem 4.7.

Theorem 4.17. *Assume that W satisfies (2.4)-(2.7) and that η satisfies (4.1)-(4.4) and let v_0 and v_ε be given in Theorems 4.6 and 4.8 respectively. Then*

$$\liminf_{\varepsilon \to 0^+} \frac{G_\varepsilon^{(1)}(v_\varepsilon) - 2c_W\eta(t_0)}{\varepsilon} \ge 2\eta'(t_0)(\tau_0 c_W + c_{\text{sym}})$$
$$+ \begin{cases} \frac{\lambda_0^2}{2W''(a)} \int_I \eta \, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases} \tag{4.71}$$

Note that Theorems 4.7 and 4.17 together provide a second-order asymptotic development by Γ-convergence for the functionals J_ε defined in (4.15). To prove Theorem 4.17 it is convenient to rescale the functionals G_ε. We define

$$H_\varepsilon(w) := \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (W(w(s)) + (w'(s))^2)\eta_\varepsilon(s) \, ds \tag{4.72}$$

for all $w \in H^1_{\eta_\varepsilon}((A\varepsilon^{-1}, B\varepsilon^{-1}))$ such that

$$\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} |w(s) - \text{sgn}_{a,b}(s)|\eta_\varepsilon(s) \, ds \le \frac{\delta}{\varepsilon},$$
$$\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w(s) - \text{sgn}_{a,b}(s))\eta_\varepsilon(s) \, ds = 0, \tag{4.73}$$

where $A = -T - t_0$, $B = T - t_0$ and

$$\eta_\varepsilon(s) := \eta(t_0 + \varepsilon s). \tag{4.74}$$

Observe that we have shifted our variables so that t_0 moves to zero and then scaled by ε^{-1}. Here we study the behavior of minimizers w_ε of H_ε, which are precisely rescaled versions of the minimizers v_ε of the functionals J_ε studied in the previous section. First we prove a bound on the locations where $w_\varepsilon = c_\varepsilon$, in the region close to $t = 0$.

Lemma 4.18. *Let w_ε be a minimizer of H_ε, and let $\tau_\varepsilon \in B(0, r_1\varepsilon^{-1})$ satisfy $w_\varepsilon(\tau_\varepsilon) = c_\varepsilon$, with r_1 as in Theorem 4.6 (i). Then we have that*

$$|\tau_\varepsilon| \le C$$

for all $\varepsilon > 0$ sufficiently small and for some constant $C > 0$ independent of ε.

39

Proof. This proof essentially combines the mass constraint with the exponential decay to obtain the desired bounds. In this proof C will denote a generic constant independent of ε.

Let s_1^ε be the first time in $[-r_1\varepsilon^{-1}, r_1\varepsilon^{-1}]$ so that $w_\varepsilon(s_1^\varepsilon) = c - \rho$, and s_4^ε be the last time in $[-r_1\varepsilon^{-1}, r_1\varepsilon^{-1}]$ so that $w_\varepsilon(s_4^\varepsilon) = c + \rho$. Then let s_2^ε and s_3^ε be the first and last times in $[-r_1\varepsilon^{-1}, r_1\varepsilon^{-1}]$ where w_ε takes the value c_ε. We note that such points exist by Theorem 4.11 (i). Furthermore, by Theorem 4.11 (ii) we know that $s_3^\varepsilon - s_2^\varepsilon \leq C$ and that $-r_1\varepsilon^{-1} < s_1^\varepsilon < s_2^\varepsilon \leq s_3^\varepsilon < s_4^\varepsilon < r_1\varepsilon^{-1}$. Furthermore, using the same argument from the proof of (4.40) we know that $w_\varepsilon([s_1^\varepsilon, s_2^\varepsilon]) = [c-\rho, c_\varepsilon]$, and that $w_\varepsilon([s_3^\varepsilon, s_4^\varepsilon]) = [c_\varepsilon, c + \rho]$. We can then estimate the following:

$$(s_2^\varepsilon - s_1^\varepsilon) \inf_{B(t_0, r_1)} \eta \inf_{(c-\rho, c+\rho)} W \leq \int_{s_1^\varepsilon}^{s_2^\varepsilon} W(w_\varepsilon)\eta_\varepsilon \, ds \leq C.$$

This, along with a similar estimate for $s_4^\varepsilon - s_3^\varepsilon$, then implies that $s_4^\varepsilon - s_1^\varepsilon \leq C$. Thus if we can prove that the s_1^ε are bounded above and that the s_4^ε are bounded below then we are done.

Suppose, for the sake of contradiction that the s_1^ε are not bounded above. By taking a subsequence as necessary we may assume that $s_1^\varepsilon \to \infty$.

By (4.40) and 4.13 we have the following bounds

$$0 < w_\varepsilon(s) - a_\varepsilon \leq 2(c - \rho - a_\varepsilon)e^{-\mu|s-s_1^\varepsilon|} \quad \text{for } s \in [-r_1\varepsilon^{-1}, s_1^\varepsilon], \tag{4.75}$$

$$0 < b_\varepsilon - w_\varepsilon(s) \leq 2(b_\varepsilon - c - \rho)e^{-\mu(s-s_4^\varepsilon)} \quad \text{for } s \in [s_4^\varepsilon, r_1\varepsilon^{-1}]. \tag{4.76}$$

By our mass constraint (4.73) we can write:

$$0 = \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds = \int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \tag{4.77}$$

$$+ \int_{s_1^\varepsilon}^{s_4^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds + \int_{s_4^\varepsilon}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds.$$

We will estimate these terms to obtain a contradiction. By (4.40) and the fact that $0 < s_4^\varepsilon - s_1^\varepsilon \leq C$ we have that

$$\left| \int_{s_1^\varepsilon}^{s_4^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| \leq C.$$

We can also calculate

$$\int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds$$

$$= \int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds + \int_{A\varepsilon^{-1}}^{s_1^\varepsilon} (a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds.$$

By (4.75) we have that

$$0 \leq \int_{-r_1\varepsilon^{-1}}^{s_1^\varepsilon} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds \leq 2(c - \rho - a_\varepsilon) \int_{-r_1\varepsilon^{-1}}^{s_1^\varepsilon} e^{-\mu|s-s_1^\varepsilon|} \, ds \leq C,$$

whereas by Theorem 4.11 (iii) and (4.40) we know that

$$\left| \int_{A\varepsilon^{-1}}^{-r_1\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon \, ds \right| \leq C\varepsilon^{k-1} + o(1).$$

Furthermore as $a_\varepsilon = a + O(\varepsilon^{1/q})$ by Theorem 4.10, we may estimate that

$$\left| \int_{A\varepsilon^{-1}}^{0} (a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon \, ds \right| \leq C\varepsilon^{\frac{1-q}{q}}.$$

40

A similar argument, and the fact that $0 < s_1^\varepsilon < s_4^\varepsilon$ shows that

$$\left| \int_{s_4^\varepsilon}^{B\varepsilon^{-1}} (w_\varepsilon - \text{sgn}_{a,b}) \eta_\varepsilon \, ds \right| \leq C.$$

Now as $s_1^\varepsilon \to \infty$ we then have that

$$\lim_{\varepsilon \to 0} \left| \int_0^{s_1^\varepsilon} (a_\varepsilon - \text{sgn}_{a,b}) \eta_\varepsilon \, ds \right| \geq \lim_{\varepsilon \to 0} \inf_{B(t_0, r_1)} \eta \left| \int_0^{s_1^\varepsilon} (a_\varepsilon - b) \, ds \right| = \infty. \tag{4.78}$$

Combining (4.77)–(4.78) gives

$$\lim_{\varepsilon \to 0} \left| \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - \text{sgn}_{a,b}) \eta_\varepsilon \, ds \right| = \infty.$$

This violates the mass constraint. Thus we must have that the s_1^ε are bounded above.

