

MORE COUNTEREXAMPLES TO REGULARITY FOR MINIMIZERS OF THE AVERAGE-DISTANCE PROBLEM

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ABSTRACT. The average-distance problem, in the penalized formulation, involves minimizing

$$(1) \quad E_\mu^\lambda(\Sigma) := \int_{\mathbb{R}^d} d(x, \Sigma) d\mu(x) + \lambda \mathcal{H}^1(\Sigma),$$

among path-wise connected, closed sets Σ with finite \mathcal{H}^1 -measure, where $d \geq 2$, μ is a given measure, λ is a given parameter and $d(x, \Sigma) := \inf_{y \in \Sigma} |x - y|$. The average-distance problem can be also considered among compact, convex sets with perimeter and/or volume penalization, i.e. minimizing

$$(2) \quad \mathcal{E}(\mu, \lambda_1, \lambda_2)(\cdot) := \int_{\mathbb{R}^d} d(x, \cdot) d\mu(x) + \lambda_1 \text{Per}(\cdot) + \lambda_2 \text{Vol}(\cdot),$$

where μ is a given measure, $\lambda_1, \lambda_2 \geq 0$ are given parameters with $\lambda_1 + \lambda_2 > 0$, and the unknown varies among compact, convex sets. Very little is known about the regularity of minimizers of (2). In particular it is unclear if minimizers of (2) are in general C^1 regular. The aim of this paper is twofold: first, we provide in \mathbb{R}^2 a second approach in constructing minimizers of (1) which are not C^1 regular; then, using the same technique, we provide an example of minimizer of (2) whose border is not C^1 regular, under perimeter penalization only.

Keywords. nonlocal variational problem, average-distance problem, regularity

Classification. 49Q20, 49K10, 49Q10, 35B65

1. INTRODUCTION

The average-distance problem was proposed by Buttazzo, Oudet and Stepanov in [2]. To guarantee well-posedness, an a priori bound on the \mathcal{H}^1 -measure of admissible minimizers was given, and this formulation is often referred as “*constrained formulation*”. To overcome the excessive rigidity imposed by hard constraints on the \mathcal{H}^1 -measure, Buttazzo, Mainini and Stepanov proposed in [1] the “*penalized formulation*”:

Problem 1.1. Given $d \geq 2$, a compactly supported, nonnegative measure μ , and $\lambda > 0$, minimize

$$\int_{\mathbb{R}^d} d(x, \Sigma) d\mu(x) + \lambda \mathcal{H}^1(\Sigma), \quad d(x, \Sigma) := \inf_{y \in \Sigma} |x - y|$$

with the unknown Σ varying in

$$\mathcal{A} := \{X \subseteq \mathbb{R}^d : X \text{ compact, path-wise connected, } \mathcal{H}^1(X) < \infty\}.$$

To simplify notations, for future reference let

$$F_\mu : \mathcal{A} \longrightarrow \mathbb{R}, \quad F_\mu(\Sigma) := \int_{\mathbb{R}^d} d(x, \Sigma) d\mu$$

$$E_\mu^\lambda : \mathcal{A} \longrightarrow \mathbb{R}, \quad E_\mu^\lambda(\Sigma) := F_\mu(\Sigma) + \lambda \mathcal{H}^1(\Sigma).$$

The functional F_μ will be often referred as “*average-distance functional*”. In the following, any considered measure will be assumed nonnegative, compactly supported probability measure. The choice of working with probability measure is for the sake of simplicity, and it is not restrictive since results proven in this paper can be easily extended to finite measures. Existence of minimizers follows from Blaschke and Gołab theorems.

In the following the expression “*average-distance problem*” will refer to Problem 1.1. Moreover, the \mathcal{H}^1 -measure of a set will be often referred as “length”. Originally this problem stemmed from mathematical modeling of optimization problems. A classic example can be found in urban planning: let

- μ be the distribution of passengers in a given region,
- Σ (the unknown) be the transport network to be built.

In this case $F_\mu(\Sigma)$ is the “average distance” of passengers from the network (thus smaller values of $F_\mu(\Sigma)$ imply that “on average, passengers are quite close to the network Σ ”, i.e. “ Σ is easily accessible”), and $\lambda \mathcal{H}^1(\Sigma)$ is the cost to build such network. Thus minimizing E_μ^λ is determining the network which “optimizes accessibility” for passengers, under cost considerations.

A more recent application can be found in data approximation: let

- μ be the distribution of data points,
- Σ (the unknown) be a one **dimensional** object which approximates the data.

In this case $F_\mu(\Sigma)$ is the error of such approximation, while $\lambda \mathcal{H}^1(\Sigma)$ is the cost associated to its complexity. Thus minimizing E_μ^λ is equivalent to determine the “best” approximation, which balances approximation error and cost.

In applications, sometimes the integrand $d(x, \Sigma)$ in $F_\mu(\Sigma)$ can be replaced by $d(x, \Sigma)^p$ for some power $p \geq 1$ (the case $p = 2$ is most common). However for the purposes of this paper the exponent p is not relevant, and we will consider only the case $p = 1$. The regularity of minimizers of Problem 1.1 is quite a delicate problem: it is known that minimizers are union of at most $[1/\lambda]$ branches, and such branches are Lipschitz regular (Buttazzo, Oudet, Paolini and Stepanov [2, 3, 4, 13]), satisfying a curvature estimate (Slepčev et al. [11]), but can fail to be C^1 regular (Slepčev [15]). Other results were proven by Santambrogio, Tilli [14, 16] and Lemenant [8]. A review is available in [7].

Average distance problem among convex sets. As proposed by Lemenant and Mainini in [9], the average-distance problem can be also considered among compact, convex sets, under perimeter and/or volume penalization:

Problem 1.2. Given $d \geq 2$, a measure μ , and parameters $\lambda_1, \lambda_2 \geq 0$ satisfying $\lambda_1 + \lambda_2 > 0$, minimize

$$\mathcal{E}(\cdot) = \mathcal{E}(\mu, \lambda_1, \lambda_2)(\cdot) := \int_{\mathbb{R}^d} d(x, \cdot) d\mu + \lambda_1 \text{Per}(\cdot) + \lambda_2 \text{Vol}(\cdot),$$

with the unknown varying in

$$\mathcal{C} := \{K \subseteq \mathbb{R}^d : K \text{ compact and convex}\}.$$

Here the “perimeter” of a set $E \subseteq \mathbb{R}^d$ is defined as the total variation (in \mathbb{R}^d) of its characteristic function χ_E , and the “volume” as its \mathcal{L}^d measure.

The motivations to study this problem are mainly theoretical, although one could easily find some applications (see [9]). Some partial results about regularity have been proven in [9]. However it is unclear if minimizers of Problem 1.2 (under only perimeter or volume penalization, not both as this case has been discussed in [9]) have C^1 regular border. This paper will be structured as follows:

- Section 2 will recall preliminary results,
- Section 3 will construct (in \mathbb{R}^2) an explicit example of minimizer of Problem 1.1 failing to be C^1 regular, using an approach different from that used in [15],
- Section 4 will construct, using techniques presented in Section 3, an explicit example of minimizer K of Problem 1.2 under perimeter penalization only, whose border ∂K is not C^1 regular.

The approach used in Section 3 uses some ideas from [15]: indeed we will approximate the reference measure μ with a sequence of discrete measures $\mu_k \xrightarrow{*} \mu$. We will use also a result similar to Lemma 11 of [15] (although with a slightly different proof), and similarly to [15], we will use the **same result** (Lemma 2.4) to pass to the limit $k \rightarrow \infty$. However the core arguments (Lemmas 3.4, 3.6 and 3.7), which prove that for infinitely many indices k there exists a minimizer $\Sigma_k \in \operatorname{argmin} E_{\mu_k}^\lambda$ containing a corner v_k with turning angle (see Definition 2.3) bounded from below (roughly corresponding to Steps 5, 6, 7 of Theorem 12 in [15]), are significantly different. These are specifically tailored for the reference measure considered in Section 3, and cannot be easily adapted for measures in Theorem 12 of [15].

It is worth noticing that this approach allows also to construct an example of minimizer of Problem 1.1 whose set of corners (i.e. points where C^1 regularity does not hold) is not closed ([10]).

2. PRELIMINARY RESULTS

The main goal of this section is to introduce some notations and recall well known results which will be used in Section 3.

The average-distance functional satisfies the following well known properties:

- (1) for any probability measure μ on \mathbb{R}^d , and $\lambda > 0$, the functional E_μ^λ is lower semicontinuous w.r.t. $d_{\mathcal{H}}$ (here, and in the following, $d_{\mathcal{H}}$ will denote the Hausdorff distance),
- (2) given $\Sigma \in \mathcal{A}$, and $\lambda > 0$, the mapping $\mu \mapsto E_\mu^\lambda(\Sigma)$ is continuous w.r.t. weak-* convergence of measures,
- (3) if $\mu_n \xrightarrow{*} \mu$, then for any $\lambda > 0$, $E_{\mu_n}^\lambda \xrightarrow{\Gamma} E_\mu^\lambda$,
- (4) consider a sequence $\mu_n \xrightarrow{*} \mu$, and for any n choose $\Sigma_n \in \operatorname{argmin} E_{\mu_n}^\lambda$. Then upon subsequence $\Sigma_n \xrightarrow{d_{\mathcal{H}}} \Sigma \in \operatorname{argmin} E_\mu^\lambda$.

For further details, we refer to [2, 3, 4, 15].

Recall that given a set of points $\Pi := \{P_1, \dots, P_j\} \subseteq \mathbb{R}^d$, the Steiner graph of Π is a pathwise connected set with minimal length containing all points of Π . The next result (from [15]) proves an intrinsic connection between Steiner graphs and minimizers of Problem 1.1, when the reference measure is discrete:

Proposition 2.1. *Given $d \geq 2$, a discrete measure $\mu := \sum_{i=1}^n a_i \delta_{x_i}$ on \mathbb{R}^d , and $\lambda > 0$, then any minimizer $\Sigma \in \operatorname{argmin}_{\mathcal{A}} E_{\mu}^{\lambda}$ is a Steiner graph.*

The following classic result (see for instance [5, 6]) proves several geometric properties about Steiner graphs:

Proposition 2.2. *Given a Steiner graph G , it holds:*

- G is a tree,
- if $\llbracket u, v \rrbracket$ and $\llbracket v, w \rrbracket$ are edges, with a common vertex v , then $\widehat{uvw} \geq 2\pi/3$,
- the maximal degree of any vertex is 3,
- if v is a vertex of degree 3, let $\llbracket u_i, v \rrbracket$, $i = 1, 2, 3$ be the three different edges containing v , then the angle between any two such edges is $2\pi/3$, and these edges are coplanar.

