Stable Disarrangement Phases of Granular Media II: Stable Phases of a Model Aggregate Cannot Support Tensile Tractions

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Abstract

We introduce in Part II the notion of a stable disarrangement phase G corresponding to a given macroscopic deformation F of an elastic aggregate. Specifically, a stable disarrangement phase G minimizes the free energy density $\Psi(G')$ among all tensors G' that satisfy the consistency relation $D\Psi(G')(F^T - G'^T) = 0$ and the accommodation inequality $0 < \det G' \leq \det F$. The classification of disarrangement phases obtained in Part I for a model elastic aggregate is employed in Part II to establish the main result of the present study: stable disarrangement phases of the model aggregate cannot support tensile tractions. This result provides an example of a no-tension material whose response in compression is non-linear, in contrast to standard descriptions of no-tension materials in which the response in compression is assumed at the outset to be linear. Moreover, our main result suggests that the present field theory of elastic aggregates provides a broad setting for the study of structures containing masonry-like elements.

1 Introduction

In Part I of the present study, we used a previously formulated theory [1] of elastic bodies undergoing disarrangements in order to provide in the context of elastic aggregates the *consistency* relation $D\Psi(G)(F^T - G^T) = 0$, a tensorial relation whose form is determined by the Helmholtz free energy response Ψ of the pieces of the aggregate, and also to provide the *accommodation* inequality, $0 < \det G \leq \det F$ that guarantees that the macroscopic deformation F provides enough volume to accommodate the submacroscopic geometrical changes G associated with the pieces of the aggregate. Together the consistency relation and accommodation inequality determine which tensors G are compatible with a given macroscopic deformation gradient F, and in Part I we defined a *disarrangement phase* corresponding to F to be a tensor G that satisfies both the consistency relation and the accommodation inequality for the given F. (A derivation of the consistency relation together with a submacroscopic interpretation are provided in the Appendix to Part II.)

For a broad class of free energy response functions Ψ the collection of disarrangement phases corresponding to F includes not only the compact phase $G = F$ in which the pieces of the aggregate all undergo the same macroscopic deformation F, but also loose phases $G = \zeta_{\min} R$ in which the pieces undergo a specific dilatation $\zeta_{\min} I$ determined by Ψ followed by an arbitrary rotation R and so attain a stress-free state, all provided that the accommodation inequality $\zeta_{\min}^3 = \det G \le \det F$ is satisfied.

The analysis given in Part I provides for a specific two-parameter class $\Psi_{\alpha\beta}$ a complete classification of all of the disarrangement phases corresponding to a given but arbitrary macroscopic deformation gradient F. That classification as well as examples presented elsewhere [1, 2] show that there are typically multiple disarrangement phases G corresponding to a given macroscopic deformation gradient F, and it is important to single out those disarrangement phases that are energetically favorable. An analogous situation arises in the statistical mechanical modelling of the macroscopic response of nematic elastomers in which a unit vector n, along with the macroscopic deformation gradient F, emerges from the statistical modelling. For a given tensor F the vector n is determined by minimizing the resulting energy with respect to that vector [3, 4, 5]. In a similar spirit within the context of micromechanics (see e.g. [6, 7, 8, 9, 10, 11]), the widely used method of representative volume elements fixes the macroscopic deformation gradient F and determines the response of the representative volume element in terms of F by solving a boundary value problem which, in some cases, amounts to finding the minima of a corresponding energy functional. Accordingly, we define here a *stable disarrangement phase* corresponding to F to be a disarrangement phase G corresponding to F that minimizes the Helmholtz free energy among all disarrangement phases corresponding to F.

Our principal goal in Part II of the present study is to establish for the two-parameter class of elastic aggregates studied in Part I that stable disarrangement phases G necessarily have a stress response that not only is non-linear in compression but also that is "no-tension" or "masonry-like": $D\Psi_{\alpha\beta}(G)F^T n \cdot n \le 0$ for all unit vectors n.

The notion of a "stable disarrangement phase" is introduced in Section 2 as an energy-minimizing disarrangement phase, and the notions of compact phase and of loose phases are reexamined in light of this notion of stability. We point out that, because disarrangements of rank one cannot increase the free energy, the class of macroscopic deformations for which the compact phase is a stable disarrangement phase may be viewed as being rather limited.

We consider in Section 3 the two-parameter class of free energy response functions $\Psi_{\alpha\beta}$ studied in Part I and that serves as the setting for the remainder of this article (with the exception of the Appendix) . We review the solutions of the consistency relation G obtained in Part I by describing the four categories that exhaust the collection of disarrangement phases: "compact," "plane-stress," "uniaxial stress." and "stress-free," the last three according to the nature of the stress response S that is calculated for each category of solutions. Because the consistency relation can be written in the tensorial form $SM^T = 0$, the disarrangement tensors $M = F - G$ for these categories turn out to have ranks 0, at most 1, at most 2, and at most 3, respectively. We close Section 3 with some comparisons of the free energy among disarrangement phases in different categories but corresponding to the same macroscopic deformation gradient F.

In Section 4 for the two-parameter class of free energy response functions $\Psi_{\alpha\beta}$ we prove the main result of this article: stable disarrangement phases necessarily have "no-tension" or "masonry-like" response that, unlike the standard setting for such material response [12, 13, 14, 15, 16, 17, 18], allows for non-linear stress-deformation relations in compression. □ Figure 1 provides a schematic, one-dimensional stress-extension curve for a no-tension material with non-linear response in compression, while □ Figure 2 depicts a no-tension material with linear response in compression. For the disarrangement phases associated with $\Psi_{\alpha\beta}$, it is noteworthy that the "no-tension" property emerges necessarily from the property of stability, and an issue for future research is the determination of broader classes of free energies for which stability implies a no-tension response.

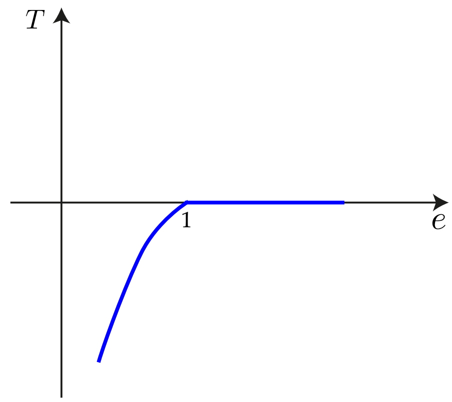

Figure 1: No-tension/non-linear in compression.

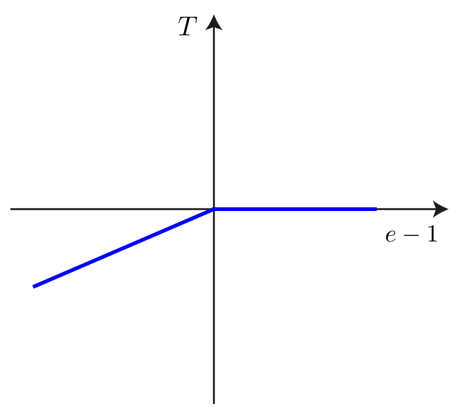

Figure 2: No tension/linear in compression.

2 Stable disarrangement phases

The analysis in Part I as well as examples available in the literature [1, 2] show that, given the free-energy response function Ψ and the macroscopic deformation gradient F, there are many disarrangement phases corresponding to F, even if one agrees to count as equivalent different disarrangement phases G and G' corresponding to the given F that deliver the same stress tensor, $D\Psi(G) = D\Psi(G')$, and the same free energy density $\Psi(G) = \Psi(G')$. The multiplicity of disarrangement phases G corresponding to a given F appearing in the contexts [1, 2] suggested additional conditions for selecting preferred disarrangement phases. A condition closely related to the present context was introduced in the paper [2] in which an "augmented energy functional" was identified as a function defined on time-parameterized families of structured deformations (g, G) and on corresponding statical environments of the body, with the property that, at each given time, the augmented energy is stationary with respect to the variable g at equilibria for the body in the given environment. Moreover, on "purely submacroscopic processes", i.e., ones that change G but not g, the rate of change of the augmented energy was shown to equal the rate of storage of energy minus the rate of dissipation of energy. This observation led in [2] to the notion that an equilibrium configuration (g, G) be submacroscopically stable if, for fixed g, the tensor field G provided an absolute minimum for the augmented energy subject to satisfaction of the consistency relation and the accommodation inequality. As a result, in order to leave a submacroscopically stable equilibrium configuration via a purely submacroscopic process, energy must be stored faster than it is dissipated.

In the present context, although no evolution in time is admitted, it is reasonable to mold the concept of "a submacroscopically stable equlilibrium configuration of the body" into one of "a stable disarrangement phase for the material." Accordingly, for a given macroscopic deformation gradient F, we say that a tensor G is a *stable disarrangement phase corresponding to* F if, not only is G a disarrangement phase corresponding to F, but also G delivers the minimum energy density $\Psi(G')$ among all disarrangement phases G' corresponding to F. Thus, each stable disarrangement phase G corresponding to F is a solution to the minimization problem :

$$\min_{G'} \Psi(G') \quad \text{subject to} \quad 0 < \det G' \leq \det F \;\; \text{and} \;\; D\Psi(G')(F^T - G'^T) = 0. \tag{1}$$

Following our discussion in Section 3 of Part I, if the free energy response function Ψ is assumed to be isotropic, smooth, rank-one convex, and to have standard growth properties, then, for each rotation R and each tensor F satisfying the accommodation inequality in the form $\varsigma_{\min}^3 \leq \det F$, the tensor $\varsigma_{\min} R$ is a stable disarrangement phase corresponding to F. Indeed, $G' = \varsigma_{\min} R$ is an absolute minimizer of the free energy response function and satisfies both relations in (1). Of course, if the tensor F does not satisfy the inequality $\varsigma_{\min}^3 \leq \det F$, then there is no loose phase corresponding to F and R, no matter what the choice of rotation tensor R. Turning to the notion of compact phase, we note that, while the tensor $G = F$ always is available as the compact phase corresponding to F, this compact phase need not be a stable disarrangement phase corresponding to F, since $G = F$ need not minimize the energy among disarrangement phases G' corresponding to F and, therefore, need not be a solution of the problem (1). We may summarize this situation through the assertions: *for arbitrary macroscopic deformation gradients F, the compact phase corresponding to F always competes for the status of a stable disarrangement phase but need not win that status. By contrast, only for F satisfying $\varsigma_{\min}^3 \leq \det F$ does a loose phase $\varsigma_{\min} R$ compete; however, when it does compete, a loose phase always achieves the status of stable disarrangement phase.*

We note that the smoothness and growth properties of Ψ introduced in [19] actually are sufficient for the existence of stable disarrangement phases for arbitrary macroscopic deformations F, because the set of tensors G' in (1) satisfying the accommodation inequality and the consistency relation form a closed subset of $Lin\mathcal{V}$, and the smoothness and growth properties of Ψ imply that there is a closed, bounded set of tensors that contains a tensor G' at which the minimum of Ψ is attained.

