Optimal flux densities for linear mappings and the multiscale geometry of structured deformations

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Abstract

We establish the unexpected equality of the optimal volume density of total flux of a linear vector field $x \longmapsto Mx$ and the least volume fraction that can be swept out by submacroscopic switches, separations, and interpenetrations associated with the purely submacroscopic structured deformation $(i, I+M)$. This equality is established first by identifying a dense set $\mathcal{S}$ of $N \times N$ matrices M for which the optimal total flux density equals $|trM|$, the absolute value of the trace of M. We then use known representation formulae for relaxed energies for structured deformations to show that the desired least volume fraction associated with $(i, I+M)$ also equals $|trM|$. We also refine the above result by showing the equality of the optimal volume density of the positive part of the flux of $x \longmapsto Mx$ and the volume fraction swept out by submacroscopic separations alone, with common value $(trM)^+$. Similarly, the optimal volume density of the negative part of the flux of $x \longmapsto Mx$ and the volume fraction swept out by submacroscopic switches and interpenetrations are shown to have the common value $(trM)^-$.

1 Introduction

Our goal in this article is to provide an unexpected connection between optimal flux densities of linear vector fields and optimal amounts of submacroscopic switching, separation, and interpenetration associated with multiscale geometrical objects called structured deformations. If M is a linear mapping on $\mathbb{R}^N$ (or, equivalently, an $N \times N$ real matrix) and $\mathcal{R}$ is a smooth region in $\mathbb{R}^N$ with volume $|\mathcal{R}|$, then the divergence theorem provides the formula

$$\frac{1}{|\mathcal{R}|}\int\limits_{\partial\mathcal{R}} Mx\cdot\nu(x)\,d\mathcal{H}^{N-1}(x) = trM \tag{1}$$

relating the outward flux of the vector field $x \longmapsto Mx$ per unit volume and the trace of the linear mapping M. Of course the right-hand side is independent of the region $\mathcal{R}$, so that the volume density of outward flux on the left-hand side is a quantity that depends on M alone.

Here, we shall be interested in what can be said when the integrand $x \longmapsto Mx\cdot\nu(x)$ is replaced by its absolute value $x \longmapsto |Mx\cdot\nu(x)|$, or by its positive part $x \longmapsto (Mx\cdot\nu(x))^+ = (|Mx\cdot\nu(x)| + Mx\cdot\nu(x))/2$, or by its negative part $x \longmapsto (Mx\cdot\nu(x))^- = (|Mx\cdot\nu(x)| - Mx\cdot\nu(x))/2$. In each case, the integrand is non-negative, so that no cancellations can arise in calculating the surface integral. For example, if the outward flux field $x \longmapsto Mx\cdot\nu(x)$ is replaced by $x \longmapsto |Mx\cdot\nu(x)|$ on the left-hand side of (1), then replacement of a region $\mathcal{R}$ by one with volume as close as desired to $|\mathcal{R}|$ but with its surface area increased by an arbitrarily large amount, can cause the the total flux per unit volume $\frac{1}{|\mathcal{R}|}\int\limits_{\partial\mathcal{R}} |Mx\cdot\nu(x)|\; d\mathcal{H}^{N-1}(x)$ to increase without bound. Nevertheless, it is easy to show that $|trM|$ provides a lower bound for the total flux per unit volume as the region $\mathcal{R}$ is varied, and a principal result that we prove in this article provides a definite class of regions $\mathcal{R}$ and a dense set $\mathcal{S}$ of linear mappings M for which the lower bound $|trM|$ is the infimum of $\frac{1}{|\mathcal{R}|}\int\limits_{\partial\mathcal{R}} |Mx\cdot\nu(x)|\; d\mathcal{H}^{N-1}(x)$ over the class of regions $\mathcal{R}$:

$$\inf_{\mathcal{R}}\frac{1}{|\mathcal{R}|}\int\limits_{\partial\mathcal{R}} |Mx\cdot\nu(x)|\; d\mathcal{H}^{N-1}(x) = |trM|\,. \tag{2}$$

It turns out that for $N = 1,2,3$, the equality (2) of the lower bound $|trM|$ and the infimum holds for arbitrary linear mappings M (the proof for $N = 2,3$ is too long to include in the present article, while the proof for $N = 1$ does not present any difficulty and hence it is omitted), and it remains unknown to us whether this stronger conclusion when $N = 1,2,3$ can be extended to arbitrary dimensions N. Because of the formula (1), all of the results that we obtain for the absolute value $|\cdot|$ remain true when in (2) $|\cdot|$ is replaced in the integrand and on the right-hand side of (2) by the positive part $(\cdot)^+$ or by the negative part $(\cdot)^-$.

The results associated with the formula (2) are stated precisely and proved in Section 2 of this article, and they permit us to interpret $|trM|$ as an optimal flux density associated with the vector field $x \longmapsto Mx$. Our proof requires generating lateral surfaces of optimizing regions through solutions of initial-value problems for the ordinary differential equation $\dot{x} = Mx$, and the stronger result available when $N = 2, 3$ results from the easier bookkeeping in these cases.

The remainder of this article is devoted to the application of our results on optimal flux densities in order to obtain an explicit formula for a relaxed energy associated with submacroscopic geometrical changes in continuous bodies. In Section 3 we review some basic concepts from the multiscale geometry of structured deformations (g, G) of a given region Ω [8] and summarize results on the relaxation of energies associated with structured deformations of continua [10]. These results address the problem of minimizing the limit inferior of an initial energy defined on sequences $n \longmapsto u_n$ of approximating deformations of a continuum with u_n converging to g and with ∇u_n converging to G in approriate senses. The resulting infimum $\mathcal{E}(g, G)$ is interpreted as the most economical way of deforming the region Ω via the macroscopic deformation g in order that small pieces of the region be deformed submacroscopically via the tensor field G. The results in [10] show that this minimization problem leads, for each point $x_0 \in \Omega$, to the simpler problem $\min_u E_{x_0}(u)$ of minimizing a related energy over mappings u of the unit cube whose boundary values agree with the linear mapping $x \longmapsto \nabla g(x_0)x$ and whose gradients ∇u on the unit cube have average value $G(x_0)$.

We show in Section 4 that, for a particular choice of initial energy, the simpler minimization problem $\min_u E_{x_0}(u)$ amounts to minimization of the flux density $\frac{1}{|\mathcal{R}|} \int\limits_{\partial\mathcal{R}} |(\nabla g(x_0) - G(x_0))x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x)$ over sets of finite perimeter $\mathcal{R}$ contained in the unit cube. Our result (2) tells us that, if a structured deformation (g, G) and a point x_0 are such that $\nabla g(x_0) - G(x_0)$ is in the dense set $\mathcal{S}$ of linear mappings, then

$$\inf_{\mathcal{R}} \frac{1}{|\mathcal{R}|} \int\limits_{\partial\mathcal{R}} |(\nabla g(x_0) - G(x_0))x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x) = |tr(\nabla g(x_0) - G(x_0))| \, .$$

Regularity results in [10] for the volume density of the relaxed energy imply in turn that, for *every* structured deformation (g, G) and point x_0, $|tr(\nabla g(x_0) - G(x_0))|$ is the volume density at x_0 of the desired relaxed energy $\mathcal{E}(g, G)$.

In the last section we use results in Section 4 to provide the alternative interpretation of the number $|tr(\nabla g(x_0) - G(x_0))|$ as the least fraction of volume at x_0 that can be swept out by switches, separations, and interpenetration of pieces of the region associated with approximating deformations u_n for the given structured deformation (g, G). When $|\cdot|$ is replaced by the positive part $(\cdot)^+$ or by the negative part $(\cdot)^-$, corresponding geometrical interpretations are justified on the basis of the results in Section 4. Specifically, $(tr(\nabla g(x_0) - G(x_0)))^+$ represents the least fraction of volume at x_0 that can be swept out by separations associated with approximating deformations u_n, while $(tr(\nabla g(x_0) - G(x_0)))^-$ is the least volume fraction that can be swept out by switches and interpenetrations associated with the approximating deformations u_n.