A similar argument shows that s_4^ε is bounded below. As $\tau_\varepsilon \in (s_1^\varepsilon, s_4^\varepsilon)$ and $s_4^\varepsilon - s_1^\varepsilon \leq C$, we then have that $|\tau_\varepsilon| \leq C$, which is the desired conclusion.

$\square$

We then prove that the functions w_ε necessarily converge.

Lemma 4.19. *Let w_ε be as in Lemma 4.18. Then (up to a subsequence, not relabeled) $\{w_\varepsilon\}$ converges in $H^1((-l,l))$ for every $l \in \mathbb{N}$ to the profile $w_0(s) := z(s - \tau_0)$, where τ_0 is determined by (4.18). Moreover, the family $\{w_\varepsilon'\}$ is bounded in $L^\infty((A\varepsilon^{-1}, B\varepsilon^{-1}))$.*

Proof. Throughout this proof we let w_ε be associated with its extension by constants outside of $[A\varepsilon^{-1}, B\varepsilon^{-1}]$. The fact the the family $\{w_\varepsilon'\}$ is uniformly bounded in $L^\infty(\mathbb{R})$ follows immediately from Lemma 4.16. Furthermore, we have that the w_ε are bounded in $L^\infty(\mathbb{R})$ by (4.40). After a diagonalization argument, this implies that for some $w_0 \in H^1_{\text{loc}}(\mathbb{R})$,

$$w_\varepsilon \rightharpoonup w_0 \text{ in } H^1_{\text{loc}}(\mathbb{R}). \tag{4.79}$$

By (4.34) and (4.41) we have that

$$\begin{cases} 2(w_\varepsilon' \eta_\varepsilon)' - W'(w_\varepsilon)\eta_\varepsilon = \varepsilon \lambda_\varepsilon \eta_\varepsilon & \text{on } (A\varepsilon^{-1}, B\varepsilon^{-1}), \\ w_\varepsilon'(A\varepsilon^{-1}) = w_\varepsilon'(B\varepsilon^{-1}) = 0. \end{cases}$$

Hence for every $\phi \in C_c^\infty(\mathbb{R})$ for ε small enough we find that

$$\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} 2w_\varepsilon' \eta_\varepsilon \phi' + W'(w_\varepsilon) \eta_\varepsilon \phi \, ds = -\int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} \varepsilon \lambda_\varepsilon \eta_\varepsilon \phi \, ds.$$

Letting $\varepsilon \to 0$ and using (4.74) and (4.79) gives

$$\int_{\mathbb{R}} 2w_0' \eta(t_0) \phi' + W'(w_0) \eta(t_0) \phi \, ds = 0,$$

which then shows that w_0 satisfies the differential equation

$$2w_0'' = W'(w_0). \tag{4.80}$$

Furthermore, by (4.40) we know that $a \leq w_0 \leq b$, which by (4.80) implies that $|w_0''| \leq C$. Also, by (4.32), and the fact that $H_\varepsilon(w_\varepsilon) = J_\varepsilon(v_\varepsilon)$, where v_ε is a minimizer of J_ε,

$$\eta(t_0) \int_{-l}^{l} (w_0')^2 + W(w_0) \, ds \leq \lim_{\varepsilon \to 0} \int_{-l}^{l} ((w_\varepsilon')^2 + W(w_\varepsilon)) \eta_\varepsilon \, ds \leq \lim_{\varepsilon \to 0} H_\varepsilon(w_\varepsilon) = 2c_W \eta(t_0)$$

41

for every $l \in \mathbb{N}$, and thus

$$\eta(t_0) \int_{\mathbb{R}} (w_0')^2 + W(w_0)\, ds \leq 2c_W \eta(t_0). \tag{4.81}$$

This combined with the fact that $|w_0''| \leq C$ (by (4.80)) implies that $\lim_{s \to \pm\infty} w_0'(s) = 0$. By then using (4.75) and (4.76) along with Lemma 4.18 we have that $\lim_{s \to -\infty} w_0(s) = a$, and that $\lim_{s \to \infty} w_0(s) = b$. Thus by integrating (4.80) we find that

$$(w_0')^2 = W(w_0). \tag{4.82}$$

We next claim that w_0 is increasing. Suppose not. Then by (4.82) there exists critical points $t_1 < t_2$ of w_0, with $w_0(t_1) = b$ and $w_0(t_2) = a$. This then implies, by Young's inequality, (4.81) and a change of variables that

$$6c_W \eta(t_0) \leq 2c_W \eta(t_0).$$

This is impossible. Thus w_0 is increasing. Moreover, by (4.43), (4.79), and Lemma 4.18, up to a subsequence, $\tau_\varepsilon \to \tau_0$ with $w_0(\tau_0) = c$. This then implies that $w_0(s) = z(s - \tau_0)$, where z is the solution of the Cauchy problem (1.17).

The only thing left to prove is that τ_0 is determined by equation (4.18). To this end, fix l large enough that $(s_1^\varepsilon, s_4^\varepsilon) \subset (-l, l)$ for all ε, where s_1^ε and s_4^ε are as in the proof of Lemma 4.18. Then by the mass constraint (4.73) we have that

$$0 = \int_{A\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds = \int_{-l}^{l} (w_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds$$

$$+ \int_{-r_1\varepsilon^{-1}}^{-l} (w_\varepsilon - a_\varepsilon + a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds + \int_{l}^{r_1\varepsilon^{-1}} (w_\varepsilon - b_\varepsilon + b_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds$$

$$+ \int_{A\varepsilon^{-1}}^{-r_1\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon + a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds + \int_{r_1\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - b_\varepsilon + b_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds.$$

By Lemma 4.13 we know that

$$\left| \int_{-r_1\varepsilon^{-1}}^{-l} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon\, ds \right| \leq 2|w_\varepsilon(-l) - a_\varepsilon| \int_0^\infty e^{-\mu s}\, ds,$$

$$\left| \int_{l}^{r_1\varepsilon^{-1}} (w_\varepsilon - b_\varepsilon)\eta_\varepsilon\, ds \right| \leq 2|w_\varepsilon(l) - b_\varepsilon| \int_0^\infty e^{-\mu s}\, ds.$$

