On Shocks in a Collisionless Plasma

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Abstract

Motivated by the problem of shocks in collisionless plasma, we consider the steady Vlasov-Maxwell system in one space dimension. It is assumed that as $x \to -\infty$ (upwind) the magnetic field approaches a nonzero constant and the particle density approaches a homogeneous state. Then under some assumptions (including that positive and negative ions have the same mass) it is shown that a steady solution must have the same behavior downwind as upwind, ruling out shock solutions.

1 Introduction

The Vlasov-Maxwell system models a collisionless plasma such as the solar wind. We will consider a plasma consisting of positive ions (with charge e, mass m_+, and number density f_+) and negative ions (with charge $-e$, mass m_-, and number density f_-). In the situation where $f_\pm$ depends on time t, the first component of position x_1, and the first two components of momentum v_1, v_2, and the electromagnetic fields are of the form

$$E = (E_1(t,x_1), E_2(t,x_1), 0)$$

$$B = (0, 0, B_3(t,x_1))$$

we have

$$(1.1)\quad \begin{cases} \partial_t f_\pm + m_\pm^{-1} v_1 \partial_{x_1} f_\pm \pm e(E_1 + c^{-1} m_\pm^{-1} v_2 B_3)\partial_{v_1} f_\pm \\ \qquad\qquad\qquad \pm e(E_2 - c^{-1} m_\pm^{-1} v_1 B_3)\partial_{v_2} f_\pm = 0, \\ \rho = e\int (f_+ - f_-)dv,\ j_k = e\int (m_+^{-1} f_+ - m_-^{-1} f_-) v_k dv, \\ \partial_t E_1 = -4\pi j_1, \partial_x E_1 = 4\pi\rho, \\ \partial_t E_2 = -c\partial_{x_1} B_3 - 4\pi j_2, \\ \partial_t B_3 = -c\partial_{x_1} E_2. \end{cases}$$

Here c is the speed of light. As a matter of convenience we will set $c = 1,\ e = 1,\ m_+ = 1$. More importantly we will also set $m_- = 1$. This allows us to consider a simpler problem as follows. Suppose $f(x_1, v_1, v_2)$ and $B_3(x_1)$ satisfy

$$(1.2)\quad \begin{cases} v_1\partial_{x_1} f + v_2 B_3 \partial_{v_1} f + (E_2 - v_1 B_3)\partial_{v_2} f = 0 \\ B_3' = -8\pi \int f v_2 dv \end{cases}$$

where E_2 is a constant. Then taking

$$\begin{cases} f_\pm(x_1, v_1, v_2) &= f(x_1, v_1, \pm v_2) \\ E(x_1) &= (0, E_2, 0) \end{cases}$$

yields a steady solution of (**??**). Although taking $m_+ = m_-$ is restrictive, this case is physically meaningful and allows for simpler analysis. Henceforth, we will drop unnecessary subscripts and write $x = x_1,\ v = (v_1, v_2),\ E = E_2,\ B = B_3$.

Our interest here is in collisionless shocks. Hence, we seek steady solutions which have different behavior as $x \to -\infty$ and as $x \to +\infty$. Since the mean free path in this collisionless model is infinite, we expect the plasma behavior to make a transition over an infinite interval. The simplest steady solution of (**??**) with a nonzero magnetic field is obtained by taking $E \neq 0$,

$$B = \text{constant } \neq 0,$$

and f of the form

$$f(v) = F\left(\sqrt{(v_1 - W)^2 + v_2^2}\right)$$

where

$$W = E/B.$$

One might hope to find a solution of (**??**) with this behavior as $x \to -\infty$ and different behavior as $x \to +\infty$, but the following theorem seriously restricts this possibility.

Theorem 1.1. *Let* $B_U > 0,\ B_D > 0,\ E > 0$ *and let*

$$\begin{aligned} W_U &= E/B_U,\ W_D = E/B_D, \\ R_U(v) &= \sqrt{(v_1 - W_U)^2 + v_2^2}\ , \\ R_D(v) &= \sqrt{(v_1 - W_D)^2 + v_2^2}, \\ B_A(x) &= \begin{cases} B_U & \text{if} \quad x \leq 0 \\ B_D & \text{if} \quad x > 0 \end{cases}. \end{aligned}$$

Assume that (f, B) *is a continuously differentiable solution of (**??**). Assume there exists* $C_0 > 0$ *and* $p > 1$ *such that*

$$|B'(x)| + |B(x) - B_A(x)| \leq C_0(1 + |x|)^{-p} \text{ for all } x. \tag{1.3}$$

Assume there exists $F_U : [0, \infty) \to [0, \infty)$ *and* $\mathcal{F}_D : \mathbb{R}^2 \to \mathbb{R}$ *with*

$$f \to F_U \circ R_U \text{ uniformly as } x \to -\infty \tag{1.4}$$

and

$$f \to \mathcal{F}_D \text{ as } x \to +\infty. \tag{1.5}$$

Further assume there exists $r_0 > 0$ such that $F_U(r) > 0$ if $r < r_0$ and $F_U(r) = 0$ if $r \geq r_0$ and that F_U constant on an interval of positive length implies $F_U = 0$ on that interval. Lastly define $(X(t,x,v), V(t,x,v))$ by

$$\begin{cases} \dfrac{dX}{dt} = V_1 & X(0,x,v) = x \\ \dfrac{dV_1}{dt} = V_2 B(X) & \\ & V(0,x,v) = v \\ \dfrac{dV_2}{dt} = E - V_1 B(X) & \end{cases}$$

and assume there exists $C_1 > 0$ and $C_2 > 0$ such that if $f(x,v) \neq 0$ then

$$\begin{cases} X(t,x,v) - x \leq C_1 t + C_2 & \text{for } t \leq 0 \\ C_1 t - C_2 \leq X(t,x,v) - x & \text{for } 0 \leq t. \end{cases} \tag{1.6}$$

Then

$$B_D = B_U, W_D = W_U,$$

and

$$\mathcal{F}_D = F_U \circ R_U.$$

Assumption (**??**) says that all charge came from "upwind" ($x \to -\infty$) and ultimately continues "downwind" ($x \to +\infty$). This is crucially important for relating F_U and $\mathcal{F}_D$. Note, though, that this does not require $v_1 \geq 0$. In fact, when B is constant V_1 can be negative but will have a positive time average (when $B > 0$ and $E > 0$).

Theorem 1.1 is in marked contrast to Theorem 3.1 of [**?**] where steady solutions of (**??**) are constructed with quite different behavior as $x \to +\infty$

and $x \to -\infty$. However, in [?] E_2 is taken to be zero, whereas this work considers $E_2 \neq 0$. Also, for the "flat-tail" solutions of [?], (??) does not hold.

Most modelling of collisionless shocks involves fluid equations, see [?] and [?] for example. References [?], [?], and [?] study collisionless shocks using kinetic models. In [?] an asymptotic expression for small amplitude soliton solutions is derived. In [?] shock solutions are assumed to exist and are approximated when $m_- << m_+$. Note that in Theorem 1.1 we take $m_+ = m_- = 1$. Aspects of unmagnetized plasma flowing into an applied field are studied in [?], [?], and [?]. When there is no magnetic field electrostatic shocks are obtained in [?]; see also [?] and [?] concerning the stability of the solutions found in [?].

The global existence in time of smooth solutions to a relativistic version of (??) was established in [?]. This was extended to two space dimensions in [?], [?], and [?]. In three dimensions global existence of smooth solutions for the relativistic Vlasov-Maxwell system is open, but it is shown in [?] that this could fail only if particle speeds approach the speed of light. For the related Vlasov-Poisson system, global existence is known in three dimensions ([?], [?]). We also mention that a variational approach to constructing steady solutions is developed in [?] and [?]. For further background on related problems see [?] and [?].

The proof of Theorem 1.1 is in Sections 2 and 3. Section 2 concerns the characteristics of the Vlasov equation. In particular, due to (??), their asymptotic behavior for $t \to -\infty$ ($t \to +\infty$) may be obtained by approximating B by $B_U(B_D)$. Then, since f remains constant on characteristics, this allows us to relate F_U and $\mathcal{F}_D$. Section 3 uses the fact that the flux of mass, momentum, and energy must be constant. These conservation laws are the primary ingredients of the proof. The main idea of the argument may be glimpsed most easily by reading only the statements of the lemmas in Section 2 (leaving the proofs for later) and then reading Section 3 fully.

