New Insights on Free Energies and Saint-Venant's Principle in Viscoelasticity

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Abstract

Explicit expressions for the minimum free energy of a linear viscoelastic material and Noll's definition of state are used here to explore spatial energy decay estimates for viscoelastic bodies, in the full dynamical case and in the quasi-static approximation.

In the inertial case, Chirita *et al.* obtained a certain spatial decay inequality for a space-time integral over a portion of the body and over a finite time interval of the total mechanical energy. This involves the work done on histories, which is not a function of state in general. Here it is shown that for free energies which are functions of state and obey a certain reasonable property, the spatial decay

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of the corresponding space-time integral is stronger than the one involving the work done on the past history. It turns out that the bound obtained is optimal for the minimal free energy.

Two cases are discussed for the quasi-static approximation. The first case deals with general states, so that general histories belonging to the equivalence class of any given state can be considered. The continuity of the stress functional with respect to the norm based on the minimal free energy is proved, and the energy measure based on the minimal free energy turns out to obey the decay inequality derived Chirita *et al.* for the quasi-static case.

The second case explores a crucial point for viscoelastic materials, namely that the response is influenced by the rate of application of loads. Quite surprisingly, the analysis of this phenomenon in the context of Saint-Venant principles has never been carried out explicitly before, even in the linear case. This effect is explored by considering states, the related histories of which are sinusoidal. The spatial decay parameter is shown to be frequency-dependent, *i.e.* it depends on the rate of load application, and it is proved that of those considered, the most conservative estimate of the frequency-dependent decay is associated with the minimal free energy. A comparison is made of the results for sinusoidal histories at low frequencies and general histories.
Keywords: Saint Venant principle, Viscoelasticity, Spatial decay, Free energy, Dissipation rate, state in viscoelasticity, residual stress decay

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Dedication

This work was conceived in 1999 and brought near completion by 2003. Giorgio Gentili was deeply involved in this research until his untimely death. He is greatly missed. Work pressures on the other authors forced a postponement of research on this topic, originally envisaged as lasting a few months but in the event it turned out to be nearly ten years. We now dedicate this work to the memory of Giorgio and to his family.

1. Introduction.

The problem of establishing Saint-Venant principles has been an important issue for bodies of different (even “arbitrary") shapes formed by a variety of materials [3], both in statics and dynamics.

Formulations of the Saint-Venant principle for linear elastic bodies in terms of stored energy go back to the pioneering work of Zanaboni [6]. Many other results have been extensively studied in subsequent research (see e.g. [7, 3]). In particular, early work of Toupin [8] yielded an exponential spatial decay estimate of the stored energy for a cylindrical semi-infinite solid, although other forms of the Saint-Venant principle have been stated [3, 4, 5]. Some results have been given also for linear viscoelastic materials ([2] and references therein) for both the inertial and the quasi-static case; for a systematic and in-depth discussion of certain aspects of this topic, see [1], chapter 20.

It is well known that in linear elasticity the state of the material is known by specifying either the strain and the tensor of elastic moduli or the stress and the compliance tensor.

As far as linear viscoelastic materials are concerned, the prevailing view was that the past strain history, the current strain and the relaxation function replace the strain and the tensor of elastic moduli to specify a viscoelastic state. However, in [9, 10, 53, 1] a different approach has been developed.

In these papers, Noll's definition of state [11] has been applied to linear viscoelasticity. This definition is in effect the statement that two histories yield the same state if the response of the material (i.e. either the stress response [10] or the work done on deformation processes [14]) is the same under any continuation of such histories. In this approach, the minimal information required to identify the state of a material is: (a) the pair formed by the current stress and strain; and (b) the future stress in any continuation obtained by holding the strain fixed at all times. This is the "minimal state" for a linear viscoelastic material. It is worth noting that knowledge of the state variables may be obtained experimentally. For example, a homogeneous sample of a material with a linear viscoelastic response under small strains may be subjected to a relaxation test: in this way the future stress under relaxation can be monitored. The strain at the beginning of the test is also easily detectable, so that the two pieces of information yield the state of the material under examination.

Dynamical processes corresponding to Noll's definition of state may be considered to be a pair formed by the prescribed state (of the material point) and the current value of the stress at that point. For our purposes, the dynamical process may be represented by a triple, in which the first two items are pairs formed by current value-past history of both the displacement field and the related strain field, in which the past strain history is any element in the equivalence class of the given state. The final item of the triple is the current value of the stress.

A further property of viscoelastic materials must be borne in mind when developing a consistent formulation of Saint-Venant principles in viscoelasticity. There is more than one definition of free energy for viscoelastic materials [45, 46, 32, 37] An extensive comparison between different available definitions has been presented in [10, 15]. Moreover, for a given definition, unlike in linear elasticity, the free energy of a viscoelastic material after any deformation process starting from a given state is not unique (see e.g. [32, 38, 42, 1]).

For the set of free energies which are functions of state in the sense of Noll, the existence of both the maximal and the minimal element is ensured; the minimum element represents the maximum recoverable work from a given state.

An explicit expression for the isothermal minimum free energy of a linear viscoelastic material has been given [12] for the case of scalar constitutive equations. A corresponding formula is given for general tensorial stresses, strains and relaxation functions in [49]. A characterization in the frequency domain for the state in the sense of Noll is also provided in [49], and the resulting expression for the minimal free energy is shown to be a quadratic form in a variable characterizing the state in the abovementioned sense. More recent work on this and related topics is presented in (see e.g. [13, 54, 55, 52, 53, 56, 57, 1, 16, 17, 18, 19, 20, 21, 22, 22, 23, 24, 25, 26, 27]).

In the light of the above discussion, two modifications will be made with respect to the case of linear elastic materials: (i) the stored energy will be replaced by a free energy, in particular, the minimal free energy, and also (ii) a definition of *linear viscoelastic state* will be chosen based on Noll's definition.

References [2, 49] form the basis of the present work, the general aims of which are:

1. to utilize the explicit expression for the minimum free energy and its properties in obtaining spatial energy decay estimates for the fully dynamical case;
2. to explore the quasi-static case for states of the material corresponding to both general and sinusoidal histories.

The case of sinusoidal histories is interesting because for rate sensitive (in particular linear viscoelastic) materials the rate of application of disturbances (displacements or tractions) on the boundary is expected to influence the spatial decay of the effects of the disturbances themselves. The one-frequency analysis does in fact yield results of this kind.

For both the inertial and quasi-static treatments, the analysis is carried out for a general body shape as in [2]. For the inertial case, it is shown that an energy measure involving the minimum free energy rather than the work done on histories obeys a spatial decay inequality that is stronger than that given in [2].

For the quasi-static case, two "energy" measures are defined, a time and space integral of a free energy, in particular the minimum free energy of the material, and the stress $\times$ strain measure used in [2]. Under a certain assumption on the relaxation properties of the material, the former is shown to be not greater than the latter. For a general history, it is shown that the above measures both obey the decay inequality derived in [2].

However, for sinusoidal histories, it is demonstrated, using arguments generalizing those in [8], that the decay parameters are frequency-dependent, *i.e.* depend on the rate of load application, and vary in magnitude in such a way that the minimum free energy measure decays more slowly than the stress $\times$ strain measure.

Various formulae are derived in Appendix 1 for the minimum free energy and related quantities, for sinusoidal histories.

2. Relaxation functions, histories and states

A linear viscoelastic material is described by the classical Boltzmann-Volterra constitutive equation relating the second order symmetric stress tensor $\mathbf{T} : \mathcal{R} \to Sym$ and the second order symmetric strain tensor $\mathbf{E} : \mathcal{R} \to Sym$:

$$\begin{aligned} \mathbf{T}(t) &= \mathbb{G}_0 \mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s) \mathbf{E}^t(s) ds \\ &= \mathbb{G}_\infty \mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s) \mathbf{E}_r^t(s) ds \end{aligned} \tag{2.1}$$

$$\mathbf{E}^t(s) := \mathbf{E}(t-s), \quad \mathbf{E}_r^t(s) := \mathbf{E}^t(s) - \mathbf{E}(t), \quad s \in \mathcal{R}^{++}.$$

The quantity $\mathbf{E}(t) \in Sym$ is the *instantaneous value* of the strain and $\mathbf{E}^t : \mathcal{R}^{++} \to Sym$ denotes the *past history*. We refer to $\mathbf{E}_r^t$ as the relative strain history. The fourth order tensor $\dot{\mathbb{G}} : \mathcal{R}^{++} \to Lin(Sym)$ is assumed to be integrable. One of its primitives, the *relaxation function* $\mathbb{G} : \mathcal{R}^{++} \to Lin(Sym)$, is a fourth order tensor defined as

$$\mathbb{G}(t) := \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)\, ds \tag{2.2}$$

where $\mathbb{G}_0 = \mathbb{G}(0)$ is the *instantaneous elastic modulus*. The material is assumed to be a solid so that there exists the limit

$$\mathbb{G}_\infty := \lim_{t\to\infty} \mathbb{G}(t) \in Lin(Sym) \tag{2.3}$$

where $\mathbb{G}_\infty$ is the *equilibrium elastic modulus*, which is assumed to be positive. It is also assumed that $\mathbb{G}$ is non-negative on $\mathcal{R}$. We require the property that [49]

$$0 < \left| \int_0^\infty s\dot{\mathbb{G}}(s)\, ds \right| < \infty. \tag{2.4}$$

The Fourier transform of $\dot{\mathbb{G}}(t)$, namely $\dot{\mathbb{G}}_F(\omega) = \dot{\mathbb{G}}_c(\omega) - i\dot{\mathbb{G}}_s(\omega)$, for real ω, belongs to $L^2(\mathcal{R})$, according to our earlier assumptions. It is clear that $\dot{\mathbb{G}}_c(\omega)$ is even as a function of ω and $\dot{\mathbb{G}}_s(\omega)$ is odd. The quantity $\dot{\mathbb{G}}_s(\omega)$ therefore vanishes at the origin. In fact, a consequence of our assumption of analyticity of Fourier transformed quantities on the real axis of Ω is that it vanishes at least linearly at the origin. The leftmost inequality in (2.4) implies that it vanishes no more strongly than linearly. The rightmost inequality follows from the assumed analyticity (and therefore differentiability) of $\dot{\mathbb{G}}_F$.

Thermodynamic properties of the linear viscoelastic materials imply that [36, 37]

$$\mathbb{G}_0 = \mathbb{G}_0^\top, \quad \mathbb{G}_\infty = \mathbb{G}_\infty^\top, \quad \dot{\mathbb{G}}_s(\omega)\mathbf{E}\cdot\mathbf{E} < 0 \quad \forall\, \mathbf{E} \in Sym \;\; \forall\, \omega \in \mathcal{R}^{++}. \tag{2.5}$$

By closing the contour on $\Omega^{(+)}$, we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\dot{\mathbb{G}}_F(\omega)}{\omega^-}d\omega = \dot{\mathbb{G}}_F(0) = \mathbb{G}_\infty - \mathbb{G}_0 \tag{2.6}$$

giving

$$\mathbb{G}_\infty - \mathbb{G}_0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\dot{\mathbb{G}}_s(\omega)}{\omega}d\omega. \tag{2.7}$$

Equations $(2.5)_3$ and (2.7) yield

$$\mathbb{G}_0\mathbf{E}\cdot\mathbf{E} > \mathbb{G}_\infty\mathbf{E}\cdot\mathbf{E} \;\; \forall\, \mathbf{E} \in Sym\setminus\{\mathbf{0}\}. \tag{2.8}$$

For simplicity, we let $\mathbb{G}(t)$ be symmetric for all values of t. An important consequence of $(2.5)_3$ is [37]

$$\dot{\mathbb{G}}(0)\mathbf{E}\cdot\mathbf{E} \le 0 \;\; \forall\, \mathbf{E} \in Sym\setminus\{\mathbf{0}\}. \tag{2.9}$$

We assume further that

$$\dot{\mathbb{G}}(t)\mathbf{E}\cdot\mathbf{E} < 0, \qquad \forall\, \mathbf{E} \in Sym\setminus\{\mathbf{0}\}, \forall\, t \in \mathcal{R}^+. \tag{2.10}$$

If the Graffi-Volterra functional, which we will use below, is required to be a free energy, it is necessary to make the further assumption:

$$\ddot{\mathbb{G}}(t)\mathbf{E}\cdot\mathbf{E} \ge 0, \;\; \forall\, \mathbf{E} \in Sym\setminus\{\mathbf{0}\}, \forall\, t \in \mathcal{R}^+. \tag{2.11}$$

This assumption is avoided in the present work, as will be noted in section 5.

We will allow the extra generality of inhomogeneity in some later sections, so that $\mathbb{G}$ may depend on $\mathbf{x}$. This dependence is omitted except where explicitly required.

Let us extend the integral in (2.1) to $\mathcal{R}$ by identifying $\dot{\mathbb{G}}$ with its odd extension while taking $\mathbf{E}^t$ to be zero on $\mathcal{R}^-$. We now apply Parseval's formula, noting that $\dot{\mathbb{G}}_F(\omega) = -2i\dot{\mathbb{G}}_s(\omega)$, to obtain[29, 1].

$$
\begin{aligned}
\mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^{\infty} \dot{\mathbb{G}}_s(\omega)\mathbf{E}^t_+(\omega)d\omega \\
&= \mathbb{G}_\infty\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^{\infty} \dot{\mathbb{G}}_s(\omega)\mathbf{E}^t_{r+}(\omega)d\omega \\
\mathbf{E}^t_{r+}(\omega) &= \mathbf{E}^t_+(\omega) - \frac{\mathbf{E}(t)}{i\omega^-}
\end{aligned}
\tag{2.12}
$$

where $\mathbf{E}^t_{r+}$ is the Fourier transform of $\mathbf{E}^t_r$, defined in (2.1), as can be seen from (A1.10). Relation $(2.12)_2$ follows from $(2.12)_1$ with the aid of (2.7). Alternatively, by choosing $\mathbf{E}^t$ on $\mathcal{R}^-$ so that $\mathbf{E}^t_F(\omega)$ is even in ω, we deduce that

$$
\begin{aligned}
\mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \frac{1}{\pi}\int_{-\infty}^{\infty} \dot{\mathbb{G}}_c(\omega)\mathbf{E}^t_+(\omega)d\omega \\
&= \mathbb{G}_\infty\mathbf{E}(t) + \frac{1}{\pi}\int_{-\infty}^{\infty} \dot{\mathbb{G}}_c(\omega)\mathbf{E}^t_{r+}(\omega)d\omega.
\end{aligned}
\tag{2.13}
$$

Further restrictions on the function $\mathbf{E}^t$ are required because we need the result

$$
\frac{d\mathbf{E}^t_+(\omega)}{dt} = -i\omega\mathbf{E}^t_+(\omega) + \mathbf{E}(t) \tag{2.14}
$$

obtained by differentiating the integral definition of $\mathbf{E}^t_+(\omega)$ and carrying out a partial integration. As well as belonging to $L^2(\mathcal{R}^+)$, we assume that $\mathbf{E}^t \in L^1(\mathcal{R}^+) \cap C^1(\mathcal{R}^+)$ and that its derivative also belongs to $L^1(\mathcal{R}^+)$ [35].

If we define the vector space

$$
\Gamma := \left\{ \mathbf{E}^t : \mathcal{R}^{++} \to Sym\ ;\ \left| \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\, ds \right| < \infty \quad \forall\, \tau \geq 0 \right\} \tag{2.15}
$$

the Boltzmann-Volterra equation (2.1) defines the linear functional $\tilde{\mathbf{T}} : Sym \times \Gamma \to Sym$ such that

$$
\tilde{\mathbf{T}}(\mathbf{E}(t), \mathbf{E}^t) = \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)ds \tag{2.16}
$$

Remark 2.1. *Given the couple* $(\mathbf{E}(t), \mathbf{E}^t)$ *and the strain continuation defined by* $\mathbf{E}(t+a) = \mathbf{E}(t)$, $\forall\, a \in \mathcal{R}^+$, *it is easy to check that the related stress is given* *by*

$$\mathbf{T}(t+a) = \mathbb{G}(a)\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbf{E}^t(s)\,ds \tag{2.17}$$

It has been shown ([10], Proposition 2.2,(ii)) that $\dot{\mathbb{G}} \in L^1$ *ensures that, for every* $\varepsilon > 0$*, there exists* $a(\varepsilon, \mathbf{E}^t)$ *sufficiently large such that*

$$\left| \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbf{E}^t(s)\,ds \right| < \varepsilon \quad , \quad \forall\, a > a(\varepsilon, \mathbf{E}^t) \tag{2.18}$$

Therefore, (2.18) *can be thought of as an expression of the fading memory property. It follows that* $\lim_{a\to\infty} \mathbf{T}(t+a) = \mathbb{G}_\infty \mathbf{E}(t)$*. The equilibrium elastic modulus is positive definite so that*

$$\mathbb{G}_\infty \mathbf{E}\cdot\mathbf{E} > 0, \qquad \forall\, \mathbf{E} \in Sym \setminus \{\mathbf{0}\}, \tag{2.19}$$

The concept of the state of a linear viscoelastic material has been discussed by various authors [38, 10, 9, 11]. We briefly recall some basic propositions.

