Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

THE STATE OF FRACTIONAL HEREDITARY MATERIALS (FHM)

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Abstract. The widespread interest on the hereditary behavior of biological and bioinspired materials motivates deeper studies on their macroscopic "minimal" state. The resulting integral equations for the detected relaxation and creep power-laws, of exponent β, are characterized by fractional operators. Here strains in SBV_{loc} are considered to account for time-like jumps. Consistently, starting from stresses in L^r_{loc}, $r \in [1, \beta^{-1}]$, $\beta \in (0, 1)$ we reconstruct the corresponding strain by extending a result in [38]. The "minimal" state is explored by showing that different histories delivering the same response are such that the fractional derivative of their difference is zero for all times. This equation is solved through a one-parameter family of strains whose related stresses converge to the response characterizing the original problem. This provides an approximation formula for the state variable, namely the residual stress associated to the difference of the histories above. Very little is known about the microstructural origins of the detected power-laws. Recent rheological models, based on a top-plate adhering and moving on functionally graded microstructures, allow for showing that the resultant of the underlying "microstresses" matches the action recorded at the top-plate of such models, yielding a relationship between the macroscopic state and the "microstresses".

1. **Introduction.** The occurrence of time-dependent power law mechanical properties has been noticed in many materials since the first half of the twentieth century [1, 2]. Macroscopic hereditariness has indeed been detected through stress relaxation and creep mechanical tests.

2010 *Mathematics Subject Classification.* Primary: 35Q74,74A60; Secondary: 76A10.

Key words and phrases. Fractional hereditary materials, power-law, functionally graded, microstuctures.

More recently the subject has been revitalized in connection with the development of novel bioinspired materials and biological structures exhibiting complex rheological properties (see e.g. [3, 4, 5, 6, 7]).

Indeed, as noticed in [3], the response of such time-dependent systems exhibiting long tail memories would entail a very large number of approximating conventional modes, namely e.g. of exponentials, that are identifiable through classical rheological models. A more appropriate and yet precise way to handle the exhibited behavior is to account for power laws, both for creep and relaxation, leading to the occurrence of fractional hereditariness. Furthermore, the same feature is observed while monitoring complex interfaces observed experimentally in thin films formed by solutions containing surface active molecules. For instance, this is the case of lipid membranes where, often times, elasticity is taken as the only feature of the effective response (see e.g. [8, 9]). The underlying nano-structure (lipids are a few nanometers long) determines the physical properties of such membranes, which are key constituents for cells.

Another example encountered in natural materials are mineralized bone tissues. They must provide load carrying capabilities and they exhibit a marked power-law time-dependent behavior under applied loads (see e.g. [10]). This arises also in mineralized tissues as ligaments and tendons. Indeed the high stiffness of the hydroxyapatite crystals encountered in such tissues is combined with the exceptional hereditariness of the collagen proteic matrix. Mineralized biological tissues exhibit a hierarchic self-organization of collagen matrix and mineral crystals that is detected at various observation scales. The presence of hierarchies in the self-assembled constituents allows for the exceptional features at the macroscale in terms of strength, stiffness and toughness. From the point of view of viscoelasticity it is even more interesting the fact that such structure assembly motivates the presence of multiple relaxation times observed in creep/relaxation tests. In this regard it has been observed with dynamical tests that continuous relaxation time spectra are in good agreement with experimental results on trabecular, as well as compact, bones (see e.g. [11], [12]).

A part from situations in which the experienced strains become very large and, hence, appropriate strain measures are required (see e.g. [13]), it is known that accurate descriptions of the results of experimental tests are reproduced by power-laws with real order exponents [14, 15, 16]. This elucidates the reason why linear hereditary equations are applicable to the analysis of complex materials whose behavior is time dependent, and it clarifies the confusion that sometimes is made between apparent material nonlinearity and power law behavior (see e.g. [17]). The motivations above regarding bone as an example explain why multiple scales present in many materials entail several multiple relaxation times, ultimately producing a macroscopic power-law hereditariness.

Although strong motivations for looking at such power laws have been discussed, neither they have been explicitly incorporated in general mathematical frameworks (see e.g. the ones developed in [18, 19, 20, 21, 22, 23, 24] among others) nor it has been shown a direct mechanical and mathematical connection between the material properties at the submacroscopic level and the observed macroscopic power law. The latter are known to yield the Boltzmann-Volterra constitutive equations, both for creep and relaxation, in terms of fractional operators (see [3, 15, 16, 25, 26, 27, 28, 29] for their applications to viscoelasticity and [30, 31, 32, 33] for applications to other mechanical contexts). This leads to refer to the constitutive properties of such media as "Fractional Hereditary Materials" (FHM).

In this paper, we focus on the characterization of the state for such materials. The particularization of the notion of state introduced by Noll in [34] to viscoelastic materials (see [22, 35, 36, 37]), one can state that if two states are different then there must be some continuation (of arbitrary duration) of such state which produces different responses with the two states as initial ones. For viscoelastic materials the conclusion that can be drawn is that two past histories verifying this condition must correspond to the same state. Whenever the strain is taken to be the independent variable of the analysis, the material response can be described either in terms of the resulting stress (through the Boltzmann-Volterra integral (27a)) or in terms of the work done by the stress on any continuation (see [37]). On the other hand, for this case a notion of state has been introduced in [36] as a pair where one of the entries is an equivalence class of strain histories and the other one is the current value of the strain. The equivalence class is established through the right-hand side of (33) (which involves a generic relaxation function), whose solution for all non negative times characterizes equivalent histories. This notion of state, eventually called "minimal state" in [22], entails the fact that a state variable can be singled out as the residual stress measurable at any further time ($\bar{t}+\tau$ in (33)) after freezing and keeping the current value of the strain to zero for any arbitrary duration τ. "Long tail" memory materials may allow for detecting such a stress for long times and this is the case for power law hereditariness. The notion of "minimality" recalled above has to do with the fact that the knowledge of such a stress is the minimum necessary information required to fully characterize the response of the material to further loadings. Obviously, the specific knowledge of the solutions of (33) are very useful, although very rarely they can be found explicitly.

In this paper, we first examine "virgin" materials characterized by power law relaxation (and hence creep) functions and we start from considering absolutely continuous strain functions in time, for which the Boltzmann-Volterra integral relation can be recast in terms of a Caputo fractional derivative, whose order matches the power encountered in the relaxation function (besides its sign). Such derivative is characterized by a hyper-singular integral, whose convergence is guaranteed whenever the strain is absolutely continuous in time (see [38], Lemma 2.2, p.35). In this case, the resulting stress is at least locally integrable and its representation formula enjoys its inverse in terms of the Riemann-Liouville integral of the same order of the derivative above. This result is generalized here to strains which are locally of Special Bounded Variations (SBV_{loc}) in time. This is done by first evaluating the contribution to the stress of the jump part of the strain, which can be represented through its Lebesgue decomposition (see e.g. [39]). In this section it is proved, both through the use of Theorem 2.4 in [38] and also by direct calculation, that the corresponding stress turns out to be invertible as well as the one associated with the absolutely continuous part of the strain. This allows for reconstructing any strain in SBV_{loc} starting from the associated stress.