The following notation will be used. The letter C denotes a positive generic constant which changes from line to line and may depend on the solution (f, B), but not on x or v. When a specific constant is chosen that must be referred to later, it will be given a subscript (for example, C_0, C_1, C_2 in Theorem 1.1). Frequently the dependence of (X, V) on (x, v) will be suppressed, so for example we may write

$$X(t) = X(t, x, v).$$

We will write

$$D_{x,v}\begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} \partial_x X & \partial_{v_1} X & \partial_{v_2} X \\ \partial_x V_1 & \partial_{v_1} V_1 & \partial_{v_2} V_1 \\ \partial_x V_2 & \partial_{v_1} V_2 & \partial_{v_2} V_2 \end{pmatrix}.$$

Also for a square matrix, M,

$$\|M\| = \max\{|Mu| : |u| = 1\}.$$

2 Characteristics

Many of the estimates of this section rely on the following:

Lemma 2.1. *For each* (x, v) *with* $f(x, v) \neq 0$ *there is* t_0 *such that*

$$|X(t, x, v)| \geq C_1|t - t_0| - C_2.$$

Hence

$$|B'(X(t,x,v))| + |B(X(t,x,v)) - B_A(X(t,x,v))| \leq C(1 + |t - t_0|)^{-p}. \tag{2.1}$$

Proof. From (**??**) it follows that there exists t_0 such that $X(t_0, x, v) = 0$. Let $z = V(t_0, x, v)$ and note that

$$X(t, x, v) = X(t - t_0, 0, z).$$

By (**??**)

$$\begin{aligned} |X(t,x,v)| &= |X(t - t_0, 0, z) - 0| \\ &\geq C_1|t - t_0| - C_2. \end{aligned}$$

For $|t - t_0| \geq 2C_2/C_1$

$$|X(t, x, v)| \geq \frac{1}{2}C_1|t - t_0|$$

so (**??**) follows from (**??**), completing the proof. □

Lemma 2.2. *There exists $C_3 > 0$ such that*

$$|V(t,x,v)| + \|D_{x,v}\begin{pmatrix} X \\ V \end{pmatrix}(t,x,v)\| \leq C_3$$

for all (t,x,v) with $f(x,v) \neq 0$.

Proof. Assume $f(x,v) \neq 0$ throughout. Choose $R_A(x,v)$ such that R_A^2 is C^1, $R_A(x,v) = R_U(v)$ if $x \leq -1$, and $R_A(x,v) = R_U(v)$ if $x \geq 1$. Note that for $X(t,x,v) \notin (-1,1)$ we have

$$\begin{aligned} \left|\frac{d}{dt}R_A^2(X,V)\right| &= \left|2EV_2\frac{B(X)-B_A(X)}{B_A(X)}\right| \\ &\leq C(1+R_A(X,V))\,|B(X)-B_A(X)|\,. \end{aligned}$$

By Lemma 2.1 it follows that

$$\left|\sqrt{R_A^2(X,V)+1}\Big|_{t_1}^{t_2}\right| \leq C \quad \forall t_1, t_2 \tag{2.2}$$

and hence there is $r_U \geq 0$ such that

$$R_A(X,V) \to r_U \quad \text{as} \quad t \to -\infty.$$

Hence $F_U(R_U(V)) \to F_U(r_U)$ as $t \to -\infty$. Using (**??**) and (**??**) it follows that

$$f(X,V) \to F_U(r_U) \quad \text{as} \quad t \to -\infty.$$

Since $f(X,V) = f(x,v)$ for all t, it follows that

$$f(x,v) = f(X,V) = F_U(r_U).$$

Since $F_U(r_U) = f(x,v) \neq 0$, $r_U \leq C$. By (**??**) for all t we have

$$R_A(X,V) \leq r_U + C \leq C.$$

It follows that $|V| \leq C$.

Let

$$(2.3) \quad M_U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & B_U \\ 0 & -B_U & 0 \end{pmatrix} \qquad M_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & B_D \\ 0 & -B_D & 0 \end{pmatrix}.$$

Consider first $x < -C_2$ and define

$$T = \sup\{t : X(s,x,v) < 0 \ \text{ for all } \ s < t\}.$$

Note that by (**??**) it follows that $T > 0$. For $t \le T$ note that

$$\frac{d}{dt} D_{x,v} \begin{pmatrix} X \\ V \end{pmatrix}(t) = M_U D_{x,v} \begin{pmatrix} X \\ V \end{pmatrix}(t) + a_U(t)$$

where

$$a_U(t) = \begin{pmatrix} 0 & 0 & 0 \\ V_2 B'(X) & 0 & B(X) - B_U \\ -V_1 B'(X) & B_U - B(X) & 0 \end{pmatrix} D_{x,v} \begin{pmatrix} X \\ V \end{pmatrix}.$$

Since $\|e^{M_U t}\| \le C$ we have

$$\begin{aligned} \|D_{x,v} \begin{pmatrix} X \\ V \end{pmatrix}(t)\| &= \|e^{M_U t} + \int_0^t e^{M_U(t-s)} a_U(s) ds\| \\ &\le C + C \left| \int_0^t \left(|B'(X)| + |B(X) - B_U|\right) \|D_{x,v} \begin{pmatrix} X \\ V \end{pmatrix}\| ds \right|. \end{aligned}$$

By Gronwall's inequality and Lemma 2.1

$$\|D_{x,v} \begin{pmatrix} X \\ V \end{pmatrix}(t)\| \le C \exp\left(C \int_{-\infty}^{T} \left(|B'(X)| + |B(X) - B_U|\right) ds \right) \le C$$

for $t \le T$. For $t > T$ let

$$a_D(t) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ V_2 B'(X) & 0 & B(X) - B_D \\ -V_1 B'(X) & B_D - B(X) & 0 \end{array} \right) D_{x,v} \left(\begin{array}{c} X \\ V \end{array} \right)$$

and note that

$$\begin{aligned} \|D_{x,v} \left(\begin{array}{c} X \\ V \end{array} \right)(t)\| &= \|e^{M_D(t-T)} D_{x,v} \left(\begin{array}{c} X \\ V \end{array} \right)(T) + \int_T^t e^{M_D(t-s)} a_D(s) ds\| \\ &\leq C + C \int_T^t (|B'(X)| + |B(X) - B_D|) \|D_{x,v} \left(\begin{array}{c} X \\ V \end{array} \right)\| ds. \end{aligned}$$

Another use of Gronwall's inequality and Lemma 2.1 yields

$$\|D_{x,v} \left(\begin{array}{c} X \\ V \end{array} \right)(t)\| \leq C$$

for all t.

We may proceed similarly for $|x| \leq C_2$ and for $x > C_2$ so the proof is complete. $\square$

We may now bound the derivatives of f. Since $f(x,v) = 0$ for $|v| > C$, $|\nabla_{x,v} f(x,v)|$ is bounded on $(x,v) \in [-1,1] \times \mathbb{R}^2$. Consider any (x,v) with $f(x,v) \neq 0$. There exists t_0 such that $X(t_0, x, v) \in (-1,1)$. Now by Lemma 2.2

$$\begin{aligned} |\partial_x f(x,v)| &= |\partial_x (f(X(t_0,x,v), V(t_0,x,v))| \\ &= |\partial_x f(X,V) \partial_x X + \nabla_v f(X,V) \cdot \partial_x V| \,|_{t_0} \leq C. \end{aligned}$$

Similarly,

$$|\partial_{v_1} f(x,v)| + |\partial_{v_2} f(x,v)| \leq C.$$

Since f is C^1 it follows that:

Lemma 2.3. *There exists $C > 0$ such that*

$$|\nabla_{x,v} f(x,v)| \leq C$$

for all (x,v).

