HIGH ORDER SCHEMES FOR WAVE EQUATIONS

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Abstract. The continuous and discontinuous Galerkin time stepping methodologies are combined to develop approximations of second order time derivatives of arbitrary order. This eliminates the doubling of the number of variables that results when a second order problem is written as a first order system. Stability, convergence, and accuracy, of these schemes is established in the context of the wave equation. It is shown that natural interpolation of non–homogeneous boundary data can degrade accuracy, and that this problem can be circumvented using interpolants matched with the time stepping scheme.

Key words. Wave Equation, Time Stepping Schemes, Non–Homogeneous Boundary Conditions.

1. Introduction. Numerical approximations the wave equation are developed which combine the continuous and discontinuous time stepping methodologies to obtain natural discretizations of the second time derivative of arbitrary order. This contrasts with the traditional approach where the equation is first decomposed into a first order system [2, 9] which has the disadvantage of doubling the number of spatial variables. This later approach can be expensive for vector valued problems, such as elastic wave propagation. Unlike schemes based upon first order systems, stability is not immediate for the new schemes; properties of Legendre polynomials established in Theorem 4.3 below are used to establish a discrete energy estimate.

In addition to the analysis of these time stepping schemes, a major focus of this work is the implementation of non–homogeneous boundary conditions and the associated error analysis. This contrasts with the majority of papers where homogeneous boundary data is considered "for simplicity". Numerical experiments are presented in Section 3 which illustrate that naive specification of boundary data degrades the rate of convergence and that this problem can be circumvented with proper treatment of the boundary terms. These technical issues are not specific to the wave equation; for example, similar treatment of non–homogeneous boundary data will be required to achieve optimal rates of convergence for parabolic problems.

1.1. Related Results. Implicit time stepping schemes are commonly used for the wave equation to avoid restrictive CFL constraints on the time step that result when local mesh refinement and higher time stepping schemes are employed. Dupont [4] developed optimal rates of convergence for an implicit scheme using the natural second order finite difference approximation of the second time derivative. While this gives a multi–step scheme, the analysis in [4] is canonical. Following the technique developed for parabolic equations, the elliptic projection is first used to estimate the error of the semi–discrete scheme where time is continuous and the spatial variables discretized. Errors due to temporal discretization are then estimated, and rates for the fully discrete scheme then follow from the triangle inequality. Using the property that temporal differentiation commutes with the elliptic projection eliminates the need to develop regularity estimates for the semi–discrete scheme. The schemes considered below are of arbitrarily high order, so regularity of the solution will be assumed as required; we note that Rauch [11] showed that derivation of the minimal regularity required to obtain optimal rates for the wave equation can be subtle.

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Subsequent to Dupont's paper, most of the time stepping schemes proposed for the wave equation were constructed by writing it as a first order system. In this context many of the time stepping schemes developed for parabolic equations, and much of the analysis, could be used for the wave equation. In particular, both continuous Galerkin (CG) and discontinuous Galerkin (DG) time stepping schemes have be proposed and analyzed for the wave equation [2, 5, 6, 9]. The CG method preserves energy equality exhibited by solutions of the wave equation [7] while the penalty term of the DG scheme is dissipative. Since the DG schemes are discontinuous it easy to formulate and analyze schemes which adapt the spatial mesh to the solution at each time step; adaptivety for the CG scheme was considered in [10].

The CG time stepping scheme involves fewer unknowns than the DG counterpart. Specifically, if approximate solutions have polynomial time dependence of degree ℓ and take values in a finite element subspace U_h, then the number of unknowns for the CG scheme is $2\ell\dim(U_h)$ while the DG scheme will have $2(\ell+1)\dim(U_h)$; the factor of two arising since the equation is posed as a first order system with twice as many variables. The time stepping scheme considered here has $\ell\dim(U_h)$ unknowns; moreover, the accuracy is similar. Letting τ and h denote the time and space step sizes, solutions the CG and DG schemes exhibit rates of convergence of order $O(\tau^{\ell+1}+h^{k+1})$ for both the error, $\|e(t)\|_{L^2(\Omega)}$, and and its derivative, $\|e_t(t)\|_{L^2(\Omega)}$, when the finite element space contains the piecewise polynomials of degree k. For the scheme analyzed below this rate is achieved for $\|e(t)\|$ at all times, and for $\|e_t(t^n_-)\|_{L^2(\Omega)}$ at the partition points. It is well known that the CG and DG time stepping schemes have a natural correspondence with collocation methods which use Gauss Lobatto and Gauss Radau quadrature points respectively. When used for ordinary differential equations these schemes have formal order $O(\tau^{2\ell})$ and $O(\tau^{2\ell+1})$ respectively [8]. French and Peterson [6] show that these super convergence rates may be achieved at the partition points by the CG time steeping scheme for the wave equation. While the time stepping scheme proposed below is not obviously a collocation scheme, it is of formal order $O(\tau^{2\ell-1})$. Example 5.6 illustrates that super convergence at this rate is observed the partition points when the solution is smooth.

Non–homogeneous boundary conditions complicate the error analysis since temporal derivatives of the boundary data appear in the stability estimate for the wave equation. Naive implementation of the boundary conditions then results in consistency errors containing time derivatives of the boundary data which converge at reduced rates; Example 3.2 illustrates this. Except for Dupont's paper [4], this issue has not been addressed; that is, homogeneous boundary data is considered ubiquitously. The time stepping scheme analyzed by Dupont was derived using finite difference methodology, and in this situation it is clear how to specify the boundary data to avoid the introduction of additional consistency errors. This issue is taken up in Section 5 where it is shown that the consistency error will not involve temporal derivatives of the boundary data if a "semi–Hermite" interpolant is used. We note that an unusual numerical schemes for the wave equation with non–smooth Dirichlet data was considered in [3]; the discrete solutions always vanished on the boundary, $u_h(t) \in H^1_0(\Omega)$. Since the exact solution does not vanishes on the boundary convergence is not possible in $H^1(\Omega)$ even when the solution is smooth; instead the authors prove convergence in weaker dual norms.

1.2. Overview and Notation. The next section introduces the abstract setting where weak and strong solutions of the wave equation are well posed, and Section 3 introduces the discrete weak statement. Stability and convergence of the numerical schemes are then established in Sections 4 and 5 respectively.

Standard notation is used for the Lebesgue spaces, $L^p(\Omega)$, and Sobolev spaces, $W^{m,p}(\Omega)$, $H^1(\Omega) = W^{1,2}(\Omega)$. Solutions of the wave equation will be functions from $[0,T]$ into these spaces and the usual notation, $L^2[0,T;H^1(\Omega)]$, $C[0,T;L^2(\Omega)]$, is used to indicate the temporal regularity of such functions. Function spaces where the abstract wave equation is posed are introduced at the beginning of the next section, and notation for the discrete function spaces appears at the beginning of Section 3. In general, a super script is used to index time, $u^n \simeq u(t^n)$, and subscripts indicate spatial approximation, $u_h(t) \in U_h \subset U$.

2. Wave Equations. We consider wave equations of the form

$$u_{tt} + Au = f, \qquad \text{on } (0,T), \tag{2.1}$$

supplemented with initial conditions for $u(0)$ and $u_t(0)$, and boundary conditions. Here $A : D(A) \to H$ is a linear, self adjoint, operator on a Hilbert space H with domain $D(A) \subset H$. To accommodate the canonical example where $Au = -\Delta u$ on a bounded domain $\Omega \subset \mathbb{R}^d$ with with boundary conditions

$$u|_{\Gamma_0} = u_0, \qquad (\partial u/\partial n)|_{\Gamma_1} = g, \qquad \text{where} \qquad \partial\Omega = \Gamma_0 \cup \Gamma_1,$$

we introduce spaces $U_0 \hookrightarrow U \hookrightarrow \mathcal{U} \hookrightarrow H$ corresponding to [12]

$$U_0 = H^1_0(\Omega), \qquad U = \{u \in H^1(\Omega) \mid u|_{\Gamma_0} = 0\}, \qquad \mathcal{U} = H^1(\Omega), \qquad H = L^2(\Omega).$$

In the general setting $A : D(A) \to H$ is determined from a continuous bilinear function $a : \mathcal{U} \times U \to \mathbb{R}$ as[1]

$$D(A) = \{u \in \mathcal{U} \mid |a(u,v)| \le C(u)\|v\|_H,\ v \in U_0\}, \qquad (Au,v)_H = a(u,v), \quad v \in U_0.$$

If $u \in D(A)$ the function $v \mapsto a(u,v) - (Au,v)_H$ is continuous on U and vanishes on U_0 so there exists $\partial_A(u) \in (U/U_0)'$ such that

$$a(u,v) - (Au,v)_H = \partial_A(u)(v), \qquad v \in U.$$

Strong solutions of the wave equation satisfy (2.1) with each term in H, and satisfy the initial and boundary conditions

$$u(0) = u^0, \quad u_t(0) = u_t^0, \qquad u \in u_0 + U, \quad \partial_A(u) = g,$$

where the initial values $u^0 \in D(A)$, $u_t^0 \in U$ and boundary values $u_0(t) \in \mathcal{U}$ and $g(t) \in (U/U_0)'$ are specified. Weak solutions of the wave equation satisfy

$$u(t) - u_0(t) \in U, \qquad (u_{tt}, v) + a(u,v) = (f,v)_H + (g,v), \qquad v \in U, \tag{2.2}$$

and the initial conditions. The weak statement is meaningful when $u_{tt}(t) \in U'$ and $u(t) \in U$, in which case the first and last terms in the weak statement are parings between U' and U.