Remark 2.2. *According to the definition in [38] and [37], a process* P *of finite duration* d*, is given by* $\dot{\mathbf{E}}_P : [0,d) \to Sym$*. Given the couple* $(\mathbf{E}(t), \mathbf{E}^t) \in Sym \times \Gamma$*, related to the strains* $\mathbf{E}(\tau)$*,* $\tau \le t$*, we associate with* P *the mapping*

$$\mathbf{E}_P : (0,d) \to Sym, \qquad \mathbf{E}_P(\tau) = \mathbf{E}(t) + \int_0^\tau \dot{\mathbf{E}}_P(s')\,ds', \quad \tau \in (0,d] \tag{2.20}$$

The strains $\mathbf{E}_f(\tau') = (\mathbf{E}_P * \mathbf{E})(\tau')$*,* $\tau' \le t+d$ *are determined by* $\mathbf{E}^t$ *and* $\dot{\mathbf{E}}_P$*, defined to be*

$$\mathbf{E}_f(t+d-s) = (\mathbf{E}_P * \mathbf{E})(t+d-s) := \begin{cases} \mathbf{E}_P(d-s) & 0 \le s < d \\ \mathbf{E}(t+d-s) & s \ge d \end{cases} \tag{2.21}$$

Thus, $\mathbf{E}_f$ *is related to the couple* $(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^{t+d})$*.*

Definition 2.1. *Two histories* $\mathbf{E}_1^t$ *and* $\mathbf{E}_2^t$ *are said to be equivalent if for every* $\mathbf{E}_P : (0,\tau] \to Sym$ *and for every* $\tau > 0$*, they satisfy [39]*

$$\tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E}_1)^{t+\tau}) = \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E}_2)^{t+\tau}). \tag{2.22}$$

As a consequence, it is easy to show that $\mathbf{E}^t$ is equivalent to the zero history

$\mathbf{0}^\dagger$ if

$$\int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\,ds = \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,ds = 0 \quad \forall\,\tau > 0 \tag{2.23}$$

Equation (2.23) defines an equivalence relation on histories. Two histories $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ are said to be equivalent if their difference $\mathbf{E}^t = \mathbf{E}_1^t - \mathbf{E}_2^t$ satisfies (2.23) [11].

Two couples $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ and $(\mathbf{E}_2(t), \mathbf{E}_2^t)$ such that $\mathbf{E}_1(t) = \mathbf{E}_2(t)$ and $\mathbf{E}_1^t - \mathbf{E}_2^t$ satisfies (2.23), are represented by the same state $\sigma(t)$ in the sense of Noll [11], and $\sigma(t)$ may be thought as the "minimum" set of variables allowing a well-defined relation between $\dot{\mathbf{E}}_P : [0,\tau) \to Sym$ and the stress $\mathbf{T}(t+\tau) = \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E})^{t+\tau})$ for every $\tau > 0$.

In other words [10, 9], denoting by Γ_0 the set of all the past histories of Γ satisfying (2.23), and by Γ/Γ_0 the usual quotient space, the state σ of a linear viscoelastic material is an element of [3]

$$\Sigma := Sym \times (\Gamma/\Gamma_0) \tag{2.24}$$

The work done on the material by the strain history $\mathbf{E}(\tau)$, $\tau \leq t$ is

$$\begin{aligned} \widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &:= \int_{-\infty}^t \mathbf{T}(\tau) \cdot \dot{\mathbf{E}}(\tau) d\tau \\ &= \frac{1}{2}\mathbb{G}_0\mathbf{E}(t) \cdot \mathbf{E}(t) + \int_{-\infty}^t \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^\tau(s) \cdot \dot{\mathbf{E}}(\tau) ds d\tau \,. \end{aligned} \tag{2.25}$$

It will be clear from the representation of $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ in the frequency domain, given below, that it is a non-negative quantity. We will restrict our considera-

[3] It is worth noting that, by virtue of (2.15) and Definition 2.1, the space of the states Σ depends on the memory kernel $\dot{\mathbb{G}}$ characterising the material by means of (2.1). This property distinguishes (2.24) from the usual fading memory spaces [40, 41]

tions to histories such that $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) < \infty$. One can show that [1]

$$\begin{aligned}
\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &= \phi(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}_r^t(s_1) \cdot \mathbb{G}_{12}(|s_1 - s_2|)\mathbf{E}_r^t(s_2) ds_1 ds_2 \\
&= S(t) + \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t(s_1) \cdot \mathbb{G}_{12}(|s_1 - s_2|)\mathbf{E}^t(s_2) ds_1 ds_2 \\
\mathbb{G}_{12}(|s_1 - s_2|) &= \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\mathbb{G}(|s_1 - s_2|) \qquad (2.26) \\
\phi(t) &:= \frac{1}{2}\mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t) \\
S(t) &:= \mathbf{T}(t) \cdot \mathbf{E}(t) - \frac{1}{2}\mathbb{G}_0 \mathbf{E}(t) \cdot \mathbf{E}(t).
\end{aligned}$$

A frequency domain representation of (2.26) is given by [29, 43, 12, 1]

$$\begin{aligned}
\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^\infty \mathbb{H}(\omega)\mathbf{E}_{r+}^t(\omega) \cdot \overline{\mathbf{E}}_{r+}^t(\omega) d\omega \\
&= S(t) + \frac{1}{2\pi}\int_{-\infty}^\infty \mathbb{H}(\omega)\mathbf{E}_{+}^t(\omega) \cdot \overline{\mathbf{E}}_{+}^t(\omega) d\omega
\end{aligned} \qquad (2.27)$$

where, for each given $\omega \in \mathcal{R}$, the fourth order tensor $\mathbb{H}(\omega) \in Lin(Sym)$ is defined as

$$\mathbb{H}(\omega) := -\omega \dot{\mathbb{G}}_s(\omega); \quad \mathbb{H}(\infty) = -\dot{\mathbb{G}}(0), \qquad (2.28)$$

The equivalence of the two forms of (2.27) follows from $(2.7)_2$ and (2.12).

The properties of the work have been extensively studied in [10]. It is shown in [49] that two couples $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ and $(\mathbf{E}_2(t), \mathbf{E}_2^t)$ are equivalent, in the sense of Definition 2.1, if and only $\mathbf{E}_1(t) = \mathbf{E}_2(t) \equiv \mathbf{E}_P(0)$ and if

$$\begin{aligned}
&\int_t^{t+d} \tilde{\mathbf{T}}(\mathbf{E}_P(\tau - t), (\mathbf{E}_P * \mathbf{E}_1)^\tau) \cdot \dot{\mathbf{E}}_P(\tau - t) d\tau \\
&= \int_t^{t+d} \tilde{\mathbf{T}}(\mathbf{E}_P(\tau - t), (\mathbf{E}_P * \mathbf{E}_2)^\tau) \cdot \dot{\mathbf{E}}_P(\tau - t) d\tau
\end{aligned} \qquad (2.29)$$

holds for every $\mathbf{E}_P : (0, d] \to Sym$ and for every $d > 0$.

3. Explicit expression for the minimum free energy

From a result in [49], based on a theorem of Gohberg and Kreĭn [28], we have that $\mathbb{H}(\omega)$ can be factorized as follows:

$$\mathbb{H}(\omega) = \mathbb{H}_+(\omega)\, \mathbb{H}_-(\omega) \tag{3.1}$$

with

$$\mathbb{H}_+(\omega) = \mathbb{H}^*_-(\omega) \tag{3.2}$$

where the matrix functions $\mathbb{H}_{(\pm)}$ admit analytic continuations which are analytic in the interior and continuous up to the boundary of the complex half planes $\Omega^{\mp}$, and are such that

$$\det\, \mathbb{H}_\pm(\zeta) \neq 0 \quad , \quad \zeta \in \Omega^{\mp} \tag{3.3}$$

Similarly $\mathbb{H}$ has a right factorization with corresponding properties [49]. The factorization is unique up to a multiplication on the left of $\mathbb{H}_-$ by a constant, unitary matrix $\in$ *Lin*(*Sym*), and multiplication of $\mathbb{H}_+$ on the right by the inverse of this matrix. Properties of the factors are discussed further in the context of (5.8) below. From $(2.28)_2$, we have that $\mathbb{H}_\pm(\infty)$ are non-zero and

$$\mathbb{H}_+(\infty)\, \mathbb{H}_-(\infty) = -\dot{\mathbb{G}}(0) \tag{3.4}$$

The notation for $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ follow the convention used in [12], *i.e.* the sign indicates the half plane where any singularities of the tensor and any zeros in the determinant of the corresponding matrix occur.

Consider now the second order symmetric tensor $\mathbf{P}^t(\omega) = \mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega)$, whose components are continuous by virtue of the properties of $\mathbb{H}_-(\omega)$ and

$\mathbf{E}^t_{r+}(\omega)$. The Plemelj formulae [44, 1] give that

$$
\begin{aligned}
\mathbf{P}^t(\omega) &:= \mathbb{H}_-(\omega)\mathbf{E}^t_{r+}(\omega) = \mathbf{p}^t_-(\omega) - \mathbf{p}^t_+(\omega) \\
\mathbf{Q}^t(\omega) &:= \mathbb{H}_-(\omega)\mathbf{E}^t_{+}(\omega) = \mathbf{q}^t_-(\omega) - \mathbf{q}^t_+(\omega)
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
\mathbf{p}^t(z) &:= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbf{P}^t(\omega)}{\omega - z} d\omega\ , \quad \mathbf{p}^t_\pm(\omega) := \lim_{\alpha\to 0^\mp} \mathbf{p}^t(\omega + i\alpha) \\
\mathbf{q}^t(z) &:= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbf{Q}^t(\omega)}{\omega - z} d\omega\ , \quad \mathbf{q}^t_\pm(\omega) := \lim_{\alpha\to 0^\mp} \mathbf{q}^t(\omega + i\alpha)
\end{aligned}
\tag{3.6}
$$

Moreover, $\mathbf{p}^t(z) = \mathbf{p}^t_+(z)$ is analytic in $z \in \Omega^{(-)}$ and $\mathbf{p}^t(z) = \mathbf{p}^t_-(z)$ is analytic in $z \in \Omega^{(+)}$. Both are analytic on the real axis (as indeed is $\mathbf{P}^t$) by virtue of the assumption at the end of section 7 on the analyticity of Fourier-transformed quantities on the real axis and an argument given in [1]. Similar statements apply to $\mathbf{q}^t$ and $\mathbf{Q}^t$. It can be shown that

$$
\mathbf{q}^t_+(\omega) = \mathbf{p}^t_+(\omega) \tag{3.7}
$$

The maximum free energy has the form

$$
\begin{aligned}
\psi_m(t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}^t_-(\omega)|^2 d\omega \\
&= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{q}^t_-(\omega)|^2 d\omega
\end{aligned}
\tag{3.8}
$$

Using an argument given in [49], section 7 (also [1], page 249), we can write $(2.12)_2$ in the form

$$
\mathbf{T}(t) = \mathbb{G}_\infty \mathbf{E}(t) - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega} \mathbf{p}^t_-(\omega) d\omega. \tag{3.9}
$$

It follows from a result in [49] (and [1], page 253) that for every viscoelastic material with a symmetric relaxation function, a given couple $(\mathbf{E}, \mathbf{E}^t)$ is equivalent to the zero couple $(\mathbf{0}, \mathbf{0}^\dagger)$ if and only if $\mathbf{p}^t_-$ related to $\mathbf{E}^t_r$ by (3.5)-(3.6), is such that

$$\mathbf{p}_-^t(\omega) = 0 \quad , \quad \forall\, \omega \in \mathcal{R} \tag{3.10}$$

and $\mathbf{E}(t) = 0$. A functional of $(\mathbf{E}, \mathbf{E}^t)$ which has the same value for all equivalent couples will be referred to as a function of state. In particular, if the dependence is only through $(\mathbf{E}, \mathbf{p}^t)$, then it follows from (3.10) that the quantity in question is a function of state. This is true in particular for ψ_m.

The main developments in [49] are expressed in terms of the history $\mathbf{E}^t$ though the result $(3.8)_2$ in terms of the relative history $\mathbf{E}_r^t$ is presented also. The representation $(3.8)_1$ has the advantage that it is explicitly positive. For fluids, $\mathbf{E}_r^t$ is in any case the natural variable [54]; the quantity $\mathbb{G}_\infty = 0$ and $(2.1)_2$ retains only the integral term.

From (2.27) and (3.8) we find that [49]

$$\begin{aligned} W(\mathbf{E}(t), \mathbf{E}^t) &= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} [|\mathbf{p}_+^t(\omega)|^2 + |\mathbf{p}_-^t \omega)|^2]\, d\omega \\ W(\mathbf{E}(t), \mathbf{E}^t) - \psi_m(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_+^t(\omega)|^2\, d\omega \geq 0. \end{aligned} \tag{3.11}$$

Also

$$\begin{aligned} W(\mathbf{E}(t), \mathbf{E}^t) &= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} [|\mathbf{q}_+^t(\omega)|^2 + |\mathbf{q}_-^t \omega)|^2]\, d\omega \\ W(\mathbf{E}(t), \mathbf{E}^t) - \psi_m(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{q}_+^t(\omega)|^2\, d\omega \geq 0. \end{aligned} \tag{3.12}$$

A free energy is a functional of the history and present value of the deformation, obeying certain properties that have been proved to hold by Coleman [48] for materials with fading memory, as a consequence of the second law of thermodynamics. Recalling Remarks 2.1 on fading memory and 2.2 on the definition of processes, a functional $\tilde{\psi} : \Gamma \times Sym \to \mathcal{R}$ is said to be a free energy in the sense of Graffi if it satisfies the following properties:

P1 *(integrated dissipation inequality)*

$$\tilde{\psi}(\mathbf{E}_P(d),\mathbf{E}_P{*}\mathbf{E}^t)-\tilde{\psi}(\mathbf{E}(t),\mathbf{E}^t)\leq\int_0^d\tilde{T}(\mathbf{E}_P(\tau),\mathbf{E}^\tau{*}\mathbf{E}^t)\cdot\dot{\mathbf{E}}_P(\tau)d\tau,\ (3.13)$$

for every pair of deformations $\mathbf{E}(t),\mathbf{E}_P(d)$, for every history $\mathbf{E}^t$, and for every segment $\mathbf{E}_P(\cdot)-\mathbf{E}(t)$ of duration d with $\mathbf{E}_P(0)=\mathbf{E}(t)$;

P2 for every deformation $\mathbf{E}(t)$ and for every history $\mathbf{E}^t$, the gradientof $\tilde{\psi}(\cdot,\mathbf{E}^t)$ (e.g. with respect to the current value of the strain $\mathbf{E}(t)$) at $\mathbf{E}(t)$ is equal to the stress $\tilde{T}(\mathbf{E}(t),\mathbf{E}^t)$;

P3 for every deformation $\mathbf{E}(t)$ and for every history $\mathbf{E}^t$,

$$\tilde{\psi}(\mathbf{E}(t),\mathbf{E}(t)^\dagger)\ \leq\ \tilde{\psi}(\mathbf{E}(t),\mathbf{E}^t) \qquad (3.14)$$

where $\mathbf{E}(t)^\dagger$ is the static history with value $\mathbf{E}(t)$;

P4 for every deformation $\mathbf{E}(t)$,

$$\tilde{\psi}(\mathbf{E}(t),\mathbf{E}^\dagger)=\phi(t) \qquad (3.15)$$

The form of ϕ is given by (2.26). If $t\mapsto\psi(t)$ is differentiable, property (P1) can be expressed in local form:

$$\mathbf{T}(t)\cdot\dot{\mathbf{E}}(t)\ \geq\ \dot{\psi}(t) \qquad (3.16)$$

which is essentially a statement that the rate of dissipation $\mathbf{T}(t)\cdot\dot{\mathbf{E}}(t)-\dot{\psi}(t)$ corresponding to $\psi(t)$ is non-negative. The quantity $\widetilde{W}(\mathbf{E}(t),\mathbf{E}^t)$ is, in some circumstances, the maximum free energy,[15, 10]. It will be denoted by ψ_M.