To analyze the downwind limit define $(Y, Z)(t, y, z)$ by

$$\begin{cases} \dfrac{dY}{dt} = Z_1 & Y(0) = y \\ \dfrac{dZ_1}{dt} = Z_2 B_D & Z_1(0) = z_1 \\ \dfrac{dZ_2}{dt} = E - Z_1 B_D & Z_2(0) = z_2 \end{cases}.$$

Note that $R_D(Z(t)) = R_D(z)$ for all t and that (using the Vlasov equation and Lemma 2.3)

$$\begin{aligned} & \left| \frac{d}{dt} \left(f(Y, Z) \right) \right| \\ = \;& |Z_1 \partial_x f + Z_2 B_D \partial_{v_1} f + (E - Z_1 B_D) \partial_{v_2} f \\ & - (Z_1 \partial_x f + Z_2 B(Y) \partial_{v_1} f + (E - Z_1 B(Y)) \partial_{v_2} f) | \, |_{(Y,Z)} \\ = \;& |(Z_2 \partial_{v_1} f - Z_1 \partial_{v_2} f) (B_D - B(Y))| \\ \leq \;& |Z| C |B(Y) - B_D| \leq C(W_D + R_D(z)) |B(Y) - B_D|. \end{aligned} \tag{2.4}$$

With this we may prove:

Lemma 2.4. *$f(x,v) \to \mathcal{F}_D(v)$ uniformly as $x \to +\infty$ and*

$$\mathcal{F}_D = F_D \circ R_D$$

where F_D is defined by

$$F_D(r) = \mathcal{F}_D(W_D + r, 0)$$

for all $r \geq 0$.

Proof. Note that

$$R_D(Z(t)) = R_D(z) \geq |z| - W_D$$

so $|z|$ sufficiently large implies

$$f(Y(t), Z(t)) = 0$$

for all t. For $|z| \leq C$, (**??**) yields

$$|f(Y(t), Z(t)) - f(y, z)| \leq C \int_0^\infty |B(Y(s)) - B_D| ds \tag{2.5}$$

for $t \geq 0$. Note (**??**) holds for all z. Since $Z(n\, 2\pi B_D^{-1}) = z$ for all positive integers, n, we have

$$\begin{aligned} |\mathcal{F}_D(z) - f(y, z)| &= \lim_{n\to\infty} \left| f(Y, Z)|_{n2\pi B_D^{-1}} - f(y, z)\right| \\ &\leq C \int_0^\infty |B(Y(s)) - B_D| ds. \end{aligned}$$

By (**??**) it follows that $f(y, z) \to \mathcal{F}_D(z)$ uniformly.

Consider any $\varepsilon > 0$. For y sufficiently large we have for all $t \in \left[0, \dfrac{2\pi}{B_D}\right]$

$$|f(y, z) - \mathcal{F}_D(z)| < \varepsilon,$$

$$|f(Y(t), Z(t)) - \mathcal{F}_D(Z(t))| < \varepsilon$$

and

$$|f(Y(t), Z(t)) - f(y, z)| < \varepsilon.$$

Hence

$$|\mathcal{F}_D(Z(t)) - \mathcal{F}_D(z)| < 3\varepsilon$$

for every $\varepsilon > 0$ so

$$\mathcal{F}_D(Z(t)) = \mathcal{F}_D(z)$$

for all $t \in \left[0, \frac{2\pi}{B_D}\right]$. The lemma follows from this.

□

Next we describe the characteristics using their asymptotic behavior as $t \to \pm\infty$. Define

$$\psi_U(t) = \int_0^t e^{M_U(t-s)} \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds, \psi_D(t) = \int_0^t e^{M_D(t-s)} \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds. \tag{2.6}$$

Lemma 2.5. *Let*

$$\begin{pmatrix} Y_U(t, x_U, v_U) \\ Z_U(t, x_U, v_U) \end{pmatrix} = e^{M_U t} \begin{pmatrix} x_U \\ v_U \end{pmatrix} + \psi_U(t). \tag{2.7}$$

For any (x_U, v_U) there is a unique solution of

$$\begin{cases} \dfrac{dY}{dt} &= Z_1 \\ \dfrac{dZ_1}{dt} &= Z_2 B(Y) \\ \dfrac{dZ_2}{dt} &= E - Z_1 B(Y) \end{cases} \tag{2.8}$$

that satisfies

$$(Y, Z) - (Y_U, Z_U) \to 0 \quad \text{as} \quad t \to -\infty. \tag{2.9}$$

Moreover,

$$f(Y(t), Z(t)) = F_U(R_U(v_U)) \tag{2.10}$$

for all t where (Y, Z) is defined. This unique solution will be denoted $(Y(t, x_U, v_U), Z(t, x_U, v_U))$. We also have $(x_U, v_U) \mapsto (Y, Z)(t, x_U, v_U)$ is continuous.

Proof. For $T < 0$ let

$$|||(Y,Z)||| = \sup\left\{|(Y(t),Z(t))|(1+|t|)^{\frac{p-1}{2}} : t \le T\right\}$$

and

$$\mathcal{C}_T = \left\{(Y,Z) : (Y,Z) \text{ is continuous and } |||(Y,Z)-(Y_U,Z_U)||| \le 1\right\}.$$

For $(Y,Z) \in \mathcal{C}_T$ define

$$\mathcal{F}(Y,Z)\big|_t = \begin{pmatrix} Y_U(t) \\ \\ Z_U(t) \end{pmatrix} + \int_{-\infty}^{t} e^{M_U(t-s)} \begin{pmatrix} 0 \\ \\ Z_2(B(Y)-B_U) \\ \\ -Z_1(B(Y)-B_U) \end{pmatrix} ds.$$

We claim that for T sufficiently negative, $\mathcal{F} : \mathcal{C}_T \to \mathcal{C}_T$ and $\mathcal{F}$ is a contraction. The fixed point is then a solution of (**??**) on $(-\infty, T]$ that satisfies (**??**). Another solution of (**??**) that satisfies (**??**) must also be a fixed point of $\mathcal{F}$ and hence the same as the previous solution.

By explicit calculation

$$\begin{pmatrix} Y_U(t) \\ \\ Z_U(t) \end{pmatrix} = \begin{pmatrix} x_U + W_U t + B_U^{-1}\left(v_{U1}\sin(B_U t) + v_{U2}(1-\cos(B_U t)) - W_U \sin(B_U t)\right) \\ \\ v_{U1}\cos(B_U t) + v_{U2}\sin(B_U t) + W_U(1-\cos(B_U t)) \\ \\ -v_{U1}\sin(B_U t) + v_{U2}\cos(B_U t) + W_U \sin(B_U t) \end{pmatrix}$$

so for any $(Y,Z) \in \mathcal{C}_T$ we have

$$Y(t) \le Y_U(t) + 1 \le x_U + W_U t + C_4(1+|v_U|)$$

and

$$|Z(t)| \le |Z_U(t)| + 1 \le C + |v_U|.$$

Requiring

$$T \le -2W_U^{-1}(|x_U| + C_4(1+|v_U|))$$

we have, for $t \le T$,

$$Y(t) \le \frac{1}{2} W_U t. \tag{2.11}$$

Hence,

$$|B(Y(t)) - B_U| \le C_0(1 + |Y(t)|)^{-p} \le C(1 + |t|)^{-p}. \tag{2.12}$$

Suppose $(\tilde{Y}, \tilde{Z}) \in \mathcal{C}_T$ also. For each $t \le T$ there is ξ between $Y(t)$ and $\tilde{Y}(t)$ such that

$$B(Y(t)) - B(\tilde{Y}(t)) = B'(\xi)(Y(t) - \tilde{Y}(t)).$$

Since (**??**) applies to both Y and $\tilde{Y}$, it applies to ξ and hence

(2.13)

$$\begin{aligned} |B(Y(t)) - B(\tilde{Y}(t))| &\le C_0(1 + |\xi|)^{-p}|Y(t) - \tilde{Y}(t)| \\ &\le C(1 + |t|)^{-p}|Y(t) - \tilde{Y}(t)| \\ &\le C(1 + |t|)^{-p}|||(Y, Z) - (\tilde{Y}, \tilde{Z})|||(1 + |t|)^{-\frac{p-1}{2}} \end{aligned} .$$

Now we may estimate

$$\begin{aligned} |\mathcal{F}(Y, Z) - (Y_U, Z_U)| &= \left| \int_{-\infty}^{t} e^{M_U(t-s)} \begin{pmatrix} 0 \\ Z_2(B(Y) - B_U) \\ -Z_1(B(Y) - B_U) \end{pmatrix} ds \right| \\ &\le \int_{-\infty}^{t} C|Z|\ |B(Y) - B_U| ds \\ &\le C(1 + |v_U|) \int_{-\infty}^{t} (1 + |s|)^{-p} ds \le C(1 + |v_U|)(1 + |t|)^{1-p}, \end{aligned}$$

so

$$(1 + |t|)^{\frac{p-1}{2}} |\mathcal{F}(Y, Z) - (Y_U, Z_U)| \le C(1 + |v_U|)(1 + |T|)^{\frac{1-p}{2}}.$$

Thus for T sufficiently negative, $\mathcal{F} : \mathcal{C}_T \to \mathcal{C}_T$.