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2 Optimal flux densities for linear mappings

Our principal relation for optimal flux densities involves the dense subset $\mathcal{S}$ of the vector space of $N \times N$ real matrices, $\mathbb{R}^{N\times N}$, defined in the following theorem.

Theorem 1 *The following sets are dense and open in $\mathbb{R}^{N\times N}$:*

(i) $\mathcal{S}_1 = \{T \in \mathbb{R}^{N\times N} : T \text{ has } N \text{ distinct eigenvalues}\}$;

(ii) $\mathcal{S}_2 = \{T \in \mathbb{R}^{N\times N} : \text{all eigenvalues of } T \text{ have nonzero real part}\}$;

(iii) $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$.

Parts (i) and (ii) of the the theorem are just Theorems 1 and 3 of Section 7.3 of [13], while part (iii) is a consequence of the Proposition contained in Section 7.3 of [13] .

The following theorem provides in a precise manner the formula (2) for optimal flux densities described in the introduction.

Theorem 2 *Let $Q := (-1/2, 1/2)^N$ be the unit cube. Let $\mathcal{S}$ be as in Theorem 1, let $M_0 \in \mathcal{S}$, and define*

$$\mathcal{A} := \{R \subset Q \;:\; R \text{ is a set of finite perimeter compactly contained in } Q \text{ with } |R| \neq 0\}. \tag{3}$$

Then

$$\inf_{R\in\mathcal{A}} \frac{1}{|R|} \int_{\partial R} |M_0 x \cdot \nu(x)|\, d\mathcal{H}^{N-1}(x) = |tr\ M_0|,$$

where ν is the outward unit normal to R.

The proof requires a lemma about Jacobians. We recall that the Jacobian J_f of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, with $m \geq n$, is

$$J_f(x) := \sqrt{\det(\nabla f(x)^T \nabla f(x))},$$

and by the Cauchy-Binet theorem (see, for instance Evans and Gariepy [12]) the square of the Jacobian is equal to the sum of the squares of the determinants of each $(n \times n)$-submatrix of the $(m \times n)$-matrix representing $\nabla f(x)$. We note that when n equals m the Jacobian is just the absolute value of the determinant of the gradient.

Lemma 1 *Let $A \subset \mathbb{R}^{N-1}$ and $I \subset \mathbb{R}$ be bounded open sets and $f : A \times I \to \mathbb{R}^N$ be a differentiable function. For each $y \in I$, let $f^y : A \to \mathbb{R}^N$ be the function defined by $f^y(\cdot) := f(\cdot, y)$. Then, if $J_{f^y} \neq 0$ in A we have*

$$J_f(\cdot, y) = J_{f^y}(\cdot) \left| \frac{\partial f}{\partial x_N}(\cdot, y) \cdot \nu^y(\cdot) \right|,$$

where ν^y is a normal to the hypersurface $f^y(A)$ in $\mathbb{R}^N$.

Proof. Before we start the proof we note that the gradient of f, ∇f, evaluated at any point of $A \times I$ is a $N \times N$ matrix, while the gradient of f^y, ∇f^y, evaluated at any point of A is an $N \times (N-1)$ matrix. Moreover, with obvious notation, we have

$$\nabla f(\cdot, y) = \left(\nabla f^y(\cdot) \left| \frac{\partial f}{\partial x_N}(\cdot, y) \right. \right).$$

In order to simplify the notation for the remainder of the proof, all the fields with domain $A \times I$ are understood to be evaluated at a point of the form $(\cdot, y)$. Then

$$\det \nabla f = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_N} \right)_i (\mathrm{cof}\ \nabla f)_{Ni} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_N} \right)_i (\mathrm{cof}\ \nabla f^y)_i = \frac{\partial f}{\partial x_N} \cdot \mathrm{cof}\ \nabla f^y,$$

where $(\mathrm{cof}\ \nabla f)_{Ni}$ is equal to $(-1)^{N+i}$ times the determinant of the $(N-1) \times (N-1)$ matrix obtained by deleting the i^{th} row and the N^{th} column from ∇f, while $(\mathrm{cof}\ \nabla f^y)_i$ is equal to $(-1)^{N+i}$ times the determinant of the $(N-1) \times (N-1)$ matrix obtained by deleting the i^{th} row from ∇f^y. Moreover we have denoted with $\mathrm{cof}\ \nabla f^y$ the vector whose i^{th} component is $(\mathrm{cof}\ \nabla f^y)_i$. From this last definition and the Cauchy-Binet theorem we have

$$\|\mathrm{cof}\ \nabla f^y\| = \sqrt{\sum_{i=1}^{N} (\mathrm{cof}\ \nabla f^y)_i^2} = J_{f^y}.$$

We now define the unit vector

$$\nu^y := \frac{1}{J_{f^y}} \mathrm{cof}\ \nabla f^y$$

so that $J_f = |\det \nabla f| = J_{f^y} \left| \frac{\partial f}{\partial x_N} \cdot \nu^y \right|$. Hence, to conclude the proof it suffices to note that for every $j = 1, \ldots, N-1$ we have

$$0 = \det \left(\nabla f^y \left| \frac{\partial f^y}{\partial x_j} \right. \right) = \frac{\partial f^y}{\partial x_j} \cdot \mathrm{cof}\ \nabla f^y,$$

and hence ν^y is orthogonal to the hypersurface $f^y(A)$ in $\mathbb{R}^N$. $\square$

In the proof of Theorem 2 we shall need the definition of the exponential of a linear operator and some of its properties. We recall that if $A \in \mathbb{R}^{N\times N}$ then the exponential of A is the linear operator defined by

$$e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k!}.$$

Note that since A commutes with each term of the series for e^{tA}, for any $t \in \mathbb{R}$, we have that

$$Ae^{tA} = e^{tA}A. \tag{4}$$

Moreover, (see Arnold [3]) we also have the following identity

$$\det e^A = e^{\mathrm{tr} A}. \tag{5}$$

The definition of the exponential of a linear operator turns out to be very important since the unique solution of the system of ordinary differential equations

$$\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0, \end{cases} \tag{6}$$

with $t_0 \in \mathbb{R}$, is

$$x(t) = e^{A(t-t_0)}x_0. \tag{7}$$

Proof of Theorem 2. Without loss of generality we assume that $M_0 \neq 0$. We start by proving that $|\mathrm{tr}\ M_0|$ is a lower bound for the infimum. This follows easily from the divergence theorem, indeed

$$\begin{aligned} \frac{1}{|R|}\int_{\partial R} |M_0 x \cdot \nu(x)|\, d\mathcal{H}^{N-1}(x) &\geq \frac{1}{|R|}\left|\int_{\partial R} M_0 x \cdot \nu(x)\, d\mathcal{H}^{N-1}(x)\right| \\ &= \frac{1}{|R|}\left|\int_R \mathrm{div}\ (M_0 x)\, d\mathcal{L}^N(x)\right| \\ &= \frac{1}{|R|}\left|\int_R \mathrm{tr}\ M_0\, d\mathcal{L}^N\right| = |\mathrm{tr}\ M_0|. \end{aligned}$$

We now prove that $|\mathrm{tr}\ M_0|$ also is an upper bound. To do this it suffices to show that for every real number $\varepsilon > 0$ there exists a region $R \in \mathcal{A}$ such that

$$\frac{1}{|R|}\int_{\partial R} |M_0 x \cdot \nu(x)|\, d\mathcal{H}^{N-1}(x) \leq |\mathrm{tr}\ M_0| + \varepsilon. \tag{8}$$

The main idea in proving (8) is to minimize the part of the boundary ∂R, of a region $R \in \mathcal{A}$, in which $M_0 x$ has a component along the normal $\nu(x)$ of the boundary. The regions R that we shall consider will be constructed as follows: we fix a subset B_1 of a hyperplane and we follow the points of B_1 along curves $t \mapsto x(t)$ with tangent $M_0 x(t)$. In this way the regions constructed have a "lateral surface" tangent to $M_0 x$ at the point x.