By (4.40) we can write:

$$\int_{A\varepsilon^{-1}}^{-l} (a_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds = -\lambda_\varepsilon |\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q-1} \int_{-T}^{t_0} \eta\, dt + o(\varepsilon^{1/q-1}),$$

$$\int_{l}^{B\varepsilon^{-1}} (b_\varepsilon - \mathrm{sgn}_{a,b})\eta_\varepsilon\, ds = -\lambda_\varepsilon |\lambda_\varepsilon|^{1/q-1}(q/\ell)^{1/q}\varepsilon^{1/q-1} \int_{t_0}^{T} \eta\, dt + o(\varepsilon^{1/q-1}).$$

Furthermore by Theorem 4.11 along with (4.40) we have that

$$\left| \int_{A\varepsilon^{-1}}^{-r_1\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon\, ds \right| = o(1),$$

$$\left| \int_{r_1\varepsilon^{-1}}^{B\varepsilon^{-1}} (w_\varepsilon - a_\varepsilon)\eta_\varepsilon\, ds \right| = o(1).$$

42

Utilizing these estimates, and taking $\varepsilon \to 0$ we find that

$$0 = \eta(t_0) \int_{-l}^{l} w_0 - \mathrm{sgn}_{a,b} \, ds - \lambda_0 |\lambda_0|^{1/q-1} (q/\ell)^{1/q} \lim_{\varepsilon \to 0} \varepsilon^{1/q-1} \int_I \eta \, dt$$
$$+ O(|a - w_0(-l)|) + O(|b - w_0(l)|).$$

Taking l to infinity, and using (2.5) then implies that

$$\eta(t_0) \int_{\mathbb{R}} w_0 - \mathrm{sgn}_{a,b} \, ds = \begin{cases} \frac{\lambda_0}{W''(a)} \int_I \eta \, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases}$$

which then implies that τ_0 has the desired form. This completes the proof.

$\square$

Next we will use the previous lemmas to derive a second-order liminf inequality, which immediately implies Theorem 4.17.

Lemma 4.20. *Let $\{w_\varepsilon\}$ be minimizers of the functionals $\{H_\varepsilon\}$. Then we have the following:*

$$\liminf_{\varepsilon \to 0} \frac{H_\varepsilon(w_\varepsilon) - 2c_W \eta(t_0)}{\varepsilon} \geq 2\eta'(t_0)(\tau_0 c_W + c_{\mathrm{sym}}) \tag{4.83}$$

$$+ \begin{cases} \frac{\lambda_0^2}{2W''(a)} \int_I \eta(s) \, ds & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases}$$

where c_W, c_{sym}, τ_0, λ_0 are given by (1.10), (1.16), (4.18) and (4.35) respectively.

Proof. Fix k to be a large integer. By (4.75) and (4.76) and the fact that s_1^ε and s_4^ε are bounded we can find $l_\varepsilon \in (s_2^\varepsilon, r_1 \varepsilon^{-1})$ such that $b_\varepsilon - w_\varepsilon(l_\varepsilon) < \varepsilon^k$ and $w_\varepsilon(-l_\varepsilon) - a_\varepsilon < \varepsilon^k$ for $\varepsilon > 0$ sufficiently small. Recall that by Corollary 4.14 we can take

$$l_\varepsilon < C|\log \varepsilon|. \tag{4.84}$$

By (4.72) we can compute

$$\frac{H_\varepsilon(w_\varepsilon) - 2c_W \eta(t_0)}{\varepsilon}$$

$$= \varepsilon^{-1} \int_{-l_\varepsilon}^{l_\varepsilon} (W^{1/2}(w_\varepsilon) - w_\varepsilon')^2 \eta_\varepsilon \, ds + 2\varepsilon^{-1} \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon) w_\varepsilon' (\eta_\varepsilon - \eta(t_0)) \, ds$$

$$+ \varepsilon^{-1} \int_{[A\varepsilon^{-1}, B\varepsilon^{-1}] \setminus (-l_\varepsilon, l_\varepsilon)} \left(W(w_\varepsilon) + (w_\varepsilon')^2 \right) \eta_\varepsilon \, ds + \varepsilon^{-1} 2\eta(t_0) \left(\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon) w_\varepsilon' \, ds - c_W \right)$$

$$\geq 2\varepsilon^{-1} \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon) w_\varepsilon' (\eta_\varepsilon - \eta(t_0)) \, ds$$

$$+ \varepsilon^{-1} \int_{[A\varepsilon^{-1}, B\varepsilon^{-1}] \setminus (-l_\varepsilon, l_\varepsilon)} W(w_\varepsilon) \eta_\varepsilon \, ds + \varepsilon^{-1} 2\eta(t_0) \left(\int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon) w_\varepsilon' \, ds - c_W \right).$$

We will examine the individual terms. The last term goes to zero as

$$\varepsilon^{-1} \left| \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon) w_\varepsilon' \, ds - c_W \right| \leq \varepsilon^{-1} \left| \int_{w_\varepsilon(-l_\varepsilon)}^{w_\varepsilon(l_\varepsilon)} W^{1/2}(r) \, dr - \int_a^b W^{1/2}(r) \, dr \right|$$

$$\leq \varepsilon^{-1} \left| \int_{a_\varepsilon}^{b_\varepsilon} W^{1/2}(r) \, dr - \int_a^b W^{1/2}(r) \, dr \right| + C\varepsilon^{k-1}$$

$$\leq C\varepsilon^{-1} \int_0^{\varepsilon^{1/q}} t^{\frac{1+q}{2}} \, dt + C\varepsilon^{k-1} = o(1), \tag{4.85}$$

43

where we have used (1.10), (2.9) and (4.40).

For $s \in [l_\varepsilon, B\varepsilon^{-1}] \cap \{w_\varepsilon \geq b_\varepsilon - \varepsilon^k\}$ by the mean value theorem we can write

$$W(w_\varepsilon(s)) = W(b_\varepsilon) + W'(\zeta_\varepsilon)(w_\varepsilon(s) - b_\varepsilon),$$

where $\zeta_\varepsilon \in [w_\varepsilon(s), b_\varepsilon]$. By (2.10) and (4.40) for such s we then have that

$$
\begin{aligned}
|W'(\zeta_\varepsilon)|(b_\varepsilon - w_\varepsilon(s)) &\leq C|\zeta_\varepsilon - b|^q (b_\varepsilon - w_\varepsilon(s)) \\
&\leq C(|\zeta_\varepsilon - b_\varepsilon|^q + |b_\varepsilon - b|^q)(b_\varepsilon - w_\varepsilon(s)) \\
&\leq C(\varepsilon^{qk} + \varepsilon^q)\varepsilon^k \leq C\varepsilon^{qk}.
\end{aligned}
$$