The state for a "non-virgin", namely prestressed (or prestrained), FHM is treated in Sect. 3. Whenever the strain is assumed as the independent variable, the equivalence of two past histories is recast in terms of the Caputo fractional derivative of their difference ε^* (see eqn (33)). An approximation of this equation is considered and its solution generates a one parameter family of strains producing a corresponding family of stresses converging in L^∞ to the response related to ε^*. In turn, it is shown that this procedure permits to single out the explicit expression of the residual stress obtained by freezing the strain to zero for any arbitrary duration starting from a conventional observation time, $\bar{t}$, before which the experienced past history is not known. For such a variable an approximation formula is provided in analogy with the one discussed above.

Two rheological models derived in [16] yielding power law creep and relaxation are discussed in Sect. 4. The first one conveys the idea of having a functionally graded submacroscopic structure, labelled with the letter (V), formed by a continuous elastic layer supported on one of its sides by a continuous sequence of dashpots whose stiffness and viscosity both decrease with depth. The second model, labelled with (E), is somehow the dual of the previous one, since it is obtained by considering a viscous layer supported on continuous springs both characterized by viscosity and stiffness decreasing with depth, respectively. In both cases the two systems are driven by an indefinite massless rigid plate on which a displacement time history is prescribed. The initial boundary value problems corresponding to both systems are summarized in Sect. 4, whose solutions (61) arise in terms of the fractional derivative of the imposed time history, leading to the force (stress) arising at the rigid plate.

On the other hand, Sect. 5 provides newer insights on the models introduced above. Indeed, the "microstresses", namely the "forces" arising in the continuous distributions of dashpots (model (V)) and springs (model (E)) are evaluated.The overall responses of both systems are shown to match the force arising at the rigid plate. This is motivated a-posteriori by exploring the balance of forces in both rheological models and it also shown how the knowledge of the state of the top plate determines the microstresses in both systems.

2. **Fractional hereditary virgin materials.** We recall that two hypotheses are considered while analyzing viscoelastic materials (see e.g. [26]): 1) invariance under time translation and 2) causality. With the first requirement we mean that time shift in the input is reflected as a same shift in the output; with the second we mean that the material response depends on previous histories only, reflecting hereditariness of such materials. In this section we shall refer to "virgin" materials, namely either the strain or the stress are known from the very beginning of the observation of their behavior, conventionally set at $t = 0$, and hence no past histories with respect to such a time need to be taken into account. From Sect. 3 on, this requirement will be relaxed and the notion of state introduced in [22, 36] will allow for characterizing the residual stress due to unknown past histories. For the sake of illustration, the whole paper is focused on one dimensional problems.

Creep and relaxation tests are performed to detect material hereditariness [40]: in the first case, the stress is held constant and the strain is measured, whereas in the second one the strain is held constant and the stress is measured. Whenever either a unit stress or a unit strain is utilized, the creep compliance $J(\circ)$ and relaxation modulus $G(\circ)$ are found as the strain and stress response to the imposed unit stress and strain respectively, i.e.

$$\varepsilon(t) = U(t) \longrightarrow \sigma(t) = G(t) \tag{1a}$$

$$\sigma(t) = U(t) \longrightarrow \varepsilon(t) = J(t), \tag{1b}$$

where $U(\circ)$ is the unit Heaviside step function. When either the creep or the relaxation function is known, the Boltzmann superposition principle allows for writing convolution-type Riemann-Stieltjies integrals to express the relationships between σ and ε. Whenever either the strain or the stress are prescribed, the constitutive relations for the corresponding derived quantities read as follows:

$$\sigma(t) = \int_{0^+}^{t} G(t-\tau)d\varepsilon(\tau), \tag{2a}$$

$$\varepsilon(t) = \int_{0^+}^{t} J(t-\tau)d\sigma(\tau). \tag{2b}$$

Smoothness assumptions on $\varepsilon(t)$ and $\sigma(t)$ will be discussed in the sequel.

Creep compliance and relaxation modulus are not independent. Indeed they are linked to each other by the relationship

$$\hat{J}_+(\omega)\hat{G}_+(\omega) = \frac{1}{(i\omega)^2}, \tag{3}$$

where the symbol ˆ denotes the right-sided Fourier transform (see Appendix A, equation (A7)).

Experiments on polymeric materials performed by Nutting [1] at beginning of the twentieth century showed that their relaxation function is well fitted by power-laws, i.e.

$$G(t) = \frac{C_\beta}{\Gamma(1-\beta)} t^{-\beta}, \tag{4}$$

where $\Gamma(\circ)$ is the Euler-Gamma function, C_β and β are characteristic constants of the material. The exponent β must be enclosed in the range $0 < \beta < 1$ because of thermodynamics restrictions [16, 28, 29]. At the extrema of the range, asymptotic behaviors are obtained: $\beta \to 0$ corresponds to purely elastic solid whereas $\beta \to 1$ to purely viscous fluid. The values of $0 < \beta < 1$ correspond to an intermediate behavior between elastic solid and viscous fluid, allowing for describing both complex-structured materials and soft matter. As we expect, the creep compliance of the given material can be determined through (3) from the relaxation modulus assumed in (4). Furthermore, the right Fourier transform of the relaxation function (4) yields

$$\hat{G}_+(\omega) = C_\beta (i\omega)^{\beta-1}, \tag{5}$$

and by substituting this expression in (3) we then obtain

$$\hat{J}_+(\omega) = \frac{1}{C_\beta (i\omega)^{\beta+1}}, \tag{6}$$

whose anti-transform reads as follows:

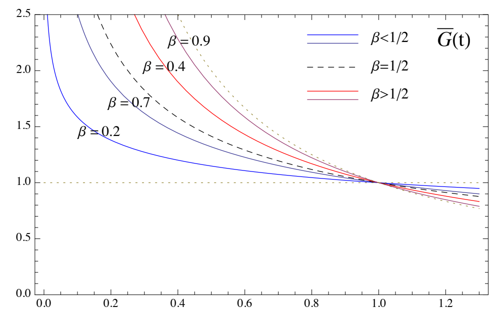
$$J(t) = \frac{1}{C_\beta \Gamma(1+\beta)} t^{\beta}. \tag{7}$$

It is worth analyzing the material behavior with the aid of normalized functions, $\overline{G}(t)$ and $\overline{J}(t)$ defined as follows:

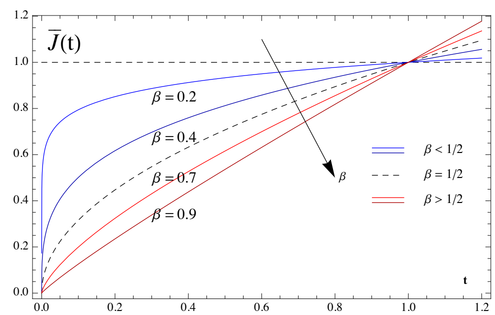
$$\overline{G}(t) := G(t)\,(C_\beta)^{-1}\,\Gamma(1-\beta) = t^{-\beta} \tag{8a}$$

$$\overline{J}(t) := J(t) C_\beta \Gamma(1+\beta) = t^{\beta} \tag{8b}$$

and showed in Figure 1. A careful observer will notice immediately that all the curves share the common point (1,1), which represents a key value. Indeed, the blue curves ($0 < \beta \leq 1/2$) show that the elastic phase prevails on viscous one with decreasing β, whereas the red ones ($1/2 \leq \beta < 1$) show that the viscous phase prevails on elastic one as increases as β. This consideration allows for identifying the former as *elastoviscous* (E) materials while the latter as *viscoelastic* (V) ones; the value $\beta = 1/2$ is clearly common to both kinds of materials, thus it may be obtained as a limiting case of both models described above. The corresponding rheological models, formed by proper arrangements of springs and dashpots, will be discussed in the sequel.