In view of Propositions 2.1 and 2.2, the following definition will be useful:

Definition 2.3. *Given a discrete measure μ , $\lambda > 0$, and $\Sigma \in \operatorname{argmin}_{\mathcal{A}} E_{\mu}^{\lambda}$, a vertex $v \in \Sigma$ is called:*

- “endpoint” if has degree 1,
- “corner” if has degree 2,
- “triple junction” if has degree 3.

If v is a corner, denoting with w, z the two vertices for which $\llbracket w, v \rrbracket$ and $\llbracket v, z \rrbracket$ are edges, the “turning angle” in v is:

$$\operatorname{TA}(v) := \pi - \widehat{wvz}.$$

Similarly, given a subset $A \subseteq \Sigma$, the turning angle of A is defined as

$$\operatorname{TA}(A) := \sum_{u \in A, u \text{ corner}} \operatorname{TA}(u).$$

Given $v \in \Sigma$ and $x \in \operatorname{supp}(\mu)$, the following expressions will be used:

- “ x talks to v ”, “ x projects on v ”, “ v talks to x ”: all these mean $d(v, \Sigma) = |x - v|$;
- “ v receives mass from A ”, where $A \subseteq \operatorname{supp}(\mu)$: there exists $x \in A$ such that x talks to v ;
- $TM(\mu, v, \Sigma)$ ($TM(v)$ when no risk of confusion arises) denotes the total mass of projecting on v , which we note may not coincide with the total mass supported on the points talking to v . The quantity $TM(\mu, v, \Sigma)$ will be often referred as “(amount of) mass projecting on v ”. For a detailed discussion see Lemma 2.1 in [11];
- “ H mass projects on v ”, where $H \geq 0$: this means $TM(\mu, v, \Sigma) = H$.

Given two points p, q the notation $\llbracket p, q \rrbracket$ will denote the straight segment $\{(1-t)p + tq : t \in [0, 1]\}$. Finally we recall **a convergence result**:

Lemma 2.4. *Given a sequence of curves $\{\gamma_k\} : [0, 1] \rightarrow K$ ($k \in \mathbb{N}$), with $K \subseteq \mathbb{R}^2$ a given compact set, satisfying*

$$\sup_k \|\gamma'_k\|_{BV} < \infty, \quad \sup_k \mathcal{H}^1(\gamma_k([0, 1])) < \infty,$$

then there exists a curve $\gamma : [0, 1] \rightarrow K$, such that (upon subsequence) it holds:

- (1) $\gamma_k \rightarrow \gamma$ in C^α , for any $\alpha \in [0, 1)$,
- (2) $\gamma'_k \rightarrow \gamma'$ in L^p , for any $p \in [1, \infty)$,
- (3) $\gamma''_k \xrightarrow{*} \gamma''$ in the space of signed Borel measures.

3. COUNTEREXAMPLE

The aim of this section is to present a different approach in constructing minimizers (of Problem 1.1) failing to be C^1 regular.

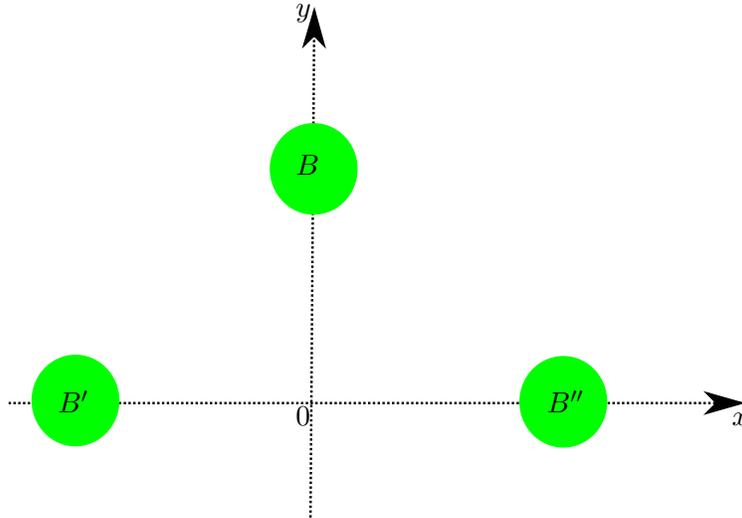


FIGURE 1. A schematic representation of the support of μ . For the sake of clarity the radius r has been chosen large.

Endow \mathbb{R}^2 with the standard Cartesian coordinate system. Let

$$(3) \quad \mu = \mu(r, \eta) := \left(\frac{1-\eta}{2\pi r^2} (\chi_{B'} + \chi_{B''}) + \frac{\eta}{\pi r^2} \chi_B \right) \cdot \mathcal{L}^2,$$

$$(4) \quad \lambda = \lambda(\eta) := \frac{1-\eta}{2} - 10^{-100},$$

where χ denotes the characteristic function (of the subscripted set), and

$$B' := B((-1, 0), r), \quad B'' := B((1, 0), r), \quad B := B((0, 1), r).$$

Parameters η and r are to be determined later (see conditions (C1) and (C2)). If $\lambda > 1/3$, any minimizer $\Sigma \in \operatorname{argmin} E_{\mu(r,\eta)}^\lambda$ is a simple curve independently of r, η . For further details we refer to [11]. For the sake of brevity, we will omit writing dependencies on η and r if no risk of confusion arises.

Lemma 3.1. *Let μ and λ be the quantities defined in (3) and (4). Then there exist $r_0, \eta_0 > 0$ such that for any $r \in (0, r_0), \eta \in (0, \eta_0)$, any minimizer $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ satisfies:*

- (i) $\Sigma \cap B((-1, 0), 0.01) \neq \emptyset, \Sigma \cap B((1, 0), 0.01) \neq \emptyset$,
- (ii) $\Sigma \subseteq \{y < 1/3\}$,
- (iii) $\Sigma \subseteq \{y \geq -r\}$.

Proof. Note that for any η such that $\lambda > 1/3$, Σ is a simple curve in view of [11, Lemma 3.1]. To prove (i), note that passing to the limit $r \rightarrow 0$ the measure $\mu = \mu(r, \eta)$ converges (w.r.t. weak-* topology) to

$$\bar{\mu} := \frac{1-\eta}{2}(\delta_{(-1,0)} + \delta_{(1,0)}) + \eta\delta_{(0,1)}.$$

Since $\bar{\mu}((\pm 1, 0)) > \lambda$, any minimizer $\bar{\Sigma} \in \operatorname{argmin}_{\mathcal{A}} E_{\bar{\mu}}^\lambda$ contains $\{(\pm 1, 0)\}$. Since for sequences $r_k \rightarrow 0$, $\{\Sigma_k \in \operatorname{argmin}_{\mathcal{A}} E_{\mu(r_k, \eta)}^\lambda\}$ it holds (upon subsequence) $\Sigma_k \xrightarrow{d_{\mathcal{A}}} \bar{\Sigma} \in \operatorname{argmin}_{\mathcal{A}} E_{\bar{\mu}}^\lambda$, statement (i) is proven.

To prove (ii) it suffices to note that any set X containing $\{p_1, p_2, q\}$ with $p_1 \in B((-1, 0), 0.01)$, $p_2 \in B((1, 0), 0.01)$ and $q \in \{y = 1/3\}$ satisfies

$$\mathcal{H}^1(X) \geq 2\sqrt{0.99^2 + (1/3 - 0.01)^2},$$

while

$$E_\mu^\lambda(\llbracket(-1, 0), (1, 0)\rrbracket) \leq \frac{1-\eta}{2}r + \eta + 2\lambda.$$

Thus for sufficiently small η, r it holds

$$E_\mu^\lambda(\llbracket(-1, 0), (1, 0)\rrbracket) < \lambda\mathcal{H}^1(X) \leq E_\mu^\lambda(X),$$

i.e. $X \notin \operatorname{argmin} E_\mu^\lambda$.

To prove (iii), let

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \pi(x, y) := (x, \max\{y, -r\}).$$

Note that:

- any $X \in \mathcal{A}$ satisfies $\mathcal{H}^1(\pi(X)) \leq \mathcal{H}^1(X)$, with equality holding only if $X \subseteq \{y \geq -r\}$,
- for any $X \in \mathcal{A}, x \in \operatorname{supp}(\mu)$ it holds $d(x, X) = d(x, \pi(X))$.

Thus any minimizer $\Sigma \in \operatorname{argmin} E_\mu^\lambda$ satisfies $\pi(\Sigma) = \Sigma$, and the proof is complete. \square

As consequence we have:

Corollary 3.2. *Let μ and λ be the quantities defined in (3) and (4). Then there exists $r_0, \eta_0 > 0$ such that for any $r \in (0, r_0), \eta \in (0, \eta_0)$, and minimizer $\Sigma \in \operatorname{argmin} E_\mu^\lambda$, it holds:*

$$\begin{aligned} (\forall z \in B) \quad & \operatorname{argmin}_{w \in \Sigma} |z - w| \subseteq \{-0.01 \leq x \leq 0.01\}, \\ (\forall z' \in B') \quad & \operatorname{argmin}_{w' \in \Sigma} |z' - w'| \subseteq \{-1.01 \leq x \leq -0.99\}, \\ (\forall z'' \in B'') \quad & \operatorname{argmin}_{w'' \in \Sigma} |z'' - w''| \subseteq \{0.99 \leq x \leq 1.01\}. \end{aligned}$$

Proof. Consider an arbitrary minimizer $\Sigma \in \operatorname{argmin} E_\mu^\lambda$. Lemma 3.1 implies the existence of a point $w \in \Sigma$ such that (for sufficiently small r)

$$(\forall z \in B) \quad |z - w| \leq 1 + 2r < \inf_{x \in B, y \in B'} |x - y| = \inf_{x \in B, y \in B''} |x - y| = \sqrt{2} - 2r.$$

Choosing sufficiently small r guarantees the existence of $w', w'' \in \Sigma$ such that

$$(\forall z' \in B') \quad |z' - w'| \leq r + \varepsilon(r), \quad (\forall z'' \in B'') \quad |z'' - w''| \leq r + \varepsilon(r),$$

where

$$\varepsilon(r) := \max\{d((-1, 0), \Sigma), d((1, 0), \Sigma)\},$$

and from the proof of Lemma 3.1 it follows $\lim_{r \rightarrow 0^+} \varepsilon(r) = 0$. Thus the proof is complete. \square

Thus choose sufficiently small r, η such that:

- (C1) $\eta \leq 10^{-100}, r/\eta \leq 10^{-100}$,
- (C2) conclusions of Lemma 3.1 and Corollary 3.2 hold.