The usual translation argument may used to establish existence for the problem with non–homogeneous boundary data. If $u_0(t) \in D(A)$, $u_{0tt}(t) \in H$, and $\partial_A(u_0) = g$, then $\tilde{u}(t) \equiv u(t) - u_0(t) \in U$ satisfies the wave equation with homogeneous boundary data,

$$(\tilde{u}_{tt}, v) + a(\tilde{u}, v) = (f - u_{0tt} - Au_0, v)_H, \qquad v \in U.$$

[1] When functions in the domain of the operator are required to satisfy homogeneous Dirichlet and Neumann boundary conditions, $D_0(A)$ is typically used to denote the domain defined here.

A proof that weak solutions of this equation are strong solutions with $\partial_A(\tilde{u}) = 0$ when the initial data and right hand side are sufficiently regular is given in [12]. The following hypotheses on $a(.,.)$ and the data were implicit in this discussion, and will be assumed below.

ASSUMPTION 2.1. *The operators and data satisfy the following continuity and coercivity properties.*

1. *(Continuity)* $a : \mathcal{U} \times U \to \mathbb{R}$ *is bilinear and continuous and the restriction to* $U \times U$ *is symmetric. Specifically, there exists a constant* $C_a > 0$ *such that*
$$|a(u,v)| \le C_a \|u\|_U \|v\|_U, \qquad u \in \mathcal{U},\ v \in U,$$
and $a(u,v) = a(v,u)$ *for* $u,\ v \in U$.
2. *(Coercivity) There exists a constant* $c_a > 0$ *such that*
$$a(u,u) \ge c_a \|u\|_U^2, \qquad u \in U.$$
The arguments below readily extend to the situation where $a(u,u)^{1/2}$ *is a semi–norm and* $a(u,u) + \|u\|_H^2 \ge c_a \|u\|_U^2$.
3. *The right hand side of the strong from satisfies* $f \in L^1[0,T;H]$.
4. *The Neumann data satisfies* $g \in W^{1,1}[0,T;U']$. *A necessary condition for weak solutions to be strong is* $g \in W^{1,1}[0,T;(U/U_0)']$; *this later condition is implicit when strong solutions are assumed.*
5. *The Dirichlet boundary data satisfies* $u_0 \in W^{2,1}[0,T;\mathcal{U}] \cap L^1[0,T;D(A)]$.

Adopting $a(.,.)$ to be the inner product on U shortens many of the estimates, so below we write $\|u\|_U = a(u,u)^{1/2}$. With this convention dual norms will be independent of $a(.,.)$ when scaled by (C_a/c_a).

2.1. Estimates. Setting $v = u_t$ in the weak statement (2.2) and integrating over $(0,t)$ gives the estimate

$$\begin{aligned}(1/2)\left(\|u_t(t)\|_H^2 + \|u(t)\|_U^2\right) &= (1/2)\left(\|u_t^0\|_H^2 + \|u^0\|_U^2\right) + \int_0^t \left\{(f,u_t) - (g_t,u)\right\} + (g,u)|_0^t \\ &\le (1/2)\left(\|u_t^0\|_H^2 + \|u^0\|_U^2\right) + \|g(0)\|_{U'}\|u^0\|_U \\ &\quad + \left(\|f\|_{L^1[0,t;H]} + \|g_t\|_{L^1[0,t;U']} + \|g(t)\|_{U'}\right) \max_{0 \le s \le t} \left(\|u_t(s)\|_H^2 + \|u(s)\|_U^2\right)^{1/2}. \end{aligned} \tag{2.3}$$

Selecting $t \in [0,T]$ where the maximum on the right occurs shows

$$\max_{0 \le s \le T} \left(\|u_t(s)\|_H + \|u(s)\|_U\right) \le C\left(\|u_t^0\|_H + \|u^0\|_U + \|f\|_{L^1[0,T;H]} + \|g\|_{W^{1,1}[0,T;U']}\right). \tag{2.4}$$

In the context of a numerical scheme an estimate of the form (2.3) only holds for discrete times t^n on the left. For a low order scheme where $u(s)$ and $u_t(s)$ at times $s \in (t^{n-1}, t^n)$ are determined from the values at their end points an estimate of the form (2.4) is immediate for the discrete scheme.

Estimates for the higher order schemes will use a discrete version of the following estimate obtained by setting the test function $v = \exp(-\lambda t) u_t(t)$ in equation (2.2),

$$
\begin{aligned}
&(e^{-\lambda t}/2)\left(\|u_t(t)\|_H^2 + \|u(t)\|_U^2\right) + (\lambda/2)\int_0^t e^{-\lambda\cdot}\left(\|u_t\|_H^2 + \|u\|_U^2\right) \\
&= (1/2)\left(\|u_t^0\|_H^2 + \|u^0\|_U^2\right) + \int_0^t e^{-\lambda\cdot}\left((f,u_t) + (\lambda g - g_t, u)\right) + e^{-\lambda\cdot}(g,u)|_0^t \\
&\le (1/2)\left(\|u_t^0\|_H^2 + \|u^0\|_U^2\right) \\
&\quad + \left(\|f\|_{L^1[0,t;H]} + (2+\lambda t)\|g\|_{C[0,t;U']} + \|g_t\|_{L^1[0,t;U']}\right)\left(\|u_t\|_{L^\infty[0,t;H]} + \|u\|_{L^\infty[0,t;U]}\right).
\end{aligned} \tag{2.5}
$$

When $\lambda t = O(1)$, so $\lambda = O(1/t)$, inverse estimates for polynomials show that the norms $\lambda\|.\|_{L^2[0,t;H]}^2$ and $\|.\|_{L^\infty[0,t,H]}^2$ are comparable, and the energy estimate (2.4) will follow.

3. Numerical Scheme. Let $0 = t^0 < t^1 < \ldots < t^N = T$ be a partition of $[0,T]$, $\mathcal{U}_h \subset \mathcal{U}$ be a subspace and set $U_h = \mathcal{U}_h \cap U$. If $u_{0h}(t) \in \mathcal{U}_h$ is an approximation the Dirichlet data, and $g^\tau(t) \in U_h'$ is an approximation of the Neumann data, we consider approximate solutions of the wave equation in the space

$$
u_h \in u_{0h} + \{u_h \in C[0,T;U_h] \mid u_h|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1}, t^n, U_h]\} \equiv u_{0h} + \mathbb{U}_h^\ell
$$

which, on each interval, satisfy

$$
\int_{t^{n-1}}^{t^n} \left\{(u_{htt}, v_h) + a(u_h, v_h)\right\} + ([u_{ht}], v_{h+})_H^{n-1} = \int_{t^{n-1}}^{t^n} (f, v_h)_H + (g^\tau, v_h), \tag{3.1}
$$

for all $v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$. Here $[u_{ht}]$ denotes the jump in the time derivative at the partition point, so this scheme can be viewed as a continuous Galerkin time stepping scheme for u coupled with a discontinuous Galerkin time stepping scheme for u_t.

EXAMPLE 3.1. If $\ell = 1$ the solution is piecewise linear in time so $u_{htt} = 0$ and $u_t = (u^n - u^{n-1})/\tau$ on (t^{n-1}, t^n) where τ is the time step. Letting $A_h u_h \in U_h$ and $F_h \in U_h$ denote the discrete spatial operator and data characterized by

$$
(A_h u_h, w_h) = a(u_h, w_h), \qquad (F_h, w_h) = (1/\tau)\int_{t^{n-1}}^{t^n} (f, w_h) + (g, w_h), \qquad w_h \in U_h,
$$

the scheme may be written as

$$
\frac{u^n - 2u^{n-1} + u^{n-2}}{\tau} + \tau A_h\left(\frac{u^n + u^{n-1}}{2}\right) = \tau F_h^{n-1/2}.
$$

Clearly this scheme is first order in time; however, this is atypical. For $\ell > 1$ the time stepping scheme is of order $\ell + 1$ for the wave equation, and is of order $2\ell - 1$ for ode's of the form $u'' = f(t, u, u_t)$. The second order scheme analyzed by Dupont [4] has the same temporal discretization but different spatial discretization:

$$
\frac{u^n - 2u^{n-1} + u^{n-2}}{\tau} + \tau A_h\left(\frac{u^n + 2u^{n-1} + u^{n-2}}{4}\right) = \frac{\tau}{4}\left(F_h^n + 2F_h^{n-1} + F_h^{n-2}\right).
$$

∎

$e \equiv u - u_h$	$\|e(1)\|_{L^2(\Omega)}$	$\|e_{t-}(1)\|_{L^2(\Omega)}$	$\lvert e(1)\rvert_{H^1(\Omega)}$
Homogeneous BC	2.9180	3.0026	2.9331
NonHomegenous BC, Lagrange interpolant	3.0084	2.5840	2.6251
NonHomegenous BC, projected data	3.0027	2.9964	3.0274

FIG. 3.1. *Rates of convergence with $k = 3$, $\ell = 2$ when $\tau \in \{1/8, 1/16, 1/32, 1/64, 1/128, 1/256\}$.*

The next example shows that specifying u_{0h} to be the usual Lagrange interpolant of the boundary data u_0 and setting $g^\tau = g$ results in a loss of accuracy. This motivated the analysis below which shows that this problem can be eliminated if the Dirichlet data for the numerical scheme is the spatial interpolant of $\mathbb{P}^\tau(u_0)$ and the Neumann data is taken to be $\mathbb{P}^\tau(g)$, where $\mathbb{P}^\tau$ is the temporal projection introduced Definition 5.1 below.