Next

$$\begin{aligned}
\left|\mathcal{F}(Y,Z) - \mathcal{F}(\tilde{Y},\tilde{Z})\right|\Big|_t &= \left| \int_{-\infty}^{t} e^{M_U(t-s)} \begin{pmatrix} 0 \\ Z_2(B(Y)-B_U) - \tilde{Z}_2(B(\tilde{Y})-B_U) \\ -Z_1(B(Y)-B_U) + \tilde{Z}_1(B(\tilde{Y})-B_U) \end{pmatrix} ds \right| \\
&\leq C \int_{-\infty}^{t} \Big(|Z-\tilde{Z}||B(Y)-B_U| + |\tilde{Z}||B(Y)-B(\tilde{Y})| \Big) ds \\
&\leq C(1+|v_U|) \int_{-\infty}^{t} |||(Y,Z)-(\tilde{Y},\tilde{Z})|||(1+|s|)^{-\frac{p-1}{2}}(1+|s|)^{-p} ds \\
&\leq C(1+|v_U|)|||(Y,Z)-(\tilde{Y},\tilde{Z})|||(1+|t|)^{-\frac{3}{2}(p-1)},
\end{aligned}$$

so

$$|||\mathcal{F}(Y,Z) - \mathcal{F}(\tilde{Y},\tilde{Z})||| \leq (C+|v_U|)|||(Y,Z)-(\tilde{Y},\tilde{Z})|||(1+|T|)^{1-p}.$$

Hence, for T sufficiently negative, $\mathcal{F}$ is a contraction and the claim is established.

Let (Y,Z) be the fixed point of $\mathcal{F}$. Then

$$\frac{d}{dt}(f(Y(t),Z(t))) = 0. \tag{2.14}$$

By explicit calculation

$$R_U(Z_U(t)) = R_U(v_U)$$

for all t, so by (**??**)

$$F_U(R_U(Z(t))) \to F_U(R_U(v_U)) \text{ as } t \to -\infty.$$

But by (**??**) and (**??**) we also have

$$f(Y(t),Z(t)) - F_U(R_U(Z(t))) \to 0 \text{ as } t \to -\infty$$

so

$$f(Y(t), Z(t)) \to F_U(R_U(v_U)) \text{ as } t \to -\infty.$$

By (**??**) it follows that

$$f(Y(t), Z(t)) = F_U(R_U(v_U)) \tag{2.15}$$

for all $t \in (-\infty, T]$.

Finally, we show the continuous dependence on (x_U, v_U). For brevity let $(Y, Z)(t) = (Y, Z)(t, x_U, v_U)$ and $(\tilde{Y}, \tilde{Z})(t) = (Y, Z)(t, \tilde{x}_U, \tilde{v}_U)$, then

$$\begin{aligned} \Big|(Y,Z) - (\tilde{Y}, \tilde{Z})\Big|\Big|_t &= \Big| e^{M_U t} \begin{pmatrix} x_U - \tilde{x}_U \\ v_U - \tilde{v}_U \end{pmatrix} \\ &+ \int_{-\infty}^t e^{M_U(t-s)} \begin{pmatrix} 0 \\ Z_2(B(Y) - B_U) - \tilde{Z}_2(B(\tilde{Y}) - B_U) \\ -Z_1(B(Y) - B_U) + \tilde{Z}_1(B(\tilde{Y}) - B_U) \end{pmatrix} ds \Big| \\ \leq\ & C|(x_U, v_U) - (\tilde{x}_U, \tilde{v}_U)| \\ &+ C \int_{-\infty}^t \Big(|Z - \tilde{Z}|\ |B(Y) - B_U| + |\tilde{Z}|\ |B(Y) - B(\tilde{Y})| \Big)\, ds. \end{aligned}$$

Working on $t \in (-\infty, T]$ with $|(x_U, v_U) - (\tilde{x}_U, \tilde{v}_U)| \leq 1$ and

$$T \leq -2W_U^{-1}(|x_U| + 1 + C_4(2 + |v_U|))$$

we may apply (**??**), (**??**), and (**??**) to obtain

$$\begin{aligned} |(Y,Z) - (\tilde{Y}, \tilde{Z})|\Big|_t &\leq C|(x_U, v_U) - (\tilde{x}_U, \tilde{v}_U)| \\ &+ C \int_{-\infty}^t (1 + |s|)^{-p} |(Y,Z) - (\tilde{Y}, \tilde{Z})| ds. \end{aligned}$$

By adapting Gronwall's inequality to $(-\infty, T]$ it follows that

$$
\begin{aligned}
|(Y,Z)-(\tilde{Y},\tilde{Z})|\Big|_t &\le C|(x_U,v_U)-(\tilde{x}_U,\tilde{v}_U)| \\
&\quad + \int_{-\infty}^{t} C|(x_U,v_U)-(\tilde{x}_U,\tilde{v}_U)|C(1+|s|)^{-p}e^{\int_s^t C(1+|\tau|)^{-p}d\tau}ds \\
&\le C|(x_U,v_U)-(\tilde{x}_U,\tilde{v}_U)|.
\end{aligned}
$$

The continuity on the full interval of existence now follows by Lemma 2.2, which completes the proof. □

Lemma 2.6. *Suppose $F_U(R_U(v_U)) \neq 0$. Then $(Y,Z)(t,x_U,v_U)$ is defined for all t. Also, there exists a unique $(x_0,v_0) \in \mathbb{R}^3$ such that*

$$
(Y,Z)-(Y_D,Z_D) \to 0 \text{ as } t \to +\infty \tag{2.16}
$$

where we define

$$
\begin{pmatrix} Y_D(t) \\ Z_D(t) \end{pmatrix} = e^{M_D t}\begin{pmatrix} x_D \\ v_D \end{pmatrix} + \psi_D(t). \tag{2.17}
$$

Moreover, (x_D,v_D) depends continuously on (x_U,v_U). Finally, for all $t \in \mathbb{R}$,

$$
F_U(R_U(v_U)) = f(Y(t),Z(t)) = F_D(R_D(v_D)). \tag{2.18}
$$

Proof. Since

$$
f(Y(t),Z(t)) = F_U(R_U(v_U)) \neq 0
$$

we may apply (**??**) to (Y,Z), hence (Y,Z) may be extended to $t \in \mathbb{R}$ with $Y(t) \to +\infty$ as $t \to +\infty$.

To construct (x_D,v_D) note that

$$
\frac{d}{dt}\begin{pmatrix} Y \\ Z \end{pmatrix} = M_D\begin{pmatrix} Y \\ Z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} + \beta_D(t) \tag{2.19}
$$

where

$$\beta_D = (B(Y) - B_D) \begin{pmatrix} 0 \\ Z_2 \\ -Z_1 \end{pmatrix}. \tag{2.20}$$

Hence

$$\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} = e^{M_D t} \begin{pmatrix} Y(0) \\ Z(0) \end{pmatrix} + \psi_D(t) + \int_0^t e^{M_D(t-s)} \beta_D(s) ds.$$

Defining

$$\begin{pmatrix} x_D \\ v_D \end{pmatrix} = \begin{pmatrix} Y(0) \\ Z(0) \end{pmatrix} + \int_0^\infty e^{-M_D s} \beta_D(s) ds \tag{2.21}$$

yields

$$\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} = e^{M_D t} \begin{pmatrix} x_D \\ v_D \end{pmatrix} + \psi_D(t) - \int_t^\infty e^{M_D(t-s)} \beta_D(s) ds \tag{2.22}$$

and (**??**) follows.

To show the uniqueness of (x_D, v_D) suppose that

$$\begin{pmatrix} Y \\ Z \end{pmatrix} - e^{M_D t} \begin{pmatrix} x_D \\ v_D \end{pmatrix} - \psi_D \to 0$$

and

$$\begin{pmatrix} Y \\ Z \end{pmatrix} - e^{M_D t} \begin{pmatrix} \tilde{x}_D \\ \tilde{v}_D \end{pmatrix} - \psi_D \to 0$$

as $t \to +\infty$. Then

$$e^{M_D t} \begin{pmatrix} x_D & - & \tilde{x}_D \\ v_D & - & \tilde{v}_D \end{pmatrix} \to 0 \text{ as } t \to +\infty$$

and $\tilde{x}_D = x_D, \tilde{v}_D = v_D$ follows.