Discrete measures. Similarly to [15], the first step involves approximating (in the weak-* topology) μ with a sequence of discrete measures. Given three points v_1, v_2, v_3 , define the “region of influence” $V(v_2)$ as follows:

- (1) if v_1, v_2, v_3 are collinear, then $V(v_2)$ is the unique line passing through v_2 and orthogonal to $v_3 - v_2$,
- (2) otherwise, let $\theta_i := \frac{v_{i+1} - v_i}{|v_{i+1} - v_i|}$ ($i = 1, 2$), $\xi := \frac{\theta_2 + \theta_1}{|\theta_2 + \theta_1|}$, $b := \frac{\theta_2 - \theta_1}{|\theta_2 - \theta_1|}$, $\beta := \operatorname{TA}(v_2)/2$, and

$$V(v_2) := v_2 + \{x \in \mathbb{R}^2 : |\langle \xi, x \rangle| \leq \langle b, x \rangle \tan \beta\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product of \mathbb{R}^2 .

Note that if $\operatorname{TA}(v_2) > 0$, $V(v_2)$ is an angle with vertex v_2 , of amplitude $\operatorname{TA}(v_2)$, and the border $\partial V(v_2)$ is union of two half-lines l^\pm starting in v_2 .

For $j = 1, 2, \dots$, define

$$\begin{aligned} \mu_j := & \frac{1 - \eta}{2} \left(\frac{1}{\#(B' \cap \frac{1}{j} \mathbb{Z}^2)} \sum_{i=1}^{\#(B' \cap \frac{1}{j} \mathbb{Z}^2)} \delta_{p'_i} \right) + \frac{1 - \eta}{2} \left(\frac{1}{\#(B'' \cap \frac{1}{j} \mathbb{Z}^2)} \sum_{i=1}^{\#(B'' \cap \frac{1}{j} \mathbb{Z}^2)} \delta_{p''_i} \right) \\ (5) \quad & + \eta \left(\frac{1}{\#(B \cap \frac{1}{j} \mathbb{Z}^2)} \sum_{i=1}^{\#(B \cap \frac{1}{j} \mathbb{Z}^2)} \delta_{p_i} \right), \end{aligned}$$

where

$$\{p'_i\} := B' \cap \frac{1}{j}\mathbb{Z}^2, \quad \{p''_i\} := B'' \cap \frac{1}{j}\mathbb{Z}^2, \quad \{p_i\} := B \cap \frac{1}{j}\mathbb{Z}^2,$$

Geometrically, this means that the mass supported in B (resp. B' , B'') is uniformly distributed on the uniform grid $B \cap \frac{1}{j}\mathbb{Z}^2$, (resp. $B' \cap \frac{1}{j}\mathbb{Z}^2$, $B'' \cap \frac{1}{j}\mathbb{Z}^2$). Note that in Corollary 3.2 replacing the reference μ with μ_j , the same conclusion holds (with the same proof). In particular any point can receive mass from at most one of the balls B, B', B'' . For the sake of brevity, in the following we will refer to Corollary 3.2 when using its conclusion, even if the reference measure of the context is μ_j instead of μ .

The next result proves that if a positive fraction of the mass supported in B projects on a point v , then $\text{TA}(v) > 0$.

Lemma 3.3. *Consider the family of measures $\{\mu_j\}$ defined in (5). Let λ be the parameter defined in (4). Then for any index j and minimizer $\Sigma \in \text{argmin } E_{\mu_j}^\lambda$, if a positive fraction of the mass supported in B projects on a point $v \in \Sigma$, then $\text{TA}(v) > 0$.*

Proof. Assume for the sake of contradiction that $\text{TA}(v) = 0$, and let $E \subseteq B$ be the set of points talking to v . Simple geometric considerations give $E \subseteq V(v)$, which (since $\text{TA}(v) = 0$) is the line through v orthogonal to $v_1 - v_2$.

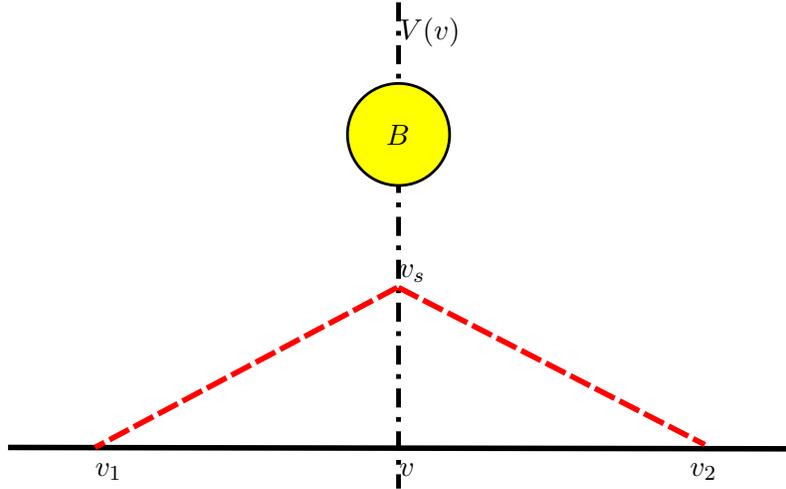


FIGURE 2. This is a schematic representation of the variation.

Consider the variation in Figure 2: define the competitor Σ_s as

$$\Sigma_s := \Sigma \setminus \llbracket v_1, v_2 \rrbracket \cup (\llbracket v_1, v_s \rrbracket \cup \llbracket v_2, v_s \rrbracket).$$

By construction

$$F_{\mu_j}(\Sigma) - F_{\mu_j}(\Sigma_s) \geq as,$$

since $E \subseteq B \subseteq \{y \geq 2/3\}$, while $\Sigma \subseteq \{y \leq 1/3\}$ (Lemma 3.1), and (for sufficiently small s)

$$\mathcal{H}^1(\Sigma_s) - \mathcal{H}^1(\Sigma) = O(s^2).$$

Thus the minimality of Σ is contradicted. \square

This result has not been used in [15], and due to the very constructions therein, it is unclear if the proof we used is valid for the reference measure in Theorem 12 of [15]. The next result proves a relation between the turning angle of a given corner and the amount of mass projecting on it.

Lemma 3.4. *Consider the family of measures $\{\mu_j\}$ defined in (5). Let λ be the parameter defined in (4). Then for any index j , minimizer $\Sigma \in \arg\min E_{\mu_j}^\lambda$, and corner $v \in \Sigma$, it holds:*

(i) *upper bound estimate on the turning angle:*

$$\text{TA}(v) \leq \frac{\pi}{2\lambda} TM(v),$$

(ii) *estimates on the curvature $\kappa(I)$ of an arbitrary subset $I \subseteq \Sigma$:*

$$\kappa(I) \leq \frac{\pi}{2\lambda} \sum_{v \in I, v \text{ corner}} TM(v),$$

(iii) *bounds for small turning angles:*

$$(6) \quad \text{TA}(v) \rightarrow 0^+ \implies \frac{\text{TA}(v)}{TM(v)/\lambda} \rightarrow 1.$$

$$\text{In particular, if } \text{TA}(v) \leq 0.01 \text{ then } \frac{\text{TA}(v)}{TM(v)/\lambda} \geq \frac{1}{2}.$$

Note that statements (i) and (ii) have been proven (or follow easily from) in [15]. However statement (iii), which will play a crucial role in the following arguments, has not been used in [15], and it has not been proven explicitly. Although it may follow from Lemma 9 of [15], our proof is somewhat easier.

Proof. Statements (i) and (ii) have been proven in [15]. To prove (iii), note that upon scaling and translation the configuration is that in Figure 3.

Consider the variations in Figure 3. The competitor

$$\Sigma_s^+ := \Sigma \setminus (\llbracket v_1, v \rrbracket \cup \llbracket v_2, v \rrbracket) \cup (\llbracket v_1, v_s^+ \rrbracket \cup \llbracket v_2, v_s^+ \rrbracket)$$

satisfies:

- simple geometric considerations give that $\min_{z \in V(v)} (|z - v| - |z - v_s^+|)$ is achieved for points $z \in \partial V(v)$, which satisfy

$$|z - v_s^+|^2 = |z - v|^2 + s^2 - 2 \cos(\text{TA}(v)/2) |z - v| s.$$

For **small values of s** , in first order approximation, this reads

$$|z - v_s^+|^2 \geq |z - v|^2 - 2 \cos(\text{TA}(v)/2) |z - v| s + O(s^2),$$

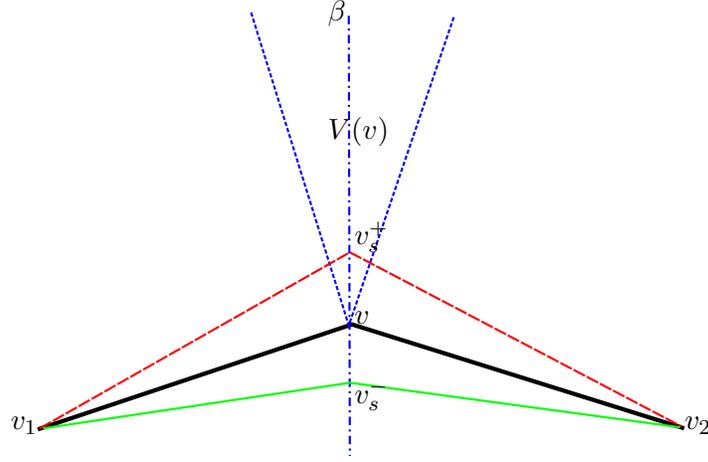


FIGURE 3. This is a schematic representation of the configuration. Here β is the bisector of $V(v)$, and $|v_1 - v| = |v_2 - v|$.