EXAMPLE 3.2. Numerical approximation of the scalar wave equation $u_{tt} - \Delta u = 0$ with solution

$$u(t, x, y) = \cos(\sqrt{2}\pi t)\cos(\pi x)\sin(\pi y)$$

is considered on the domain $\Omega = (-1, 1)^2$. Notice that

- $u|_{y=\pm 1} = 0$ and $\partial u/\partial n|_{x=\pm 1} = 0$, but
- $u|_{x=\pm 1} \neq 0$ and $\partial u/\partial n|_{y=\pm 1} \neq 0$,

so homogeneous boundary data will result if $\Gamma_0 = \{(x, y) \in \partial\Omega \mid y = \pm 1\}$ and $\Gamma_1 = \{(x, y) \in \partial\Omega \mid x = \pm 1\}$, and interchanging the two gives non–homogeneous boundary data.

Approximate solutions were computed on uniform square meshes with fixed time steps. To illustrate the role of the time stepping scheme, serendipity elements containing the piecewise polynomials of degree $k = 3$ were used for for the spatial variables, and piecewise polynomials of degree $\ell = 2$ were used for the time dependence.

The solution was evolved until a time $T = 1$ using the same number of elements in space and time ($h = 2\tau$) and rates of convergence for the errors $\|(u - u_h)(1)\|_{L^2(\Omega)}$, $\|(u - u_h)_t(1_-)\|_{L^2(\Omega)}$, and $|(u - u_h)(1)|_{H^1(\Omega)}$, tabulated in Figure 3.1. The middle row shows the rates of convergence when the Dirichlet data for the numerical scheme on each interval (t^{n-1}, t^n) was the Lagrange interpolant using the end points and the mid point, and boundary integrals for the Neumann data were computed "exactly" (high order quadrature). This implementation of the non–homogeneous boundary data clearly results in a degradation of the rate. The third row of the table illustrates that there is no degradation of the rates when the Dirichlet data for the numerical scheme is the spatial interpolant of $\mathbb{P}^\tau(u_0) \in \mathcal{P}_\ell[t^{n-1}, t^n, \mathcal{U}]$ and Neumann data $g^\tau = \mathbb{P}^\tau(g) \in \mathcal{P}_\ell[t^{n-1}, t^n, U']$ is specified. $\square$

4. Stability. Setting $v_h = u_{ht}$ in the discrete weak statement (3.1) and summing, and integrating the last term by parts, shows

$$E(u^n, u^n_{t-}) + (1/2)\sum_{m=0}^{n-1} \|[u_{ht}]\|_H^2 = E(u^0, u^0_t) + \int_0^{t^n} \{(f, u_{ht}) - (g^\tau_t, u_h)\} + (g^\tau, u_h)|_{t=0}^{t^n} \tag{4.1}$$

$$\leq E(u^0, u^0_t) + \left(\|f\|_{L^1[0,t^n;H]} + \|g^\tau_t\|_{L^1[0,t^n;U']} + 2\|g^\tau\|_{C[0,t^n;U']}\right) \max_{0 \leq t \leq t^n} E\left(u_h(t), u_{ht}(t)\right)^{1/2}.$$

This is the analog of equation (2.3); however, it does not immediately bound the solution since the left hand side only estimates the energy at the discrete times $\{t^n\}_{n=0}^N$, while the right hand side involves the energy at all times $0 \leq t \leq T$. This issue is circumvented by developing an analog of equation (2.5).

4.1. Preliminaries. To develop a discrete analog of (2.5) properties of polynomials will be exploited. When the target space is a polynomial subspace the L^2 projection is well defined for all integrable functions.

DEFINITION 4.1. *Let U be a Banach space, $0 = t^0 < t^1 < \ldots < t^N = T$ be a partition of $[0,T]$, and $\ell \geq 1$ an integer. For $w \in L^1[0,T;U]$, let $\bar{w}$ denote the function in*

$$\{\bar{w} \in L^1[0,T;U] \mid \bar{w}|_{(t^{n-1},t^n)} \in \mathcal{P}_{\ell-1}[t^{n-1},t^n;U],\ 1 \leq n \leq N\},$$

satisfying on each interval

$$\int_{t^{n-1}}^{t^n} p(t)\bar{w}(t)\,dt = \int_{t^{n-1}}^{t^n} p(t)w(t)\,dt, \qquad p \in \mathcal{P}_{\ell-1}(t^{n-1},t^n).$$

Stability and approximation properties of this projection follow from elementary parent element arguments.

LEMMA 4.2. *Let U be a Banach space, $\ell \geq 1$ be an integer, and $0 = t^0 < t^1 < \ldots t^N = T$ be a partition of $[0,T]$. Then there exists $C = C(\ell) > 0$ depending only upon ℓ such that the projection $u \mapsto \bar{u}$ characterized in Definition 4.1 satisfies*

$$\|\bar{u}\|_{L^r[t^{n-1},t^n;U]} \leq C\|u\|_{L^r[t^{n-1},t^n;U]}, \qquad u \in L^r[0,T;U],$$
$$\|\bar{u}'\|_{L^r[t^{n-1},t^n;U]} \leq C\|u'\|_{L^r[t^{n-1},t^n;U]}, \qquad u' \in L^r[0,T;U],$$

for all $1 \leq r \leq \infty$, and

$$\|\bar{u} - u\|_{C[t^{n-1},t^n;U]} \leq C\|u'\|_{L^1[t^{n-1},t^n;U]}, \quad \text{and} \quad \|[\bar{u}^n]\|_U \leq C\|u'\|_{L^1[t^{n-1},t^{n+1};U]},$$

when $u' \in L^1[0,T;U]$.

The construction of test functions presents a major difficulty for the analysis of Galerkin schemes. The following lemma shows that the (in)equality

$$\int_0^\tau (1-\lambda t)(u_t,u)_U \geq (1/2)(1-\lambda\tau)\|u(\tau)\|_U^2 - (1/2)\|u(0)\|_U^2 + (\lambda/2)\int_0^\tau \|u(t)\|_U^2\,dt,$$

remains valid when $u \in \mathcal{P}_\ell[0,\tau;U]$ is replaced by $\bar{u} \in \mathcal{P}_{\ell-1}[0,\tau;U]$ on the left; the later being a valid test function for the numerical scheme.

THEOREM 4.3. *Let U be a (semi) inner product space, $\ell \geq 1$ be an integer, and $\tau > 0$. Let $p_\ell(t) = \sqrt{2\ell+1}L_\ell(-1+2t/\tau)$ where $L_\ell(\xi)$ is the Legendre polynomial on $[-1,1]$ normalized so that $L(1) = 1$.*

Let $u \in \mathcal{P}_\ell[0,\tau;U]$ and $\bar{u}$ be the projection of u onto $\mathcal{P}_{\ell-1}[0;\tau;U]$ given in Definition 4.1, and let

$$u_\ell = (1/\tau)\int_0^\tau p_\ell(t)u(t)\,dt.$$

Then for any $\lambda \in \mathbb{R}$,

$$\int_0^\tau (1-\lambda t)(u_t(t),\bar{u}(t))_U\,dt = (1/2)(1-\lambda\tau)\|u(\tau)\|_U^2 - (1/2)\|u(0)\|_U^2 + \lambda\tau\ell\|u_\ell\|_U^2 + (\lambda/2)\int_0^\tau \|u(t)\|_U^2\,dt.$$

Before starting the proof, we recall that the Legendre polynomial $L_\ell(\xi)$ is orthogonal to $\mathcal{P}_{\ell-1}(-1,1)$, and when when normalized by $L_\ell(1) = 1 = \|L_\ell\|_{C[-1,1]}$ has norm $\|L_\ell\|^2_{L^2(-1,1)} = 2/(2\ell+1)$. The scaled Legendre polynomial $p_\ell(t)$ in the theorem then has norm $\|p_\ell\|_{L^2(0,\tau)} = \sqrt{\tau}$.

Proof. By construction, $p_\ell(t)$ is orthogonal to $\mathcal{P}_{\ell-1}(0,\tau)$ and $\|p_\ell\|_{L^2(0,\tau)} = \sqrt{\tau}$. It follows from Definition 4.1 that $u(t) = \bar{u}(t) + p_\ell(t)u_\ell$.