To show the continuity define

$$\sigma(x,v) = \begin{pmatrix} x \\ v \end{pmatrix} + \int_0^\infty e^{-M_D s}(B(X(s,x,v)) - B_D) \begin{pmatrix} 0 \\ V_2(s,x,v) \\ -V_1(s,x,v) \end{pmatrix} ds.$$

Then by (**??**) and (**??**) we have

$$\begin{pmatrix} x_D \\ v_D \end{pmatrix} = \sigma\left(Y(0,x_U,v_U), Z(0,x_U,v_U)\right),$$

so by Lemma 2.5 it suffices to show σ is continuous at $(Y(0,x_U,v_U), Z(0,x_U,v_U))$. Consider (x,v) with $f(x,v) \neq 0$ and $(\tilde{x},\tilde{v})$ with $f(\tilde{x},\tilde{v}) \neq 0$ and $|(\tilde{x},\tilde{v}) - (x,v)| \leq 1$. Denote $(X,V)(t) = (X,V)(t,x,v)$ and $(\tilde{X},\tilde{V})(t) = (X,V)(t,\tilde{x},\tilde{v})$. By (**??**)

$$\tilde{X}(t) \geq \tilde{x} + C_1 t - C_2 \geq C_1 t - (C_2 + 1 + |x|) \tag{2.23}$$

for $t \geq 0$. By the mean value theorem there is ξ between $X(t)$ and $\tilde{X}(t)$ such that

$$B(X(t)) - B(\tilde{X}(t)) = B'(\xi)(X(t) - \tilde{X}(t)).$$

For $t \geq T = 2C_1^{-1}(C_2+1+|x|)$, (**??**) forces $X(t), \tilde{X}(t)$, and hence ξ to exceed $\frac{1}{2}C_1 t$ so by (**??**) we have

$$\begin{aligned}
&\left|(B(X) - B_D)\begin{pmatrix} 0 \\ V_2 \\ -V_1 \end{pmatrix} - B\left(\tilde{X} - B_D\right)\begin{pmatrix} 0 \\ \tilde{V}_2 \\ -\tilde{V}_1 \end{pmatrix}\right| \\
\leq\ & \left|V - \tilde{V}\right| |B(X) - B_D| + \left|\tilde{V}\right| \ \left|B(X) - B(\tilde{X})\right| \\
\leq\ & \left|V - \tilde{V}\right| C_0(1+|X|)^{-p} + C\,|B'(\xi)| \left|X - \tilde{X}\right| \\
\leq\ & C\left((1+X)^{-p} + (1+\xi)^{-p}\right) \left|(X,V) - (\tilde{X},\tilde{V})\right| \\
\leq\ & C(1 + \frac{1}{2}C_1 t)^{-p} \left|(X,V) - (\tilde{X},\tilde{V})\right|.
\end{aligned}$$

Similarly, for $0 \leq t \leq T$ we have

$$\left|(B(X) - B_D)\begin{pmatrix} 0 \\ V_2 \\ -V_1 \end{pmatrix} - (B(\tilde{X}) - B_D)\begin{pmatrix} 0 \\ \tilde{V}_2 \\ -\tilde{V}_1 \end{pmatrix}\right|$$

$$\leq \quad C\left|(X, V) - (\tilde{X}, \tilde{V})\right|.$$

Hence, by Lemma 2.2, we have

$$\begin{aligned}
&|\sigma(x, v) - \sigma(\tilde{x}, \tilde{v})| = |(x, v) - (\tilde{x}, \tilde{v}) \\
&+ \int_0^\infty e^{-M_D s}\left((B(X) - B_D)\begin{pmatrix} 0 \\ V_2 \\ -V_1 \end{pmatrix} - (B(\tilde{X}) - B_D)\begin{pmatrix} 0 \\ \tilde{V}_2 \\ -\tilde{V}_1 \end{pmatrix}\right) ds\Bigg| \\
\leq \quad & |(x, v) - (\tilde{x}, \tilde{v})| + \int_0^T C|(X, V) - (\tilde{X}, \tilde{V})| ds \\
&+ \int_T^\infty C\left(1 + \frac{1}{2}C_1 s\right)^{-p}\left|(X, V) - (\tilde{X}, \tilde{V})\right| ds \\
\leq \quad & \left(1 + C\,TC_3 + C\int_T^\infty \left(1 + \frac{1}{2}C_1 s\right)^{-p} ds\, C_3\right)|(x, v) - (\tilde{x}, \tilde{v})|
\end{aligned}$$

and the continuity follows.

Finally

$$f(Y(t), Z(t)) = F_D(R_D(v_D))$$

may be shown in the same manner as (**??**) and the proof is complete. □

By reversing the roles of (x_U, v_U) and (x_D, v_D) in Lemmas 2.5 and 2.6 we immediately have:

Lemma 2.7. *For any (x_D, v_D) there is a unique solution of (**??**) that satisfies (**??**) where (Y_D, Z_D) is defined by (**??**). It depends continuously on (x_D, v_D). If $F_D(R_D(v_D)) \neq 0$ then this solution is defined for all t and there exists a* *unique $(x_U, v_U) \in \mathbb{R}^3$ such that (??) holds where (Y_U, Z_U) is defined by (??). (x_U, v_U) depends continuously on (x_D, v_D) and (??) holds.*

Definition. Since the choice of (x_D, v_D) in Lemma 2.6 is unique, we may define X_D, V_D by

$$(X_D(x_U, v_U), V_D(x_U, v_U)) = (x_D, v_D)$$

for (x_U, v_U) with $F_U(R_U(v_U)) \neq 0$. Similarly for (x_D, v_D) with $F_D(R_D(v_D)) \neq 0$ we may define X_U, V_U by

$$(X_U(x_D, v_D), V_U(x_D, v_D)) = (x_U, v_U).$$

Then (X_D, V_D) and (X_U, V_U) are inverses.

Let us define

$$A(x) = \int_0^x B(y)dy,$$

then

$$\frac{d}{dt}(Z_2 + A(Y)) = E.$$

This allows us to derive one equation relating (x_U, v_U) and (x_D, v_D).

Lemma 2.8. *For any (x_U, v_U) with $F_U(R_U(v_U)) \neq 0$ let $(x_D, v_D) = (X_D(x_U, v_U), V_D(x_U, v_U))$. Then*

$$(v_{D2} + B_D x_D) - (v_{U2} + B_U x_U) = -\int (B(y) - B_A(y))dy.$$

Proof. Let $0_+(1)(0_-(1))$ denote terms which tend to zero as $t \to +\infty$ ($t \to -\infty$). By explicit calculation on (??) and (??) we find that

$$\begin{aligned} Y(t) &= x_D + v_{D1}B_D^{-1}\sin(B_D t) + v_{D2}B_D^{-1}(1 - \cos(B_D t)) \\ &\quad + W_D(t - B_D^{-1}\sin(B_D t)) + 0_+(1) \end{aligned}$$

and

$$Z_2(t) = -v_{D1}\sin(B_D t) + v_{D2}\cos(B_D t) + W_D\sin(B_D t) + 0_+(1).$$

Also,

$$A(x) = B_D x + \int_0^\infty (B - B_D) dy - \int_x^\infty (B - B_D) dy$$

so it follows that

$$Z_2(t) + A(Y(t)) = v_{D2} + B_D x_D + Et + \int_0^\infty (B - B_D) dy + 0_+(1).$$

A very similar computation yields

$$Z_2(t) + A(Y(t)) = v_{U2} + B_U x_U + Et - \int_{-\infty}^0 (B - B_U) dy + 0_-(1). \tag{2.24}$$

Hence, for $T > 0$

$$\begin{aligned} 2ET &= \int_{-T}^{T} \frac{d}{ds}(Z_2 + A(Y)) ds \\ &= (v_{D2} + B_D x_D) - (v_{U2} + B_U x_U) + 2ET \\ &\quad + \int (B - B_A) dy + 0(1), \end{aligned}$$

where $0(1)$ denotes terms that tend to zero as $T \to +\infty$. Letting $T \to +\infty$ completes the proof. □

We now restrict the mapping between (x_U, v_U) and (x_D, v_D) to get a function of v only.

Definition. For $F_U(R_U(v_U)) \neq 0$ define

$$\mathcal{D}(v_U) = V_D(-B_U^{-1} v_{U2}, v_U).$$

For $F_D(R_D(v_D)) \neq 0$ define

$$\mathcal{U}(v_D) = V_U(-B_D^{-1}(v_{D2} + \int (B - B_A) dy),\ v_D).$$

Comment. Given v_U, taking $x_U = -B_U^{-1} v_{U2}$ forces $v_{U2} + B_U x_U = 0$. Given v_D, taking $x_D = -B_D^{-1}(v_{D2} + \int (B - B_A) dy)$ forces $v_{D2} + B_D x_D = -\int (B - B_A) dy$.