i.e.

$$|z - v| - |z - v_s^+| \geq \frac{2 \cos(\text{TA}(v)/2) |z - v| s + O(s^2)}{|z - v| + |z - v_s^+|} = s \cos(\text{TA}(v)/2) + O(s^2).$$

Thus

$$(7) \quad F_{\mu_j}(\Sigma) - F_{\mu_j}(\Sigma_s^+) \geq TM(v) s \cos(\text{TA}(v)/2) + O(s^2).$$

- For length, direct computation gives

$$|v_1 - v_s^+|^2 = |v_1 - v|^2 + s^2 - 2 \cos \frac{\pi - \text{TA}(v)}{2} s |v_1 - v|,$$

which for **small values of s** gives

$$(8) \quad \mathcal{H}^1(\Sigma_s^+) - \mathcal{H}^1(\Sigma) = 2(|v_1 - v| - |v_1 - v_s^+|) + O(s^2) = 2 \cos \frac{\pi - \text{TA}(v)}{2} s + O(s^2) = 2 \sin \frac{\text{TA}(v)}{2} s + O(s^2).$$

Combining estimates (7), (8) and minimality condition $E_{\mu_j}^\lambda(\Sigma) \leq E_{\mu_j}^\lambda(\Sigma_s^+)$ (**for any $s > 0$**) yields

$$(9) \quad TM(v) \cos \frac{\text{TA}(v)}{2} \leq 2\lambda \sin \frac{\text{TA}(v)}{2}.$$

The competitor

$$\Sigma_s^- := \Sigma \setminus ([v_1, v] \cup [v_2, v]) \cup ([v_1, v_s^-] \cup [v_2, v_s^-])$$

satisfies:

- for any z it holds $|z - v_s^-| \leq |z - v| + s$, i.e.

$$(10) \quad F_{\mu_j}(\Sigma_s^-) - F_{\mu_j}(\Sigma) \leq TM(v) s.$$

- For length, direct computation gives

$$|v_1 - v_s^+|^2 = |v_1 - v|^2 + s^2 - 2 \cos \frac{\text{TA}(v)}{2} s |v_1 - v|,$$

i.e.

$$(11) \quad \mathcal{H}^1(\Sigma_s^+) - \mathcal{H}^1(\Sigma) = 2 \sin \frac{\text{TA}(v)}{2} s + O(s^2).$$

Combining estimates (10), (11) and minimality condition $E_{\mu_j}^\lambda(\Sigma) \leq E_{\mu_j}^\lambda(\Sigma_s^-)$ yields

$$(12) \quad TM(v) \geq 2\lambda \sin \frac{\text{TA}(v)}{2}.$$

Combining (9) and (12) proves (6). The implication

$$\text{TA}(v) \leq \frac{1}{100} \implies \frac{\text{TA}(v)}{TM(v)/\lambda} \geq \frac{1}{2}$$

follows immediately from (9): under assumption $\text{TA}(v) \leq 0.01$, inequality (9) reads

$$TM(v) \leq 2\lambda \tan \frac{\text{TA}(v)}{2} \stackrel{\text{TA}(v) \leq 0.01}{\leq} 2\lambda \text{TA}(v),$$

hence $\frac{1}{2} \leq \frac{\text{TA}(v)}{TM(v)/\lambda}$, and the proof is complete. \square

Lemma 3.5. Consider the family of measures $\{\mu_j\}$ defined in (5). Let λ be the parameter defined in (4). Then for any index j , minimizer $\Sigma \in \text{argmin } E_{\mu_j}^\lambda$, and corner $v \in \Sigma$ receiving mass from B , it holds $V(v) \cap \Sigma = \{v\}$.

For future reference, the notation $\|\cdot\|_{TV}$ will denote the total variation semi-norm. We will omit writing the domain if no risk of confusion arises.

Proof. Let $f : [0, 1] \rightarrow \Sigma$ be a constant speed bijective parameterization, and denote with $t_v := f^{-1}(v)$. Assume there exists another point $w := f(t_w) \in V(v) \cap \Sigma$, $w \neq v$. Recall that by construction, the border $\partial V(v)$ is union of half-lines l^\pm starting in v and orthogonal to the left/right tangent vector $\tau^\pm := \lim_{t \rightarrow t_v^\pm} f'(t)$. Since the amplitude of $V(v)$ is $\text{TA}(v) \leq \frac{\pi\eta}{2\lambda}$ (in view of Corollary 3.2 and Lemma 3.4), it follows $\angle(w-v)l^- \leq \text{TA}(v)$, i.e. $\angle(w-v)\tau^- \in [\pi/2 - \text{TA}(v), \pi/2 + \text{TA}(v)]$, thus $\|f'\|_{TV} \geq \pi/4$. Since Lemma 3.4 gives $\|f'\|_{TV} \leq \frac{\pi}{2\lambda}(1 - 2\lambda)$, a contradiction has been achieved, concluding the proof. \square

The next result proves that given distinct corners $v_1 \neq v_2$, then the intersection $V(v_1) \cap V(v_2)$ is empty.

Lemma 3.6. Consider the family of measures $\{\mu_j\}$ defined in (5). Let λ be the parameter defined in (4). Then for any index j , minimizer $\Sigma \in \text{argmin } E_{\mu_j}^\lambda$, and distinct corners v_i, v'_i receiving mass from B , it holds $V(v_i) \cap V(v'_i) = \emptyset$.

The arguments we use in this proof strongly rely on Lemma 3.1, whose proof uses the particular construction of μ , and cannot be extended (at least without very significant modifications) to measures considered in Theorem 12 of [15].

Proof. For the sake of brevity, given a point p , the notations p_x (resp. p_y) will denote the x (resp. y) coordinate of p . Assume for the sake of contradiction there exist distinct corners v_1, v_2 such that $V(v_1) \cap V(v_2) \ni v$.

Lemma 3.5 implies $v \notin \{v_1, v_2\}$, $V(v_1) \not\subseteq V(v_2)$ and $V(v_2) \not\subseteq V(v_1)$. Lemma 3.1 gives $\Sigma \subseteq \{y < 1/3\}$, while $B \subseteq \{y > 2/3\}$.

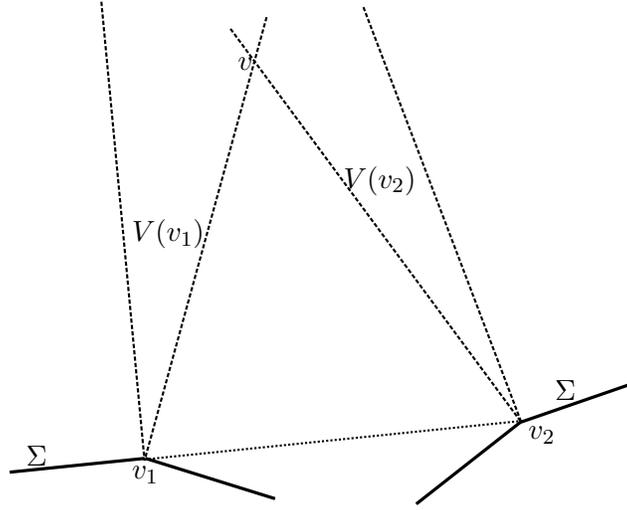


FIGURE 4. This is a schematic representation of the configuration.

Since Σ is a simple curve, let $f : [0, 1] \rightarrow \Sigma$ be a constant speed bijective parameterization. Let

$$t_1 = f^{-1}(v_1), \quad t_2 = f^{-1}(v_2),$$

and assume $t_1 < t_2$. Note that the triangle $\Delta v_1 v v_2$ is non degenerate, thus $\min\{\widehat{v_2 v_1 v}, \widehat{v v_2 v_1}\} < \pi/2$. Assume (by symmetry) $\widehat{v_2 v_1 v} < \pi/2$. Thus

$$(13) \quad \{\varepsilon > 0 : (\forall t \in (t_1, t_1 + \varepsilon))(\exists z \in \llbracket v_1, v_2 \rrbracket \cap \{x = (f(t))_x\}) : z_y > (f(t))_y\} \neq \emptyset,$$

and let

$$\varepsilon^* := \sup\{\varepsilon > 0 : (\forall t \in (t_1, t_1 + \varepsilon))(\exists z \in \llbracket v_1, v_2 \rrbracket \cap \{x = (f(t))_x\}) : z_y > (f(t))_y\}.$$

Clearly $\varepsilon^* \leq t_2 - t_1$. Consider the competitor

$$\tilde{\Sigma} := \Sigma \setminus f([t_1, t_1 + \varepsilon^*]) \cup \llbracket f(t_1), f(t_1 + \varepsilon^*) \rrbracket.$$

By construction it holds $\mathcal{H}^1(\tilde{\Sigma}) < \mathcal{H}^1(\Sigma)$. Let $q \in B$ be an arbitrary point. Choose an arbitrary $t_w \in (t_1, t_1 + \varepsilon^*)$ such that $|q - f(t_w)| = d(q, \Sigma)$, and by definition there exists $\tilde{w} \in \llbracket f(t_1), f(t_1 + \varepsilon^*) \rrbracket$ satisfying $\tilde{w}_x = (f(t_w))_x$, $\tilde{w}_y > (f(t_w))_y$. Thus

$$z \in \{y > (\tilde{w}_y + (f(t_w))_y)/2\} \implies |z - \tilde{w}_y| < |z - f(t_w)|.$$

Since any point of $f([t_1, t_1 + \varepsilon^*])$ can only talk to masses supported in $B \subseteq \{y > 2/3\}$, while $\llbracket f(t_1), f(t_1 + \varepsilon^*) \rrbracket \subseteq \{y < 1/2\}$, it follows

$$(\forall z \in B)(\forall t \in [t_1, t_1 + \varepsilon^*]) \quad |z - f(t)| \geq |z - \tilde{w}_t|,$$

where \tilde{w}_t is the unique point satisfying

$$(\tilde{w}_t)_x = (f(t))_x, \quad (\tilde{w}_t)_y > (f(t))_y, \quad \tilde{w}_t \in \llbracket f(t_1), f(t_1 + \varepsilon^*) \rrbracket.$$

Thus it follows $F_\mu(\tilde{\Sigma}) \leq F_\mu(\Sigma)$. Since $\mathcal{H}^1(\tilde{\Sigma}) < \mathcal{H}^1(\Sigma)$, the minimality of Σ is contradicted. Thus such a point v cannot exist. \square

The next result is the core argument of our construction.