Indeed, we shall prove that, for any $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, there exists a region $R \in \mathcal{A}$ such that:

$$\frac{1}{|R|} \int_{\partial R} |M_0 x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x) = \frac{\text{tr } M_0}{e^{(\text{tr } M_0)(t_2 - t_1)} - 1} + \frac{\text{tr } M_0}{1 - e^{-(\text{tr } M_0)(t_2 - t_1)}}. \tag{9}$$

Relation (8) follows directly from this equation. In fact, since $M_0 \in \mathcal{S}$ we have that $\text{tr } M_0 \neq 0$ and for $\text{tr } M_0 > 0$ the limit for $t_1 \to -\infty$ of the right hand side of (9) is equal to $\text{tr } M_0 = |\text{tr } M_0|$, while if $\text{tr } M_0 < 0$ the limit for $t_2 \to +\infty$ of the same quantity is $-\text{tr } M_0 = |\text{tr } M_0|$. Thus in both cases we can choose t_1 and t_2 in such a way to make the right hand side of (9) less than $|\text{tr } M_0| + \varepsilon$.

We now verify equation (9). Let $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$ be fixed. Let $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1}$ be points in Q, and let $x_i(t)$ be the unique solution of the system of ordinary equations

$$\begin{cases} \dot{x}_i = M_0 x_i \\ x_i(t_1) = \bar{x}_i, \end{cases} \tag{10}$$

for every $i = 0, 1, \ldots, N-1$. For any vector $\alpha \in \mathbb{R}^{N-1}$ with components $\alpha_i \in [0,1]$, for $i = 1, \ldots, N-1$, we define the function $f : [0,1]^{N-1} \times (t_1, t_2) \to \mathbb{R}^N$ defined by

$$f(\alpha, t) := x_0(t) + \sum_{i=1}^{N-1} \alpha_i (x_i(t) - x_0(t)). \tag{11}$$

Clearly the function f depends on the chosen points $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1}$.

Claim: For every $\varepsilon > 0$ there exist points $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{N-1} \in Q$ for which the function f is injective, and $\|\bar{x}_i\| < \varepsilon$.

We postpone the proof of the claim and note that f is a solution of the following system of ordinary differential equations and initial condition:

$$\begin{cases} \dot{f} = M_0 f \\ f(\alpha, t_1) = \bar{x}_0 + \sum_{i=1}^{N-1} \alpha_i (\bar{x}_i - \bar{x}_0). \end{cases} \tag{12}$$

Let us denote by R the image of $[0,1]^{N-1} \times (t_1, t_2)$ under the function f and by B_1 and B_2, respectively, the surfaces $f([0,1]^{N-1}, t_1)$ and $f([0,1]^{N-1}, t_2)$, each of which is contained in a hyperplanes in $\mathbb{R}^N$ (see Figure 1). Note that by using the solution

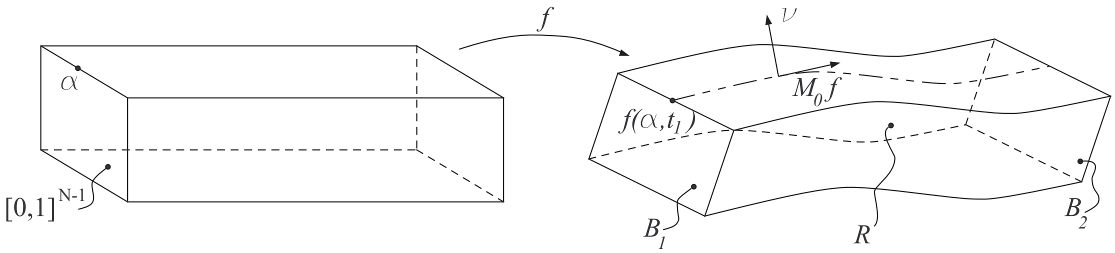

Figure 1: The region R.

of the system (10), written in the form (7), and the property of the exponential function given by (4) we have

$$\begin{aligned}\dot f(\alpha,t) &= M_0 f(\alpha,t) = M_0 e^{M_0(t-t_1)} f(\alpha,t_1) = e^{M_0(t-t_1)} M_0 f(\alpha,t_1)\\ &= e^{M_0(t-t_1)} \dot f(\alpha,t_1)\\ \frac{\partial f}{\partial \alpha_i}(\alpha,t) &= x_i(t) - x_0(t) = e^{M_0(t-t_1)}(\bar x_i - \bar x_0) = e^{M_0(t-t_1)} \frac{\partial f}{\partial \alpha_i}(\alpha,t_1),\end{aligned}$$

and, therefore,

$$(\nabla_\alpha f(\alpha,t)|\dot f(\alpha,t)) = \nabla f(\alpha,t) = e^{M_0(t-t_1)} \nabla f(\alpha,t_1). \tag{13}$$

From the equation above and (5) we find

$$J_f(\alpha,t) = |\det \nabla f(\alpha,t)| = \det e^{M_0(t-t_1)} |\det \nabla f(\alpha,t_1)| = e^{(\mathrm{tr}\ M_0)(t-t_1)} J_f(\alpha,t_1). \tag{14}$$

Denoting by $f^{t_1} : [0,1]^{N-1} \to \mathbb{R}^N$ the map defined by

$$f^{t_1}(\cdot) := f(\cdot, t_1),$$

and applying Lemma 1 we obtain

$$J_f(\alpha,t) = e^{(\mathrm{tr}\ M_0)(t-t_1)} |M_0 f(\alpha,t_1) \cdot \nu^{t_1}| J_{f^{t_1}}(\alpha) = e^{(\mathrm{tr}\ M_0)(t-t_1)} |M_0 f^{t_1}(\alpha) \cdot \nu^{t_1}| J_{f^{t_1}}(\alpha),$$

where ν^{t_1} is a normal to the hyperplane B_1. Then

$$\begin{aligned}|R| &= \int_{t_1}^{t_2} \int_{[0,1]^{N-1}} J_f(\alpha,t)\, d\alpha dt \qquad (15)\\ &= \int_{t_1}^{t_2} e^{(\mathrm{tr}\ M_0)(t-t_1)}\, dt \int_{[0,1]^{N-1}} |M_0 f^{t_1}(\alpha) \cdot \nu^{t_1}| J_{f^{t_1}}(\alpha)\, d\alpha.\end{aligned}$$

Then by the change of variables formula and by setting $\nu(x) = \nu^{t_1}$ we obtain

$$|R| = \frac{e^{(\text{tr } M_0)(t_2 - t_1)} - 1}{\text{tr } M_0} \int_{B_1} |M_0 x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x). \tag{16}$$

In a similar way we can show that

$$|R| = \frac{1 - e^{-(\text{tr } M_0)(t_2 - t_1)}}{\text{tr } M_0} \int_{B_2} |M_0 x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x). \tag{17}$$

Let us denote the lateral boundary of ∂R by $\partial^\ell R$, i.e., $\partial^\ell R = \partial R \setminus (B_1 \cup B_2)$. Then for $x(t) \in \partial^\ell R$ we have $M_0 x(t) \cdot \nu(x(t)) = \dot{x}(t) \cdot \nu(x(t)) = 0$ and thus

$$\int_{\partial R} |M_0 x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x) = \int_{B_1 \cup B_2} |M_0 x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x). \tag{18}$$

Since tr $M_0 \neq 0$, from Eq.s (16), (17) and (18) we obtain (9).