Lemma 2.9. *$\mathcal{D}$ and $\mathcal{U}$ are continuous and are inverses. Also, if $v_D = \mathcal{D}(v_U)$ (with $F_U(R_U(v_U)) \neq 0$) then*

$$F_U(R_U(v_U)) = F_D(R_D(v_D)). \tag{2.25}$$

Proof. $\mathcal{D}$ and $\mathcal{U}$ continuous follows from Lemmas 2.5 and 2.7. Let v_U with $F_U(R_U(v_U)) \neq 0$ be given. Take

$$\begin{aligned} x_U &= -B_U^{-1} v_{U2}, \\ x_D &= X_D(x_U, v_U), v_D = V_D(x_U, v_U). \end{aligned}$$

Then by Lemma 2.8

$$x_D = -B_D^{-1} \left(v_{D2} + \int (B - B_A) dy \right)$$

so

$$v_D = V_D(x_U, v_U) = \mathcal{D}(v_U).$$

But since (X_U, V_U) and (X_D, V_D) are inverses we have

$$x_U = X_U(x_D, v_D), \; v_U = V_U(x_D, v_D).$$

Hence,

$$\mathcal{U}(v_D) = V_U \left(-B_D^{-1} \left(v_{D2} + \int (B - B_A) dy \right), \; v_D \right) = V_U(x_D, v_D) = v_U.$$

Finally, (**??**) follows from (**??**) and the proof is complete. □

Lemma 2.10. *If F_D is constant on an interval of positive length then F_D is zero on this interval.*

Proof. Suppose F_D is a nonzero constant on some interval of positive length. Choose v_D so that $R_D(v_D)$ is in the interior of this interval and $v_D \neq \mathcal{D}(W_U, 0)$. Let $v_U = \mathcal{U}(v_D)$ and note that $R_U(v_U) \neq 0$. For λ near one let

$$v_U^{(\lambda)} = (W_U, 0) + \lambda(v_U - (W_U, 0))$$

and

$$v_D^{(\lambda)} = \mathcal{D}(v_U^{(\lambda)}),$$

then

$$F_D(R_D(v_D^{(\lambda)})) = F_U(R_U(v_U^{(\lambda)})) = F_U(\lambda R_U(v_U))$$

must be constant on some interval with $\lambda = 1$ in its interior. This contradicts an assumption on F_U made in Theorem 1.1 so the proof is complete. □

Now we can show that $R_D(v_D)$ is determined by $R_U(v_U)$.

Lemma 2.11. *Suppose that $r_U \geq 0$ with $F_U(r_U) \neq 0$. If $r_U = R_U(v_U) = R_U(\tilde{v}_U)$ then $R_D(\mathcal{D}(v_U)) = R_D(\mathcal{D}(\tilde{v}_U))$. Similarly, if $F_D(r_D) \neq 0$ and $r_D = R_D(v_D) = R_D(\tilde{v}_D)$ then $R_U(\mathcal{U}(v_D)) = R_U(\mathcal{U}(\tilde{v}_D))$.*

Proof. Let

$$v_U^{(\sigma)} = (W_U, 0) + \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} (v_U - (W_U, 0))$$

and note that

$$R_U(v_U^{(\sigma)}) = r_U$$

for all σ. Let $v_D^{(\sigma)} = \mathcal{D}(v_U^{(\sigma)})$, then

$$F_D(R_D(v_D^{(\sigma)})) = F_U(R_U(v_U^{(\sigma)})) = F_U(r_U).$$

If there exists σ such that $R_D(v_D^{(\sigma)}) \neq R_D(\mathcal{D}(v_U))$ then F_D would be constant on the interval with endpoints $R_D(v_D^{(\sigma)})$ and $R_D(\mathcal{D}(v_U))$. This would contradict Lemma 2.10, so

$$R_D(v_D^{(\sigma)}) = R_D(\mathcal{D}(v_U))$$

for all σ. The first half of the lemma follows. The second is very similar and it's proof is omitted. $\square$

Comment. The mapping

$$r_U = R_U(v_U) \mapsto r_D = R_D(v_D) = R_D(\mathcal{D}(v_U))$$

is a bijection between $\{r \geq 0 : F_U(r) > 0\}$ and $\{r \geq 0 : F_D(r) > 0\}$. From (**??**)

$$F_U(r_U) = F_D(r_D). \tag{2.26}$$

Note that in the next lemma $f(x, v) \neq 0$ is not required and hence (**??**) may not be used. Also, C_3 was chosen in Lemma 2.2.

Lemma 2.12. *There exists $L_- < 0$ such that for any (x, v) with $x \leq L_-$ and $|v| \leq C_3$ there is a solution of (**??**), $t_0 \in \mathbb{R}$, and $v_U \in \mathbb{R}^2$ such that*

$$(Y(t_0), Z(t_0)) = (x, v) \tag{2.27}$$

and

$$(Y, Z) - e^{M_U t} \begin{pmatrix} x_U \\ v_U \end{pmatrix} - \psi_U \to 0 \text{ as } t \to -\infty \tag{2.28}$$

where

$$x_U = -B_U^{-1} v_{U2}.$$

Also,

$$|R_U(v) - R_U(v_U)| \leq C|L_-|^{1-p}. \tag{2.29}$$

*Similarly, there exists $L_+ > 0$ such that for any (x, v) with $x \geq L_+$ and $|v| \leq C_3$ there is a solution of (**??**), $t_0 \in \mathbb{R}$, and $v_D \in \mathbb{R}^2$ such that*

$$(Y(t_0), Z(t_0)) = (x, v)$$

and

$$(Y,Z) - e^{M_D t}\begin{pmatrix} x_D \\ v_D \end{pmatrix} - \psi_D \to 0 \text{ as } t \to +\infty$$

where

$$x_D = -B_D^{-1}(v_{D2} + \int (B - B_A)dy).$$

Also,

$$|R_D(v) - R_D(v_D)| \le CL_+^{1-p}.$$

Proof. Writing $(X,V)(t) = (X,V)(t,x,v)$ define

$$(Y,Z)(t) = (X,V)(t - t_0)$$

where

$$t_0 = E^{-1}\left(v_2 + A(x) + \int_{-\infty}^{0}(B - B_U)dy\right).$$

Then (Y,Z) is a solution of (**??**) that satisfies (**??**). Also

$$\frac{d}{dt}\begin{pmatrix} Y \\ Z \end{pmatrix} = M_U\begin{pmatrix} Y \\ Z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} + \beta_U$$

where

$$\beta_U = (B(Y) - B_U)\begin{pmatrix} 0 \\ Z_2 \\ -Z_1 \end{pmatrix}$$

so

$$\begin{pmatrix} Y \\ Z \end{pmatrix}(t) = e^{M_U(t-t_0)}\begin{pmatrix} X \\ V \end{pmatrix} + \int_{t_0}^{t} e^{M_U(t-s)}\left(\begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} + \beta_U(s)\right)ds. \tag{2.30}$$

Let

$$S(t) = \left| \int_{t_0}^{t} e^{M_U(t-s)} \beta_U(s) ds \right|$$

and

$$T = \inf \left\{ t < t_0 : S \leq 1 \text{ on } [t, t_0] \right\}.$$

Then on $[T, t_0]$

$$Y(t) \leq C(t - t_0) + C_5 + x \leq C(t - t_0) + C_5 + L_-$$

and

$$|Z(t)| \leq C$$

so taking $L_- < -2C_5$

$$\begin{aligned} S(t) &\leq \int_t^{t_0} C|\beta_U(s)| ds \leq C \int_t^{t_0} |B(Y) - B_U| ds \\ &\leq C \int_t^{t_0} (1 + |Y|)^{-p} ds \qquad (2.31) \\ &\leq C \int_t^{t_0} (1 + C(t_0 - S) + \frac{1}{2}|L_-|)^{-p} ds \leq C_6 |L_-|^{1-p}. \end{aligned}$$

Taking $C_6 |L_-|^{1-p} < 1$ forces $T = -\infty$.

Next let

$$\begin{pmatrix} x_U \\ v_U \end{pmatrix} = e^{-M_U t_0} \begin{pmatrix} x \\ v \end{pmatrix} - \int_0^{t_0} e^{-M_U s} \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds - \int_{-\infty}^{t_0} e^{-M_U s} \beta_U(s) ds,$$

then (**??**) may be written

$$
\begin{aligned}
\begin{pmatrix} Y \\ Z \end{pmatrix}(t) &= e^{M_U(t-t_0)}\begin{pmatrix} x \\ v \end{pmatrix} + \psi_U(t) - \int_0^{t_0} e^{M_U(t-s)}\begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds \\
&\quad - \int_{-\infty}^{t_0} e^{M_U(t-s)}\beta_U(s)ds + \int_{-\infty}^{t} e^{M_U(t-s)}\beta_U(s)ds \\
&= e^{M_U t}\begin{pmatrix} x_U \\ v_U \end{pmatrix} + \psi_U(t) + \int_{-\infty}^{t} e^{M_U(t-s)}\beta_U(s)ds.
\end{aligned}
$$

(**??**) now follows.

Next using (**??**) we have

$$
\begin{aligned}
E(t_0 - t) &= \int_t^{t_0} \frac{d}{ds}(Z_2 + A(Y))ds \\
&= v_2 + A(x) - (Z_2(t) + A(Y(t))) \\
&= v_2 + A(x) - (v_{U2} + B_U x_U + Et - \int_{-\infty}^{0}(B - B_U)ds) + 0_-(1).
\end{aligned}
$$

Using the definition of t_0 it follows that

$$x_U = -B_U v_{U2}.$$

Finally

$$\frac{d}{ds}(R_U^2(Z)) = 2Z_2 W_U(B_U - B(Y))$$

so by (**??**) it follows that for $t \leq t_0$

$$|R_U(Z(t)) - R_U(v)| \leq C|L_-|^{1-p}.$$

Let

$$\begin{pmatrix} Y_U \\ Z_U \end{pmatrix} = e^{M_U t}\begin{pmatrix} x_U \\ v_U \end{pmatrix} + \psi_U,$$

then

$$R_U(Z_U(t)) = R_U(v_U)$$

for all t and by (**??**)

$$R_U(Z(t)) \to R_U(v_U) \text{ as } t \to -\infty.$$

(**??**) now follows.

The other assertions of the lemma may be shown in a similar manner.

□

3 Conservation Laws

We first use the fact that the mass flux is constant for any solution of the Vlasov equation.

Lemma 3.1. *Let $r_U > 0$ with $F_U(r_U) \neq 0$ and define*

$$r_D = R_D(\mathcal{D}(v_U))$$

for any v_U with $R_U(v_U) = r_U$. Then

$$W_U r_U^2 = W_D r_D^2.$$

Proof. We claim first that if $g(x,v)$ is continuous, satisfies $g(X(t,x,v), V(t,x,v)) = g(x,v)$ for all t, x, v, and there is $C > 0$ such that $|v| > C \Rightarrow g(x,v) = 0$ then

$$\int g(x,v) v_1 dv = \text{ constant.}$$

To show this note that

$$\frac{d}{dt}[V_2 + A(X)] = E$$

so for any (x,v) there is exactly one value of t when $V_2 + A(X) = 0$. Define

$$G(x, v_1) = g(x, v_1, -A(x))$$

and note that G is continuous and compactly supported. Let G_ε be a sequence of smooth functions with uniformly bounded support and $G_\varepsilon \to G$ as $\varepsilon \to 0^+$. Then define

$$g_\varepsilon(x,v) = G_\varepsilon\ (X(t,x,v), V_1(t,x,v))|_{t=-E^{-1}(v_2+A(x))}\,.$$

Note that

$$(V_2 + A(X))|_t = v_2 + A(x) + Et$$

so

$$(V_2 + A(X))|_{t=-E^{-1}(v_2+A(x))} = 0.$$

Then g_ε is smooth and constant on the characteristics so

$$v_1\partial_x g_\varepsilon + v_2 B(x)\partial_{v_1} g_\varepsilon + (E - v_1 B(x))\partial_{v_2} g_\varepsilon = 0.$$

Hence,

$$\int g_\varepsilon v_1 dv = \text{ constant in } x.$$

But

$$\begin{aligned} g_\varepsilon(x,v) &= G_\varepsilon(X, V_1)|_{-E^{-1}(v_2+A(x))} \\ &\to G(X, V_1)|_{-E^{-1}(v_2+A(x))} = g(x,v) \end{aligned}$$

as $\varepsilon \to 0$ so

$$\int g_\varepsilon dv \to \int g\, dv \text{ as } \varepsilon \to 0.$$

The claim now follows.

Next consider any $\varepsilon \in (0, r_U)$ with $F_U(r_U + 2\varepsilon) \neq 0$ (i.e. $r_U + 2\varepsilon < r_0$). Let $\sigma : [0,\infty) \to [0,1]$ be smooth with $\sigma(r) = 1$ if $r \leq r_U$ and $\sigma(r) = 0$ if $r \geq r_U + \varepsilon$. Define

$$g(x,v) = \begin{cases} \sigma(R_U(v_U)) & \text{if } f(x,v) \neq 0 \\ 0 & \text{else.} \end{cases}$$

Then g is continuous and has bounded v support. We claim that g is constant on characteristics. To check this suppose that

$$(x,v) = (X,V)(T,y,w).$$

Then $f(x,v) = f(y,w)$. If $f(x,v) = 0$ then $g(x,v)$ and $g(y,w)$ are both zero so consider $f(x,v) \neq 0$. We have

$$\begin{pmatrix} X \\ V \end{pmatrix} (t,y,w) = e^{M_U t} \begin{pmatrix} y_U \\ w_U \end{pmatrix} + \psi_U(t) + 0_-(1)$$

and

$$e^{M_U t} \begin{pmatrix} x_U \\ v_U \end{pmatrix} + \psi_U(t) + 0_-(1) = \begin{pmatrix} X \\ V \end{pmatrix} (t,x,v)$$

$$= \begin{pmatrix} X \\ V \end{pmatrix} (t+T,y,w) = e^{M_U (t+T)} \begin{pmatrix} y_U \\ w_U \end{pmatrix} + \psi_U(t+T) + 0_-(1).$$

But

$$\begin{aligned} \psi_U(t+T) &= \int_0^{t+T} e^{M_U(t+T-s)} \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds \\ &= \int_0^{T} e^{M_U(t+T-s)} \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds + \int_T^{t+T} e^{M_U(t+T-s)} \begin{pmatrix} 0 \\ 0 \\ E \end{pmatrix} ds \\ &= e^{M_U t}\psi_U(T) + \psi_U(t) \end{aligned}$$

so

$$e^{M_U t} \begin{pmatrix} x_U \\ v_U \end{pmatrix} + \psi_U(t) = e^{M_U (t+T)} \begin{pmatrix} y_U \\ w_U \end{pmatrix} + e^{M_U t}\psi_U(T) + \psi_U(t) + 0_-(1).$$

It follows that

$$\begin{pmatrix} x_U \\ v_U \end{pmatrix} = e^{M_U T} \begin{pmatrix} y_U \\ w_U \end{pmatrix} + \psi_U(T) + 0_-(1)$$

and hence,

$$\begin{pmatrix} x_U \\ v_U \end{pmatrix} = e^{M_U T} \begin{pmatrix} y_U \\ w_U \end{pmatrix} + \psi_U(T).$$

By explicit calculation it follows that

$$R_U(v_U) = R_U(w_U)$$

so $g(x,v) = g(y,w)$. Therefore

$$\int g v_1 dv = \text{ constant.} \tag{3.1}$$

By Lemma 2.12 for $|v| \le C_3$ and x sufficiently negative we have

$$|R_U(v) - R_U(v_U)| < \varepsilon.$$

So if $g(x,v) \neq 0$ then $\sigma(R_U(v_U)) \neq 0$, so $R_U(v_U) < r_U + \varepsilon$ and

$$R_U(v) < R_U(v_U) + \varepsilon < r_U + 2\varepsilon.$$

If $R_U(v) < r_U - \varepsilon$ then

$$R_U(v_U) < R_U(v) + \varepsilon < r_U$$

and hence, $g(x,v) = 1$. Thus

$$I_{\{v:R_U(v)<r_U-\varepsilon\}} \le g(x,v) \le I_{\{v:R_U(v)<r_U+2\varepsilon\}}.$$

It follows that

$$\begin{aligned} &\int g v_1 dv - \pi W_U r_U^2 \\ = &\int g v_1 dv - \int_{\{v:R_U(v)<r_U\}} v_1 dv \to 0 \text{ as } \varepsilon \to 0^+. \end{aligned} \tag{3.2}$$

By a similar argument (using Lemma 2.12)

$$\int g v_1 dv \to \pi W_D r_D^2 \text{ as } \varepsilon \to 0^+.$$

The lemma now follows by (**??**) and (**??**). □

Define

$$\lambda = \sqrt{\frac{W_D}{W_U}}. \tag{3.3}$$

If $F_D(r_D) \neq 0$ then by Lemma 3.1 and (**??**)

$$F_D(r_D) = F_U(r_U) = F_U(\lambda r_D).$$

If $F_D(r) = 0$ then $F_U(\lambda r) = 0$. Hence,

$$F_D(r) = F_U(\lambda r) \text{ for all } r \geq 0. \tag{3.4}$$

Note also that

$$B_D = \frac{E}{W_D} = \frac{B_U W_U}{W_D} = B_U \lambda^{-2}. \tag{3.5}$$

Next we use the fact that the energy flux is constant.