In [15] it has been pointed out that there are two available definitions of free energy in viscoelasticity. One is due to Coleman and Owen [32], and it has been specialized to linear viscoelasticity in [15, 10], and the other one, structured in $P1\div P4$ is due to Graffi (see e.g. [45, 46], [10, 15]). It is shown in [49] that the minimum free energy ψ_m, given by (3.8), is a free energy according to both of the definition above.

The rate of dissipation corresponding to the minimum free energy is given by (*e.g.* [49])

$$\begin{aligned} D_m(t) &= \mathbf{T}(t)\cdot\dot{\mathbf{E}}(t) - \dot{\psi}_m(t) \\ &= \frac{1}{2\pi}\frac{d}{dt}\int_{-\infty}^{\infty}|\mathbf{p}_+^t(\omega)|d\omega = |\mathbf{K}(t)|^2 \end{aligned} \tag{3.17}$$

where

$$\mathbf{K}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{H}_-(\omega)\mathbf{E}_{r+}^t(\omega)d\omega. \tag{3.18}$$

This can be shown with the aid of the relationships

$$\begin{aligned} \frac{d}{dt}\mathbf{p}_+^t(\omega) &= -i\omega\mathbf{p}_+^t(\omega) - \mathbf{K}(t) \\ \frac{d}{dt}\mathbf{p}_-^t(\omega) &= -i\omega\mathbf{p}_-^t(\omega) - \mathbf{K}(t) - \frac{\mathbb{H}_-(\omega)\dot{\mathbf{E}}(t)}{i\omega} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &\lim_{|\omega|\to\infty}\omega\mathbf{p}_\pm^t(\omega) = i\mathbf{K}(t) \\ \frac{1}{2\pi}\int_{-\infty}^{\infty}&\mathbf{p}_\pm^t(-\omega)d\omega = \mp\frac{1}{2}\mathbf{K}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{p}_\pm^t(\omega)d\omega. \end{aligned} \tag{3.20}$$

Certain relations which will be relevant in later sections are now derived. If the explicit form of $\mathbf{q}_-^t$ is substituted into $(3.8)_2$, the integration over ω can be carried out and we obtain (see also [1], page 250 for analogous results in relation to relative histories)

$$\begin{aligned} Q_-(t) &:= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{q}_-^t(\omega)|^2 d\omega = \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{A^t(\omega_1,\omega_2)}{\omega_1^+ - \omega_2^-}d\omega_1 d\omega_2 \\ A^t(\omega_1,\omega_2) &:= \bar{\mathbf{E}}_+^t(\omega_1)\cdot\ \mathbb{H}_+(\omega_1)\ \mathbb{H}_-(\omega_2)\mathbf{E}_+^t(\omega_2). \end{aligned} \tag{3.21}$$

The notation in the denominator of the right-most integrand is discussed in [1].

Also, in the same way, we obtain

$$Q_+(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{q}_+^t(\omega)|^2 d\omega = -\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{A^t(\omega_1,\omega_2)}{\omega_1^- - \omega_2^+} d\omega_1 d\omega_2 \tag{3.22}$$

Relation (3.12) follows from $(2.27)_2$, (3.21), (3.22) and the Plemelj formulae. One can furthermore show that

$$\begin{aligned} R_-(t) &:= \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{B^t(\omega_1,\omega_2)}{\omega_1^+ - \omega_2^-} d\omega_1 d\omega_2 = 0 \\ B^t(\omega_1,\omega_2) &:= \bar{\mathbf{E}}_+^t(\omega_1)\cdot\, \mathbb{H}_+(\omega_2)\, \mathbb{H}_-(\omega_1)\mathbf{E}_+^t(\omega_2) \end{aligned} \tag{3.23}$$

by integrating over ω_2 for example and closing the contour on $\Omega^{(-)}$. Also,

$$\begin{aligned} R_+(t) &:= -\frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{B^t(\omega_1,\omega_2)}{\omega_1^- - \omega_2^+} d\omega_1 d\omega_2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{E}}_+^t(\omega)\cdot\, \mathbb{H}(\omega)\mathbf{E}_+^t(\omega) d\omega \\ &= Q_-(t) + Q_+(t) \end{aligned} \tag{3.24}$$

by virtue of $(2.27)_2$ and (3.12). Relation (3.23) allows us to write (3.21) in the explicitly convergent form

$$Q_-(t) = \frac{i}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{A^t(\omega_1,\omega_2) - B^t(\omega_1,\omega_2)}{\omega_1 - \omega_2} d\omega_1 d\omega_2 \tag{3.25}$$

which is convenient for numerical evaluation. We can replace the $(\omega_1 - \omega_2)$ in the denominator by $(\omega_1^+ - \omega_2^-)$ which gives (3.21), or by $(\omega_1^- - \omega_2^+)$ which gives the same result by way of (3.24) and (3.22). Relation (3.25) implies that the quantity

$$\mathbb{D}(\omega_1,\omega_2) := i\frac{(\,\mathbb{H}_+'(\omega_1)\,\mathbb{H}_-(\omega_2) - \,\mathbb{H}_+(\omega_2)\,\mathbb{H}_-'(\omega_1))}{\omega_1 - \omega_2} \tag{3.26}$$

is a non-negative kernel (in the sense that the integral, as given by (3.25), is non-negative) . By using very localized choices of $\mathbf{E}_+^t(\omega)$, we deduce that the "diagonal elements" of $\mathbb{D}(\omega_1,\omega_2)$ are non-negative. This is a statement about $\mathbb{D}(\omega_1,\omega_2)$ as a function on $\mathcal{R}\times\mathcal{R}$. Using a prime to denote differentiation, we can write these diagonal elements as

$$\mathbb{D}(\omega) := i(\, \mathbb{H}'_+(\omega)\, \mathbb{H}_-(\omega) - \,\mathbb{H}_+(\omega)\, \mathbb{H}'_-(\omega)) \geq 0 \quad \omega \in \mathcal{R}. \tag{3.27}$$

Proposition 3.1. *Let $Q_\pm, R_+ \in L^1((-\infty, t))$ for all finite times. Then*

$$\begin{aligned}\int_{-\infty}^{t} Q_+(u)du &:= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{A^t(\omega_1,\omega_2)}{(\omega_1^- - \omega_2^+)^2} d\omega_1 d\omega_2 \\ &= -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \frac{A^t(\omega_1,\omega_2)}{(\omega_1^+ - \omega_2^-)^2} + \frac{B^t(\omega_1,\omega_2)}{(\omega_1^- - \omega_2^+)^2} \right\} d\omega_1 d\omega_2 \\ &\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{E}}_+^t(\omega) \cdot\, \mathbb{D}(\omega)\mathbf{E}_+^t(\omega) d\omega \end{aligned} \tag{3.28}$$

where $\mathbb{D}$ *is defined by* (3.27).

Proof. Relation $(3.28)_1$ follows immediately, by time differentiation, using (2.14) and Cauchy's theorem. Equation $(3.28)_2$ can be verified similarly, on noting a cancellation between the derivatives of the first and second terms. Relations such as

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{H}_-(\omega_2)}{(\omega_1^+ - \omega_2^-)^2} d\omega_2 = \,\mathbb{H}'_-(\omega_1) \tag{3.29}$$

are required. □

Remark 3.1. *The assumption that $Q_\pm, R_+ \in L^1((-\infty, t))$ implies of course that the strain history vanishes in the distant past.*

Remark 3.2. *In consequence of* (3.7) *and* (3.17)*, the quantity Q_+ is the integral of D_m over past history, or the total dissipation up to the present time, associated with the minimum free energy. It is not less than the total dissipation corresponding to any other free energy.*

Let us define

$$\mathbb{M}(\omega) := \mathbb{G}_0 + \dot{\mathbb{G}}_F(\omega) = \,\mathbb{R}(\omega) + i\frac{\mathbb{H}(\omega)}{\omega} \tag{3.30}$$

and refer to it as the complex modulus tensor. Note that

$$\mathbb{R}(\omega) = \mathbb{G}_0 + \dot{\mathbb{G}}_c(\omega) = \,\mathbb{R}(-\omega). \tag{3.31}$$

This quantity is not required to be positive by thermodynamics. However, in many situations, and in particular for relaxation functions given by sums or integrals of decaying exponentials with positive coefficients/density functions, it is a positive definite tensor [49, 59].

Proposition 3.2. *Let us assume that*

$$F(t) := \int_{-\infty}^{t} \mathbf{T}(u) \cdot \mathbf{E}(u) du \tag{3.32}$$

exists for all finite values of t*. Then*

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\mathbf{E}}_+^t(\omega) \cdot \mathbb{R}(\omega) \mathbf{E}_+^t(\omega) d\omega \tag{3.33}$$

and is non-negative if $\mathbb{R} \geq 0$ *for all* $\omega \in \mathcal{R}$*.*

Proof. Let $\mathbf{E}(u) = 0, \quad u > t$ and we write

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{T}_F(\omega) \cdot \bar{\mathbf{E}}_F(\omega) d\omega \tag{3.34}$$

by virtue of Parseval's formula. Now

$$\mathbf{E}_F(\omega) = \bar{\mathbf{E}}_+^t(\omega) e^{-i\omega t}. \tag{3.35}$$

Writing $(2.1)_1$ in the form

$$\mathbf{T}(u) = \mathbb{G}_0 \mathbf{E}(u) + \int_{-\infty}^{u} \dot{\mathbb{G}}(u-s) \mathbf{E}(s) ds \tag{3.36}$$

we see that the Faltung theorem gives, remembering that $\dot{\mathbb{G}}$ is a causal function [58],

$$\mathbf{T}_F(\omega) = \mathbb{M}(\omega) \mathbf{E}_F(\omega) \tag{3.37}$$

so that

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\mathbf{E}}_+^t(\omega) \cdot \mathbb{M}(\omega) \mathbf{E}_+^t(\omega) d\omega. \tag{3.38}$$

The result (3.33) follows from the requirement that F be real, or alternatively from the oddness of $\mathbb{H}(\omega)/\omega$. The non-negativity of F follows immediately.

□

Differentiation of (3.33) with respect to t gives $\mathbf{T}(t) \cdot \mathbf{E}(t)$, with the aid of (2.13), (2.14) and the relationship

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{E}_+^t(\omega) d\omega = \frac{1}{2} \mathbf{E}(t), \tag{3.39}$$

which follows from the fact that $\mathbf{E}^t$, defined on $\mathcal{R}$, belongs to $L^1(\mathcal{R})$ and there is a discontinuity at the origin [34]. The existence assumption on F implies in particular that the strain history tends to zero in the distant past.

4. Dynamical viscoelasticity

In this section, we derive certain spatial decay results for dynamical linear viscoelasticity. Consider a regular open bounded region $\mathcal{B}$ which is occupied by an anisotropic and inhomogeneous medium with relaxation tensor $\mathbb{G}(\mathbf{x}, t)$. It is assumed that $\mathbb{G}$ satisfies the thermodynamic restrictions outlined in section 2; and also that $\mathbb{G}_0(\mathbf{x})$ and $\mathbb{G}_\infty(\mathbf{x})$ are continuous on $\bar{\mathcal{B}}$, the closure of $\mathcal{B}$. The boundary of $\mathcal{B}$ is denoted by $\partial\mathcal{B}$. We further assume that the mass density ρ is strictly positive, continuous and bounded on $\bar{\mathcal{B}}$. Let us set

$$\rho_0 = \text{ess} \inf_{\bar{\mathcal{B}}} \rho(\mathbf{x}) \tag{4.1}$$

It is proved in [2] that

$$\begin{aligned} |\mathbf{T}(t)|^2 &\leq 2c_0\psi(t), \qquad \psi = \psi_M \\ c_0(\mathbf{x}) &= 2k \max\left(|\mathbb{G}_\infty(\mathbf{x})|, |\mathbb{G}_\infty(\mathbf{x}) - \mathbb{G}_0(\mathbf{x})|\right); \\ c_0 &= \text{ess} \sup_{\mathbf{x}\in\mathcal{B}} c_0(\mathbf{x}) \end{aligned} \tag{4.2}$$

where k is introduced in (A1.6). The following result is now proved.

Proposition 4.1. *The bound* (4.2) *holds for* $\psi = \psi_m$ *and indeed for all free energies because of the minimal property of* ψ_m.

Proof. Relation (3.9) yields

$$|\mathbf{T}(t)|^2 \leq \left(|\mathbb{G}_\infty \mathbf{E}(t)| + \left|\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)\right|\right)^2. \tag{4.3}$$

Using $|a+b|^2 \leq (|a|+|b|)^2 \leq 2(|a|^2+|b|^2)$, for any $a, b \in \mathcal{V}$, we obtain

$$|\mathbf{T}(t)|^2 \leq 2|\mathbb{G}_\infty \mathbf{E}(t)|^2 + 2\left|\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)d\omega\right|^2. \tag{4.4}$$

From (A1.4) and $(2.5)_2$

$$|\mathbb{G}_\infty \mathbf{E}(t)|^2 = \mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbb{G}_\infty \mathbf{E}(t) \leq |\mathbb{G}_\infty|(\mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t)). \tag{4.5}$$

Also

$$\begin{aligned} \left|\frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)d\omega\right|^2 &= \overline{\frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)d\omega} \cdot \int_{-\infty}^{\infty} \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)d\omega \\ &\leq \frac{1}{\pi^2}\int_{-\infty}^{\infty} \left|\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)\right|^2 d\omega. \end{aligned} \tag{4.6}$$

From the identity

$$|\,\mathbb{H}_+(\omega)\mathbf{p}^t_-(\omega)|^2 = \,\mathbb{H}(\omega)\mathbf{p}^t_-(\omega)\cdot\bar{\mathbf{p}}^t_-(\omega) \tag{4.7}$$

we have

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty}\left|\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{p}^t_-(\omega)\right|^2 d\omega = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\mathbb{H}(\omega)}{\omega^2}\mathbf{p}^t_-(\omega)\cdot\bar{\mathbf{p}}^t_-(\omega)d\omega. \tag{4.8}$$

Recalling that $\mathbb{H}(\omega)$ is a real symmetric fourth order tensor, the function inside the integral in (4.8) is real valued. Because of the positive definiteness of $\mathbb{H}(\omega)$ we can use (A1.5). Thus (4.4), (4.5), (4.6) and (4.8) yield the following inequality for the square of the magnitude of the stress $\mathbf{T}(t)$:

$$|\mathbf{T}(t)|^2 \leq 2|\mathbb{G}_\infty|(\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)) + \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{tr(\,\mathbb{H}(\omega))}{\omega^2}d\omega\;\frac{1}{\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_-(\omega)|^2 d\omega. \tag{4.9}$$

Using $(2.7)_2$ and $(2.28)_1$ we deduce that

$$\begin{aligned}|\mathbf{T}(t)|^2 &\leq 2|\mathbb{G}_\infty|(\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t)) + 2\;\; tr(\mathbb{G}_0 - \mathbb{G}_\infty)\,\frac{1}{\pi}\int_{-\infty}^{\infty}|\mathbf{p}^t_-(\omega)|^2 d\omega \\ &\leq 2c_0\psi_m(\sigma(t))\end{aligned} \tag{4.10}$$

where c_0 is given by (4.2) and ψ_m by $(3.8)_1$.