Thus we can write, after applying (2.9), part (iii) of Theorem 4.11, (4.40), and (4.84),

$$
\begin{aligned}
\varepsilon^{-1} \int_{l_\varepsilon}^{B\varepsilon^{-1}} W(w_\varepsilon)\eta_\varepsilon \, ds &\geq \varepsilon^{-1} W(b_\varepsilon) \left(\int_{l_\varepsilon}^{B\varepsilon^{-1}} \eta_\varepsilon \, ds + o(1) \right) + O(\varepsilon^{qk-2}) \\
&= \varepsilon^{-1} \left(\frac{\ell}{q(1+q)} |b_\varepsilon - b|^{1+q} + o(|b_\varepsilon - b|^{1+q}) \right) \left(\varepsilon^{-1} \int_{t_0}^{T} \eta \, dt + O(|\log \varepsilon|) \right) + O(\varepsilon^{qk-2}) \\
&= \left(\frac{q^{1/q} |\lambda_\varepsilon|^{1+1/q}}{(1+q)\ell^{1/q}} + o(1) \right) \left(\varepsilon^{1/q-1} \int_{t_0}^{T} \eta \, dt + O(\varepsilon^{1/q}|\log \varepsilon|) \right) + O(\varepsilon^{qk-2}).
\end{aligned}
$$

An analogous bound will hold on the interval $[A\varepsilon^{-1}, -l_\varepsilon]$. Hence

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \int_{[A\varepsilon^{-1}, B\varepsilon^{-1}] \setminus (-l_\varepsilon, l_\varepsilon)} W(w_\varepsilon)\eta_\varepsilon \, ds = \begin{cases} \frac{\lambda_0^2}{2W''(a)} \int_I \eta \, dt & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases} \tag{4.86}$$

For the first term we use assumption (4.1) to estimate $\eta_\varepsilon(s) - \eta(t_0) = \varepsilon s \eta'(t_0) + O(\varepsilon^{1+\beta}|s|^{1+\beta})$. Using (4.40) and 4.19 and (4.84) we have that

$$\left| \varepsilon^{-1} \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' O(\varepsilon^{1+\beta}|s|^{1+\beta}) \, ds \right| \leq C\varepsilon^\beta |\log \varepsilon|^{2+\beta} \to 0.$$

Thus we find that:

$$\lim_{\varepsilon \to 0} 2\varepsilon^{-1} \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon'(\eta_\varepsilon - \eta(t_0)) \, ds = 2\eta'(t_0) \lim_{\varepsilon \to 0} \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s \, ds.$$

Now for any fixed l by (4.79) and the fact that $w_0(s) = z(s - \tau_0)$, we can write

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \int_{-l}^{l} W^{1/2}(w_\varepsilon)w_\varepsilon' s \, ds &= \int_{-l}^{l} W^{1/2}(w_0)w_0' s \, ds \\
&= \int_{-l-\tau_0}^{l-\tau_0} W^{1/2}(z(t))z'(t)(t + \tau_0) \, dt \\
&= \tau_0 \Phi(z(l - \tau_0)) - \tau_0 \Phi(z(-l - \tau_0)) + \int_{-l-\tau_0}^{l-\tau_0} W^{1/2}(z(t))z'(t)t \, dt,
\end{aligned}
$$

where we recall that $\Phi(s) = \int_a^s W^{1/2}(r) \, dr$. Furthermore we can establish the following bound using (2.9), (4.76) and Lemma 4.19:

$$
\begin{aligned}
\left| \int_{l}^{l_\varepsilon} W^{1/2}(w_\varepsilon)w_\varepsilon' s \, ds \right| &\leq C \int_{l}^{l_\varepsilon} |b - w_\varepsilon|^{\frac{1+q}{2}} s \, ds \\
&\leq C(|b_\varepsilon - c - \rho|^{\frac{1+q}{2}} + |b_\varepsilon - b|^{\frac{1+q}{2}}) \int_{l}^{\infty} e^{-\frac{1+q}{2}\mu(s - s_4^\varepsilon)} s \, ds,
\end{aligned}
$$

44

provided $l > s_4^\varepsilon$. Thus we can write

$$\lim_{\varepsilon \to 0} \int_{-l_\varepsilon}^{l_\varepsilon} W^{1/2}(w_\varepsilon) w_\varepsilon' s \, ds = \tau_0 \Phi(z(l - \tau_0)) - \tau_0 \Phi(z(-l - \tau_0))$$

$$+ \int_{-l-\tau_0}^{l-\tau_0} W^{1/2}(z(s)) z'(s) s \, ds + O(l e^{-\frac{1+q}{2}\mu l}).$$

Taking l to ∞, combined with (4.85) and (4.86) gives the desired claim, namely, (4.83). $\qquad\square$

We now give the proof of Theorem 4.17.

Proof of Theorem 4.17. By changing variables it is immediate that $H(w_\varepsilon) = G_\varepsilon^{(1)}(v_\varepsilon)$. Lemma 4.20 then immediately implies (4.71). This concludes the proof. $\qquad\square$

5 Proofs of Main Theorems

With our tools in hand, we now can approach the problem of proving Theorems 1.1 and 1.2. We begin by proving the Γ-$\liminf$ inequalities from Theorems 1.1 and 1.2. Precisely, we prove the following theorem:

Theorem 5.1. *Assume that Ω satisfies (2.1), m satisfies (2.2), $\mathcal{I}_\Omega$ satisfies (2.3) and that W satisfies (2.4)-(2.7). Let $\{u_n\} \subset L^1(\Omega)$ converge to u in $L^1(\Omega)$. Then*

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq \mathcal{F}^{(2)}(u),$$

where $\mathcal{F}_\varepsilon^{(2)}$ is defined by (1.3) and $\mathcal{F}^{(2)}$ is defined in Theorems 1.1 and 1.2.

Proof. If $\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) = \infty$ then there is nothing to prove. Thus, passing to a subsequence, if necessary, we can assume that

$$\sup_\varepsilon \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) < \infty. \tag{5.1}$$

By standard results on compactness and lower semicontinuity for the Cahn–Hilliard functional $\mathcal{F}_\varepsilon^{(1)}$ (see, e.g., [43] and the references therein), it follows from (1.3), (1.4) and (5.1) that u must be a minimizer of $\mathcal{F}^{(1)}$. This implies that the set $E := \{u = a\}$ is a minimizer of (1.11), and its mean curvature is given by (1.14). Using $\mathcal{I}_\Omega^*$ from Proposition 3.1 as in Section 3, we then have that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_I (W(f_{u_\varepsilon}) + \varepsilon^2 (f'_{u_\varepsilon})^2) \mathcal{I}_\Omega^*(V_\Omega) \, dt, \quad m = \int_\Omega u_\varepsilon \, dx = \int_I f_{u_\varepsilon} \mathcal{I}_\Omega^*(V_\Omega) \, dt.$$

where V_Ω and f_u are defined in Section 3 (see (3.9), (3.16) and Remark 3.11). We then set $\eta := \mathcal{I}_\Omega^*(V_\Omega)$. This η will satisfy all of the assumptions in Section 4. Indeed, since $V_\Omega > 0$ in $(-T, \infty)$ and $V_\Omega(-T) = 0$, by (3.5) and (3.9), $V_\Omega(t) = [C_0/n(t + T)]^n$ near $-T$, and so $\eta = C_0^n[\frac{1}{n}(t + T)]^{n-1}$, which shows that (4.2) and (4.4) hold for t close to $-T$. On the other hand, since $V_\Omega(t) = 1 - V_\Omega(-t)$ (by (3.2) and (3.9)), for t close to T we have that $\eta(t) = C_0^n[\frac{1}{n}(T - t)]^{n-1}$ and thus (4.3) and (4.4) hold close to T. Since $\mathcal{I}_\Omega^* \in C_{loc}^{1,\beta}(0, 1)$, by (3.9) we have that $V_\Omega \in C_{loc}^{2,\beta}(I)$, and in turn $\eta \in C_{loc}^{1,\beta}(I)$. Thus (4.1) is satisfied. Finally, since $\mathcal{I}_\Omega^* > 0$ in $(0, 1)$ we have by (3.3) that $\eta > 0$ in I, and thus (4.4) holds on any compact subset of I by uniform continuity.