Using the identity $(u_t, u)_U = d/dt(\|u\|^2_U/2)$, integration by parts shows

$$\int_0^\tau (1-\lambda t)\left(u_t(t), \bar{u}_h(t)\right)_U \, dt = \int_0^\tau (1-\lambda t)\left(u_t(t), u(t) - p_\ell(t)u_\ell\right)_U \, dt$$

$$= (1/2)(1-\lambda\tau)\|u(\tau)\|^2_U - (1/2)\|u(0)\|^2_U + \int_0^\tau (\lambda/2)\|u(t)\|^2_U - (1-\lambda t)\left(u_t(t), p_\ell(t)u_\ell\right)_U \, dt,$$

It remains to show that the last term takes the form stated. First,

$$\int_0^\tau (1-\lambda t)\left(u_t(t), p_\ell(t)u_\ell\right) \, dt = \int_0^\tau (-\lambda t)\left(u_t(t), p_\ell(t)u_\ell\right) \, dt,$$

since $u_t \in \mathcal{P}_{\ell-1}[0,\tau;U]$ and p_ℓ is orthogonal to $\mathcal{P}_{\ell-1}(0,\tau)$. Next, write $u_t(t) = \bar{u}_{ht}(t) + p'_\ell u_\ell$, and use the property that $t\bar{u}_t(t)$ has degree bounded by $\ell-1$ so is orthogonal to p_ℓ, to conclude that the last term may be written as

$$\int_0^\tau t\left(u_t(t), p_\ell(t)u_\ell\right)_U \, dt = \int_0^\tau t\left(p'_\ell(t)u_\ell, p_\ell(t)u_\ell\right)_U \, dt$$

$$= (\tau/2)p_\ell(\tau)^2\|u_\ell\|^2_U - (1/2)\int_0^\tau p_\ell(t)^2 \, dt \, \|u_\ell\|^2_U$$

$$= (\tau/2)(p_\ell(\tau)^2 - 1)\|u_\ell\|^2_U.$$

The lemma now follows since $p_\ell(\tau)^2 = 2\ell+1$. ∎

It now follows that the projection of $(1-\lambda t)u_t(t)$ onto $\mathcal{P}_{\ell-1}[0,\tau;U]$ will give a discrete analog of the test function $v = \exp(-\lambda t)u_t(t)$ used in the derivation of equation (2.5).

COROLLARY 4.4. *Let $U \hookrightarrow H$ be an embedding of Hilbert spaces, $u \in \mathcal{P}_\ell[0,\tau;U]$, $v = (1-\lambda t)u_t(t)$, and $\bar{v}$ be the projection of v onto $\mathcal{P}_{\ell-1}[0,\tau;U]$ characterized in Definition 4.1. Then*

$$\int_0^\tau (u_{tt}, \bar{v})_H = (1/2)\left((1-\lambda\tau)\|u_t(\tau)\|^2_H - \|u_t(0)\|^2_H\right) + (\lambda/2)\int_0^\tau \|u_t\|^2_H,$$

and

$$\int_0^\tau (u, \bar{v})_U = (1/2)\left((1-\lambda\tau)\|u(\tau)\|^2_U - \|u(0)\|^2_U\right) + \lambda\ell\tau\|u_\ell\|^2_U + (\lambda/2)\int_0^\tau \|u\|^2_U,$$

where $u_\ell \in U$ is as in the theorem. Moreover,

$$\|u_t(0) - \bar{v}(0)\|_H \le |\lambda|\sqrt{(2\ell+1)\tau}\|u_t\|_{L^2[0,\tau;H]}.$$

Proof. Since $u_{tt} \in \mathcal{P}_{\ell-2}[0,\tau;U]$ it follows that

$$\int_0^\tau (u_{tt}, \bar{v})_H = \int_0^\tau (u_{tt}, v)_H = \int_0^\tau (1-\lambda t)(u_{tt}, u_t)_H \, dt = \int_0^\tau (1-\lambda t)(\|u_t\|^2_H/2)_t \, dt,$$

and the first identity follows upon integration by parts. To establish the second identity write

$$\int_0^\tau (u, \bar{v})_U = \int_0^\tau (\bar{u}, v)_U = \int_0^\tau (1 - \lambda t)(\bar{u}, u_t)_U \, dt,$$

and apply of the theorem. For the final estimate write $v(t) - \bar{v}(t) = p_\ell(t) v_\ell$. Then $v(0) = u_t(0)$ and

$$u_t(0) - \bar{v}(0) = p_\ell(0)(1/\tau) \int_0^\tau p_\ell(t)(1 - \lambda t) u_t(t) \, dt = p_\ell(0)(1/\tau) \int_0^\tau -p_\ell(t) \lambda t u_t(t) \, dt,$$

where the second equality follows since p_ℓ is orthogonal to $\mathcal{P}_{\ell-1}(0, \tau)$ and $u_t \in \mathcal{P}_{\ell-1}[0, \tau; U]$. Then

$$\|u_t(0) - \bar{v}(0)\|_H \leq |p_\ell(0)| \lambda \|p_\ell\|_{L^2(0,\tau)} \|u_t\|_{L^2[0,\tau;H]} = \sqrt{2\ell + 1} |\lambda| \sqrt{\tau} \|u_t\|_{L^2[0,\tau;H]}.$$

$\square$

4.2. Stability. Stability of the numerical scheme (3.1) with homogeneous Dirichlet data can now be established. The usual translation argument then bounds solutions with non–homogeneous Dirichlet data.

THEOREM 4.5. *Let $U \hookrightarrow H$ be an embedding of Hilbert spaces, $U_h \subset U$ be a subspace, and $0 = t^0 < t^1 < \ldots t^N = T$ be a partition of $[0, T]$ with maximal time step $\tau = \max_{1 \leq n \leq N} \Delta t^n \leq 1$. Assume that the bilinear function $a : U_h \times U_h \to \mathbb{R}$ satisfies Assumptions 2.1, $f \in L^1[0, T; H]$, and $g^\tau \in W^{1,1}[0, T; U']$. Then there exists $C = C(\ell) > 0$ such that solutions of the numerical scheme (3.1) with initial data u_h^0 and $u_{ht-}^0 \in U_h$ satisfy*

$$\max_{0 \leq t \leq T} E(u_h^n, u_{ht-}^n) + \sum_{m=0}^{n-1} \|[u_{ht}^m]\|_H^2 \leq E(u_h^0, u_{ht-}^0) + C \left(\|f\|_{L^1[0,T;H]} + \|g^\tau\|_{W^{1,1}[0,T;U']} \right)^2,$$

where $E(u, v) \equiv (1/2) \left(\|u\|_U^2 + \|v\|_H^2 \right)$, $u_h^n = u_h(t^n)$ and $[u_t^m] = u_{t+}^m - u_{t-}^m$.

Proof. Fix $\lambda = 1/(4(2\ell + 1)\Delta t^n)$ and set $v_h(t) = \overline{(1 - \lambda(t - t^{n-1})) u_{ht}(t)}$ in the discrete weak statement (3.1) to obtain

$$\begin{aligned}
(1 - \lambda \Delta t^n) E(u^n, u_{t-}^n) + (\lambda/2) \int_{t^{n-1}}^{t^n} \left(\|u_t\|_H^2 + \|u_h\|_U^2 \right) + \lambda \Delta t^n \ell \|u_\ell^{n+1/2}\|_U^2 + (1/2) \|[u_t^{n-1}]\|_H^2 \\
= E(u^{n-1}, u_{t-}^{n-1}) + ([u_t^{n-1}], u_{t+}^{n-1} - v_+^{n-1})_H + \int_{t^{n-1}}^{t^n} (1 - \lambda(. - t^{n-1})) \left((\bar{f}, u_t) + (\bar{g}, u_t) \right). \quad (4.2)
\end{aligned}$$

Here and below the subscript on u_h and superscript on g^τ are omitted. Corollary 4.4 was used to write the left hand side in the form shown with $u_\ell^{n+1/2}$ denoting the "hight frequency" component of u_h. We consider each of the terms on the right.

1. The jump term is bounded using Corollary 4.4.

$$\begin{aligned}
([u_t^{n-1}], u_{t+}^{n-1} - v_+^{n-1})_H &\leq \|[u_t^{n-1}]\|_H \|u_{t+}^{n-1} - v_+^{n-1}\|_H \\
&\leq \lambda \sqrt{(2\ell + 1)\Delta t^n} \|[u_t^{n-1}]\|_H \|u_t\|_{L^2[t^{n-1}, t^n; H]} \\
&\leq \sqrt{(2\ell + 1)\lambda \Delta t^n} \left((1/2) \|[u_t^{n-1}]\|_H^2 + (\lambda/2) \|u_t\|_{L^2[t^{n-1}, t^n; H]}^2 \right) \\
&= (1/2) \left((1/2) \|[u_t^{n-1}]\|_H^2 + (\lambda/2) \|u_t\|_{L^2[t^{n-1}, t^n; H]}^2 \right);
\end{aligned}$$

the last step following since $\lambda = 1/(4(2\ell + 1)\Delta t^n)$. This shows the jump term can be absorbed into the left hand side of (4.2).