□

In what follows, for a given material point $\mathbf{x}$ and a time t, we consider a state $\sigma(t)$ (the dependence upon $\mathbf{x}$ is omitted for the sake of brevity). We shall consider a dynamical (linear viscoelastic) process formed by the triple $\{(\mathbf{u}(t),\mathbf{u}^t),(\mathbf{E}(t),\mathbf{E}^t),\mathbf{T}(t)\}$, in which $(\mathbf{E}(t),\mathbf{E}^t)\sim\sigma(t)$, $\mathbf{E} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^\top]$ and the stress $\mathbf{T}(t)$ is assumed to satisfy the constitutive equation (2.1) and the balance of linear momentum

$$\nabla\cdot\mathbf{T}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) = \rho\ddot{\mathbf{u}}(\mathbf{x},t), \quad (\mathbf{x},t)\in\mathcal{B}\times\mathcal{R}^+, \tag{4.11}$$

where $\mathbf{b}(\mathbf{x},t)$ is the body force. We shall refer to the *dynamical process* just introduced as being *relative to the given state* $\sigma(t)$. There may be more than one dynamical process relative to a given state, depending on whether or not (2.23) has more than one solution.

It is assumed that the material is undisturbed for $t\in\mathcal{R}^{--}$. Following [2], with minor simplifications, we now define certain subsets of $\bar{\mathcal{B}}$. Let T be a given positive time and let $\mathcal{D}_T$ denote a subset of $\bar{\mathcal{B}}$ such that:

1. if $\mathbf{x}\in\mathcal{B}$ then

$$\mathbf{u}(\mathbf{x},0)\neq\mathbf{0} \text{ or } \dot{\mathbf{u}}(\mathbf{x},0)\neq\mathbf{0}; \tag{4.12}$$

or

$$\mathbf{b}(\mathbf{x},\tau)\neq\mathbf{0} \text{ for some } \tau\in[0,T]; \tag{4.13}$$

2. if $\mathbf{x} \in \partial\mathcal{B}$ then

$$\mathbf{s}(\mathbf{x},\tau)\cdot\dot{\mathbf{u}}(\mathbf{x},\tau) \neq \mathbf{0} \text{ for some } \tau \in [0,T] \tag{4.14}$$

where

$$\mathbf{s}(\mathbf{x},\tau) := \mathbf{T}(\mathbf{x},\tau)\mathbf{n}(\mathbf{x}) \tag{4.15}$$

the vector $\mathbf{n}$ being the unit outward normal on $\partial\mathcal{B}$. Thus, $\mathcal{D}_T$ represents the support of the initial and boundary data and the body force. If the region $\mathcal{B}$ is unbounded, then we assume that $\mathcal{D}_T$ is a bounded region. Furthermore, let $\mathcal{D}_T^*$ be a bounded, regular region such that $\mathcal{D}_T \subseteq \mathcal{D}_T^* \subseteq \bar{\mathcal{B}}$.

Let the set $\mathcal{D}_r$ consist of all points of $\bar{\mathcal{B}}$ that can be reached by signals propagating from $\mathcal{D}_T^*$ with speeds less than or equal to the speed of propagation $r/T, r > 0$, namely

$$\mathcal{D}_r := \left\{\mathbf{x} \in \bar{\mathcal{B}} : \mathcal{D}_T^* \cap \overline{\mathbf{O}(\mathbf{x},r)} \neq \Phi\right\} \tag{4.16}$$

where $\mathbf{O}(\mathbf{x},r)$ is the open ball with radius r and centre at $\mathbf{x}$ and Φ is the empty set. We put

$$\mathcal{B}_r = \mathcal{B}\backslash\mathcal{D}_r \tag{4.17}$$

and denote by $\mathcal{S}_r$ the surface separating $\mathcal{D}_r$ and $\mathcal{B}$. This surface is inside $\bar{\mathcal{B}}$, with its boundary in $\partial\mathcal{B}$.

The $\mathbf{x}$ dependence of various quantities will be understood rather than explicitly indicated in many formulae below.

We set the stage here for a dynamical Saint-Venant principle by introducing the total mechanical energy contained in $\mathcal{B}_r$ at time t; this is given by

$$I(r,t) := \int_{\mathcal{B}_r} \left\{\frac{1}{2}\rho|\dot{\mathbf{u}}(t)|^2 + \psi_M(t)\right\} dV \tag{4.18}$$

where $\psi_M(t)$ is defined in (2.25) and (2.26). The total mechanical energy is then the sum of the kinetic energy in the dynamical (linear viscoelastic) process under examination and of work done on such a dynamical process. Unfortunately, different dynamical processes related to the same given state may produce different values of the work. This is the case because the work done on histories is not in general a function of state (see e.g. [10]). Since there is no disturbance of the medium before time $t = 0$ we have

$$\psi_M(t) = \int_0^t \mathbf{T}(s)\cdot\dot{\mathbf{E}}(s)ds, \quad \mathbf{x} \in \mathcal{B}_r. \tag{4.19}$$

It is shown in [2] that

$$I(r,t) = -\int_0^t \int_{\mathcal{S}_r} \mathbf{s}(\tau)\cdot\dot{\mathbf{u}}(\tau)dSd\tau \tag{4.20}$$

where $\mathbf{s}$ is defined by (4.15) with the outward normal pointing into $\mathcal{B}_r$. We define also the energy measure

$$U(r,t) := \int_{\mathcal{B}_r} \left\{ \frac{1}{2}\rho|\dot{\mathbf{u}}(t)|^2 + \psi(t) \right\} dV \leq I(r,t) \tag{4.21}$$

where $\psi(t)$ is any free energy of the system, for example, the minimum free energy $\psi_m(t)$ given by (3.8). In general, we have

$$\dot{\psi}(t) + D(t) = \mathbf{T}(t) \cdot \dot{\mathbf{E}}(t), \quad D(t) \geq 0 \ \ \forall\, t \in \mathcal{R} \tag{4.22}$$

which is essentially (3.16). Using (4.19) and the integrated form of (4.22) in (4.21), one obtains

$$I(r,t) = U(r,t) + \int_0^t D_{\mathcal{B}}(r,\tau)d\tau, \quad D_{\mathcal{B}}(r,t) := \int_{\mathcal{B}_r} D(\mathbf{x},t)dV \geq 0. \tag{4.23}$$

We shall refer to $D_{\mathcal{B}}$ as the bulk dissipation. Following the developments in [2], we have

$$\frac{\partial}{\partial r}I(r,t) = -\int_{\mathcal{S}_r} \left\{ \frac{1}{2}\rho|\dot{\mathbf{u}}(t)|^2 + \psi_M(t) \right\} dS \tag{4.24}$$

and analogously

$$\begin{aligned}
\frac{\partial}{\partial r}U(r,t) &= -\int_{\mathcal{S}_r} \left\{ \frac{1}{2}\rho|\dot{\mathbf{u}}(t)|^2 + \psi(t) \right\} dS \\
&= \frac{\partial}{\partial r}I(r,t) + D_{\mathcal{S}}(r,t) \\
D_{\mathcal{S}}(r,t) &:= \int_0^t \int_{\mathcal{S}_r} D(\mathbf{x},t)dS \geq 0,
\end{aligned} \tag{4.25}$$

where we assume sufficient smoothness in the displacement field $\mathbf{u}$ so that the surface integrals exist. Furthermore,

$$\frac{\partial}{\partial t}I(r,t) = -\int_{\mathcal{S}_r} \mathbf{s}(t) \cdot \dot{\mathbf{u}}(t)dS. \tag{4.26}$$

From (4.23), we have

$$\frac{\partial}{\partial t}I(r,t) = \frac{\partial}{\partial t}U(r,t) + D_{\mathcal{B}}(r,t). \tag{4.27}$$

Let

$$I^*(r_1,t) := \int_0^t I(r_1,\tau)d\tau \tag{4.28}$$

be the time integral over a finite interval of the total mechanical energy expended in any dynamical process relative to a given state and

$$U^*(r_1,t) := \int_0^t U(r_1,\tau)d\tau, \tag{4.29}$$

be the corresponding part without dissipation. It is possible to give precise estimates of the spatial decay of these time integrated quantities according to the following proposition.

Proposition 4.2. *Let $\sigma(t)$ be a given state and let us consider any dynamical process $\{(\mathbf{u}(t),\mathbf{u}^t),(\mathbf{E}(t),\mathbf{E}^t),\mathbf{T}(t)\}$ relative to $\sigma(t)$. Then*

$$\begin{aligned} &I(r,t)=0, \quad D_{\mathcal{S}}(r,t)=0 \ \ \forall\, r\geq ct.\\ &I^*(r,t)\leq\left(1-\frac{r}{ct}\right)\left[\int_0^t I(0,s)ds\right.\\ &\qquad\left.-\int_0^t\int_0^r D_{\mathcal{S}}\left(r',\left(1-\frac{r'}{ct}\right)s+\frac{r'}{c}\right)dr'ds\right], \quad \forall\, r\leq ct, \end{aligned} \tag{4.30}$$

where $c=\sqrt{c_0/\rho_0}$. The energy measure U and the bulk dissipation in such a process are such that the relations

$$\begin{aligned} &U(r,t)=0, \quad D_{\mathcal{B}}(r,t)=0 \ \ \forall\, r\geq ct\\ &U^*(r,t)\leq\left(1-\frac{r}{ct}\right)\left[\int_0^t U(0,s)ds\right.\\ &\qquad\left.-\frac{1}{ct}\int_0^t\int_0^r (t-s)D_{\mathcal{B}}\left(r',\left(1-\frac{r'}{ct}\right)s+\frac{r'}{c}\right)dr'ds\right]\\ &\qquad\forall\, r\leq ct. \end{aligned} \tag{4.31}$$

hold.

Proof. Applying Schwartz's inequality (twice: to the integral and to obtain $|\mathbf{s}|^2\leq|\mathbf{T}|^2$) and the arithmetic-geometric inequality to (4.26) we have that

$$\left|\frac{\partial}{\partial t}I(r,t)\right|\leq\frac{1}{2}\int_{\mathcal{S}_r}[\epsilon|\dot{\mathbf{u}}(t)|^2+\epsilon^{-1}|\mathbf{T}(t)|^2]dS \tag{4.32}$$

where ϵ is an arbitrary positive number which will be assigned a value below. Invoking Proposition 4.1, we deduce that

$$\left|\frac{\partial}{\partial t}I(r,t)\right|\leq\int_{\mathcal{S}_r}[\frac{1}{2}\epsilon|\dot{\mathbf{u}}(t)|^2+\epsilon^{-1}c_0\psi(t)]dS. \tag{4.33}$$

Setting $\epsilon=\sqrt{c_0\rho_0}$ and $c=\sqrt{c_0/\rho_0}$, it follows that

$$\left|\frac{\partial}{\partial t}I(r,t)\right|+c\frac{\partial}{\partial r}U(r,t)\leq 0\ \forall\, t\in[0,T] \tag{4.34}$$

or, using (4.25)

$$\left|\frac{\partial}{\partial t}I(r,t)\right| + c\frac{\partial}{\partial r}I(r,t) \leq -D_{\mathcal{S}}(r,t)\ \forall\, t \in [0,T]. \tag{4.35}$$

The term on the right is in general non-positive and may be non-zero. Equation (4.35) differs from the partial differential inequality derived in [2] in that this term is present. We wish to explore the constraints imposed on $I(r,t)$ by (4.35) and in particular, how they differ from those established in [2]. We also present constraints on $U(r,t)$. The technique used is essentially the same as in [2].

The inequality (4.35) is equivalent to following two simultaneous differential inequalities:

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}I(r,t) + \frac{\partial}{\partial r}I(r,t) &\leq -D_{\mathcal{S}}(r,t), \\ -\frac{1}{c}\frac{\partial}{\partial t}I(r,t) + \frac{\partial}{\partial r}I(r,t) &\leq -D_{\mathcal{S}}(r,t). \end{aligned} \tag{4.36}$$

Before considering (4.36) in detail, we note that on using (4.27), they may also be written in the form

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}U(r,t) + \frac{\partial}{\partial r}U(r,t) &\leq -\frac{D_{\mathcal{B}}(r,t)}{c} \\ -\frac{1}{c}\frac{\partial}{\partial t}U(r,t) + \frac{\partial}{\partial r}U(r,t) &\leq \frac{D_{\mathcal{B}}(r,t)}{c}. \end{aligned} \tag{4.37}$$

Multiplying the two relations in (4.36) by arbitrary positive numbers and adding them, we deduce that

$$\frac{1}{\kappa}\frac{\partial}{\partial t}I(r,t) + \frac{\partial}{\partial r}I(r,t) \leq -D_{\mathcal{S}}(r,t), \quad |\kappa| \geq c. \tag{4.38}$$

Similarly, (4.37) gives

$$\frac{1}{\kappa}\frac{\partial}{\partial t}U(r,t) + \frac{\partial}{\partial r}U(r,t) \leq -\frac{1}{\kappa}D_{\mathcal{B}}(r,t), \quad |\kappa| \geq c. \tag{4.39}$$

Let (r_0, t_0) be a point on the rt plane and we consider two lines through this point with slopes c^{-1} and $-c^{-1}$, where r is the independent variable. Next we consider a line through (r_0, t_0) with slope κ^{-1} where $|\kappa| \geq c$. This line intersects the t axis between the points of intersection of the two lines just defined. We choose $\kappa \geq c$ and utilise a line integral along this line to write

$$I(r, t_0 + \frac{r-r_0}{\kappa}) = I(r_0,t_0) + \int_{r_0}^{r} dr' \left(\frac{1}{\kappa}\frac{\partial}{\partial t'} + \frac{\partial}{\partial r'}\right) I(r',t')\,\Big|_{t'=t_0+\frac{r'-r_0}{\kappa}} \tag{4.40}$$

For $r \geq r_0$ we have

$$I(r, t_0 + \frac{r - r_0}{\kappa}) \leq I(r_0, t_0) - \int_{r_0}^{r} D_{\mathcal{S}}(r', t_0 + \frac{r' - r_0}{\kappa})dr' \tag{4.41}$$

so that $I(r,t)$ declines in value as r increases, within this region; while for $r \leq r_0$

$$\begin{aligned} I(r, t_0 + \frac{r - r_0}{\kappa}) &= I(r_0, t_0) - \int_{r}^{r_0} dr' \left(\frac{1}{\kappa}\frac{\partial}{\partial t'} + \frac{\partial}{\partial r'}\right) I(r', t') \mid_{t'=t_0 + \frac{r' - r_0}{\kappa}} \\ &\geq I(r_0, t_0) + \int_{r}^{r_0} D_{\mathcal{S}}(r', t_0 + \frac{r' - r_0}{\kappa})dr' \end{aligned} \tag{4.42}$$

which indicates that it increases in value as r decreases. The quantity $I(r,t)$ is non-negative and vanishes as $t \to 0^+$. Thus, if we let $r_0 \to 0^+$ and $t_0 \to 0^+$, (4.41) becomes

$$I(r, \frac{r}{\kappa}) \leq -\int_{0}^{r} D_{\mathcal{S}}(r', \frac{r'}{\kappa})dr', \quad r > 0 \tag{4.43}$$

which implies that both sides vanish. Therefore

$$I(r,t) = 0, \quad D_{\mathcal{S}}(r,t) = 0 \;\; \forall\, r \geq ct. \tag{4.44}$$

which is $(4.30)_1$. The second relation, which is not given in [2], is however not a new consequence of the argument, since the first relation implies that all stresses and displacements are zero on $\mathcal{B}_r$ for $r \geq ct$, which in turn gives that there can be no dissipation in that region at such times.