Lemma 3.7. *Consider the family of measures $\{\mu_j\}$ defined in (5). Let λ be the parameter defined in (4). Then for any sufficiently large index j and $\Sigma_j \in \operatorname{argmin} E_{\mu_j}^\lambda$, there exists a corner $v_j \in \Sigma_j$ such that $TM(\mu_j, v_j, \Sigma_j) \geq \eta/4$. Moreover, $\operatorname{TA}(v_j) \geq \eta/6$.*

Since we will use Lemma 3.6, this proof cannot be used for measures considered in Theorem 12 of [15]. Note also that the choice of the denominator in $\operatorname{TA}(v_j) \geq \eta/6$ is quite arbitrary (and certainly not optimal), but acceptable for the purposes of this section.

Proof. Fix an index j , and choose a minimizer $\Sigma \in \operatorname{argmin} E_{\mu_j}^\lambda$. Let $f : [0, 1] \rightarrow \Sigma$ be a constant speed bijective parameterization, and let $\{v_i\}_{i=1}^H$ be the set of **corners receiving positive mass** from B . Recall that Corollary 3.2 implies that such $\{v_i\}$ can talk only to mass supported in B .

Let $t_i := f^{-1}(v_i)$ and $M_i := TM(\mu_j, v_i, \Sigma)$. If there exist two indices i_1, i_2 such that $M_{i_1} + M_{i_2} \geq \eta/2$, then the proof is complete. Thus in the following we will assume

$$(14) \quad (\forall i_1, i_2, i_1 \neq i_2) \quad M_{i_1} + M_{i_2} \leq \eta/2.$$

The goal is to prove that this assumption leads to a contradiction.

Lemma 3.4 gives

$$\frac{M_i}{2\lambda} \leq \operatorname{TA}(v_i) \leq \frac{M_i}{\lambda}, \quad i = 1, \dots, H,$$

and combining with Lemma 3.1 gives

$$(15) \quad d(v_i, B) \geq \frac{1}{3} \sin \operatorname{TA}(v_i) \geq \frac{1}{6} \operatorname{TA}(v_j).$$

Let l_i^\pm be the two half-lines forming the border $\partial V(v_i)$, Lemma 3.6 proves that $V(v_{i_1}) \cap V(v_{i_2}) = \emptyset$ whenever $i_1 \neq i_2$.

- **Claim:** for any corner v_i , except at most two, both half-lines l_i^\pm must intersect the border ∂B .

Let v_{i_1}, v_{i_2} be the two corners for which (upon renaming) $l_{i_1}^+ \cap \partial B = l_{i_2}^+ \cap \partial B = \emptyset$ (clearly if such a couple v_{i_1}, v_{i_2} does not exist, then the claim is true). The goal is to prove that it does not exist a third corner v_{i_3} for which (upon renaming) $l_{i_3}^+ \cap \partial B = \emptyset$.

Assume (upon renaming) $i_1 < i_2$, and both $l_{i_1}^-$ and $l_{i_2}^-$ intersect ∂B since:

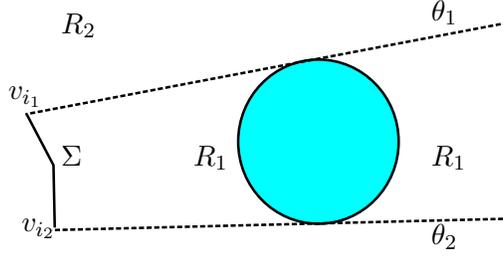


FIGURE 5. This is a schematic representation of the configuration.

- v_{i_1} and v_{i_2} receive mass from B , thus $V(v_{i_1}) \cap B$ and $V(v_{i_2}) \cap B$ are both non empty,
- $l_{i_1}^+$ and $l_{i_2}^+$ do not intersect ∂B ,
- if $l_{i_1}^-$ (resp. $l_{i_2}^-$) does not intersect ∂B , then $B \subseteq V(v_{i_1})$ (resp. $B \subseteq V(v_{i_2})$) and Lemma 3.6 implies $B \cap V(v_{i_2}) = \emptyset$ (resp. $B \cap V(v_{i_1}) = \emptyset$). This is a contradiction.

Thus there exist half-lines $\theta_1 \subseteq V(v_{i_1})$ (resp. $\theta_2 \subseteq V(v_{i_2})$) starting in v_1 (resp. v_2) and tangent to ∂B . Note that

$$\mathbb{R}^2 \setminus (f([t_{i_1}, t_{i_2}]) \cup \theta_1 \cup \theta_2)$$

is divided in two connected components R_1 and R_2 , of which (upon renaming) R_1 contains B . Note also that any half-line contained in R_1 must intersect ∂B .

Choose another corner v_{i_3} : since it talks to some mass in B , the intersection $V(v_{i_3}) \cap B$ is not empty, thus there exists a half-line $\phi \subseteq V(v_{i_3})$. Lemma 3.5 implies $V(v_{i_3}) \cap \Sigma = \{v_3\}$, and since $V(v_{i_3})$ is connected, it intersects B , but not $\theta_1 \cup \theta_2$ (Lemma 3.6). Thus it holds $V(v_{i_3}) \setminus \{v_{i_3}\} \subseteq R_1$. Since any half-line contained in R_1 must intersect ∂B , we conclude that both $l_{i_3}^\pm$ intersect ∂B , and the claim is proven.

Using (15) gives that any corner v_i such that both l_i^\pm intersect ∂B satisfies

$$(16) \quad \min_{z \in l_i^-, |z-v_i| \geq 1/3} d(z, l_i^+) \geq \frac{1}{3} \sin \text{TA}(v_i) \geq \frac{1}{6} \text{TA}(v_i),$$

since Lemma 3.4 gives $\text{TA}(v_i) \leq \pi\eta 2\lambda$. Since for any index i except at most two (which will be denoted by i' and i''), both l_i^\pm intersect ∂B , choose $w_i^\pm \in l_i^\pm \cap \partial B$, and clearly $V(v_i) \cap \partial B$ is an arc connecting w_i^- and w_i^+ . Combining with (16) gives

$$\mathcal{H}^1(V(v_i) \cap \partial B) \geq \min_{z \in l_i^-, |z-v_i| \geq 1/3} d(z, l_i^+) \geq \frac{1}{6} \text{TA}(v_i),$$

and using Lemma 3.4 gives $\text{TA}(v_i) \geq \frac{M_i}{2\lambda}$, i.e.

$$(17) \quad \mathcal{H}^1(V(v_i) \cap \partial B) \geq \frac{1}{6} \text{TA}(v_i) \geq \frac{1}{6} \frac{M_i}{2\lambda} \geq \frac{1}{12} M_i.$$

Recalling that $V(v_{i_1}) \cap V(v_{i_2}) = \emptyset$ whenever $i_1 \neq i_2$, summing over indices $i \in \{1, \dots, H\} \setminus \{i', i''\}$ gives

$$\mathcal{H}^1(\partial B) \geq \sum_{\substack{i=1 \\ i \neq i', i''}}^H \mathcal{H}^1(V(v_i) \cap \partial B) \stackrel{(17)}{\geq} \sum_{\substack{i=1 \\ i \neq i', i''}}^H \frac{1}{12} M_i \stackrel{(14)}{\geq} \frac{\eta}{24} \stackrel{(C1)}{>} 2\pi r = \mathcal{H}^1(\partial B),$$

which is a contradiction.

Thus there exist indices i', i'' such that $M_{i'} + M_{i''} \geq \eta/2$, i.e. $\max\{M_{i'}, M_{i''}\} \geq \eta/4$ independently of j . Using Lemma 3.4, we conclude that $\max\{\text{TA}(v_{i'}), \text{TA}(v_{i''})\} \geq \frac{\eta}{8\lambda}$, and since $8\lambda < 6$, the proof is complete. \square

Passing to the limit. Now we can pass to the limit $j \rightarrow \infty$. The arguments we use are quite **standard**, and similar to those used in [15] (mainly Step 8 of Theorem 12). For any j choose $\Sigma_j \in \text{argmin } E_{\mu_j}^\lambda$, and since $\mu_j \xrightarrow{*} \mu$, upon subsequence it holds (using Lemma 2.4)

$$\Sigma_j \xrightarrow{d_{\mathcal{H}}} \Sigma \in \text{argmin } E_\mu^\lambda.$$

Let

$$(j = 1, 2, \dots) \quad f_j : [0, 1] \rightarrow \Sigma_j, \quad f : [0, 1] \rightarrow \Sigma$$

be constant speed bijective parameterizations, such that $f_j \rightarrow f$ uniformly. Lemma 3.7 proves that for any j there exists a corner $v_j = f_j^{-1}(t_j)$ such that $\text{TA}(v_j) \geq \eta/6$. In other words, the measure f_j'' has an atom of measure at least $\eta/6$ in t_j . **Note that conditions of Lemma 2.4 are satisfied:**

- $\sup_j \|f_j'\|_{TV} \leq 1/\lambda$ in view of [11, Theorem 5.1],
- the minimality condition $\Sigma_j \in \text{argmin}_{\mu_j}^\lambda$, $j = 1, 2, \dots$ implies $\sup_j \mathcal{H}^1(\Sigma_j) < +\infty$ (since the opposite would imply the existence of a subsequence $\{\Sigma_{j(k)}\}$ satisfying $\lambda \mathcal{H}^1(\Sigma_{j(k)}) \rightarrow \infty$) and the existence of a compact set K containing $\bigcup_j \Sigma_j$ (since the opposite, i.e. there exists a subsequence $\{\Sigma_{j(k)}\}$ "escaping to infinity", would imply $F_{\mu_{j(k)}}(\Sigma_{j(k)}) \rightarrow \infty$).

Thus upon subsequence $t_j \rightarrow t$, and the convergence $f_j'' \xrightarrow{*} f''$ (given by Lemma 2.4), implies that the measure f'' has an atom of size at least $\eta/6$ in t . Corollary 3.2 gives $f(t) \in \{-0.01 \leq x \leq 0.01\}$, thus $t \neq 0, 1$. Since an atom for the measure f'' corresponds to a jump for the tangent derivative f' , we conclude that Σ admits a corner in $f(t)$, with $\text{TA}(f(t)) \geq \eta/6$.

Thus we have proven:

Theorem 3.8. *Let μ be the measure defined in (3) and λ the parameter defined in (4). Then there exists a minimizer $\Sigma \in \text{argmin } E_\mu^\lambda$ containing a corner v with $\text{TA}(v) \geq \eta/6$.*

Corollary 3.9. *The minimizer Σ from Theorem 3.8 is also minimizer for the constrained problem*

$$(18) \quad \min_{\mathcal{H}^1(\cdot) \leq \mathcal{H}^1(\Sigma)} \int_{\mathbb{R}^2} d(x, \cdot) d\mu.$$

Proof. In [2] it has been proven that any minimizer $\tilde{\Sigma}$ of (18) satisfies $\mathcal{H}^1(\tilde{\Sigma}) = \mathcal{H}^1(\Sigma)$, thus if Σ is not a minimizer of (18), choosing Σ^* minimizer of (18) would give

$$\int_{\mathbb{R}^2} d(x, \Sigma^*) d\mu < \int_{\mathbb{R}^2} d(x, \Sigma) d\mu, \quad \mathcal{H}^1(\Sigma^*) = \mathcal{H}^1(\Sigma),$$

contradicting $\Sigma \in \operatorname{argmin} E_\mu^\lambda$. \square

4. AVERAGE DISTANCE PROBLEM AMONG CONVEX SETS

The aim of this section is to analyze regularity properties of minimizers of Problem 1.2. In particular we construct a minimizer failing to be C^1 regular, **under perimeter penalization only**. Unfortunately, the arguments we use cannot be extended to the case of volume penalization. We recall if both perimeter and volume are penalized, it has been proven in [9] that minimizers can fail to be C^1 .

The considered energy will be

$$\mathcal{E} = \mathcal{E}(\mu, \lambda) : \mathcal{C} \longrightarrow [0, \infty), \quad \mathcal{E}(\mu, \lambda)(K) := \int_{\mathbb{R}^2} d(x, K) d\mu + \lambda \operatorname{Per}(K),$$

where \mathcal{C} and $\operatorname{Per}(\cdot)$ have been defined in Problem 1.2, μ is a given measure and $\lambda > 0$ a given parameter. For the sake of brevity, we will omit writing the dependencies on μ, λ when no risk of confusion arises. Existence of minimizers, as proven in [9], follows from Blaschke and Gołab theorems.

Let

$$(19) \quad \mu_{r,a,\eta} := \begin{cases} \frac{1-\eta}{2} \left(\frac{1}{\pi r^2} \mathcal{L}_{\perp B(p_1,r)}^2 + \frac{1}{\pi r^2} \mathcal{L}_{\perp B(p_2,r)}^2 \right) + \eta \left(\frac{1}{\pi r^2} \mathcal{L}_{\perp B(p,r)}^2 \right) & \text{if } r > 0 \\ \frac{1-\eta}{2} (\delta_{p_1} + \delta_{p_2}) + \eta \delta_p & \text{if } r = 0. \end{cases}$$

Here for given point q , the notation " δ_q " denotes the Dirac measure in q , and a, η, r are parameters to be determined later. Note that for $r < \delta/4$ the balls $B(p_1, r), B(p_2, r), B(p, r)$ are mutually disjoint. The construction of the counterexample will be achieved over two steps:

- (1) first, prove that for suitable choices of parameters λ, η, a , any minimizer of $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ contains $\{p_1, p_2\}$ but not p (Lemma 4.1),
- (2) then, choose sufficiently small r , approximate $\mu_{r,a,\eta}$ with a sequence of discrete measures $\mu_j \xrightarrow{*} \mu_{r,a,\eta}$ and prove that minimizers of $\mathcal{E}(\mu_j, \lambda)$ contain a corner with uniformly bounded amplitude (Lemma 4.7),
- (3) finally, take the limit $j \rightarrow \infty$.

The choice to approximate $\mu_{r,a,\eta}$ is advantageous since:

- (i) given a measure ν , sum of finitely many Dirac measures, and parameter λ , there exists a polygon $K \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\nu, \lambda)$,

(ii) given sequences $\nu_j \xrightarrow{*} \nu$, $\{C_j \in \operatorname{argmin}_C \mathcal{E}(\nu_j, \lambda)\}$, it holds (upon subsequence)

$$C_j \xrightarrow{d_H} C \in \operatorname{argmin}_C \mathcal{E}(\nu, \lambda).$$

The proof is identical to the case of average distance problem among trees (Problem 1.1), noting that the convex hull of finitely many points is a convex polygon.

Basic configuration. A key result is:

Lemma 4.1. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in (19). Then there exist $\lambda, \eta > 0$ and $a > 1$ such that the unique minimizer of $\mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is an isosceles triangle $\Delta p_1 p_2 q$, with base $\llbracket p_1, p_2 \rrbracket$ and $q = (0, q_y) \in \mathbb{R}^2$, $q_y \in (0, a)$.*

The proof will be split over several lemmas.

Lemma 4.2. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in (19). Then for any $a \geq 1$, there exist $\lambda_0, \eta_0 > 0$ such that for any $\lambda \in (0, \lambda_0)$, $\eta \in (0, \eta_0)$, satisfying $\lambda > \eta/2$, any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ contains $\{p_1, p_2\}$.*

Condition $\lambda > \eta/2$ will be crucial for the proof of Lemma 4.4.

Proof. Note that

$$\mathcal{E}(\mu_{0,a,\eta}, \lambda)(\llbracket p_1, p_2 \rrbracket) = a\eta + \lambda\delta,$$

thus for any minimizer K it holds

$$(20) \quad \lambda \operatorname{Per}(K) \leq \mathcal{E}(\mu_{0,a,\eta}, \lambda)(K) \leq \mathcal{E}(\mu_{0,a,\eta}, \lambda)(\llbracket p_1, p_2 \rrbracket) = a\eta + \lambda\delta.$$

Let $\pi : \mathbb{R}^2 \rightarrow K$ be the projection map, and assume (for the sake of contradiction) $p_1 \notin K$, i.e. $\pi(p_1) \neq p_1$. Let $e_1 := \frac{p_1 - \pi(p_1)}{|p_1 - \pi(p_1)|}$, and let e_2 be a unit vector orthogonal to e_1 . Consider the family of linear applications

$$T_\varepsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T_\varepsilon e_i = (1 + \varepsilon)e_i, \quad i = 1, 2.$$

By construction it holds

$$(\forall \varepsilon > 0)(\forall E \subseteq \mathbb{R}^2) \quad E \text{ convex} \implies T_\varepsilon E \text{ convex}.$$

Moreover

$$(21) \quad F_{\mu_0}(K) - F_{\mu_0}(T_\varepsilon K) \geq \mu_0(p_1)\varepsilon = \frac{1-\eta}{2}\varepsilon, \quad \operatorname{Per}(T_\varepsilon K) - \operatorname{Per}(K) = \varepsilon \operatorname{Per}(K) + o(\varepsilon)$$

This yields

$$\lambda \operatorname{Per}(T_\varepsilon K) - \lambda \operatorname{Per}(K) \leq \lambda \varepsilon \operatorname{Per}(K) \leq (a\eta + \lambda\delta)\varepsilon,$$

thus for sufficiently small λ, η it holds

$$(22) \quad (a\eta + \lambda\delta)\varepsilon \leq 0.3\varepsilon,$$

Combining (22) and (21) contradicts the minimality of K , concluding the proof. \square

Corollary 4.3. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in (19). Then for suitable choice of parameters η, λ, a , satisfying $\lambda > \eta/2$, any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is an isosceles triangle with base $\llbracket p_1, p_2 \rrbracket$.*

Proof. Lemma 4.2 implies that for suitable choice of parameters η, λ, a , any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ contains $\{p_1, p_2\}$. Since for any convex set E the projection map $\pi_E : \mathbb{R}^2 \rightarrow E$ is well defined, it follows that any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ should be the convex hull of three points (namely p_1, p_2 and $\pi_K(p)$), i.e. a triangle with an edge $\llbracket p_1, p_2 \rrbracket$. Since p lies on the axis of $\llbracket p_1, p_2 \rrbracket$, and for any triangle with fixed base and height the isosceles one minimizes the perimeter, the proof is complete. \square

Lemma 4.4. *Consider the family of measures $\{\mu_{r,a,\eta}\}$ defined in (19). Then for suitable choice of parameters η, λ, a , satisfying $\lambda > \eta/2$ and (22), any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is a non degenerate triangle not containing the point $p = (0, a)$.*

Before the proof, note that condition (22) is satisfied if

$$a\eta < 0.15, \quad \lambda\delta < 0.15,$$

and since $\delta = 10^{-100}$ is fixed, while $\lambda < 1$, condition $\lambda\delta < 0.15$ is satisfied. A potential issue can be present when choosing η and a , since it is required that “choosing η small does not force to choose a large”. This will be essentially the main point of this lemma.

Proof. Corollary 4.3 implies that for suitable choice of parameters η, λ, a , any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ is a non degenerate isosceles triangle with base $\llbracket p_1, p_2 \rrbracket$. Let h be its height (relative to the base $\llbracket p_1, p_2 \rrbracket$), and let $q_h := (0, h) \in \mathbb{R}^2$. Direct computation gives

$$\psi(h) := \mathcal{E}(\mu_{0,a,\eta}, \lambda, \alpha)(\Delta p_1 p_2 q_h) = (a - h)\eta + \lambda(\delta + \sqrt{\delta^2 + 4h^2})$$

and

$$(23) \quad \frac{d}{dh}\psi(h) = -\eta + \frac{4h\lambda}{\sqrt{\delta^2 + 4h^2}},$$

thus the optimal value for h is

$$h^* = 4\delta \left(\frac{4\lambda^2}{\eta^2} - 1 \right).$$

Condition $\lambda > \eta/2$ guarantees $h^* > 0$. Note that further imposing $4\lambda^2/\eta^2 \leq 100$ yields $h^* \leq 400\delta$, thus any choice $a \in [3, 5]$ is acceptable (the extremes 3 and 5 are arbitrary, but acceptable for the purposes of the proof), and the compatibility with (22) can be satisfied (upon choosing sufficiently small η). \square

Proof. (of Lemma 4.1) The proof follows by combining Lemmas 4.2, 4.4 and Corollary 4.3, and noting that these are valid if λ, η, a satisfy:

$$a\eta < 0.15, \quad \lambda\delta < 0.15, \quad 3 \leq a \leq 5, \quad 0.7 \leq \frac{\lambda}{\eta} \leq 5 < \frac{1}{2}\sqrt{1 + \frac{1}{2\delta}}.$$

Since there exist triplets (λ, η, a) satisfying these conditions, the proof is complete. \square

Remark I. Note that it is possible to further impose that parameters λ, a, η must satisfy $\frac{\eta}{8\lambda} \in (0, \pi)$. This condition will be used in Lemma 4.7. Moreover, note that it is possible to choose $\eta > 0$ arbitrarily small.