On the other hand, if Ψ is smooth and rank-one convex, Remark 1 in Part I and the notion of stable disarrangement phases corresponding to F now tell us: if the compact phase for F is a stable disarrangement phase corresponding to F, then so are all disarrangement phases G for F having $F - G$ of rank one.

3 Classification from Part I of the disarrangement phases for the model free energy $\Psi_{\alpha\beta}$

In Part I we illustrated the richness of possibilities for disarrangement phases of elastic aggregates through the specific choice of free energy response

$$\Psi_{\alpha\beta}(G) = \frac{1}{2}\alpha(\det G)^{-2} + \frac{1}{2}\beta tr(GG^T) = \frac{1}{2}\beta(\frac{r}{\det B_G} + trB_G) \tag{2}$$

where $B_G := GG^T$ is a Cauchy-Green tensor corresponding to G and $r := \alpha/\beta$. The numbers α and β represent "elastic constants" for the pieces of the aggregate, and they determine the stress response in the reference configuration through the relation

$$\beta^{-1}S = \beta^{-1}D\Psi_{\alpha\beta}(G) = -\frac{r}{(\det G)^2}G^{-T} + G. \tag{3}$$

The consistency relation $D\Psi(G)(F^T - G^T) = 0$ here is equivalent to

$$(G - \frac{r}{(\det G)^2}G^{-T})(F^T - G^T) = 0 \tag{4}$$

or, in terms of the polar decomposition $G = V_G R_G$ of G with $V_G^2 = B_G$, in the form

$$(B_G - \frac{r}{\det B_G}I)(R_G F^T - V_G) = 0. \tag{5}$$

We display below all of the disarrangement phases for this model aggregate as classified in Part I.

3.1 $G = F$ (compact phase)

$$\beta^{-1}(\det F)T = FF^T - \frac{r}{(\det F)^2}I \tag{6}$$

$$2\beta^{-1}\Psi_{\alpha\beta}(G) = \frac{r}{(\det F)^2} + tr(FF^T) \tag{7}$$

$$0 < \det G = \det F \tag{8}$$

3.2 $G \neq F$ (non-compact phases)

When $G \neq F$, the nullspace of $B_G - \frac{r}{\det B_G} I$ is non-trivial and the number $r/\det B_G$ must be one of the eigenvalues λ_1^G, λ_2^G, λ_3^G of B_G, say (without loss of generality) λ_1^G. We represent V_G and $B_G = V_G^2$ in terms of an orthonormal basis e_1^G ,e_2^G, e_3^G of eigenvectors corresponding to the eigenvalues λ_1^G, λ_2^G, λ_3^G of B_G :

$$B_G = \sum_{i=1}^{3} \lambda_i^G e_i^G \otimes e_i^G \text{ and } V_G = \sum_{i=1}^{3} (\lambda_i^G)^{1/2} e_i^G \otimes e_i^G \tag{9}$$

and assume without loss of generality that $e_1^G = e_2^G \times e_3^G$. In Part I we showed: *if* $G \neq F$, *then without loss of generality* $(\lambda_1^G)^2 \lambda_2^G \lambda_3^G = r$, *and the consistency relation is equivalent to*

$$(\lambda_i^G - \lambda_1^G)(F R_G^T - (\lambda_i^G)^{1/2} I) e_i^G = 0 \quad \text{for} \quad i = 2, 3, \tag{10}$$

3.2.1 Plane stress: $(\lambda_1^G)^2 \lambda_2^G \lambda_3^G = r$ and $\lambda_i^G \neq \lambda_1^G$ for $i = 2, 3$

All of the solutions of the consistency relation in this case are given through the following result from Part I:

Remark 1 *Let orthogonal unit vectors e and f and a linear mapping F with* $\det F > 0$ *be given satisfying*

$$F^{-1}e \cdot F^{-1}f = 0, \quad r^{1/2} \left|F^{-1}e\right|^3 \left|F^{-1}f\right| \neq 1, \quad r^{1/2} \left|F^{-1}e\right| \left|F^{-1}f\right|^3 \neq 1. \tag{11}$$

Then the tensor

$$\begin{aligned} G &= r^{1/4} \left|F^{-1}e\right|^{1/2} \left|F^{-1}f\right|^{1/2} (e \times f) \otimes \left(\frac{F^{-1}e}{|F^{-1}e|} \times \frac{F^{-1}f}{|F^{-1}f|}\right) + \\ &\quad + \left|F^{-1}e\right|^{-1} e \otimes \frac{F^{-1}e}{|F^{-1}e|} + \left|F^{-1}f\right|^{-1} f \otimes \frac{F^{-1}f}{|F^{-1}f|} \end{aligned} \tag{12}$$

is a solution of the consistency relation (10), and the solution (12) equals F if and only if

$$B_F(e \times f) = \frac{r}{\det B_F} e \times f. \tag{13}$$

*Moreover, every solution $G \neq F$ of the consistency relation (**??**) in the case $\lambda_i^G \neq \lambda_1^G$ for $i = 2, 3$ is of the form (12) for some choice of the orthogonal unit vectors e and f satisfying (11), and this formula for G implies that*

$$\begin{aligned} V_G &= r^{1/4} \left|F^{-1}e\right|^{1/2} \left|F^{-1}f\right|^{1/2} (e \times f) \otimes (e \times f) + \\ &\quad + \left|F^{-1}e\right|^{-1} e \otimes e + \left|F^{-1}f\right|^{-1} f \otimes f, \end{aligned} \tag{14}$$

$$\begin{aligned} R_G &= (e \times f) \otimes \left(\frac{F^{-1}e}{|F^{-1}e|} \times \frac{F^{-1}f}{|F^{-1}f|}\right) + \\ &\quad + e \otimes \frac{F^{-1}e}{|F^{-1}e|} + f \otimes \frac{F^{-1}f}{|F^{-1}f|}, \end{aligned} \tag{15}$$

$$\det G = r^{1/4}\left|F^{-1}e\right|^{-1/2}\left|F^{-1}f\right|^{-1/2}. \tag{16}$$

In addition, if $T = (\det F)^{-1} D\Psi_{\alpha\beta}(G)F^T$ *is the Cauchy stress, then*

$$\begin{aligned}\beta^{-1}(\det F)T &= \left|F^{-1}e\right|^{-2}(1-\ r^{1/2}\left|F^{-1}e\right|^3\left|F^{-1}f\right|)\,e\otimes e+\\ &\quad+\left|F^{-1}f\right|^{-2}(1-\ r^{1/2}\left|F^{-1}e\right|\left|F^{-1}f\right|^3)\,f\otimes f,\end{aligned} \tag{17}$$

and the free energy $\Psi_{\alpha\beta}(G)$ *is given by*

$$\frac{2}{\beta}\Psi_{\alpha\beta}(G) = 2r^{1/2}\left|F^{-1}e\right|\left|F^{-1}f\right| + \left|F^{-1}e\right|^{-2} + \left|F^{-1}f\right|^{-2}. \tag{18}$$

The formula (17) for the Cauchy stress implies that the traction $T(e\times f)$ on a plane with normal $e\times f$ is zero and that every traction vector Tn lies in the plane determined by e and f. Moreover, both Te and Tf are non-zero. It is then appropriate to use the attribute *plane-stress* to describe the solutions G (12) of the consistency relation in the present case $\lambda_i^G \neq \lambda_1^G$ for $i = 2, 3$, and we use as in Part I the term *plane-stress disarrangement phases corresponding to* F in referring to such tensors G that also satisfy the accommodation inequality $0 < \det G \leq \det F$ that now takes the form:

$$0 < r^{1/4}\left|F^{-1}e\right|^{-1/2}\left|F^{-1}f\right|^{-1/2} \leq \det F. \tag{19}$$

3.2.2 Uniaxial stress: $(\lambda_1^G)^2\lambda_2^G\lambda_3^G = r$ and $\lambda_2^G = \lambda_1^G$, $\lambda_3^G \neq \lambda_1^G$

All of the solutions of the consistency relation in this case are given through the following result from Part I:

Remark 2 *Let a unit vector* e*, a proper orthogonal tensor* R*, and a linear mapping* F *with* $\det F > 0$ *be given satisfying*

$$R^T e = \frac{F^{-1}e}{|F^{-1}e|} \qquad and \qquad r^{1/8}\left|F^{-1}e\right| \neq 1. \tag{20}$$

Then the tensor G *given by*

$$G = r^{1/6}\left|F^{-1}e\right|^{1/3}(I - e\otimes e)\,R + \left|F^{-1}e\right|^{-1} e\otimes\frac{F^{-1}e}{|F^{-1}e|} \tag{21}$$

is a solution of the consistency relation (10) for the case $(\lambda_1^G)^2\lambda_2^G\lambda_3^G = r$ *and* $\lambda_2^G = \lambda_1^G$*,* $\lambda_3^G \neq \lambda_1^G$*. The solution (21) equals* F *if and only if* $R_F = R$ *and, for all vectors* v *perpendicular to* e*,*

$$B_F\,v = \frac{r}{\det B_F}v. \tag{22}$$

Moreover, every solution of the consistency relation for this case is of the form (21) with R *and* e *satisfying (20), and the following relations hold:*

$$V_G = r^{1/6}\left|F^{-1}e\right|^{1/3}(I - e\otimes e) + \left|F^{-1}e\right|^{-1} e\otimes e \tag{23}$$

$$R_G = R \tag{24}$$

$$\det G = \det V_G = r^{1/3}\left|F^{-1}e\right|^{-1/3} \tag{25}$$

In addition, if $T = (\det F)^{-1} D\Psi_{\alpha\beta}(G)F^T$ *is the Cauchy stress, then*

$$\beta^{-1}(\det F)T = \frac{1 - r^{1/3}\left|F^{-1}e\right|^{8/3}}{\left|F^{-1}e\right|^2} e \otimes e, \tag{26}$$

and the free energy $\Psi_{\alpha\beta}(G)$ *is given by*

$$\frac{2}{\beta}\Psi_{\alpha\beta}(G) = 3r^{1/3}\left|F^{-1}e\right|^{2/3} + \left|F^{-1}e\right|^{-2}. \tag{27}$$