Let us consider the integrated total mechanical energy $I^*(r_1, t)$ defined by (4.28). From (4.44) we see that

$$I^*(r_1, t) := \int_0^t I(r_1, \tau)d\tau = \int_{\frac{r_1}{c}}^{t} I(r_1, \tau)d\tau, \tag{4.45}$$

Putting

$$\tau = \left(1 - \frac{r_1}{ct}\right) s + \frac{r_1}{c} \tag{4.46}$$

we obtain

$$I^*(r_1, t) = \left(1 - \frac{r_1}{ct}\right) \int_0^t I\left(r_1, \left(1 - \frac{r_1}{ct}\right) s + \frac{r_1}{c}\right) ds \tag{4.47}$$

Letting

$$r_0 = r_1, \quad t_0 = \left(1 - \frac{r_1}{ct}\right) s + \frac{r_1}{c}, \quad r = 0, \quad \kappa = \frac{ct}{t - s} \tag{4.48}$$

we deduce from (4.42) that

$$I(0, s) \geq I\left(r_1, \left(1 - \frac{r_1}{ct}\right) s + \frac{r_1}{c}\right) + \int_0^{r_1} D_{\mathcal{S}}\left(r', \left(1 - \frac{r'}{ct}\right) s + \frac{r'}{c}\right) dr' \tag{4.49}$$

so that, replacing r_1 by r the inequality $(4.30)_2$ follows for $r \leq ct$. A similar line of reasoning can be applied to (4.39). Taking $\kappa \geq c$ we deduce analogously to (4.44) that

$$U(r,t) = 0, \quad D_{\mathcal{B}}(r,t) = 0 \ \ \forall\, r \geq ct. \tag{4.50}$$

which is $(4.31)_1$. Recalling (4.29), we see that (4.50) gives

$$U^*(r_1,t) := \int_0^t U(r_1,\tau)d\tau = \int_{\frac{r_1}{c}}^t U(r_1,\tau)d\tau. \tag{4.51}$$

Carrying through the argument, we find that (4.49) is replaced by

$$U(0,s) \geq U\left(r_1, \left(1-\frac{r_1}{ct}\right)s + \frac{r_1}{c}\right) + \frac{t-s}{ct}\int_0^{r_1} D_{\mathcal{B}}\left(r', \left(1-\frac{r'}{ct}\right)s + \frac{r'}{c}\right)dr' \tag{4.52}$$

and inequality $(4.31)_2$ holds. □

It it worth noting that inequality $(4.30)_2$ provides a stronger bound than that given in [2] and is the central result of this section. The bound becomes smaller as the dissipation rate increases.

In particular, if ψ is equal to the minimum free energy ψ_m then $D(t) = D_m(t)$ given by (3.17). The quantity $D_{\mathcal{B}}$, given by (4.23), is a volume integral of this quantity, while $D_{\mathcal{S}}$, defined by $(4.25)_3$, is a surface integral of the quantity (see (3.7))

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{p}_+^t(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{q}_+^t(\omega)|^2\, d\omega = Q_+(t) \tag{4.53}$$

or in the more explicit form given by (3.22). The time integral of $D_{\mathcal{S}}$ in (4.30) is given by (*cf* (4.45), (4.47))

$$\int_0^t D_{\mathcal{S}}\left(r', \left(1-\frac{r'}{ct}\right)s + \frac{r'}{c}\right)ds = \left(1-\frac{r'}{ct}\right)^{-1}\int_{\frac{r'}{c}}^t D_{\mathcal{S}}\left(r',\tau\right)d\tau. \tag{4.54}$$

The time integral on the right may be extended to $-\infty$ by virtue of (4.44), and the last term in (4.30) is a surface integral of a time integral over Q_+, expressible in two different forms as given by (3.28) and proved by Proposition 3.1.

Similarly, in (4.31), using (4.44), we have

$$\begin{aligned}\int_0^t (t-s)D_{\mathcal{B}}\left(r', \left(1-\frac{r'}{ct}\right)s + \frac{r'}{c}\right)ds &= \left(1-\frac{r'}{ct}\right)^{-2}\int_{\frac{r'}{c}}^t (t-\tau)D_{\mathcal{B}}(r',\tau)d\tau \\ &= \left(1-\frac{r'}{ct}\right)^{-2}\int_{\frac{r'}{c}}^t\int_{\frac{r'}{c}}^{\tau} D_{\mathcal{B}}(r',u)du d\tau.\end{aligned} \tag{4.55}$$

The integrals can be extended to $-\infty$ and this quantity is given by the volume integral of a time integral over Q_+, given as before by (3.28).

5. Preliminary Results for the non-inertial case

Before considering the non-inertial case, we deduce in this section certain inequalities which will be required. Consider the functional

$$\begin{aligned}\psi_G(t) &= \phi(t) - \frac{1}{2}\int_0^\infty \mathbf{E}_r^t(s)\cdot\dot{\mathbb{G}}(s)\mathbf{E}_r^t(s)ds \\ &= S(t) - \frac{1}{2}\int_0^\infty \mathbf{E}^t(s)\cdot\dot{\mathbb{G}}(s)\mathbf{E}^t(s)ds.\end{aligned} \tag{5.1}$$

This functional is non-negative by virtue of (2.10) and (2.26); ψ_G is also a free energy in the sense of Graffi(see e.g. [15]), the Graffi-Volterra free energy, if the relaxation tensor obeys (2.10) and the further condition (2.11). We will not assume (2.11). The quantity ψ_G will be referred to as the Graffi-Volterra functional. It can be shown [2] that

$$\mathbf{T}(t)\cdot\mathbf{E}(t) = \psi_G(t) + \frac{1}{2}\frac{d}{dt}\int_0^\infty \mathbf{E}^t(s)\cdot\mathbb{G}(s)\mathbf{E}^t(s)ds \tag{5.2}$$

giving

$$\begin{aligned}F(t) := \int_{-\infty}^t \mathbf{T}(s)\cdot\mathbf{E}(s)ds &= \int_{-\infty}^t \psi_G(s)ds + \frac{1}{2}\int_0^\infty \mathbf{E}^t(s)\cdot\mathbb{G}(s)\mathbf{E}^t(s)ds \\ &\geq \int_{-\infty}^t \psi_G(s)ds \geq 0\end{aligned} \tag{5.3}$$

under the assumption that F exists.

We now consider free energies of the general form

$$\begin{aligned}\psi(t) &= \phi(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \mathbf{E}_r^t(s_1)\cdot\mathbb{G}_{12}(s_1,s_2)\mathbf{E}_r^t(s_2)ds_1ds_2 \\ &= S(t) + \frac{1}{2}\int_0^\infty\int_0^\infty \mathbf{E}^t(s_1)\cdot\mathbb{G}_{12}(s_1,s_2)\mathbf{E}^t(s_2)ds_1ds_2\end{aligned} \tag{5.4}$$

where $\mathbb{G}_{12}(s_1,s_2)\in Lin(Sym)$ has the properties

$$\begin{aligned}\mathbb{G}_{12}^\top(s_1,s_2) &= \mathbb{G}_{12}(s_2,s_1); \\ \mathbb{G}_{12}(s_1,s_2) &= \frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\mathbb{G}(s_1,s_2); \\ \mathbb{G}(0,s_1) &= \mathbb{G}(s_1,0) = \mathbb{G}(s_1); \\ \mathbb{G}_1(s_1,\infty) &= \mathbb{G}_2(\infty,s_2) = 0, \quad \forall\, s_1,s_2\in\mathcal{R}^+ \\ \mathbb{G}(s_1,\infty) &= \mathbb{G}(\infty,s_2) = \mathbb{G}_\infty, \quad \forall\, s_1,s_2\in\mathcal{R}^+\end{aligned} \tag{5.5}$$

where $\phi(t)$ and $S(t)$ are defined by (2.26). The two forms can be shown to be equivalent with the aid of the given constraints on $\mathbb{G}$. Also, property P2 after (3.13) can be demonstrated. From the first form of (5.4) and (3.14), it follows that $\mathbb{G}_{12}$ must be a positive-definite operator. We will assume that it is a symmetric tensor so that, by $(5.5)_1$ we have

$$\mathbb{G}_{12}(s_1, s_2) = \mathbb{G}_{12}(s_2, s_1) \quad \forall\, s_1, s_2 \in \mathcal{R}^+. \tag{5.6}$$

It is further assumed that

$$|\mathbb{G}_{12}(s_1, s_2)| < \infty \quad \forall\, s_1, s_2 \in \mathcal{R}^+ \otimes \mathcal{R}^+. \tag{5.7}$$

It follows from the time domain representation of the minimum free energy given in [1], chapter 11 (see also [53]) that it can be expressed in the form (5.4). Similarly, the family of free energies derived in [55] and in [1], chapters 15, 16 can also be expressed in this form.

A restriction on the choice of the relaxation function $\mathbb{G}(s)$ was considered in [49] (see also [1]) in which it was assumed that its eigenspaces do not depend on time. The factorization problem for the tensor relaxation function then reduces to that for a scalar relaxation function [12] and allows explicit forms of the minimum free energy to be written down. In particular, it was shown that, under this assumption, $\mathbb{H}_\pm(\omega)$ also have this property and that they commute. It will be true if $\mathbb{G}$ can be expanded as follows:

$$\mathbb{G}(s_1, s_2) = \sum_{k=1}^{6} G_k(s_1, s_2)\, \mathbb{B}^k \tag{5.8}$$

where $\mathbb{B}^k = \mathbf{B}^k \otimes \mathbf{B}^k \quad k = 1, ...6$ are the projectors on the 6 constant eigenspaces of $\mathbb{G}$ and $\{\mathbf{B}^k\}$ are its normal eigenvectors, which constitute an orthonormal basis of Sym. The quantities G_k are scalars. This is a special case of (A1.2). The tensor $\mathbb{G}_{12}$ also has property (5.8). Note that (5.8) implies (5.6).

Proposition 5.1. *If $\mathbb{G}_{12}(s_1, s_2)$ is a positive semi-definite tensor for all $s_1, s_2 \in \mathcal{R}^+$ then*

$$\psi(t) \leq \psi_G(t), \quad t \in \mathcal{R} \tag{5.9}$$

where ψ_G is the Graffi-Volterra functional (5.1) *and ψ is given by* (5.4).

Proof. Consider the identity

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_0^\infty \left(\mathbf{E}^t(s_1) - \mathbf{E}^t(s_2)\right) \cdot \mathbb{G}_{12}(s_1, s_2) \left(\mathbf{E}^t(s_1) - \mathbf{E}^t(s_2)\right) ds_1 ds_2 \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t(s_1) \cdot \mathbb{G}_{12}(s_1, s_2)\mathbf{E}^t(s_1) ds_1 ds_2 \\
&+ \frac{1}{2}\int_0^\infty \int_0^\infty \mathbf{E}^t(s_2) \cdot \mathbb{G}_{12}(s_1, s_2)\mathbf{E}^t(s_2) ds_1 ds_2 \\
&- \int_0^\infty \int_0^\infty \mathbf{E}^t(s_1) \cdot \mathbb{G}_{12}(s_1, s_2)\mathbf{E}^t(s_2) ds_1 ds_2.
\end{aligned} \tag{5.10}$$

The left-hand side is non-negative by virtue of the positivity assumption on

$\mathbb{G}_{12}$. The first two terms on the right yield the integral terms in (5.1) and the last term is the integral term in $(5.4)_2$. Relation (5.9) follows immediately.

□

In particular, we have

$$\psi_m(t) \leq \psi_G(t), \quad t \in \mathcal{R}. \tag{5.11}$$

if the assumption of Proposition 5.1 hold. The same argument applies to the family of free energies derived in [55]. It should be pointed out that if (2.11) holds, relation (5.11) in fact follows from the minimum property of ψ_m and the fact that it and ψ_G, now a free energy, have the Graffi properties P1-P4 [49].

If $\ddot{\mathbb{G}}$ is not assumed to be non-negative, we rely on Proposition 5.1 to prove (5.11). The question therefore arises: is $\mathbb{G}_{12}$ a positive, semi-definite tensor for the minimum free energy? The answer is in the affirmative for all cases where explicit forms have been obtained, namely where the relaxation function is a sum of decaying exponentials in the scalar case [12] and in the tensor case under the assumption of time-independent eigenspaces as outlined before (5.8); the answer is affirmative also for the case where the relaxation function is completely monotonic, so that the Bernstein representation formula [31] allows it to be represented as an integral over decaying exponentials with a non-negative density function [59].

The quantity ψ_M, given by (2.26), has the form (5.4) but where $\mathbb{G}_{12}$ is not bounded on $\mathcal{R}^+ \otimes \mathcal{R}^+$. In fact [30]

$$\mathbb{G}_{12}(|s_1 - s_2|) = -2\delta(s_1 - s_2)\dot{\mathbb{G}}(|s_1 - s_2|) - \ddot{\mathbb{G}}(|s_1 - s_2|), \tag{5.12}$$

in terms of the singular Dirac measure. In this case, Proposition 5.1 does not hold. In fact, we see that the left-hand side of (5.10) is non-positive if (2.11) holds, since the delta-function term yields zero. Therefore

$$\psi_M(t) \geq \psi_G(t). \quad t \in \mathcal{R} \tag{5.13}$$

which is consistent with the fact that ψ_M is maximal, a property that holds whenever the state of the material can be identified with the pair current strain-past strain history.

For the remaining sections, we suppose that the relaxation tensor $\mathbb{G}(t)$ satisfies the condition (2.10) and that (5.9) holds.

6. The non-inertial case for general histories

We consider the region $\mathcal{B}$ and the subsets as defined in section 4, except that $\dot{\mathbf{u}}$ is omitted from (4.12). Also, $\mathcal{D}_r$ is defined by (4.16) but interpreted simply as the set of points within a distance r of $\mathcal{D}_T^*$. The parameter r ranges over the interval $[0, L]$.

In what follows, for a material point $\mathbf{x}$ and time t we consider a state $\sigma(t)$. Let us denote by the triple $\{(\mathbf{u}(t), \mathbf{u}^t), (\mathbf{E}(t), \mathbf{E}^t), \mathbf{T}(t)\}$ a quasi-static (linear viscoelastic) process, where $(\mathbf{E}(t), \mathbf{E}^t) \sim \sigma(t)$, $\mathbf{E} = \nabla \mathbf{u}$ and the stress $\mathbf{T}(t)$ satisfies the constitutive equation (2.1) together with the balance of linear momentum:

$$\nabla \cdot \mathbf{T}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t) = 0, \quad (\mathbf{x},t) \in \mathcal{B} \times \mathcal{R}^{+}. \tag{6.1}$$

with body force $\mathbf{b}(\mathbf{x},t)$.

The total load and moment acting on $\mathcal{S}_0$ are denoted by $\mathbf{R}(t)$ and $\mathbf{M}(t)$. The necessary conditions for the equilibrium of $\mathcal{D}_0$ are given by

$$\begin{aligned} \mathbf{R}(t) &= \int_{\mathcal{S}_0} \mathbf{s}\, dS = -\int_{\mathcal{D}_0} \mathbf{b}\, dV - \int_{\partial\mathcal{D}_0 \setminus \mathcal{S}_0} \mathbf{s}\, dS \\ \mathbf{M}(t) &= \int_{\mathcal{S}_0} \mathbf{x} \times \mathbf{s}\, dS = -\int_{\mathcal{D}_0} \mathbf{x} \times \mathbf{b}\, dV - \int_{\partial\mathcal{D}_0 \setminus \mathcal{S}_0} \mathbf{x} \times \mathbf{s}\, dS \end{aligned} \tag{6.2}$$

where $\mathbf{s}$ is defined by (4.15) in which $\mathbf{n}$ is the normal on $\mathcal{S}_0$ pointing out of $\mathcal{D}_0$. Necessary conditions for the equilibrium of $\mathcal{D}_r$ are

$$\int_{\mathcal{S}_r} \mathbf{s}\, dS = \mathbf{R}(t), \qquad \int_{\mathcal{S}_r} \mathbf{x} \times \mathbf{s}\, dS = \mathbf{M}(t) \tag{6.3}$$

where $\mathbf{n}$ (in the definition of $\mathbf{s}$) is in the increasing direction of r.

Saint-Venant's principle deals with the difference in behaviour of the family of stress fields yielding the same $\mathbf{R}(t)$ and $\mathbf{M}(t)$. This leads us to consider a stress field that is the difference between any two members of this family, which, in view of the linearity of the governing equations, will be characterized by null global load and moment $\mathbf{R}(t), \mathbf{M}(t)$, null body force and surface loads non-zero only on $\partial\mathcal{D}_0$.