To conclude the verification of (9) we only need to check that the region R considered above is in the set $\mathcal{A}$ defined in equation (3). We first show that $|R| \neq 0$. From equation (15) it suffices to show that

$$\int_{[0,1]^{N-1}} |M_0 f^{t_1}(\alpha) \cdot \nu^{t_1}| J_{f^{t_1}}(\alpha) \, d\alpha \neq 0.$$

From (12) we have that

$$M_0 f^{t_1}(\alpha) \cdot \nu^{t_1} = M_0 \bar{x}_0 \cdot \nu^{t_1} + \sum_{i=1}^{N-1} \alpha_i M_0 (\bar{x}_i - \bar{x}_0) \cdot \nu^{t_1}.$$

The first term on the right hand side is different from zero by the choice of the points $\bar{x}_0, \ldots, \bar{x}_{N-1}$. Consequently the set of α's in $[0,1]^{N-1}$ for which the right hand side will vanish has null $(N-1)$-Hausdorff measure. The choice of the points $\bar{x}_0, \ldots, \bar{x}_{N-1}$ also implies that $J_{f^{t_1}}(\alpha) \neq 0$.

Since the region $[0,1]^{N-1} \times (t_1, t_2)$ is of finite perimeter, so also is the region $R = f([0,1]^{N-1} \times (t_1, t_2))$. Indeed, this follows from the proof of the chain rule in BV (see Ambrosio et. al. [2] proof of theorem 3.96), once we notice that f^{-1} is Lipschitzian. Moreover, since

$$\|x_i(t)\| = \left\| e^{M_0(t - t_1)} \bar{x}_i \right\| \leq e^{\|M_0\|(t_2 - t_1)} \|\bar{x}_i\|,$$

for every $i = 0, 1, \ldots, N-1$, it is always possible to choose the points $\bar{x}_i$, for $i = 0, 1, \ldots, N-1$, so that $\|x_i(t)\| < 1/2$, for any $t \in (t_1, t_2)$, and hence $R \subset Q$.

It only remains to prove the claim.

Proof of the Claim.

Suppose that M_0 has a real eigenvalue λ which, by hypothesis, must be non-zero. Then there are M_0-invariant subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{R}^N$, with $\mathcal{S}$ the eigenspace corresponding to λ, such that $\mathbb{R}^N = \mathcal{S}\oplus \mathcal{T}$, see [13]. Since M_0 has distinct eigenvalues the subspace $\mathcal{T}$ has dimension $N-1$. Choose $\bar{x}_0 \in \mathcal{S}\backslash\{0\}$ with $\|\bar{x}_0\| < \frac{\varepsilon}{2}$, so that $M_0\bar{x}_0 = \lambda\bar{x}_0$, and choose a basis $u_1, \dots u_{N-1}$ of $\mathcal{T}$ with $\|u_i\| < \frac{\varepsilon}{2}$ for $i = 1, \dots, N-1$. We put $\bar{x}_i = \bar{x}_0 + u_i$ for $i = 1, \dots, N-1$, and we note that $\|\bar{x}_i\| < \varepsilon$ for $i = 1, \dots, N-1$. Moreover, $e^{tM_0}\bar{x}_0 = e^{t\lambda}\bar{x}_0$ and $\mathcal{T}$ is e^{tM_0}-invariant. Let (t,α), $(\tilde{t},\tilde{\alpha}) \in \mathbb{R}\times\mathbb{R}^{N-1}$ be given such that $f(t,\alpha) = f(\tilde{t},\tilde{\alpha})$, so that

$$e^{t\lambda}\bar{x}_0 - e^{\tilde{t}\lambda}\bar{x}_0 = e^{\tilde{t}M_0}\sum_{i=1}^{N-1}\tilde{\alpha}_i(\bar{x}_i - \bar{x}_0) - e^{tM_0}\sum_{i=1}^{N-1}\alpha_i(\bar{x}_i - \bar{x}_0). \tag{19}$$

The left-hand side of this relation is in $\mathcal{S}$ and the right-hand side is in the complementary subspace $\mathcal{T}$, because $\bar{x}_i - \bar{x}_0 = u_i$ for $i = 1, \dots, N-1$. Therefore, both sides are zero, so that $(e^{t\lambda} - e^{\tilde{t}\lambda})x_0 = 0$ and, therefore, since $x_0 \neq 0$, $e^{t\lambda} = e^{\tilde{t}\lambda}$. Consequently, $t = \tilde{t}$. This relation, the vanishing of the right-hand side of (19), and the invertibility of e^{tM_0} imply

$$\sum_{i=1}^{N-1}(\tilde{\alpha}_i - \alpha_i)u_i = 0.$$

The linear independence of $u_1, \dots u_{N-1}$ imply that $\alpha = \tilde{\alpha}$, and this completes the proof of injectivity when M_0 has a real eigenvalue.

Suppose now that M_0 has no real eigenvalues. We may choose a and b in $\mathbb{R}$ such that $a+bi$ and $a-bi$ are eigenvalues of M_0, and, since each eigenvalue has non-zero real part and non-zero imaginary part, we may assume without loss of generality that $ab > 0$. Then there are M_0-invariant subspaces $\mathcal{S}_1$ and $\mathcal{T}$ of $\mathbb{R}^N$ such that $\mathbb{R}^N = \mathcal{S}_1\oplus \mathcal{T}$, and there is a basis u_0, $u_1 \in \mathbb{R}^N$ of $\mathcal{S}_1$ such that the matrix of the restriction $M_0\,|_{\mathcal{S}_1}$of M_0 to $\mathcal{S}_1$, relative to this basis, is $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, see [13]. Choose a basis $u_2, \dots, u_{N-1}$ of $\mathcal{T}$ and $d \geq 1$ such that $\|u_i\| < \varepsilon/2d$ for $i = 0, 1, \dots, N-1$. We put $\bar{x}_0 := du_0$, $\bar{x}_i := \bar{x}_0 + u_i$ for $i = 1, \dots, N-1$, and note that $\|\bar{x}_i\| < \varepsilon$ for $i = 0, 1, \dots, N-1$. For each $\tau \in \mathbb{R}$ the M_0-invariance of the subspaces $\mathcal{S}_1$ and $\mathcal{T}$ imply that the matrix of the restriction $e^{\tau M_0}\,|_{\mathcal{S}_1}$ of $e^{\tau M_0}$ to $\mathcal{S}_1$, relative to the basis u_0, u_1, is $e^{a\tau}\begin{bmatrix} \cos b\tau & \sin b\tau \\ -\sin b\tau & \cos b\tau \end{bmatrix}$. Let (t,α), $(\tilde{t},\tilde{\alpha}) \in \mathbb{R}\times\mathbb{R}^{N-1}$ be given such that

$f(t,\alpha) = f(\tilde{t},\tilde{\alpha})$, so that

$$e^{tM_0}(du_0+\alpha_1 u_1)-e^{\tilde{t}M_0}(du_0+\tilde{\alpha}_1 u_1) = e^{\tilde{t}M_0}\sum_{i=2}^{N-1}\tilde{\alpha}_i(\bar{x}_i-\bar{x}_0)-e^{tM_0}\sum_{i=2}^{N-1}\alpha_i(\bar{x}_i-\bar{x}_0). \quad (20)$$