(a) $\overline{G}(t)$



(b) $\overline{J}(t)$

FIGURE 1. Normalized (a) relaxation and (b) creep functions.

In order to introduce the appropriate functional setting characterizing stress and strain we start from (2a), i.e. when ε is assumed to be the control variable. Lemma 2.2 in [38] (p.35) assures that if $\varepsilon \in AC_{\text{loc}}$, where AC_{loc} denotes the set of locally absolutely continuous functions, then the integral (2a) exists almost everywhere for $0 < \beta < 1$ and takes the following form[1]:

$$\sigma(t) = \frac{C_\beta}{\Gamma(1-\beta)} \int_{0^+}^{t} \frac{\dot{\varepsilon}(\tau)}{(t-\tau)^\beta} d\tau. \tag{9}$$

Lemma 2.2 also guarantees that $\sigma \in L^r_{\text{loc}}$, where $1 \leq r \leq \beta^{-1}$. Nevertheless, in reality, the strain may exhibit localized jumps, i.e. strain discontinuities could be present and localized in sets of measure zero, namely in specific locations in time. Estimates regarding the properties of the generalization of (2a) to the tensor valued case and to $a \to -\infty$ have been provided in [22, 36]. There strains were taken in BV_{loc} and $\text{SBV}_{loc} \cup \text{L}^2_{\text{loc}}$ respectively[2]. Obviously, any absolutely continuous

[1]Here $\varepsilon(0) = 0$

[2]Here BV_{loc} and SBV_{loc} denote the sets of functions of bounded and special bounded variations on bounded sets, respectively

function has bounded total variation[3], since the following chain of inclusion holds: $\mathrm{AC}_{\mathrm{loc}} \subset \mathrm{SBV}_{\mathrm{loc}} \subset \mathrm{BV}_{\mathrm{loc}}$. In particular any $\varepsilon_{SBV} \in SBV_{loc}$ is such that

$$\varepsilon_{SBV}(t) = \int_{0^+}^{t} \dot{\varepsilon}_{ac}(\tau) d\tau + \varepsilon_{\mathcal{J}}(t), \tag{10}$$

where ε_{ac} is the absolutely continuous part of ε and the second term $\varepsilon_{\mathcal{J}}$ represents the so called jump part of ε, i.e.

$$\varepsilon_{\mathcal{J}}(t) := \sum_{t > t_i \in \mathcal{J}(\varepsilon)} [\![\varepsilon]\!](t_i) U(t - t_i), \tag{11}$$

where $\mathcal{J}(\varepsilon)$ is the jump set of ε, t_i are the locations of the jumps, $U(\circ - t_i)$ are unit step Heaviside functions located at t_i and $[\![\varepsilon]\!](t_i) := \varepsilon(t_i^+) - \varepsilon(t_i^-)$ is the jump experienced by ε at t_i. It is worth remarking that any $\dot{\varepsilon}_{ac} \in L^1_{loc}$. It is well known that $\varepsilon_{BV} \in BV_{loc}$ can be decomposed in the following way:

$$\varepsilon_{BV}(t) = \varepsilon_{SBV}(t) + \mathcal{C}(\varepsilon(t)), \tag{12}$$

where $\mathcal{C}$ represents the so called Cantor part of ε at t. Relation (12) is the Lebesgue decomposition of ε for BV_{loc} functions and (10) is its particularization to the case of $\varepsilon \in SBV_{loc}$ (see e.g. [39]). Henceforth, by considering strains in such space, the stress can still be computed by means of (9), by understanding that the following representation holds for the strain rate:

$$\dot{\varepsilon}(\tau) = \dot{\varepsilon}_{ac}(\tau) + \sum_{t > t_i \in \mathcal{J}(\varepsilon)} [\![\varepsilon]\!](t_i) \delta(t - t_i), \tag{13}$$

where $\delta(\circ - t_i)$ are Dirac delta masses located at $t_i \in \mathcal{J}(\varepsilon)$. Because $\varepsilon \in SBV_{loc}$ its total variation $\underset{loc}{\mathrm{Var}}(\varepsilon)$ is finite, and hence the strain measure[4] $d\varepsilon = \dot{\varepsilon}(t)dt$, with $\dot{\varepsilon}(t)$ as in (13), is bounded from above by $\underset{loc}{\mathrm{Var}}(\varepsilon)$. It is worth noting that the power law relaxation function (4) is such that $\dot{G} \in L^1_{loc/\{0\}}$ and that $G(0)$ is unbounded (for thermodynamic restriction about such function see [41]). Henceforth, as long as $t = 0$ is excluded, estimates of the stress (see Sec. 3 in the sequel) can be done by using the fact that (2a) is a Riemann-Stieltjies integral and hence (see Theorem 2 in [39] p. 368) the following inequality holds, i.e.

$$\left| \int_a^t G(t - \tau) d\varepsilon(\tau) \right| \leq \sup_{r \in (a,b)} |G(t - r)| \underset{r \in (a,b)}{\mathrm{Var}}(\varepsilon(r)) \quad \forall a \neq 0, \forall t \leq b. \tag{14}$$

Finally, we recognize that relation (9) may be recast in terms of the Caputo fractional derivative of order β of ε (see Appendix A). i.e

$$\sigma(t) = C_\beta \left({}_C\mathrm{D}^{\beta}_{0^+} \varepsilon \right)(t). \tag{15}$$

This notation must be interpreted bearing in mind that every straining $\dot{\varepsilon}$ appearing in (9) has the representation (13); in the sequel of this section the contribution of the Dirac delta masses to the stress will be singled out. Whenever σ is assumed as

[3]The total variation Var of a function ε is defined as:

$$\underset{loc}{\mathrm{Var}}(\varepsilon) := \underset{r \in (a,b)}{\mathrm{Var}}(\varepsilon(r)) = \sup \left\{ \sum_{i=1}^{n} |\varepsilon(r_i) - \varepsilon(r_{i-1})| : \{r_0, r_1, \ldots, r_n\} \in (a,b) \right\}$$

[4]It is worth noting that $d\varepsilon$ can be seen as a Radon measure and, hence, (13) represents the Radon-Nykodim derivative of $\varepsilon(t)$ with respect to the one-dimensional Lebesgue measure dt.