Thus, we consider the balance of linear momentum with no body forces

$$\nabla \cdot \mathbf{T}(\mathbf{x},t) = 0, (\mathbf{x},t) \in \mathcal{B}_0 \times [0,T], \tag{6.4}$$

where

$$\mathbf{s}(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\mathcal{B}_0 \setminus \mathcal{S}_0, \quad t \in [0,T], \tag{6.5}$$

and such that the overall balance of forces and moments hold:

$$\begin{aligned} \int_{\mathcal{S}_0} \mathbf{s}\, dS &= \int_{\mathcal{S}_r} \mathbf{s}\, dS = 0 \\ \int_{\mathcal{S}_0} \mathbf{x} \times \mathbf{s}\, dS &= \int_{\mathcal{S}_r} \mathbf{x} \times \mathbf{s}\, dS = 0, \quad t \in [0,T]. \end{aligned} \tag{6.6}$$

We define the following "energy" measures on $\mathcal{B}_r$:

$$U_E(r) = \int_0^T \int_{\mathcal{B}_r} \mathbf{T}(t) \cdot \mathbf{E}(t)\, dV dt; \quad U_\psi(r) = \int_0^T \int_{\mathcal{B}_r} \psi(t)\, dV dt \le U_E(r) \tag{6.7}$$

where ψ is any free energy obeying (5.9), in particular the minimum free energy. The inequality follows from (5.3) and (5.9). The quantity T is for present purposes any positive time.

The quantity $U(r)$ will indicate any one of these two measures.

We use the result of Berdichevskii [50] that, for all vector fields $\mathbf{v}$ on a bounded domain Γ which satisfy the constraints

$$\int_\Gamma \mathbf{v}\, dV = 0, \qquad \int_\Gamma \mathbf{x} \times \mathbf{v}\, dV = 0, \tag{6.8}$$

the inequality

$$b \int_\Pi |\mathbf{v}|^2\, dS \leq \int_\Gamma \mathbf{E} \cdot \mathbb{C}\mathbf{E}\, dV \tag{6.9}$$

holds, where $\mathbf{E} = sym\nabla\mathbf{v} \in Sym$, Π is any surface such that $\Pi \subset \partial\Gamma$ and b is a constant depending on Γ, Π and the positive-definite tensor $\mathbb{C} \in Lin(Sym)$.

Proposition 6.1. *Let $\sigma(t)$ be a given state and let $\{(\mathbf{u}(t), \mathbf{u}^t), (\mathbf{E}(t), \mathbf{E}^t), \mathbf{T}(t)\}$ be any quasi-static process related to $\sigma(t)$ such that* (6.4) - (6.6) *hold. Then*

$$\begin{aligned} U(r) &\leq U(0)e^{-r/\alpha}, \quad 0 \leq r \leq L - l \\ \alpha &= \frac{4c_0}{\beta}, \quad \beta = \min_{0\leq r\leq L-l} b(r), \quad l > 0, \end{aligned} \tag{6.10}$$

on such a process, where c_0 is defined by (4.2) *and $b(r)$ is the optimal choice of the constant in* (6.9) *for $\mathbb{C} = \mathbb{G}_\infty$, $\Pi = \mathcal{S}_r$ and $\Gamma = \mathcal{B}_r$.*

Proof. We firstly change the displacement vector field by replacing $\mathbf{u}$ with

$$\tilde{\mathbf{u}} = \mathbf{u}_0 + \mathbf{u} \tag{6.11}$$

where $\mathbf{u}_0$ is a rigid motion (translation and rigid rotation) chosen so as to satisfy the equations

$$\int_{\mathcal{B}_r} \tilde{\mathbf{u}}\, dV = 0, \quad \int_{\mathcal{B}_r} \mathbf{x} \times \tilde{\mathbf{u}}\, dV = 0. \tag{6.12}$$

It is shown in [8] that this is always possible. From (6.9), we have the inequality

$$\int_{\mathcal{S}_r} |\tilde{\mathbf{u}}|^2\, dS \leq \frac{1}{b(r)} \int_{\mathcal{B}_r} \mathbf{E} \cdot \mathbb{G}_\infty \mathbf{E}\, dV \tag{6.13}$$

This change in $\mathbf{u}$ does not alter $\mathbf{E}$ or $\mathbf{T}$. Note that from (3.14) and (3.15),

$$\frac{1}{2} \int_0^T \int_{\mathcal{B}_r} \mathbf{E} \cdot \mathbb{G}_\infty \mathbf{E}\, dV dt \leq U_\psi(r) \leq U_E(r). \tag{6.14}$$

Applying the divergence theorem to $\mathcal{B}_r$, we obtain from (6.7) that

$$U(r) \leq U_E(r) = -\int_0^T \int_{\mathcal{S}_r} \mathbf{s} \cdot \tilde{\mathbf{u}}\, dS dt \tag{6.15}$$

and Schwartz's inequality gives

$$U(r) \leq \left(\int_0^T \int_{\mathcal{S}_r} |\mathbf{T}(t)|^2 \, dSdt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathcal{S}_r} |\tilde{\mathbf{u}}(t)|^2 \, dSdt \right)^{\frac{1}{2}}. \tag{6.16}$$

Relations (6.13) and (6.14) yield

$$\int_0^T \int_{\mathcal{S}_r} |\tilde{\mathbf{u}}(t)|^2 \, dSdt \leq \frac{2}{b(r)} U(r) \tag{6.17}$$

which, on substitution into (6.16), and squaring both sides, results in

$$U(r) \leq \frac{2}{b(r)} \int_0^T \int_{\mathcal{S}_r} |\mathbf{T}(t)|^2 \, dSdt. \tag{6.18}$$

By virtue of Proposition 4.1, we deduce that

$$U(r) \leq \alpha \int_0^T \int_{\mathcal{S}_r} \psi(t) \, dSdt. \tag{6.19}$$

where in fact $\psi(t)$ could be replaced by $\mathbf{T}(t) \cdot \mathbf{E}(t)$. Noting that

$$\frac{dU(r)}{dr} = - \int_0^T \int_{\mathcal{S}_r} \psi(t) \, dSdt. \tag{6.20}$$

we have the differential inequality

$$U(r) + \alpha \frac{dU(r)}{dr} \leq 0 \tag{6.21}$$

the solution of which yields (6.10). The quantity l in (6.10) must be taken to be strictly positive if β is to be non-zero [2, 50].

□

This result was presented in [2] for $U = U_E$, given by $(6.7)_1$; see also [1], page 458 for the case of a cylindrical body. Proposition 6.1 generalizes the estimate to a family of energy measures involving a class of free energies with the property required by Proposition 5.1. As noted earlier, the minimum free energy is in this class for general categories of relaxation tensors.

The spatial decay of the states (*i.e.* the level of disturbance) of the material points located inwards from the loaded boundary may also be explored. To this end, we recall that in [49], Sect. 9, the following L^2-norm is introduced in the state space:

$$\|\sigma(\mathbf{x}, t)\|^2 := |\mathbf{E}(\mathbf{x}, t)|^2 + \int_{-\infty}^{\infty} |\mathbf{q}_-^t(\mathbf{x}, \omega)|^2 d\omega, \tag{6.22}$$

where $\mathbf{q}_-^t$ is defined by $(3.5)_2$ and $(3.6)_2$. Proposition 9.2 in [49] shows the equivalence of the norm defined by (6.22) and the norm based on the minimal free energy ψ_m. It is worth recalling that such an equivalence yields a different way to get the coarsest possible L^2-type norm in the state space. The measure $\Sigma(r,t)$ for the state of the points encountered by moving from the loaded boundary into the body may be defined as follows:

$$\Sigma(r,t) := \int_0^T \int_{\mathcal{B}_r} \|\sigma(\mathbf{x},t)\|^2 dV dt. \tag{6.23}$$

From (6.22) and Proposition 9.2 in [49] it then follows that the measure $\Sigma(r,t)$ obeys the inequality (6.10). This ensures that any other measure of the state of the points in $\mathcal{B}_r$ finer than $\Sigma(r,t)$ also decays at the same spatial rate as $\Sigma(r,t)$. This conclusion could not be drawn by exploring the decay of U_E which may not even induce a norm in the state space. We have proved the following result.

Proposition 6.2. *The measure $\Sigma(r,t)$ of the state of material points at time t located in the region $\mathcal{B}_r$ spatially decay according to*

$$\begin{aligned} &\Sigma(r,t) \leq \Sigma(0,t)e^{-r/\alpha}, \quad 0 \leq r \leq L-l \\ &\alpha = \frac{4c_0}{\beta}, \quad \beta = \min_{0\leq r\leq L-l} b(r), \quad l > 0. \end{aligned} \tag{6.24}$$

It is worth noting that the latter proposition leads to an important conclusion.

Indeed, after dividing both sides of (6.24) by $T\,vol(\mathcal{B}_r)$, the obtained result is showing that an averaged (space-time) measure of the residual stress $\sigma(\mathbf{x},t)$ over the region $\mathcal{B}_r$ is spatially decaying. This could not have been proved unless a one to one relation between states $(\sigma(\mathbf{x},t))$ and free energy would not have been established This proves that in linear viscoelastic solids not only we can show a decay in energy, but also we have a stress measure that spatially decays too at some very definite rate. In other words, at a sufficient distance from the applied loads, the state of the material, and hence the residual stress inherited from past histories, does not depend on the specific application of the tractions, but only on the resultants. In general getting information on the stress decay is the hardest part of a Saint-Venant's-like result, being the decay of the energy easier to obtain [51].

7. The non-inertial case for sinusoidal histories

We now consider states σ_ω for the linear viscoelastic material such that the equivalence class is represented by sinusoidal histories with frequency ω (for a definition of such histories see Appendix 1). An equivalence class of such histories may be defined using (2.23). It is easy to show that the equivalence class so defined is a singleton and we denote the corresponding state by σ_m.

Such states in a body may be caused either by applied tractions or displacements, or both, which are sinusoidal with frequency ω. In such cases, the spatial decay of energy measures will depend on the frequency. We seek here to study this dependence. The energy measure $U(r)$ introduced in the previous section is replaced by $U(r, \omega)$, which can be either of the measures in (6.7). The assumption that the material is undisturbed for $t < 0$ must now be dropped.

Proposition 7.1. *Let* $\{(\mathbf{u}(t), \mathbf{u}^t), (\mathbf{E}(t), \mathbf{E}^t), \mathbf{T}(t)\}$ *be a quasi-static (linear viscoelastic) process related to the state* σ_ω*, so that the histories* $\mathbf{u}^t$*,* $\mathbf{E}^t := \frac{1}{2}[\nabla \mathbf{u}^t + (\nabla \mathbf{u}^t)^\top]$ *are sinusoidal satisfying* (6.4) - (6.6) *over the interval* $(-\infty, t]$*. Then*

$$U(r, \omega) \leq U(0) e^{-r/\alpha_U(\omega)}, \quad 0 \leq r \leq L - l \tag{7.1}$$

where $\alpha(\omega)$ *depends upon the load application frequency and the choice of measure from* (6.7).

Proof. Let $(\mathbf{E}, \mathbf{E}^t)$ be given by (A2.1), where the amplitude $\mathbf{C} \in Sym$ contains the space dependence, and let T be a multiple of π/ω. It follows from (5.3) and (5.9) that, before taking the limit $\eta \to 0$,

$$\int_{-\infty}^{T} \mathbf{T}(t) \cdot \mathbf{E}(t) dt \geq \int_{-\infty}^{T} \psi_G(t) dt \geq \int_{-\infty}^{T} \psi(t) dt. \tag{7.2}$$

Now

$$\int_{-\infty}^{T} \mathbf{T}(t) \cdot \mathbf{E}(t) dt = \frac{1}{1 - e^{-2\eta T}} \int_{0}^{T} \mathbf{T}(t) \cdot \mathbf{E}(t) dt \tag{7.3}$$

with similar relations for other quantities. Then for any finite η we have

$$\int_{0}^{T} \mathbf{T}(t) \cdot \mathbf{E}(t) dt \geq \int_{0}^{T} \psi(t) dt. \tag{7.4}$$

This relationship will therefore hold in the limit $\eta \to 0$ since the integrals exist and are continuous at $\eta = 0$. Thus we have, as in (6.7)

$$U_E(r, \omega) \geq U_\psi(r, \omega). \tag{7.5}$$

The measure $U(r, \omega)$ can be expressed in the form

$$\frac{1}{T} U(r, \omega) = \int_{\mathcal{B}_r} \bar{\mathbf{C}}(\mathbf{x}) \cdot \mathbb{K}(\mathbf{x}, \omega) \mathbf{C}(x) \, dV. \tag{7.6}$$

where $\mathbb{K} : \mathcal{R} \times \mathcal{R}^3 \mapsto Lin(Sym)$ is a positive-definite tensor, the forms of which, for the two measures, will be discussed later.

Rather than use Proposition 4.1 where the constant c_0 does not depend on the measure used, we follow the line of reasoning of Toupin [8] in his original work on Saint-Venant's principle in linear elasticity, to replace c_0 with a parameter that depends on the energy measure. Furthermore, we adopt a different form of (6.13). The dependence on $\mathbf{x}$ will be indicated only when necessary.

The tensor $\mathbb{K}(\mathbf{x},\omega)$ is Hermitean and thus has real eigenvalues which must also be positive since $\mathbb{K}$ is positive-definite. Let $\lambda_m^U(\mathbf{x},\omega)$ and $\lambda_M^U(\mathbf{x},\omega)$ be the minimum and maximum eigenvalues. Then

$$\lambda_m^U(\mathbf{x},\omega)|\mathbf{C}(\mathbf{x})|^2 \leq \bar{\mathbf{C}}(\mathbf{x})\cdot\mathbb{K}(\mathbf{x},\omega)\mathbf{C}(\mathbf{x}) \leq \lambda_M^U(\mathbf{x},\omega)|\mathbf{C}(\mathbf{x})|^2 \tag{7.7}$$

Also, from (A2.5),

$$\frac{1}{T}\int_0^T |\mathbf{T}(t)|^2 dt = \bar{\mathbf{C}}(\mathbf{x})\cdot\mathbf{W}(\mathbf{x},\omega)\mathbf{C}(\mathbf{x}), \quad \mathbf{W}(\omega) = 2\ \mathbb{M}^*\mathbf{x},\omega)\ \mathbb{M}(\mathbf{x},\omega) \tag{7.8}$$

Let $\mu_M^2(\mathbf{x},\omega)$ be the largest eigenvalue of the positive-definite tensor $\mathbf{W}(\mathbf{x},\omega)$. Then, for almost all points $\mathbf{x}\in\bar{\mathcal{B}}$

$$\frac{1}{T}\int_0^T |\mathbf{T}(t)|^2 dt \leq \mu_M^2(\mathbf{x},\omega)|\mathbf{C}(\mathbf{x})|^2 \leq \kappa_U(\omega)\bar{\mathbf{C}}(\mathbf{x})\cdot\mathbb{K}(\mathbf{x},\omega)\mathbf{C}(\mathbf{x}) \tag{7.9}$$

where

$$\kappa_U(\omega) = \operatorname*{ess\,sup}_{\mathbf{x}\in\bar{\mathcal{B}}} \frac{\mu_M^2(\mathbf{x},\omega)}{\lambda_m^U(\mathbf{x},\omega)} \tag{7.10}$$

so that

$$\frac{1}{T}\int_0^T\int_{\mathcal{S}_r} |\mathbf{T}(t)|^2\ dSdt \leq \kappa_U(\omega)\int_{\mathcal{S}_r} \bar{\mathbf{C}}(\mathbf{x})\cdot\mathbb{K}(\mathbf{x},\omega)\mathbf{C}(\mathbf{x})\ dS. \tag{7.11}$$

Equation (7.9) replaces Proposition 4.1. From (7.6), we have

$$\frac{1}{T}\frac{dU(\mathbf{x},\omega)}{dr} = -\int_{\mathcal{S}_r} \bar{\mathbf{C}}(\mathbf{x})\cdot\mathbb{K}(\mathbf{x},\omega)\mathbf{C}(\mathbf{x})\ dS. \tag{7.12}$$

Let

$$\tilde{\mathbf{u}}(\mathbf{x},t) = \mathbf{d}(\mathbf{x})e^{i\omega t} + \bar{\mathbf{d}}(\mathbf{x})e^{-i\omega t} \tag{7.13}$$

which yields (A2.1) (in the real frequency limit) provided that

$$\mathbf{C} = sym\nabla\mathbf{d}. \tag{7.14}$$

Inequality (6.9) gives that

$$\begin{aligned}\frac{1}{T}\int_0^T\int_{\mathcal{S}_r} |\tilde{\mathbf{u}}(t)|^2\ dSdt &= 2\int_{\mathcal{S}_r} |\mathbf{d}(\mathbf{x})|^2\ dS \\ &\leq \frac{2}{b(r,\omega)}\int_{\mathcal{B}_r} \bar{\mathbf{C}}(\mathbf{x})\cdot\mathbb{K}(\mathbf{x},\omega)\mathbf{C}(\mathbf{x})\ dV\end{aligned} \tag{7.15}$$

where $b(r,\omega)$ for any $\mathcal{S}_r$, $\mathcal{B}_r$, depends on $\mathbb{K}$. It is greatest for the largest energy measure. Thus, we obtain

$$\int_0^T \int_{\mathcal{S}_r} |\tilde{\mathbf{u}}(t)|^2 \, dSdt \leq \frac{2}{b(r,\omega)} U(r,\omega) \tag{7.16}$$

by virtue of (7.6). Using (7.16) in (6.16), we obtain, instead of (6.18),

$$U(r,\omega) \leq \frac{2}{b(r,\omega)} \int_0^T \int_{\mathcal{S}_r} |\mathbf{T}(t)|^2 \, dSdt \leq \alpha_U(\omega) T \int_{\mathcal{S}_r} \bar{\mathbf{C}}(\mathbf{x}) \cdot \mathbb{K}(\mathbf{x},\omega) \mathbf{C}(\mathbf{x}) \, dS \tag{7.17}$$

where

$$\alpha_U(\omega) = \frac{2\kappa_U(\omega)}{\beta(\omega)}, \quad \beta(\omega) = \min_{0 \leq r \leq L-l} b(r,\omega). \tag{7.18}$$

Equation (7.11) has been invoked to provide the second inequality. Using (7.12), we find that (6.21) is replaced by

$$U(r,\omega) + \alpha_U(\omega) \frac{dU(r,\omega)}{dr} \leq 0 \tag{7.19}$$

the solution of which yields (7.1).

□

We see that the larger the measure chosen for U_ψ the faster the decay. The choice of $U_E(r,\omega)$ provides the most rapid decay, while $U_\psi(r,\omega)$ falls off more slowly. This means that the measure $U_{\psi_m}(r,\omega)$ based on the minimal free energy yields the *most conservative estimate* in terms of the frequency dependent spatial decay of the energy. Indeed, for the given frequency of application of external loads, more distance is required for the energy to decay to its asymptotic value.

Following the same reasoning of the previous section we may infer some information about the decay of a suitable measure of the state of points encountered moving from the loaded boundary (at the given frequency ω) into the body.

Proposition 7.2. *Let $\sigma_\omega(\mathbf{x})$ be the state of the point $\mathbf{x}$ at the prescribed frequency ω and let $\|\cdot\|$ be the norm defined by (6.22). The measure*

$$\Sigma(r,\omega) := \int_0^T \int_{\mathcal{B}_r} \|\sigma_\omega(\mathbf{x})\|^2 dV dt \tag{7.20}$$

of the state of material points located in the region $\mathcal{B}_r$ spatially decays as in (7.1) with U given by U_{ψ_m} where $U(r,\omega)$ and $U(0,\omega)$ are replaced by $\Sigma(r,\omega)$ and $\Sigma(0,\omega)$ respectively.

The forms of the tensor $\mathbb{K}(\omega)$ in (7.6) for the two choices of U in (7.5) are given as follows. For $U = U_E$, we determine from (A2.13) that

$$\mathbb{K}_E(\omega) = 2\, \mathbb{R}(\omega) \tag{7.21}$$

where $\mathbb{R}$ is the Hermitean part of the complex modulus tensor, defined in (3.30). For $U = U_{\psi_m}$ where the minimum free energy ψ_m is used, we have

$$\mathbb{K}_{\psi_m}(\omega) = \mathbb{B}(\omega) \tag{7.22}$$

from (A2.24), (A2.23) and (3.27) in terms of the factors of $\mathbb{H}$. Note that (7.5) gives that

$$2\,\mathbb{R}(\omega) \geq \mathbb{B}(\omega) \geq 0 \quad \forall\, \omega \in \mathcal{R}$$

or

$$\begin{aligned} \mathbb{R}(\omega) &\geq \mathbb{D}(\omega) - \omega\, \mathbb{R}'(\omega) \\ &\geq \omega\, \mathbb{R}'(\omega) - \mathbb{D}(\omega) \quad \forall\, \omega \in \mathcal{R}. \end{aligned} \tag{7.23}$$

It should be observed that for exponential models with non-negative coefficients or density functions [49, 59], we have

$$\omega\, \mathbb{R}'(\omega) \geq 0.$$

The rate of decay depends on the rate of application of the load as reflected in the frequency. In the low frequency limit, $\mathbb{B}(\omega)$ tends to $\mathbb{R}(\omega)$ and we have

$$\mathbb{K}_E(0) = 2\,\mathbb{R}(0) = 2\mathbb{G}_\infty; \quad \mathbb{K}_{\psi_m}(0) = \mathbb{R}(0) = \mathbb{G}_\infty. \tag{7.24}$$

Also, as ω gets larger, $\mathbb{K}_E(\omega)$ increases to $2\mathbb{G}_0$. Since $\mathbb{H}'_\pm$ tend to zero at large ω, we have that $\mathbb{K}_{\psi_m}(\omega)$ tends to $\mathbb{G}_0$ at large ω.

In the case of the exponential models referred to above, for example, we see that $\mathbb{K}_E$ may increase reasonably smoothly. Though there may be complicated behaviour at intermediate frequencies, particularly in $\mathbb{K}_{\psi_m}$, both $\mathbb{K}_E$ and $\mathbb{K}_{\psi_m}$ are always non-negative. Broadly, therefore, as the rate of load application increases, the larger U_E, U_{ψ_m} become and, referring to the statement after (7.19), the larger their rates of decay with r.

It must be noted however that the validity of the quasi-static approximation comes into question in the high frequency limit.

While a precise comparison of the results for a general history given by (6.10) and the results for a sinusoidal history is not possible, some observations can be made. We compare the sinusoidal results for very low frequencies and the results for a general history, since both involve the equilibrium modulus $\mathbb{G}_\infty$ (though of course the sinusoidal history in the limit $\omega = 0$ is not of great interest, being in fact the stationary history). Inhomogeneity effects are neglected. We assume that $|\mathbb{G}_\infty| \geq |\mathbb{G}_\infty - \mathbb{G}_0|$ in (4.2) and, remembering (A1.6), replace $k|\mathbb{G}_\infty|$ by $tr\,\mathbb{G}_\infty$. Thus α in (6.10) becomes

$$\alpha = \frac{8\, tr\, \mathbb{G}_\infty}{\beta}. \tag{7.25}$$

Let the eigenvalues of $\mathbb{G}_\infty$ be $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_6$. Then

$$\mu_M^2(0) = 2\gamma_1^2. \tag{7.26}$$

Also

$$\begin{aligned} \lambda_m(0) &= n\gamma_6 \\ n &= 2, \quad U = U_E \\ &= 1, \quad U = U_\psi. \end{aligned} \tag{7.27}$$

Then,

$$\kappa(0) = \frac{2\gamma_1^2}{n\gamma_6}. \tag{7.28}$$

Now, from a comparison of (6.13) and (7.15), and noting the definition of β and $\beta(\omega)$ in (6.10) and (7.18) , we find that

$$\beta(0) = n\beta. \tag{7.29}$$

Therefore

$$\alpha(0) = \frac{4\gamma_1^2}{n^2\gamma_6\beta}. \tag{7.30}$$

Thus

$$\frac{\alpha}{\alpha(0)} = \frac{2n^2\gamma_6}{\gamma_1^2} tr\,\mathbb{G}_\infty. \tag{7.31}$$

For the case of an isotropic solid, $\mathbb{G}_\infty$ has two distinct eigenvalues, $3\lambda + 2\mu$ with multiplicity one and 2μ with multiplicity five, where λ, μ are the Lamé constants. In this case, we find

$$\frac{\alpha}{\alpha(0)} = \frac{6n^2(1-2\nu)(2-3\nu)}{(1+\nu)^2}. \tag{7.32}$$

where ν is the equilibrium Poisson's ratio. For an incompressible medium, the estimate (7.1) yields no decay. For $\nu \leq 1/3$, we see that (7.1) near $\omega = 0$ gives faster spatial decay than (6.10).

The exponential decay exhibited in (6.10) and (7.1) express the content of the Saint-Venant Principle, which states that any solution of the problem specified by (6.1) - (6.3) is well approximated by a solution of the relaxed Saint-Venant problem, namely that for which the stress and moment on $\mathcal{S}_0$ are independent of space coordinates, while obeying (6.2) and (6.3).

The forms of solutions of the relaxed problem are discussed in detail in Chapter 20 of [1].

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Appendix 1: Notation and basic assumptions for a linear viscoelastic solid

Let Sym be the space of symmetric second order tensors acting on $\mathcal{R}^3$ viz. $Sym := \{\mathbf{M} \in Lin(\mathcal{R}^3) : \mathbf{M} = \mathbf{M}^\top\}$, where the superscript "$^\top$" denotes the transpose. Operating on Sym is the space of fourth order tensors $Lin(Sym)$.

It is well known that Sym is isomorphic to $\mathcal{R}^6$. In particular, for every $\mathbf{L}, \mathbf{M} \in Sym$, if $\mathbf{C}_i$, $i = 1, ..., 6$ is an orthonormal basis of Sym with respect to the usual inner product in $Lin(\mathcal{R}^3)$, namely $tr(\mathbf{L}\mathbf{M}^\top)$, it is clear that the representation

$$\mathbf{L} = \sum_{i=1}^{6} L_i \mathbf{C}_i \ , \qquad \mathbf{M} = \sum_{i=1}^{6} M_i \mathbf{C}_i \tag{A1.1}$$

yields $tr(\mathbf{L}\mathbf{M}^\top) = \sum_{i=1}^{6} L_i M_i$. Therefore, we can treat each tensor of Sym as a vector in $\mathcal{R}^6$ and denote by $\mathbf{L} \cdot \mathbf{M}$ the inner product between elements of Sym, viz.

$$\mathbf{L} \cdot \mathbf{M} = tr(\mathbf{L}\mathbf{M}^\top) = tr(\mathbf{L}\mathbf{M}) = \sum_{i=1}^{6} L_i M_i$$

and $|\mathbf{M}|^2 = \mathbf{M} \cdot \mathbf{M}$. Consequently [33] any fourth order tensor $\mathbb{K} \in Lin(Sym)$ will be identified with an element of $Lin(\mathcal{R}^6)$ by the representation

$$\mathbb{K} = \sum_{i,j=1}^{6} K_{ij} \mathbf{C}_i \otimes \mathbf{C}_j \tag{A1.2}$$

and $\mathbb{K}^\top$ means the transpose of $\mathbb{K}$ as an element of $Lin(\mathcal{R}^6)$. According to (A1.2), the norm $|\mathbb{K}|$ of $\mathbb{K} \in Lin(Sym)$ may be given by

$$|\mathbb{K}|^2 = tr\left(\mathbb{K}\mathbb{K}^\top\right) = \left(\sum_{i,j=1}^{6} K_{ij} K_{ij}\right) \ .$$

In the sequel we deal with complex valued tensors. Denoting by Ω the complex plane and by $Sym(\Omega)$ and $Lin(Sym(\Omega))$ respectively the tensors represented by the forms (A1.1) and (A1.2) with $L_i, M_i, K_{ij} \in \Omega$, then the norms $|\mathbf{M}|$ and $|\mathbb{K}|$ of $\mathbf{M} \in Sym(\Omega)$ and $\mathbb{K} \in Lin(Sym(\Omega))$ will be given respectively by

$$|\mathbf{M}|^2 = \left(\mathbf{M} \cdot \overline{\mathbf{M}}\right) \ , \qquad |\mathbb{K}|^2 = tr\left(\mathbb{K}\mathbb{K}^*\right) = \left(\sum_{i,j=1}^{6} K_{ij}\overline{K}_{ij}\right) \ , \tag{A1.3}$$

where the overhead bar indicates complex conjugate and $\mathbb{K}^* = \overline{\mathbb{K}}^\top$ is the hermitian conjugate.

The following result will be required. For $\mathbf{L} \in Sym$ and the real symmetric positive-definite tensor $\mathbb{K} \in Lin(Sym)$

$$\mathbb{K}\mathbf{L} \cdot \mathbb{K}\mathbf{L} \leq |\mathbb{K}|\mathbf{L} \cdot \mathbb{K}\mathbf{L} \tag{A1.4}$$

Also, for $\mathbf{L} \in Sym(\Omega)$,

$$\sup_{\mathbf{L} \in Sym(\Omega)} \mathbb{K}\mathbf{L} \cdot \bar{\mathbf{L}} \leq tr(\mathbb{K})|\mathbf{L}|^2. \tag{A1.5}$$

Note that $tr\mathbb{K}$ is the sum of the (real) eigenvalues of $\mathbb{K}$. We have

$$tr\mathbb{K} \leq k|\mathbb{K}|. \tag{A1.6}$$

where $k > 1$ depends on the dimensions of the normed space.

The symbols $\mathcal{R}^+$ and $\mathcal{R}^{++}$ denote the non-negative reals and the strictly positive reals, respectively, while $\mathcal{R}^-$ and $\mathcal{R}^{--}$ denote the non-positive and strictly negative reals.

For function $f : \mathcal{R} \to \mathcal{V}$, where $\mathcal{V}$ is a finite-dimensional vector space, in particular in the present context Sym or $Lin(Sym)$, let f_F, denote its *Fourier transform* viz. $f_F(\omega) = \int_{-\infty}^{\infty} f(s)e^{-i\omega s}ds$. Also, we define

$$\begin{aligned} f_+(\omega) &= \int_0^\infty f(s)e^{-i\omega s}ds, \qquad & f_-(\omega) &= \int_{-\infty}^0 f(s)e^{-i\omega s}ds \\ f_s(\omega) &= \int_0^\infty f(s)\sin\omega s\, ds, \qquad & f_c(\omega) &= \int_0^\infty f(s)\cos\omega s\, ds \end{aligned} \tag{A1.7}$$

The relations defining f_F and (A1.7) are to be understood as applying to each component of the tensor quantities involved. Some constraint must be placed on these components to ensure that the Fourier transforms exist. It is assumed that all components of tensors in the time domain belong to $L^2(\mathcal{R})$ (or $L^2(\mathcal{R}^\pm)$ in the case of $f_\pm$) so that in the frequency domain, they belong to $L^2(\mathcal{R})$ [34, 35]. Further restrictions on the allowed function spaces will be imposed below.

When $f : \mathcal{R}^+ \to \mathcal{V}$ we can always extend the domain of f to $\mathcal{R}$, by considering its *causal* extension *viz.*

$$f(s) = \begin{cases} f(s) & \text{for } s \geq 0 \\ \\ 0 & \text{for } s < 0 \end{cases} \tag{A1.8}$$

in which case

$$f_F(\omega) = f_+(\omega) = f_c(\omega) - i f_s(\omega) \tag{A1.9}$$

We shall need to consider the Fourier transform of functions that do not go to zero at large times and thus do not belong to L^2 for the appropriate domain. In particular, let $f(s)$ in (A1.8) be given by a constant a for all s. The standard procedure is adopted of introducing an exponential decay factor, calculating the Fourier transform and then letting the time decay constant tend to infinity. Thus, we obtain

$$\begin{aligned} f_+(\omega) &= \frac{a}{i\omega^-} \\ \omega^- &= \lim_{\alpha \to 0} (\omega - i\alpha) \end{aligned} \tag{A1.10}$$

The corresponding result for a constant function defined on R^- is obtained by taking the complex conjugates of this relationship. Also, if f is a function defined on $\mathcal{R}^-$ and if $\lim_{s \to -\infty} f(s) = b$ where the components of the function $g : R^- \to \mathcal{V}$ defined by $g(s) = f(s) - b$ belong to $L^2(\mathcal{R}^-)$, then

$$f_-(\omega) = g_-(\omega) - \frac{b}{i\omega^+} \tag{A1.11}$$

Again, taking complex conjugates gives the result for functions defined on $\mathcal{R}^+$. This procedure amounts to defining the Fourier transform of such functions as the limit of the transforms of a sequence of functions in L^2. The limit is to be taken after integrations over ω are carried out if the ω^{-1} results in a singularity in the integrand. Generally, in the present application, the ω^{-1} produces no such singularity and the limiting process is redundant.

The complex frequency plane Ω will play an important role in our discussions. We define the following sets:

$$\Omega^+ = \{\omega \in \Omega : Im\omega \geq 0\}\ , \quad \Omega^{(+)} = \{\omega \in \Omega : Im\omega > 0\}\,. \tag{A1.12}$$

Analogous meanings are assigned to Ω^- and $\Omega^{(-)}$.

The quantities $f_\pm$ defined by (A1.7) are analytic in $\Omega^{(\mp)}$ respectively. This analyticity is extended by assumption to an open set containing the real axis and therefore to $\Omega^\mp$. The function f_+ may be defined by (A1.7) and analytic on a portion of Ω^+ if for example f decays exponentially at large times. Over the remaining portion of Ω^+, on which the integral definition is meaningless, f_+ is defined by analytic continuation.

Appendix 2: Sinusoidal histories

This topic is discussed in a more general context in [1], page 258.

Consider a current value and history of strain $(\mathbf{E}, \mathbf{E}^t)$ defined by

$$\mathbf{E}(t) = \mathbf{C}e^{i\omega_- t} + \bar{\mathbf{C}}e^{-i\omega_+ t}, \quad \mathbf{E}^t(s) = \mathbf{E}(t-s) \tag{A2.1}$$

where $\mathbf{C} \in Sym$ is an amplitude and $\bar{\mathbf{C}}$ its complex conjugate, both of which may depend on $\mathbf{x}$ in the present application.. Also

$$\omega_- = \omega_0 - i\eta, \quad \omega_+ = \bar{\omega}_-, \quad \omega \in \mathcal{R}, \quad \eta \in \mathcal{R}^{++}. \tag{A2.2}$$

The quantity η is introduced to ensure finite results. The quantity $\mathbf{E}^t_+$ has the form

$$\mathbf{E}^t_+(\omega) = \mathbf{C}\frac{e^{i\omega_- t}}{i(\omega + \omega_-)} + \bar{\mathbf{C}}\frac{e^{-i\omega_+ t}}{i(\omega - \omega_+)}. \tag{A2.3}$$

The stress, given by (2.1), has the form

$$\begin{aligned}
\mathbf{T}(t) &= \mathbb{G}_0\left\{\mathbf{C}e^{i\omega_- t} + \bar{\mathbf{C}}e^{-i\omega_+ t}\right\} + \mathbf{T}_1(t), \\
\mathbf{T}_1(t) &= \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\, ds \\
&= \dot{\mathbb{G}}_F(\omega_-)\mathbf{C}e^{i\omega_- t} + \dot{\mathbb{G}}_F(-\omega_+)\bar{\mathbf{C}}e^{-i\omega_+ t}
\end{aligned} \tag{A2.4}$$

or, in terms of the tensor complex modulus (3.30), we have

$$\mathbf{T}(t) = \mathbb{M}(\omega_-)\mathbf{C}e^{i\omega_- t} + \mathbb{M}(-\omega_+)\bar{\mathbf{C}}e^{-i\omega_+ t}. \tag{A2.5}$$

Note that, in view of (2.5) and the remark after (2.8), we have $\mathbb{M}^\top = \mathbb{M}$, so that

$$\mathbb{M}(-\omega_+) = \overline{\mathbb{M}(\omega_-)} = \mathbb{M}^*(\omega_-) \tag{A2.6}$$

where $\mathbb{M}^*$ is the Hermitean conjugate of $\mathbb{M}$. Alternatively, we find, from (2.12) and (A2.3), that the stress has the form

$$\begin{aligned}
\mathbf{T}(t) &= \mathbb{G}_0\mathbf{E}(t) + \mathbf{N}(-\omega_-)\mathbf{C}e^{i\omega_- t} + \mathbf{N}(\omega_+)\bar{\mathbf{C}}e^{-i\omega_- t}, \\
\mathbf{N}(z) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\dot{\mathbb{G}}_s(\omega')}{\omega' - z}d\omega', \quad z \in \Omega^{(+)}
\end{aligned} \tag{A2.7}$$

and comparison with (A2.5) yields that

$$\mathbb{G}_0 + \mathbf{N}(z) = \mathbb{M}(-z), \quad z \in \Omega^{(+)} \tag{A2.8}$$

which can be shown to be equivalent to a "Dispersion Relation", ([58] for example). Using the relations $\mathbb{M}(-z) = \mathbb{M}^*(\bar{z})$ (see (A2.6)) $\mathbf{N}^*(z) = \mathbf{N}(\bar{z})$, $z \in$

$\Omega^{(+)}$ together with (A2.8), we obtain

$$\mathbb{G}_0 + \mathbf{N}(z) = \mathbb{M}(z), \quad z \in \Omega^{(-)}. \tag{A2.9}$$

The work $W(t)$ done on the material to achieve the state $(\mathbf{E}(t), \mathbf{E}^t)$ is given by (2.25). Equations (A2.1) and (A2.5) yield

$$\begin{aligned} W(t) = \frac{1}{2}\left[\mathbf{C}\cdot \mathbb{M}(\omega_-)\mathbf{C}e^{2i\omega_- t} + \bar{\mathbf{C}}\cdot \mathbb{M}(-\omega_+)\bar{\mathbf{C}}e^{-2i\omega_+ t}\right] \\ + \bar{\mathbf{C}}\cdot\left[\omega_- \mathbb{M}(-\omega_+) - \omega_+ \mathbb{M}(\omega_-)\right]\mathbf{C}\frac{e^{i(\omega_- - \omega_+)t}}{(\omega_- - \omega_+)} \end{aligned} \tag{A2.10}$$

where the symmetry of $\mathbb{M}$ has been used. It will be observed that the last term diverges in the limit $\eta \to 0$, which is entirely reasonable from a physical point of view.

The Fourier transform of the relative history $\mathbf{E}^t_r(s) = \mathbf{E}^t(s) - \mathbf{E}(t)$, namely $\mathbf{E}^t_{r+}(\omega)$ has the form

$$\mathbf{E}^t_{r+}(\omega) = \mathbf{E}^t_+(\omega) - \frac{\mathbf{E}(t)}{i\omega^-} = -\mathbf{C}\frac{\omega_-}{\omega^-}\frac{e^{i\omega_- t}}{i(\omega+\omega_-)} + \bar{\mathbf{C}}\frac{\omega_+}{\omega^-}\frac{e^{-i\omega_+ t}}{i(\omega-\omega_+)} \tag{A2.11}$$

using the notation (A1.10). The quantity $\mathbf{T}(t)\cdot\mathbf{E}(t)$ has the form

$$\begin{aligned} \mathbf{T}(t)\cdot\mathbf{E}(t) = \mathbf{C}\cdot \mathbb{M}(\omega_-)\mathbf{C}e^{2i\omega_- t} + \bar{\mathbf{C}}\cdot \mathbb{M}(-\omega_+)\bar{\mathbf{C}}e^{-2i\omega_+ t} \\ + \bar{\mathbf{C}}\cdot(\mathbb{M}(\omega_-) + \mathbb{M}(-\omega_+))\mathbf{C}e^{i(\omega_- - \omega_+)t} \end{aligned} \tag{A2.12}$$

where the symmetry of $\mathbb{M}$ has again been used. Thus, in the limit $\eta \to 0$

$$\int_0^T \mathbf{T}(t)\cdot\mathbf{E}(t)dt = 2T\bar{\mathbf{C}}\cdot \mathbb{R}(\omega)\mathbf{C}. \tag{A2.13}$$

Observe that the generalization of (3.2) to the complex plane is

$$\mathbb{H}_+(\omega) = \mathbb{H}^*_-(\bar{\omega}). \tag{A2.14}$$

From the properties $\overline{\mathbb{H}(\omega)} = \mathbb{H}(\bar{\omega})$ and $\mathbb{H}(-\omega) = \mathbb{H}(\omega)$, it follows that we can choose $\mathbb{H}_\pm$ such that

$$\overline{\mathbb{H}_\pm(\omega)} = \mathbb{H}_\pm(-\bar{\omega}) \tag{A2.15}$$

giving

$$\mathbb{H}^\top_\pm(\omega) = \mathbb{H}_\mp(-\omega). \tag{A2.16}$$

If $\mathbb{H}_\pm$ commute (see the discussion before (5.8)) we have further that

$$\overline{\mathbb{H}_\pm(\omega)} = \mathbb{H}_\mp(\bar{\omega}). \tag{A2.17}$$

Also [49] $\mathbb{H}_\pm$ are symmetric for all frequencies as are products of these factors at the same or different frequencies.

The minimum free energy $\psi_m(t)$ is given by (3.8). We evaluate the integrals in (3.6) by closing the contours on $\Omega^{(+)}$ to obtain

$$\mathbf{p}_+^t(\omega) = -\left[\frac{e^{i\omega_- t}}{i(\omega+\omega_-)}\ \mathbb{H}_-(-\omega_-)\mathbf{C} + \frac{e^{-i\omega_+ t}}{i(\omega-\omega_+)}\ \mathbb{H}_-(\omega_+)\bar{\mathbf{C}}\right] \tag{A2.18}$$

and

$$\mathbf{p}_-^t(\omega) = \ \mathbb{H}_-(\omega)\mathbf{E}_r^t(\omega) + \mathbf{p}_+^t(\omega)$$

The expression for $\psi_m(t)$ can be obtained from $(3.11)_2$ combined with (A2.10) and (A2.18). From (A2.18) we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{p}_+^t(\omega)|^2 d\omega = &-\frac{ie^{2i\omega_- t}}{2\omega_-}\mathbf{C}\cdot\ \mathbb{H}_-^*(\omega_+)\ \mathbb{H}_-(-\omega_-)\mathbf{C}\\ &+\frac{ie^{-2i\omega_+ t}}{2\omega_+}\bar{\mathbf{C}}\cdot\ \mathbb{H}_-^*(-\omega_-)\ \mathbb{H}_-(\omega_+)\bar{\mathbf{C}}\\ &-\frac{2ie^{i(\omega_- -\omega_+)t}}{(\omega_- - \omega_+)}\bar{\mathbf{C}}\cdot\ \mathbb{H}_+(-\omega_+)\ \mathbb{H}_-(-\omega_-)\mathbf{C}\end{aligned} \tag{A2.19}$$

where (A2.14) - (A2.16) have been used. It will be observed that the last term diverges in the limit $\eta \to 0$. The quantity given by (A2.19) in the limit $\eta \to 0$ is in fact the total dissipation over history (its derivative is the rate of dissipation (3.17) so this divergence is an expression of a physically obvious fact. Equation (A2.19) can also be deduced from (3.22) and (A2.3). From (A2.10), (A2.19) and $(3.11)_2$ we obtain

$$\begin{aligned}\psi_m(t) = \mathbf{C}\cdot\ \mathbb{B}_1(\omega,\eta)\mathbf{C}\frac{e^{2i\omega_- t}}{2\omega_-} &+ \bar{\mathbf{C}}\cdot\ \mathbb{B}_1^*(\omega,\eta)\bar{\mathbf{C}}\frac{e^{-2i\omega_+ t}}{2\omega_+}\\ &+\bar{\mathbf{C}}\cdot\ \mathbb{B}_2(\omega,\eta)\mathbf{C}\frac{e^{i(\omega_- -\omega_+)t}}{\omega_- - \omega_+}\end{aligned} \tag{A2.20}$$

where

$$\begin{aligned}\mathbb{B}_1(\omega,\eta) &= \omega_-\ \mathbb{M}(\omega_-) - i\ \mathbb{H}_+(\omega_-)\ \mathbb{H}_-(-\omega_-)\\ \mathbb{B}_2(\omega,\eta) &= \omega_-\ \mathbb{M}(-\omega_+) - \omega_+\ \mathbb{M}(\omega_-) + 2i\ \mathbb{H}_+(-\omega_+)\ \mathbb{H}_-(-\omega_-).\end{aligned} \tag{A2.21}$$

This can also be shown by starting from (3.21), with the aid of (A2.8), (A2.9) and judicious use of partial fractions. In the limit $\eta \to 0$ we obtain (replacing ω_0 by $-\omega$)

$$\psi_m(t) = \mathbf{C}\cdot\ \mathbb{B}_1(\omega)\mathbf{C}e^{-2i\omega t} + \bar{\mathbf{C}}\cdot\ \mathbb{B}_1^*(\omega)\bar{\mathbf{C}}e^{2i\omega t} + \bar{\mathbf{C}}\cdot\ \mathbb{B}_2(\omega)\mathbf{C} \tag{A2.22}$$

where

$$
\begin{aligned}
\mathbb{B}_1(\omega) &= \frac{1}{2}\left[\, \mathbb{M}(-\omega) - \frac{i}{\omega}\, \mathbb{H}_+(-\omega)\, \mathbb{H}_-(\omega)\right] \\
\mathbb{B}_2(\omega) &= \mathbb{B}(\omega) = \mathbb{R}(\omega) - \omega\, \mathbb{R}'(\omega) + \mathbb{D}(\omega)
\end{aligned}
$$

where $\mathbb{D}$ is defined by (3.27). A prime denotes differentiation. If $\mathbb{H}_\pm$ commute then $\mathbb{B}_1$ simplifies to

$$
\mathbb{B}_1(\omega) = \frac{1}{2}\left[\, \mathbb{M}(-\omega) - \frac{i}{\omega}\, \mathbb{H}_+^2(-\omega)\right]. \tag{A2.23}
$$

We have

$$
\int_0^T \psi_m(t)dt = T\bar{\mathbf{C}}\, \mathbb{B}(\omega)\mathbf{C}. \tag{A2.24}
$$

Note that $\mathbb{B}$ must be a non-negative quantity in general for all $\omega \in \mathcal{R}$. We recall from (3.27) that $\mathbb{D}$ is non-negative for all $\omega \in \mathcal{R}$.

The rate of dissipation is given by (3.17) and (3.18). Closing on Ω^+, we find that

$$
\mathbf{K}(t) = \mathbb{H}_-(-\omega_-)\mathbf{C}e^{i\omega_- t} + \mathbb{H}_-(\omega_+)\bar{\mathbf{C}}e^{-i\omega_+ t} \tag{A2.25}
$$

and

$$
\begin{aligned}
D(t) = \mathbf{C}\cdot\, \mathbb{H}_+(\omega_-)\, \mathbb{H}_-(-\omega_-)\mathbf{C}e^{2i\omega_- t} + \bar{\mathbf{C}}\cdot\, \mathbb{H}_+(-\omega_+)\, \mathbb{H}_-(\omega_+)\bar{\mathbf{C}}e^{-2i\omega_+ t} \\
+2\bar{\mathbf{C}}\cdot\, \mathbb{H}_+(-\omega_+)\, \mathbb{H}_-(-\omega_-)\mathbf{C}e^{i(\omega_- - \omega_+)t}
\end{aligned} \tag{A2.26}
$$

As $\eta \to 0$, replacing ω_0 by $-\omega$ we obtain

$$
\begin{aligned}
D(t) = \mathbf{C}\cdot\, \mathbb{H}_+(-\omega)\, \mathbb{H}_-(\omega)\mathbf{C}e^{-2i\omega t} + \bar{\mathbf{C}}\cdot\, \mathbb{H}_+(\omega)\, \mathbb{H}_-(-\omega)\bar{\mathbf{C}}e^{2i\omega t} \\
+2\bar{\mathbf{C}}\cdot\, \mathbb{H}(\omega)\mathbf{C}.
\end{aligned} \tag{A2.27}
$$

If $\mathbb{H}_\pm$ commute then the operators in the first two terms by $\mathbb{H}_-^2(\omega)$ and $\mathbb{H}_+^2(\omega)$ respectively

One may check that (4.22) holds, using (A2.1), (A2.5), (A2.23) and (A2.27).

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