Next observe that since $u \in BV(\Omega, \{a, b\})$ and (1.2) holds, by Lemma 3.3 we have that f_u only takes the values a and b and $\int_I f_u \eta \, dt = \int_\Omega u \, dx = m$. Since f_u is increasing, this implies that $f_u(t) = \text{sgn}_{a,b}(t - t_0)$ for some $t_0 \in I$ and all $t \in I$. It follows from Theorem 4.6 that f_u is a local minimizer of the functional $G^{(1)}$ defined in (4.8). Moreover, by Lemma 3.4 we have that $u_\varepsilon \to u$ in

45

$L^1(\Omega)$ implies that $f_{u_\varepsilon} \to f_u$ in $L^1_\eta(I)$. Hence, $\|f_{u_\varepsilon} - f_u\|_{L^1_\eta} \leq \delta$ for all ε sufficiently small, where $\delta > 0$ is the number given in Theorem 4.6 (with $v_0 = f_u$). In turn choosing v_ε to be minimizers of the function J_ε defined in (4.15), by Corollary 3.12 we have that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \geq G_\varepsilon(f_{u_\varepsilon}) = J_\varepsilon(f_{u_\varepsilon}) \geq J_\varepsilon(v_\varepsilon). \tag{5.2}$$

Since $\int_I f_u \eta \, dt = m$, it follows from the fact that (see (2.1) and Lemma 3.3)

$$1 = \mathcal{L}^n(\Omega) = \int_I \eta \, dt \tag{5.3}$$

and (3.9) that

$$r_m = \frac{b-m}{b-a} = \mathcal{L}^n(\{u = a\}) = \int_{-T}^{t_0} \eta \, dt = \int_{-T}^{t_0} \frac{d}{dt} V_\Omega \, dt = V_\Omega(t_0). \tag{5.4}$$

In turn, by (3.4),

$$\eta(t_0) = \mathcal{I}_\Omega^*(r_m) = \mathcal{I}_\Omega(r_m) = \mathrm{P}(\{u = a\}; \Omega),$$

which shows that $\mathcal{F}^{(1)}(u) = G^{(1)}(f_u)$. Hence by (5.2) we have that

$$\mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) = \frac{\mathcal{F}_\varepsilon^{(1)}(u_\varepsilon) - \mathcal{F}^{(1)}(u)}{\varepsilon} \geq \frac{J_\varepsilon^{(1)}(v_\varepsilon) - J^{(1)}(f_u)}{\varepsilon} = J_\varepsilon^{(2)}(v_\varepsilon).$$

By applying Theorem 4.20 we thus have that

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq 2\eta'(t_0)(\tau_0 c_W + c_{\mathrm{sym}}) + \begin{cases} \frac{\lambda_0^2}{2W''(a)} & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases} \tag{5.5}$$

where we have used (5.3). By (3.9) we have that $\eta'(t) = (\mathcal{I}_\Omega^*)'(V_\Omega(t))\eta(t)$, and so by (1.14), (3.4) and (5.4),

$$\eta'(t_0) = \mathcal{I}_\Omega'(r_m)\mathcal{I}_\Omega(r_m) = (n-1)\kappa_u \, \mathrm{P}(\{u = a\}; \Omega).$$

In turn by (1.20) and (4.19),

$$\lambda_0 = \frac{2(n-1)c_W}{(b-a)}\kappa_u = \Lambda_u, \tag{5.6}$$

and so by (4.18) the number τ_0 coincides with the number τ_u in (1.18). Combining (5.5)-(5.6) gives

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \geq 2(\tau_u c_W + c_{\mathrm{sym}})(n-1)\kappa_u \, \mathrm{P}(\{u = a\}; \Omega) + \begin{cases} \frac{\Lambda_u^2}{2W''(a)} & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases}$$

This completes the proof.

$\square$

Remark 5.2. *We note that the assumption that Ω is C^2 is not truly necessary to prove the previous theorem. A Lipschitz domain satisfying (2.3) would actually be sufficient. However, for a Lipschitz domain it is not clear that (2.3) need hold, and furthermore for the Γ-lim sup inequality we directly utilize the fact that $\Omega \in C^2$.*

Now we prove the corresponding Γ-lim sup inequality.

Theorem 5.3. *Assume that Ω satisfies (2.1), m satisfies (2.2) and that W satisfies (2.4)-(2.7). Let $u \in L^1(\Omega)$. Then there exists a sequence $\{u_n\} \subset L^1(\Omega)$ that converges to u in $L^1(\Omega)$ such that*

$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^{(2)}(u_\varepsilon) \leq \mathcal{F}^{(2)}(u),$$

where $\mathcal{F}_\varepsilon^{(2)}$ is defined by (1.3) and $\mathcal{F}^{(2)}$ is defined in Theorems 1.1 and 1.2.

We note that Theorems 5.1 and 5.3 together establish Theorems 1.1 and 1.2. To prove this theorem, we primarily utilize the approach from previous works (see [47, 57]), while leveraging the insight regarding skew in transition layers that we have developed over the course of this work. We begin with the following lemma:

Lemma 5.4. *Suppose that $E \subset \Omega$ is a volume-constrained perimeter minimizer in Ω. Define the function $\eta(s) := \mathcal{H}^{n-1}(\{d_E(x) = s\})$, where d_E is the signed distance function (see (2.13)). Then η is twice differentiable at zero and satisfies*

$$\eta(0) = \mathrm{P}(E; \Omega), \tag{5.7}$$

$$\eta'(0) = (n-1)\kappa_E \, \mathrm{P}(E; \Omega), \tag{5.8}$$

where κ_E is the mean curvature of E. Furthermore, the function η is bounded.