2. Since $\lambda \Delta t^n \le 1/4 \le 1$ the term involving f is bounded as

$$\begin{aligned}\int_{t^{n-1}}^{t^n} (1-\lambda(.-t^{n-1}))(\bar{f}, u_t) &\le \|\bar{f}\|_{L^1[t^{n-1},t^n;H]} \|u_t\|_{L^\infty[t^{n-1},t^n;H]} \\ &\le C\|f\|_{L^1[t^{n-1},t^n;H]} \|u_t\|_{L^\infty[t^{n-1},t^n;H]},\end{aligned}$$

where $C = C(\ell)$ is the constant from Lemma 4.2.

3. The term involving g is integrated by parts to give

$$\begin{aligned}&\int_{t^{n-1}}^{t^n} (1-\lambda(.-t^{n-1}))(\bar{g}, u_t) \\ &= \int_{t^{n-1}}^{t^n} \left\{ \lambda(\bar{g}, u_h) - (1-\lambda(.-t^{n-1}))(\bar{g}_t, u_h) \right\} + (1-\lambda(.-t^n))(\bar{g}, u_h)\Big|_{t=t_+^{n-1}}^{t_-^n} \\ &\le \left((2+\lambda\Delta t^n)\|\bar{g}\|_{C[t^{n-1},t^n;U']} + \|\bar{g}_t\|_{L^1[t^{n-1},t^n;U']} \right) \|u_h\|_{C[t^{n-1},t^n;U]} \\ &\le C\left(\|g\|_{C[t^{n-1},t^n;U']} + \|g_t\|_{L^1[t^{n-1},t^n;U']} \right) \|u_h\|_{C[t^{n-1},t^n;U]}.\end{aligned}$$

The last step used Lemma 4.2 to bound the terms involving $\bar{g}$ by the corresponding quantities in g and the property $\lambda \Delta t^n \le 1/4$.

Collecting the above gives the estimate

$$\begin{aligned}&(3/4)(E(u^n, u_{t-}^n) + (\lambda/4)\int_{t^{n-1}}^{t^n} \left(\|u_t\|_H^2 + \|u_h\|_U^2 \right) + (1/4)\|[u_t^{n-1}]\|_H^2 \\ &\le (E(u^{n-1}, u_{t-}^{n-1}) + C\left(\|f\|_{L^1[t^{n-1},t^n;U']} + \|g\|_{C[t^{n-1},t^n;U']} + \|g_t\|_{L^1[t^{n-1},t^n;U']} \right) \max_{t_+^{n-1} \le t \le t_-^n} E(u_h(t), u_t(t))^{1/2},\end{aligned}$$

where the constant on the right depends only upon ℓ. Combining this with the estimate of equation (4.1) then shows

$$\begin{aligned}&(3/4)E(u^n, u_{t-}^n) + (\lambda/4)\int_{t^{n-1}}^{t^n} \left(\|u_t\|_H^2 + \|u_h\|_U^2 \right) + (1/4)\sum_{m=0}^{n-1} \|[u_t^m]\|_H^2 \\ &\le E(u_h^0, u_t^0) + C\left(\|f\|_{L^1[0,t^n;H]} + \|g\|_{C[0,t^n;U']} + \|g_t\|_{L^1[0,t^n;U']} \right) \max_{0_+ \le t \le t_-^n} E(u_h(t), u_t(t))^{1/2}.\end{aligned}$$

Since $\lambda = 1/(4(2\ell+1)\Delta t^n) = O(1/\Delta t^n)$ the inverse inequality for polynomials of degree ℓ shows there exists $c = c_\ell > 0$ such that

$$\begin{aligned}&c_\ell \max_{t_+^{n-1} \le t \le t_-^n} E(u_h(t), u_t(t)) + (1/4)\sum_{m=0}^{n-1} \|[u_t^m]\|_H^2 \\ &\le E(u_h^0, u_t^0) + C\left(\|f\|_{L^1[0,t^n;U']} + \|g\|_{C[0,t^n;U']} + \|g_t\|_{L^1[0,t^n;U']} \right) \max_{0_+ \le t \le t_-^n} E(u_h(t), u_t(t))^{1/2}.\end{aligned}$$

The theorem then follows upon selecting n to be the interval where $E(u_h(t), u_t(t))$ achieves its maximum on $[0, T]$. □

5. Error Estimate. The numerical experiments in Section 3 showed that numerical solutions of the wave equation do not exhibit optimal rates of convergence if classical Lagrange interpolation is used to approximate the boundary data. This section introduces a temporal interpolant,

$$\mathbb{P}^\tau : C^1[0,T] \to \{w \in C[0,T] \mid w|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n)\},$$

and establishes optimal rates when the Neumann data is approximated by $\mathbb{P}^\tau(g)$ and the Dirichlet data is approximated as $u_{0h} = \mathbb{P}^\tau \circ I_h(u_0)$, where $I_h : D(I_h) \subset \mathcal{U} \to \mathcal{U}_h$ is a spatial interpolation operator.

When $g^\tau = \mathbb{P}^\tau(g)$ the orthogonality condition for the error $e = u - u_h$ takes the form

$$\int_{t^{n-1}}^{t^n} (e_{tt}, v_h) + a(e, v_h) + ([e_t], v_+)^{n-1} = \int_{t^{n-1}}^{t^n} \langle g - \mathbb{P}^\tau(g), v_h\rangle, \quad v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h],$$

where the continuity of the time derivative, $[u_t(t^{n-1})] = 0$, was used to write the jump term in the form shown. Letting $u_p \in u_{0h} + \mathbb{U}_h^\ell$ be a projection of the solution into the discrete space, write $e = u - u_h$ as $e = (u - u_p) + (u_p - u_h) \equiv e_p - e_h$ to get

$$\begin{aligned} &\int_{t^{n-1}}^{t^n} \{(e_{htt}, v_h)_H + a\,(e_h, v_h)\} + ([e_{ht}], v_+)_H^{n-1} \qquad (5.1)\\ &= \int_{t^{n-1}}^{t^n} \left\{-\,(e_{ptt}, v_h)_H - a\,(e_p, v_h) + \langle g - \mathbb{P}^\tau(g), v_h\rangle\right\} - ([e_{pt}], v_+)_H^{n-1}. \end{aligned}$$

This equation shows that the consistency error $e_h \in \mathbb{U}_h^\ell$ satisfies the scheme (3.1) with homogeneous Dirichlet data so can be estimated using Theorem 4.5 upon establishing bounds for the right hand side.

5.1. Temporal Projection. The temporal projection defined next is well defined for functions $w \in C^1[0,T;W]$ taking values in an arbitrary Banach space W; for example, $\mathbb{P}^\tau(g)$ where g takes values in $(U/U_0)'$ and $\mathbb{P}^\tau(u)$ where u takes values in $\mathcal{U}$.

DEFINITION 5.1. *Given a partition $0 = t^0 < t^1 < \ldots < t^N = T$ of $[0,T]$, an integer $\ell \geq 2$, and a Banach space W, the projection*

$$\mathbb{P}^\tau : C^1[0,T;W] \to \{u \in C[0,T;W] \mid u|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;W]\}$$

is characterized by by $\mathbb{P}^\tau(w)(0) = w(0)$, and on each interval (t^{n-1}, t^n)

$$\mathbb{P}^\tau(w)_t(t^n_-) = w_t(t^n), \qquad \int_{t^{n-1}}^{t^n} p(w_t - \mathbb{P}^\tau(w)_t) = 0, \qquad p \in \mathcal{P}_{\ell-2}(t^{n-1},t^n). \tag{5.2}$$

Also, define $\mathbb{P}^\tau(w)_t(0_-) = w_t(0)$ so that the jumps in the derivative, $[\mathbb{P}^\tau(w)_t](t^n)$, are defined for $n = 0, 1, \ldots, N-1$.

This projection, which is only defined for $\ell \geq 2$, will be used to establish estimates for the high order schemes which do not to hold for the lowest order scheme. Similar projections have been used in the context of continuous and discontinuous Galerkin schemes [1, 13]. Note too that $\mathbb{P}^\tau(w)$ can be computed explicitly when $\ell = 2$ and quadrature is required if $\ell \geq 3$; this topic is taken up in Section 5.3.

The following properties of $\mathbb{P}^\tau$ are immediate.

1. Setting $p(t) = 1$ in the definition shows $\mathbb{P}^\tau(w)(t^n) = w(t^n)$ at each partition point. Integration by parts then shows that an alternative (local) characterization of the projection is: $\mathbb{P}^\tau(w)(t^{n-1}) = w(t^{n-1})$, $\mathbb{P}^\tau(w)(t^n) = w(t^n)$, $\mathbb{P}^\tau(w)_t(t^n_-) = w_t(t^n)$, and
$$\int_{t^{n-1}}^{t^n} p\,(w - \mathbb{P}^\tau(w)) = 0, \qquad p \in \mathcal{P}_{\ell-3}(t^{n-1}, t^n), \quad n = 1, 2, \ldots, N,$$
where the last condition is omitted if $\ell = 2$.
2. If $p \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n)$, integration by parts, and the identity $\mathbb{P}^\tau(u)(t^{n-1}_-) = u(t^{n-1})$, shows
$$\int_{t^{n-1}}^{t^n} p(u_{tt} - \mathbb{P}^\tau(u)_{tt}) = -p(t^{n-1})(u - \mathbb{P}^\tau(u))_{t+}(t^{n-1}) = p(t^{n-1})[\mathbb{P}^\tau(u)_t]^{n-1},$$
so
$$\int_{t^{n-1}}^{t^n} p u_{tt} = \int_{t^{n-1}}^{t^n} p\,\mathbb{P}^\tau(u)_{tt} + p(t^{n-1})[\mathbb{P}^\tau(u)_t]^{n-1}, \qquad p \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n).$$
Similarly, if $u_{tt} \in L^1[0,T;U']$ then $u \in C^1[0,T;U']$ and
$$\int_{t^{n-1}}^{t^n} (u_{tt}, v) = \int_{t^{n-1}}^{t^n} (\mathbb{P}^\tau(u)_{tt}, v) + ([\mathbb{P}^\tau(u)_t], v_+)^{n-1}, \qquad v \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U]. \tag{5.3}$$
This is a crucial identity; it will be used to show that the temporal error vanishes in the corresponding terms of the orthogonality relation (5.1).
3. If $A : D(A) \to H$ is linear and closed and $u \in C^1[0,T;D(A)]$, then $\mathbb{P}^\tau(Au) = A\mathbb{P}^\tau(u)$; that is, closed spatial operators compute with the temporal operator $\mathbb{P}^\tau$.

The following lemma summarizes the stability and approximation properties of $\mathbb{P}^\tau$. The proof of these properties is standard; $\mathbb{P}^\tau(u) = u$ when u piecewise polynomial of degree ℓ so the usual parent element construction is applicable.

Lemma 5.2. *Let W be a Banach space, $0 = t^0 < t^1 < \ldots < t^N = T$ be a partition of $[0,T]$, $\ell \geq 2$ an integer, and let*
$$\mathbb{P}^\tau : C^1[0,T;W] \to \{u \in C[0,T;W] \mid u|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1}, t^n; W]\}$$
be the projection in Definition 5.1. Then there exists a constant $C = C(\ell)$ depending only upon ℓ such that
$$\|\mathbb{P}^\tau(u)_t\|_{L^p[0,T;W]} \leq C T^{1/p} \|u_t\|_{C[0,T;W]},$$
$$\|\mathbb{P}^\tau(u)\|_{L^p[0,T;W]} \leq C T^{1/p} \left(\|u\|_{C[0,T;W]} + \tau \|u_t\|_{C[0,T;W]}\right),$$
for all $1 \leq p \leq \infty$, where $\tau = \max_{1 \leq n \leq N}(t^n - t^{n-1})$. Moreover,
$$\|(I - \mathbb{P}^\tau)(u)\|_{L^p[0,T;W]} \leq C |u|_{W^{m+1,p}[0,T;W]}\, \tau^{m+1},$$
$$\|(I - \mathbb{P}^\tau)(u)_t\|_{L^p[0,T;W]} \leq C |u|_{W^{m+1,p}[0,T;W]}\, \tau^{m},$$
whenever $u \in W^{m+1,p}[0,T;W]$ with $1 \leq m \leq \ell$.

5.2. Estimate. It will be assumed that the Dirichlet data for the scheme takes the form $u_{0h} = \mathbb{P}^\tau \circ I_h(u_0)$ where $I_h : D(I_h) \subset \mathcal{U} \to \mathcal{U}_h$ is a projection or an interpolant with the property that it also maps the subspace $U \subset \mathcal{U}$ to itself; that is, $I_h : D(I_h) \cap U \to \mathcal{U}_h \cap U \equiv U_h$. Let $\Pi_h : \mathcal{U} \to U_h$ denote the elliptic projection,

$$\Pi_h(u) \in U_h, \qquad a(\Pi_h(u), v_h) = a(u, v_h), \qquad v_h \in U_h. \tag{5.4}$$

The proof of the following lemma uses the projection $u_p = u_{0p} + \mathbb{P}^\tau \circ \Pi_h(u - u_{0p})$ of u where $u_{0p} = \mathbb{P}^\tau \circ I_h(u)$. Since $(u - u_0)(t) \in U$ it follows that $\mathbb{P}^\tau \circ I_h(u - u_0) \in \mathbb{U}_h^\ell$, so

$$u_{0h} + \mathbb{U}_h^\ell = \mathbb{P}^\tau \circ I_h(u) + \mathbb{P}^\tau \circ I_h(u_0 - u) + \mathbb{U}_h^\ell = u_{0p} + \mathbb{U}_h^\ell,$$

which shows u_h and u_p have the same Dirichlet boundary data; in particular $e_h \equiv u_p - u_h \in \mathbb{U}_h^\ell$, so

$$e_h = (u_{0p} - u_h) + \mathbb{P}^\tau \circ \Pi_h(u - u_{0p}) = \mathbb{P}^\tau \circ \Pi_h(u - u_h).$$

Lemma 5.3. *Let $U_0 \subset U \subset \mathcal{U}$ be subspaces of the Hilbert space $\mathcal{U}$ and $U \hookrightarrow H \hookrightarrow U'$ be continuous embeddings, and assume that U_0 is dense in H. Assume that the bilinear form $a : \mathcal{U} \times U \to \mathbb{R}$ and data f, g, and u_0 satisfy Assumptions 2.1.*

Let $\mathcal{U}_h \subset \mathcal{U}$ be a closed subspace, $U_h = U \cap \mathcal{U}_h$, and $0 = t^0 < t^1 < \dots t^N = T$, be a partition of $[0,T]$ with maximal time step $\tau = \max_{1 \le n \le N}(t^n - t^{n-1})$. Let $\Pi_h : U \to U_h$ denote the elliptic projection characterized in equation (5.4) and $\mathbb{P}^\tau$ be the temporal projection characterized in Definition 5.1, and let $I_h : D(I_h) \subset \mathcal{U} \to \mathcal{U}$ be linear and assume that its restriction to U takes values in U_h.

Let $u \in L^2[0,T;\mathcal{U}] \cap H^2[0,T;U']$ be a solution of the wave equation with data (f, g, u_0) and $u_{tt} \in L^1[0,T;D(I_h)]$ and $u \in C^1[0,T;D(A)]$; in particular, $u_0 \in C^1[0,T;\mathcal{U}]$ and $g \in C^1[0,T;(U/U_0)']$. Let u_h denote the approximate solution of the wave equation computed using the scheme (3.1) with $\ell \ge 2$, Dirichlet data $u_{0h} = \mathbb{P}^\tau \circ I_h(u_0)$, and Neumann data $g = \mathbb{P}^\tau(g)$. Then there exists a constant $C = C(\ell, c_a, C_a) > 0$ such that the error $e_h = \mathbb{P}^\tau \circ \Pi_h(u - u_h)$ satisfies

$$\begin{aligned}
&\max_{0 \le t \le T} E(e_h(t), e_{ht-}(t)) + \sum_{m=1}^{n} \|[e_{ht}^m]\|_H^2 \\
&\le E(e_h(0), e_{ht-}(0)) + C \left(\|(I - \Pi_h)(I - I_h)u_{tt}\|_{L^1[0,T;H]} + \|(I - \mathbb{P}^\tau)Au\|_{L^1[0,T;H]} \right)^2,
\end{aligned}$$

where $E(u,v) = (1/2)(\|u\|_U^2 + \|v\|_H^2)$.

Proof. The lemma will follow from the orthogonality condition upon selecting $u_p = u_{0p} + \mathbb{P}^\tau \circ \Pi_h(u - u_{0p})$, where $u_{0p} = \mathbb{P}^\tau \circ I_h(u)$. Using the property that the spatial and temporal projections commute, the projection error $e_p = u - u_p$ may be written as

$$e_p = (u - u_{0p}) - \Pi_h \circ \mathbb{P}^\tau(u - u_{0p}) = u - \mathbb{P}^\tau \left(I_h(u) + \Pi_h(u - I_h(u)) \right).$$

The first expression for e_p will be used to simplify the spatial terms, and the second to simplify the temporal terms.

1. Substituting the above expression of e_p into equation (5.1) the spatial terms become

$$\begin{aligned}\int_{t^{n-1}}^{t^n} a\left(e_p, v_h\right) - (g - \mathbb{P}^\tau(g), v_h) &= \int_{t^{n-1}}^{t^n} a\big((I - \Pi_h \circ \mathbb{P}^\tau)(u - u_{0p}), v_h\big) - ((I - \mathbb{P}^\tau)(g), v_h) \\ &= \int_{t^{n-1}}^{t^n} a\big((I - \mathbb{P}^\tau)(u - u_{0p}), v_h\big) - ((I - \mathbb{P}^\tau)(g), v_h) \\ &= \int_{t^{n-1}}^{t^n} a\big((I - \mathbb{P}^\tau)(u), v_h\big) - ((I - \mathbb{P}^\tau)(g), v_h) \\ &= \int_{t^{n-1}}^{t^n} ((I - \mathbb{P}^\tau)(Au), v_h)_H \,.\end{aligned}$$

The term involving u_{0p} vanishes since it has polynomial time dependence of degree ℓ; $(I - \mathbb{P}^\tau)(u_{0p}) = 0$. The last step used the assumption $u \in C^1[0,T; D(A)]$ which guarantees $a(u, v) = (Au, v)_H + (g, v)$ and $\mathbb{P}^\tau(Au)$ is well defined.

2. Equation (5.3) shows that (operationally) $\mathbb{P}^\tau$ becomes the identity operator when acting on the temporal terms,

$$e_p = u - \mathbb{P}^\tau\left(I_h(u) + \Pi_h(u - I_h(u))\right) \mapsto u - I_h(u) - \Pi_h(u - I_h(u)) = (I - \Pi_h)(I - I_h)(u).$$

The corresponding terms in the orthogonality relation (5.1) then become

$$\begin{aligned}&\int_{t^{n-1}}^{t^n} (e_{ptt}, v_h)_H + ([e_{pt}], v_+)_H^{n-1} \\ &= \int_{t^{n-1}}^{t^n} ((I - \Pi_h)(I - I_h)(u)_{tt}, v_h)_H + ([u_t], v_+)_H^{n-1} \\ &= \int_{t^{n-1}}^{t^n} ((I - \Pi_h)(I - I_h)(u_{tt}), v_h)_H \,.\end{aligned}$$

The last step used the property that the spatial operators commute with temporal differentiation when $u_{tt}(t) \in D(I_h) \subset \mathcal{U}$, and the jump term vanishes since u_t is continuous.

Using these identities the orthogonality relation (5.1) becomes

$$\begin{aligned}&\int_{t^{n-1}}^{t^n} \left\{(e_{htt}, v_h)_H + a\left(e_h, v_h\right)\right\} + ([e_{ht}], v_+)_H^{n-1} \\ &= \int_{t^{n-1}}^{t^n} \big((I - \Pi_h)(I - I_h)(u_{tt}), v_h\big)_H + \big((I - \mathbb{P}^\tau)(Au), v_h\big)_H \,.\end{aligned}$$

The estimate for e_h now follows from the stability estimate, Theorem 4.5. □

When U_h is a classical finite element space, approximation theory for Sobolev spaces provides rates of convergence for the numerical approximations of the spatial error $(I - \Pi_h)(I - I_h)(u_{tt})$. In this setting the Aubin Nitche technique establishes stability of the elliptic projection in the pivot space; $\|\Pi_h(u)\|_{L^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + h\|u\|_{H^1(\Omega)})$.

Theorem 5.4. *Let $U_0 = H^1_0(\Omega)$, $U_0 \subset U \subset \mathcal{U} \equiv H^1(\Omega)$, and $H = L^2(\Omega)$ (or $H^1_0(\Omega)^d$, $H^1(\Omega)^d$ and $L^2(\Omega)^d$). Assume that the bilinear form and data satisfy Assumptions 2.1 and that solutions $u \in U$ of the elliptic problem, $a(u, v) = (f, v)_H$ for each $v \in U$, exhibit H^2 regularity; $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$.*

Let and $\{\mathcal{U}_h\}_{h>0} \subset \mathcal{U}$ be classical finite element spaces containing the piecewise polynomials of degree less than or equal to $k \geq 1$ constructed over a family of regular meshes indexed by the maximal element diameter h, and let $0 = t^0 < t^1 < \dots t^N = T$, be a partition of $[0,T]$ with maximal time step $\tau = \max_{1 \leq n \leq N}(t^n - t^{n-1})$ and $\ell \geq 2$. Assume that Ω is piecewise polygonal and the mesh resolves interfaces where the boundary data changes type.

Let $u \in W^{\ell+1,1}[0,T; H^2(\Omega)] \cap W^{2,1}[0,T; H^{k+1}(\Omega)]$ be a solution of the wave equation and u_h be the solution of the numerical scheme (3.1) with initial data $(u_h^0, u_{ht}^0) = (\Pi_h(u(0)), I_h(u_t(0)))$, Dirichlet data $\mathbb{P}^\tau \circ I_h(u_0)$, and Neumann data $\mathbb{P}^\tau(g)$, where where $I_h : C(\bar{\Omega}) \to U_h$ is the Lagrange interpolant, $\Pi_h : H^1(\Omega) \to U_h$ is the elliptic projection, and $\mathbb{P}^\tau$ is the temporal projection of Definition 5.1. Then the discrete error $e_h = \Pi_h \circ \mathbb{P}^\tau(u - u_h)$ satisfies

$$\max_{0 \leq t \leq T} E(e_h(t), e_{ht-}(t)) + \sum_{m=0}^{n-1} \|[e_{ht}^m]\|_H^2 \leq C\left(\tau^{\ell+1} + h^{k+1}\right)^2.$$

If in addition $u \in C^{\ell+1}[0,T; H^1(\Omega)] \cap C^1[0,T; H^{k+1}(\Omega)]$ the error $e = u - u_h$ satisfies

$$\begin{aligned}
\|e\|_{C[0,T;U]} &\leq C\left(\tau^{\ell+1} + h^k\right), \\
\|e\|_{C[0,T;H]} &\leq C\left(\tau^{\ell+1} + h^{k+1}\right), \\
\|e_t\|_{L^\infty[0,T;H]} &\leq C\left(\tau^{\ell} + h^{k+1}\right), \\
\max_{1 \leq n \leq N} \|e_{t-}^n\|_H &\leq C\left(\tau^{\ell+1} + h^{k+1}\right).
\end{aligned}$$

Proof. The rate of convergence for e_h follows by bounding each term on the right hand side of the estimate in Lemma 5.3.

1. Using the stability estimate $\|\Pi_h(u)\|_H \leq C(\|u\|_H + h\|u\|_U)$, the first term is bounded as
$$\begin{aligned}
\|(I - \Pi_h)(I - I_h)u_{tt}\|_{L^1[0,T;H]} &\leq C\left(\|(I - I_h)u_{tt}\|_{L^1[0,T;H]} + h\|(I - I_h)u_{tt}\|_{L^1[0,T;U]}\right) \\
&\leq C|u_{tt}|_{L^1[0;T;H^{k+1}(\Omega)]} h^{k+1}.
\end{aligned}$$
2. The estimates for $I - \mathbb{P}^\tau$ in Lemma 5.2 show
$$\|(I - \mathbb{P}^\tau)Au\|_{L^1[0,T;H]} \leq C\|Au^{(\ell+1)}\|_{L^1[0,T;H]}\tau^{\ell+1} \leq C|u|_{W^{\ell+1,1}[0,T;H^2(\Omega)]}\tau^{\ell+1},$$
since A commutes with time differentiation, $(Au)^{(\ell+1)} = A(u^{(\ell+1)})$.
3. By assumption, $u_h(0) = \Pi_h u(0) = u_p(0)$, and $u_{pt-}^0 = \Pi_h u_t(0)$, so
$$E(e_h(0), e_{ht-}(0)) = (1/2)\|(\Pi_h - I_h)u_t(0)\|_H^2 \leq \left(C|u_t(0)|_{H^{k+1}(\Omega)} h^{k+1}\right)^2.$$

This establishes the rates of convergence for e_h.

Estimates on the error e now follow from the triangle inequality and estimates for e_p. Notice that $e_p = (I - \mathbb{P}^\tau \circ \Pi_h)(I - \mathbb{P}^\tau \circ I_h)(u)$ vanishes when $u \in \mathcal{P}_\ell[t^{n-1}, t^n; \mathcal{U}_h]$ so

$$\|e_p\|_{C[0,T;H]} \leq C\left(\tau^{\ell+1} + h^{k+1}\right), \quad \|e_p\|_{C[0,T;U]} \leq C\left(\tau^{\ell+1} + h^k\right), \quad \|e_{pt}\|_{L^\infty[0,T;H]} \leq C\left(\tau^{\ell} + h^{k+1}\right).$$

Super–convergence of e_{t-}^n with respect to τ follows since $\mathbb{P}^\tau(u)_{t-}(t^n) = u_t(t^n)$ which shows

$$e_{pt-}^n = \left((I - \mathbb{P}^\tau \circ \Pi_h)(I - \mathbb{P}^\tau \circ I_h)(u)\right)_{t-}(t^n) = (I - \Pi_h)(I - I_h)(u_t(t^n)),$$

is independent of the time step τ. ∎

5.3. Computing the Projection $\mathbb{P}^\tau$. Optimal rates of convergence of the numerical scheme were established under the assumption that the Dirichlet data for the scheme was $u_{0h} = \mathbb{P}^\tau \circ I_h$, where $I_h : \mathcal{U} \to U_h$ is a spatial interpolant, and the Neumann data for the scheme was $\mathbb{P}^\tau(g)$. The temporal projection $\mathbb{P}^\tau$ can only be computed explicitly when $\ell = 2$; in this section we discuss the construction of approximations of $\mathbb{P}^\tau$ which do not degrade the rate of convergence.