The formula (26) and the restriction (20) show that the state of stress in the deformed configuration of the aggregate is uniaxial and non-zero for every solution G of the consistency relation in the present case $(\lambda_1^G)^2\lambda_2^G\lambda_3^G = r$ and $\lambda_2^G = \lambda_1^G$, $\lambda_3^G \neq \lambda_1^G$. It is then appropriate to use the attribute *uniaxial stress* to describe the solutions G and the term *uniaxial stress disarrangement phases corresponding to* F in referring to such tensors G that also satisfy the accommodation inequality $0 < \det G \leq \det F$ in the form:

$$0 < r^{1/3}\left|F^{-1}e\right|^{-1/3} \leq \det F. \tag{28}$$

3.2.3 Loose phase: the case $(\lambda_1^G)^2\lambda_2^G\lambda_3^G = r$ and $\lambda_1^G = \lambda_2^G = \lambda_3^G$

The relations $(\lambda_1^G)^2\lambda_2^G\lambda_3^G = r$ and $\lambda_1^G = \lambda_2^G = \lambda_3^G$ immediately yield $\lambda_1^G = \lambda_2^G = \lambda_3^G = r^{1/4}$, so that

$$B_G = r^{1/4}I \quad \text{and} \quad G = r^{1/8}R, \tag{29}$$

Of course, in this case we also have

$$\det G = r^{3/8}, \tag{30}$$

and we note that this case recovers precisely those tensors G that render $\Psi_{\alpha\beta}$ a minimum. Moreover, we have:

$$T = 0 \quad \textit{and} \quad 2\beta^{-1}\Psi_{\alpha\beta}(G) = 4r^{1/4},$$

and the accommodation inequality takes the form

$$r^{3/8} \leq \det F. \tag{31}$$

3.3 Some energy comparisons among disarrangement phases

The categories of solutions of the consistency relation and the associated versions of the accommodation inequality just provided permit us to compare the energies of competing disarrangement phases corresponding to a given F. The first comparison relates the energies for the compact and plane-stress disarrangement phases: *if* G_p *is a plane-stress disarrangement phase corresponding to* F with $G_p \neq F$, *then the energy associated with* G_p *is strictly less than that for the compact phase* $G = F$:

$$\Psi_{\alpha\beta}(G_p) < \Psi_{\alpha\beta}(F). \tag{32}$$

Indeed, the stress in a plane-stress disarrangement phase has rank two, and the consistency relation then tells us that the range of the transpose $F^T - G_p^T$ of the disarrangement tensor $F - G_p$ is included in the one-dimensional nullspace of the stress. Therefore, since $G_p \neq F$, both $F^T - G_p^T$ and $F - G_p$ have rank 1. Consequently, Remark 1 of Part I and the strict rank-one convexity of $\Psi_{\alpha\beta}$ yield (32).

The second comparison involves the plane-stress and uniaxial stress disarrangement phases: *if G_p is a plane-stress phase with corresponding unit vectors e and f as in (12) and if G_u is a uniaxial stress phase as in (21), with e the same unit vector as in (12), then*

$$\Psi_{\alpha\beta}(G_u) < \Psi_{\alpha\beta}(G_p). \tag{33}$$

To verify this inequality, we put $x = B_F^{-1}e \cdot e$ and $y = B_F^{-1}f \cdot f$ in the formulas (18) and (27) for the free energy in the two phases to obtain the relations

$$\begin{aligned} \frac{2}{\beta}\Psi_{\alpha\beta}(G_p) &= 2r^{1/2}x^{1/2}y^{1/2} + x^{-1} + y^{-1} \\ \frac{2}{\beta}\Psi_{\alpha\beta}(G_u) &= 3r^{1/3}x^{1/3} + x^{-1}. \end{aligned}$$

It then suffices to compare $\frac{2}{\beta}\Psi_{\alpha\beta}(G_p) - x^{-1}$ and $\frac{2}{\beta}\Psi_{\alpha\beta}(G_u) - x^{-1}$, and we may use the arithmetic-geometric mean inequality to write

$$\begin{aligned} \frac{2}{\beta}\Psi_{\alpha\beta}(G_p) - x^{-1} &= r^{1/2}x^{1/2}y^{1/2} + r^{1/2}x^{1/2}y^{1/2} + y^{-1} \\ &\geq 3(r^{1/2}x^{1/2}y^{1/2}r^{1/2}x^{1/2}y^{1/2}y^{-1})^{1/3} \\ &= \frac{2}{\beta}\Psi_{\alpha\beta}(G_u) - x^{-1}. \end{aligned}$$

Equality holds in this relation if and only if $r^{1/2}x^{1/2}y^{1/2} = y^{-1}$, i.e., $r^{1/2}x^{1/2}y^{3/2} = 1$, which is ruled out by $(11)_3$, and this verifies (33).

Similar arguments based on the fact that loose phases achieve the absolute minimum value $2\beta r^{1/4} = 2\beta^{3/4}\alpha^{1/4}$ of $\Psi_{\alpha\beta}$ easily deliver three additional comparisions: if G_p and G_u are plane-stress and uniaxial stress disarrangement phases corresponding to F (with or without any equality of the unit vectors associated with the two phases) and if $\det F \geq r^{3/8}$, then for any loose phase $G_l = r^{1/8}R$ corresponding to F we have

$$\Psi_{\alpha\beta}(G_p) > \Psi_{\alpha\beta}(G_l), \;\; \Psi_{\alpha\beta}(G_u) > \Psi_{\alpha\beta}(G_l), \text{ and } \;\Psi_{\alpha\beta}(F) \geq \Psi_{\alpha\beta}(G_l) \tag{34}$$

with equality holding in the last inequality if and only if F, itself, is of the form $r^{1/8}R'$ for some rotation R'.

4 Stable disarrangement phases associated with $\Psi_{\alpha\beta}$ cannot support tensile tractions

The previous section provides four categories of disarrangement phases, compact, plane-stress, uniaxial stress, and loose, and it is easy to show that the consistency relation (4) implies that the disarrangement tensor M has rank 0, has rank at most 1, rank at most 2, and rank at most 3, respectively. These categories and the explicit solutions obtained in each then provide the starting point for studying the stable disarrangement phases associated with $\Psi_{\alpha\beta}$ for arbitrary macroscopic deformation F.

The principal result of the present study is the assertion that the stable disarrangement phases associated with $\Psi_{\alpha\beta}$ cannot support tensile tractions and, therefore, provide a no-tension material response that is non-linear in compression.

Remark 3 *Let a tensor F with $\det F > 0$ and a stable disarrangement phase G corresponding to F be given. It follows that the normal tractions provided by G are never positive, i.e., the Cauchy stress $T = (\det F)^{-1} D\Psi_{\alpha\beta}(G)F^T$ satisfies the "no-tension" condition:*

$$Tn \cdot n \leq 0. \qquad \textit{for every unit vector } n. \tag{35}$$

We establish this result by considering in turn the four categories of disarrangement phases to which the given tensor G can belong and by showing that, in each category, the conditions "stability" and "violation of the no-tension condition (35)" are contradictory. In doing so, it is convenient to denote by $\lambda_M \geq \lambda_{med} \geq \lambda_m > 0$ the eigenvalues of $B_F = FF^T$ and by e_M, e_{med}, and e_m a corresponding orthonormal basis of eigenvectors of B_F.

Suppose first that G is a compact phase corresponding to F that is stable and violates the no-tension condition (35). In this case, $G = F$ and, since G is stable, we have

$$\Psi_{\alpha\beta}(G) \leq \Psi_{\alpha\beta}(G') \tag{36}$$

for every disarrangement phase G' corresponding to F. The "no-tension" condition (35) is violated if and only if the largest principal stress associated with $T = (\det F)^{-1} D\Psi_{\alpha\beta}(G)F^T$ is positive. From the catalog in Section 3, the formula (6) for T in the compact phase provides a formula for the largest principal stress and yields the following equivalent statement that the no-tension condition is violated:

$$\lambda_M - \frac{r}{\lambda_M\, \lambda_{med}\, \lambda_m} = \lambda_M - \frac{r}{(\det F)^2} > 0. \tag{37}$$

i.e.,

$$\lambda_M{}^2 \lambda_{med}\, \lambda_m > r. \tag{38}$$

If $\lambda_M = \lambda_m$ then all eigenvalues of FF^T equal λ_M, the relation (38) reduces to $\lambda_M > r^{1/4}$, and we have

$$\det F = (\lambda_M^3)^{1/2} > (r^{3/4})^{1/2} = r^{3/8}.$$

We may then conclude from (31) that the accommodation inequality for loose phases $\tilde{G} = r^{1/8}\tilde{R}$ corresponding to F is satisfied. Moreover, the given compact phase satisfies: $G = F = \lambda_M^{1/2} R$ for some rotation R, from which it follows that

$$\frac{2}{\beta}\Psi_{\alpha\beta}(G) = \frac{r}{\lambda_M^3} + 3\lambda_M.$$

By the Arithmetic-Geometric Mean Inequality, there follows

$$\frac{2}{\beta}\Psi_{\alpha\beta}(G) \geq 4(\frac{r}{\lambda_M^3}\lambda_M^3)^{1/4} = 4r^{1/4} = \frac{2}{\beta}\Psi_{\alpha\beta}(\tilde{G})$$

with equality holding if and only if $\frac{r}{\lambda_M^3} = \lambda_M$, i.e. $r = \lambda_M^4$, which is excluded by (38) . Consequently, $\Psi_{\alpha\beta}(G) > \Psi_{\alpha\beta}(\tilde{G})$, contradicting (36) when $G' = \tilde{G}$.