Construction of the counterexample. Choose parameters λ, η, a such that any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{0,a,\eta}, \lambda)$ satisfies $0 < d(p, K) =: b$ (this choice is possible due to Lemma 4.1). Choose sufficiently small radius $r \in (0, 10^{-10}\eta)$ such that for any minimizer $K \in \operatorname{argmin} \mathcal{E}(\mu_{r,a,\eta}, \lambda)$ it holds:

- distance estimate:

$$(24) \quad d_{\mathcal{H}}(B(p, r), K) \geq b/4,$$

- for any minimizer $C_j \in \operatorname{argmin}_C \mathcal{E}(\mu_j, \lambda)$, the sets

$$\bigcup_{z \in B(p, r)} \operatorname{argmin}_{w \in \partial K} |z - w|, \quad \bigcup_{z' \in B(p', r)} \operatorname{argmin}_{w \in \partial K} |z' - w|, \quad \bigcup_{z'' \in B(p'', r)} \operatorname{argmin}_{w \in \partial K} |z'' - w|$$

are mutually disjoint. This is possible in view of Lemma 4.1.

Choose λ, η, a, r such that all the conditions and results mentioned (until now) in this section hold. Upon choosing sufficiently small r , assume also

$$(25) \quad 2\pi r < \frac{b\eta}{48\lambda}.$$

This condition will be useful for Lemma 4.7. From now parameters λ, η, a, r will be fixed.

Similarly to (5), let

$$(26) \quad \begin{aligned} \mu_j := & \frac{1-\eta}{2} \left(\frac{1}{\#(B(p', r) \cap \frac{1}{j}\mathbb{Z}^2)} \sum_{x \in B(p', r) \cap \frac{1}{j}\mathbb{Z}^2} \delta_x \right) + \frac{1-\eta}{2} \left(\frac{1}{\#(B(p'', r) \cap \frac{1}{j}\mathbb{Z}^2)} \sum_{x \in B(p'', r) \cap \frac{1}{j}\mathbb{Z}^2} \delta_x \right) \\ & + \eta \left(\frac{1}{\#(B(p, r) \cap \frac{1}{j}\mathbb{Z}^2)} \sum_{x \in B(p, r) \cap \frac{1}{j}\mathbb{Z}^2} \delta_x \right). \end{aligned}$$

The results we use to analyze minimizers of $\mathcal{E}(\mu_j, \lambda)$ are adapted versions of Lemmas 3.4, 3.6 and 3.7.

Lemma 4.5. *Consider the family of measures $\{\mu_j\}$ defined in (26). Then for any index j , there exists a convex polygon K minimizing $\mathcal{E}(\mu_j, \lambda)$ and satisfying*

$$(\forall v \in \bigcup_{z \in B(p, r)} \operatorname{argmin}_{w \in \partial K} |z - w|) \quad \frac{M_v}{2\lambda} \leq \operatorname{TA}(v) \leq \frac{\pi}{2} \frac{M_v}{\lambda},$$

where $M_v := TM(\mu, v, \partial K)$.

Proof. The proof is done by adapting the arguments from Lemma 3.4, to deal with the convexity constraint. Let K be a convex polygon minimizing $\mathcal{E}(\mu_j, \lambda)$, $v \in K$ be an arbitrary corner receiving mass from $B(p, r)$. Choose $v_1, v_2 \in \partial K$ such that $[[v_1, v]]$, $[[v_2, v]]$ are straight segments and $|v_1 - v| = |v_2 - v| > 0$. Note that it is possible to choose such points v_1, v_2 exactly because K is a convex polygon.

- Upper bound estimate $\operatorname{TA}(v) \leq \frac{\pi M_v}{2\lambda}$.

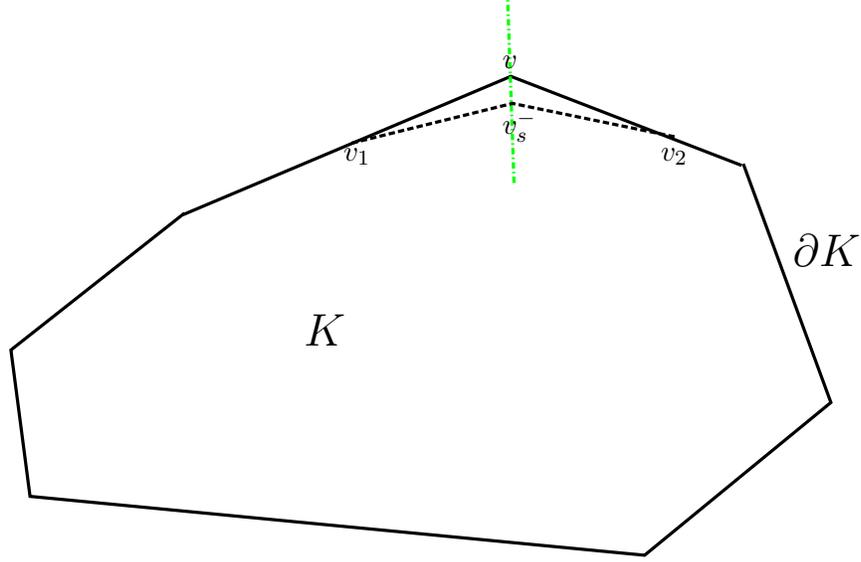


FIGURE 6. This is a schematic representation of the considered variation. The green dash-dotted line is the bisector of the angle $\widehat{v_1 v v_2}$.

Consider the modification in Figure 6. The point $v_s^- \in K$ is chosen on the bisector of the angle $\widehat{v_1 v v_2}$ such that $|v - v_s^-| = s$ (s is a free parameter). Define the competitor

$$K_s := \text{conv} \left((\partial K \setminus ([v_1, v] \cup [v_2, v])) \cup [[v_1, v_s^-] \cup [v_2, v_s^-]] \right),$$

where $\text{conv}(\cdot)$ denotes the convex hull. By construction

$$(27) \quad \partial K_s = \partial K \setminus ([v_1, v] \cup [v_2, v]) \cup ([v_1, v_s^-] \cup [v_2, v_s^-]).$$

Direct computation gives

$$\text{Per}(K) - \text{Per}(K_s) = s \sin \frac{\text{TA}(v)}{2} + o(s), \quad \int_{\mathbb{R}^2} d(x, K_s) d\mu_j \leq \int_{\mathbb{R}^2} d(x, K) d\mu_j + M_v s,$$

and using the minimality of K gives the upper bound estimate.

- Lower bound estimate $\text{TA}(v) \geq \frac{M_v}{2\lambda}$.

Consider the modification in Figure 7. The point $v_s^+ \notin K$ is chosen on the bisector of the angle $\widehat{v_1 v v_2}$ such that $|v - v_s^+| = s$ (s is a free parameter). Let

$$\tilde{K}_s := \text{conv} \left((\partial K \setminus ([v_1, v] \cup [v_2, v])) \cup [[v_1, v_s^+] \cup [v_2, v_s^+]] \right).$$

For the sake of brevity, given points $w, z \in \partial K$, the notation $[[w, z]]_{\partial K}$ will denote the unique *clockwise* (this is well defined since we endowed \mathbb{R}^2 with an orthogonal coordinate system, and

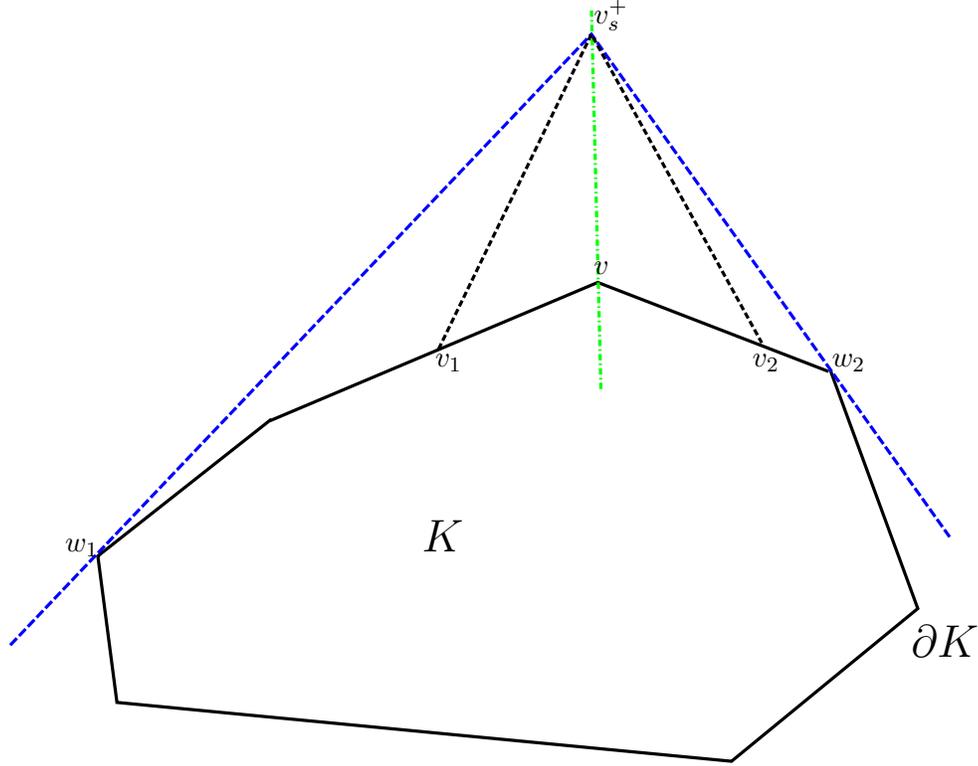


FIGURE 7. This is a schematic representation of the considered variation. The quantity $|v_s^+ - v| = s$ has been purposely exaggerated for the sake of clarity. The green dash-dotted line is the bisector of the angle $\widehat{v_1 v v_2}$.