If I^τ is a temporal interpolation operator and if the Dirichlet and Neumann data are approximated by $I^\tau \circ I_h(u_0)$ and $I^\tau(g)$ respectively, additional terms involving $\mathbb{P}^\tau - I^\tau$ appear in the analysis of the error.

1. The Neumann data on right hand side of the orthogonality condition (5.1) becomes

$$\int_{t^{n-1}}^{t^n} ((I - I^\tau)g, v_h) = \int_{t^{n-1}}^{t^n} ((I - \mathbb{P}^\tau)g + (\mathbb{P}^\tau - I^\tau)g, v_h)\,.$$

   This term gets integrated by parts which gives rise to an additional error of the form $\|(\mathbb{P}^\tau - I^\tau)(g)_t\|_{L^1[t^{n-1},t^n;U']}$. This term will be of order $O(\tau^{\ell+1})$ provided $\mathbb{P}^\tau - I^\tau$ vanishes on polynomials of degree $\ell + 1$.
2. The additional temporal term $(I^\tau - \mathbb{P}^\tau)(u)_{tt}$ and jumps in $(I^\tau - \mathbb{P}^\tau)(u)_t$ will be of order $O(\tau^{\ell+1})$ when the difference vanishes on polynomials of degree $\ell + 2$.

This motivates development of a "semi-Hermite" interpolant

$$I^\tau : C^1[0,T;U] \to \{u \in C[0,T;U] \mid u|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell[t^{n-1},t^n;U]\}.$$

which, on each interval (t^{n-1}, t^n) of the partition, satisfies

1. $I^\tau u(t^{n-1}) = u(t^{n-1}), \quad I^\tau u(t^n) = u(t^n), \quad (I^\tau u)_{t-}(t^n) = u(t^n).$
2. $\displaystyle\int_{t^{n-1}}^{t^n} p\left(u - I^\tau(u)\right) = 0, \qquad p \in \mathcal{P}_{\ell-3}(t^{n-1},t^n), \quad u \in \mathcal{P}_{\ell+2}(t^{n-1},t^n).$

These two conditions guarantee that $\mathbb{P}^\tau(u) = I^\tau(u)$ for $u \in \mathcal{P}_{\ell+2}(t^{n-1},t^n)$. If $Q : C[t^{n-1},t^n] \to \mathbb{R}$ is a quadrature rule exact on $\mathcal{P}_{\ell+1}(t^{n-1},t^n)$, then (2) will be satisfied if the $\ell - 2$ coefficients of $I^\tau(u)$ not determined by (1) are selected so that

$$Q(pI^\tau(u)) = Q(pu), \qquad p \in \mathcal{P}_{\ell-3}(t^{n-1},t^n).$$

The following example illustrates this.

EXAMPLE 5.5. For the cubic case $\ell = 3$ on the interval $[-1,1]$, selecting the internal interpolation point to be $\xi = -1/5$ (the root of the linear polynomial orthogonal to constants with respect to the weight $(\xi+1)(\xi-1)^2$) gives the interpolant

$$\begin{aligned} I^\tau(u)(t) = -\frac{5}{16}\,(\xi+1/5)\,(\xi-1)^2\,u\,(-1) + (\xi+1)\,(\xi+1/5)\left(\frac{35}{36} - \frac{5}{9}\,\xi\right) u\,(1) \\ + \frac{5}{12}\,(\xi+1)\,(\xi+1/5)\,(\xi-1)\,u'\,(1) + \frac{125}{144}\,(\xi+1)\,(\xi-1)^2\,I^\tau(u)(-1/5). \end{aligned}$$

The value of $I^\tau(u)(-1/5)$ is determined from the condition that u and $I^\tau(u)$ have the same average when $u \in \mathcal{P}_5(-1,1)$. Integrating the expression expression for $I^\tau(u)$ shows

$$\int_{-1}^{1} I^\tau(u)(\xi)\,d\xi = \frac{1}{4}\,u\,(-1) + \frac{16}{27}\,u\,(1) - \frac{1}{9}\,u'\,(1) + \frac{125}{108}\,I^\tau(u)(-1/5).$$

τ	$\|e(1)\|_{L^2(\Omega)}$	$\|e_t(1_-)\|_{L^2(\Omega)}$	$\|e(1)\|_{H^1(\Omega)}$	$\|e\|_{L^\infty[0,T;L^2(\Omega)]}$	$\|e\|_{L^\infty[0,T;H^1(\Omega)]}$	$\|e_t\|_{L^\infty[0,T;L^2(\Omega)]}$
1/8	4.66251e-05	7.17469e-04	1.64095e-03	2.777284e-04	5.247327e-03	2.493516e-02
1/16	1.62419e-06	2.55427e-05	1.14498e-04	1.688954e-05	3.667632e-04	3.137673e-03
1/32	5.06401e-08	7.91559e-07	7.14651e-06	1.045848e-06	2.287154e-05	3.933005e-04
1/64	1.50948e-09	2.34317e-08	4.33516e-07	6.524833e-08	1.386393e-06	4.917484e-05
1/128	2.25954e-10	1.30960e-09	2.64466e-08	4.154366e-09	8.434562e-08	6.148737e-06
Norm	0.995993	5.963905	3.826756	1.127867	3.826755	6.137173
Rate	4.9748	4.9719	3.9887	4.0073	3.9897	2.9967

FIG. 5.1. $u(t,x) = \phi(r-ct)/r$, $h = 2\tau$: $k = 4$, $\ell = 3$.

The quadrature rule using the function values and derivatives at $\xi = -1, 1, -1/5$ is

$$Q(u) = \frac{13}{24}\,u\,(-1) + \frac{1}{12}\,u'\,(-1) + \frac{40}{81}\,u\,(1) - \frac{2}{27}\,u'\,(1) + \frac{625}{648}\,u\,(-1/5) + \frac{25}{108}\,u'\,(-1/5)\,,$$

and is exact on $\mathcal{P}_5(-1,1)$. Equating the above gives

$$\begin{aligned} I^\tau(u)(-1/5) &= \frac{63}{250}\,u\,(-1) + \frac{9}{125}\,u'\,(-1) - \frac{32}{375}\,u\,(1) + \frac{4}{125}\,u'\,(1) + \frac{5}{6}\,u\,(-1/5) + \frac{1}{5}\,u'\,(-1/5) \\ &= u(-1/5) + \frac{24}{15625}u^{(5)}(-1/5) + \dots \end{aligned}$$

Using this formula gives an interpolant I^τ which agrees with $\mathbb{P}^\tau(u)$ for $u \in \mathcal{P}_5[0,T;\mathcal{U}]$. Setting $I^\tau(-1/5) = u(-1/5)$ gives an interpolant which agrees with $\mathbb{P}^\tau$ on $\mathcal{P}_4(-1,1)$ which could be used to interpolate the Neumann data. ∎

The following example illustrates that using this interpolation scheme for the boundary values will yield the optimal rates predicted by Theorem 5.4 and super convergence of the errors $\|e(t^n)\|_{L^2(\Omega)}$ and $\|e_t(t^n)\|_{L^2(\Omega)}$ at the partition points.

EXAMPLE 5.6. Setting $u(t,x) = \phi(r-ct)/r$ where $r = |x - x_0|$ and $x_0 = (1/2, 3/2)$, a solution of the wave equation was manufactured on $\Omega = (-1,1)^2$ by setting the right hand side to be $f = u_{tt} - \Delta u$ with $\phi(\xi) = \cos(\pi\xi + \pi/3)$ and $c = \sqrt{3}$. Dirichlet data is specified on $\Gamma_0 = \{(x,y) \in \partial\Omega \mid y = \pm 1\}$ and Neumann data is specified on the complement, $\Gamma_1 = \{(x,y) \in \partial\Omega \mid x = \pm 1\}$.

Approximate solutions were computed on uniform square meshes with fixed time steps. Serendipity elements containing the piecewise polynomials of degree $k = 4$ were used for for the spatial variables, and piecewise polynomials of degree $\ell = 3$ were used for the time dependence. The solution was evolved until a time $T = 1$ using the same number of elements in space and time ($h = 2\tau$). Rates of convergence for the errors at time $t = 1$ and the $L^\infty[0,T;L^2(\Omega)]$ space–time errors are tabulated in Figure 5.1. The predicted fourth order rates of convergence are achieved for $\|e(1)\|_{H^1(\Omega)}$, $\|e\|_{L^\infty[0,T;L^2(\Omega)]}$, and $\|e\|_{L^\infty[0,T;H^1(\Omega)]}$, and the predicted third order rate of convergence observed for $\|e_t\|_{L^\infty[0,T;L^2(\Omega)]}$. A super convergent fifth order rate of convergence is achieved by $\|e(1)\|_{L^2(\Omega)}$ and $\|e_t(1_-)\|_{L^2(\Omega)}$ until the square of error becomes comparable with the machine precision at $\tau = 1/128$. This illustrates the effectiveness of high order methods; very accurate approximations of smooth solutions are achieved on modest meshes. ∎

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