Again, this relation implies that each side of this relation separately vanishes and, in particular, that the matrix equation

$$e^{at}\begin{bmatrix}\cos bt & \sin bt\\ -\sin bt & \cos bt\end{bmatrix}\begin{bmatrix}d\\ \alpha_1\end{bmatrix} = e^{a\tilde{t}}\begin{bmatrix}\cos b\tilde{t} & \sin b\tilde{t}\\ -\sin b\tilde{t} & \cos b\tilde{t}\end{bmatrix}\begin{bmatrix}d\\ \tilde{\alpha}_1\end{bmatrix} \quad (21)$$

is satisfied. We shall show that, when d is chosen to be sufficiently large, the only solution of (21) is $\tilde{t}=t$ and $\tilde{\alpha}_1=\alpha_1$; consequently, the vanishing of the right-hand side of (20) then implies that $\tilde{\alpha}_i=\alpha_i$ for $i=2,\ldots,N-1$. Assume without loss of generality that $a\tilde{t}\geq at$, put $\tau := \tilde{t}-t$, and note that (21) is equivalent to

$$e^{a\tau}\begin{bmatrix}\cos b\tau & \sin b\tau\\ -\sin b\tau & \cos b\tau\end{bmatrix}\begin{bmatrix}d\\ \tilde{\alpha}\end{bmatrix} = \begin{bmatrix}d\\ \alpha\end{bmatrix} \quad (22)$$

where, for convenience, we have dropped the subscript 1. If $\sin b\tau = 0$, then $\cos b\tau = \pm 1$ and the system becomes

$$e^{a\tau}\cos b\tau\begin{bmatrix}d\\ \tilde{\alpha}\end{bmatrix} = \begin{bmatrix}d\\ \alpha\end{bmatrix},$$

which implies $\tau=0$ and $\tilde{\alpha}=\alpha$. If $a\tau=0$ then $\tau=0$ and the orthogonality of the 2×2 matrix in (22) yields $\tilde{\alpha}=\alpha$. We then consider the case $a\tau>0$ and $\sin b\tau\neq 0$, which yields from (22) the formulas

$$\tilde{\alpha} = \frac{e^{-a\tau}-\cos b\tau}{\sin b\tau}d,\ \alpha = \frac{\cos b\tau - e^{a\tau}}{\sin b\tau}d. \quad (23)$$

If $\sin b\tau>0$, then $\alpha\notin[0,1]$, because the numerator $\cos b\tau - e^{a\tau}$ in (23) is negative and the denominator is positive. We assume that $\sin b\tau<0$. The formulas (23) imply

$$d^{-2}\sin^2(b\tau)\,\tilde{\alpha}\,\alpha = -(\cos b\tau-1)^2+\cos b\tau(e^{a\tau}+e^{-a\tau}-2)\,,$$

and, since $e^{a\tau}+e^{-a\tau}-2\geq 0$, we conclude that $\tilde{\alpha}\,\alpha<0$ if $\cos b\tau\leq 0$. Thus, the inequality $\cos b\tau\leq 0$ rules out solutions $\tilde{\alpha}$, α , both in the interval $[0,1]$. Henceforth, we assume that $\sin b\tau<0$ and $\cos b\tau>0$. The solutions α and $\tilde{\alpha}$ of (23) must be in the interval $[0,1]$, so that

$$0\leq\frac{\cos b\tau - e^{-a\tau}}{-\sin b\tau}d\leq 1,\quad 0\leq\frac{e^{a\tau}-\cos b\tau}{-\sin b\tau}d\leq 1,$$

or, equivalently,

$$\cos b\tau + \frac{\sin b\tau}{d} \le e^{-a\tau} \le \cos b\tau \le e^{a\tau} \le \cos b\tau - \frac{\sin b\tau}{d}. \tag{24}$$

It is convenient in what follows to put $x := a\tau > 0$ and $y := b\tau \in (\frac{3\pi}{2}, 2\pi)$. The inequality chain becomes

$$\cos y + \frac{\sin y}{d} \le e^{-x} \le \cos y \le e^{x} \le \cos y - \frac{\sin y}{d}. \tag{25}$$

This chain of inequalities is equivalent to

$$\max\{\cos y + \frac{\sin y}{d}, \frac{1}{\cos y - \frac{\sin y}{d}}\} \le e^{-x} \le \cos y. \tag{26}$$

However, noting that

$$\cos y + \frac{\sin y}{d} = (\cos y + \frac{\sin y}{d})\frac{\cos y - \frac{\sin y}{d}}{\cos y - \frac{\sin y}{d}} = \frac{\cos^2 y - (\frac{\sin y}{d})^2}{\cos y - \frac{\sin y}{d}} \le \frac{1}{\cos y - \frac{\sin y}{d}}$$

we find that (26) also is equivalent to

$$\frac{1}{\cos y - \frac{\sin y}{d}} \le e^{-x} \le \cos y. \tag{27}$$

A necessary and sufficient condition that there exist $x \ge 0$ and $y \in (\frac{3\pi}{2}, 2\pi)$ satisfying (27) is

$$\frac{1}{\cos y - \frac{\sin y}{d}} \le \cos y$$

or, equivalently, $1 \le \cos y(\cos y - \frac{\sin y}{d}) = 1 - \sin^2 y - \cos y \frac{\sin y}{d}$. This, in turn, is equivalent to

$$0 \le -\sin y(\frac{\cos y}{d} + \sin y),$$

and the condition $y \in (\frac{3\pi}{2}, 2\pi)$ then yields the equivalent condition $\frac{\cos y}{d} + \sin y \ge 0$, i.e., $\tan y \ge -\frac{1}{d}$. This inequality holds for all y in the interval $(2\pi - \tan^{-1}\frac{1}{d}, 2\pi)$, we conclude that the condition $y \in (2\pi - \tan^{-1}\frac{1}{d}, 2\pi)$ is a sufficient condition that there exists $x \ge 0$ for which (25) holds; from (27) we have

$$-\ln \cos y \le x \le \ln(\cos y - \frac{\sin y}{d}). \tag{28}$$

We conclude: necessary and sufficient conditions that there exist $\tau \neq 0$ and $\tilde{\alpha}$, $\alpha \in [0,1]$ satisfying (22) are the existence of $n_d \in \mathbb{Z}$, $y_d \in (2\pi + 2\pi n_d - \tan^{-1}\frac{1}{d}, 2\pi + 2\pi n_d)$ and $x_d \in [-\ln\cos y_d, \ln(\cos y_d - \frac{\sin y_d}{d})]$ such that

$$\frac{b}{a} = \frac{y_d}{x_d}. \tag{29}$$

In this case, $\tau = x_d/a = y_d/b$, x_d is positive, $\tilde{\alpha}$, α are given by (23). Moreover, because $ab > 0$, it follows from (29) that y_d also must be positive, and we can require that the integer n_d is non-negative. We note that, as d increases, points in the interval $[2\pi + 2\pi n_d - \tan^{-1}\frac{1}{d}, 2\pi + 2\pi n_d]$ differ from $2\pi + 2\pi n_d$ by at most $\tan^{-1}\frac{1}{d}$, which tends to zero as d increases without bound; similarly, for $y_d \in (2\pi + 2\pi n_d - \tan^{-1}\frac{1}{d}, 2\pi + 2\pi n_d)$, points in the interval $[-\ln\cos y_d, \ln(\cos y_d - \frac{\sin y_d}{d})]$ differ from 0 by an amount that tends to zero as d increases without bound. Consequently, for each family of pairs $d \longmapsto (x_d, y_d) \in [-\ln\cos y_d, \ln(\cos y_d - \frac{\sin y_d}{d})] \times (2\pi + 2\pi n_d - \tan^{-1}\frac{1}{d}, 2\pi + 2\pi n_d)$ with n_d non-negative integers, we have

$$\lim_{d \longrightarrow \infty} \frac{y_d}{x_d} = +\infty. \tag{30}$$

Therefore, for d sufficiently large, the condition (29) cannot be satisfied when $a\tau$ is positive, and we conclude that $\tau = 0$ and, as a consequence, that also $\tilde{\alpha} = \alpha$. $\square$