Proof. By classical results (see [26]), we have that (5.7) holds. To prove that η is twice differentiable at 0, and that (5.8) holds we are primarily concerned with possible interactions with $\partial\Omega$. By [33] we know that ∂E is a $C^{2,\alpha}$ surface, that intersects $\partial\Omega$ orthogonally. By appropriately reflecting E outside of Ω we may assume that E is a $C^{2,\alpha}$ set in $\mathbb{R}^n$. Since ∂E is of class $C^{2,\alpha}$ for every $x \in \partial E$ there exist a ball $B(x, r_x)$, with local coordinates $y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that x corresponds to $y = 0$ and a function g of class $C^{2,\alpha}(\mathbb{R}^{n-1})$ such that $g(0) = 0$, $\frac{\partial g}{\partial y_i}(0) = 0$ for all $i = 1, \ldots, n-1$, and

$$E \cap B(x, r_x) = \{y \in B(0, r_x) : y_n < g(y')\},$$
$$\partial E \cap B(x, r_x) = \{y \in B(0, r_x) : y_n = g(y')\}.$$

In what follows we use local coordinates and we set $y'' := (y_2, \ldots, y_{n-2})$. In particular, since ∂E meets the boundary of Ω transversally, if $x \in \partial E \cap \partial\Omega$, by a rotation and by taking r_x smaller, we can assume that

$$\Omega \cap B(x, r_x) = \{y \in B(0, r_x) : y_1 < f(y'', y_n)\},$$
$$\partial\Omega \cap B(x, r_x) = \{y \in B(0, r_x) : y_1 = f(y'', y_n)\}$$

for some function $f(\mathbb{R}^{n-1})$ of class $C^{2,\alpha}$. Setting $F(y) := y_1 - f(y'', y_n)$ and $G(y) := y_n - g(y'', y_n)$, by the transversality condition we have that

$$0 = \nabla F(y) \cdot \nabla G(y) = -\frac{\partial g}{\partial y_1}(y') \tag{5.9}$$

$$+ \sum_{k=2}^{n-1} \frac{\partial f}{\partial y_k}(y'', y_n) \frac{\partial g}{\partial y_i}(y') - \frac{\partial f}{\partial y_n}(y'', y_n)$$

for all $y \in \partial E \cap \partial\Omega \cap B(0, r_x)$. For $|s|$ sufficiently small the set

$$E_s := \{z \in \mathbb{R}^n : \mathrm{d}_E(z) = s\}$$

in a neighborhood of x is given by an $n-1$-dimensional manifold parametrized by

$$\varphi_i(y', s) = y_i - s \frac{\partial g}{\partial y_i}(y') \left(1 + |\nabla g(y')|^2_{n-1}\right)^{-1/2} \tag{5.10}$$

$$\varphi_n(y', s) = g(y') + s \left(1 + |\nabla g(y')|^2_{n-1}\right)^{-1/2} \tag{5.11}$$

for all $i = 1, \ldots, n-1$ and for $y' \in Q_{n-1}(0, r'_x)$, where $0 < r'_x < r_x$ and we are using local coordinates (see [40]). We are interested in in the surface area of E_s contained in Ω, or in other words, the area of the region characterized by

$$\varphi_1(y', s) > f(\varphi_2(y', s), \ldots, \varphi_n(y', s)). \tag{5.12}$$

Consider the function

$$H(y', s) := \varphi_1(y', s) - f(\varphi_2(y', s), \dots, \varphi_n(y', s)). \tag{5.13}$$

By (5.10) and (5.11) we have

$$\frac{\partial H}{\partial y_1}(y', s) = \frac{\partial \varphi_1}{\partial y_1}(y', s) - \sum_{k=2}^{n} \frac{\partial f}{\partial y_k}(\varphi_2(y', s), \dots, \varphi_n(y', s)) \frac{\partial \varphi_k}{\partial y_1}(y', s).$$

Taking $y' = 0$ and recalling that $\frac{\partial g}{\partial y_i}(0) = 0$ for all $i = 1, \dots, n-1$ gives

$$\frac{\partial H}{\partial y_1}(0, s) = 1 - s \frac{\partial^2 g}{\partial y_1^2}(0) + s \sum_{k=2}^{n-1} \frac{\partial f}{\partial y_k}(0) \frac{\partial^2 g}{\partial y_k^2}(0) > 0$$

provided we take

$$1 > s(1 + \|\nabla f\|_{L^\infty(Q_{n-1}(0, r'_x))}) \left\|\nabla^2 g\right\|_{L^\infty(Q_{n-1}(0, r'_x))}.$$

By taking $r'_x > 0$ smaller, it follows by the implicit function theorem that there exists a function $h(y'', s)$ of class $C^{1,\alpha}$ such that the condition (5.12) is equivalent to

$$y_1 > h(y'', s)$$

for all for $y' = (y_1, y'') \in Q_{n-1}(0, r'_x)$ and all $s > 0$ sufficiently small. Moreover, since f is of class $C^{2,\alpha}$ and the functions φ_i and $\frac{\partial \varphi_i}{\partial y_k}$ are infinitely differentiable in the variable s (by (5.10) and (5.11)), we have that $\frac{\partial^2 h}{\partial s^2}$ and $\frac{\partial^2 h}{\partial s \partial y_i}$ exist and are continuous. By (5.9), (5.10), (5.11), and (5.13),

$$\frac{\partial H}{\partial s}(y', 0) = \frac{\partial \varphi_1}{\partial s}(y', 0) - \sum_{k=2}^{n} \frac{\partial f}{\partial y_k}(\varphi_2(y', 0), \dots, \varphi_n(y', 0)) \frac{\partial \varphi_k}{\partial s}(y', 0)$$

$$= - \left(1 + |\nabla g(y')|_{n-1}^2\right)^{-1/2} \frac{\partial g}{\partial y_1}(y')$$

$$+ \left(1 + |\nabla g(y')|_{n-1}^2\right)^{-1/2} \sum_{k=2}^{n-1} \frac{\partial f}{\partial y_k}(y'', g(y')) \frac{\partial g}{\partial y_i}(y')$$

$$- \left(1 + |\nabla g(y')|_{n-1}^2\right)^{-1/2} \frac{\partial f}{\partial y_n}(y'', g(y')) = 0.$$

It follows that

$$\frac{\partial h}{\partial s}(y'', 0) = \frac{\partial H}{\partial s}(h(y'', 0), y'', 0) \left(\frac{\partial H}{\partial s}(h(y'', 0), y'', 0)\right)^{-1} = 0. \tag{5.14}$$

In a neighborhood of x the surface area of E_s inside Ω is given by the surface integral

$$A_x(s) := \int_{Q_{n-2}(0, r'_x)} \int_{h(y'', s)}^{r'_x/2} \sqrt{\sum_{\alpha \in \Xi} \left[\det \frac{\partial (\varphi_{\alpha_1}, \dots, \varphi_{\alpha_{n-1}})}{\partial (y_1, \dots, y_{n-1})}(y', s)\right]^2} \, dy_1 dy'',$$

where

$$\Xi := \left\{\alpha \in \mathbb{N}^{n-1} : 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} \le n\right\}.$$

By standard theorems of differentiation under the integral sign, we have that A_x is of class C^2.