control variable (2b) must be considered. By substituting relation (7) in (2b) we get:

$$\varepsilon(t) = (C_\beta \Gamma(1+\beta))^{-1} \int_{0^+}^{t} (t-\tau)^{\beta} \dot{\sigma}(\tau) d\tau. \tag{16}$$

A simple integration by parts yields the following representation formula:

$$\varepsilon(t) = \frac{1}{C_\beta} \left(\frac{\sigma(0) t^{\beta}}{\Gamma(1+\beta)} + \left(\mathrm{I}^{\beta}_{0^+} \sigma \right)(t) \right) \tag{17}$$

after setting

$$\left(\mathrm{I}^{\beta}_{0^+} \sigma \right)(t) := \frac{1}{C_\beta \Gamma(1-\beta)} \int_{0^+}^{t} (t-\tau)^{\beta-1} \sigma(\tau) d\tau; \tag{18}$$

this defines the Riemann-Liouville fractional integral of order β of the stress. It is worth noting that this is well defined since $\sigma \in L^{r}_{loc}$, $1 \le r \le \beta^{-1}$, $0 < \beta < 1$. Furthermore, the fact that σ is at least L^{1}_{loc} is a necessary and sufficient condition for the resulting ε to be AC_{loc} (see Theorem 2.3 in [38] p. 43).

Moreover, Theorem 2.4 in [38] (p.44) ensures that if σ is represented through (15), where $\dot{\varepsilon}$ is replaced by $\dot{\varepsilon}_{ac}$, because $\dot{\varepsilon}_{ac} \in L^{1}_{loc}$ then

$$\mathrm{I}^{\beta}_{0^+} \left(\mathrm{D}^{\beta}_{0^+} \varepsilon_{ac} \right)(t) = \varepsilon_{ac}(t). \tag{19}$$

The jump part $\varepsilon_{\mathcal{J}}$ defined in (11) of ε must then be treated separately. By denoting with $\sigma_{\mathcal{J}}(t)$ the contribution to the stress $\sigma(t)$ of $\varepsilon_{\mathcal{J}}(t)$, the direct evaluation of (9) (or (15)) can be done bearing in mind that

$$\dot{\varepsilon}_{\mathcal{J}}(t) := \sum_{t > t_i \in \mathcal{J}(\varepsilon)} [\![\varepsilon]\!](t_i) \delta(t - t_i). \tag{20}$$

Such a contribution to the total stress reads as follows

$$\sigma_{\mathcal{J}}(t) = \frac{C_\beta}{\Gamma(1-\beta)} \sum_{t > t_i \in \mathcal{J}(\varepsilon)} \frac{[\![\varepsilon]\!](t_i)}{(t-t_i)^{\beta}}. \tag{21}$$

We are now in the position to check whether or not the analog result for ε_{ac}, granted by Theorem 2.4 in [38], holds also for $\varepsilon_{\mathcal{J}}$. In other words, we conjecture that the relationship (19) above holds in the form

$$\left(\mathrm{I}^{\beta}_{0^+} \sigma_{\mathcal{J}} \right)(t) = \varepsilon_{\mathcal{J}}(t). \tag{22}$$

Actually, the fact that this may be the case follows from the property of $\sigma_{\mathcal{J}}$ in (21) which is certainly L^{1}_{loc}. Henceforth, Theorem 2.4 in [38] applies. Nevertheless, a direct proof of the validity of relation (22) can be obtained by substitution of (21) into (18). Indeed, a direct inspection shows that the fractional integral at the left hand side of (22) can be written as follows:

$$\left(\mathrm{I}^{\beta}_{0^+} \sigma_{\mathcal{J}} \right)(t) = \frac{(C_\beta)}{\Gamma(\beta)\Gamma(1-\beta)} \sum_{t > t_i \in \mathcal{J}(\varepsilon)} [\![\varepsilon_{\mathcal{J}}]\!](t_i) \int_{t_i}^{t} (t-\tau)^{\beta-1} (\tau - t_i)^{-\beta} d\tau \tag{23}$$

Each integral on the right hand side may be easily evaluated by setting $\alpha := 1 - \beta$ and by making use of procedure highlighted in [38] (p.29) namely:

$$\int_{t_i}^{t} (t-\tau)^{-\alpha} (\tau - t_i)^{\alpha - 1} d\tau = B(\alpha, 1-\alpha), \tag{24}$$

where $B(z,w)$ is the Euler Beta function defined as follows:

$$B(z,w) = \int_0^1 x^{z-1}(1-x)^{w-1}dx. \tag{25}$$

By noting that $\Gamma(\beta)\Gamma(1-\beta) = B(\beta, 1-\beta)$ and $B(\beta, 1-\beta) = B(1-\beta, \beta)$, the solution of (23) may be recast in the following form:

$$\left(\mathrm{I}^{\beta}_{0^+}\sigma_{\mathcal{J}}\right)(t) = C_\beta \sum_{t>t_i\in\mathcal{J}(\varepsilon)} [\![\varepsilon_{\mathcal{J}}]\!](t_i)U(t-t_i) \tag{26}$$

This result shows that (19) holds true even when jumps are present in the strain ε as long as $\varepsilon \in SBV_{loc}$.

3. **The state of pre-stressed fractional hereditary materials.** In [22, 36, 24] a definition of state introduced by Noll [34] has been established for viscoelastic materials whenever the strain is the control variable. This allows for defining equivalence classes of past-strain histories which, together with the current value of the strain, characterize the state in such context. Two histories are equivalent if any arbitrary continuation (often called either process or segment) of finite arbitrary duration yields to the same response. The latter can be either measured in terms of the stress or through the work done on processes [24].

On the other hand, whenever the stress is assumed to be the control variable, then the response is measured in terms of the strain. Here an equivalence class of stress histories could be singled out for the given material so that its state can be expressed in terms of any representative of such a class and of the current value of the stress (this belongs to a work in progress [42], although the logic followed in [22, 36, 24] applies with appropriate substitutions and considerations). In either case, here we completely characterize the state of FHM by making use of the analog of (2a) and (2b), both extended to $(-\infty, t)$, i.e.

$$\sigma(t) = \int_{-\infty}^t G(t-\tau)d\varepsilon(\tau) = \frac{C_\beta}{\Gamma(1-\beta)}\int_{-\infty}^t \frac{\dot\varepsilon(\tau)}{(t-\tau)^\beta}d\tau = C_\beta\left({}_C\mathrm{D}^\beta_+\varepsilon\right)(t)\,, \tag{27a}$$

$$\varepsilon(t) = \int_{-\infty}^t J(t-\tau)d\sigma(\tau) = \frac{(C_\beta)^{-1}}{\Gamma(1+\beta)}\int_{-\infty}^t (t-\tau)^\beta\dot\sigma(\tau)d\tau = \frac{1}{C_\beta}\left(\mathrm{I}^\beta_+\sigma\right)(t)\,, \tag{27b}$$

where $\dot\varepsilon$ in (27a) and $\dot\sigma$ in (27b) make sense because of the smoothness assumptions listed above. We bear in mind that $\sigma(-\infty) = 0$ and, hence, by extending (16), (17) and (18) when 0^+ is replaced by $-\infty$ we get (27b). We start by considering (27a) and two functions $\dot\varepsilon_{(i)}$, $i = 1, 2$ to be defined as follows:

$$\dot\varepsilon_{(i)}(r) := \dot\varepsilon^*_{(i)}(r) + U(r-\bar t)\dot\varepsilon^p(r) \quad i = 1,2 \text{ and } r \in (-\infty, t) \tag{28}$$

where:

$$\dot\varepsilon^*_{(i)}(r) := \begin{cases} \dot\varepsilon^{\bar t}_{(i)}(\tau) & r \in (-\infty, \bar t) \\ 0 & r \in [\bar t, \bar t+\tau) \end{cases} \qquad i = 1,2 \tag{29}$$

represents the null extension of $\dot\varepsilon^{\bar t}_{(i)}(\circ)$ and where

$$\dot\varepsilon^{\bar t}_{(i)}(r) := \dot\varepsilon_{(i)}(\bar t - r) \quad i = 1,2. \tag{30}$$

The latter represent two past straining histories with respect to the initial observation time (conventionally labeled with $\bar t$), namely the instant in which the common straining continuation $\dot\varepsilon^p$ is applied. In the sequel we shall refer to $\varepsilon^*_{(i)}$, $i = 1, 2$, as *extended histories*, namely the null extensions of finite duration, τ, of the given

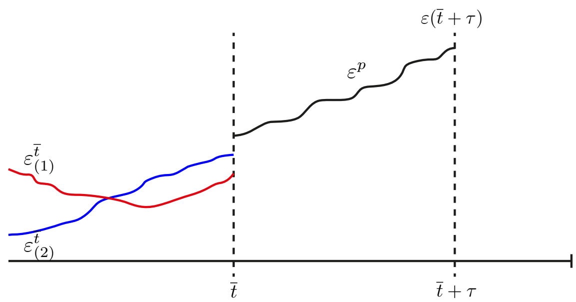


FIGURE 2. Past histories $\varepsilon^{\bar{t}}_{(1)}$, $\varepsilon^{\bar{t}}_{(2)}$ and deformation process ε^p.

histories $\varepsilon^{\bar{t}}_{(i)}$. Henceforth, by substituting (30) in (29) and the result in (28) and, finally, by evaluating the stress response through (27a) of both $\dot{\varepsilon}_{(i)}$ $i=1,2$ we have:

$$\begin{aligned}\sigma_{(i)}(\bar{t}+\tau) :&= \int_{-\infty}^{\bar{t}} G(\tau+\bar{t}-r)\dot{\varepsilon}^{\bar{t}}_{(i)}(r)dr + \int_0^{\tau} G(t-r')\dot{\varepsilon}^p(r')dr' \\ &= \int_{-\infty}^{\bar{t}+\tau} G(\tau+\bar{t}-r)\dot{\varepsilon}^*_{(i)}(r)dr + C_\beta \left({}_C\mathrm{D}^\beta_{0+}\varepsilon^p\right)(\tau) \\ &= C_\beta\left(\left({}_C\mathrm{D}^\beta_{+}\varepsilon^*_{(i)}\right)(\bar{t}+\tau) + \left({}_C\mathrm{D}^\beta_{0+}\varepsilon^p\right)(\tau)\right) \quad i=1,2\end{aligned} \tag{31}$$

where ε^p represents the prescribed strain during the interval $[0,\tau]$. Following the definition of state given above, we say that $\dot{\varepsilon}^*_{(1)}$ and $\dot{\varepsilon}^*_{(2)}$ are equivalent if

$$\sigma_{(1)}(\bar{t}+\tau) = \sigma_{(2)}(\bar{t}+\tau) \quad \forall\tau\geq 0 \tag{32}$$

on any process ε^p. In other words, this condition implies that

$$\int_{-\infty}^{\bar{t}+\tau} G(\bar{t}+\tau-r)\dot{\varepsilon}^*(r)dr \equiv C_\beta\left({}_C\mathrm{D}^\beta_{+}\varepsilon^*\right)(\bar{t}+\tau) = 0 \quad \forall\tau\geq 0 \tag{33}$$

where $\varepsilon^* := \varepsilon^*_{(2)} - \varepsilon^*_{(1)}$. Henceforth, an equivalence class on the set of extended histories may be considered. The state of the material at the time $\bar{t}+\tau$ may be determined by any of such extended histories, say e.g. ε^*, in the equivalence class characterized by (33) together with the current value of the strain $\varepsilon(\bar{t}+\tau)$.

It is worth noting that a more realistic condition for (33) is when we impose that such a relationship holds for all $\tau>0$ which, in practice, would imply that there is always a nonzero unloading time. In order to show that the equivalence class of extended histories does not contain the zero strain only we need to study the solutions of (33). This may not be done directly, but rather by constructing a one parameter family of strains approximating such a solutions. For this purpose, it may also be considered the function ${}_a\varepsilon^*(\circ)$ as the restriction of $\varepsilon^*(\circ)$ to the domain $[a,\bar{t}+\tau)$ namely (see Figure 3):

$${}_a\varepsilon^*(r) := \begin{cases} 0 & r<a \\ \varepsilon(r) & a\leq r<\bar{t} \\ 0 & \bar{t}\leq r<\bar{t}+\tau\ . \end{cases} \tag{34}$$

where a is finite time. In this case relation (33) yields:

$$\left({}_{C}\mathrm{D}^{\beta}_{a+}\varepsilon^*\right)(\bar{t}+\tau) = 0 \quad \forall\tau > 0, \tag{35}$$

where

$$\left({}_{C}\mathrm{D}^{\beta}_{a+}\varepsilon^*\right)(\bar{t}+\tau) = \frac{1}{\Gamma(1+\beta)}\int_a^{\bar{t}+\tau} \frac{\dot{\varepsilon}(r)}{(\bar{t}+\tau-r)^{\beta}dr}. \tag{36}$$

The result 2.27 in [38] (p.36) may be useful; its translation to the current notation allows for stating that solution of (35) are such that

$$\varepsilon^*_{(2)}(\bar{t}+\tau) - \varepsilon^*_{(1)}(\bar{t}+\tau) = \frac{\varepsilon_0}{(\bar{t}+\tau-a)^{1-\beta}}. \tag{37}$$

It is worth noting that whenever (A3b) (see Appendix A) is considered, the rela-

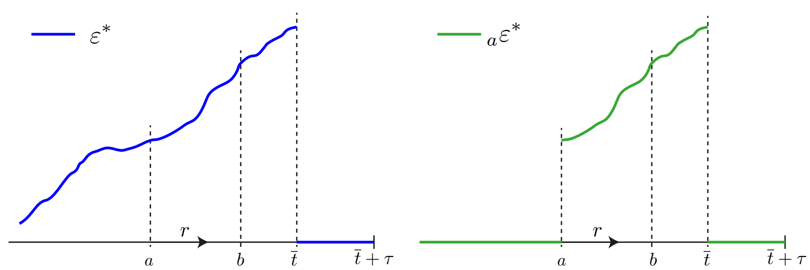


FIGURE 3. Restriction ${}_a\varepsilon^*$ to a prescribed domain of the strain ε^*.

tionship (36) may be rewritten in the form

$$\left({}_{C}\mathrm{D}^{\beta}_{a+}\varepsilon^*\right)(\bar{t}+\tau) = \left(\mathrm{D}^{\beta}_{a+}\varepsilon^*\right)(\bar{t}+\tau) - \frac{\varepsilon^*(a)}{\Gamma(1-\beta)(\bar{t}+\tau-a)^{\beta}} \tag{38}$$

where $\left(\mathrm{D}^{\beta}_{a+}\varepsilon^*\right)(\bar{t}+\tau)$ is the Riemann-Lioville fractional integral defined in (A2b) (see Appendix A).

In order to show that the left-hand side of (35) approximates (33) we first show that the stress response associated with ${}_a\varepsilon^*$ approximates the stress caused by ε^*. To this end we consider relations (27a) and (29) yielding the following expression for the stress at $\bar{t}+\tau$:

$$\begin{aligned}\sigma(\bar{t}+\tau) = -G(t)\varepsilon(\bar{t}) + \int_b^{\bar{t}} G(\tau+\bar{t}-r)\dot{\varepsilon}(r)dr + \int_a^{b} G(\tau+\bar{t}-r)\dot{\varepsilon}(r)dr+ \\ + \int_{-\infty}^{a} G(\tau+\bar{t}-r)\dot{\varepsilon}(r)dr,\end{aligned} \tag{39}$$

where b is such that $a < b < \bar{t}$ is a fixed arbitrary time. The stress ${}_a\sigma(\bar{t}+\tau)$ corresponding to ${}_a\varepsilon^*$ defined by (34) is obtained from (39) by neglecting the last integral, i.e.

$$\begin{aligned}{}_a\sigma(\bar{t}+\tau) = G(\tau+\bar{t}-a)\varepsilon(a) - G(\tau)\varepsilon(\bar{t}) + \int_b^{\bar{t}} G(\tau+\bar{t}-r)d\varepsilon(r)+ \\ + \int_a^{b} G(\tau+\bar{t}-r)d\varepsilon(r).\end{aligned} \tag{40}$$

By computing the difference between (39) and (40) we obtain:

$$\sigma(\bar{t}+r) -_a \sigma(\bar{t}+r) = \int_{-\infty}^{a} G(\tau+\bar{t}-r)d\varepsilon(r) - G(\tau+\bar{t}-a)\varepsilon(a), \tag{41}$$

whose magnitude may be estimated as follows

$$\left|\sigma(\bar{t}+r) - {}_a\sigma(\bar{t}+r)\right| \leq \left|G(\tau+\bar{t}-a)\right| |\varepsilon(a)| + \sup_{r\in(-\infty,a]} |G(\tau+\bar{t}-r)| \operatorname*{Var}_{r\in(-\infty,a)}(\varepsilon(r)). \tag{42}$$

The following extension of (14)

$$\left|\int_{-\infty}^{a} G(\tau+\bar{t}-r)d\varepsilon(r)\right| \leq \sup_{r\in(-\infty,a)} |G(\tau+\bar{t}-r)| \quad \operatorname*{Var}_{r\in(-\infty,a)}(\varepsilon(r)) \tag{43}$$

holds true since, by inspection of (4), it can be shown that $G(\tau+\bar{t}-\circ)$ is absolutely continuous and, hence, bounded in $(-\infty, a)$. Indeed, because of relation (4), if we let $\delta > 0$ there exists at least an a_δ for which if we chose an $a < a_\delta$ we have that $G(\tau+\bar{t}-a)C_\beta^{-1} = \Gamma(1-\beta)^{-1}(\tau+\bar{t}-a)^{-\beta} < \delta$. Finally, by extending $\varepsilon \in SBV_{(-\infty,a)}$ we can find an $M > 0$ such that $\operatorname*{Var}_{r\in(-\infty,a)}(\varepsilon(r)) < M$, so that $\left|\sigma(\bar{t}+r) - {}_a\sigma(\bar{t}+r)\right| < M\delta$. Hence

$$\lim_{a\to\infty} {}_a\sigma(\bar{t}+\tau) = \sigma(\bar{t}+\tau), \tag{44}$$

in other words the stress ${}_a\sigma$ converges to σ in L^∞. Obviously the same conclusions could have been drawn starting from (31) written for ε^* and for its restriction ${}_a\varepsilon^*$ instead of $\varepsilon^*_{(i)}$ and by setting $\varepsilon^p = 0$.

It is worth noting that the integral appearing in (41) represents the residual stress at time $\bar{t}+\tau$ caused by past strain histories truncated at the time a

$$\mathcal{I}^a(\tau+\bar{t}) := \int_{-\infty}^{a} G(\tau+\bar{t}-r)d\varepsilon(r). \tag{45}$$

If one considers the difference

$$\varepsilon_a^* := \varepsilon^* \; - \; {}_a\varepsilon^* \tag{46}$$

then (45) can be expressed as follows:

$$\mathcal{I}^a(\tau+\bar{t}) = C_\beta \left({}_C\mathrm{D}_+^\beta \varepsilon_a^*\right)(\tau+\bar{t}). \tag{47}$$

Upon considering the intermediate arbitrary time $b \in (a, \bar{t})$ introduced in (39) and the equality:

$${}_a\mathcal{I}^b(\tau+\bar{t}) := \int_a^b G(\tau+\bar{t}-r)d\varepsilon(r) + G(\tau+\bar{t}-b)\varepsilon(b) - G(\tau+\bar{t}-a)\varepsilon(a), \tag{48}$$

the reasoning leading to (44) allows for proving that (48) approximates the state variable (45), i.e.

$$\lim_{a\to-\infty} {}_a\mathcal{I}^b(\tau+\bar{t}) = \mathcal{I}^b(\tau+\bar{t}). \tag{49}$$

As a first step, we may consider the stresses ${}_b\sigma(\bar{t}+\tau)$ and ${}_a\sigma(\bar{t}+\tau)$ whenever ${}_b\varepsilon$ and ${}_a\varepsilon$ defined by (34) are assumed to be experienced at the material point; the difference between such stresses reads as follows:

$${}_b\sigma(\bar{t}+\tau) - {}_a\sigma(\bar{t}+\tau) = G(\tau+\bar{t}-b)\varepsilon(b) - G(\tau+\bar{t}-a)\varepsilon(a) - \int_a^b G(\tau+\bar{t}-r)d\varepsilon(r). \tag{50}$$

An estimate of the magnitude of such a difference may be delivered by noting that:

$$\begin{aligned}\left|{}_b\sigma(\bar{t}+\tau)\ -\ {}_a\sigma(\bar{t}+\tau)\right| &\leq \left|G(\tau+\bar{t}-b)\varepsilon(b)-G(\tau+\bar{t}-a)\varepsilon(a)\right|+\\ &+\left|\int_a^b G(\tau+\bar{t}-r)d\varepsilon(r)\right|.\end{aligned} \tag{51}$$

Henceforth, we also note that whenever $\delta > 0$ is given such that there exists at least a $\Delta_\delta > 0$ for which if $|b-a| < \Delta_\delta$ we have $\left|G(\tau+\bar{t}-b)-G(\tau+\bar{t}-a)\right| < \omega\frac{C_\beta}{\Gamma(1-\beta)}\Delta_\delta$, with $\omega > 0$ and $\underset{r\in(a,b)}{\mathrm{Var}}(\varepsilon(r)) < \delta$, hence we obtain

$$\left|G(\tau+\bar{t}-b)\varepsilon(b)-G(\tau+\bar{t}-a)\varepsilon(a)\right| < \frac{C_\beta}{\Gamma(1-\beta)}\left((\tau+\bar{t})^{-\beta}\delta+\omega\left|\varepsilon(a)\right|\Delta_\delta\right) \tag{52}$$

and

$$\left|\int_a^b G(\tau+\bar{t}-r)d\varepsilon(r)\right| \leq \left|G(\tau+\bar{t}-a)\right|\delta < \frac{C_\beta}{\Gamma(1-\beta)(\tau+\bar{t})^\beta}\ \delta. \tag{53}$$

Therefore it turns out that the following result holds

$$\lim_{b\to a}\ {}_b\sigma(\bar{t}+\tau) =\ {}_a\sigma(\bar{t}+\tau). \tag{54}$$

Of course the stress ${}_a\sigma(\bar{t}+\tau)$ may be expressed in terms of a fractional derivative, simply by recalling relation (34) for ${}_a\varepsilon^*$, which allows for writing

$${}_a\sigma(\bar{t}+\tau) = \frac{C_\beta}{\Gamma(1-\beta)}\left(\frac{\varepsilon(a)}{(\bar{t}+\tau-a)^\beta}-\frac{\varepsilon(\bar{t})}{\tau^\beta}+\left({}_C\mathrm{D}^\beta_{a^+}\ {}_a\varepsilon^*\right)(\bar{t}+\tau)\right). \tag{55}$$

On the other hand the stress response evaluated for ε^* (see (31) with $\varepsilon^*_{(i)}$ replaced by ε^* and $\varepsilon = 0$) reads as follows

$$\sigma(\bar{t}+\tau) = C_\beta\left({}_C\mathrm{D}^\beta_+\varepsilon^*\right)(\bar{t}+\tau)-\frac{C_\beta\varepsilon(\bar{t})}{\Gamma(1-\beta)\tau^\beta}. \tag{56}$$

By comparing the procedures used for showing that (44) and (54) hold, it is not difficult to show that the approximation formula (49) for the state variable holds. Hence (37) allows for providing an approximation to equivalent histories for FHMs. Physically, if a in (37) is interpreted as the instant in which the virgin material can bear loading, then (37) gives the exact characterization of the state of the material, in terms of histories that can equivalently bring the material to the same stress.

4. **Rheological models for fractional hereditariness.** The design of novel materials is strictly related to the capability to obtain the desired behavior at the macroscale starting from both the knowledge of mechanical properties of basic constituents and their proper arrangement at the nano-to-microscale. Often times real materials have intermediate behavior between elastic materials and viscous fluids, showing hereditary features. Micromechanics provides a way, to deliver macroscopic properties starting from basic elements at the submacroscopic level. For this reason, a rheological model explaining fractional hereditariness is required.

In this regard, the exact mechanical model of fractional hereditary materials was recently proposed in [16]. The authors separated the behavior of elasto-viscous material from visco-elastic one: both are ruled by β-order differential equation, but in the former case $0 < \beta < 1/2$ while in the latter $1/2 \leq \beta < 1$. The different range of fractional-order involved in constitutive equations is linked to a different mechanical model. In this section we show two different mechanical arrangements to describe the material behavior.

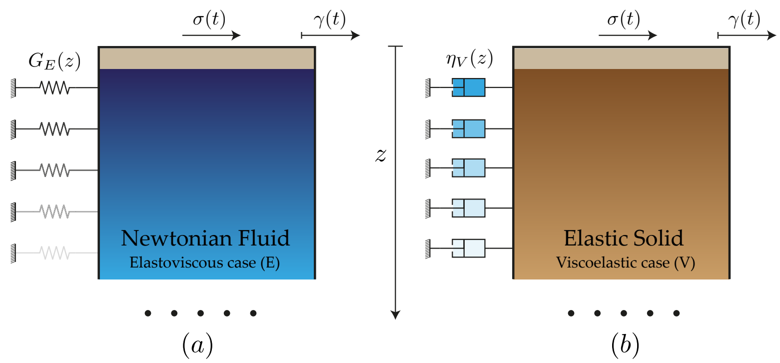


FIGURE 4. Rheological continuum models: (a) elastoviscous (E) and (b) viscoelastic (V).

The rheological scheme of *Elasto-Viscous* material (E) is represented by an indefinite massless plate resting on a column of Newtonian fluid supported on a side by a "bed" by way of independent elastic springs, whereas the model of *Visco-Elastic* material (V) is represented by an indefinite massless plate resting on a column of elastic solid linked to a rigid support through independent viscous dashpots, as depicted in Figure 4. In both cases, we consider a cross-section with area A; moreover, we assume that the material elastic modulus $k(z)$ and viscous coefficient $c(z)$ spatially decay with a power-law, resembling a functionally graded microstructure. Thus, in the case of elastoviscous material, they read as follows:

$$k_E(z) = AG_E(z) = A\frac{G_0}{\Gamma(1+\alpha)}z^{-\alpha} \tag{57a}$$

$$c_E(z) = A\eta_E(z) = A\frac{\eta_0}{\Gamma(1-\alpha)}z^{-\alpha} \quad , \tag{57b}$$

whereas in the case of viscoelastic material they become:

$$k_V(z) = AG_V(z) = A\frac{G_0}{\Gamma(1-\alpha)}z^{-\alpha} \tag{58a}$$

$$c_V(z) = A\eta_V(z) = A\frac{\eta_0}{\Gamma(1+\alpha)}z^{-\alpha}. \tag{58b}$$

In equations (57), (58) the subscripts E and V indicate Elastoviscous and Viscoelastic case, respectively, and $0 \le \alpha \le 1$ is the decay parameter. In these models the equilibrium is governed by a differential equation in the following form:

$$\text{(EV)}: \quad \frac{\partial}{\partial z}\left[c_E(z)\frac{\partial \dot{\gamma}}{\partial z}\right] = k_E(z)\gamma(z,t) \tag{59a}$$

$$\text{(VE)}: \quad \frac{\partial}{\partial z}\left[k_V(z)\frac{\partial \gamma}{\partial z}\right] = c_V(z)\dot{\gamma}(z,t), \tag{59b}$$

where $\gamma(z,t)$ represent the transverse displacement imposed to the shear layer at depth z and $\dot{\gamma}(z,t) := \frac{\partial}{\partial t}\gamma(z,t)$ is its time rate of change. In order to solve the problems above, we make use of the boundary conditions related to mechanical schemes in Figure 4, expressed in the form of limits as follows:

$$\begin{cases} \lim_{z\to 0} \gamma(z,t) = \gamma(t) \\ \lim_{z\to\infty} \gamma(z,t) = 0. \end{cases} \tag{60}$$