It suffices then to consider the remaining case $\lambda_M > \lambda_m$. Recalling that e_{med} and e_m denote a pair of orthogonal unit eigenvectors of FF^T corresponding to λ_{med} and λ_m, respectively, we have

$$\begin{aligned} & (\det B_F)^2 (B_F^{-1} e_{med} \cdot e_{med})(B_F^{-1} e_m \cdot e_m) \\ = \quad & (\lambda_M \, \lambda_{med} \, \lambda_m)^2 \lambda_{med}{}^{-1} \lambda_m^{-1} = \lambda_M{}^2 \lambda_{med} \, \lambda_m > r, \end{aligned}$$

and this inequality tells us that the accommodation inequality (19) for the plane-stress category corresponding to F is satisfied with strict inequality when $e := e_{med}$ and $f := e_m$, i.e.,

$$(\det B_F)^2 (B_F^{-1} e_{med} \cdot e_{med})(B_F^{-1} e_m \cdot e_m) > r. \tag{39}$$

Consequently, if we replace $f = e_m$ by $f_\varepsilon := (e_m + \varepsilon e_M)/(1+\varepsilon^2)^{1/2}$, then for ε sufficiently small we have

$$(\det B_F)^2 (B_F^{-1} e_{med} \cdot e_{med})(B_F^{-1} f_\varepsilon \cdot f_\varepsilon) > r \tag{40}$$

as well as

$$e_{med} \cdot f_\varepsilon = B_F^{-1} e_{med} \cdot f_\varepsilon = 0. \tag{41}$$

Moreover, the formula

$$\left| F^{-1} f_\varepsilon \right|^2 = B_F^{-1} f_\varepsilon \cdot f_\varepsilon = \frac{\lambda_m^{-1} + \varepsilon^2 \lambda_M^{-1}}{1+\varepsilon^2}$$

and the inequality $\lambda_m < \lambda_M$ tell us that, as ε varies in a small open interval containing 0, the pairs $(r^{1/2} \left| F^{-1} e_{med} \right| \left| F^{-1} f_\varepsilon \right|^3, r^{1/2} \left| F^{-1} e_{med} \right|^3 \left| F^{-1} f_\varepsilon \right|)$ trace out a smooth, non-trivial curve, and we may conclude that there are infinitely many points (x', y') on that curve such that $x' \neq 1$ or $y' \neq 1$. Therefore, in addition to the accommodation inequality in the form (40), the conditions $(11)_{2,3}$ are satisfied with $e = e_{med}$ and $f = f_\varepsilon$ for an appropriate choice of ε. Moreover, because the spectrum of G_ε varies with ε while that of V_F does not, ε can be chosen so that, in addition, $G_\varepsilon \neq F$. Consequently, this choice of the pair of unit vectors determines through (12) a plane-stress disarrangement phase G_ε corresponding to F. The fact that G_ε is in the plane-stress category with $G_\varepsilon \neq F$ implies that we may use the inequality (32) with $G_p := G_\varepsilon$ to conclude that

$$\Psi_{\alpha\beta}(G) = \Psi_{\alpha\beta}(F) > \Psi_{\alpha\beta}(G_\varepsilon),$$

and this inequality contradicts the assumption that the compact phase $G = F$ is stable. This completes the proof for a compact phase that stability and violation of the no-tension condition are contradictory.

Suppose that G is a plane-stress phase corresponding to F that is stable and violates the no-tension condition (35). In this case, G is given by (12), and, since G is stable, we have

$$\Psi_{\alpha\beta}(G) \leq \Psi_{\alpha\beta}(G') \tag{42}$$

for every disarrangement phase G' corresponding to F. The no-tension condition (35) is violated if and only if the largest principal stress associated with $T = (\det F)^{-1} D\Psi_{\alpha\beta}(G) F^T$ is positive. From the catalog in Section 6, the formula (17) for T in plane-stress phases can be written:

$$\beta^{-1} (\det F) T = x^{-1}(1 - r^{1/2} x^{3/2} y^{1/2}) e \otimes e + y^{-1}(1 - r^{1/2} x^{1/2} y^{3/2}) f \otimes f \tag{43}$$

with

$$x = B_F^{-1} e \cdot e, \quad y = B_F^{-1} f \cdot f, \tag{44}$$

and e, f orthogonal unit vectors satisfying the constraints (11). Consequently, the no-tension condition (35) is violated if and only if at least one of the scalar coefficients on the right-hand side is positive or, equivalently,

$$rx^3y < 1 \quad \text{or} \quad rxy^3 < 1. \tag{45}$$

Moreover, the accommodation inequality (19) for the plane-stress phase G can be written now as

$$r \leq (\det F)^4 xy. \tag{46}$$

If $\lambda_m = \lambda_M$, then $B_F = \lambda_M I$, $F = \lambda_M^{1/2} R$ for some rotation R, $\det F = \lambda_M^{3/2}$, $x = y = \lambda_M^{-1}$, and the last inequality becomes

$$r \leq (\lambda_M^{3/2})^4 (\lambda_M^{-1})^2 = \lambda_M^4 = ((\det F)^{2/3})^4 = (\det F)^{8/3}.$$

In other words, the accommodation inequality (31) for loose phases corresponding to $F = \lambda_M^{1/2} R$ is satisfied. Therefore, there is a loose phase $G' = r^{1/8} R'$ corresponding to F and we have from (18)

$$\begin{aligned} \frac{2}{\beta}\Psi_{\alpha\beta}(G) &= 2r^{1/2}x^{1/2}y^{1/2} + x^{-1} + y^{-1} \\ &= 2r^{1/2}\lambda_M^{-1} + 2\lambda_M \geq 4r^{1/4} = \frac{2}{\beta}\Psi_{\alpha\beta}(G') \end{aligned} \tag{47}$$

with equality holding if and only if $r^{1/2}\lambda_M^{-1} = \lambda_M$, i.e., $r = \lambda_M^4$, which is excluded by (45), the violation of the no-tension condition. Consequently, if $\lambda_m = \lambda_M$, the disarrangement phase G corresponding to F is not stable, and the assumptions that G is stable and that its stress violates the no-tension condition are contradictory.

In the alternative case $\lambda_m < \lambda_M$, we shall use the following consequence of the formula (47) for $\frac{2}{\beta}\Psi_{\alpha\beta}(G)$ in terms of x and y:

$$\begin{aligned} -x\frac{\partial}{\partial x}(\frac{2}{\beta}\Psi_{\alpha\beta}(G)) &= x^{-1}(1 - r^{1/2}x^{3/2}y^{1/2}) \\ -y\frac{\partial}{\partial y}(\frac{2}{\beta}\Psi_{\alpha\beta}(G)) &= y^{-1}(1 - r^{1/2}x^{1/2}y^{3/2}). \end{aligned} \tag{48}$$

Comparing the right-hand sides of these relations with the inequalities (45) we conclude that the no-tension condition is violated for the given pair (x, y) associated with G if and only if

$$\frac{\partial}{\partial x}\Psi_{\alpha\beta}(G) < 0 \quad \text{or} \quad \frac{\partial}{\partial y}\Psi_{\alpha\beta}(G) < 0. \tag{49}$$

We denote by $\mathcal{R}$ the range of the mapping $\Pi := (e', f') \longmapsto (x', y') = (B_F^{-1}e' \cdot e', B_F^{-1}f' \cdot f')$ with domain the set of pairs of orthogonal unit vectors satisfying (11) (with e replaced by e' and f replaced by f'), and we note that $\mathcal{R}$ is contained in the square in the $x' - y'$ plane determined by the minimum and maximum eigenvalues of B_F^{-1}, λ_M^{-1} and λ_m^{-1}, respectively (see Fig. 3):

$$\mathcal{R} \subset [\lambda_M^{-1}, \lambda_m^{-1}] \times [\lambda_M^{-1}, \lambda_m^{-1}].$$

The definition of $\mathcal{R}$ and the nature of the constraints (11) tell us that $\mathcal{R}$ is symmetric under

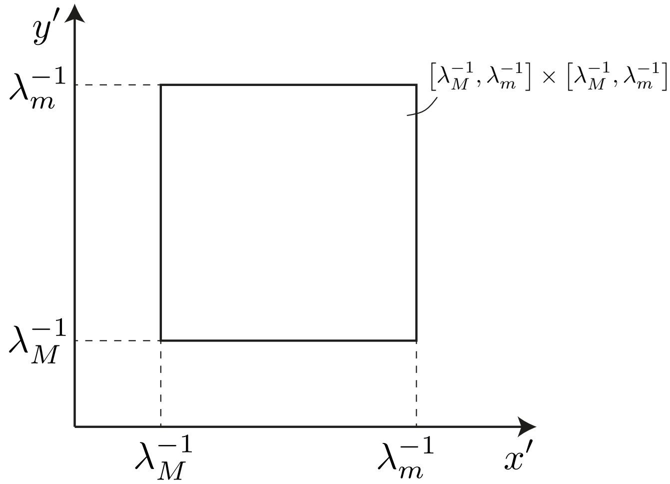

Figure 3: The square that contains the range of Π in all cases.

interchange of x' and y'. Moreover, the "$\neq$" parts of the constraints (11) along with (48) imply that at every point (x', y') in $\mathcal{R}$, both of the partial derivatives $\Psi_{\alpha\beta}$ are non-zero. It is clear then that the pair (x, y) corresponding to the given stable disarrangement phase G cannot be an interior point of $\mathcal{R}$. In fact, were this so, the partial derivatives $\frac{\partial}{\partial x'}\Psi_{\alpha\beta}\mid_{(x,y)}$ and $\frac{\partial}{\partial y'}\Psi_{\alpha\beta}\mid_{(x,y)}$ both would vanish, contradicting the previous statement. Consequently, the given minimizing point (x, y) associated with G lies in $\partial\mathcal{R}$, the boundary of $\mathcal{R}$, which is also symmetric under interchange of x' and y'.

It is convenient to consider three mutually exclusive and exhaustive cases that may arise. Firstly, we assume $\lambda_m = \lambda_{med} < \lambda_M$, so that the minimum eigenvalue λ_m of $B_F = FF^T$ has a two-dimensional eigenspace E_m orthogonal to the one-dimensional eigenspace E_M associated with the maximum eigenvalue λ_M. Because we may represent each pair of orthogonal unit vectors e' and f' that form the domain of the mapping Π in terms of these eigenspaces, we have for some $\varphi' \in [0, 2\pi]$ and for some unit vector $e'_\perp$ perpendicular to e_M: $e' = \cos\varphi' e'_\perp + \sin\varphi' e_M$ and, therefore,

$$x' = B_F^{-1}e' \cdot e' = \lambda_m^{-1}\cos^2\varphi' + \lambda_M^{-1}\sin^2\varphi'.$$

Recalling from $(11)_1$ that $B_F^{-1}e' \cdot f' = 0$, we obtain in the following table shows all the essential possibilities for e', for f', and for $(x', y') = \Pi(e', f')$ that can occur, taking into account the symmetry of $\mathcal{R}$ noted above:

e'	f'	x'	y'
$\in E_M$	$\in E_m$	λ_M^{-1}	λ_m^{-1}
$\in E_m$	$\in E_m$	λ_m^{-1}	λ_m^{-1}
$\notin E_M$ and $\notin E_m$	$\pm e' \times B_F^{-1}e'$	$\lambda_m^{-1}\cos^2\varphi' + \lambda_M^{-1}\sin^2\varphi'$	λ_m^{-1}

As φ' varies in the interval $[0, 2\pi]$, the pairs $(x', y') \in \mathcal{R}$ explicitly covered in the table form the horizontal line line segment $[\lambda_M^{-1}, \lambda_m^{-1}] \times \{\lambda_m^{-1}\}$ in the x'–y'plane, with the exception of at most four points (x_*, y_*) satisfying $r^{1/2}x_*'^{3/2}y_*^{1/2} = 1$ or $r^{1/2}x_*^{1/2}y_*^{3/2} = 1$.

The symmetry of $\mathcal{R}$ under interchange of x' and y' then yields for the present case $\lambda_m = \lambda_{med} < \lambda_M$:

$$\mathcal{R} \cup \{(x_*, y_*)\} = [\lambda_M^{-1}, \lambda_m^{-1}] \times \{\lambda_m^{-1}\} \ \cup \ \{\lambda_m^{-1}\} \times [\lambda_M^{-1}, \lambda_m^{-1}] \tag{50}$$

where $\{(x_*, y_*)\}$ represents a set of points with at most four elements (see Fig. 4). Thus,

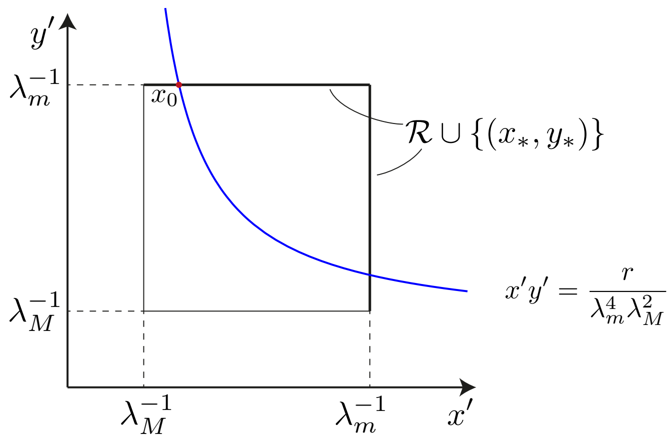

Figure 4: The range of Π for $\lambda_m = \lambda_{med} < \lambda_M$.

$\mathcal{R} \cup \{(x_*, y_*)\}$ is the union of a horizontal line segment and its vertical reflection about $x' = y'$, the two segments meeting at the point $(\lambda_m^{-1}, \lambda_m^{-1})$. Moreover, $\mathcal{R} \cup \{(x_*, y_*)\} = \partial\mathcal{R}$ in this case, and since the given minimizing point (x, y) lies in $\mathcal{R}$ and represents a disarrangement phase, the accommodation inequality (46) is satisfied at least at one point in $\mathcal{R}$.

It is convenient to assume first that every point in $\mathcal{R}$ satisfies the accommodation inequality (46), so that the pair of segments $\mathcal{R} \cup \{(x_*, y_*)\}$ lies entirely on or above the curve $(\det F)^4 x'y' = r$ in the x'–y' plane. Because both of the partial derivatives of $\Psi_{\alpha\beta}$ are non-zero at every point of $\mathcal{R}$, we may conclude that the given minimizing point (x, y) cannot lie in the interior of either segment and so must lie at one of the three endpoints $(\lambda_M^{-1}, \lambda_m^{-1})$, $(\lambda_m^{-1}, \lambda_m^{-1})$, or $(\lambda_m^{-1}, \lambda_M^{-1})$ of the two segments that form $\mathcal{R} \cup \{(x_*, y_*)\}$.

Suppose now that the minimizing point (x, y) associated with the given G equals $(\lambda_M^{-1}, \lambda_m^{-1})$, the left endpoint of the horizontal segment in $\mathcal{R} \cup \{(x_*, y_*)\}$. We cannot have the free energy decreasing on points (x', λ_m^{-1}) as x' increases from λ_M^{-1}, and it follows that $\frac{\partial}{\partial x'} \Psi_{\alpha\beta} \mid_{(\lambda_M^{-1}, \lambda_m^{-1})} > 0$ (the value 0 is ruled out as noted above). The formulas (48) then tell us that $1 - r^{1/2} x^{3/2} y^{1/2} < 0$ and, by (46), we have

$$\begin{aligned} \lambda_M^3 \lambda_m &= x^{-3} y^{-1} < r \leq (\det F)^4 xy = (\det B_F)^2 xy \\ &= \lambda_M^2 \lambda_m^4 \lambda_M^{-1} \lambda_m^{-1} = \lambda_M \lambda_m^3, \end{aligned}$$

so that $\lambda_M < \lambda_m$, a contradiction. We conclude that the possibility $(x, y) = (\lambda_M^{-1}, \lambda_m^{-1})$ is ruled out. By (47), the free energy is unchanged when x and y are interchanged, thus ruling out also the possibility $(x, y) = (\lambda_m^{-1}, \lambda_M^{-1})$. For the remaining possibility $(x, y) = (\lambda_m^{-1}, \lambda_m^{-1})$ the minimization property of this point implies that both partial derivatives of the free energy must be negative.

The relation (48) then yield $1 - r^{1/2}x^{3/2}y^{1/2} = 1 - r^{1/2}x^{1/2}y^{3/2} > 0$, or, equivalently, $\lambda_m^4 > r$. This relation then yields

$$\begin{aligned} (\det F)^{8/3} &= (\det B_F)^{4/3} = (\lambda_M \lambda_m^2)^{4/3} \\ &> (\lambda_m \lambda_m^2)^{4/3} = \lambda_m^4 > r, \end{aligned} \tag{51}$$

which shows that F admits the loose disarrangement phases $G' = r^{1/8}R'$ with R' an arbitrary rotation. The inequality $(34)_1$ at the end of Section 6, with $G_p = G$ and $G_l = G'$, then tells us that $\Psi_{\alpha\beta}(G) > \Psi_{\alpha\beta}(G')$, and this inequality completes the proof that the assertions "G is a stable disarrangement phase corresponding to F in the plane-stress category" and "G violates the no-tension condition" are contradictory for the case in which $\lambda_m = \lambda_{med} < \lambda_M$ and in which no portion of $\mathcal{R}$ lies below the curve $(\det F)^4 x'y' = r$.

Turning to the case in which $\lambda_m = \lambda_{med} < \lambda_M$ and some portion of $\mathcal{R}$ lies below the curve $(\det F)^4 x'y' = r$, we note that the symmetry of both $\mathcal{R}$ and of the curve $(\det F)^4 x'y' = r$ under reflection through the line $x' = y'$ permits us to consider only the portion of the horizontal segment

$$\mathcal{H} = [\lambda_M^{-1}, \lambda_m^{-1}] \times \left\{\lambda_m^{-1}\right\}$$

that lies on or above $(\det F)^4 x'y' = r$ (see Fig. 6). If the only such portion is the single point $(\lambda_m^{-1}, \lambda_m^{-1})$, then the analysis for the previous case shows that a contradiction arises if $(x, y) = (\lambda_m^{-1}, \lambda_m^{-1})$. The only remaining case to consider is when the portion of the horizontal segment $\mathcal{H}$ lying on or above the curve has the form $[x_0, \lambda_m^{-1}] \times \left\{\lambda_m^{-1}\right\}$ with $\lambda_M^{-1} < x_0 < \lambda_m^{-1}$ and with (x_0, λ_m^{-1}) lying on the curve $(\det F)^4 x'y' = r$, so that

$$x_0 = \lambda_M^{-2}\lambda_m^{-3} r \quad \text{and} \quad (x_0, \lambda_m^{-1}) = (\lambda_M^{-2}\lambda_m^{-3} r, \lambda_m^{-1}). \tag{52}$$

As in earlier cases, we know that the minimizing point (x, y) must be an endpoint of the segment $[x_0, \lambda_m^{-1}] \times \left\{\lambda_m^{-1}\right\}$. The earlier analysis shows again that a contradiction arises if $(x, y) = \left\{\lambda_m^{-1}\right\} \times \left\{\lambda_m^{-1}\right\}$, and we consider now the remaining alternative: $(x, y) = (\lambda_M^{-2}\lambda_m^{-3} r, \lambda_m^{-1})$. This left-hand endpoint of the segment minimizes the free energy only if $\frac{\partial}{\partial x'}\Psi_{\alpha\beta} \mid_{(x,y)} > 0$, and (49) at (x, y) implies $\frac{\partial}{\partial y'}\Psi_{\alpha\beta} \mid_{(x,y)} < 0$, which by (48) means that $1 - r^{1/2}x^{1/2}y^{3/2} > 0$. Using the formula $(x, y) = (\lambda_M^{-2}\lambda_m^{-3} r, \lambda_m^{-1})$ we conclude that $r < \lambda_M \lambda_m^3$. On the other hand, the inequality $\lambda_M^{-1} < x_0 = \lambda_M^{-2}\lambda_m^{-3} r$ implies $\lambda_M \lambda_m^3 < r$, a contradiction. This completes the analysis of the case in which $\lambda_m = \lambda_{med} < \lambda_M$.

Of the two remaining cases $\lambda_m < \lambda_{med} = \lambda_M$ and $\lambda_m < \lambda_{med} < \lambda_M$ we now treat the former : $\lambda_m < \lambda_{med} = \lambda_M$, so that $\det B_F = \lambda_M^2 \lambda_m$. With reasoning along the lines that led to the table above, it is easy to show now that the $\mathcal{R}$ range of the mapping Π again is, with the exception of at most four points (x_*, y_*) satisfying $r^{1/2}x_*^{3/2}y_*^{1/2} = 1$ or $r^{1/2}x_*^{1/2}y_*^{3/2} = 1$, the union of two perpendicular segments

$$\mathcal{R} \cup \{(x_*, y_*)\} = ([\lambda_M^{-1}, \lambda_m^{-1}] \times \left\{\lambda_M^{-1}\right\}) \cup (\left\{\lambda_M^{-1}\right\} \times [\lambda_M^{-1}, \lambda_m^{-1}]) \tag{53}$$

that now meet at the point $(\lambda_M^{-1}, \lambda_M^{-1})$. Here, again, $\{(x_*, y_*)\}$ represents a set of points with at most four elements. Following the argument in the previous case, we assume first that all points of $\mathcal{R}$ lie on or above the curve $(\det F)^4 x'y' = r$ and conclude that the assumed minimizing point (x, y) must be among the three endpoints $(\lambda_M^{-1}, \lambda_m^{-1})$, $(\lambda_M^{-1}, \lambda_M^{-1})$, $(\lambda_m^{-1}, \lambda_M^{-1})$ of the two segments of $\mathcal{R}$. By symmetry, we again may limit our attention to the two endpoints $(\lambda_M^{-1}, \lambda_M^{-1})$ and $(\lambda_m^{-1}, \lambda_M^{-1})$ of

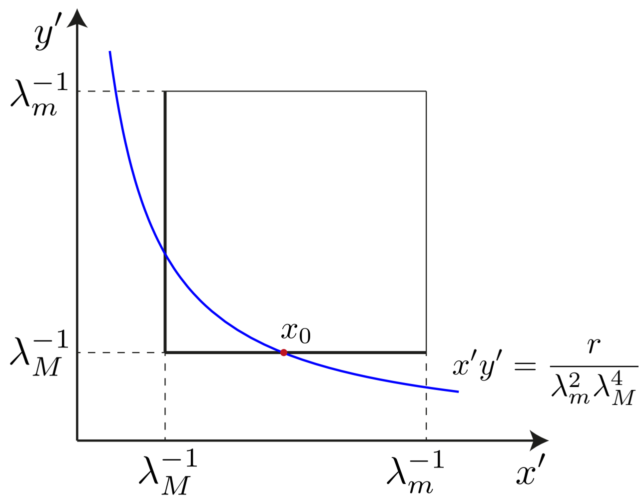

Figure 5: The range of Π for $\lambda_m < \lambda_{med} = \lambda_M$.

the horizontal segment of $\mathcal{R}$. If $(x, y) = (\lambda_M^{-1}, \lambda_M^{-1})$, then the minimizing property implies that both partial derivatives of $\Psi_{\alpha,\beta}$ must be positive and, therefore, both principal stresses must be negative. Consequently, the stability of the plane-stress phase G and the violation of the no-tension condition are contradictory for $(x, y) = (\lambda_M^{-1}, \lambda_M^{-1})$. If $(x, y) = (\lambda_m^{-1}, \lambda_M^{-1})$, then the minimizing property of (x, y) implies that $\frac{\partial}{\partial x'}\Psi_{\alpha\beta}\mid_{(x,y)}< 0$, so that $1 - r^{1/2}x^{3/2}y^{1/2} > 0$, or, equivalently, $\lambda_M\lambda_m^3 > r$. We then have

$$r^{3/4} < \lambda_M^{3/4}\lambda_m^{9/4} = \lambda_M^{3/4}\lambda_m^{1/4}\lambda_m^2 < \lambda_M\lambda_m^2 < \lambda_M^2\lambda_m = \det B_F,$$

so that $r^{3/8} < \det F$, and the loose phases $G' = r^{1/8}R'$ corresponding to F are admissible as competitors for disarrangement phases. Comparing energies by means of $(34)_1$ as in previous cases, we conclude that the stability of the phase G and the violation of the no-tension condition are contradictory in the case where $\lambda_m < \lambda_{med} = \lambda_M$ and where all points of $\mathcal{R}$ lie on or above the curve $(\det F)^4x'y' = r$.

For the case $\lambda_m < \lambda_{med} = \lambda_M$ with some points of $\mathcal{R}$ lying below the curve $(\det F)^4x'y' = r$ as in Fig. 5, we need only consider points (x, y) on the horizontal segment $[x_0, \lambda_m^{-1}] \times \{\lambda_M^{-1}\}$, where the point (x_0, λ_M^{-1}) lies on $(\det F)^4x'y' = r$. Consequently, we have

$$x_0 = \lambda_M^{-3}\lambda_m^{-2}r \quad \text{and} \quad (x_0, \lambda_M^{-1}) = (\lambda_M^{-3}\lambda_m^{-2}r, \lambda_M^{-1}), \tag{54}$$

and the segment under consideration may be written as $[\lambda_M^{-3}\lambda_m^{-2}r, \lambda_m^{-1}] \times \{\lambda_M^{-1}\}$. The interior points of this segment, as usual, are not candidates for the minimizer (x, y) and, as in the case where no points of $\mathcal{R}$ lie below $(\det F)^4x'y' = r$, the right-hand endpoint $(\lambda_m^{-1}, \lambda_M^{-1})$ can be shown not to be a minimizer that violates the no-tension condition. The only remaining possibility to consider is $(x, y) = (x_0, \lambda_M^{-1}) = (\lambda_M^{-3}\lambda_m^{-2}r, \lambda_M^{-1})$. This left-hand endpoint of the segment minimizes the free energy only if $\frac{\partial}{\partial x'}\Psi_{\alpha\beta}\mid_{(x,y)}> 0$, and (48) then tells us that $1 - r^{1/2}x^{3/2}y^{1/2} < 0$. Consequently, because $(x, y) = (x_0, \lambda_M^{-1}) = (\lambda_M^{-3}\lambda_m^{-2}r, \lambda_M^{-1})$ in the present case, we find that $\lambda_M^{5/2}\lambda_m^{3/2} < r$. The relation (49) at (x, y) implies $\frac{\partial}{\partial y'}\Psi_{\alpha\beta}\mid_{(x,y)}< 0$, which by (48) means that $1 - r^{1/2}x^{1/2}y^{3/2} > 0$.

Using the formula $(x,y) = (\lambda_M^{-3}\lambda_m^{-2}r, \lambda_M^{-1})$ we conclude that $r < \lambda_M^3\lambda_m$, and, combining the last two inequalities, we may write

$$\lambda_M^{5/2}\lambda_m^{3/2} < r < \lambda_M^3\lambda_m. \tag{55}$$

Thus, the *plane-stress* disarrangement phase G corresponding to F associated with the point $(B_F^{-1}e_0 \cdot e_0, B_F^{-1}f_0 \cdot f_0) = (x_0, \lambda_M^{-1}) = (\lambda_M^{-3}\lambda_m^{-2}r, \lambda_M^{-1})$ is stable and violates the no-tension condition only if the inequalities (55) hold. We now show that these inequalities are sufficient to provide a *uniaxial stress* disarrangement phase $\tilde{G}$ corresponding to F that has lower energy than does G. According to (20) and (28) in the catalog of disarrangement phases, the determination of $\tilde{G}$ requires that we find a unit vector $\tilde{e}$ such that the number $\tilde{x} = B_F^{-1}\tilde{e} \cdot \tilde{e}$ satisfies

$$\tilde{x} \neq r^{-1/4} \quad \text{and} \quad r \leq (B_F^{-1}\tilde{e} \cdot \tilde{e})^{1/2}(\det F)^3 = \tilde{x}^{1/2}(\lambda_M^2\lambda_m)^{3/2}. \tag{56}$$

We show now that the choice $\tilde{e} = e_0$, which implies that $\tilde{x} = x_0 = \lambda_M^{-3}\lambda_m^{-2}r$, satisfies these conditions. First, we note that $\tilde{x} = r^{-1/4}$ is equivalent to the relation $\lambda_M^{12/5}\lambda_m^{8/5} = r$. Substituting this relation into the left-hand inequality of (55), we find that $\tilde{x} = r^{-1/4}$ implies that $\lambda_M^{5/2}\lambda_m^{3/2} < \lambda_M^{12/5}\lambda_m^{8/5}$ which in turn is equivalent to $\lambda_M < \lambda_m$. The last relation is false, and we conclude that $\tilde{x} \neq r^{-1/4}$. Similarly, when $\tilde{x} = \lambda_M^{-3}\lambda_m^{-2}r$ is substituted into the right-hand side of the second inequality in (56), we obtain the inequality $r \leq \lambda_M^3\lambda_m$. Thus, according to the right-hand inequality in (55), the inequality $r \leq \lambda_M^3\lambda_m$ indeed is satisfied, and we have verified (56). We may now use (34), with $G_p = G$ and $G_u = \tilde{G}$, to show that the uniaxial stress phase $\tilde{G}$ has lower energy than does the plane-stress phase G and, thus, that the conditions of stabilty and of violation of the no-tension condition are contradictory when G is a plane-stress phase, when $\lambda_m < \lambda_{med} = \lambda_M$, and when some points of $\mathcal{R}$ lying below the curve $(\det F)^4 x'y' = r$. The treatment of the case $\lambda_m < \lambda_{med} = \lambda_M$ now is complete.

We turn finally to the remaining case $\lambda_m < \lambda_{med} < \lambda_M$, and we note here that the range of the mapping $\Pi = (e', f') \longmapsto (x', y') = (B_F^{-1}e' \cdot e', B_F^{-1}f' \cdot f')$ consists of the rectangle $\mathcal{R}_L$ defined by

$$\mathcal{R}_L = [\lambda_{med}^{-1}, \lambda_m^{-1}] \times [\lambda_M^{-1}, \lambda_{med}^{-1}], \tag{57}$$

without the points on the two curves $r^{1/2}x'^{\,3/2}y'^{\,1/2} = 1$ and $r^{1/2}x'^{\,1/2}y'^{\,3/2} = 1$, together with its reflection about the line $x' = y'$ (see Fig. 6). The symmetry of the condition (45) for violation of the no-tension condition and the invariance of the free energy in (47) under interchange of x and y permits us to consider only the points in the range of Π with $x' \geq y'$, and we first consider the case where all of the points of $\mathcal{R}_L$ lie on or above the hyperbola $(\det F)^4 x'y' = r$. The interior points of $\mathcal{R}_L$ are ruled out as minimizers, because both of the partial derivatives of the free energy would have to be zero at such every point, and we need only consider the cases where the minimizing point (x, y) equals one of the four vertices of $\mathcal{R}_L$. Because the details of the reasoning for each of the vertices are similar to those considered in previous cases, we only summarize the nature of the argument that excludes each of the four vertices. For the upper left-hand vertex $(\lambda_{med}^{-1}, \lambda_{med}^{-1})$, we note that *(1)* the partial derivatives of the free energy do not vanish and *(2)* points on the horizontal segment $[\lambda_{med}^{-1} - \varepsilon, \lambda_{med}^{-1} + \varepsilon] \times \{\lambda_{med}^{-1}\}$ and on the vertical segment $\{\lambda_{med}^{-1}\} \times [\lambda_{med}^{-1} - \varepsilon, \lambda_{med}^{-1} + \varepsilon]$ lie in the range of Π for $\varepsilon > 0$ sufficiently small. Observations *(1)* and *(2)* together rule out $(\lambda_{med}^{-1}, \lambda_{med}^{-1})$ being a minimizing point. For the lower left-hand vertex $(\lambda_{med}^{-1}, \lambda_M^{-1})$ to be a minimizing point, both partial derivatives of the free energy would have to be positive, and this contradicts the violation of the no-tension condition (49). The lower right-hand vertex $(\lambda_m^{-1}, \lambda_M^{-1})$ would be a minimum only if $\frac{\partial}{\partial x'}\Psi_{\alpha\beta}(\lambda_m^{-1}, \lambda_M^{-1}) < 0$ and $\frac{\partial}{\partial y'}\Psi_{\alpha\beta}(\lambda_m^{-1}, \lambda_M^{-1}) > 0$, and these two inequalities are easily shown to lead to the contradiction: $\lambda_m^3 \lambda_M > r > \lambda_m \lambda_M^3$. Finally, the upper right-hand vertex $(\lambda_m^{-1},\ \lambda_{med}^{-1})$ is ruled out by showing that both the partial derivatives of $\Psi_{\alpha\beta}$ at this point must be negative and, therefore, the determinant of F then is large enough to admit loose phases corresponding to F having free energy strictly lower than that of the assumed plane-stress phase minimizer.

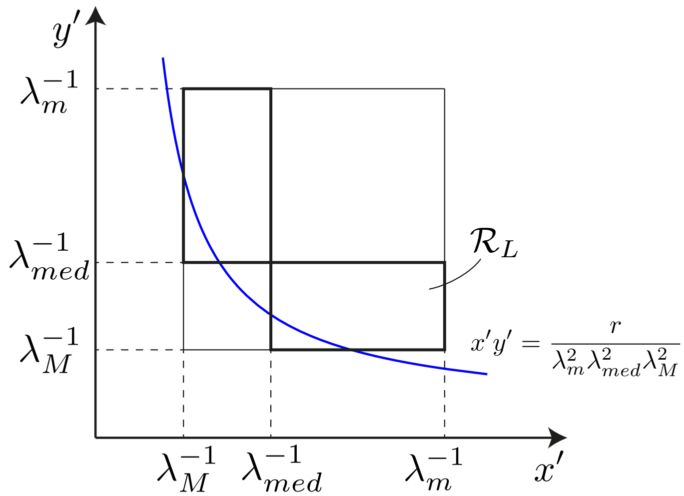

Figure 6: The range of Π for $\lambda_m < \lambda_{med} < \lambda_M$.

For disarrangement phases G in the plane-stress category, the final case to consider is where $\lambda_m < \lambda_{med} < \lambda_M$ and where some of the points of the rectangle $\mathcal{R}_L$ defined in (57) lie below the hyperbola $(\det F)^4 x'y' = r$ in the $x'{-}y'$ plane. In this case, all of the points of $\mathcal{R}_L$ lying below the hyperbola are removed from consideration, and only the remaining points (x, y) on the boundary of $\mathcal{R}_L$ as well as those points (x, y) of $(\det F)^4 x'y' = r$ that intersect $\mathcal{R}_L$ need be considered. It is easy to show that a point on the hyperbola $(\det F)^4 x'y' = r$ that also is an interior point $\mathcal{R}_L$ cannot be both a minimizer and violate the no-tension condition, since both of the partial derivatives of $\Psi_{\alpha\beta}$ at such a point must be positive. There remain to be considered (a) the points of intersection of two remaining perpendicular segments of edges of $\mathcal{R}_L$ and (b) the points (x, y) that are on the intersection of the hyperbola $(\det F)^4 x'y' = r$ and one of the sides of $\mathcal{R}_L$. The collection (a) includes only vertices of $\mathcal{R}_L$, at most three of them, and all such vertices can be ruled out as candidates for minimizers in the manner described in the previous paragraph. The collection (b) has exactly two points and may include vertices of $\mathcal{R}_L$, with the exception of $(\lambda_{med}^{-1}, \lambda_M^{-1})$, as well as other points on the sides of $\mathcal{R}_L$. Ruling out the vertices of $\mathcal{R}_L$ in the collection (b) as candidates for minimizers that violate the no-tension condition follows in a manner analogous to that described in the previous paragraph. The remaining points (x, y) in the collection (b) are not vertices of $\mathcal{R}_L$ and lie on one of the four sides of $\mathcal{R}_L$ as well as on the hyperbola $(\det F)^4 x'y' = r$. If such a point (x, y) lies on either of the two horizontal sides $[\lambda_{med}^{-1}, \lambda_m^{-1}] \times \{\lambda_{med}^{-1}\}$ and $[\lambda_{med}^{-1}, \lambda_m^{-1}] \times \{\lambda_M^{-1}\}$ or on the vertical side $\{\lambda_{med}^{-1}\} \times [\lambda_M^{-1}, \lambda_{med}^{-1}]$, then reasoning similar to that used in the case $\lambda_m < \lambda_{med} = \lambda_M$ permits one to show that there is a uniaxial phase corresponding to F that has lower energy than the given plane-stres phase. Finally, if (x, y) lies on the remaining vertical side $\{\lambda_m^{-1}\} \times [\lambda_M^{-1}, \lambda_{med}^{-1}]$

of $\mathcal{R}_L$, then it is a routine matter to show that the assumed minimization property and the assumed violation of the no-tension condition together lead to the inequality $\lambda_m > \lambda_{med}$, a contradiction. The case where the disarrangement phase under consideration is in the plane-stress category has now been completed.

Suppose next that G is a uniaxial stress phase corresponding to F that is stable and violates the no-tension condition (35). Then there is a unit vector e and a rotation R satisfying (20) such that G is given by (21), the free energy is given by (27), and the Cauchy stress is given by (26). If we put $x := B_F^{-1}e \cdot e$, then $(20)_2$, (27), and (26) here read

$$x \neq r^{-1/4} \tag{58}$$

$$\frac{2}{\beta}\Psi_{\alpha\beta} = 3r^{1/3}x^{1/3} + x^{-1} \tag{59}$$

$$\frac{\det F}{\beta}T = \frac{1-(rx^4)^{1/3}}{x}e \otimes e, \tag{60}$$

and the accommodation inequality (28) becomes

$$r \leq (\det F)^3 x^{1/2} = \lambda_M^{3/2}\lambda_{med}^{3/2}\lambda_m^{3/2}x^{1/2}. \tag{61}$$

We denote by $\mathcal{I}$ the set of numbers $x' = B_F^{-1}e' \cdot e'$ with e' a unit vector vector satisfying $B_F^{-1}e' \cdot e' \neq r^{-1/4}$, and we note that $\mathcal{I}$ differs from the interval $[\lambda_M^{-1}, \lambda_m^{-1}]$ by at most the singleton $\{r^{-1/4}\}$. Because there holds

$$\frac{d}{dx}\frac{2}{\beta}\Psi_{\alpha\beta} = r^{1/3}x^{-2/3} - x^{-2} = x^{-2}(r^{1/3}x^{4/3} - 1),$$

the relation (60) can be written

$$\frac{\det F}{\beta}T = -\,x\frac{d}{dx}(\frac{2}{\beta}\Psi_{\alpha\beta})\, e \otimes e. \tag{62}$$

Suppose now that x is in the interior of the set $\mathcal{I}$. Since by assumption the vector e yields the minimum free energy over all disarrangement phases corresponding to F, the interior point x minimizes the free energy over the set $\mathcal{I}$, so that the derivative of the free energy at x vanishes. The formula (62) for T then tells us that the Cauchy stress vanishes and, therefore, the no-tension condition is satisfied. Because we assumed that the no-tension condition is violated, we conclude that x must be a boundary point of $\mathcal{I}$ that is in the set $\mathcal{I}$. However, the boundary of the set $\mathcal{I}$ is the set $\{\lambda_M^{-1}, \lambda_m^{-1}, r^{-1/4}\}$ or the set $\{\lambda_M^{-1}, \lambda_m^{-1}\}$, depending upon whether or not $r^{-1/4} \in (\lambda_M^{-1}, \lambda_m^{-1})$. The relation (58) tells us that $x = \lambda_M^{-1}$ or $x = \lambda_m^{-1}$, and the violation of the no-tension condition along with (62) tell us that $\frac{d}{dx}\frac{2}{\beta}\Psi_{\alpha\beta} < 0$. Since x minimizes $\Psi_{\alpha\beta}$ on the set $\mathcal{I}$, we may conclude that $x = \lambda_m^{-1}$ as well as $r^{1/3}\lambda_m^{-4/3} - 1 = r^{1/3}x^{4/3} - 1 < 0$. Consequently, this inequality and (61) yield the inequalities

$$r < \lambda_m^4 \quad \text{and } r \leq \lambda_M^{3/2}\lambda_{med}^{3/2}\lambda_m^{3/2}\lambda_m^{-1/2} = \lambda_M^{3/2}\lambda_{med}^{3/2}\lambda_m$$

and we conclude that

$$\begin{aligned} r^3 &= r^2 r < (\lambda_M^{3/2}\lambda_{med}^{3/2}\lambda_m)^2\lambda_m^4 \\ &= (\lambda_M\lambda_{med}\lambda_m)^3\lambda_m^3 \leq ((\det F)^2)^3(\det F)^2 \\ &= (\det F)^8. \end{aligned}$$

This inequality permits us to conclude from (31) that F admits not only the given uniaxial stress disarrangment phase G, but also loose phases $G' = r^{1/8}R'$ that, according to the comparisons made earlier, have lower energy than G. Thus, a uniaxial stable disarrangement phase cannot violate the no-tension condition.

Our argument is completed by noting that every loose disarrangement phase has zero stress and, hence, cannot violate the no-tension condition.

5 Conclusions and outlook

We have provided here in the context of elastic aggregates undergoing purely dissipative disarrangements a notion of disarrangment phase G corresponding to a given macroscopic deformation F. Given the free energy response function $G \longmapsto \Psi(G)$, the disarrangement phases G are solutions of the consistency relation

$$D\Psi(G)(F^T - G^T) = 0 \tag{63}$$

that also satisfy the accommodation inequality

$$0 < \det G \leq \det F. \tag{64}$$

We pointed out previously (see [19]) that both relations are satisfied with $G = F$, the compact phase corresponding to F, in which the pieces of the aggregate undergo on average the macroscopic deformation F, and we also showed there that, under mild restrictions on Ψ, there also exist loose phases $G = \varsigma_{\min}R$, provided that $\varsigma_{\min}^3 \leq \det F$. For the loose phases, the pieces of the aggregate on average undergo an expansion $\varsigma_{\min}I$ followed by a rotation R that achieve a state of zero stress and (globally) minimum free energy.

In addition, we have provided here a notion of material stability by considering *stable* disarrangement phases corresponding to F, i.e., minimizers G of the free energy $\Psi(G)$ subject to the constraints (63) and (64), and we point out that the loose phases corresponding to F, when they exist, always are stable. We focussed our considerations in this article on the two-parameter free energy response

$$\Psi_{\alpha\beta}(G) = \tfrac{\alpha}{2}(\det G)^{-2} + \tfrac{\beta}{2}tr(GG^T), \tag{65}$$

and we found for an arbitrary macroscopic deformation F all solutions G of (63) and (64) and, hence, all disarrangement phases corresponding to F. Included in this portfolio of disarrangement phases are not only the compact and loose phases, but also a one-parameter family of phases G_u in which the Cauchy stress $T = (\det F)^{-1}D\Psi(G_u)F^T$ is uniaxial, as well as a two-parameter family of phases G_p in which $T = (\det F)^{-1}D\Psi(G_p)F^T$ is planar (see Remarks 2 and 3). We employed our catalog of the disarrangement phases for $\Psi_{\alpha\beta}$ to obtain our main result in this paper, that every *stable* disarrangement phase has the no-tension property:

$$Ta \cdot a \leq 0 \;\text{ for all vectors } a. \tag{66}$$

This result provides an unexpected connection to the widely studied class of no-tension materials, in which the stress by assumption is a linear function of the infinitesimal elastic strain (which corresponds here to our $\frac{1}{2}(G+G^T)-I$) and in which, also by assumption, T has the no-tension property (66). Such no-tension materials are used to model structures composed of masonry-like elements that can support large compressive tractions but that separate or crack under tensile tractions, Our result "stability implies no-tension" shows that the present setting in which elastic aggregates undergo purely dissipative disarrangements provides an alternative and broader perspective for the important subject of no-tension materials.

In a paper under preparation we shall determine for several families $\{F_\lambda\}$ all of the stable disarrangement phases corresponding to F_λ for the free energy response $\Psi_{\alpha\beta}$. These results will provide confirmation of the present result "stability implies no-tension" for the particular macroscopic deformations F_λ and also provide an explicit catalogue of unstable disarrangement phases. Unstable phases are important in the statics and dynamics of elastic aggregates, because in some circumstances unstable phases can satisfy prescribed boundary conditions while stable phases cannot. For example, loose phases, when they can form, are always stable and stress-free. Moreover, in some circumstances loose phases are the only stable disarrangement phases available. Consequently, non-zero prescribed boundary tractions cannot be supported by these particular stable phases. We expect that solutions of boundary value problems in the present context will consist typically of coexistent disarrangement phases, some stable and some not. This situation requires that the static compatibility condition at phase boundaries be formulated and coupled with the differential equations that govern the macroscopic placement field corresponding to each phase present in the body.

The notions of disarrangement phase and of stable disarrangement phase are meaningful in the broader context of the field theory of elasticity with disarrangements and were used implicitly in the article [2] in defining and analyzing "submacroscopically stable equilibria" of elastic bodies. Examples there provide not only a catalogue of disarrangement phases for two model free energies but also the partial differential equations satisfied by the macroscopic placement field g in each stable disarrangement phase. The coexistence of both stable and unstable disarrangement phases alluded to above for elastic aggregates also will be the subject of further study in the broader context of elasticity with disarrangements.

6 Appendix: Derivation and submacroscopic interpretation of the consistency relation

The consistency relation (63) is a key ingredient in formulating the concepts of disarrangement phase and stable disarrangement phase. The form given in (63) is the result of specialization of the original consitency relation derived in detail in [1] to the case of purely dissipative disarrangements $D_M\Psi = 0$, and we sketch here for the convenience of the reader the principal ideas in that derivation, as well as a submacroscopic interpretation available in the case of purely dissipative disarrangements . Central to the derivation is the factorization of a structured deformation (g, G)

$$(g, G) = (g, \nabla g) \circ (i, K) \tag{67}$$

in which i is the identity mapping and $K := (\nabla g)^{-1} G$. The definition of the composition operation $\circ$ on the right-hand side requires that, in the usual manner, g be composed with the identity mapping i and that the tensor field ∇g and the tensor field K be composed pointwise as linear mappings. The result of these compositions is the pair (g, G) on the left-hand side. The structured deformation (i, K) is called the purely submacroscopic part of (g, G): it does not move material points but does introduce purely submacroscopic disarrangements when $I = \nabla i$ differs from K. The structured deformation $(g, \nabla g)$ introduces no disarrangements and is called the classical part of (g, G). Since the classical part $(g, \nabla g)$ describes the deformation from the reference configuration to the deformed configuration, we may view the purely submacroscopic part (i, K) as taking the body from a "virgin" configuration (in which no disarrangements have been produced) into the reference configuration, and the consistency relation centers on the tractions that can arise through (i, K) in terms of the Piola-Kirchhoff stress S (defined on the reference configuration).

We consider the vector calculus formula:

$$\begin{aligned}\det K \ \operatorname{div} S &= \operatorname{div}\left((\det K)\ SK^{-T}\right) + \operatorname{div}\left((\det K)\ S - (\det K)\ SK^{-T}\right) \\ &\quad -S\left(\nabla \det K\right) \end{aligned} \tag{68}$$

obtained by calculating $\operatorname{div}((\det K)S)$ using a product rule, adding and subtracting $\operatorname{div}\left((\det K)\ SK^{-T}\right)$, and then solving for $\det K \ \operatorname{div} S$. Application of the Approximation Theorem to the structured deformation (i, K) and repeated use of the divergence theorem and change of variables formulas in connection with the approximatations permit us ([20], Part Two, Section 1) to identify the terms in this identity in the following way:

terms in (68)	**interpretation**
$\det K \ \operatorname{div} S$	volume density of tractions Sn on prescribed surfaces and on disarrangement sites
$\operatorname{div}((\det K)\ SK^{-T})$	volume density of tractions Sn on prescribed surfaces only

These interpretations permit us to call $S_\backslash := (\det K)\ SK^{-T}$ the stress without disarrangements and $S_d = (\det K)\ S - (\det K)\ SK^{-T}$ the stress due to disarrangments. They also yield multiplicative and additive formulas involving the stresses S, $S_\backslash$, and S_d:

$$\begin{aligned} S &= (\det K)^{-1} S_\backslash K^T \\ (\det K)S &= S_\backslash + S_d. \end{aligned}$$

Elimination of $(\det K)S$ from these two formulas leads to the relation $S_\backslash K^T = S_\backslash + S_d$ which is equivalent to the consistency relation in the useful form [1]:

$$S_\backslash M^T + S_d F^T = 0. \tag{69}$$

This relation is based on the multiscale geometry of structured deformations and on the notion of contact forces, and it does not entail any constitutive assumptions. However, the constitutive assumptions [1]:

$$\psi = \Psi(G, M), \ \ S_\backslash = (\det K) D_G \Psi(G, M), \text{ and } S_d = (\det K) D_M \Psi(G, M)$$

permit one to write the consistency relation (69) in the final form

$$D_G \Psi(G, M) M^T + D_M \Psi(G, M) F^T = 0 \tag{70}$$

and the assumption that the free energy does not depend upon M (purely dissipative disarrangements) leads to the consistency relation (63) introduced at the beginning of this article.

The definition $M = F - G$ of the disarrangement tensor M, the consistency relation (63), and the stress relation $S = D_G \Psi + D_M \Psi$ imply for the case of purely dissipative disarrangements $D_M \Psi = 0$ that the consistency relation takes the simple form

$$SM^T = 0. \tag{71}$$

The analysis in [21] provides the following "identification relation" for the tensor $M(X)$ in terms of an appropriate sequence $n \longmapsto f_n$ of piecewise smooth, injective functions:

$$M(X) = \lim_{r \longrightarrow 0} \lim_{n \longrightarrow \infty} vol(B_r(X))^{-1} \int_{B_r(X) \cap S(f_n)} [f_n](Y) \otimes N_Y \, dA_Y \tag{72}$$

where $B_r(X)$ denotes the ball centered at X of radius r, $S(f_n)$ denotes the jump set of f_n, $[f_n](Y)$ denotes the jump of f_n at Y, and N_Y denotes the normal at Y to the jump set. The identification relation (72) for $M(X)$ tells us that this tensor is a limit of averages of the jumps of approximating deformations f_n as n tends to ∞ and as the balls over which the averages are taken shrink to the point X. Consequently, the left-hand side of the consistency relation in the form (71) when evaluated at the point X possesses the following identification relation:

$$S(X)M(X)^T = \lim_{r \longrightarrow 0} \lim_{n \longrightarrow \infty} vol(B_r(X))^{-1} \int_{B_r(X) \cap S(f_n)} S(X)N_Y \otimes [f_n](Y) dA_Y \tag{73}$$

in which the integrand $S(X)N_Y \otimes [f_n](Y)$ is the tensor product of the traction $S(X)N_Y$ vector at the point Y in $B_r(X) \cap S(f_n)$, calculated using the stress at the center of the ball, and of the jump $[f_n](Y)$. This identification relation permits us to give the following interpretation to the consistency relation in the form (71): *for the case of purely dissipative disarrangements the consistency relation (71) is the assertion that, on average, the tractions vanish at points of the body where submacroscopic slips or separations occur.*

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