Lemma 3.2. *Let*

$$M_k = 2\pi \int_0^\infty F_U(r) r^{k+1} dr$$

for $k = 0, 2$. *Then*

$$M_0 W_U^2 + 2M_2 + \frac{1}{4\pi} B_U^2 = M_0 W_U^2 \lambda^4 + (2M_2 + \frac{1}{4\pi} B_U^2)\lambda^{-2}. \tag{3.6}$$

Proof. It is easy to check that

$$\frac{d}{dx}\left(\int f|v|^2 v_1 dv + \frac{1}{4\pi} EB\right) = 0,$$

so

$$\int f|v|^2 v_1 dv + \frac{1}{4\pi} EB = \text{ constant.}$$

Letting $x \to -\infty$ and letting $x \to +\infty$ and using (**??**), (**??**), and (**??**) yields

$$
\begin{aligned}
&\int F_U(R_U(v))|v|^2 v_1 dv + \frac{1}{4\pi} EB_U \\
(3.7) \qquad = &\int F_D(R_D(v))|v|^2 v_1 dv + \frac{1}{4\pi} EB_D.
\end{aligned}
$$

Now using (**??**) and (**??**) we compute

$$
\begin{aligned}
\int F_D(R_D(v))|v|^2 v_1 dv &= \int F_U(\lambda R_D(v))|v|^2 v_1 dv \\
&= \int_0^\infty \int_0^{2\pi} F_U(\lambda r)((W_D + r\cos\theta)^2 + (r\sin\theta)^2) \\
&\quad (W_D + r\cos\theta) d\theta r\, dr \\
&= 2\pi \int_0^\infty F_U(\lambda r)(W_D^3 + 2W_D r^2) r\, dr \\
&= 2\pi \int_0^\infty F_U(r)(W_D^3 + 2W_D(\lambda^{-1}r)^2)(\lambda^{-1}r)\lambda^{-1} dr \\
&= 2\pi \int_0^\infty F_U(r)((W_U\lambda^2)^3 + 2(W_U\lambda^2)\lambda^{-2}r^2)\lambda^{-2} r dr \\
&= 2\pi \int_0^\infty F_U(r)(W_U^3\lambda_2^4 + 2W_U\lambda^{-2}r^2) r\, dr \\
&= M_0 W_U^3 \lambda^4 + 2M_2 W_U \lambda^{-2}.
\end{aligned}
$$

Similarly,

$$
\int F_U(R_U(v))|v|^2 v_1 dv = M_0 W_U^3 + 2M_2 W_U.
$$

Now (**??**) becomes

$$
M_0 W_U^3 + 2M_2 W_U + \frac{1}{4\pi} EB_U = M_0 W_U^3 \lambda^4 + 2M_2 W_U \lambda^{-2} + \frac{1}{4\pi} EB_D.
$$

Using

$$E = W_U B_U = W_D B_D$$

it follows that

$$M_0 W_U^3 + 2M_2 W_U + \frac{1}{4\pi} W_U B_U^2 = M_0 W_U^3 \lambda^4 + 2M_2 W_U \lambda^{-2} + \frac{1}{4\pi} W_D B_D^2.$$

Finally using (**??**) and (**??**) yield (**??**). □

Next we use the fact that the momentum flux is constant.

Lemma 3.3. *We have*

$$M_0 W_U^2 + \frac{1}{2} M_2 + \frac{1}{16\pi} B_U^2 = M_0 W_U^2 \lambda^2 + \left(\frac{1}{2} M_2 + \frac{1}{16\pi} B_U^2 \right) \lambda^{-4}. \tag{3.8}$$

Proof. We have

$$\int f v_1^2 dv + \frac{1}{16\pi} B^2 = \text{constant}$$

so by (**??**) and (**??**) it follows that

$$\int F_U(R_U(v)) v_1^2 dv + \frac{1}{16\pi} B_U^2 = \int F_D(R_D(v)) v_1^2 dv + \frac{1}{16\pi} B_D^2. \tag{3.9}$$

Using (**??**), (**??**), and (**??**) we compute

$$
\begin{aligned}
\int F_D(R_D(v_1))v_1^2 dv &= \int F_U(\lambda R_D(v))v_1^2 dv \\
&= \int_0^\infty \int_0^{2\pi} F_U(\lambda r)(W_D + r\cos\theta)^2 d\theta r\, dr \\
&= 2\pi \int_0^\infty F_U(\lambda r)(W_D^2 + \frac{1}{2}r^2) r\, dr \\
&= 2\pi \int_0^\infty F_U(r) \left(W_D^2 + \frac{1}{2}(\lambda^{-1}r)^2 \right) \lambda^{-1} r \lambda^{-1} dr \\
&= 2\pi \int_0^\infty F_U(r) \left((W_U\lambda^2)^2 + \frac{1}{2}\lambda^{-2}r^2 \right) \lambda^{-2} r\, dr \\
&= M_0 W_U^2 \lambda^2 + \frac{1}{2} M_2 \lambda^{-4}.
\end{aligned}
$$

Similarly,

$$\int F_U(R_U(v))v_1^2 dv = M_0 W_U^2 + \frac{1}{2} M_2.$$

Now (**??**) becomes

$$M_0 W_U^2 + \frac{1}{2} M_2 + \frac{1}{16\pi} B_U^2 = M_0 W_U^2 \lambda^2 + \frac{1}{2} M_2 \lambda^{-4} + \frac{1}{16\pi} B_D^2.$$

Using (**??**) yields (**??**) completing the proof. □

We may now prove Theorem 1.1. Let

$$\alpha = \frac{\frac{1}{2}M_2 + \frac{1}{16\pi}B_U^2}{M_0 W_U^2},$$

then (**??**) and (**??**) may be written as

$$\lambda^4 + 4\alpha\lambda^{-2} = 1 + 4\alpha \qquad (3.10)$$

and

$$\lambda^2 + \alpha\lambda^{-4} = 1 + \alpha. \tag{3.11}$$

Then (**??**) yields

$$0 = \lambda^2(\lambda^4 - 1) - 4\alpha(\lambda^2 - 1) = (\lambda^2 - 1)(\lambda^2(\lambda^2 + 1) - 4\alpha)$$

and (**??**) yields

$$0 = \lambda^4(\lambda^2 - 1) - \alpha(\lambda^4 - 1) = (\lambda^2 - 1)(\lambda^4 - \alpha(\lambda^2 + 1)).$$

Suppose $W_U \neq W_D$. Then $\lambda^2 \neq 1$ and we have

$$\lambda^4 + \lambda^2 - 4\alpha = 0 \tag{3.12}$$

and

$$\lambda^4 - \alpha\lambda^2 - \alpha = 0. \tag{3.13}$$

Subtracting (**??**) from (**??**) yields

$$\lambda^2 = \frac{3\alpha}{1 + \alpha}. \tag{3.14}$$

Multiplying (**??**) by 4 and subtracting (**??**) yields

$$\lambda^2 = \frac{4\alpha + 1}{3}.$$

Hence,

$$\frac{3\alpha}{1 + \alpha} = \frac{4\alpha + 1}{3}$$

which implies $\alpha = \frac{1}{2}$. Now $\lambda^2 = 1$ follows from (**??**). This is a contradiction so

$$W_U = W_D.$$

By (**??**)

$$F_U = F_D.$$

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