Moreover, by (5.14),

$$A'_x(0) = \int_{Q_{n-1}(0,r'_x)} \frac{\partial}{\partial s} \left(\sqrt{\sum_{\alpha \in \Xi} \left[\det \frac{\partial (\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_{n-1}})}{\partial (y_1, \ldots, y_{n-1})} (y', s) \right]^2} \right)_{s=0} dy'$$

$$- \frac{\partial h}{\partial s}(y'', 0) \int_{Q_{n-2}(0,r'_x)} \sqrt{\sum_{\alpha \in \Xi} \left[\det \frac{\partial (\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_{n-1}})}{\partial (y_1, \ldots, y_{n-1})} (h(y'', 0,), y'', 0) \right]^2} dy''$$

$$= \int_{Q_{n-1}(0,r'_x)} \frac{\partial}{\partial s} \left(\sqrt{\sum_{\alpha \in \Xi} \left[\det \frac{\partial (\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_{n-1}})}{\partial (y_1, \ldots, y_{n-1})} (y', s) \right]^2} \right)_{s=0} dy'.$$

As the boundary term has dropped out we can then obtain (5.8) using a partition of unity and classical formulas (see, e.g., [45]).

The fact that η is bounded follows from [52]. This completes the proof.

$\square$

We then prove our Γ-lim sup inequality.

Proof of Theorem 5.3. If $u \notin \mathcal{U}_1$ the inequality is trivial. Thus, let $u \in \mathcal{U}_1$. Then we have that u is of the form $a\chi_E + b\chi_{E^c}$. Define

$$\eta(t) := \mathcal{H}^{n-1}(\{x : d_E(x) = t\}). \tag{5.15}$$

By Lemma 5.4 we have that η satisfies the assumptions of Theorem 4.7. Let g_ε be the one-dimensional function constructed in Theorem 4.7, using η chosen via (5.15). Define $u_\varepsilon(x) := g_\varepsilon(d_E(x))$. By the coarea formula for Lipschitz functions we have that

$$F_\varepsilon^{(2)}(u_\varepsilon) = \frac{1}{\varepsilon} \left(\int_{\mathbb{R}} (\varepsilon^{-1} W(g_\varepsilon(t)) + \varepsilon(g'_\varepsilon)^2) \mathcal{H}^{n-1}(\{x : d_E(x) = t\}) \, dt - 2c_W \eta(0) \right).$$

Applying Theorem 4.7 then gives the desired result.

$\square$

Remark 5.5. *We note that our recovery sequence construction does not require any smoothness properties on $\mathcal{I}_\Omega$, such as (2.3). The technique given in this section thus establishes the Γ-lim sup inequality at any mass level. Although we do not currently have a liminf inequality to support it, we suspect that the Γ-limit we establish in this paper should hold at all mass levels. This would imply that surfaces with higher magnitude mean curvature are (slightly) energetically favored.*

Next we prove Corollary 1.3.

Proof of Corollary 1.3. If W is symmetric about $\frac{a+b}{2}$ then $W \circ z$ will be an even function. Thus (see (1.16))

$$c_{\text{sym}} = \int_{\mathbb{R}} W(z(t)) t \, dt = 0.$$

Furthermore if W is symmetric then z will satisfy $b - z(t) = z(-t) - a$ for all $t > 0$. In turn, this implies that

$$\int_{\mathbb{R}} \text{sgn}_{a,b}(t) - z(t) \, dt = 0,$$

and hence

$$\int_{\mathbb{R}} z(t - \tau_u) - \text{sgn}_{a,b}(t) \, dt = \int_{\mathbb{R}} z(t - \tau_u) - z(t) \, dt.$$

Utilizing the fundamental theorem of calculus and Fubini's theorem

$$\int_{\mathbb{R}} z(t-\tau_u) - z(t)\,dt = -\int_{\mathbb{R}} \int_{t-\tau_u}^{t} z'(s)\,ds\,dt = -\tau_u(b-a).$$

Recalling (1.20) we then find that

$$\tau_u = \begin{cases} -\dfrac{\Lambda_u}{W''(a)\,P(\{u=a\};\Omega)(b-a)} & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases}$$

This then gives

$$\mathcal{F}^{(2)}(u) = \begin{cases} \frac{1}{W''(a)}\left(\frac{\Lambda_u^2}{2} - 2\kappa_u(n-1)c_W\frac{\Lambda_u}{(b-a)}\right) & \text{if } q = 1, \\ 0 & \text{if } q < 1. \end{cases}$$

Again recalling (1.20) we find that

$$\mathcal{F}^{(2)}(u) = \begin{cases} -\frac{\Lambda_u^2}{2W''(a)} & \text{if } q = 1, \\ 0 & \text{if } q < 1, \end{cases}$$

as desired. $\qquad\square$

Acknowledgments

References

[1] ALBERICO, A., AND CIANCHI, A. Borderline sharp estimates for solutions to Neumann problems. *Ann. Acad. Sci. Fenn. Math. 32*, 1 (2007), 27–53.

[2] ALIKAKOS, N., BRONSARD, L., AND FUSCO, G. Slow motion in the gradient theory of phase transitions via energy and spectrum. *Calc. Var. Partial Differential Equations 6*, 1 (1998), 39–66.

[3] ALIKAKOS, N., AND FUSCO, G. Slow dynamics for the Cahn-Hilliard equation in higher space dimensions. I. Spectral estimates. *Comm. Partial Differential Equations 19*, 9-10 (1994), 1397–1447.

[4] ALIKAKOS, N., AND FUSCO, G. Slow dynamics for the Cahn-Hilliard equation in higher space dimensions: the motion of bubbles. *Arch. Rational Mech. Anal. 141*, 1 (1998), 1–61.

[5] ALIKAKOS, N. D., BATES, P. W., AND CHEN, X. Convergence of the cahn-hilliard equation to the hele-shaw model. *Archive for rational mechanics and analysis 128*, 2 (1994), 165–205.

[6] AMBROSIO, L., FUSCO, N., AND PALLARA, D. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[7] ANZELLOTTI, G., AND BALDO, S. Asymptotic development by Γ-convergence. *Appl. Math. Optim. 27*, 2 (1993), 105–123.

[8] ANZELLOTTI, G., BALDO, S., AND ORLANDI, G. Γ-asymptotic developments, the Cahn-Hilliard functional, and curvatures. *J. Math. Anal. Appl. 197*, 3 (1996), 908–924.

[9] BAVARD, C., AND PANSU, P. Sur le volume minimal de $\mathbf{R}^2$. *Ann. Sci. École Norm. Sup. (4) 19*, 4 (1986), 479–490.

[10] BELLETTINI, G., HASSEM NAYAM, A., AND NOVAGA, M. Γ-type estimates for the one-dimensional allen-cahn's action, 2013.

[11] BRAIDES, A. *Local minimization, variational evolution and Γ-convergence*, vol. 2094 of *Lecture Notes in Mathematics.* Springer, Cham, 2014.

[12] BRAIDES, A., AND TRUSKINOVSKY, L. Asymptotic expansions by Γ-convergence. *Contin. Mech. Thermodyn. 20*, 1 (2008), 21–62.

[13] BRONSARD, L., AND KOHN, R. On the slowness of phase boundary motion in one space dimension. *Comm. Pure Appl. Math. 43*, 8 (1990), 983–997.

[14] BRONSARD, L., AND KOHN, R. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Differential Equations 90*, 2 (1991), 211–237.

[15] BRONSARD, L., AND STOTH, B. Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation. *SIAM J. Math. Anal. 28*, 4 (1997), 769–807.

