STRESS REGULARITY FOR A NEW QUASISTATIC EVOLUTION MODEL OF PERFECTLY PLASTIC PLATES

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Abstract. We study some properties of solutions to a quasistatic evolution problem for perfectly plastic plates, that has been recently derived from three-dimensional Prandtl-Reuss plasticity. We prove that the stress tensor has locally square-integrable first derivatives with respect to the space variables. We also exhibit an example showing that the model under consideration has in general a genuinely three-dimensional nature and cannot be reduced to a two-dimensional setting.

1. Introduction

In this paper we continue the study of the quasistatic evolution model for perfectly plastic plates, that has been derived in [5] starting from three-dimensional Prandtl-Reuss plasticity. Under suitable regularity assumptions for the applied body forces, we prove $W^{1,2}_{loc}$ regularity of the stress with respect to the space variables.

Let ω be a bounded domain in $\mathbb{R}^2$ with a C^2 boundary. The set $\Omega := \omega\times(-\frac{1}{2},\frac{1}{2})$ represents the reference configuration of a three-dimensional plate. The current configuration of the plate at time t is described by a triple $(u(t), e(t), p(t))$, where $u(t)$ is the displacement, $e(t)$ is the elastic strain tensor, and $p(t)$ is the plastic strain tensor, satisfying the following conditions:

(sf1) *kinematic admissibility:* $Eu(t) = e(t) + p(t)$ in Ω, $u(t) = w(t)$ on Γ_d, and $e_{i3}(t) = p_{i3}(t) = 0$ in Ω for $i = 1, 2, 3$.

Here $Eu(t)$ denotes the infinitesimal strain tensor, given by the symmetric part of $Du(t)$, while $w(t)$ is a prescribed boundary condition on $\Gamma_d := \gamma_d\times(-\frac{1}{2},\frac{1}{2})$, γ_d being a subset of $\partial\omega$. These conditions imply that $u(t)$ is a Kirchhoff-Love displacement, that is, the vertical displacement $u_3(t)$ is independent of the out-of-plane variable x_3 and the horizontal displacement takes the form

$$u_\alpha(t,x) = \bar{u}_\alpha(t,x') - x_3\partial_\alpha u_3(t,x') \quad \text{for } x = (x', x_3) \in \Omega,\ \alpha = 1, 2. \tag{1.1}$$

In particular,

$$(Eu)_{\alpha\beta}(t,x) = (E\bar{u})_{\alpha\beta}(t,x') - x_3\partial^2_{\alpha\beta}u_3(t,x') \quad \text{for } x = (x', x_3) \in \Omega,\ \alpha,\beta = 1, 2.$$

From a mechanical point of view this structure guarantees that straight fibers that are normal to the mid-surface of the plate in the reference configuration, stay straight and normal after the deformation, within the first order (see, e.g., [3]).

Condition (sf1) does not imply, in general, that $e(t)$ and $p(t)$ are affine with respect to x_3. However, one can prove (Proposition 3.1) that $e(t)$ and $p(t)$ admit the following decomposition:

$$e(t,x) = \bar{e}(t,x') + x_3\hat{e}(t,x') + e_\perp(t,x), \qquad p(t,x) = \bar{p}(t,x') + x_3\hat{p}(t,x') - e_\perp(t,x),$$

where the zero-th order moments $\bar{e}(t)$ and $\bar{p}(t)$ satisfy

$$E\bar{u}(t) = \bar{e}(t) + \bar{p}(t) \quad \text{in } \omega$$

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while the first order moments $\hat{e}(t)$ and $\hat{p}(t)$ are such that

$$-D^2 u_3(t) = \hat{e}(t) + \hat{p}(t) \quad \text{ in } \omega.$$

In the above identities and in the following we identify $e(t)$, $p(t)$, and their moments with functions taking values in $\mathbb{M}^{2\times 2}_{sym}$, since their third row and column are zero by condition (sf1).

The strong formulation of the quasistatic evolution problem on a time interval $[0,T]$ consists in finding $u(t)$, $e(t)$, and $p(t)$ such that for every $t \in [0,T]$ equation (sf1) is satisfied, together with the following conditions:

(sf2) *constitutive equation:* $\sigma(t) = \mathbb{C}_r e(t)$ in Ω, where $\mathbb{C}_r$ is the elasticity tensor;

(sf3) *equilibrium:* $-\mathrm{div}_{x'}\bar{\sigma}(t) = f(t)$ and $-\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = g(t)$ in ω, together with suitable Neumann boundary conditions on $\partial\omega \setminus \gamma_d$;

(sf4) *stress constraint:* $\sigma(t) \in K_r$, where K_r is a given convex and compact set, representing the set of admissible stresses;

(sf5) *flow rule:* $\dot{p}(t) = 0$ if $\sigma(t) \in \mathrm{int}\, K_r$, while $\dot{p}(t)$ belongs to the normal cone to K_r at $\sigma(t)$ if $\sigma(t) \in \partial K_r$.

Here $f(t) : \omega \to \mathbb{R}^2$ and $g(t) : \omega \to \mathbb{R}$ represent the applied body forces at time t, while $\bar{\sigma}(t) := \mathbb{C}_r \bar{e}(t)$ and $\hat{\sigma}(t) := \mathbb{C}_r \hat{e}(t)$ are the stretching and bending components of the stress, respectively. Condition (sf5) can also be written in the equivalent form:

(sf5′) *maximum dissipation principle:* $H_r(\dot{p}(t)) = \sigma(t) : \dot{p}(t)$, where H_r is the support function of K_r, i.e., $H_r(p) := \sup\{\sigma : p \ : \ \sigma \in K_r\}$,

or alternatively,

(sf5″) *maximum plastic work condition:* $(\theta - \sigma(t)) : \dot{p}(t) \leq 0$ for every $\theta \in K_r$.

In [5] this model has been rigorously justified via Γ-convergence techniques, starting from the three-dimensional Prandtl-Reuss quasistatic evolution model. In other words the system (sf1)–(sf5) describes (up to a suitable scaling) the asymptotic behaviour of the quasistatic evolutions in a three-dimensional plate, when the plate thickness approaches zero.

We note that the equilibrium conditions are purely two-dimensional, while the stress constraint and the flow rule (which are the main ingredients of the plastic reponse) involve the whole stress $\sigma(t)$, whose dependence on the thickness variable x_3 may be not trivial (because of the component $\sigma_\perp(t) := \mathbb{C}_r e_\perp(t)$). Thus, the problem has in general a genuinely three-dimensional nature and differs from the classical two-dimensional plastic plate model that has been extensively studied in the literature [2, 6, 9, 10]. This comparison is discussed in the last section of the paper, where an explicit solution to (sf1)–(sf5) is shown for a specific choice of data.

Existence of a solution to (sf1)–(sf5) can be proved by setting the problem within the variational framework for rate-independent processes, developed in [14]. This accounts to approximating the problem by time discretization: the interval $[0,T]$ is subdivided into k subintervals by means of points

$$0 = t_k^0 < t_k^1 < \cdots < t_k^{k-1} < t_k^k = T\,,$$

and the approximate solution u_k^i, e_k^i, p_k^i at time t_k^i is defined by induction as a minimizer of the energy functional

$$\frac{1}{2}\int_\Omega \mathbb{C}_r e : e\, dx + \int_\Omega H_r(p - p_k^{i-1})\, dx - \langle \mathcal{L}(t_k^i), u\rangle \tag{1.2}$$

among all triples (u, e, p) that are kinematically admissible at time t_k^i, where

$$\langle \mathcal{L}(t), u\rangle := \int_\omega f(t)\cdot \bar{u}\, dx' - \frac{1}{12}\int_\omega g(t) u_3\, dx'.$$

Because of the linear growth of H_r, the energy functional in (1.2) is not coercive in any Sobolev norm. The natural setting for a weak formulation is the space $BD(\Omega)$ of functions with bounded deformation in Ω for the displacement $u(t)$ and the space $M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym})$ of bounded Borel measures on $\Omega \cup \Gamma_d$ for the plastic strain $p(t)$. This is also natural from a mechanical point of view, because it is well known that in the absence of hardening displacements may develop jump discontinuities along so-called slip surfaces, on which plastic strain concentrates.

Since $p(t) \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym})$, the functional

$$\int_\Omega H_r(p(t))\,dx$$

has to be interpreted according to the theory of convex functions of measures, developed in [12, 15] (see also Section 2), as

$$\mathcal{H}_r(p(t)) := \int_{\Omega\cup\Gamma_d} H_r\Big(\frac{dp(t)}{d|p(t)|}\Big)\,d|p(t)|,$$

where $dp(t)/d|p(t)|$ is the Radon-Nicodym derivative of $p(t)$ with respect to its total variation $|p(t)|$. Moreover, the boundary condition is relaxed by requiring that

$$p(t) = (w(t) - u(t)) \odot \nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d, \tag{1.3}$$

where $\odot$ denotes the symmetrized tensor product and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. The mechanical interpretation of (1.3) is that $u(t)$ may not attain the boundary condition: in this case a plastic slip is developed along Γ_d, whose amount is proportional to the difference between the prescribed boundary value and the actual value.

Combining these remarks with the kinematic admissibility condition (sf1), we see that $u(t)$ is a Kirchhoff-Love displacement in $BD(\Omega)$, that is, $u_3(t)$ belongs to the space $BH(\omega)$ of functions with bounded Hessian in ω and the averaged tangential displacement $\bar{u}(t)$ in (1.1) belongs to $BD(\omega)$. Therefore, $\bar{u}(t)$ may exhibit jump discontinuities, while, because of the embedding of $BH(\omega)$ into $C(\overline{\omega})$, the normal displacement $u_3(t)$ is continuous, but its gradient may have jump discontinuities. Since the dependence of u on x_3 is affine, we can conclude that slip surfaces are vertical surfaces whose projection on ω is the union of the jump set of $\bar{u}$ and the jump set of ∇u_3.

Moreover, writing condition (1.3) in terms of moments yields

$$\begin{gathered}\bar{p}(t) = (\bar{w}(t) - \bar{u}(t)) \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \gamma_d,\\ u_3(t) = w_3(t), \qquad \hat{p}(t) = (\nabla u_3(t) - \nabla w_3(t)) \odot \nu_{\partial\omega}\mathcal{H}^1 \quad \text{on } \gamma_d.\end{gathered}$$

In this setting the flow rule is proved to hold in the form

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t)\rangle,$$

where the product at the right-hand side is meant in the sense of the stress-strain duality introduced in [5] (see also Section 3).

In this paper we focus on the spatial regularity of the stress component $\sigma(t)$ for solutions of the quasistatic evolution problem (sf1)–(sf5) in its weak formulation. We restrict to the case where the yield criterion in the fully three-dimensional Prandtl-Reuss problem is that of von Mises, often used for metals (see [13]). In other words, the set of admissible stresses for the fully three-dimensional Prandtl-Reuss problem is a cylinder $B_{\alpha_0} + \mathbb{R}I_{3\times 3}$, where B_{α_0} is a ball of radius α_0 in the space of trace-free $\mathbb{M}^{3\times 3}_{sym}$ matrices and $I_{3\times 3}$ is the identity matrix in $\mathbb{M}^{3\times 3}_{sym}$. By the characterization in [5] this implies that the set K_r is an ellipsoid of the form

$$K_r = \{\xi \in \mathbb{M}^{2\times 2}_{sym} : \ |\xi|_r \le \alpha_0\},$$

where

$$|\xi|_r^2 := \frac{1}{6}(\xi_{11} + \xi_{22})^2 + \frac{1}{2}(\xi_{11} - \xi_{22})^2 + 2\xi_{12}^2 = |\xi|^2 - \frac{1}{3}(\operatorname{tr}\xi)^2.$$

Our main result is that for the solutions of the quasistatic evolution problem under consideration the stress component is locally $W^{1,2}$ with respect to space variables (Theorem 4.6).

More precisely, we show that for every open set ω' compactly contained in ω there exists a positive constant $C_1(\omega')$ such that

$$\sup_{t\in[0,T]} \|D_\alpha \sigma(t)\|_{L^2(\omega'\times(-\frac{1}{2},\frac{1}{2});\mathbb{M}^{2\times 2}_{sym})} \leq C_1(\omega') \quad \text{for } \alpha = 1,2 \tag{1.4}$$

and for every open set Ω' compactly contained in Ω there exists a positive constant $C_2(\Omega')$ such that

$$\sup_{t\in[0,T]} \|D_3 \sigma(t)\|_{L^2(\Omega';\mathbb{M}^{2\times 2}_{sym})} \leq C_2(\Omega'). \tag{1.5}$$

This implies in particular that both the stretching component $\bar{\sigma}$ and the bending component $\hat{\sigma}$ are in $L^\infty(0,T;W^{1,2}_{loc}(\omega;\mathbb{M}^{2\times 2}_{sym}))$, while $\sigma_\perp \in L^\infty(0,T;W^{1,2}_{loc}(\Omega;\mathbb{M}^{2\times 2}_{sym}))$.

Local regularity of stresses in the fully three-dimensional Prandtl-Reuss plasticity has been proved in [1, 8], see also [11] for a recent global regularity result for the stress velocity. For the classical two-dimensional plastic plate model local regularity has been established in [10], using different techniques and assuming K_r to be a ball, $\mathbb{C}_r$ to be the identity tensor, and $\gamma_d = \partial\omega$.

The strategy of our proof is inspired by that of [1]. We consider an equivalent formulation of problem (sf1)–(sf5) in terms of a parabolic variational inequality for the stress variable and we construct some approximating problems of Norton-Hoff type, where the constraint (sf4) is replaced by a penalization term. These approximating problems involve a monotone differential equation in the stress variable; the displacement and the plastic strain are then indirectly recovered a posteriori. We first establish regularity for the approximating problems with uniform estimates with respect to the approximation parameter and then prove convergence to the parabolic variational inequality formulation.

The main novelty with respect to [1] is that in the approximating model the equilibrium equations are expressed in terms of the moments $\bar{\sigma}$ and $\hat{\sigma}$, while the nonlinearity in the monotone operator involve the whole stress σ. For this reason we obtain a slightly better regularity of the stress with respect to the in-plane variables: the regularity estimate (1.4) is indeed global in the out-of-plane direction x_3, whereas (1.5) is local with respect to both in-plane and out-of-plane variables. In particular, we observe that (1.4) cannot be deduced from the regularity estimates in [1] for the fully three-dimensional Prandtl-Reuss problem, using the convergence result of [5]; indeed, the estimates of [1] (whose dependence on the domain should be explicited if one wished to pass to the limit as the thickness of the plate tends to zero) are local in all directions.

The plan of the paper is as follows. In Section 2 we recall some mathematical preliminaries. The setting of the problem is detailed in Section 3. The existence and regularity results are the subject of Section 4. In Section 5 an explicit example is discussed.

2. Mathematical preliminaries

In this section we recall some notions from measure theory that we will use throughout the article.

Measures. Given a Borel set $B \subset \mathbb{R}^n$ and a finite dimensional Hilbert space X, $M_b(B;X)$ denotes the space of all bounded Borel measures on B with values in X, endowed with the norm $\|\mu\|_{M_b} := |\mu|(B)$, where $|\mu| \in M_b(B;\mathbb{R})$ is the variation of the measure μ. If μ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, we always identify μ with its density with respect to $\mathcal{L}^n$, which is a function in $L^1(B;X)$.

If the relative topology of B is locally compact, by Riesz representation Theorem the space $M_b(B;X)$ can be identified with the dual of $C_0(B;X)$, which is the space of all continuous functions $\varphi : B \to X$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $M_b(B;X)$ is defined using this duality.

Convex functions of measures. For every $\mu \in M_b(B;X)$ let $d\mu/d|\mu|$ be the Radon-Nicodym derivative of μ with respect to its variation $|\mu|$. Let $H : X \to [0,+\infty)$ be a convex and positively one-homogeneous function such that

$$\alpha_H|\xi| \leq H(\xi) \leq \beta_H|\xi| \quad \text{for every } \xi \in X,$$

where α_H and β_H are two constants, with $0 < \alpha_H \leq \beta_H$. According to the theory of convex functions of measures, developed in [12], we introduce the nonnegative Radon measure $H(\mu) \in M_b(B)$ defined by

$$H(\mu)(A) := \int_A H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every Borel set $A \subset B$. We also consider the functional $\mathcal{H} : M_b(B;X) \to [0,+\infty)$ defined by

$$\mathcal{H}(\mu) := H(\mu)(B) = \int_B H\Big(\frac{d\mu}{d|\mu|}\Big)\, d|\mu|$$

for every $\mu \in M_b(B;X)$. One can prove that $H(\mu)$ coincides with the measure studied in [15, Chapter II, Section 4]. Hence,

$$\mathcal{H}(\mu) = \sup\Big\{ \int_B \varphi : d\mu \ :\ \varphi \in C_0(B;X),\ \varphi(x) \in K \text{ for every } x \in B \Big\}, \tag{2.1}$$

where $K := \partial H(0)$ is the subdifferential of H at 0. Moreover, $\mathcal{H}$ is lower semicontinuous on $M_b(B;X)$ with respect to weak* convergence.

Functions with bounded deformation. Let U be an open set of $\mathbb{R}^n$. The space $BD(U)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(U;\mathbb{R}^n)$ whose symmetric gradient $Eu := \operatorname{sym} Du$ (in the sense of distributions) belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BD(U)$ is a Banach space endowed with the norm

$$\|u\|_{BD} := \|u\|_{L^1} + \|Eu\|_{M_b}.$$

We say that a sequence (u^k) converges to u weakly* in $BD(U)$ if $u^k \rightharpoonup u$ weakly in $L^1(U;\mathbb{R}^n)$ and $Eu^k \rightharpoonup Eu$ weakly* in $M_b(U;\mathbb{M}^{n\times n}_{sym})$. Every bounded sequence in $BD(U)$ has a weakly* converging subsequence. If U is bounded and has a Lipschitz boundary, $BD(U)$ can be embedded into $L^{n/(n-1)}(U;\mathbb{R}^n)$ and every function $u \in BD(U)$ has a trace, still denoted by u, which belongs to $L^1(\partial U;\mathbb{R}^n)$. Moreover, if Γ is a nonempty open subset of ∂U, there exists a constant $C > 0$, depending on U and Γ, such that

$$\|u\|_{L^1(\Omega)} \leq C\|u\|_{L^1(\Gamma)} + C\|Eu\|_{M_b} \tag{2.2}$$

(see [15, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(U)$ we refer to [15].

Functions with bounded Hessian. The space $BH(U)$ of functions with *bounded Hessian* is the space of all functions $u \in W^{1,1}(U)$ whose Hessian D^2u (in the sense of distributions) belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. It is easy to see that $BH(U)$ is a Banach space endowed with the norm

$$\|u\|_{BH} := \|u\|_{L^1} + \|\nabla u\|_{L^1} + \|D^2u\|_{M_b}.$$

If U has the cone property, then $BH(U)$ coincides with the space of functions in $L^1(U)$ whose Hessian belongs to $M_b(U;\mathbb{M}^{n\times n}_{sym})$. If U is bounded and has a Lipschitz boundary, $BH(U)$ can be embedded into $W^{1,n/(n-1)}(U)$. If U is bounded and has a C^2 boundary, then for every function $u \in BH(U)$ one can define the traces of u and of ∇u, still denoted by u and ∇u; they satisfy $u \in W^{1,1}(\partial U)$, $\nabla u \in L^1(\partial U;\mathbb{R}^n)$, and $\frac{\partial u}{\partial \tau} = \nabla u \cdot \tau$ in $L^1(\partial U)$, where τ is any tangent vector to ∂U. If, in addition, $n = 2$, then $BH(U)$ embeds into $C(\overline{U})$, which is the space of all continuous functions on $\overline{U}$. For the general properties of the space $BH(U)$ we refer to [7].

Notation. The symmetrized tensor product $a \odot b$ of two vectors $a, b \in \mathbb{R}^n$ is the symmetric matrix with entries $(a_ib_j + a_jb_i)/2$. The brackets $\langle\cdot,\cdot\rangle$ denote the duality pairing between conjugate L^p spaces, as well as between other pairs of spaces, according to the context.

3. Setting of the problem

Throughout the paper Ω is an open subset of $\mathbb{R}^3$ of the form $\Omega = \omega \times (-\frac{1}{2}, \frac{1}{2})$, where ω is a bounded and connected open set of $\mathbb{R}^2$ with a C^2 boundary. We suppose that the boundary $\partial\omega$ is partitioned into two disjoint open subsets γ_d, γ_n and their common boundary $\partial\lfloor_{\partial\omega}\gamma_d = \partial\lfloor_{\partial\omega}\gamma_n$ (topological notions refer here to the relative topology of $\partial\omega$). We assume that $\gamma_d \neq \emptyset$ and that $\partial\lfloor_{\partial\omega}\gamma_d$ is made of two points in $\partial\omega$. The outer unit normal to $\partial\omega$ is denoted by $\nu_{\partial\omega}$ and the outer unit normal to $\partial\Omega$ by $\nu_{\partial\Omega}$. Moreover, we set $\Gamma_d := \gamma_d \times (-\frac{1}{2}, \frac{1}{2})$.

The elasticity tensor and its inverse. Let $\mathbb{C}_r$ be the *elasticity tensor*, considered as a symmetric positive definite linear operator $\mathbb{C}_r : \mathbb{M}^{2\times2}_{sym} \to \mathbb{M}^{2\times2}_{sym}$ and let $\mathbb{A}_r : \mathbb{M}^{2\times2}_{sym} \to \mathbb{M}^{2\times2}_{sym}$ be its inverse $\mathbb{A}_r := \mathbb{C}_r^{-1}$. It follows that there exist two constants $\alpha_{\mathbb{A}}$ and $\beta_{\mathbb{A}}$, with $0 < \alpha_{\mathbb{A}} \leq \beta_{\mathbb{A}}$, such that

$$\alpha_{\mathbb{A}}|\xi|^2 \leq \tfrac{1}{2}\mathbb{A}_r\xi : \xi \leq \beta_{\mathbb{A}}|\xi|^2 \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{3.1}$$

These inequalities imply

$$|\mathbb{A}_r\xi| \leq 2\beta_{\mathbb{A}}|\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{3.2}$$

The set of admissible stresses. Let K_r be a closed convex set of $\mathbb{M}^{2\times2}_{sym}$ such that there exist two constants α_H and β_H, with $0 < \alpha_H \leq \beta_H$, such that

$$\{\xi \in \mathbb{M}^{2\times2}_{sym} : \ |\xi| \leq \alpha_H\} \subset K_r \subset \{\xi \in \mathbb{M}^{2\times2}_{sym} : \ |\xi| \leq \beta_H\}.$$

The boundary of K_r is interpreted as the *yield surface*. We define the set

$$\mathcal{K}_r(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{2\times2}_{sym}) : \ \sigma(x) \in K_r \text{ for a.e. } x \in \Omega\}.$$

The *plastic dissipation potential* is given by the support function $H_r : \mathbb{M}^{2\times2}_{sym} \to [0, +\infty)$ of K_r, defined as

$$H_r(\xi) := \sup_{\sigma \in K_r} \sigma : \xi \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}.$$

It follows that H_r is a convex and positively one-homogeneous function such that

$$\alpha_H|\xi| \leq H_r(\xi) \leq \beta_H|\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym}. \tag{3.3}$$

In particular, H_r satisfies the triangle inequality

$$H_r(\xi + \zeta) \leq H_r(\xi) + H_r(\zeta) \quad \text{for every } \xi, \zeta \in \mathbb{M}^{2\times2}_{sym}. \tag{3.4}$$

In [5] it is proved that, if $K \subset \mathbb{M}^{3\times3}_{sym}$ is the convex set of admissible stresses for the three-dimensional Prandtl-Reuss plasticity problem, then K_r can be characterized as follows:

$$\xi \in K_r \quad \Leftrightarrow \quad \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{12} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr}\xi)\, I_{3\times3} \in K. \tag{3.5}$$

Thus, in particular, if

$$K = \Big\{\xi \in \mathbb{M}^{3\times3}_{sym} : \ \Big|\xi - \frac{1}{3}(\operatorname{tr}\xi)\, I_{3\times3}\Big| \leq \alpha_0\Big\}$$

for some $\alpha_0 > 0$, then (3.5) implies that

$$K_r = \{\xi \in \mathbb{M}^{2\times2}_{sym} : \ |\xi|_r \leq \alpha_0\} \tag{3.6}$$

and

$$H_r(\xi) = \alpha_0 \left|\xi + (\operatorname{tr}\xi) I_{2\times2}\right|_r \quad \text{for every } \xi \in \mathbb{M}^{2\times2}_{sym},$$

where

$$|\xi|_r^2 := \frac{1}{6}(\xi_{11} + \xi_{22})^2 + \frac{1}{2}(\xi_{11} - \xi_{22})^2 + 2\xi_{12}^2 = |\xi|^2 - \frac{1}{3}(\operatorname{tr}\xi)^2. \tag{3.7}$$

Zero-th and first order moments. For $f \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ we denote by $\bar{f}, \hat{f} \in L^2(\omega;\mathbb{M}^{2\times 2}_{sym})$ and by $f_\perp \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ the following orthogonal components (in the sense of $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$) of f:

$$\bar{f}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x', x_3)\, dx_3, \qquad \hat{f}(x') := 12 \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 f(x', x_3)\, dx_3$$

for a.e. $x' \in \omega$, and

$$f_\perp(x) := f(x) - \bar{f}(x') - x_3 \hat{f}(x')$$

for a.e. $x \in \Omega$. The component $\bar{f}$ is called the *zero-th order moment* of f, while $\hat{f}$ is called the *first order moment* of f.

Analogously, if $q \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times 2}_{sym})$, the *zero-th order moment* of q is the measure $\bar{q} \in M_b(\omega \cup \gamma_d;\mathbb{M}^{2\times 2}_{sym})$ defined by

$$\int_{\omega\cup\gamma_d} \varphi : d\bar{q} := \int_{\Omega\cup\Gamma_d} \varphi : dq$$

for every $\varphi \in C_0(\omega \cup \gamma_d;\mathbb{M}^{2\times 2}_{sym})$, while the *first order moment* of q is the measure $\hat{q} \in M_b(\omega \cup \gamma_d;\mathbb{M}^{2\times 2}_{sym})$ defined by

$$\int_{\omega\cup\gamma_d} \varphi : d\hat{q} := 12 \int_{\Omega\cup\Gamma_d} x_3\varphi : dq$$

for every $\varphi \in C_0(\omega\cup\gamma_d;\mathbb{M}^{2\times 2}_{sym})$. We also define $q_\perp \in M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ as the measure given by

$$q_\perp := q - \bar{q} \otimes \mathcal{L}^1 - \hat{q} \otimes x_3\mathcal{L}^1,$$

where the symbol $\otimes$ denotes the usual product of measures.

Kirchhoff-Love admissible triples. We introduce the set of *Kirchhoff-Love displacements*, defined as

$$KL(\Omega) := \big\{u \in BD(\Omega) :\ (Eu)_{i3} = 0 \quad \text{for } i = 1,2,3\big\}.$$

We note that $u \in KL(\Omega)$ if and only if $u_3 \in BH(\omega)$ and there exists $\bar{u} \in BD(\omega)$ such that

$$u_\alpha = \bar{u}_\alpha - x_3\partial_\alpha u_3, \quad \alpha = 1,2. \tag{3.8}$$

In particular, if $u \in KL(\Omega)$, then Eu can be identified with a 2×2 matrix and $(Eu)_{\alpha\beta} = (E\bar{u})_{\alpha\beta} - x_3\partial^2_{\alpha\beta}u_3$ for $\alpha,\beta = 1,2$. If, in addition, $u \in W^{1,p}(\Omega;\mathbb{R}^3)$, then $\bar{u} \in W^{1,p}(\omega;\mathbb{R}^2)$ and $u_3 \in W^{2,p}(\omega)$. We call $\bar{u}, u_3$ the *Kirchhoff-Love components* of u.

For every $w \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ we define the class $\mathcal{A}_{KL}(w)$ of *Kirchhoff-Love admissible triples* for the boundary datum w as the set of all triples $(u,e,p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{sym}) \times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{3\times 3}_{sym})$ satisfying

$$Eu = e + p \quad \text{in } \Omega, \qquad p = (w-u)\odot\nu_{\partial\Omega}\mathcal{H}^2 \quad \text{on } \Gamma_d,$$
$$e_{i3} = 0 \quad \text{in } \Omega, \quad p_{i3} = 0 \quad \text{in } \Omega\cup\Gamma_d, \quad i = 1,2,3,$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. Note that the space

$$\big\{\xi \in \mathbb{M}^{3\times 3}_{sym} :\ \xi_{i3} = 0 \text{ for } i = 1,2,3\big\}$$

is canonically isomorphic to $\mathbb{M}^{2\times 2}_{sym}$. Therefore, in the following, given a triple $(u,e,p) \in \mathcal{A}_{KL}(w)$ we will systematically identify e with a function in $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ and p with a measure in $M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$. Note also that the class $\mathcal{A}_{KL}(w)$ is always nonempty as it contains the triple $(w, Ew, 0)$.

Let $(u,e,p) \in \mathcal{A}_{KL}(w)$. By definition u is a Kirchhoff-Love displacement, hence $u_3 \in BH(\omega)$ and u_α, $\alpha = 1,2$, is affine in the x_3 variable (see (3.8)). In general, one cannot conclude that e and p are affine in x_3, too. However, some conditions on the structure of e and p can be deduced.

Proposition 3.1. *Let $w \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ and $(u,e,p) \in KL(\Omega) \times L^2(\Omega;\mathbb{M}^{2\times2}_{sym}) \times M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times2}_{sym})$. Let $\bar{u} \in BD(\omega)$, $u_3 \in BH(\omega)$, and $\bar{w} \in W^{1,2}(\omega;\mathbb{R}^2)$, $w_3 \in W^{2,2}(\omega)$ be the Kirchhoff-Love components of u and w, respectively. Finally, let $\bar{e}, \hat{e} \in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$, $e_\perp \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $\bar{p}, \hat{p} \in M_b(\omega \cup \gamma_d;\mathbb{M}^{2\times2}_{sym})$, and $p_\perp \in M_b(\Omega \cup \Gamma_d;\mathbb{M}^{2\times2}_{sym})$ be the moments of e and p. Then $(u,e,p) \in \mathcal{A}_{KL}(w)$ if and only if the following three conditions are satisfied:*

(i) *$E\bar{u} = \bar{e} + \bar{p}$ in ω and $\bar{p} = (\bar{w} - \bar{u}) \odot \nu_{\partial\omega}\mathcal{H}^1$ on γ_d;*
(ii) *$D^2 u_3 = -(\hat{e} + \hat{p})$ in ω, $u_3 = w_3$ on γ_d, and $\hat{p} = (\nabla u_3 - \nabla w_3) \odot \nu_{\partial\omega}\mathcal{H}^1$ on γ_d;*
(iii) *$p_\perp = -e_\perp$ in Ω and $p_\perp = 0$ on Γ_d,*

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure.

Spaces of stresses. We will also use the set

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym}) : \ \mathrm{div}_{x'}\bar{\sigma} \in L^2(\omega;\mathbb{R}^2), \ \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} \in L^2(\omega)\},$$

where $\bar{\sigma}, \hat{\sigma} \in L^2(\omega;\mathbb{M}^{2\times2}_{sym})$ are the zero-th and first order moments of σ.

For every $\sigma \in \Sigma(\Omega)$ we can define the trace $[\bar{\sigma}\nu_{\partial\omega}] \in H^{-\frac{1}{2}}(\partial\omega;\mathbb{R}^2)$ of its zero-th order moment $\bar{\sigma}$ through the formula

$$\langle [\bar{\sigma}\nu_{\partial\omega}], \varphi \rangle := \int_\omega \mathrm{div}_{x'}\bar{\sigma} \cdot \varphi \, dx' + \int_\omega \bar{\sigma} : E\varphi \, dx' \tag{3.9}$$

for every $\varphi \in W^{1,2}(\omega;\mathbb{R}^2)$. Note that, if $\sigma \in \Sigma(\Omega) \cap L^\infty(\Omega;\mathbb{M}^{2\times2}_{sym})$, then $\bar{\sigma} \in L^\infty(\omega;\mathbb{M}^{2\times2}_{sym})$ and equation (3.9) makes sense for every $\varphi \in W^{1,1}(\omega;\mathbb{R}^2)$ (since by Sobolev embedding any such φ belongs to $L^2(\omega;\mathbb{R}^2)$), so that $[\bar{\sigma}\nu_{\partial\omega}]$ can be identified in this case with an element of $L^\infty(\partial\omega;\mathbb{R}^2)$.

We can also give a meaning to the traces of the first order moments of elements in $\Sigma(\Omega)$. More precisely, for every $\sigma \in \Sigma(\Omega)$ there exist $b_0(\hat{\sigma}) \in H^{-\frac{3}{2}}(\partial\omega)$ and $b_1(\hat{\sigma}) \in H^{-\frac{1}{2}}(\partial\omega)$ such that

$$-\langle b_0(\hat{\sigma}), \psi \rangle + \big\langle b_1(\hat{\sigma}), \frac{\partial\psi}{\partial\nu_{\partial\omega}} \big\rangle = \int_\omega \hat{\sigma} : D^2\psi \, dx' - \int_\omega \psi \, \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} \, dx' \tag{3.10}$$

for every $\psi \in W^{2,2}(\omega)$. Moreover, if $\hat{\sigma} \in C^2(\overline{\omega};\mathbb{M}^{2\times2}_{sym})$, then

$$\begin{aligned} b_0(\hat{\sigma}) &= \mathrm{div}_{x'}\hat{\sigma} \cdot \nu_{\partial\omega} + \frac{\partial}{\partial\tau_{\partial\omega}}(\hat{\sigma}\nu_{\partial\omega} \cdot \tau_{\partial\omega}), \\ b_1(\hat{\sigma}) &= \hat{\sigma}\nu_{\partial\omega} \cdot \nu_{\partial\omega}, \end{aligned}$$

where $\tau_{\partial\omega}$ is the tangent vector to $\partial\omega$ (see, e.g., [6, Théorème 2.1]). Note that, if $\sigma \in \Sigma(\Omega) \cap L^\infty(\Omega;\mathbb{M}^{2\times2}_{sym})$, then $\hat{\sigma} \in L^\infty(\Omega;\mathbb{M}^{2\times2}_{sym})$ and the right-hand side of (3.10) makes sense for every $\psi \in W^{2,1}(\omega)$, so that $b_0(\hat{\sigma})$ can be identified in this case with an element of $(T(W^{2,1}(\omega)))'$, the dual of the space of traces of $W^{2,1}(\omega)$ functions, and $b_1(\hat{\sigma})$ with an element of $L^\infty(\partial\omega)$ (see [6, Théorèm 2.3]).

For $h \in H^{-\frac{1}{2}}(\partial\omega;\mathbb{R}^2)$ and $m = (m_0, m_1) \in H^{-\frac{3}{2}}(\partial\omega) \times H^{-\frac{1}{2}}(\partial\omega)$ we define $\Theta(\gamma_n, h, m)$ as the class of all $\sigma \in \Sigma(\Omega)$ such that

$$\langle [\bar{\sigma}\nu_{\partial\omega}] - h, \varphi \rangle = 0 \tag{3.11}$$

for every $\varphi \in H^{\frac{1}{2}}(\partial\omega;\mathbb{R}^2)$ satisfying $\varphi = 0$ on γ_d, and

$$\langle b_0(\hat{\sigma}) - m_0, \psi_0 \rangle = \langle b_1(\hat{\sigma}) - m_1, \psi_1 \rangle = 0 \tag{3.12}$$

for every $\psi_0 \in H^{\frac{3}{2}}(\partial\omega)$ satisfying $\psi_0 = 0$ on γ_d and every $\psi_1 \in H^{\frac{1}{2}}(\partial\omega)$ satisfying $\psi_1 = 0$ on γ_d.

In the next proposition we prove the closure of the class $\Theta(\gamma_n, h, m)$ with respect to weak convergence.

Proposition 3.2. *Let $h \in H^{-\frac{1}{2}}(\partial\omega;\mathbb{R}^2)$ and let $m = (m_0, m_1) \in H^{-\frac{3}{2}}(\partial\omega) \times H^{-\frac{1}{2}}(\partial\omega)$. Let $(\sigma^k)_k$ be a sequence in $\Theta(\gamma_n, h, m)$ such that $\sigma^k \rightharpoonup \sigma$ weakly in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$, $\mathrm{div}_{x'}\bar{\sigma}^k \rightharpoonup f$ weakly in $L^2(\omega;\mathbb{R}^2)$, and $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}^k \rightharpoonup g$ weakly in $L^2(\omega)$, as $k \to \infty$. Then $\sigma \in \Theta(\gamma_n, h, m)$ and $\mathrm{div}_{x'}\bar{\sigma} = f$, $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} = g$ in ω.*

Proof. It is immediate to see that $\sigma \in \Sigma(\Omega)$ and $\mathrm{div}_{x'}\bar\sigma = f$, $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma = g$ in ω. Passing to the limit in (3.9), we deduce that $[\bar\sigma^k \nu_{\partial\omega}] \rightharpoonup [\bar\sigma\nu_{\partial\omega}]$ weakly in $H^{-\frac12}(\partial\omega;\mathbb{R}^2)$. Therefore, (3.11) is satisfied.

For every $\psi_0 \in H^{\frac32}(\partial\omega)$ we can construct $\psi \in W^{2,2}(\omega)$ such that $\psi = \psi_0$ on $\partial\omega$ and $\partial\psi/\partial\nu_{\partial\omega} = 0$ on $\partial\omega$. Passing to the limit in (3.10) with this choice of ψ, we obtain

$$\langle b_0(\hat\sigma^k), \psi_0\rangle \to \langle b_0(\hat\sigma), \psi_0\rangle$$

for every $\psi_0 \in H^{\frac32}(\partial\omega)$. Arguing analogously, we can prove that

$$\langle b_1(\hat\sigma^k), \psi_1\rangle \to \langle b_1(\hat\sigma), \psi_1\rangle$$

for every $\psi_1 \in H^{\frac12}(\partial\omega)$. Therefore, (3.12) is satisfied. □

We also have the following characterization.

Proposition 3.3. *Let $\sigma \in L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$. Then*

$$\int_\Omega \sigma : Ev\,dx = 0 \tag{3.13}$$

for every $v \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ such that $v = 0$ on Γ_d if and only if $\sigma \in \Theta(\gamma_n, 0, 0)$ and $\mathrm{div}_{x'}\bar\sigma = 0$, $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma = 0$ *in ω.*

Proof. Since

$$\int_\Omega \sigma : Ev\,dx = \int_\omega \bar\sigma : E\bar v\,dx' - \frac{1}{12}\int_\omega \hat\sigma : D^2 v_3\,dx'$$

for every $v \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$, condition (3.13) is equivalent to the two following conditions:

(a) for every $\varphi \in W^{1,2}(\omega;\mathbb{R}^2)$ with $\varphi = 0$ on γ_d

$$\int_\omega \bar\sigma : E\varphi\,dx' = 0;$$

(b) for every $\psi \in W^{2,2}(\omega)$ with $\psi = 0$ and $\nabla\psi = 0$ on γ_d

$$\int_\omega \hat\sigma : D^2\psi\,dx' = 0.$$

By (3.9) condition (a) is equivalent to $\mathrm{div}_{x'}\bar\sigma = 0$ in ω and $\langle[\bar\sigma\nu_{\partial\omega}], \varphi\rangle = 0$ for every $\varphi \in H^{\frac12}(\partial\omega;\mathbb{R}^2)$ satisfying $\varphi = 0$ on γ_d. Similarly, by (3.10) condition (b) is equivalent to $\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma = 0$ in ω and

$$\langle b_0(\hat\sigma), \psi\rangle - \Big\langle b_1(\hat\sigma), \frac{\partial\psi}{\partial\nu_{\partial\omega}}\Big\rangle = 0 \tag{3.14}$$

for every $\psi \in W^{2,2}(\omega)$ with $\psi = 0$ and $\nabla\psi = 0$ on γ_d. Since for every $\psi_0 \in H^{\frac32}(\partial\omega)$ there exists $\psi \in W^{2,2}(\omega)$ such that $\psi = \psi_0$ and $\partial\psi/\partial\nu_{\partial\omega} = 0$ on $\partial\omega$ and for every $\psi_1 \in H^{\frac12}(\partial\omega)$ there exists $\psi \in W^{2,2}(\omega)$ such that $\psi = 0$ and $\partial\psi/\partial\nu_{\partial\omega} = \psi_1$ on $\partial\omega$, condition (3.14) is in turn equivalent to

$$\langle b_0(\hat\sigma), \psi_0\rangle = \langle b_1(\hat\sigma), \psi_1\rangle = 0$$

for every $\psi_0 \in H^{\frac32}(\partial\omega)$ satisfying $\psi_0 = 0$ on γ_d and every $\psi_1 \in H^{\frac12}(\partial\omega)$ satisfying $\psi_1 = 0$ on γ_d. □

Remark 3.4. Let $\sigma \in \Theta(\gamma_n, 0, 0) \cap L^p(\Omega;\mathbb{M}^{2\times2}_{sym})$ for some $2 \le p \le \infty$. Then

$$\int_\Omega \sigma : Ev\,dx = -\int_\omega \mathrm{div}_{x'}\bar\sigma \cdot \bar v\,dx' + \frac{1}{12}\int_\omega v_3\,\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma\,dx' \tag{3.15}$$

for every $v \in W^{1,p'}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ such that $v = 0$ on Γ_d, where p' is the conjugate exponent of p. Note that the right-hand side is well defined for such a regularity of v, since $\bar v \in L^2(\omega;\mathbb{R}^2)$ and $v_3 \in W^{1,2}(\omega)$ by Sobolev embedding.

Equation (3.15) clearly holds for every $v \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ such that $v = 0$ on Γ_d by the definition of $\Theta(\gamma_n, 0, 0)$ and can be extended to functions v of $W^{1,p'}$ regularity by approximation (see, e.g., [5, Lemma 7.10]).

Stress-strain duality. In the following we will consider the space $\Pi_{\Gamma_d}(\Omega)$ of admissible plastic strains, defined as the class of all $p \in M_b(\Omega \cup \Gamma_d; \mathbb{M}^{2\times 2}_{sym})$ for which there exist $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$, and $w \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ such that $(u, e, p) \in \mathcal{A}_{KL}(w)$. Following [5, Section 7.1], for every $p \in \Pi_{\Gamma_d}(\Omega)$ and $\sigma \in \Sigma(\Omega) \cap L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$ we can give a meaning to the product $[\sigma : p]$ as a measure in $M_b(\Omega \cup \Gamma_d)$. We refer to [5, Section 7.1] for the precise definition and the main properties of this duality product. Here we just note that, if $\sigma \in \Sigma(\Omega) \cap L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$ is such that $\bar\sigma, \hat\sigma \in C(\overline{\omega}; \mathbb{M}^{2\times 2}_{sym})$, then

$$\int_{\Omega\cup\Gamma_d} \varphi\, d[\sigma : p] = \int_{\omega\cup\gamma_d} \varphi\bar\sigma : d\bar p + \frac{1}{12}\int_{\omega\cup\gamma_d} \varphi\hat\sigma : d\hat p + \int_\Omega \varphi\sigma_\perp : p_\perp\, dx$$

for every $\varphi \in C(\overline{\omega})$. The last integral makes sense since $p_\perp \in L^2(\Omega; \mathbb{M}^{2\times 2}_{sym})$ by Proposition 3.1-(iii).

For every $p \in \Pi_{\Gamma_d}(\Omega)$ and $\sigma \in \Sigma(\Omega)\cap L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$ the duality product $\langle\sigma, p\rangle$ is defined as

$$\langle\sigma, p\rangle := [\sigma : p](\Omega \cup \Gamma_d).$$

Using this notion of duality, a variant of equality (2.1) can be proved. More precisely, by [5, Proposition 7.8] we have that for every $p \in \Pi_{\Gamma_d}(\Omega)$

$$\mathcal{H}_r(p) = \sup\{\langle\sigma, p\rangle : \ \sigma \in \Sigma(\Omega) \cap \mathcal{K}_r(\Omega)\}. \tag{3.16}$$

Moreover, the following integration by parts formula holds.

Proposition 3.5. *Let $\sigma \in \Sigma(\Omega) \cap L^\infty(\Omega; \mathbb{M}^{2\times 2}_{sym})$, $w \in W^{1,2}(\Omega;\mathbb{R}^2) \cap KL(\Omega)$, and $(u, e, p) \in \mathcal{A}_{KL}(w)$. Then*

$$\begin{aligned}
&\int_{\Omega\cup\Gamma_d} \varphi\, d[\sigma : p] + \int_\Omega \varphi\,\sigma : (e - Ew)\, dx \\
&= \ -\int_\omega \bar\sigma : \big(\nabla\varphi \odot (\bar u - \bar w)\big)\, dx' - \int_\omega \mathrm{div}_{x'}\bar\sigma \cdot \varphi\,(\bar u - \bar w)\, dx' + \int_{\gamma_n} [\bar\sigma\nu_{\partial\omega}] \cdot \varphi\,(\bar u - \bar w)\, d\mathcal{H}^1 \\
&\quad + \frac{1}{12}\int_\omega \hat\sigma : (u_3 - w_3)D^2\varphi\, dx' + \frac{1}{6}\int_\omega \hat\sigma : \big(\nabla\varphi \odot (\nabla u_3 - \nabla w_3)\big)\, dx' \\
&\quad - \frac{1}{12}\int_\omega (u_3 - w_3)\varphi\, \mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma\, dx' + \frac{1}{12}\langle b_0(\hat\sigma), \varphi(u_3 - w_3)\rangle \\
&\quad - \frac{1}{12}\int_{\gamma_n} b_1(\hat\sigma)\frac{\partial\big(\varphi(u_3 - w_3)\big)}{\partial\nu_{\partial\omega}}\, d\mathcal{H}^1
\end{aligned} \tag{3.17}$$

for every $\varphi \in C^2(\overline{\omega})$.

Proof. Note that the duality product on the right-hand side of (3.17) is well defined since $T(W^{2,1}(\omega)) = T(BH(\omega))$ (see, e.g., [7, Section 2]). For the proof we refer to [5, Propositions 7.2 and 7.6]. □

4. The quasistatic evolution problem: regularity

In this section we describe the quasistatic evolution problem and prove local regularity of the stress. The data of the problem are the prescribed boundary displacement and the applied forces. More precisely, for every $t \in [0, T]$ we prescribe a boundary displacement $w(t) \in KL(\Omega)\cap W^{1,2}(\Omega;\mathbb{R}^3)$ on Γ_d. We assume that $t \mapsto w(t)$ is a $W^{1,2}$ map from $[0, T]$ into $W^{1,2}(\Omega;\mathbb{R}^3)$. For every $t \in [0, T]$ we consider a body force $(f(t), g(t)) \in L^2(\omega;\mathbb{R}^2) \times L^2(\omega)$ and surface forces $h(t) \in L^\infty(\partial\omega;\mathbb{R}^2)$ and $m(t) = (m_0(t), m_1(t)) \in (T(W^{2,1}(\omega))' \times L^\infty(\partial\omega)$. We assume that $t \mapsto (f(t), g(t))$, $t \mapsto h(t)$, and $t \mapsto m(t)$ are $W^{1,2}$ maps from $[0, T]$ into their respective spaces. Moreover, the following *uniform safe-load condition* is assumed: there exist a map $\varrho \in W^{1,2}(0,T;L^\infty(\Omega;\mathbb{M}^{2\times 2}_{sym}))$ and a constant $\alpha_1 > 0$ such that for every $t \in [0,T]$

$$\begin{gathered} -\mathrm{div}_{x'}\bar{\varrho}(t) = f(t), \quad -\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\varrho}(t) = g(t) \quad \text{in } \omega, \\ \varrho(t) \in \Theta(\gamma_n, h(t), m(t)), \end{gathered} \tag{4.1}$$

and

$$\varrho(t,x) + \xi \in K_r \tag{4.2}$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{M}^{2\times 2}_{sym}$ with $|\xi|_r \le \alpha_1$.

Definition 4.1. Let $t \mapsto (u(t), e(t), p(t))$ be a function from $[0,T]$ into $BD(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}) \times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times 2}_{sym})$ and let $\sigma(t) := \mathbb{C}_r e(t)$. We say that $t \mapsto (u(t), e(t), p(t))$ is a *quasistatic evolution* if the following three conditions are satisfied:

(qs1) *regularity:* $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous;

(qs2) *equilibrium:* for every $t \in [0,T]$ we have $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$,

$$\begin{gathered} \sigma(t) \in \mathcal{K}_r(\Omega) \cap \Theta(\gamma_n, h(t), m(t)), \\ -\mathrm{div}_{x'}\bar{\sigma}(t) = f(t), \quad -\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = g(t) \text{ in } \omega; \end{gathered}$$

(qs3) *flow rule:* for a.e. $t \in [0,T]$ there holds

$$\mathcal{H}_r(\dot{p}(t)) = \langle \sigma(t), \dot{p}(t)\rangle.$$

We observe that by (3.16) the flow rule is equivalent to the following *maximum plastic work condition*: for a.e. $t \in [0,T]$

$$\langle \vartheta - \sigma(t), \dot{p}(t)\rangle \le 0 \tag{4.3}$$

for every $\vartheta \in \mathcal{K}_r(\Omega) \cap \Sigma(\Omega)$.

In [5] existence of a quasistatic evolution is proved assuming the body and surface forces to be zero. Under the assumptions (4.1)–(4.2), existence of a quasistatic evolution in presence of applied forces can be proved by applying the abstract method for rate-independent processes [14], namely by discretizing time and by solving suitable incremental minimum problems. This method leads to a weaker notion of quasistatic evolution, which can be proved to be equivalent to that of Definition 4.1 arguing as in [4, Sections 5 and 6].

In this paper we focus on the spatial regularity for quasistatic evolutions in case of smooth applied forces, under the additional assumption that the set K_r of admissible stresses is of the form (3.6)–(3.7). We note that $|\cdot|_r$ is an anisotropic norm on $\mathbb{M}^{2\times 2}_{sym}$ satisfying

$$\frac{1}{\sqrt{3}}\,|\xi| \le |\xi|_r \le |\xi| \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{4.4}$$

We also consider the inner product associated with this norm:

$$(\xi,\zeta)_r := \xi : \zeta - \frac{1}{3}\mathrm{tr}\,\xi\,\mathrm{tr}\,\zeta \quad \text{for every } \xi,\zeta \in \mathbb{M}^{2\times 2}_{sym}.$$

We now introduce some approximating problems of (qs1)–(qs3). Let $N \in \mathbb{N}$, $N \ge 4$ be a fixed parameter. We consider the function $\phi_N : \mathbb{M}^{2\times 2}_{sym} \to [0,+\infty)$ defined by

$$\phi_N(\xi) := \frac{1}{N\alpha_0^{N-1}}|\xi|_r^N \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

The function ϕ_N is clearly convex and continuously differentiable with differential

$$D\phi_N(\xi) = \frac{1}{\alpha_0^{N-1}}|\xi|_r^{N-2}\Big(\xi - \frac{1}{3}(\mathrm{tr}\,\xi)\,I_{2\times 2}\Big) \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}.$$

Moreover, we have that

$$D\phi_N(\xi) : \xi = \frac{1}{\alpha_0^{N-1}}|\xi|_r^N \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{sym}. \tag{4.5}$$

Through ϕ_N we define some approximating problems of Norton-Hoff type, for which existence of solutions is proved in the following theorem.

Theorem 4.2. *Let K_r be of the form (3.6)–(3.7). Let $w \in W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3)\cap KL(\Omega))$, $f \in W^{1,2}(0,T;L^2(\omega;\mathbb{R}^2))$, $g \in W^{1,2}(0,T;L^2(\omega))$, $h \in W^{1,2}(0,T;L^\infty(\partial\omega;\mathbb{R}^2))$, and $m \in W^{1,2}(0,T;(T(W^{2,1}(\omega))' \times L^\infty(\partial\omega))$. Assume conditions (4.1) and (4.2) with $\varrho \in W^{1,\infty}(0,T;L^\infty(\Omega;\mathbb{M}^{2\times2}_{sym}))$. Let $\sigma_0 \in L^\infty(\Omega;\mathbb{M}^{2\times2}_{sym})$ be such that*

$$-\mathrm{div}_{x'}\bar{\sigma}_0 = f(0), \quad -\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}_0 = g(0) \quad \text{in } \omega,$$
$$\sigma_0 \in \mathcal{K}_r(\Omega)\cap\Theta(\gamma_n, h(0), m(0)).$$

Then for every $N\in\mathbb{N}$, $N\geq 4$, the problem

$$\begin{cases} \mathbb{A}_r\dot{\sigma}(t) + D\phi_N(\sigma(t)) = Ev(t) & \text{in } \Omega,\\ -\mathrm{div}_{x'}\bar{\sigma}(t) = f(t), \quad -\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = g(t) & \text{in } \omega,\\ \sigma(t)\in\Theta(\gamma_n, h(t), m(t)), & \\ v(t) = \dot{w}(t) & \text{on } \Gamma_d, \end{cases} \tag{4.6}$$

has one and only one solution (σ^N, v^N) with

$$\sigma^N \in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym}))\cap L^\infty(0,T;L^N(\Omega;\mathbb{M}^{2\times2}_{sym})),$$
$$v^N \in L^{N/(N-1)}(0,T;W^{1,N/(N-1)}(\Omega;\mathbb{R}^2)\cap KL(\Omega))$$

and satisfying $\sigma^N(0)=\sigma_0$.

Moreover, the following estimates hold:

$$\sup_{t\in[0,T]}\|\sigma^N(t)\|_{L^2}\leq C, \qquad \int_0^T \|\dot{\sigma}^N(t)\|^2_{L^2}\,dt \leq C, \tag{4.7}$$

$$\sup_{t\in[0,T]}\int_\Omega |\sigma^N(t)|^N_r\,dx \leq CN\alpha_0^{N-1}, \qquad \int_0^T\int_\Omega |\sigma^N(t)|^N_r\,dx\,dt\leq C\alpha_0^{N-1}, \tag{4.8}$$

and

$$\|v^N\|_{L^2(0,T;BD(\Omega))}\leq C, \tag{4.9}$$

where C is a constant independent of N.

Proof. Let us fix $N\in\mathbb{N}$, $N\geq 4$. For every $\lambda>0$ we introduce the functions $\psi_\lambda : \mathbb{M}^{2\times2}_{sym}\to[0,+\infty)$ defined by

$$\psi_\lambda(\xi) := \frac{1}{N\alpha_0^{N-1}}(|\xi|^N_r\wedge\lambda^N) + \frac{1}{2\alpha_0^{N-1}}\lambda^{N-2}(|\xi|^2_r-\lambda^2)^+$$

for every $\xi\in\mathbb{M}^{2\times2}_{sym}$. Note that ψ_λ is strictly convex and C^1. Moreover, we have that

$$D\psi_\lambda(\xi) = \frac{1}{\alpha_0^{N-1}}(|\xi|^{N-2}_r\wedge\lambda^{N-2})\Big(\xi - \frac{1}{3}(\mathrm{tr}\,\xi)\,I_{2\times2}\Big)$$

for every $\xi\in\mathbb{M}^{2\times2}_{sym}$, hence

$$D\psi_\lambda(\xi):\zeta = \frac{1}{\alpha_0^{N-1}}\big(|\xi|^{N-2}_r\wedge\lambda^{N-2}\big)\,(\xi,\zeta)_r, \tag{4.10}$$

$$|D\psi_\lambda(\xi)|^{N/(N-1)} \leq \frac{1}{\alpha_0}D\psi_\lambda(\xi):\xi, \tag{4.11}$$

for every $\xi,\zeta\in\mathbb{M}^{2\times2}_{sym}$. Finally, we observe that the functions $D\psi_\lambda$ are Lipschitz continuous on $\mathbb{M}^{2\times2}_{sym}$ with

$$|D^2\psi_\lambda(\xi)|\leq\frac{N+1}{\alpha_0^{N-1}}\lambda^{N-2} \tag{4.12}$$

for a.e. $\xi\in\mathbb{M}^{2\times2}_{sym}$.

Lemma 4.3 below guarantees that for every $\lambda>0$ there exists a unique pair

$$(\sigma^\lambda, v^\lambda)\in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym}))\times L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)\cap KL(\Omega))$$

satisfying

$$\begin{cases} \mathbb{A}_r\dot{\sigma}^\lambda(t) + D\psi_\lambda(\sigma^\lambda(t)) = Ev^\lambda(t) & \text{in } \Omega, \\ -\mathrm{div}_{x'}\bar{\sigma}^\lambda(t) = f(t), \quad -\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}^\lambda(t) = g(t) & \text{in } \omega, \\ \sigma^\lambda(t) \in \Theta(\gamma_n, h(t), m(t)), & \\ v^\lambda(t) = \dot{w}(t) & \text{on } \Gamma_d, \\ \sigma^\lambda(0) = \sigma_0. & \end{cases} \tag{4.13}$$

We now want to pass to the limit in (4.13), as $\lambda \to +\infty$. This will provide us with a solution to problem (4.6). In the following estimates we will stress the dependence of the involved constants on the parameters λ and N. Unless otherwise stated, C denotes a positive constant independent of λ and N.

Multiplying the first equation in (4.13) by $\sigma^\lambda(t) - \varrho(t)$ and integrating over Ω yield

$$\langle \mathbb{A}_r\dot{\sigma}^\lambda(t), \sigma^\lambda(t) - \varrho(t)\rangle + \langle D\psi_\lambda(\sigma^\lambda(t)), \sigma^\lambda(t) - \varrho(t)\rangle = \langle E\dot{w}(t), \sigma^\lambda(t) - \varrho(t)\rangle, \tag{4.14}$$

where the brackets $\langle\cdot,\cdot\rangle$ denote the scalar product in $L^2(\Omega;\mathbb{M}^{2\times2}_{sym})$ and we used that

$$\langle Ev^\lambda(t), \sigma^\lambda(t) - \varrho(t)\rangle = \langle E\dot{w}(t), \sigma^\lambda(t) - \varrho(t)\rangle$$

by Proposition 3.3 and (4.1). Integrating with respect to time, (4.14) implies that

$$\begin{aligned} &\frac{1}{2}\langle \mathbb{A}_r\big(\sigma^\lambda(t) - \varrho(t)\big), \sigma^\lambda(t) - \varrho(t)\rangle + \int_0^t \langle D\psi_\lambda(\sigma^\lambda(s)), \sigma^\lambda(s) - \varrho(s)\rangle\, ds \\ &\qquad = \frac{1}{2}\langle \mathbb{A}_r\big(\sigma_0 - \varrho(0)\big), \sigma_0 - \varrho(0)\rangle + \int_0^t \langle E\dot{w}(s) - \mathbb{A}_r\dot{\varrho}(s), \sigma^\lambda(s) - \varrho(s)\rangle\, ds. \end{aligned} \tag{4.15}$$

Since ψ_λ is convex, we have

$$\big(D\psi_\lambda(\xi) - D\psi_\lambda(\zeta)\big) : (\xi - \zeta) \geq 0 \quad \text{for every } \xi, \zeta \in \mathbb{M}^{2\times2}_{sym}. \tag{4.16}$$

Therefore, we infer

$$\begin{aligned} &\frac{1}{2}\langle \mathbb{A}_r\big(\sigma^\lambda(t) - \varrho(t)\big), \sigma^\lambda(t) - \varrho(t)\rangle \\ &\quad \leq \frac{1}{2}\langle \mathbb{A}_r\big(\sigma_0 - \varrho(0)\big), \sigma_0 - \varrho(0)\rangle + \int_0^t \langle E\dot{w}(s) - \mathbb{A}_r\dot{\varrho}(s) - D\psi_\lambda(\varrho(s)), \sigma^\lambda(s) - \varrho(s)\rangle\, ds. \end{aligned}$$

Note that, by (4.2), the term $D\psi_\lambda(\varrho(s))$ is uniformly bounded independently of λ and N for $\lambda \geq \alpha_0$. Thus, by Cauchy's inequality and (3.1) we deduce

$$\sup_{t\in[0,T]} \|\sigma^\lambda(t)\|_{L^2} \leq C \quad \text{and} \quad \int_0^T \langle D\psi_\lambda(\sigma^\lambda(t)), \sigma^\lambda(t) - \varrho(t)\rangle\, dt \leq C. \tag{4.17}$$

From the second estimate above and (4.10) it follows

$$\frac{1}{\alpha_0^{N-1}}\int_0^T\int_\Omega \big(|\sigma^\lambda(t)|_r^{N-1} \wedge \lambda^{N-2}|\sigma^\lambda(t)|_r\big)\big(|\sigma^\lambda(t)|_r - |\varrho(t)|_r\big)\, dx\, dt \leq C. \tag{4.18}$$

Let now $A_\lambda(t) := \{x \in \Omega : \ |\sigma^\lambda(t)|_r \geq \alpha_0\}$. Condition (4.2) guarantees that $|\sigma^\lambda(t)|_r - |\varrho(t)|_r \geq \alpha_1$ on $A_\lambda(t)$, thus, since the integrand is uniformly bounded from below on $\Omega\backslash A_\lambda(t)$, we have

$$\frac{\alpha_1}{\alpha_0^{N-1}}\int_0^T\int_{A_\lambda(t)} \big(|\sigma^\lambda(t)|_r^{N-1} \wedge \lambda^{N-2}|\sigma^\lambda(t)|_r\big)\, dx\, dt \leq C.$$

Combining this inequality with (4.18) and the fact that $|\varrho(t)|_r \leq \alpha_0$, we obtain

$$\frac{1}{\alpha_0^{N-1}}\int_0^T\int_{A_\lambda(t)} \big(|\sigma^\lambda(t)|_r^{N} \wedge \lambda^{N-2}|\sigma^\lambda(t)|_r^2\big)\, dx\, dt \leq C.$$

By definition of $A_\lambda(t)$ the integrand is clearly bounded on the complement of $A_\lambda(t)$; thus, we conclude that

$$\frac{1}{\alpha_0^{N-1}}\int_0^T\int_\Omega \big(|\sigma^\lambda(t)|_r^N\wedge\lambda^{N-2}|\sigma^\lambda(t)|_r^2\big)\,dx\,dt\le C. \tag{4.19}$$

We now derive a bound on $\dot\sigma^\lambda$. We test the first equation in (4.13) with $\dot\sigma^\lambda(t)-\dot\varrho(t)$:

$$\langle\mathbb{A}_r\dot\sigma^\lambda(t),\dot\sigma^\lambda(t)-\dot\varrho(t)\rangle+\langle D\psi_\lambda(\sigma^\lambda(t)),\dot\sigma^\lambda(t)-\dot\varrho(t)\rangle=\langle E\dot w(t),\dot\sigma^\lambda(t)-\dot\varrho(t)\rangle.$$

Integrating with respect to time, we obtain

$$\begin{aligned}\int_0^t\langle\mathbb{A}_r(\dot\sigma^\lambda(s)-\dot\varrho(s)),\dot\sigma^\lambda(s)-\dot\varrho(s)\rangle\,ds+\int_\Omega\psi_\lambda(\sigma^\lambda(t))\,dx\\ =\int_\Omega\psi_\lambda(\sigma_0)\,dx+\int_0^t\langle D\psi_\lambda(\sigma^\lambda(s)),\dot\varrho(s)\rangle\,ds+\int_0^t\langle E\dot w(s)-\mathbb{A}_r\dot\varrho(s),\dot\sigma^\lambda(s)-\dot\varrho(s)\rangle\,ds.\end{aligned} \tag{4.20}$$

Note that, since $\sigma_0\in K_r$ a.e. in Ω, for $\lambda\ge\alpha_0$ the first term on the right-hand side is uniformly bounded independently of λ and N. Moreover, from (4.10) and (4.11) it follows immediately that

$$\int_\Omega|D\psi_\lambda(\sigma^\lambda(t))|^{N/(N-1)}\,dx\le\frac{1}{\alpha_0^N}\int_\Omega(|\sigma^\lambda(t)|_r^N\wedge\lambda^{N-2}|\sigma^\lambda(t)|_r^2)\,dx,$$

so that (4.19) implies that the sequence $\big(D\psi_\lambda(\sigma^\lambda)\big)$ is uniformly bounded with respect to λ in $L^{N/(N-1)}(0,T;L^{N/(N-1)}(\Omega;\mathbb{M}^{2\times2}_{sym}))$. This fact, together with the assumption that $\dot\varrho\in L^\infty((0,T)\times\Omega;\mathbb{M}^{2\times2}_{sym})$, guarantees that the second term on the right-hand side of (4.20) is uniformly bounded, as well. Thus, by Cauchy's inequality we have

$$\int_0^T\|\dot\sigma^\lambda(t)\|_{L^2}^2\,dt\le C\quad\text{and}\quad\sup_{t\in[0,T]}\int_\Omega\psi_\lambda(\sigma^\lambda(t))\,dx\le C. \tag{4.21}$$

By Ascoli-Arzelà Theorem we deduce from the first estimates in (4.17) and (4.21) that there esists a subsequence (not relabelled) and a function $\sigma^N\in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym}))$ such that

$$\sigma^\lambda(t)\rightharpoonup\sigma^N(t)\quad\text{weakly in }L^2(\Omega;\mathbb{M}^{2\times2}_{sym}), \tag{4.22}$$

as $\lambda\to+\infty$, for every $t\in[0,T]$ and

$$\sigma^\lambda\rightharpoonup\sigma^N\quad\text{weakly in }W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym})), \tag{4.23}$$

as $\lambda\to+\infty$. By (4.22) it is clear that $\sigma^N(0)=\sigma_0$ and, in view of Proposition 3.2, that $-\mathrm{div}_{x'}\bar\sigma^N(t)=f(t)$, $-\mathrm{div}_{x'}\mathrm{div}_{x'}\hat\sigma^N(t)=g(t)$ in ω, and $\sigma^N(t)\in\Theta(\gamma_n,h(t),m(t))$ for every $t\in[0,T]$.

Setting $\tau^\lambda(t):=\chi_{\{|\sigma^\lambda(t)|_r\le\lambda\}}\sigma^\lambda(t)$, the second estimate in (4.21) yields

$$\sup_{t\in[0,T]}\int_\Omega|\tau^\lambda(t)|_r^N\,dx\le CN\alpha_0^{N-1}, \tag{4.24}$$

while by (4.19) we have

$$\int_0^T\int_\Omega|\sigma^\lambda(t)-\tau^\lambda(t)|_r^2\,dx\le C\frac{\alpha_0^{N-1}}{\lambda^{N-2}}. \tag{4.25}$$

Therefore, $\sigma^\lambda-\tau^\lambda\to0$ strongly in $L^2(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym}))$, as $\lambda\to+\infty$. Together with (4.24) and (4.23), this implies that $\sigma^N\in L^\infty(0,T;L^N(\Omega;\mathbb{M}^{2\times2}_{sym}))$, the first inequality in (4.8) is satisfied, and

$$\tau^\lambda\rightharpoonup\sigma^N\quad\text{weakly}^*\text{ in }L^\infty(0,T;L^N(\Omega;\mathbb{M}^{2\times2}_{sym})), \tag{4.26}$$

as $\lambda\to+\infty$. Moreover, by (4.19) we deduce the second inequality in (4.8).

Finally, the uniform bound of $\big(D\psi_\lambda(\sigma^\lambda)\big)$ in $L^{N/(N-1)}(0,T;L^{N/(N-1)}(\Omega;\mathbb{M}^{2\times 2}_{sym}))$ with respect to λ implies that, up to subsequences,

$$D\psi_\lambda(\sigma^\lambda) \rightharpoonup \gamma^N \quad \text{weakly in } L^{N/(N-1)}(0,T;L^{N/(N-1)}(\Omega;\mathbb{M}^{2\times 2}_{sym})), \tag{4.27}$$

as $\lambda \to \infty$, for some function $\gamma^N \in L^{N/(N-1)}(0,T;L^{N/(N-1)}(\Omega;\mathbb{M}^{2\times 2}_{sym}))$.

We now want to prove that $\gamma^N = D\phi_N(\sigma^N)$. To this purpose we proceed as follows. We multiply the first equation in (4.13) by $\sigma^N(t) - \varrho(t)$. Integration in space and time yields

$$\int_0^T \langle \mathbb{A}_r \dot\sigma^\lambda(t), \sigma^N(t) - \varrho(t)\rangle\,dt + \int_0^T \langle D\psi_\lambda(\sigma^\lambda(t)), \sigma^N(t) - \varrho(t)\rangle\,dt = \int_0^T \langle E\dot w(t), \sigma^N(t) - \varrho(t)\rangle\,dt.$$

Passing to the limit as $\lambda \to +\infty$, we deduce

$$\int_0^T \langle \mathbb{A}_r \dot\sigma^N(t), \sigma^N(t) - \varrho(t)\rangle\,dt + \int_0^T \langle \gamma^N(t), \sigma^N(t) - \varrho(t)\rangle\,dt = \int_0^T \langle E\dot w(t), \sigma^N(t) - \varrho(t)\rangle\,dt. \tag{4.28}$$

We now go back to identity (4.15) for $t = T$ and observe that by (4.23) the right-hand side of (4.15) converges, as $\lambda \to +\infty$, to

$$\frac{1}{2}\langle \mathbb{A}_r\big(\sigma_0 - \varrho(0)\big), \sigma_0 - \varrho(0)\rangle + \int_0^T \langle E\dot w(s) - \mathbb{A}_r\dot\varrho(s), \sigma^N(s) - \varrho(s)\rangle\,ds.$$

Thus, we deduce

$$\begin{aligned}
&\limsup_{\lambda\to+\infty} \int_0^T \langle D\psi_\lambda(\sigma^\lambda(s)), \sigma^\lambda(s) - \varrho(s)\rangle\,ds \\
&\quad\le \frac{1}{2}\langle \mathbb{A}_r\big(\sigma_0 - \varrho(0)\big), \sigma_0 - \varrho(0)\rangle - \liminf_{\lambda\to+\infty} \frac{1}{2}\langle \mathbb{A}_r\big(\sigma^\lambda(T) - \varrho(T)\big), \sigma^\lambda(T) - \varrho(T)\rangle \\
&\qquad + \int_0^T \langle E\dot w(s) - \mathbb{A}_r\dot\varrho(s), \sigma^N(s) - \varrho(s)\rangle\,ds.
\end{aligned}$$

On the other hand, by (4.22) we obtain

$$\begin{aligned}
&-\liminf_{\lambda\to+\infty} \frac{1}{2}\langle \mathbb{A}_r\big(\sigma^\lambda(T) - \varrho(T)\big), \sigma^\lambda(T) - \varrho(T)\rangle \\
&\quad\le\ -\frac{1}{2}\langle \mathbb{A}_r\big(\sigma^N(T) - \varrho(T)\big), \sigma^N(T) - \varrho(T)\rangle \\
&\quad=\ -\frac{1}{2}\langle \mathbb{A}_r\big(\sigma_0 - \varrho(0)\big), \sigma_0 - \varrho(0)\rangle - \int_0^T \langle \mathbb{A}_r\big(\dot\sigma^N(s) - \dot\varrho(s)\big), \sigma^N(s) - \varrho(s)\rangle\,ds
\end{aligned}$$

Combining the two previous inequalities with (4.28), we conclude that

$$\begin{aligned}
&\limsup_{\lambda\to+\infty} \int_0^T \langle D\psi_\lambda(\sigma^\lambda(s)), \sigma^\lambda(s) - \varrho(s)\rangle\,ds \\
&\quad\le\ -\int_0^T \langle \mathbb{A}_r\dot\sigma^N(s), \sigma^N(s) - \varrho(s)\rangle\,ds + \int_0^T \langle E\dot w(s), \sigma^N(s) - \varrho(s)\rangle\,dt \\
&\quad=\ \int_0^T \langle \gamma^N(s), \sigma^N(s) - \varrho(s)\rangle\,ds.
\end{aligned} \tag{4.29}$$

Let now $\tau \in L^\infty((0,T)\times\Omega;\mathbb{M}^{2\times 2}_{sym})$. By (4.16) we have

$$\int_0^T \langle D\psi_\lambda(\sigma^\lambda(t)) - D\psi_\lambda(\tau(t)), \sigma^\lambda(t) - \tau(t)\rangle\,dt \ge 0,$$

hence, using the fact that $D\psi_\lambda(\tau(t)) = D\phi_N(\tau(t))$ for $\lambda > \|\tau\|_{L^\infty}$,

$$\liminf_{\lambda\to+\infty} \int_0^T \langle D\psi_\lambda(\sigma^\lambda(t)), \sigma^\lambda(t)\rangle\,dt \ge \int_0^T \langle \gamma^N(t), \tau(t)\rangle\,dt + \int_0^T \langle D\phi_N(\tau(t)), \sigma^N(t) - \tau(t)\rangle\,dt.$$

Combining this inequality with (4.29), we conclude that

$$\int_0^T \langle \gamma^N(t) - D\phi_N(\tau(t)), \sigma^N(t) - \tau(t)\rangle \, dt \geq 0$$

for every $\tau \in L^\infty((0,T)\times\Omega;\mathbb{M}^{2\times 2}_{sym})$, hence $\gamma^N(t) = D\phi_N(\sigma^N(t))$.

Finally, we establish some compactness for the sequence (v^λ). From the first equation in (4.13), combined with (4.21) and (4.27), it follows that the sequence (v^λ) is uniformly bounded in $L^{N/(N-1)}(0,T;W^{1,N/(N-1)}(\Omega;\mathbb{R}^3))$, hence

$$v^\lambda \rightharpoonup v^N \quad \text{weakly in } L^{N/(N-1)}(0,T;W^{1,N/(N-1)}(\Omega;\mathbb{R}^3)),$$

for some $v^N \in L^{N/(N-1)}(0,T;W^{1,N/(N-1)}(\Omega;\mathbb{R}^3)\cap KL(\Omega))$. It is easy to check that v^N satisfies the boundary condition on Γ_d. This allows us to pass to the limit in the first equation of (4.13) and thus establish the existence of a solution to (4.6).

It remains to prove (4.9). This will follow from the first equation in (4.6) and from (4.7), once we show that $(D\phi_N(\sigma^N))$ is bounded in $L^2(0,T;L^1(\Omega;\mathbb{M}^{2\times 2}_{sym}))$. Multiplying the first equation in (4.6) by $\sigma^N(t) - \varrho(t)$, integrating over Ω, and using the first estimate in (4.7) yield

$$\langle D\phi_N(\sigma^N(t)), \sigma^N(t) - \varrho(t)\rangle \leq C\big(\|E\dot{w}(t)\|_{L^2} + \|\dot{\sigma}^N(t)\|_{L^2}\big)$$

for a.e. $t\in[0,T]$. On the other hand, setting $A_N(t) := \{x \in \Omega : \ |\sigma^N(t)|_r \geq \alpha_0\}$ and using (4.2) and the expression of $D\phi_N$, we have

$$\int_{A_N(t)} D\phi_N(\sigma^N(t)) : \big(\sigma^N(t) - \varrho(t)\big)\, dx \geq \alpha_1 \int_{A_N(t)} |D\phi_N(\sigma^N(t))|\, dx,$$

while

$$\int_{\Omega\setminus A_N(t)} D\phi_N(\sigma^N(t)) : \big(\sigma^N(t) - \varrho(t)\big)\, dx \geq -\int_{\Omega\setminus A_N(t)} |D\phi_N(\sigma^N(t))|\,|\varrho(t)|\, dx \geq -C\alpha_0.$$

Therefore, we have

$$\alpha_1 \int_{A_N(t)} |D\phi_N(\sigma^N(t))|\, dx \leq C\big(\|E\dot{w}(t)\|_{L^2} + \|\dot{\sigma}^N(t)\|_{L^2} + 1\big).$$

Since $|D\phi_N(\sigma^N(t))| \leq 1$ on $\Omega\setminus A_N(t)$, we conclude that

$$\int_\Omega |D\phi_N(\sigma^N(t))|\, dx \leq C\big(\|E\dot{w}(t)\|_{L^2} + \|\dot{\sigma}^N(t)\|_{L^2} + 1\big),$$

thus, the second inequality in (4.7) implies that the sequence $(D\phi_N(\sigma^N))$ is bounded in $L^2(0,T;L^1(\Omega;\mathbb{M}^{2\times 2}_{sym}))$, as claimed.

We now prove uniqueness of solutions. Let (σ^N, v^N) and (τ^N, u^N) be solutions of (4.6) with the same initial datum $\sigma^N(0) = \tau^N(0) = \sigma_0$. We test the first equation in (4.6) for (σ^N, v^N) and (τ^N, u^N) with $\sigma^N - \tau^N$ and take the difference:

$$\begin{aligned}\langle \mathbb{A}_r(\dot{\sigma}^N(t) - \dot{\tau}^N(t)), \sigma^N(t) - \tau^N(t)\rangle + \langle D\phi_N(\sigma^N(t)) - D\phi_N(\tau^N(t)), \sigma^N(t) - \tau^N(t)\rangle \\ = \langle Ev^N - Eu^N, \sigma^N(t) - \tau^N(t)\rangle.\end{aligned}$$

Integrating by parts the right-hand side and using the convexity of ϕ_N yield

$$\langle \mathbb{A}_r(\dot{\sigma}^N(t) - \dot{\tau}^N(t)), \sigma^N(t) - \tau^N(t)\rangle \leq 0.$$

Owing to the initial condition, this implies

$$\frac{1}{2}\langle \mathbb{A}_r(\sigma^N(t) - \tau^N(t)), \sigma^N(t) - \tau^N(t)\rangle \leq 0,$$

hence $\sigma^N(t) = \tau^N(t)$ for every $t\in[0,T]$. From the first equation in (4.6) we deduce that $Ev^N(t) = Eu^N(t)$, hence by the boundary condition on Γ_d we conclude that $v^N(t) = u^N(t)$ for every $t \in [0,T]$. □

We now prove two lemmas, that were used in the proof of Theorem 4.2.

Lemma 4.3. *Let $w \in W^{1,2}(0,T;W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega))$, $f \in W^{1,2}(0,T;L^2(\omega;\mathbb{R}^2))$, $g \in W^{1,2}(0,T;L^2(\omega))$, $h \in W^{1,2}(0,T;H^{-\frac{1}{2}}(\omega;\mathbb{R}^2))$, and $m \in W^{1,2}(0,T;H^{-\frac{1}{2}}(\omega) \times H^{-\frac{3}{2}}(\omega))$. Assume (4.1) with $\varrho \in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}))$. Let $\sigma_0 \in \Theta(\gamma_n, h(0), m(0))$ be such that $-\mathrm{div}_{x'}\bar{\sigma}_0 = f(0)$, $-\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}_0 = g(0)$ in ω. Finally, let $\Psi : \mathbb{M}^{2\times 2}_{sym} \to \mathbb{M}^{2\times 2}_{sym}$ be a Lipschitz continuous function. Then the problem*

$$\begin{cases} \mathbb{A}_r\dot{\sigma}(t) + \Psi(\sigma(t)) = Ev(t) & \text{in } \Omega, \\ -\mathrm{div}_{x'}\bar{\sigma}(t) = f(t), \quad -\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = g(t) & \text{in } \omega, \\ \sigma(t) \in \Theta(\gamma_n, h(t), m(t)), \\ v(t) = \dot{w}(t) & \text{on } \Gamma_d, \\ \sigma(0) = \sigma_0 \end{cases} \tag{4.30}$$

has a unique solution $(\sigma, v) \in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})) \times L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega))$.

Proof. On $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ we consider the scalar product

$$\langle \sigma, \tau \rangle_{\mathbb{A}_r} := \langle \mathbb{A}_r\sigma, \tau \rangle \quad \text{for every } \sigma, \tau \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym}), \tag{4.31}$$

which is topologically equivalent to the standard scalar product owing to (3.1). We introduce the set

$$\Sigma^* := \{\sigma \in \Theta(\gamma_n, 0, 0) : \ \mathrm{div}_{x'}\bar{\sigma} = 0, \ \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma} = 0 \text{ in } \omega\},$$

which is a closed subspace of $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$, and we denote the projection onto Σ^* with respect to the scalar product (4.31) by P_*.

We consider the following problem: to find $\theta \in W^{1,2}(0,T;\Sigma^*)$ such that

$$\begin{cases} \dot{\theta}(t) + P_*\big(\mathbb{A}_r^{-1}\Psi(\theta(t) + \varrho(t))\big) = P_*\big(\mathbb{A}_r^{-1}E\dot{w}(t) - \dot{\varrho}(t)\big) & \text{in } \Sigma^*, \\ \theta(0) = \sigma_0 - \varrho(0). \end{cases} \tag{4.32}$$

Let

$$\Lambda : [0,T] \times \Sigma^* \to \Sigma^* : (t, \theta) \mapsto P_*\big(\mathbb{A}_r^{-1}\Psi(\theta + \varrho(t))\big) - P_*\big(\mathbb{A}_r^{-1}E\dot{w}(t) - \dot{\varrho}(t)\big).$$

Since $\Lambda(t, \cdot)$ is Lipschitz continuous for a.e. $t \in [0,T]$ and $\Lambda(\cdot, \theta) \in L^2(0,T;\Sigma^*)$ for every $\theta \in \Sigma^*$, existence and uniqueness of solutions to problem (4.32) follow from the Cauchy-Lipschitz Theorem. Now, the first equation in (4.32) implies that

$$\langle \dot{\theta}(t), \tau \rangle_{\mathbb{A}_r} + \langle \mathbb{A}_r^{-1}\Psi(\theta(t) + \varrho(t)), \tau \rangle_{\mathbb{A}_r} = \langle \mathbb{A}_r^{-1}E\dot{w}(t) - \dot{\varrho}(t), \tau \rangle_{\mathbb{A}_r}$$

for every $\tau \in \Sigma^*$, that is,

$$\langle \mathbb{A}_r\dot{\theta}(t), \tau \rangle + \langle \Psi(\theta(t) + \varrho(t)), \tau \rangle = \langle E\dot{w}(t) - \mathbb{A}_r\dot{\varrho}(t), \tau \rangle$$

for every $\tau \in \Sigma^*$. By Lemma 4.4 below for a.e. $t \in [0,T]$ there exists $z(t) \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ such that $z(t) = 0$ on Γ_d and

$$\mathbb{A}_r\dot{\theta}(t) + \Psi(\theta(t) + \varrho(t)) - E\dot{w}(t) + \mathbb{A}_r\dot{\varrho}(t) = Ez(t).$$

We set $\sigma(t) := \theta(t) + \varrho(t)$ and $v(t) := z(t) + \dot{w}(t)$. We observe that $v \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega))$ by construction. Thus, we have found a pair (σ, v) satisfying (4.30).

On the other hand, if (σ, v) is a solution to (4.30), then $\theta(t) := \sigma(t) - \varrho(t)$ satisfies (4.32) and is therefore uniquely determined. Uniqueness of v follows from Lemma 4.4. □

Lemma 4.4. *Let $\sigma \in L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$ be such that*

$$\int_\Omega \sigma : \tau \, dx = 0 \quad \textit{for every } \tau \in \Sigma^*, \tag{4.33}$$

where $\Sigma^* := \{\tau \in \Theta(\gamma_n, 0, 0) : \ \mathrm{div}_{x'}\bar{\tau} = 0, \ \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\tau} = 0$ in $\omega\}$. Then there exists a unique function $u \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ such that $u = 0$ on Γ_d and $\sigma = Eu$ in Ω.

Proof. We consider the set

$$E_0 := \{Ev : \ v \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega), \ v = 0 \text{ on } \Gamma_d\},$$

which is a closed subspace of $L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})$, owing to Korn-Poincaré inequality. Let P_0 be the orthogonal projection onto E_0 and let $Eu := P_0(\sigma)$. By definition u satisfies

$$\int_\Omega Eu : Ev\, dx = \int_\Omega \sigma : Ev\, dx \tag{4.34}$$

for every $v \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ with $v = 0$ on Γ_d.

We now set $\theta := \sigma - Eu$. From (4.34) it follows that

$$\int_\Omega \theta : Ev\, dx = 0$$

for every $v \in W^{1,2}(\Omega;\mathbb{R}^3) \cap KL(\Omega)$ with $v = 0$ on Γ_d. By Proposition 3.3 we deduce that $\theta \in \Sigma^*$. On the other hand, since $u = 0$ on Γ_d, integration by parts yields

$$\int_\Omega Eu : \tau\, dx = 0$$

for every $\tau \in \Sigma^*$. Therefore, we have by (4.33) that

$$\int_\Omega \theta : \tau\, dx = 0 \quad \text{for every } \tau \in \Sigma^*.$$

We conclude that $\theta = 0$, hence $\sigma = Eu$.

Uniqueness of u is straightforward. □

We now prove additional regularity of σ^N.

Proposition 4.5. *In addition to the assumptions of Theorem 4.2, suppose that* $\sigma_0 \in W^{1,2}(\Omega;\mathbb{M}^{2\times 2}_{sym})$ *and that*

$$\mathrm{div}_{x'}\bar{\varrho} \in L^\infty(0,T;W^{2,2}(\Omega;\mathbb{R}^2)), \quad \mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\varrho} \in L^\infty(0,T;W^{1,2}(\Omega)). \tag{4.35}$$

For every $N \in \mathbb{N}$, $N \geq 4$, *let* σ^N *be the stress component of the solution of* (4.6). *Then the following estimates are satisfied:*

- *for every open set* ω' *compactly contained in* ω *there exists a constant* $C_1(\omega') > 0$, *depending on* ω' *but independent of* N, *such that for* $\alpha = 1,2$
$$\sup_{t\in[0,T]} \|D_\alpha \sigma^N(t)\|_{L^2(\omega'\times(-\frac{1}{2},\frac{1}{2});\mathbb{M}^{2\times 2}_{sym})} \leq C_1; \tag{4.36}$$
- *for every open set* Ω' *compactly contained in* Ω *there exists a constant* $C_2(\Omega') > 0$, *depending on* Ω' *but independent of* N, *such that*
$$\sup_{t\in[0,T]} \|D_3 \sigma^N(t)\|_{L^2(\Omega';\mathbb{M}^{2\times 2}_{sym})} \leq C_2. \tag{4.37}$$

Proof. Let $N \in \mathbb{N}$, $N \geq 4$. We first prove higher regularity for σ^λ, where σ^λ are the approximating solutions constructed in the proof of Theorem 4.2.

For $i = 1,2,3$ let D^h_i be the difference quotient operator defined by

$$D^h_i \tau(x) = \frac{1}{h}(\tau(x + he_i) - \tau(x))$$

for every function $\tau : \mathbb{R}^3 \to \mathbb{M}^{2\times 2}_{sym}$.

Let $\alpha = 1,2$ and let $\varphi \in C^\infty_c(\omega)$. Multiplying the first equation in (4.13) by the term $D^{-h}_\alpha(\varphi^2 D^h_\alpha \sigma^\lambda(t))$, we obtain

$$\langle \varphi^2 \mathbb{A}_r D^h_\alpha \dot{\sigma}^\lambda(t), D^h_\alpha \sigma^\lambda(t)\rangle + \langle \varphi^2 D^h_\alpha\big(D\psi_\lambda(\sigma^\lambda(t))\big), D^h_\alpha \sigma^\lambda(t)\rangle = \langle \varphi^2 D^h_\alpha Ev^\lambda(t), D^h_\alpha \sigma^\lambda(t)\rangle.$$

Integrating this equation with respect to time and using (3.1), we deduce

$$\begin{aligned}&\alpha_{\mathbb{A}}\int_\Omega \varphi^2|D^h_\alpha\sigma^\lambda(t)|^2\,dx-\beta_{\mathbb{A}}\int_\Omega\varphi^2|D^h_\alpha\sigma_0|^2\,dx\\&\qquad+\int_0^t\int_\Omega\int_0^1\varphi^2D^2\psi_\lambda(\sigma^\lambda(s)+rhD^h_\alpha\sigma^\lambda(s))\,D^h_\alpha\sigma^\lambda(s):D^h_\alpha\sigma^\lambda(s)\,dr\,dx\,ds\\&\leq\ \int_0^t\langle\varphi^2D^h_\alpha Ev^\lambda(s),D^h_\alpha\sigma^\lambda(s)\rangle\,ds.\end{aligned}\tag{4.38}$$

Note that in the third integral on the left-hand side we used the chain rule, which holds since $D\psi_\lambda$ is a composition of smooth functions with a truncation.

Let us focus on the right-hand side of (4.38). We express it in terms of the Kirchhoff-Love components of v^λ and use integration by parts, together with the fact that $\operatorname{div}_{x'}\bar\sigma^\lambda(t)=\operatorname{div}_{x'}\bar\varrho(t)$ and $\operatorname{div}_{x'}\operatorname{div}_{x'}\hat\sigma^\lambda(t)=\operatorname{div}_{x'}\operatorname{div}_{x'}\hat\varrho(t)$ in ω. By the first equation in (4.13) we deduce that for a.e. $t\in[0,T]$ we have

$$\begin{aligned}&\langle\varphi^2D^h_\alpha Ev^\lambda(t),D^h_\alpha\sigma^\lambda(t)\rangle\\&=\ -2\sum_{\beta,\gamma}\int_\Omega\int_0^1D_\beta\varphi^2(x')\,(\mathbb{A}_r\dot\sigma^\lambda+D\psi_\lambda(\sigma^\lambda))_{\alpha\gamma}(t,x+rhe_\alpha)\,D^h_\alpha\sigma^\lambda_{\beta\gamma}(t,x)\,dr\,dx\\&\quad+\sum_{\beta,\gamma}\int_\Omega\int_0^1v^\lambda_\alpha(t,x+rhe_\alpha)\,D^{-h}_\alpha D^2_{\beta\gamma}\varphi^2(x')\,\sigma^\lambda_{\beta\gamma}(t,x)\,dr\,dx\\&\quad+\sum_{\beta,\gamma}\int_\Omega\int_0^1\int_0^1(\mathbb{A}_r\dot\sigma^\lambda+D\psi_\lambda(\sigma^\lambda))_{\alpha\alpha}(t,x+(r-\tilde r)he_\alpha)\,D^2_{\beta\gamma}\varphi^2(x')\,\sigma^\lambda_{\beta\gamma}(t,x)\,d\tilde r\,dr\,dx\\&\quad-\sum_{\beta,\gamma}\int_\omega\int_0^1\bar v^\lambda_\alpha(t,x+rhe_\alpha)\,D_\beta\varphi^2(x')\,D^h_\alpha D_\gamma\bar\varrho_{\beta\gamma}(t,x)\,dr\,dx\\&\quad-\frac{1}{12}\sum_{\beta,\gamma}\int_\omega\int_0^1D_\alpha v^\lambda_3(t,x+rhe_\alpha)\,\varphi^2(x')\,D^h_\alpha D^2_{\beta\gamma}\hat\varrho_{\beta\gamma}(t,x)\,dr\,dx\\&\quad+\sum_{\beta,\gamma}\langle D^{-h}_\alpha\varphi^2\,\bar v^\lambda_\gamma(t),D^h_\alpha D_\beta\bar\varrho_{\beta\gamma}(t)\rangle+\sum_{\beta,\gamma}\langle\varphi^2\bar v^\lambda_\gamma(t),D^{-h}_\alpha D^h_\alpha D_\beta\bar\varrho_{\beta\gamma}(t)\rangle.\end{aligned}\tag{4.39}$$

We now combine (4.38) and (4.39). From the definition of ψ_λ we deduce the following estimate:

$$D^2\psi_\lambda(\xi)\,\zeta:\zeta\geq\frac{1}{\alpha_0^{N-1}}\big(|\xi|_r^{N-2}\wedge\lambda^{N-2}\big)|\zeta|_r^2\tag{4.40}$$

for a.e. $\xi\in\mathbb{M}^{2\times2}_{sym}$ and every $\zeta\in\mathbb{M}^{2\times2}_{sym}$. In particular, this implies that the third term on the left-hand side of (4.38) is non negative. Moreover, using the inequality $|D\psi_\lambda(\xi)|\leq\frac{1}{\alpha_0^{N-1}}\lambda^{N-2}|\xi|$ for every $\xi\in\mathbb{M}^{2\times2}_{sym}$, we obtain

$$\begin{aligned}&\alpha_{\mathbb{A}}\int_\Omega\varphi^2|D^h_\alpha\sigma^\lambda(t)|^2\,dx-\beta_{\mathbb{A}}\int_\Omega\varphi^2|D^h_\alpha\sigma_0|^2\,dx\\&\leq\ C\int_0^t\big(\|\varphi\,D^h_\alpha\sigma^\lambda(s)\|_{L^2}+\|\sigma^\lambda(s)\|_{L^2}\big)\Big(\|\dot\sigma^\lambda(s)\|_{L^2}+\frac{\lambda^{N-2}}{\alpha_0^{N-1}}\|\sigma^\lambda(s)\|_{L^2}\Big)\,ds\\&\qquad+C\int_0^t\|v^\lambda(s)\|_{L^2}\big(\|\sigma^\lambda(s)\|_{L^2}+1\big)\,ds,\end{aligned}$$

where we used (4.35). Since $\sigma^\lambda\in W^{1,2}(0,T;L^2(\Omega;\mathbb{R}^3))$ and $v^\lambda\in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, the previous inequality implies that $D_\alpha\sigma^\lambda\in L^2_{loc}(\omega\times[-\frac12,\frac12];\mathbb{M}^{2\times2}_{sym})$ for $\alpha=1,2$. Therefore, we can pass to the limit in (4.38)–(4.39), as $h \to 0$, and, using also (4.40), we obtain

$$\begin{aligned}
&\alpha_{\mathbb{A}} \int_\Omega \varphi^2 |D_\alpha \sigma^\lambda(t)|^2\,dx + \frac{1}{\alpha_0^{N-1}} \int_0^t \int_\Omega \varphi^2 \left(|\sigma^\lambda(s)|_r^{N-2} \wedge \lambda^{N-2}\right) |D_\alpha \sigma^\lambda(s)|_r^2\,dx\,ds \\
&\quad \leq \beta_{\mathbb{A}} \int_\Omega \varphi^2 |D_\alpha \sigma_0|^2\,dx - 2\sum_{\beta,\gamma} \int_0^t \int_\Omega D_\beta \varphi^2 \left(\mathbb{A}_r \dot\sigma^\lambda(s) + D\psi_\lambda(\sigma^\lambda(s))\right)_{\alpha\gamma} D_\alpha \sigma^\lambda_{\beta\gamma}(s)\,dx\,ds \\
&\qquad + \sum_{\beta,\gamma} \int_0^t \int_\Omega (\mathbb{A}_r \dot\sigma^\lambda(s) + D\psi_\lambda(\sigma^\lambda(s)))_{\alpha\alpha} D^2_{\beta\gamma}\varphi^2\, \sigma^\lambda_{\beta\gamma}(s)\,dx\,ds \\
&\qquad + \sum_{\beta,\gamma} \int_0^t \int_\Omega v^\lambda_\alpha(s)\, D^3_{\alpha\beta\gamma}\varphi^2\, \sigma^\lambda_{\beta\gamma}(s)\,dx\,ds - \sum_{\beta,\gamma} \int_0^t \int_\omega \bar v^\lambda_\alpha(s)\, D_\beta \varphi^2\, D^2_{\alpha\gamma} \bar\varrho_{\beta\gamma}(s)\,dx\,ds \\
&\qquad - \frac{1}{12} \sum_{\beta,\gamma} \int_0^t \int_\omega D_\alpha v^\lambda_3(s)\, \varphi^2\, D^3_{\alpha\beta\gamma} \hat\varrho_{\beta\gamma}(s)\,dx\,ds \\
&\qquad + \sum_{\beta,\gamma} \langle D_\alpha \varphi^2\, \bar v^\lambda_\gamma(t), D^2_{\alpha\beta} \bar\varrho_{\beta\gamma}(t)\rangle + \sum_{\beta,\gamma} \langle \varphi^2 \bar v^\lambda_\gamma(t), D^3_{\alpha\alpha\beta} \bar\varrho_{\beta\gamma}(t)\rangle. \qquad (4.41)
\end{aligned}$$

From this inequality we will deduce a uniform bound for $(D_\alpha \sigma^\lambda)$ with respect to λ. We consider the second term on the right-hand side of (4.41). Using the expression of $D\psi_\lambda$, we have

$$\begin{aligned}
&\int_0^t \int_\Omega \left| D_\beta \varphi^2 (\mathbb{A}_r \dot\sigma^\lambda(s) + D\psi_\lambda(\sigma^\lambda(s)))_{\alpha\gamma} D_\alpha \sigma^\lambda_{\beta\gamma}(s)\right| dx\,ds \\
&\quad \leq C \int_0^t \|\varphi D_\alpha \sigma^\lambda(s)\|_{L^2} \|\dot\sigma^\lambda(s)\|_{L^2}\,ds \\
&\qquad + \frac{C}{\alpha_0^{N-1}} \int_0^t \int_\Omega \left(|\sigma^\lambda(s)|_r^{N-2} \wedge \lambda^{N-2}\right) |\sigma^\lambda(s)|_r\, |\varphi D_\alpha \sigma^\lambda(s)|\,dx\,ds \\
&\quad \leq C \Big(\int_0^t \|\varphi D_\alpha \sigma^\lambda(s)\|^2_{L^2}\,ds \Big)^{\frac12} + \frac{C}{\alpha_0^{\frac{N-1}{2}}} \Big(\int_0^t \left\| \left(|\sigma^\lambda(s)|_r^{N-2} \wedge \lambda^{N-2}\right)^{1/2} \varphi D_\alpha \sigma^\lambda(s)\right\|^2_{L^2}\,ds \Big)^{\frac12},
\end{aligned}$$

where we have used (4.21) and (4.19).

Analogously, the third term on the right-hand side of (4.41) can be estimated as follows:

$$\begin{aligned}
&\int_0^t \int_\Omega |D^2_{\beta\gamma}\varphi^2 \left(\mathbb{A}_r \dot\sigma^\lambda(s) + D\psi_\lambda(\sigma^\lambda(s))\right)_{\alpha\alpha} \sigma^\lambda_{\beta\gamma}(s)|\,dx\,ds \\
&\quad \leq C \int_0^t \|\sigma^\lambda(s)\|_{L^2} \|\dot\sigma^\lambda(s)\|_{L^2}\,ds + \frac{C}{\alpha_0^{N-1}} \int_0^t \int_\Omega \left(|\sigma^\lambda(s)|_r^{N-2} \wedge \lambda^{N-2}\right) |\sigma^\lambda(s)|_r^2\,dx\,ds,
\end{aligned}$$

where the right-hand side is uniformly bounded with respect to λ, owing to (4.17), (4.21), and (4.19).

As for the remaining terms on the right-hand side of (4.41), we recall that (v^λ) is uniformly bounded in $L^{N/(N-1)}(0,T;W^{1,N/(N-1)}(\Omega;\mathbb{R}^3))$, with respect to λ. Since v^λ is a Kirchhoff-Love displacement for every λ, this implies that the sequence $(\bar v^\lambda)$ is uniformly bounded in $L^{N/(N-1)}(0,T;W^{1,N/(N-1)}(\omega;\mathbb{R}^2))$, while the sequence of vertical displacements (v^λ_3) is uniformly bounded in $L^{N/(N-1)}(0,T;W^{2,N/(N-1)}(\omega))$. By Sobolev embedding we have that $(\bar v^\lambda)$ is uniformly bounded in $L^{N/(N-1)}(0,T;L^2(\omega;\mathbb{R}^2))$ and (v^λ_3) is uniformly bounded in $L^{N/(N-1)}(0,T;W^{1,2}(\omega))$. Therefore, by (4.17)

$$\begin{aligned}
&\Big| \int_0^t \int_\Omega D^3_{\alpha\beta\gamma}\varphi^2\, v^\lambda_\alpha(s)\, \sigma^\lambda_{\beta\gamma}(s)\,dx\,ds \Big| \\
&\qquad\qquad \leq C \int_0^t \|\bar v^\lambda(s)\|_{L^2} \|\bar\sigma^\lambda(s)\|_{L^2}\,ds + C \int_0^t \|D_\alpha v^\lambda_3(s)\|_{L^2} \|\hat\sigma^\lambda(s)\|_{L^2}\,ds \leq C_N,
\end{aligned}$$

where the last constant depends on N via the Sobolev embedding constants. The remaining terms in (4.41) can be estimated in a similar way, using the assumptions (4.35) on ϱ.

Combining all these estimates with (4.41), we conclude that for every open set ω' compactly contained in ω and for $\alpha = 1, 2$ the sequence $(D_\alpha \sigma^\lambda)$ is uniformly bounded in $L^\infty(0,T; L^2(\omega' \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{M}^{2\times 2}_{sym}))$, with respect to λ. By (4.23) this implies that $D_\alpha \sigma^N$ belongs to $L^\infty(0,T; L^2(\omega' \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{M}^{2\times 2}_{sym}))$ for $\alpha = 1, 2$.

To conclude the proof of (4.36), it remains to show that the norm of $D_\alpha \sigma^N$ in the space $L^\infty(0,T; L^2(\omega' \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{M}^{2\times 2}_{sym}))$ is uniformly bounded with respect to N. To this purpose, arguing exactly as in the proof of (4.41), we obtain

$$\begin{aligned}
\alpha_{\mathbb{A}} \int_\Omega \varphi^2 |D_\alpha \sigma^N(t)|^2 \, dx &+ \int_0^t \int_\Omega \varphi^2 D^2 \phi_N(\sigma^N(s)) \, D_\alpha \sigma^N(s) : D_\alpha \sigma^N(s) \, dx \, ds \\
\leq \;& \beta_{\mathbb{A}} \int_\Omega \varphi^2 |D_\alpha \sigma_0|^2 \, dx - 2 \sum_{\beta,\gamma} \int_0^t \int_\Omega D_\beta \varphi^2 \, (\mathbb{A}_r \dot\sigma^N(s) + D\phi_N(\sigma^N(s)))_{\alpha\gamma} \, D_\alpha \sigma^N_{\beta\gamma}(s) \, dx \, ds \\
& + \sum_{\beta,\gamma} \int_0^t \int_\Omega (\mathbb{A}_r \dot\sigma^N(s) + D\phi_N(\sigma^N(s)))_{\alpha\alpha} \, D^2_{\beta\gamma} \varphi^2 \, \sigma^N_{\beta\gamma}(s) \, dx \, ds \\
& + \sum_{\beta,\gamma} \int_0^t \int_\Omega v^N_\alpha(s) \, D^3_{\alpha\beta\gamma} \varphi^2 \, \sigma^N_{\beta\gamma}(s) \, dx \, ds - \sum_{\beta,\gamma} \int_0^t \int_\omega \bar v^N_\alpha(s) \, D_\beta \varphi^2 \, D^2_{\alpha\gamma} \bar\varrho_{\beta\gamma}(s) \, dx \, ds \\
& - \frac{1}{12} \sum_{\beta,\gamma} \int_0^t \int_\omega D_\alpha v^N_3(s) \, \varphi^2 \, D^3_{\alpha\beta\gamma} \hat\varrho_{\beta\gamma}(s) \, dx \, ds \\
& + \sum_{\beta,\gamma} \langle D_\alpha \varphi^2 \, \bar v^N_\gamma(t), D^2_{\alpha\beta} \bar\varrho_{\beta\gamma}(t) \rangle + \sum_{\beta,\gamma} \langle \varphi^2 \bar v^N_\gamma(t), D^3_{\alpha\alpha\beta} \bar\varrho_{\beta\gamma}(t) \rangle. \qquad (4.42)
\end{aligned}$$

We note that the second term on the left-hand side of (4.42) satisfies the following coercivity inequality:

$$D^2 \phi_N(\sigma^N(s)) \, D_\alpha \sigma^N(s) : D_\alpha \sigma^N(s) \geq \frac{1}{\alpha_0^{N-1}} |\sigma^N(s)|_r^{N-2} |D_\alpha \sigma^N(s)|_r^2.$$

As for the right-hand side of (4.42), using the expression of $D\phi_N$, we have

$$\begin{aligned}
\int_0^t \int_\Omega & \big| D_\beta \varphi^2 (\mathbb{A}_r \dot\sigma^N(s) + D\phi_N(\sigma^N(s)))_{\alpha\gamma} \, D_\alpha \sigma^N_{\beta\gamma}(s) \big| \, dx \, ds \\
\leq \;& C \int_0^t \|\varphi \, D_\alpha \sigma^N(s)\|_{L^2} \, \|\dot\sigma^N(s)\|_{L^2} \, ds + \frac{C}{\alpha_0^{N-1}} \int_0^t \int_\Omega |\sigma^N(s)|_r^{N-1} \, |\varphi \, D_\alpha \sigma^N(s)| \, dx \, ds \\
\leq \;& C \Big(\int_0^t \|\varphi \, D_\alpha \sigma^N(s)\|^2_{L^2} \, ds \Big)^{\frac{1}{2}} + \frac{C}{\alpha_0^{\frac{N-1}{2}}} \Big(\int_0^t \| |\sigma^N(s)|_r^{\frac{N-2}{2}} \varphi \, D_\alpha \sigma^N(s) \|^2_{L^2} \, ds \Big)^{\frac{1}{2}},
\end{aligned}$$

where we have used (4.7) and (4.8). Analogously, the third term on the right-hand side of (4.42) can be estimated as follows:

$$\begin{aligned}
\int_0^t \int_\Omega & |D^2_{\beta\gamma} \varphi^2 \, (\mathbb{A}_r \dot\sigma^N(s) + D\phi_N(\sigma^N(s)))_{\alpha\alpha} \, \sigma^N_{\beta\gamma}(s)| \, dx \, ds \\
& \leq \; C \int_0^t \|\sigma^N(s)\|_{L^2} \, \|\dot\sigma^N(s)\|_{L^2} \, ds + \frac{C}{\alpha_0^{N-1}} \int_0^t \int_\Omega |\sigma^N(s)|_r^N \, dx \, ds,
\end{aligned}$$

where the right-hand side is uniformly bounded with respect to N, owing to (4.7) and (4.8). As for the remaining terms on the right-hand side of (4.41), we observe that, in view of (4.9), the sequence (v^N) is uniformly bounded in $L^2(0,T; BD(\Omega))$. Since v^N is a Kirchhoff-Love displacement, this implies that $(\bar v^N)$ is uniformly bounded in $L^2(0,T; BD(\omega))$ and (v^N_3) is uniformly bounded in $L^2(0,T; BH(\omega))$. By Sobolev embedding $(\bar v^N)$ is uniformly bounded in $L^2(0,T; L^2(\omega; \mathbb{R}^2))$ and (v^N_3) is uniformly bounded in $L^2(0,T; W^{1,2}(\omega))$. Therefore, we

have
$$\int_0^t\int_\Omega |D^3_{\alpha\beta\gamma}\varphi^2\, v^N_\alpha(s)\,\sigma^N_{\beta\gamma}(s)|\,dx\,ds\le C\int_0^t \|v^N(s)\|_{L^2}\,\|\sigma^N(s)\|_{L^2}\,ds,$$
which is uniformly bounded with respect to N by (4.7) and (4.9). The remaining terms in (4.42) can be estimated similarly.

Combining these estimates with (4.42), we conclude that (4.36) is satisfied.

We now prove higher regularity with respect to x_3. Let $\varphi\in C^\infty_c(\Omega)$. Multiplying the first equation in (4.6) by $D_3^{-h}(\varphi^2 D_3^h\sigma^N(t))$, we obtain
$$\langle\varphi^2\mathbb{A}_r D_3^h\dot\sigma^N(t), D_3^h\sigma^N(t)\rangle+\langle\varphi^2 D_3^h\big(D\phi_N(\sigma^N(t))\big), D_3^h\sigma^N(t)\rangle=-\langle\varphi^2 D^2v_3^N(t), D_3^h\sigma^N(t)\rangle.$$
Integrating this equation with respect to time and using (3.1), we deduce
$$\begin{aligned}\alpha_{\mathbb{A}}\int_\Omega\varphi^2|D_3^h\sigma^N(t)|^2\,dx-\beta_{\mathbb{A}}\int_\Omega\varphi^2|D_3^h\sigma_0|^2\,dx\\ +\int_0^t\int_\Omega\int_0^1\varphi^2D^2\phi_N(\sigma^N(s)+rhD_3^h\sigma^N(s))\,D_3^h\sigma^N(s):D_3^h\sigma^N(s)\,dr\,dx\,ds\\ \le-\int_0^t\langle\varphi^2D^2v_3^N(s), D_3^h\sigma^N(s)\rangle\,ds.\end{aligned}\tag{4.43}$$
By integration by parts the right-hand side can be written as
$$\begin{aligned}&-\int_0^t\langle\varphi^2D^2v_3^N(s), D_3^h\sigma^N(s)\rangle\,ds\\ &\quad=-\int_0^t\langle D_3^{-h}(\varphi^2)\,\nabla v_3^N(s),\mathrm{div}_{x'}\sigma^N(s)\rangle\,ds+\int_0^t\langle D_3^{-h}(\nabla_{x'}\varphi^2)\odot\nabla v_3^N(s),\sigma^N(s)\rangle\,ds.\end{aligned}$$
Combining the first estimate in (4.7), (4.36), and the uniform bound of (v_3^N) in the space $L^2(0,T;W^{1,2}(\omega))$, we deduce that the right-hand side of (4.43) is uniformly bounded with respect to h and N. Moreover, since ϕ_N is convex, the last term on the left-hand side of (4.43) is non-negative. Therefore, we have
$$\alpha_{\mathbb{A}}\int_\Omega\varphi^2|D_3^h\sigma^N(t)|^2\,dx\le\beta_{\mathbb{A}}\int_\Omega\varphi^2|D_3^h\sigma_0|^2\,dx+C,\tag{4.44}$$
where the constant C is independent of h and N. Using the assumptions on σ_0, this inequality implies that $D_3\sigma^N\in L^\infty(0,T;L^2_{loc}(\Omega;\mathbb{M}^{2\times2}_{sym}))$ for every N. Therefore, we can pass to the limit in (4.44), as $h\to0$, and we obtain
$$\alpha_{\mathbb{A}}\int_\Omega\varphi^2|D_3\sigma^N(t)|^2\,dx\le\beta_{\mathbb{A}}\int_\Omega\varphi^2|D_3\sigma_0|^2\,dx+C,$$
where the constant C is independent of N. This completes the proof of (4.37) and of the proposition. □

We are now in a position to prove the main result of the paper, namely higher regularity for the stress component of the quasistatic evolutions. This will be established by showing convergence of the solutions to the Norton-Hoff problems (4.6) to a solution of the quasistatic evolution problem. As a by-product, we also prove existence of solutions to (qs1)–(qs3), under the assumptions (3.6)–(3.7) on the set K_r of admissible stresses.

Theorem 4.6. *Let the assumptions of Theorem 4.2 be satisfied with $\sigma_0=\mathbb{C}_re_0$, where $(u_0,e_0,p_0)\in\mathcal{A}_{KL}(w(0))$. Then there exists a solution*
$$(u,e,p)\in W^{1,2}(0,T;BD(\Omega)\times L^2(\Omega;\mathbb{M}^{2\times2}_{sym})\times M_b(\Omega\cup\Gamma_d;\mathbb{M}^{2\times2}_{sym}))$$
of the quasistatic evolution problem (qs1)–(qs3) with $(u(0),e(0),p(0))=(u_0,e_0,p_0)$. The stress component $\sigma(t):=\mathbb{C}_re(t)$ is unique and, under the assumptions of Proposition 4.5, it satisfies the following estimates:

- *for every open set ω' compactly contained in ω there exists a constant $C_1(\omega') > 0$ such that for $\alpha = 1, 2$*

$$\sup_{t\in[0,T]} \|D_\alpha \sigma(t)\|_{L^2(\omega'\times(-\frac{1}{2},\frac{1}{2});\mathbb{M}^{2\times2}_{sym})} \le C_1; \tag{4.45}$$

- *for every open set Ω' compactly contained in Ω there exists a constant $C_2(\Omega') > 0$ such that*

$$\sup_{t\in[0,T]} \|D_3 \sigma(t)\|_{L^2(\Omega';\mathbb{M}^{2\times2}_{sym})} \le C_2. \tag{4.46}$$

Proof. By applying Ascoli-Arzelà Theorem we deduce from (4.7) that there exists $\sigma \in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym}))$ such that, up to subsequences,

$$\sigma^N(t) \rightharpoonup \sigma(t) \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{2\times2}_{sym}), \tag{4.47}$$

as $N \to \infty$, for every $t \in [0,T]$ and

$$\sigma^N \rightharpoonup \sigma \quad \text{weakly in } W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym})), \tag{4.48}$$

as $N \to \infty$. By (4.47) it is clear that $\sigma(0) = \sigma_0$ and, in view of Proposition 3.2, that $-\mathrm{div}_{x'}\bar{\sigma}(t) = f(t)$, $-\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\sigma}(t) = g(t)$ in ω, and $\sigma(t) \in \Theta(\gamma_n, h(t), m(t))$ for every $t \in [0,T]$. From Proposition 4.5 and (4.48) it follows that (4.45) and (4.46) are satisfied. Passing to the limit in the first inequality of (4.8), as $N \to \infty$, we deduce that $\sigma(t) \in \mathcal{K}_r(\Omega)$ for every $t \in [0,T]$.

We now set

$$u^N(t) := u_0 + \int_0^t v^N(s)\,ds \qquad \text{for every } t \in [0,T].$$

By (4.9) the sequences (u^N) and $(\dot{u}^N)$ are uniformly bounded in $L^2(0,T;BD(\Omega))$. Thus, there exists $u \in W^{1,2}(0,T;BD(\Omega))$ such that, up to subsequences, $u^N \rightharpoonup u$ and $\dot{u}^N \rightharpoonup \dot{u}$ weakly* in $L^2(0,T;BD(\Omega))$, as $N \to \infty$. In particular, using the Kirchhoff-Love structure,

$$\begin{aligned} \bar{v}^N = \dot{\bar{u}}^N \rightharpoonup \dot{\bar{u}} \quad &\text{weakly in } L^2(0,T;L^2(\omega;\mathbb{R}^2)), \\ v_3^N = \dot{u}_3^N \rightharpoonup \dot{u}_3 \quad &\text{weakly in } L^2(0,T;W^{1,2}(\omega)), \end{aligned} \tag{4.49}$$

as $N \to \infty$.

We define $e(t) := \mathbb{A}_r\sigma(t)$ and $p(t) := Eu(t) - e(t)$ in Ω, $p(t) := (w(t) - u(t)) \odot \nu_{\partial\Omega}\mathcal{H}^2$ on Γ_d. It is thus clear that $(u(t), e(t), p(t)) \in \mathcal{A}_{KL}(w(t))$.

It remains to prove that the flow rule is satisfied. To this purpose, we will show that (4.3) holds. Let $\theta \in W^{1,2}(0,T;L^2(\Omega;\mathbb{M}^{2\times2}_{sym})) \cap L^\infty(0,T;L^N(\Omega;\mathbb{M}^{2\times2}_{sym}))$ and let $\varphi \in C^2(\overline{\omega})$ with $\varphi \ge 0$. Multiplying the first equation in (4.6) by $\varphi\,(\theta(t) - \sigma^N(t))$, we obtain

$$\langle \mathbb{A}_r\dot{\sigma}^N(t) + D\phi_N(\sigma^N(t)) - Ev^N(t), \varphi\,(\theta(t) - \sigma^N(t))\rangle = 0.$$

Using the convexity of ϕ_N, this equality can be interpreted as the following minimality condition:

$$\begin{aligned} \langle \mathbb{A}_r\dot{\sigma}^N(t) - Ev^N(t), \varphi\,\sigma^N(t)\rangle &+ \int_\Omega \varphi\,\phi_N(\sigma^N(t))\,dx \\ &\le \langle \mathbb{A}_r\dot{\sigma}^N(t) - Ev^N(t), \varphi\,\theta(t)\rangle + \int_\Omega \varphi\,\phi_N(\theta(t))\,dx. \end{aligned} \tag{4.50}$$

We now choose $\theta = \sigma$ and $\varphi \equiv 1$, and integrate the inequality with respect to time on a time interval $[0, t_1]$. Using Remark 3.4 and the fact that $\phi_N \ge 0$, we obtain

$$\begin{aligned} \frac{1}{2}\langle \mathbb{A}_r\big(\sigma^N(t_1) - \sigma(t_1)\big), \sigma^N(t_1) - \sigma(t_1)\rangle \le \int_0^{t_1} &\langle \sigma^N(t) - \sigma(t), E\dot{w}(t) - \mathbb{A}_r\dot{\sigma}(t)\rangle\,dt \\ &+ \int_0^{t_1}\int_\Omega \phi_N(\sigma(t))\,dx\,dt, \end{aligned}$$

hence, by the coercivity (3.1) of $\mathbb{A}_r$ we deduce

$$\alpha_{\mathbb{A}}\|\sigma^N(t_1)-\sigma(t_1)\|_{L^2}^2 \leq \int_0^{t_1}\langle\sigma^N(t)-\sigma(t), E\dot{w}(t)-\mathbb{A}_r\dot{\sigma}(t)\rangle\,dt+\int_0^{t_1}\int_\Omega \phi_N(\sigma(t))\,dx\,dt.$$

Since $\sigma(t)\in\mathcal{K}_r(\Omega)$ for every $t\in[0,T]$, the last term in the previous expression tends to zero, as $N\to\infty$. Together with (4.47) and the first estimate in (4.7), this implies that

$$\sigma^N\to\sigma \quad \text{strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{2\times 2}_{sym})), \tag{4.51}$$

as $N\to\infty$.

We now go back to equation (4.50), where we choose $\theta\in\mathcal{K}_r(\Omega)\cap\Sigma(\Omega)$ independent of time and $\varphi\in C^2(\overline{\omega})$, $\varphi\geq 0$, with $\varphi=0$ in a neighbourhood of γ_n. Since $\varphi(v^N(t)-\dot{w}(t))=0$ on $\partial\omega\times(-\frac{1}{2},\frac{1}{2})$, integration by parts yields

$$\begin{aligned}
&\langle Ev^N(t),\varphi\,(\sigma^N(t)-\theta)\rangle=\langle E\dot{w}(t),\varphi\,(\sigma^N(t)-\theta)\rangle\\
&\quad+\int_\omega\varphi\,(\bar{v}^N(t)-\dot{\bar{w}}(t))\cdot(f(t)+\mathrm{div}_{x'}\bar{\theta})\,dx'-\int_\omega\nabla\varphi\odot(\bar{v}^N(t)-\dot{\bar{w}}(t)):(\bar{\sigma}^N(t)-\bar{\theta})\,dx'\\
&+\frac{1}{12}\int_\omega\varphi\,(v_3^N(t)-\dot{w}_3(t))(g(t)+\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\theta})\,dx'+\frac{1}{12}\int_\omega(v_3^N(t)-\dot{w}_3(t))D^2\varphi:(\hat{\sigma}^N(t)-\hat{\theta})\,dx'\\
&\qquad+\frac{1}{6}\int_\omega\nabla\varphi\odot(\nabla v_3^N(t)-\nabla\dot{w}_3(t)):(\hat{\sigma}^N(t)-\hat{\theta})\,dx'.
\end{aligned}\tag{4.52}$$

We use this expression in (4.50), integrate with respect to time on an arbitrary time interval $[t_1,t_2]$, and observe that the established convergences (4.48), (4.49), and (4.51) are enough to pass to the limit, as $N\to\infty$. In this way we deduce

$$\begin{aligned}
&\int_{t_1}^{t_2}\langle\mathbb{A}_r\dot{\sigma}(t)-E\dot{w}(t),\varphi\,(\sigma(t)-\theta)\rangle\,dt\\
&\leq\int_{t_1}^{t_2}\int_\omega\varphi\,(\dot{\bar{u}}(t)-\dot{\bar{w}}(t))\cdot(f(t)+\mathrm{div}_{x'}\bar{\theta})\,dx'\,dt\\
&\quad-\int_{t_1}^{t_2}\int_\omega\nabla\varphi\odot(\dot{\bar{u}}(t)-\dot{\bar{w}}(t)):(\bar{\sigma}(t)-\bar{\theta})\,dx'\,dt\\
&\quad+\frac{1}{12}\int_{t_1}^{t_2}\int_\omega\varphi\,(\dot{u}_3(t)-\dot{w}_3(t))(g(t)+\mathrm{div}_{x'}\mathrm{div}_{x'}\hat{\theta})\,dx'\,dt\\
&\quad+\frac{1}{12}\int_{t_1}^{t_2}\int_\omega(\dot{u}_3(t)-\dot{w}_3(t))D^2\varphi:(\hat{\sigma}(t)-\hat{\theta})\,dx'\,dt\\
&\quad+\frac{1}{6}\int_{t_1}^{t_2}\int_\omega\nabla\varphi\odot(\nabla\dot{u}_3(t)-\nabla\dot{w}_3(t)):(\hat{\sigma}(t)-\hat{\theta})\,dx'\,dt.
\end{aligned}$$

By the integration by parts formula (3.17) this is equivalent to

$$\int_{t_1}^{t_2}\int_{\Omega\cup\Gamma_d}\varphi\,d[(\theta-\sigma(t)):\dot{p}(t)]\,dt\leq 0 \tag{4.53}$$

for every $\theta\in\mathcal{K}_r(\Omega)\cap\Sigma(\Omega)$ and every $\varphi\in C^2(\overline{\omega})$, $\varphi\geq 0$, with $\varphi=0$ in a neighbourhood of γ_n. For every $\delta>0$ let now $\varphi_\delta\in C^2(\overline{\omega})$ be such that $0\leq\varphi_\delta\leq 1$, $\varphi_\delta=0$ on the set $\{x'\in\overline{\omega}:\ \mathrm{dist}(x',\gamma_n)<\delta\}$, and $\varphi_\delta=1$ on the set $\{x'\in\overline{\omega}:\ \mathrm{dist}(x',\gamma_n)>2\delta\}$. Using φ_δ as test function in (4.53) and sending δ to zero, we obtain

$$\int_{t_1}^{t_2}\langle\theta-\sigma(t),\dot{p}(t)\rangle\,dt\leq 0.$$

Since the time interval $[t_1,t_2]$ is arbitrary, this is equivalent to the flow rule in the form (4.3).

□

5. An example

In this section we show an explicit example of quasistatic evolution where the stress component $\sigma_\perp$ is different from zero.

We assume that the three-dimensional elasticity tensor $\mathbb{C}$ is isotropic, that is, of the form

$$\mathbb{C}\xi := 2\mu\,\xi + \lambda(\operatorname{tr}\xi)I_{3\times 3}$$

for some λ, μ satisfying $\mu > 0$, $\lambda + \mu \geq 0$, and every $\xi \in \mathbb{M}^{3\times 3}_{sym}$. From the results of [5, Subsection 3.2] it follows that the elasticity tensor of the reduced problem takes the form

$$\mathbb{C}_r\xi = 2\mu\,\xi + \frac{2\lambda\mu}{\lambda + 2\mu}(\operatorname{tr}\xi)I_{2\times 2}$$

for every $\xi \in \mathbb{M}^{2\times 2}_{sym}$. We also assume that K_r is of the form (3.6)–(3.7). Finally we consider the boundary condition

$$w(t,x) := \begin{pmatrix} -tx_1x_3 \\ -tx_2x_3 \\ \frac{t}{2}(x_1^2 + x_2^2) \end{pmatrix},$$

for $t \in [0,T]$, prescribed on the whole lateral boundary $\partial\omega\times(-\frac{1}{2}, \frac{1}{2})$; hence, $\gamma_d = \partial\omega$ and $\gamma_n = \emptyset$. We assume the body forces to be zero, that is, $f(t) = 0$ and $g(t) = 0$. We consider as initial datum $(u_0, e_0, p_0) = (0, 0, 0)$.

Let now

$$t_0 := \sqrt{\frac{3}{2}}\frac{\lambda + 2\mu}{\mu(3\lambda + 2\mu)}\alpha_0.$$

For $t \leq t_0$ we define

$$u(t,x) := w(t,x), \qquad e(t,x) = -tx_3I_{2\times 2}, \qquad p(t,x) := 0$$

for every $t \in [0,T]$ and $x \in \Omega$. For $t > t_0$ we define

$$u(t,x) := w(t,x), \qquad e(t,x) = \begin{cases} \frac{t_0}{2}I_{2\times 2} & \text{for } x_3 < -\frac{t_0}{2t}, \\ -tx_3I_{2\times 2} & \text{for } |x_3| \leq \frac{t_0}{2t}, \\ -\frac{t_0}{2}I_{2\times 2} & \text{for } x_3 > \frac{t_0}{2t}, \end{cases} \qquad p(t,x) := Eu(t,x) - e(t,x).$$

We claim that $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution, that is, satisfies conditions (qs1)–(qs3) in Definition 4.1.

It is easy to see that $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous, that is, condition (qs1) holds. Clearly $(u(t), e(t), p(t))$ belongs to $\mathcal{A}_{KL}(w(t))$ for every $t \in [0,T]$. Setting $\sigma(t,x) := \mathbb{C}_r e(t,x)$, we have that

$$\sigma(t,x) = -2\mu\frac{3\lambda + 2\mu}{\lambda + 2\mu}tx_3I_{2\times 2}$$

for $t \leq t_0$, while

$$\sigma(t,x) = \begin{cases} \mu\dfrac{3\lambda + 2\mu}{\lambda + 2\mu}t_0I_{2\times 2} & \text{for } x_3 < -\frac{t_0}{2t}, \\ -2\mu\dfrac{3\lambda + 2\mu}{\lambda + 2\mu}tx_3I_{2\times 2} & \text{for } |x_3| \leq \frac{t_0}{2t}, \\ -\mu\dfrac{3\lambda + 2\mu}{\lambda + 2\mu}t_0I_{2\times 2} & \text{for } x_3 > \frac{t_0}{2t} \end{cases}$$

for $t > t_0$. Using this expression and the definition of p, it is easy to check that also conditions (qs2) and (qs3) are satisfied. Thus, $t \mapsto (u(t), e(t), p(t))$ is a quasistatic evolution.

Note that $\bar{\sigma}(t) = 0$ for every $t \in [0,T]$. For $t \leq t_0$ we have $\sigma(t) = x_3\hat{\sigma}(t)$, while for $t > t_0$ we have $\sigma(t) = x_3\hat{\sigma}(t) + \sigma_\perp(t)$ with $\sigma_\perp(t) \neq 0$. Since the stress component is unique by Theorem 4.6, this is the expression of the stress for any solution to the quasistatic evolution problem with this choice of the data.

This example shows that the problem has a genuinely three-dimensional nature. Since the location of the plastic zone (that is, the region where the stress is on the yield surface) depends on the thickness variable x_3, reducing the problem to a two-dimensional setting is not possible. In particular, applying the classical plastic plate model to this set of data would mean to look for a solution that is linear with respect to x_3 both on e and p, and thus would lead to a wrong description of the plastic response.

We also point out that this example is a counterexample to the result of [5, Proposition 7.17], which is therefore false. It is in fact not true, in general, that if $\sigma \in K_r$ a.e. in Ω, then $\hat{\sigma} \in K_r$ a.e. in ω.

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