3 Relaxed bulk and interfacial energy densities for structured deformations

The need in continuum mechanics to deal with multiscale geometrical changes has led Del Piero and Owen, [8], to the concept of structured deformations. Roughly speaking we can say that a structured deformation is a triplet (κ, g, G), where the injective, piecewise continuosly differentiable field g describes the macroscopic changes in the geometry of a continuous body, G is a piecewise continuous tensor field satisfying the "accommodation inequality" $0 < \det G(x) \leq \det \nabla g(x)$ at each point x, and κ is a surface-like subset of the body at which g and ∇g can have jump discontinuities. Here, ∇ denotes the classical gradient operator. An interpretation of the tensor field G is provided by the Approximation Theorem [8]: for every structured deformation (κ, g, G) there exists a sequence $n \longmapsto (\kappa_n, f_n)$ of deformations, with f_n injective and piecewise continuosly differentiable, such that

$$g = \lim_{n \to \infty} f_n, \tag{31}$$

$$G = \lim_{n \to \infty} \nabla f_n, \tag{32}$$

and

$$\kappa = \liminf_{n \to \infty} \kappa_n. \tag{33}$$

In (31) and (32), the limits are taken in the sense of L^∞ convergence, and in (33) $\liminf_{n\to\infty} \kappa_n := \bigcup_{n=1}^{\infty} \bigcap_{p=n}^{\infty} \kappa_p$. Therefore, $G(x)$ represents the local deformation at x without including the effects of discontinuities of f_n at the disarrangement sites κ_n for the approximating deformations (κ_n, f_n), and the Approximation Theorem motivates calling the field G the *deformation without disarrangements*.

In order to study the energy of a structured deformation, Choksi and Fonseca [10] were led to broaden the setting of structured deformations from piecewise smooth fields to fields in SBV. In this setting a structured deformation is a pair (g, G) where

$$g \in SBV(\Omega; \mathbb{R}^N) \qquad \text{and} \qquad G \in L^1(\Omega; \mathbb{R}^{N\times N}).$$

In this definition the set κ is identified with the jump set, $S(g)$ of g, $\kappa = S(g)$. Using a Lusin-type result due to Alberti [1], Choksi and Fonseca [10] obtained an analogue of the Approximation Theorem of Del Piero and Owen [8]: for each structured deformation (g, G) there exists a sequence $n \mapsto f_n$ in $SBV(\Omega; \mathbb{R}^N)$ such that

$$f_n \to g \ \text{ in } L^1(\Omega; \mathbb{R}^N) \qquad \text{and} \qquad \nabla f_n \overset{*}{\rightharpoonup} G \ \text{ in the sense of measures.} \tag{34}$$

Here, ∇f_n denotes the absolutely continuous part of the distributional derivative of f_n. (See also [15, 4] for alternative settings for the Approximation Theorem, and [14] for the approximation of second order structured deformations.)

For each "simple" deformation f_n it is natural to consider a total energy which is sum of a bulk term and of an interfacial term. Then, since each structured deformation is a limit of simple deformations, one might define the energy of the structured deformation (g, G) as the limit of the sequence of energies associated to a sequence $n \longmapsto f_n$ of simple deformations whose limit is the given structured deformation (g, G). However, since an approximating sequence is far from being unique, the energy of a structured deformation is defined by means of the energetically least costly sequences $n \mapsto f_n$ determining (g, G). This definition and the characterization of these energies have been given in Choksi and Fonseca [10]. For further studies in one and multidimensional settings see Del Piero [6], [7]. The following theorem is the starting point for application of our formula for optimal flux densities to structured deformations.

Theorem 3 *Let $S^{N-1} = \{\nu \in \mathbb{R}^N : |\nu| = 1\}$. Let Ω be a bounded open subset of $\mathbb{R}^N$ and $\psi : \Omega \times \mathbb{R}^N \times S^{N-1} \to [0, +\infty)$ be such that*
(H1) there exists a constant $C > 0$ such that

$$0 \leq \psi(x, \xi, \nu) \leq C|\xi|$$

for all $(x,\xi,\nu) \in \Omega \times \mathbb{R}^N \times S^{N-1}$,
(H2) for every $x_0 \in \Omega$ *and for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|x - x_0| < \delta \Rightarrow |\psi(x,\xi,\nu) - \psi(x_0,\xi,\nu)| \leq \varepsilon C|\xi|$$

for all $(x,\xi,\nu) \in \Omega \times \mathbb{R}^N \times S^{N-1}$,
(H3) $\psi(x,\cdot,\nu)$ *is positively homogeneous of degree 1:*

$$\psi(x,t\xi,\nu) = t\psi(x,\xi,\nu)$$

for all $t > 0$ *and* $(x,\xi,\nu) \in \Omega \times \mathbb{R}^N \times S^{N-1}$,
(H4) ψ *is subadditive, i.e., for all* $\xi_1, \xi_2 \in \mathbb{R}^N$ *and* $(x,\nu) \in \Omega \times S^{N-1}$,

$$\psi(x,\xi_1+\xi_2,\nu) \leq \psi(x,\xi_1,\nu) + \psi(x,\xi_2,\nu).$$

Then, for any $p > 1$, *if*

$$\begin{aligned} I(g,G) \;:=\; & \inf_{\{u_n\}}\{\liminf_{n\to+\infty}\int_{S(u_n)\cap\Omega}\psi(x,[u_n],\nu_{u_n})\,d\mathcal{H}^{N-1} : u_n \in SBV(\Omega,\mathbb{R}^N), \\ & u_n \to g \text{ in } L^1(\Omega,\mathbb{R}^N), \nabla u_n \stackrel{*}{\rightharpoonup} G, \\ & \sup_n \big(|\nabla u_n|_{L^p(\Omega,\mathbb{R}^{N\times N})} + |Du_n|(\Omega)\big) < +\infty\} \end{aligned}$$

we have

$$I(g,G) = \int_\Omega H(x,\nabla g(x),G(x))\,d\mathcal{L}^N + \int_{S(g)\cap\Omega} h(x,[g](x),\nu_g(x))\,d\mathcal{H}^{N-1}(x),$$

where

$$\begin{aligned} H(x_0,A,B) \;:=\; & \inf_{\{u\}}\{\int_{S(u)\cap Q}\psi(x_0,[u],\nu_u)\,d\mathcal{H}^{N-1} : u \in SBV(Q,\mathbb{R}^N), \\ & u|_{\partial Q} = Ax, |\nabla u| \in L^p(Q), \int_Q \nabla u\,d\mathcal{L}^N = B\}, \end{aligned}$$

and

$$\begin{aligned} h(x_0,\xi,\eta) \;:=\; & \inf_{\{u\}}\{\int_{S(u)\cap Q_\eta}\psi(x_0,[u],\nu_u)\,d\mathcal{H}^{N-1} \\ & : u \in SBV(Q_\eta,\mathbb{R}^N), u|_{\partial Q_\eta} = u_{\xi,\eta}, \int_{Q_\eta}\nabla u\,d\mathcal{L}^N = 0\}, \end{aligned}$$

where

$$u_{\xi,\eta}(x) := \begin{cases} 0 & \text{if } -\frac{1}{2} \leq x\cdot\eta < 0 \\ \xi & \text{if } 0 \leq x\cdot\eta < \frac{1}{2}. \end{cases} \tag{35}$$

Here Q *equals* $(-1/2,1/2)^N$ *and* Q_η *denotes the unit cube centered at the origin and with two faces normal to* η.

The theorem above, with ψ independent of the variable x, is based on (2.15), (2.16), and (a corrected version of) (2.17) in Theorem 2.17 of Choksi and Fonseca [10], taking into account their Remark 3.3 and with their bulk energy W set equal to zero. The dependence of ψ on x was not included in [10] but can be handled, thanks to assumption $(H2)$, as in Barroso et. al. [5] again taking into account Remark 3.3 of [10].

4 Relaxation of a specific interfacial energy density

Let f be a real valued function, then we denote by f^+ and by f^- the positive and the negative part of f, i.e.,

$$f^{\pm} = \frac{|f| \pm f}{2}.$$

Our main result on structured deformations is

Theorem 4 *With the notation of Theorem 3, if $L : \Omega \to \mathbb{R}^{N\times N}$ is a continuous tensor field such that $L(x)$ is invertible for every $x \in \bar{\Omega}$ and if the initial energy density has the specific form*

$$\psi(x, \xi, \nu) := (\xi \cdot L(x)\nu)^{\pm},$$

then we have

$$\begin{aligned} H(x, A, B) &= (L(x) \cdot (A - B))^{\pm} \\ h(x, \xi, \nu) &= (\xi \cdot L(x)\nu)^{\pm}. \end{aligned}$$

The proof of the Theorem will rely on the following Lemma.

Lemma 2 *Under the assumptions and with the notation of Theorem 3, if $L : \bar{\Omega} \to \mathbb{R}^{N\times N}$ is a continuous tensor such that $L(x)$ is invertible for every $x \in \bar{\Omega}$ and*

$$\psi(x, \xi, \nu) := |\xi \cdot L(x)\nu|,$$

we have

$$\begin{aligned} H(x, A, B) &= |L(x) \cdot (A - B)| \\ h(x, \xi, \eta) &= |\xi \cdot L(x)\eta|. \end{aligned}$$

Proof. It is an easy matter to check that the function $\psi(x,\xi,\nu) := |\xi \cdot L(x)\nu|$ satisfies the assumptions of Theorem 3. We now prove the representation for H. To show that $|L(x_0)\cdot(B-A)|$ is a lower bound for $H(x_0,A,B)$, we note that for any $u \in SBV(Q,\mathbb{R}^N)$ we have

$$Du = \nabla u\, \mathcal{L}^N \lfloor Q + [u]\otimes \nu_u\, \mathcal{H}^{N-1} \lfloor S(u)\cap Q,$$

and from the trace theorem, see Evans and Gariepy [12],

$$Du(Q) = \int_{\partial Q} u \otimes \nu \, d\mathcal{H}^{N-1}.$$

Hence, from the two equations above, we find

$$\int_{\partial Q} u\otimes \nu\, d\mathcal{H}^{N-1} = Du(Q) = \int_Q \nabla u\, d\mathcal{L}^N + \int_{S(u)\cap Q} [u]\otimes \nu_u\, d\mathcal{H}^{N-1}. \tag{36}$$

For given $A, B \in \mathbb{R}^{N\times N}$ and for any $u \in SBV(Q,\mathbb{R}^N)$ such that $u|_{\partial Q} = Ax$ and $\int_Q \nabla u\, d\mathcal{L}^N = B$ we have

$$\begin{aligned}
\int_{S(u)\cap Q}[u]\otimes \nu\, d\mathcal{H}^{N-1} &= -\int_Q \nabla u\, d\mathcal{L}^N + \int_{\partial Q} u\otimes\nu\, d\mathcal{H}^{N-1} \\
&= -B + \int_Q \nabla(Ax)\, d\mathcal{L}^N \\
&= -B + A,
\end{aligned} \tag{37}$$

and hence

$$\begin{aligned}
\int_{S(u)\cap Q} |[u]\cdot L(x_0)\nu|\, d\mathcal{H}^{N-1} &\geq \left| L(x_0)\cdot \int_{S(u)\cap Q}[u]\otimes\nu\, d\mathcal{H}^{N-1}\right| \\
&= |L(x_0)\cdot(B-A)|\,.
\end{aligned}$$

In order to prove that $|L(x_0)\cdot(B-A)|$ is an upper bound for $H(x_0,A,B)$, we employ the formula for optimal flux densities in Theorem 2. Indeed, let $\mathcal{A}$ be the set defined in (3) and R any region in $\mathcal{A}$. For given $A, B \in \mathbb{R}^{N\times N}$ define the function, see Figure 2,

$$u_R(x) := \begin{cases} Ax & \text{if } x \in Q\setminus R \\ |R|^{-1}(B - A(1-|R|))x & \text{if } x \in R. \end{cases}$$

It is easily checked that $u_R \in SBV(Q,\mathbb{R}^N)$, $u_R|_{\partial Q} = Ax$ and $\int_Q \nabla u_R\, d\mathcal{L}^N = B$. After noticing that $S(u_R) \subset \partial R$ and that $[u_R](x) = |R|^{-1}(A-B)x$, and by taking

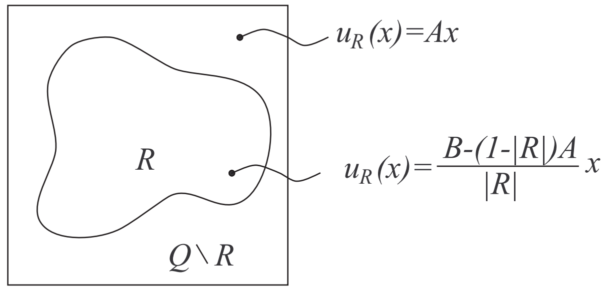

Figure 2: The function u_R.

u_R as a test function, we conclude that

$$H(x_0, A, B) \leq \frac{1}{|R|} \int_{\partial R} |(A-B)x \cdot L(x_0)\nu| \, d\mathcal{H}^{N-1}.$$

Since the above equation holds for any $R \in \mathcal{A}$ we have

$$H(x_0, A, B) \leq \inf_{R \in \mathcal{A}} \frac{1}{|R|} \int_{\partial R} |L^T(x_0)(A-B)x \cdot \nu| \, d\mathcal{H}^{N-1}. \tag{38}$$

for every $A, B \in \mathbb{R}^{N \times N}$.

For given $A, B \in \mathbb{R}^{N \times N}$, let $M_0 := L^T(x_0)(A-B)$. For every $\varepsilon > 0$, let $M_\varepsilon \in \mathcal{S}$, with $\mathcal{S}$ as in Theorem 1, be such that $M_\varepsilon \to M_0$ as ε approaches zero. Let $C_\varepsilon := L^{-T}(x_0)M_\varepsilon$ and $A_\varepsilon := C_\varepsilon + B$. Then $C_\varepsilon \to A - B$, and $A_\varepsilon \to A$ as ε approaches zero. Since $M_\epsilon = L^T(x_0)(A_\epsilon - B)$, from (38) and Theorem 2 we deduce that

$$H(x_0, A_\varepsilon, B) \leq \inf_{R \in \mathcal{A}} \frac{1}{|R|} \int_{\partial R} |L^T(x_0)(A_\varepsilon - B)x \cdot \nu| \, d\mathcal{H}^{N-1} = |\mathrm{tr}\ M_\varepsilon|,$$

and by the continuity of H with respect to the two last variables, see Prop. 5.2 of [10], we find that

$$H(x_0, A, B) \leq |\mathrm{tr}\ M_0| = |\mathrm{tr}\ (L^T(x_0)(A-B))|.$$

We now prove the representation formula for h. Let $(x_0, \xi, \eta) \in Q \times \mathbb{R}^N \times S^{N-1}$ be given. Let also $\{\eta^1, \eta^2, \ldots, \eta^{N-1}, \eta\}$ be an orthonormal basis of $\mathbb{R}^N$. For $\alpha \in (-1/2, 1/2)$ we define

$$S_\alpha := \{x \in \bar{Q}_\eta : x \cdot \eta = \alpha\},$$

and, for every $i = 1, \ldots, N-1$,

$$S^i_+ := \{x \in \bar{Q}_\eta : x \cdot \eta > 0, x \cdot \eta^i = 1/2\}, \quad S^i_- := \{x \in \bar{Q}_\eta : x \cdot \eta > 0, x \cdot \eta^i = -1/2\}.$$

Then for $u \in SBV(Q_\eta, \mathbb{R}^N)$ such that $u|_{\partial Q_\eta} = u_{\xi,\eta}$, recall definition (35) of $u_{\xi,\eta}$, and $\int_{Q_\eta} \nabla u \, d\mathcal{L}^N = 0$ we obtain, using equation (36) and the divergence theorem

$$\begin{aligned}
\int_{S(u)\cap Q_\eta} |[u]\cdot L(x_0)\nu| \, d\mathcal{H}^{N-1} &\geq \left| L(x_0) \cdot \int_{S(u)\cap Q_\eta} [u]\otimes \nu \, d\mathcal{H}^{N-1} \right| \\
&\geq \left| L(x_0) \cdot \left(\int_{Q_\eta} \nabla u \, d\mathcal{L}^N - \int_{\partial Q_\eta} u \otimes \nu \, d\mathcal{H}^{N-1} \right) \right| \\
&= \left| L(x_0) \cdot \left(\sum_{i=1}^{N-1} \left(\int_{S_+^i} \xi \otimes \eta^i \, d\mathcal{H}^{N-1} - \int_{S_-^i} \xi \otimes \eta^i \, d\mathcal{H}^{N-1} \right) \right.\right. \\
&\qquad \left.\left. + \int_{S_{1/2}} \xi \otimes \eta \, d\mathcal{H}^{N-1} \right) \right| \\
&= \left| L(x_0) \cdot \left(\sum_{i=1}^{N-1} \frac{1}{2} \left(\xi \otimes \eta^i - \xi \otimes \eta^i \right) + \xi \otimes \eta \right) \right| \\
&= |\xi \cdot L(x_0)\eta| \, .
\end{aligned}$$

The upper bound simply follows by taking $u_{\xi,\eta}$ as test function. In fact

$$\begin{aligned}
h(x_0, \xi, \eta) &\leq \int_{S(u_{\xi,\eta})\cap Q_\eta} |[u_{\xi,\eta}] \cdot L(x_0)\eta| \, d\mathcal{H}^{N-1} \\
&= \int_{S_0} |\xi \cdot L(x_0)\eta| \, d\mathcal{H}^{N-1} = |\xi \cdot L(x_0)\eta|. \qquad \square
\end{aligned}$$

Proof of Theorem 4. Let $A, B \in \mathbb{R}^{N\times N}$ and $x_0 \in Q$ be fixed. For any $u \in SBV(Q, \mathbb{R}^N)$ such that $u|_{\partial Q} = Ax$ and $\int_Q \nabla u \, d\mathcal{L}^N = B$ we have, taking into account (37), that

$$\begin{aligned}
&2 \int_{S(u)\cap Q} ([u]\cdot L(x_0)\nu)^{\pm} \, d\mathcal{H}^{N-1} \\
&\qquad = \int_{S(u)\cap Q} |[u]\cdot L(x_0)\nu| \, d\mathcal{H}^{N-1} \pm \int_{S(u)\cap Q} [u]\cdot L(x_0)\nu \, d\mathcal{H}^{N-1} \\
&\qquad = \int_{S(u)\cap Q} |[u]\cdot L(x_0)\nu| \, d\mathcal{H}^{N-1} \pm L(x_0)\cdot(-B + A).
\end{aligned}$$

Hence taking the infimum of both sides of the equation above we find

$$2H(x_0, A, B) = |L(x_0)\cdot(A - B)| \pm L(x_0)\cdot(-B + A) = 2(L(x_0)\cdot(A - B))^{\pm},$$

where, to evaluate the right hand side, we have used Lemma 2.

The representations for h in Theorem 4 follow in a similar manner. $\square$

5 Multiscale geometrical interpretations of optimal flux densities

We begin by providing geometrical interpretations for the formulas $H(x, A, B) = (L(x) \cdot (A - B))^{\pm}$ for the volume density of the relaxed energy obtained in Theorem 4. These interpretations rest on the interpretations of the initial energy $\int_{S(u)\cap Q}([u] \cdot L(x_0)\nu)^{\pm}\, d\mathcal{H}^{N-1}$. We take the simple case in which L is the constant mapping with value I, the identity matrix, and in which the positive part $([u] \cdot \nu)^{+}$ of the normal component of the jump appears in the initial interfacial energy. The integral in the initial energy then represents the volume swept out by jumps in u that cause separation at the disarrangement site $S(u) \cap Q$, and the corresponding bulk relaxed density $(I \cdot (A-B))^{+} = tr(A-B)^{+}$ represents the least volume fraction in the reference region Ω that can arise from separations associated with simple deformations that converge to a structured deformation (g, G) with constant $\nabla g = A$ and with $G = B$. In particular, for a given $N \times N$ matrix M_0, we may take g to be the identity mapping $x \longmapsto x$, so that $A = I$, and we may take G to be $I + M_0$. For these choices, the structured deformation $(i, I + M_0)$ is "purely submacroscopic" in that it produces no macroscopic changes while guaranteeing that, in a precise limiting sense, small pieces of the reference region Ω are deformed by an average amount $I + M_0$ with each piece being centered in its original location. The bulk relaxed density $tr(A-B)^{+} = (tr M_0)^{+}$ then represents the least volume fraction in the reference region swept out by separations alone in producing the purely submacroscopic deformation $(i, I + M_0)$. Thus, the relaxed volume density of the particular initial energy considered in this example provides purely geometrical information about the submacroscopic separations associated with structured deformations and has relevance in describing bodies that accumulate submacroscopic separations during changes in shape.

Alternatively, when the negative part $([u] \cdot \nu)^{-}$ of the normal component of the jump is taken in the initial interfacial energy density, then the initial energy tracks the volume swept out in the reference region due to switching or, when u is not injective, interpenetration of pieces of the reference region. For the structured deformation $(i, I + M_0)$ the corresponding relaxed volume density $tr(M_0)^{-}$ provides geometrical information about the volume fraction of material in the reference region that is swept out by submacroscopic switches and interpenetrations across disarrangement sites. This bulk relaxed density has relevance in the description of bodies that experience diffused submacroscopic defects that can switch places with one another or with normal pieces of the body (as in the case of dislocation motion) or in the case of bodies in which two distinct lattices interpenetrate.

Of course, the relation

$$([u] \cdot \nu)^{+} + ([u] \cdot \nu)^{-} = |[u] \cdot \nu|$$

tells us that when $|[u] \cdot \nu|$ is taken as the initial interfacial density, then both the initial energy and the relaxed bulk energy density $|tr(A - B)|$ track separations, switches and interpenetrations. In particular, the relaxed volume density $|tr(M_0)|$ for the purely submacroscopic structured deformation $(i, I + M_0)$ provides the volume fraction of volume swept out by separations, switches, and interpenetrations.

When the matrix M_0 is in the set $\mathcal{S}$ in Theorem 1, then we have from Theorem 2:

$$\inf_{R \in \mathcal{A}} \frac{1}{|R|} \int_{\partial R} |M_0 x \cdot \nu(x)| \, d\mathcal{H}^{N-1}(x) = |tr M_0|,$$

and we conclude that the infimum on the left-hand side of this formula not only represents the optimal total density of flux arising from the linear mapping $x \longmapsto M_0 x$, but also represents the least volume fraction swept out by separations, switches, and interpenetrations arising from simple deformations u_n converging to $x \longmapsto x$ and with ∇u_n converging to $I + M_0$ in the sense described in (34). It is easy to show from Theorem 2 that for $M_0 \in \mathcal{S}$

$$\inf_{R \in \mathcal{A}} \frac{1}{|R|} \int_{\partial R} (M_0 x \cdot \nu(x))^{\pm} \, d\mathcal{H}^{N-1}(x) = (tr M_0)^{\pm},$$

from which we conclude that, for each choice of $\pm$, the optimal flux density on the left-hand side also represents the least volume fraction swept out by separations (for the case of "$+$") or by switches and interpenetrations (for the case of "$-$") arising in simple deformations u_n converging to $x \longmapsto x$ and with ∇u_n converging to $I + M_0$ in the sense described in (34). In this manner, we have established the desired connections between optimal flux densities and optimal measures of submacroscopic geometrical changes associated with structured deformations.

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