[16] CAHN, J., AND HILLIARD, J. Free energy of a nonuniform system. i. interfacial free energy. *The Journal of chemical physics 28*, 2 (1958), 258–267.

[17] CARR, J., GURTIN, M., AND SLEMROD, M. Structured phase transitions on a finite interval. *Arch. Rational Mech. Anal. 86*, 4 (1984), 317–351.

[18] CARR, J., AND PEGO, R. Metastable patterns in solutions of ut= 2uxx- f (u). *Communications on pure and applied mathematics 42*, 5 (1989), 523–576.

[19] CHEN, X. Generation and propagation of interfaces for reaction-diffusion equations. *Journal of Differential equations 96*, 1 (1992), 116–141.

[20] CIANCHI, A., ESPOSITO, L., FUSCO, N., AND TROMBETTI, C. A quantitative Pólya-Szegö principle. *J. Reine Angew. Math. 614* (2008), 153–189.

[21] CIANCHI, A., AND FUSCO, N. Functions of bounded variation and rearrangements. *Arch. Ration. Mech. Anal. 165*, 1 (2002), 1–40.

[22] CIANCHI, A., AND MAZ'YA, V. Neumann problems and isocapacitary inequalities. *J. Math. Pures Appl. (9) 89*, 1 (2008), 71–105.

[23] CRANDALL, M., AND TARTAR, L. Some relations between nonexpansive and order preserving mappings. *Proc. Amer. Math. Soc. 78*, 3 (1980), 385–390.

[24] DAL MASO, G. *An introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser Boston, Inc., Boston, MA, 1993.

[25] DAL MASO, G., FONSECA, I., AND LEONI, G. Second order asymptotic development for the cahn-hilliard functional. *To Appear* (2013).

[26] EVANS, L., AND GARIEPY, R. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[27] FOCARDI, M. Γ-convergence: a tool to investigate physical phenomena across scales. *Mathematical Methods in the Applied Sciences 35*, 14 (2012), 1613–1658.

[28] FONSECA, I., AND TARTAR, L. The gradient theory of phase transitions for systems with two potential wells. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics 111*, 1-2 (1989), 89–102.

[29] FUSCO, G., AND HALE, J. Slow-motion manifolds, dormant instability, and singular perturbations. *J. Dynam. Differential Equations 1*, 1 (1989), 75–94.

[30] GARCKE, H. Curvature driven interface evolution. *Jahresber. Dtsch. Math.-Ver. 115*, 2 (2013), 63–100.

[31] GONZALEZ, E., MASSARI, U., AND TAMANINI, I. On the regularity of boundaries of sets minimizing perimeter with a volume constraint. *Indiana Univ. Math. J. 32*, 1 (1983), 25–37.

[32] GRANT, C. Slow motion in one-dimensional Cahn-Morral systems. *SIAM J. Math. Anal. 26*, 1 (1995), 21–34.

[33] GRÜTER, M. Boundary regularity for solutions of a partitioning problem. *Arch. Rational Mech. Anal. 97*, 3 (1987), 261–270.

[34] GURTIN, M. On a theory of phase transitions with interfacial energy. *Arch. Rational Mech. Anal. 87*, 3 (1985), 187–212.

[35] GURTIN, M., AND MATANO, H. On the structure of equilibrium phase transitions within the gradient theory of fluids. *Quart. Appl. Math. 46*, 2 (1988), 301–317.

[36] HIRIART-URRUTY, J.-B., AND LEMARÉCHAL, C. *Convex analysis and minimization algorithms. I*, vol. 305 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1993. Fundamentals.

[37] KAWOHL, B. *Rearrangements and convexity of level sets in PDE*, vol. 1150 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[38] KESAVAN, S. *Symmetrization & applications*, vol. 3 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[39] KOHN, R., AND STERNBERG, P. Local minimisers and singular perturbations. *Proc. Roy. Soc. Edinburgh Sect. A 111*, 1-2 (1989), 69–84.

[40] KRANTZ, S. G., AND PARKS, H. R. Distance to c k hypersurfaces. *Journal of Differential Equations 40*, 1 (1981), 116–120.

[41] KURATA, K., AND SHIBATA, M. On a one-dimensional variational problem related to the Cahn–Hilliard energy in a bent strip-like domain. *Nonlinear Analysis: Theory, Methods & Applications 47*, 2 (2001), 1059–1068.

[42] LEONI, G. *A first course in Sobolev spaces*, vol. 105 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.

[43] LEONI, G. A remark on the compactness for the Cahn-Hilliard functional. *ESAIM Control Optim. Calc. Var. 20*, 2 (2014), 517–523.

[44] LUCKHAUS, S., AND MODICA, L. The Gibbs-Thompson relation within the gradient theory of phase transitions. *Arch. Rational Mech. Anal. 107*, 1 (1989), 71–83.

[45] MAGGI, F. *Sets of finite perimeter and geometric variational problems*, vol. 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[46] MAZ'YA, V. *Sobolev spaces with applications to elliptic partial differential equations*, augmented ed., vol. 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[47] MODICA, L. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal. 98*, 2 (1987), 123–142.

[48] MODICA, L., AND MORTOLA, S. Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B (5) 14*, 1 (1977), 285–299.

[49] MURRAY, R., AND RINALDI, M. Slow motion for the nonlocal Allen–Cahn equation in n dimensions. *In Preparation* (2015).

[50] NI, W. M. On the positive radial solutions of some semilinear elliptic equations on $\mathbf{R}^n$. *Appl. Math. Optim. 9*, 4 (1983), 373–380.

[51] NIETHAMMER, B. Existence and uniqueness of radially symmetric stationary points within the gradient theory of phase transitions. *European J. Appl. Math. 6*, 1 (1995), 45–67.

[52] OLEKSIV, I. Y., AND PESIN, N. Finiteness of Hausdorff measure of level sets of bounded subsets of Euclidean space. *Mathematical Notes 37*, 3 (1985), 237–242.

[53] OTTO, F., AND REZNIKOFF, M. Slow motion of gradient flows. *Journal of Differential Equations 237*, 2 (2007), 372–420.

[54] PEGO, R. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. London Ser. A 422*, 1863 (1989), 261–278.

[55] ROS, A. The isoperimetric problem. *Global theory of minimal surfaces 2* (2001), 175–209.

[56] SPECTOR, D. Simple proofs of some results of Reshetnyak. *Proc. Amer. Math. Soc. 139*, 5 (2011), 1681–1690.

[57] STERNBERG, P. The effect of a singular perturbation on nonconvex variational problems. *Arch. Rational Mech. Anal. 101*, 3 (1988), 209–260.

[58] STERNBERG, P., AND ZUMBRUN, K. Connectivity of phase boundaries in strictly convex domains. *Archive for Rational Mechanics and Analysis 141*, 4 (1998), 375–400.

[59] STERNBERG, P., AND ZUMBRUN, K. On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint. *Comm. Anal. Geom. 7*, 1 (1999), 199–220.

[60] TALENTI, G. Elliptic equations and rearrangements. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 3*, 4 (1976), 697–718.

[61] ZIEMER, W. *Weakly differentiable functions*, vol. 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation.