A Note on Two Scale Compactness for p=1

Laura Bufford

Department of Mathematical Sciences
Carnegie Mellon University, Pittsburgh, PA 15213, USA
lbufford@andrew.cmu.edu

Irene Fonseca

Department of Mathematical Sciences
Carnegie Mellon University, Pittsburgh, PA 15213, USA
fonseca@andrew.cmu.edu

Abstract. In this paper the notion of two-scale convergence introduced by G. Nguetseng and G. Allaire is extended to the case of bounded sequences in $L^1(\Omega)$, where Ω is any open subset of $\mathbb{R}^N$. Three different approaches will be considered: an adaptation of the method used in $L^p(\Omega)$ with $p > 1$, a measure-theoretic argument, and the periodic unfolding technique.

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1 Introduction

The method of two-scale convergence, introduced by G. Nguetseng in [18] and further developed by G. Allaire in [1], is an important tool in the study of homogenization theory. Although periodicity poses constraints on physically realistic models, it is generally agreed that understanding the effective behavior of periodically structured composite materials may aid in the study of more complex media. Accordingly, the theory of two-scale convergence has played an important role in the study of PDEs and their applications in homogenization.

Both Nguetseng and Allaire restricted most of their interest to the case of two-scale convergence in $L^2(\Omega)$. The proof by Allaire in [1] of two-scale compactness in $L^2(\Omega)$ relies on duality and the separability of $L^2(\Omega)$. As stated in his paper [1], this proof easily extends to the case of two-scale compactness in $L^p(\Omega)$ with $1 < p \leq +\infty$. This is the form of two-scale compactness that is most commonly used in the literature. Unfortunately, the arguments used for the case when $1 < p \leq +\infty$ cannot be applied to the case when $p = 1$ due to a lack of separability of the dual of $L^1(\Omega)$, $L^\infty(\Omega)$. The case of $p = 1$ is rarely mentioned explicitly.

A few authors have touched on the problem, including Holmbom, Silfver, Svanstedt and Wellander in [16] and A. Visintin in [19], although detailed arguments seem to be unavailable in the literature. In [5, 6] the authors address a related case of two-scale convergence in generalized Besicovitch spaces where there is also lack of separability.

In this paper we present three proofs for the two-scale compactness of bounded sequences in $L^1(\Omega)$ under appropriate assumptions. To be precise,

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^N$. Let $\{u_\varepsilon\} \subset L^1(\Omega)$ be a bounded sequence in $L^1(\Omega)$, equi-integrable, and assume that for all $\eta > 0$ there exists an open set $E \subset \Omega$ such that $|E| < +\infty$ and*

$$\sup_{\varepsilon>0} \int_{\Omega\setminus E} |u_\varepsilon(x)| dx < \eta.$$

Then there exists a subsequence (not relabled) such that $\{u_\varepsilon\}$ two-scale converges to some $u_0 \in L^1(\Omega \times Y)$. In particular, $u_\varepsilon \rightharpoonup \bar{u}_0$ in $L^1(\Omega)$, with $\bar{u}_0(x) := \int_Y u_0(x, y)dy$.

The first proof of this theorem uses a truncation argument in order to apply two-scale compactness results for $p > 1$. The second makes use of the two-scale compactness proved for Radon measures by M. Amar in [2]. The last approach relies on the periodic unfolding characterization of two-scale limits, as introduced in [7] (see also [9, 10, 19]). The latter proof is the simplest and most intuitive, due to the fact that the periodic unfolding method reduces two-scale convergence in $L^p(\Omega)$ to standard weak L^p convergence in $\Omega \times Y$ (where Y is the period of the oscillations) of the unfolded functions, thus allowing us to replace rapidly oscillating test functions with non-oscillatory test functions. This method has been used in many contexts, including electromagnetism, homogenization in a domain with oscillating boundaries, and thin junctions in linear elasticity [3, 4, 8, 11, 12, 14, 15, 17].

2 Preliminaries

In this paper $\{\varepsilon\} = \{\varepsilon_n\}_{n=1}^{\infty}$ stands for a generic decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$.

We recall the definition of two-scale convergence [1]. Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $Y := (0, 1)^N$. We denote by $C^\infty_\#(Y)$ the set of smooth, periodic functions on $\mathbb{R}^N$ with period Y. In the following, for E a measurable set in $\mathbb{R}^N$, $|E|$ denotes the N-dimensional Lebesgue measure of E .

Definition 2.1. *Let $\{u_\varepsilon\} \subset L^p(\Omega)$ where $1 \le p \le +\infty$. Then $\{u_\varepsilon\}$ two-scale converges to a function $u_0 \in L^p(\Omega \times Y)$ if*

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x, y)\, \psi(x, y)\, dydx \tag{2.1}$$

for all $\psi \in C^\infty_c[\Omega; C^\infty_\#(Y)]$.

We will denote two-scale convergence by $u_\varepsilon \overset{2-s}{\rightharpoonup} u_0$. If a sequence $\{u_\varepsilon\}$ in $L^p(\Omega)$ two-scale converges for $1 < p \le +\infty$, since we may consider $C^\infty_c(\Omega)$ as a subset of $C^\infty_c[\Omega; C^\infty_\#(Y)]$ and as $C^\infty_c(\Omega)$ is dense in $L^{p'}(\Omega)$, then we also know, from [1], that $u_\varepsilon \rightharpoonup \int_Y u_0(x, y)dy$

in $L^p(\Omega)$ ($\overset{\star}{\rightharpoonup}$ if $p = +\infty$). This argument does not apply to $p = 1$, because $C_c^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$. However, this is a property we would like to preserve for two-scale compactness theorem for L^1 functions. Hence, we will assume the Dunford-Pettis criterion for weak sequential compactness in $L^1(\Omega)$ (see [13]) on $\{u_\varepsilon\}$ in our main theorem. We recall the definition of equi-integrability.

Definition 2.2. *A family $\mathcal{F}$ of measurable functions $f : \Omega \to [-\infty, +\infty]$ is said to be* equi-integrable *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\int_E |f| dx \leq \varepsilon$$

for all $f \in \mathcal{F}$ and for every measurable set $E \subset \Omega$ such that $|E| \leq \delta$

Theorem 2.3 (Dunford-Pettis)**.** *A family $\mathcal{F} \subset L^1(\Omega)$ is weakly sequentially precompact if and only if*

(i) $\mathcal{F}$ is bounded in $L^1(\Omega)$,

(ii) $\mathcal{F}$ is equi-integrable,

(iii) for every $\eta > 0$ there exists a measurable set $E \subset \Omega$ with $|E| < +\infty$ such that

$$\sup_{f \in \mathcal{F}} \int_{\Omega \setminus E} |u| dx \leq \eta. \tag{2.2}$$

Remark 2.4. *By the regularity properties of $\mathcal{L}^N$, it can be shown that assuming conditions (ii) and (iii) is equivalent to assuming (ii) and (iii'), where in (iii') E is an open bounded set of finite measure.*

3 Method Using Two-Scale Compactness for $p > 1$

Our first proof relies on the two-scale compactness result for $p > 1$, as proved by Allaire in Corollary 1.15 in [1]. To be precise,

Theorem 3.1. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, with $1 < p \leq +\infty$. There exists a function $u_0(x, y)$ in $L^p(\Omega \times Y)$ such that, up to a subsequence, $\{u_\varepsilon\}$ two-scale converges to u_0.*

We are able to use this result to prove the following:

Proposition 3.2. *Let Ω be an open subset of $\mathbb{R}^N$ with $|\Omega| < +\infty$. If $\{u_\varepsilon\} \subset L^1(\Omega)$ is a bounded equi-integrable sequence, then there exists a subsequence (not relabled) such that $\{u_\varepsilon\}$ two-scale converges to $u_0 \in L^1(\Omega \times Y)$.*

Proof. For $M > 0$, let τ_M be the truncating operator $\tau_M : L^1(\Omega) \to L^1(\Omega)$ defined by

$$\tau_M u(x) := \begin{cases} u(x) & \text{if } |u(x)| \leq M, \\ M & \text{if } u(x) > M, \\ -M & \text{if } u(x) < -M, \end{cases}$$

where $u \in L^1(\Omega)$. Since $|\Omega| < +\infty$, if $u \in L^1(\Omega)$ then $\tau_M u \in \cap_{1<p<+\infty} L^p(\Omega)$ and (iii) in Theorem 2.3 is trivially satisfied. Equi-integrability of $\{u_\varepsilon\}$ and Theorem 2.3 imply that there exists a weakly convergent subsequence and so, without loss of generality, we assume that $u_\varepsilon \rightharpoonup \bar{u}$ in $L^1(\Omega)$ for some $\bar{u} \in L^1(\Omega)$.

Step 1: Consider first the case in which $u_\varepsilon \geq 0$ a.e. in Ω and for all $\varepsilon > 0$. Fix $M > 0$. By Theorem 1.2 in [1] we know that, up to a subsequence (not relabeled), $\{\tau_M u_\varepsilon\} \overset{2-s}{\rightharpoonup} u_M$ for some $u_M \in L^2(\Omega \times Y)$, i.e.

$$\lim_{\varepsilon \to 0} \int_\Omega \tau_M u_\varepsilon(x)\, \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_M(x,y)\, \psi(x,y)\, dy dx$$

for all $\psi \in C_c^\infty[\Omega; C_\#^\infty(Y)]$. From this we deduce that $u_M \geq 0$.

We will extract a two-scale convergent subsequence as follows: For $M = 1$ let $\{u_{\varepsilon(1)}\}$ be a subsequence of $\{u_\varepsilon\}$ such that $\{\tau_1 u_{\varepsilon(1)}\}$ two-scale converges to a function $u_1 \in L^2(\Omega \times Y)$. Recursively, for $M > 1$, $M \in \mathbb{N}$, apply the compactness theorem for $p = 2$ in [1] to the sequence $\{\tau_M u_{\varepsilon(M-1)}\}$ to obtain $\{u_{\varepsilon(M)}\} \subset \{u_{\varepsilon(M-1)}\}$ such that $\{\tau_M u_{\varepsilon(M)}\}$ two-scale converges to some $u_M \in L^2(\Omega \times Y)$. Since $\{\varepsilon^{(M+1)}\}$ is a subsequence of $\{\varepsilon^{(M)}\}$, we have that $\tau_M u_{\varepsilon(M+1)} \overset{2-s}{\rightharpoonup} u_M$. In turn, as $u_\varepsilon \geq 0$ a.e., then $\tau_{M+1} u_{\varepsilon(M+1)} \geq \tau_M u_{\varepsilon(M+1)}$, and thus, passing the the two-scale limit we conclude that $u_{M+1} \geq u_M$ a.e. Let

$$u^+(x,y) := \sup_M u_M(x,y) = \lim_{M \to +\infty} u_M(x,y). \tag{3.3}$$

Next we show that $u^+ \in L^1(\Omega \times Y)$. Consider an increasing sequence of test functions $\{\varphi_n\} \subset C_0^\infty(\Omega; [0,1])$ such that $\varphi_n \equiv 1$ in $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\} \cap B(0,n)$ and $\varphi_n \equiv 0$ in $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq \frac{1}{2n}\} \cup (\mathbb{R}^N \setminus B(0, n+1))$. Then $\varphi_n \in C_c^\infty[\Omega; C_\#(Y)]$ and

$$\begin{aligned} +\infty > \limsup_{\varepsilon \to 0} \int_\Omega u_\varepsilon dx &\geq \limsup_{\varepsilon^{(M)} \to 0} \int_\Omega \tau_M u_{\varepsilon(M)}(x) dx \\ &\geq \lim_{\varepsilon^{(M)} \to 0} \int_\Omega \tau_M u_{\varepsilon(M)}(x) \varphi_n(x) dx = \int_\Omega \int_Y u_M(x,y) \varphi_n(x) dy dx \end{aligned}$$

for all $M, n \in \mathbb{N}$. Taking the limit first in n as $n \to +\infty$, and applying the Monotone Convergence Theorem, we obtain

$$+\infty > \limsup_{\varepsilon \to 0} \int_\Omega u_\varepsilon dx \geq \int_\Omega \int_Y u_M(x,y) dy dx,$$

and next taking the limit in M as $M \to +\infty$, by (3.3) we deduce

$$+\infty > \limsup_{\varepsilon \to 0} \int_\Omega u_\varepsilon dx \geq \int_\Omega \int_Y u^+(x,y) dy dx. \tag{3.4}$$

Hence, as u^+ is non-negative, $u^+ \in L^1(\Omega \times Y)$.

We claim that, up to a subsequence, for all $\psi \in C_c^\infty \left[\Omega; C_\#^\infty(Y)\right]$

$$\lim_{\varepsilon\to 0^+} \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u^+(x,y)\psi(x,y)dydx.$$

Fix $\psi \in C_c^\infty[\Omega; C_\#^\infty(y)]$. We have

$$\begin{aligned}
&\int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \int_Y u^+(x,y)\psi(x,y)dydx \\
&\quad = \int_\Omega \tau_M u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}\right) dx - \int_\Omega \int_Y u_M(x,y)\psi(x,y)dydx \\
&\quad + \int_\Omega \int_Y \left(u_M(x,y) - u^+(x,y)\right)\psi(x,y)dydx \\
&\quad + \int_\Omega \left(u_\varepsilon(x) - \tau_M u_\varepsilon(x)\right)\psi\left(x, \frac{x}{\varepsilon}\right) dx.
\end{aligned}$$

First, we analyze the convergence of the first difference in the right hand side above. Consider the diagonalizing sequence $\{\hat{\varepsilon}\}$ where $\hat{\varepsilon}_j := \varepsilon_j^{(j)}$, the jth element of the subsequence $\{\varepsilon^{(j)}\}$. We claim that

$$\lim_{\hat{\varepsilon}\to 0^+} \left| \int_\Omega \tau_M u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_\Omega \int_Y u_M(x,y)\psi(x,y)dydx \right| = 0 \tag{3.5}$$

for all M. This can be easily seen by observing that for $j > M$, $\{\hat{\varepsilon}_j\}$ is a subsequence of $\{\varepsilon^{(M)}\}$. Hence,

$$\begin{aligned}
\lim_{\hat{\varepsilon}\to 0^+} \int_\Omega \tau_M u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx &= \lim_{\varepsilon^{(M)}\to 0^+} \int_\Omega \tau_M u_{\varepsilon^{(M)}}(x)\psi\left(x, \frac{x}{\varepsilon^{(M)}}\right) dx \\
&= \int_\Omega \int_Y u_M(x,y)\psi(x,y)\, dxdy,
\end{aligned}$$

proving (3.5). We conclude that

$$\begin{aligned}
&\lim_{\hat{\varepsilon}\to 0^+} \left| \int_\Omega u_{\hat{\varepsilon}}(x)\psi\left(x, \frac{x}{\hat{\varepsilon}}\right) dx - \int_\Omega \int_Y u^+(x,y)\psi(x,y)dydx \right| \\
&\quad \le ||\psi||_{L^\infty(\Omega\times Y)} \lim_{M\to\infty} \left[\int_\Omega \int_Y \left(u^+(x,y) - u_M(x,y)\right) dydx + \sup_{\varepsilon>0} \int_\Omega \left(u_\varepsilon(x) - \tau_M u_\varepsilon(x)\right) dx \right].
\end{aligned} \tag{3.6}$$

From (3.3), taking into account that $u^+ \in L^1(\Omega \times Y)$, we have by the Monotone Convergence Theorem,

$$\lim_{M\to\infty} \int_\Omega \int_Y \left(u^+(x,y) - u_M(x,y)\right) dydx = 0.$$

Also,

$$\sup_{\varepsilon>0} \int_\Omega (u_\varepsilon(x) - \tau_M u_\varepsilon(x))dx \le \sup_{\varepsilon>0} \int_{\{u_\varepsilon > M\}} u_\varepsilon(x)dx,$$

so using the equi-integrability of $\{u_\varepsilon\}$ and the fact that $|\Omega| < +\infty$, we conclude that

$$\lim_{M\to\infty} \int_\Omega (u_\varepsilon(x) - \tau_M u_\varepsilon(x))dx = 0.$$

By (3.6), this concludes the proof.

Step 2: In this case the sequence $\{u_\varepsilon\}$ may take both positive and negative values. The positive and negative parts of these functions can be considered separately, precisely, let $u_\varepsilon^+ := u_\varepsilon \chi_{\{u_\varepsilon \geq 0\}}$ and $u_\varepsilon^- := -u_\varepsilon \chi_{\{u_\varepsilon \leq 0\}}$. Then for all ε, $u_\varepsilon = u_\varepsilon^+ - u_\varepsilon^-$, where $u_\varepsilon^+ \geq 0$ and $u_\varepsilon^- \geq 0$. From the previous step, there exists a subsequence $\{\hat{\varepsilon}^+\} \subset \{\varepsilon\}$ such that $\{u_{\hat{\varepsilon}^+}\}$ two-scale converges to some $u^+ \in L^1(\Omega \times Y)$. Applying that step again we can extract an additional subsequence $\{\hat{\varepsilon}^-\} \subset \{\hat{\varepsilon}^+\}$ such that $\{u_{\hat{\varepsilon}^-}\}$ two-scale converges to some $u^- \in L^1(\Omega \times Y)$. Let $u_0 := u^+ - u^-$. Then $u_0 \in L^1(\Omega \times Y)$, and for all $\psi \in C_c^\infty[\Omega; C_\#^\infty(Y)]$

$$
\begin{aligned}
&\lim_{\hat{\varepsilon}^- \to 0^+} \left| \int_\Omega u_{\hat{\varepsilon}^-}(x) \psi\left(x, \frac{x}{\hat{\varepsilon}^-}\right) dx - \int_\Omega \int_Y u_0(x,y) \psi(x,y) dy dx \right| \\
&\quad \leq \lim_{\hat{\varepsilon}^- \to 0^+} \left| \int_\Omega u_{\hat{\varepsilon}^-}^+ \psi\left(x, \frac{x}{\hat{\varepsilon}^-}\right) dx - \int_\Omega \int_Y u^+(x,y) \psi(x,y) dy dx \right| \\
&\qquad + \lim_{\hat{\varepsilon}^- \to 0^+} \left| \int_\Omega u_{\hat{\varepsilon}^-}^-(x) \psi\left(x, \frac{x}{\hat{\varepsilon}^-}\right) dx - \int_\Omega \int_Y u^-(x,y) \psi(x,y) dy dx \right| \\
&\quad = 0.
\end{aligned}
$$

■

Now that we have established two-scale compactness assuming that Ω is of finite measure, we may use this result in order to prove Theorem 1.1.

First Proof of Theorem 1.1. For each $k \in \mathbb{N}$ let $E_k \subset \Omega$ be open and such that $|E_k| < +\infty$, $E_{k-1} \subset E_k$, and

$$\sup_{\varepsilon > 0} \int_{\Omega \setminus E_k} |u_\varepsilon(x)| dx < \frac{1}{k}.$$

Step 1: Again, we first address the case in which $u_\varepsilon \geq 0$ a.e. in Ω. By Proposition 3.2, for $k = 1$ there exists a subsequence $\{\varepsilon^{(1)}\} \subset \{\varepsilon\}$ such that $\{u_{\varepsilon^{(1)}}\}$ two scale converges in L^1 to some $u_1^+ \in L^1(E_1 \times Y)$ with

$$\int_{E_1 \times Y} u_1^+(x,y) dy dx \leq \limsup_{\varepsilon \to 0} \int_{E_1} u_\varepsilon dx \leq \limsup_{\varepsilon \to 0} \int_\Omega u_\varepsilon dx =: C < +\infty$$

where we have used (3.4). For $k > 1$ extract $\{\varepsilon^{(k)}\} \subset \{\varepsilon^{(k-1)}\}$ such that $u_{\varepsilon^{(k)}} \overset{2-s}{\rightharpoonup} u_k^+$ for some $u_k^+ \in L^1(E_k \times Y)$ with

$$\int_{E_k \times Y} u_k^+(x,y) dy dx \leq C. \tag{3.7}$$

The function u_k^+ can be extended to be a function in $L^1(\Omega \times Y)$ by setting it to be zero on $(\Omega \setminus E_k) \times Y$. Consider the diagonalizing subsequence $\{\hat{\varepsilon}\}$ where $\hat{\varepsilon}_j := \varepsilon_j^{(j)}$, the jth element of the subsequence $\{\varepsilon^{(j)}\}$. We prove that $u_k^+ \leq u_j^+$ if $k \leq j$. Let $\psi \in C_c^\infty\left[E_k; C_\#^\infty(Y)\right]$ and define $\hat{\psi} \in C_c^\infty[E_j; C_\#^\infty(Y)]$ by $\hat{\psi} = \psi$ in E_k and $\hat{\psi} = 0$ on $E_j \setminus E_k$. Then

$$\begin{aligned}\int_{E_k}\int_Y u_j^+(x,y)\psi(x,y)dydx &= \int_{E_j}\int_Y u_j^+(x,y)\hat{\psi}(x,y)dydx \\ &= \lim_{\hat{\varepsilon}\to 0}\int_{E_j} u_{\hat{\varepsilon}}(x)\hat{\psi}\left(x,\frac{x}{\hat{\varepsilon}}\right)dx = \lim_{\hat{\varepsilon}\to 0}\int_{E_k} u_{\hat{\varepsilon}}(x)\psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx \\ &= \int_{E_k}\int_Y u_k^+(x,y)\psi(x,y)dydx\end{aligned}$$

and we conclude that $u_j^+ = u_k^+$ a.e. in $E_k \times Y$. The claim now follows by observing that $0 = u_k^+ \le u_j^+$ on $\Omega \setminus E_k$. Set $u^+ := \sup_k u_k^+ = \lim_{k\to\infty} u_k^+$. By (3.7),

$$\begin{aligned}\int_{\Omega\times Y} u^+(x,y)dydx &= \lim_{k\to\infty}\int_{\Omega\times Y} u_k^+(x,y)dydx \\ &= \lim_{k\to\infty}\int_{E_k\times Y} u_k^+(x,y)dydx \le C < +\infty.\end{aligned}$$

Hence $u^+ \in L^1(\Omega \times Y)$.

We claim that $\{u_{\hat{\varepsilon}}\}$ two-scale converges to u^+. Let $\psi \in C_c^\infty[\Omega; C_\#^\infty]$ be given. We have for all $k \in \mathbb{N}$

$$\begin{aligned}&\left|\int_\Omega u_{\hat{\varepsilon}}(x)\ \psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx - \int_\Omega\int_Y u^+(x,y)\,\psi(x,y)\,dydx\right| \\ &\quad\le \left|\int_{E_k} u_{\hat{\varepsilon}}(x)\psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx - \int_{E_k}\int_Y u_k^+(x,y)\,\psi(x,y)\,dydx\right| \\ &\qquad+ \left|\int_{E_k}\int_Y u^+(x,y) - u_k^+(x,y)dydx\right| \\ &\qquad+ \left|\int_{\Omega\setminus E_k} u_{\hat{\varepsilon}}(x)\psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx - \int_{\Omega\setminus E_k}\int_Y u^+(x,y)\,\psi(x,y)\,dydx\right|. \qquad (3.8)\end{aligned}$$

For a fixed $k \in \mathbb{N}$, recall that $\{\hat{\varepsilon}_j\}_{j=k}^\infty \subset \{\varepsilon_j^{(k)}\}_{j=k}^\infty$ so $u_{\hat{\varepsilon}} \overset{2-s}{\rightharpoonup} u_k^+$, and thus

$$\lim_{\hat{\varepsilon}\to 0}\left|\int_{E_k} u_{\hat{\varepsilon}}(x)\psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx - \int_{E_k}\int_Y u_k^+(x,y)\,\psi(x,y)\,dydx\right| = 0.$$

Also, recall that $u_k^+ = u^+$ in E_k, therefore

$$\left|\int_{E_k}\int_Y u^+(x,y) - u_k^+(x,y)dydx\right| = 0.$$

Moreover, by the Monotone Convergence Theorem and as $u_{\hat{\varepsilon}} \rightharpoonup u_j^+$ in $L^1(E_j)$,

$$\begin{aligned}\int_{\Omega\setminus E_k}\int_Y u^+(x,y)dydx &= \lim_{j\to\infty}\int_{(\Omega\setminus E_k)\cap E_j}\int_Y u_j^+(x,y)dydx \\ &= \lim_{j\to\infty}\lim_{\hat{\varepsilon}\to 0}\int_{(\Omega\setminus E_k)\cap E_j} u_{\hat{\varepsilon}}(x)dx \le \sup_\varepsilon \int_{\Omega\setminus E_k} u_\varepsilon(x)dx \le \frac{1}{k}.\end{aligned}$$

Also,

$$\left|\int_{\Omega\setminus E_k} u_{\hat{\varepsilon}}(x)\psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx - \int_{\Omega\setminus E_k}\int_Y u^+(x,y)\,\psi(x,y)\,dydx\right|$$
$$\leq 2||\psi||_{L^\infty(\Omega\times Y)}\sup_{\varepsilon>0}\int_{\Omega\setminus E_k} u_\varepsilon(x)dx \leq \frac{2||\psi||_{L^\infty(\Omega\times Y)}}{k},$$

By first taking the limit $\hat{\varepsilon}\to\infty$ and then the limit $k\to\infty$ in (3.8) we obtain

$$\lim_{\hat{\varepsilon}\to\infty}\left|\int_\Omega u_{\hat{\varepsilon}}(x)\psi\left(x,\frac{x}{\hat{\varepsilon}}\right)dx - \int_\Omega\int_Y u^+(x,y)\,\psi(x,y)\,dydx\right| = 0,$$

and thus $\{u_{\hat{\varepsilon}}\}$ two-scale converges in L^1 to u^+.

Step 2: A similar argument as in the previous proof can be used for the case in which u_ε may also take negative values.

Lastly, we notice that should a weak limit, $\bar{u}_0$, of $\{u_\varepsilon\}$ exist, then for all $\varphi\in C_c^\infty(\Omega)$

$$\int_\Omega \varphi(x)\bar{u}_0(x)dx = \lim_{\varepsilon\to 0}\int_\Omega \varphi(x)u_\varepsilon(x)dx = \int_\Omega\varphi(x)\int_Y u_0(x,y)dydx. \tag{3.9}$$

From this we see that $\bar{u}_0(x) = \int_Y u_0(x,y)dy$ a.e. $x\in\Omega$. From Theorem 2.3 we know that $\{u_\varepsilon\}$ is weakly sequentially precompact and from (3.9) it is easy to see that $\bar{u}_0(x)\rightharpoonup \int_Y u_0(x,y)dy$. ■

4 The Measure Approach

In [2] Amar defines two-scale convergence of measures. We will denote by $\mathcal{M}(\Omega)$ the set of all signed Radon measures on Ω, $C_0(\Omega;\mathbb{R}^N)$ is the space of all continuous functions that vanish on the boundary of Ω, and $C_0[\Omega;C_\#(Y)]$ is the space of all continuous functions $\varphi:\Omega\to C_\#(Y)$ that vanish on $\partial\Omega$ (see [2]).

Definition 4.1. *A sequence of measures $\{\mu_\varepsilon\}\subset\mathcal{M}(\Omega)$ is said to two-scale converge to a measure $\mu_0\in\mathcal{M}(\Omega\times Y)$ if for any function $\varphi\in C_0[\Omega;C_\#(Y)]$ we have*

$$\lim_{\varepsilon\to 0^+}\int_\Omega\varphi\left(x,\frac{x}{\varepsilon}\right)d\mu_\varepsilon(x) = \int_{\Omega\times Y}\varphi(x,y)d\mu_0(x,y). \tag{4.10}$$

Using an argument similar to that of Allaire in [1], if Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$, then the following compactness result for measures is obtained in Theorem 3.5 in [2].

Theorem 4.2. *Every bounded sequence of measures $\{\mu_\varepsilon\}_\varepsilon\in\mathcal{M}(\Omega)$, admits a subsequence $\{\mu_{\varepsilon_h}\}_h$ which two-scale converges to a measure $\mu_0\in\mathcal{M}(\Omega\times Y)$.*

We remark that the boundedness of Ω and the Lipschitz continuity of its boundary are not used in the proof of this result in Theorem 3.5 in [2]. Using this theorem we provide an alternate proof for the two-scale compactness of sequences bounded in $L^1(\Omega)$. We will use the following lemma.

Lemma 4.3. *Let λ be a positive Radon measure on an open subset $U \subset \mathbb{R}^N$. Then λ is absolutely continuous with respect to $\mathcal{L}^N$ ($\lambda << \mathcal{L}^N$) if and only if for all $\varepsilon > 0$ there exists a $\delta < 0$ such that for all $\varphi \in C_c^\infty(U;[0,1])$ with $|\mathrm{supp}(\varphi)| < \delta$*

$$\int_U \varphi(x)d\lambda(x) < \varepsilon.$$

Proof. First assume that $\lambda << \mathcal{L}^N$. Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda(A) < \varepsilon$ for all measurable sets A such that $|A| < \delta$. Let $\varphi \in C_c^\infty(U;[0,1])$ be such that $|\mathrm{supp}(\varphi)| < \delta$, and let $A := \mathrm{supp}(\varphi)$. Then

$$\int_U \varphi(x)d\lambda(x) \leq \int_A d\lambda(x) = \lambda(A) < \varepsilon.$$

Now, assume the alternate condition. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that for all $\varphi \in C_c^\infty(U;[0,1])$ with $|\mathrm{supp}(\varphi)| < \delta$

$$\int_U \varphi(x)d\lambda(x) < \frac{\varepsilon}{2}.$$

As λ is a Radon measure, every Borel set in U is outer regular and every open set in U is inner regular. Let $A \subseteq U$ be such that $|A| < \frac{\delta}{2}$. We claim that $\lambda(A) \leq \varepsilon$. By the outer regularity of λ, there exists an open set E, with $A \subseteq E$ and $|E| < \frac{2}{3}\delta$. We may use the inner regularity of λ to find a compact set $K \subset E$ such that

$$\lambda(E) \leq \lambda(K) + \frac{\varepsilon}{2}. \tag{4.11}$$

As K is compact and E is open, there exists a function $\varphi \in C_c^\infty(U;[0,1])$ such that $\varphi = 1$ on K and $\varphi = 0$ outside E. Then

$$\lambda(K) \leq \int_U \varphi(x)d\lambda(x) \leq \frac{\varepsilon}{2}, \tag{4.12}$$

where in the second inequality we have used the assumption. Hence, by (4.11) and (4.12),

$$\lambda(A) \leq \lambda(E) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and this concludes the proof that λ is absolutely continuous with respect to $\mathcal{L}^N$. ■

Second Proof of Theorem 1.1. ***Step 1:*** Assume that $u_\varepsilon \geq 0$ a.e. in Ω. By Theorem 4.2 there exists a Radon measure λ such that $u_\varepsilon \mathcal{L}^N \lfloor \Omega \overset{2-s}{\rightharpoonup} \lambda$. Note that, by (4.10), $\int_{\Omega\times Y} \varphi(x,y)d\lambda(x,y) \geq 0$ for all $\varphi \in C_c^\infty[\Omega; C_\#(Y)]$ such that $\varphi \geq 0$, so λ is a positive Radon measure.

We claim that λ is absolutely continuous with respect to $\mathcal{L}^{2N}\lfloor\Omega\times Y$, and for this purpose we will use Lemma 4.3. Fix $\eta>0$. By equi-integrability there exists $\delta>0$ such that for all measurable $A\subset\Omega$ such that $|A|<\delta$,

$$\sup_{\varepsilon>0}\int_A u_\varepsilon(x)dx<\eta. \tag{4.13}$$

Let $\varphi\in C_c^\infty(\Omega\times Y;[0,1])$.

First let us assume that $\mathrm{supp}(\varphi)\subseteq A\times B\subseteq\Omega\times Y$ where $A:=x_0+(-\rho,\rho)^N$ for some $x_0\in\Omega$, $B:=y_0+(-\rho,\rho)^N$ for some $y_0\in Y$, and $\rho>0$. We can take, without loss of generality, $\rho>\varepsilon$. Note also that $|A|=(2\rho)^N=2^N\rho^N$. Extend φ periodically in the variable y with period Y so that φ is an admissible test function for two-scale convergence. Then, for any $\varepsilon>0$

$$\mathrm{supp}\left(\varphi\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right)\subseteq\left\{x\in\Omega: x\in A \text{ and } \frac{x}{\varepsilon}\in k+B, k\in\mathbb{Z}^N\right\},$$

so

$$\left|\mathrm{supp}\ \varphi\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right|\leq\sum_{\{k\in\mathbb{Z}^N:A\cap\{\varepsilon k+\varepsilon B\}\neq\emptyset\}}|\varepsilon K+\varepsilon B|=\sum_{\{k\in\mathbb{Z}^N:A\cap\{\varepsilon k+\varepsilon B\}\neq\emptyset\}}\varepsilon^N|B|. \tag{4.14}$$

Now, if $k=(k_1,k_2,...,k_N)\in\mathbb{Z}^N$ is such that $A\cap\{\varepsilon k+\varepsilon B\}\neq\emptyset$, then there exists $b\in B$, with $b=(b_1,...,b_N)$ such that $\varepsilon k\in A-\varepsilon b$ or, equivalently, there exist $s_i\in(-\rho,\rho)$ (where $b_i=y_{0,i}+s_i$) such that

$$\varepsilon k_i\in\left(x_{0,i}-\rho-\varepsilon(y_{0,i}+s_i),x_{0,i}+\rho-\varepsilon(y_{0,i}+s_i)\right).$$

This is equivalent to

$$k_i\in X_{i,\varepsilon}+\left(\frac{-\rho}{\varepsilon}-s_i,\frac{\rho}{\varepsilon}-s_i\right)\qquad\text{with}\qquad X_{i,\varepsilon}:=\frac{x_{0,i}}{\varepsilon}-y_{0,i}.$$

Therefore, the number of integer valued vectors in $\mathbb{Z}^N$ such that $A\cap\{\varepsilon k+\varepsilon B\}\neq\emptyset$ is at most the number of integer valued vectors in $X+\left(-\frac{\rho}{\varepsilon}-\rho,\frac{\rho}{\varepsilon}+\rho\right)^N$ with $X\in\mathbb{R}^N$, which is the number of $k\in\mathbb{Z}^N$ such that $k\in\left(-\frac{\rho}{\varepsilon}-\rho,\frac{\rho}{\varepsilon}+\rho\right)^N\subset\left(\frac{-2\rho}{\varepsilon},\frac{2\rho}{\varepsilon}\right)^N$ (here we used the fact that $\rho>\varepsilon$), and this is at most $\left(\frac{4\rho}{\varepsilon}\right)^N$. In view of (4.14), we deduce that

$$\left|\mathrm{supp}\left(\varphi\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right)\right|\leq 4^N\frac{\rho^N}{\varepsilon^N}\varepsilon^N|B|=2^N|A||B|. \tag{4.15}$$

Now let φ be a function in $C_c^\infty(\Omega\times Y;[0,1])$ such that for $K:=\mathrm{supp}(\varphi)$, $|K|<\frac{\delta}{2^{N+1}}$. By the construction of the Lebesgue measure and the compactness of K, there exists a finite cover of K by sets of the type $A_i\times B_i:=(x_i+(-\rho_i,\rho_i)^N)\times(y_i+(-\rho_i,\rho_i)^N)$, $\rho_i>0$, $x_i\in\Omega$, $y_i\in Y$, such that $E\subseteq\bigcup_{i=1}^m A_i\times B_i\subset\Omega\times Y$ and

$$\sum_{i=1}^m|A_i\times B_i|=\sum_{i=1}^m|A_i||B_i|<\frac{\delta}{2^N}.$$

Set $\rho_0 := \min\{\rho_i : i = 1, 2, ..., m\}$. Then, for $\varepsilon < \rho_0$ and by (4.15),

$$\left|\operatorname{supp}\left(\varphi\left(\cdot, \frac{\cdot}{\varepsilon}\right)\right)\right| \le \sum_{i=1}^{m} \left|\{z \in \Omega : z \in A_i \text{ and } z \in \varepsilon k + \varepsilon B_i, k \in \mathbb{Z}^N\}\right|$$
$$\le 2^N \sum_{i=1}^{m} |A_i||B_i| < \delta, \tag{4.16}$$

and so, in view of (4.13) and (4.16), we have

$$\int_{\Omega\times Y} \varphi(x,y)d\lambda(x,y) = \lim_{\varepsilon\to 0} \int_{\Omega} u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)dx$$
$$\le \limsup_{\varepsilon\to 0} \int_{\operatorname{supp}\left(\varphi\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right)} u_\varepsilon(x)dx < \eta.$$

By Lemma 4.3 we deduce $\lambda << \mathcal{L}^{2N}\lfloor\Omega \times Y$, and so by the Radon-Nikodym Theorem there exists a function $u_0 \in L^1(\Omega \times Y)$ such that $\lambda = u_0\mathcal{L}^{2N}\lfloor\Omega \times Y$. Thus, for all $\varphi \in C_c^\infty[\Omega; C_\#(Y)]$

$$\lim_{\varepsilon\to 0} \int_{\Omega} \varphi\left(x, \frac{x}{\varepsilon}\right) u_\varepsilon(x)dx = \int_{\Omega\times Y} \varphi(x,y)u_0(x,y)dydx,$$

and $\{u_\varepsilon\}$ two scale converges in L^1 to u_0.

Step 2: The proof for the general case in which u_ε are allowed to take both positive and negative values can be completed in the same manner as Step 2 of the proof of Proposition 3.2. From here, it is also easy to see that $\bar{u}_0(x) = \int_Y u_0(x,y)dy$. ∎

5 Periodic Unfolding Approach

Recall that $\{u_\varepsilon\}$ is a family of functions in $L^1(\Omega)$. For the following we will extend u_ε by zero outside of Ω for convenience of notation. An alternate approach to the study of two-scale convergence, the periodic unfolding introduced in [7], involves defining a family of scale transformations $S_\varepsilon : \mathbb{R}^N \times [0,1)^N \to \mathbb{R}^N$ which, for $\varepsilon > 0$, are defined by

$$S_\varepsilon(x,y) := \varepsilon\mathcal{N}(x/\varepsilon) + \varepsilon y \quad \text{for } (x,y) \in \mathbb{R}^N \times [0,1)^N, \tag{5.17}$$

where

$$\mathcal{N}(x) := (\hat{n}(x_1), ..., \hat{n}(x_N)) \quad \text{for } x = (x_1, ..., x_N) \in \mathbb{R}^N,$$

and

$$\hat{n}(s) := \max\{n \in \mathbb{Z} : n \le s\} \quad \text{for } s \in \mathbb{R}.$$

Furthermore, let

$$\hat{r}(s) := s - \hat{n}(s) \in [0,1) \quad \text{for } s \in \mathbb{R}$$

and

$$\mathcal{R}(x) := x - \mathcal{N}(x) \in [0,1)^N \quad \text{for } x \in \mathbb{R}^N.$$

Using these scale transformations it is possible to define obtain a characterization of two-scale convergence as follows (see [19] Proposition 2.5 and (1.9)).

Proposition 5.1. *Let* $\{u_\varepsilon\} \subset L^1(\Omega)$. *Then*

$$u_\varepsilon \overset{2-s}{\rightharpoonup} u_0 \qquad \textit{if and only if} \quad u_\varepsilon \circ S_\varepsilon \rightharpoonup u_0 \quad \textit{in } L^1(\mathbb{R}^N \times [0,1)^N). \tag{5.18}$$

Additionally, we will use the following result in [19]:

Lemma 5.2. *Let* f *be a function in* $L^1(\mathbb{R}^N)$. *Then for any* $\varepsilon > 0$

$$\int_{\mathbb{R}^N} f(x)\, dx = \int_{\mathbb{R}^N \times [0,1)^N} f(S_\varepsilon(x,y))\, dxdy. \tag{5.19}$$

For our last proof of Theorem 1.1, we present a modified version of the proof of two-scale compactness by Visintin in [19], Proposition 3.2 (iii).

Third Proof of Theorem 1.1. From the periodic unfolding characterization (5.18) of two-scale limits, it is sufficient to prove that if $\{u_\varepsilon\}$ is weakly sequentially precompact in $L^1(\mathbb{R}^N)$ then so is $\{u_\varepsilon \circ S_\varepsilon\}$ in $L^1(\mathbb{R}^N \times [0,1)^N)$. In turn, the latter condition is equivalent to $\{u_\varepsilon \circ S_\varepsilon\}$ being weakly sequentially precompact in $L^1(\mathbb{R}^N \times Y)$, therefore, in view of Theorem 2.3 it is sufficient to check that conditions (i), (ii) and (iii) hold for the sequence $\{u_\varepsilon \circ S_\varepsilon\}$. Additionally, we may assume, without loss of generality, that $0 < \varepsilon < 1$.

(i) From (5.19) it is easily seen that if $\{u_\varepsilon\}$ is bounded in $L^1(\Omega)$ then so is $\{u_\varepsilon \circ S_\varepsilon\}$ in $L^1(\mathbb{R}^N \times Y)$.

(ii) By the classic de la Vallée-Poussin criterion, $\{u_\varepsilon\}$ is equi-integrable if and only if there exists a Borel function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{t\to+\infty} \frac{\varphi(t)}{t} = +\infty \qquad \text{and} \qquad \sup_\varepsilon \int_{\mathbb{R}^N} \varphi(|u_\varepsilon(x)|)dx < +\infty. \tag{5.20}$$

By (5.19),

$$\int_{\mathbb{R}^N} \varphi(|u_\varepsilon(x)|)dx = \int_{\mathbb{R}^N \times Y} \varphi(|u_\varepsilon(S_\varepsilon(x,y))|)dydx.$$

Therefore, the criterion (5.20) holds for $\{u_\varepsilon\}$ in $\mathbb{R}^N$ if and only if it holds for $\{u_\varepsilon \circ S_\varepsilon\}$ in $\mathbb{R}^n \times Y$.

(iii) Last we show that $\{u_\varepsilon \circ S_\varepsilon\}$ also inherits property (iii) from $\{u_\varepsilon\}$, i.e., we claim that for all $\eta > 0$ there exists a set $E \subset \mathbb{R}^N \times Y$ such that $|E| < +\infty$

$$\sup_\varepsilon \int_{(\mathbb{R}^N \times Y)\setminus E} |u_\varepsilon(S_\varepsilon(x,y))|dydx < \eta.$$

Fix $\eta > 0$ and in view of Remark 2.4 let E_0 be open, bounded and such that

$$\sup_\varepsilon \int_{\mathbb{R}^N \setminus E_0} |u_\varepsilon(x)|dydx < \eta. \tag{5.21}$$

Let $E := (E_0 + [-1,1]^N) \times Y$. Clearly $|E| < +\infty$, and we show that if $(x,y) \in (\mathbb{R}^N \times Y) \setminus E$ then $S_\varepsilon(x,y) \in \mathbb{R}^N \setminus E_0$. Indeed, $(\mathbb{R}^N \times Y) \setminus E = (\mathbb{R}^N \setminus (E_0 + [-1,1]^N)) \times Y$, thus the claim reduces to proving that if $x \in \mathbb{R}^N \setminus (E_0 + [-1,1]^N)$ then $S_\varepsilon(x,y) \in \mathbb{R}^N \setminus E_0$, or, equivalently,

$$\text{if} \qquad S_\varepsilon(x,y) \in E_0 \qquad \text{then} \qquad x \in E_0 + [-1,1]^N. \tag{5.22}$$

From the definition of S_ε we know that

$$x = S_\varepsilon(x, y) - \varepsilon \left[y - \mathcal{R}\left(\frac{x}{\varepsilon}\right)\right],$$

and $\varepsilon[y - \mathcal{R}(\frac{x}{\varepsilon})] \in [-1,1]^N$, and this asserts (5.22). We conclude that

$$\begin{aligned}
\int_{(\mathbb{R}^N \times Y)\setminus E} |u_\varepsilon(S_\varepsilon(x,y))| dydx &= \int_{(\mathbb{R}^N \setminus (E_0 + [-1,1]^N))\times Y} |u_\varepsilon(S_\varepsilon(x,y))| dydx \\
&\leq \int_{\mathbb{R}^N \times Y} \chi_{\mathbb{R}^N \setminus E_0}(S_\varepsilon(x,y))\, |u_\varepsilon(S_\varepsilon(x,y))|\, dydx \\
&= \int_{\mathbb{R}^N \setminus E_0} |u_\varepsilon(x)| dx < \eta
\end{aligned}$$

where in the last equality we have used (5.19) and (5.21).

We have shown that $\{u_\varepsilon \circ S_\varepsilon\}$ is relatively weakly sequentially compact in $L^1(\mathbb{R}^N)$, therefore it admits a subsequence that converges weakly in $L^1(\mathbb{R}^N)$ which, by (5.18), is equivalent to $\{u_\varepsilon\}$ admitting, up to a subsequence, a two-scale limit. ∎

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