$\partial \tilde{K}_s$ is homeomorphic to the unit circle S^1 path in ∂K with endpoints in w and z . In this case

$$\partial \tilde{K}_s = \partial K \setminus (\llbracket w_1, v \rrbracket_{\partial K} \cup \llbracket v, w_2 \rrbracket_{\partial K}) \cup \llbracket w_1, v_s^+ \rrbracket_{\partial K} \cup \llbracket v_s^+, w_2 \rrbracket_{\partial K},$$

where w_1 and w_2 are the intersections between ∂K and the two half-lines lines starting in v_s^+ and tangent to K . Direct computation gives

$$(28) \quad \int_{\mathbb{R}^2} d(x, \tilde{K}_s) d\mu_j \leq \int_{\mathbb{R}^2} d(x, K) - M_v s \cos \frac{\text{TA}(v)}{2}.$$

By construction it holds

$$(29) \quad \text{Per}(\tilde{K}_s) \leq \mathcal{H}^1 \left((\partial K \setminus (\llbracket v_1, v \rrbracket \cup \llbracket v_2, v \rrbracket)) \cup \llbracket v_1, v_s^+ \rrbracket \cup \llbracket v_2, v_s^+ \rrbracket \right),$$

and direct computation gives

$$(30) \quad \mathcal{H}^1 \left((\partial K \setminus (\llbracket v_1, v \rrbracket \cup \llbracket v_2, v \rrbracket)) \cup \llbracket v_1, v_s^+ \rrbracket \cup \llbracket v_2, v_s^+ \rrbracket \right) - \text{Per}(K) = s \sin \frac{\text{TA}(v)}{2} + o(s).$$

Combining (28), (29), (30) with the minimality of K (compared against \tilde{K}_s) gives the desired inequality. \square

Lemma 4.6. *Consider the family of measures $\{\mu_j\}$ defined in (26). Then for any sufficiently large index j , there exists a polygon $K_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)$ satisfying:*

- given distinct corners $v_1, v_2 \in \bigcup_{z \in B(p, r)} \operatorname{argmin}_{w \in \partial C_j} |z - w|$, i.e. v_1, v_2 receive mass only from $B(p, r)$, it holds $V(v_1) \cap V(v_2) = \emptyset$.

Proof. Note that

- r has been chosen (sufficiently small) such that (24) holds,
- for any sequence of minimizers $\{C_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)\}$ it holds (upon subsequence)

$$C_j \xrightarrow{d_{\mathcal{H}}} C \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_{r, a, \eta}, \lambda).$$

Thus for sufficiently large j , any minimizer $C_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)$ satisfies $d_{\mathcal{H}}(C_j, B(p, r)) \geq b/8$ (for the definition of b , see the arguments immediately before (24)). Note also that (upon choosing sufficiently large index j), for any minimizer $C_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)$, the sets

$$\bigcup_{z \in B(p, r)} \operatorname{argmin}_{w \in \partial C_j} |z - w|, \quad \bigcup_{z' \in B(p', r)} \operatorname{argmin}_{w \in \partial C_j} |z' - w|, \quad \bigcup_{z'' \in B(p'', r)} \operatorname{argmin}_{w \in \partial C_j} |z'' - w|$$

are mutually disjoint. Intuitively, this implies that any point of ∂C_j receives mass from at most one of the balls $B(p, r)$, $B(p', r)$, $B(p'', r)$. Then the conclusion follows by using the same construction from the proof of Lemma 3.6, which preserves convexity. \square

Lemma 4.7. *Consider the family of measures $\{\mu_j\}$ defined in (26). Then for any sufficiently large index j , there exists a minimizer $K_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)$ satisfying:*

- there exists a corner $v_j \in \partial K_j$, receiving mass from $B(p, r)$, such that $\operatorname{TA}(v_j) \geq \eta/(8\lambda)$.

Again the denominator 8λ is quite arbitrary, but sufficient for the purposes of this section (indeed any positive lower bound to $\operatorname{TA}(v_j)$ independent of j is sufficient). The proof follows by applying straightforwardly the same argument from the proof of Lemma 3.7, with the roles of Lemmas 3.4 and 3.6 replaced by Lemmas 4.5 and 4.6. However, since this result is crucial for the purposes of this section, we will report its proof.

Proof. Let $B := B(p, r)$. Consider an index j , a polygon $K_j \in \operatorname{argmin}_{\mathcal{C}} \mathcal{E}(\mu_j, \lambda)$, and let $\{v_i\}_{i \in \mathcal{I}} \subseteq K_j$ be the (finite) set of corners receiving mass from B , with \mathcal{I} a suitable set of indices. Similarly to the proof of Lemma 3.7, for any index $i \in \mathcal{I}$ let $V(v_i)$ be the wedge of v_i , and let l_i^{\pm} the two half-lines (the order is not relevant) forming the border $\partial V(v_i)$.

Again it holds (with the same proof from Lemma 3.7):

- for any index $i \in \mathcal{I}$, except at most two, both half-lines l_i^{\pm} must intersect ∂B .

Let $M_i := \operatorname{TM}(\mu_j, v_i, \partial K_j)$ ($i \in \mathcal{I}$). If there exists a couple of indices $i', i'' \in \mathcal{I}$ such that $M_{i'} + M_{i''} \geq \eta/2$, then the proof is complete. Thus assume:

$$(31) \quad (\forall i', i'' \in \mathcal{I}, i' \neq i'') \quad M_{i'} + M_{i''} \leq \eta/2.$$

The goal is to prove that (31) gives a contradiction.

Let $\mathcal{J} \subseteq \mathcal{I}$ be the set of indices i such that both half-lines l_i^\pm intersect ∂B . This implies that there exist points $p_i^\pm \in l_i^\pm \cap \partial B$; for any index $i \in \mathcal{J}$ choose an arc of minimal length $\phi_i \subseteq \partial B \cap V(v_i)$ connecting p_i^- and p_i^+ . Clearly $\mathcal{H}^1(\phi_i) \geq |p_i^- - p_i^+|$. However, since $d_{\mathcal{H}}(K_j, B) \geq b/8$ and $\text{TA}(v_i) \geq M_i/(2\lambda)$ (Lemma 4.5), elementary geometry (combined with the fact that M_i and $\text{TA}(v_i)$ are very small) gives

$$|p_i^- - p_i^+| \geq |v_i - p_i^-| \sin \text{TA}(v_i) \geq \frac{b}{8} \sin \text{TA}(v_i) \geq \frac{b}{12} \text{TA}(v_i) \geq \frac{bM_i}{24\lambda}.$$

Lemma 4.6 gives that the wedge of distinct corners are disjoint, and in particular the arcs ϕ_i ($i \in \mathcal{J}$) are mutually disjoint. Thus summing over indices $i \in \mathcal{J}$ gives

$$(32) \quad \sum_{i \in \mathcal{J}} \mathcal{H}^1(\phi_i) \geq \frac{b}{24\lambda} \sum_{i \in \mathcal{J}} M_i \stackrel{(31)}{\geq} \frac{b\eta}{48\lambda},$$

while by construction it holds $\phi_i \subseteq \partial B$ ($i \in \mathcal{J} \subseteq \mathcal{I}$), yielding

$$(33) \quad \sum_{i \in \mathcal{J}} \mathcal{H}^1(\phi_i) \leq 2\pi r.$$

Combining inequalities (32) and (33) gives

$$2\pi r = \mathcal{H}^1(\partial B) \geq \sum_{i \in \mathcal{J}} \mathcal{H}^1(\phi_i) \geq \frac{b\eta}{48\lambda},$$

which contradicts condition (25). Thus there exists a couple of indices $i', i'' \in \mathcal{I}$ such that $M_{i'} + M_{i''} \geq \eta/2$, and using Lemma 4.5 concludes the proof. \square

Now it is possible to pass to the limit: for any j choose a minimizer $K_j \in \text{argmin } \mathcal{E}(\mu_j, \lambda)$ such that the conclusion of Lemma 4.7 holds. Let $\varphi_j : [0, 1] \rightarrow \partial K_j$ be a constant speed parameterization, and it is clear that

$$\sup_j \|\varphi_j\|_{L^1} < \infty, \quad \sup_j \|\varphi_j\|_{TV} < \infty,$$

since the former follows from the minimality of K_j , and the latter follows from the convexity of K_j . Upon subsequence $K_j \xrightarrow{d_{\mathcal{H}}} K$, with K convex, thus $\partial K_j \xrightarrow{d_{\mathcal{H}}} \partial K$. Lemma 2.4 gives (upon subsequence) the existence of a limit curve $\varphi = \lim_j \varphi_j$ (this limit is taken in the C^0 topology) parameterizing ∂K . Since for any j , the measure φ_j'' (which does not change sign due to the convexity of K_j) has an atom of measure at least $\eta/(8\lambda)$ at some time t_j , and (upon subsequence) $t_j \rightarrow t$, the convergence (upon subsequence) $\varphi_j'' \xrightarrow{*} \varphi''$ implies that the measure φ'' has an atom of size at least $\eta/(8\lambda)$ at time t . Since an atom for the curvature measure φ'' is equivalent to a jump for the tangent derivative φ' , it follows:

Theorem 4.8. *Let $\mu_{r,a,\eta}$ be the measure defined in (19). Then for suitable choice of parameters λ, a, η, r , there exists a minimizer $K \in \text{argmin } \mathcal{E}(\mu_{r,a,\eta}, \lambda)$ whose border ∂K is not C^1 regular.*

Acknowledgments. The author warmly thanks the Center of Nonlinear Analysis (NSF grant DMS-0635983), where part of this research was carried out. The author acknowledges the support by ICTI and FCT (grant UTA_CMU/MAT/0007/2009), and is grateful to Dejan Slepčev for useful comments and suggestions, and Universidade Nova de Lisboa for its hospitality.

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