By using such boundary conditions in [16] it is shown that (59a) (or (59a)) delivers a relationship between the force σ arising in the top layer in both models and the Caputo fractional derivative of displacement γ, i.e.

$$\sigma(t) = C_\beta \left({}_C\mathrm{D}^\beta_{0+}\gamma \right)(t) \tag{61}$$

where we assumed the parameters as:

$$C_\beta := C^E_\beta = \frac{G_0 \Gamma(\beta)}{\Gamma(2-2\beta)\Gamma(1-\beta)2^{1-2\beta}} \left(\tau_\alpha^{(E)}\right)^\beta \tag{62a}$$

$$\tau_\alpha^{(E)} = \frac{\eta_0}{G_0} \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} \tag{62b}$$

and $\alpha = 1 - 2\beta$ for the EV material, whereas:

$$C_\beta := C^V_\beta = \frac{G_0 \Gamma(1-\beta)}{\Gamma(2-2\beta)\Gamma(\beta)2^{2\beta-1}} \left(\tau_\alpha^{(V)}\right)^\beta \tag{63a}$$

$$\tau_\alpha^{(V)} = \frac{\eta_0}{G_0} \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \tag{63b}$$

and $\alpha = 2\beta - 1$ for the VE material. The terms $\tau_E(\alpha)$, $\tau_V(\alpha)$ are relaxation times. The result expressed by (61) highlights that these rheological models are capable to yield a force on the top layer relaxing with a power-law, ultimately resembling the macroscopic material behavior. In this respect, the boundary of such rheological models reproduces the material response.

5. Overall response from the rheological model: "micro" and "macro" state.

The forces $\sigma_m^{(E)}(z,t)$ exerted either on the springs (E model) or $\sigma_m^{(V)}(z,t)$ on the dashpots (V model) at z may be computed as follows:

$$\sigma_m^{(E)}(z,t) = k(z)\ \gamma^{(E)}(z,t), \tag{64a}$$

$$\sigma_m^{(V)}(z,t) = c(z)\ \dot{\gamma}^{(V)}(z,t). \tag{64b}$$

By denoting by $\gamma^{(E)}$ and $\gamma^{(V)}$ the solution of the field equations (59a) and (59b), the knowledge of transfer functions $H^{(E)}(z,\circ)$ and $H^{(V)}(z,\circ)$, relative to (59a) and (59b) yield the following representation formulas:

$$\gamma^{(E)}(z,t) = \int_{-\infty}^t H^{(E)}(z, t-r)\gamma(r)dr, \tag{65a}$$

$$\gamma^{(V)}(z,t) = \int_{-\infty}^t H^{(V)}(z, t-r)\gamma(r)dr, \tag{65b}$$

where

$$H^{(E)}(z,t) = \mathcal{F}_+^{-1}\left[\hat{H}^{(E)}(z,\circ)\right](t) \tag{66a}$$

$$H^{(V)}(z,t) = \mathcal{F}_+^{-1}\left[\hat{H}^{(E)}(z,\circ)\right](t) \tag{66b}$$

and

$$\hat{H}^{(E)}(z,\omega) = \frac{G_0}{\Gamma(1-\beta)\Gamma(2-2\beta)2^{-\beta}} \left(\tau_\alpha^E i\omega\right)^{\frac{\beta}{2}} z^\beta \; \mathrm{K}_\beta \left(\frac{z}{\sqrt{\tau_\alpha^E i\omega}}\right) \tag{67a}$$

$$\hat{H}^{(V)}(z,\omega) = \frac{G_0}{\Gamma(\beta)\Gamma(2-2\beta)2^{\beta-1}} \left(\tau_\alpha^V i\omega\right)^{\frac{1+\beta}{2}} z^{1-\beta} \; \mathrm{K}_{1-\beta} \left(z\sqrt{\tau_\alpha^V i\omega}\right). \tag{67b}$$

Let us assume that the displacement of the rigid top plate of both models (E) and

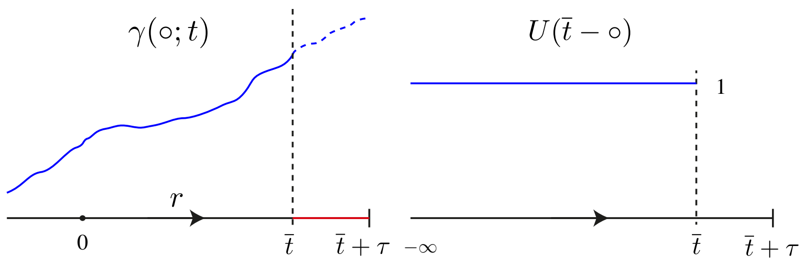


FIGURE 5. Displacement at the rigid top plate.

(V) is prescribed up to the time $\bar{t}$ and it is held constant to zero (Figure 5), i.e.

$$\gamma(r;\bar{t}) := \gamma(r)U(\bar{t}-r). \tag{68}$$

In other words, the prescribed displacement history for the top plate is continued by the null process (see Figure 5). The displacement recorded at z in the rheological model at time $\tau+\bar{t}$ ($\tau > 0$) is simply calculated through (65a) and (65b) respectively by setting $t = \tau + \bar{t}$:

By substituting the resulting expressions in (64a) and (64b) respectively, the "micro" stresses $\sigma_m^{(E)}$, $\sigma_m^{(V)}$ are recovered. In both cases the resultants of those forces can be evaluated by integrating the equation across the semi-infinite sequence of springs and dashpots respectively. Such resultants read as follows:

$$\int_0^\infty \sigma_m^{(E)}(z,\tau+\bar{t})dz = \int_{-\infty}^{\tau+\bar{t}} \int_0^\infty k(z)H^{(E)}(z,\tau+\bar{t}-r)\; \gamma(r)\; dz\; dr \tag{69a}$$

$$\int_0^\infty \sigma_m^{(V)}(z,\tau+\bar{t})dz = \int_{-\infty}^{\tau+\bar{t}} \int_0^\infty c(z)H^{(V)}(z,\tau+\bar{t}-r)\; \dot{\gamma}(r)\; dz\; dr, \tag{69b}$$

where the order of integration has been interchanged thanks to Fubini's Theorem. The upper limit on the integrals in r can be truncated at $\bar{t}$ because of (68), although it is kept as it is for the sake of convenience. For the purpose of obtaining the overall response of the two rheological models there is no need to produce (66a) and (66b). We may indeed evaluate the right Fourier transforms of (69a) and (69b), and make use of (67a) and (67b) respectively in order to perform the explicit calculations. Upon introducing

$$\hat{\sigma}_m^{(E)}(z,\omega) := \mathcal{F}_+\left\{\sigma_m^{(E)}(z,\circ)\right\}(\omega) \tag{70a}$$

$$\hat{\sigma}_m^{(V)}(z,\omega) := \mathcal{F}_+\left\{\sigma_m^{(V)}(z,\circ)\right\}(\omega), \tag{70b}$$

and

$$\hat{\gamma}^{(E)}(z,\omega) := \mathcal{F}_+\left\{\gamma^{(E)}(z,\circ)\right\}(\omega) \tag{71a}$$

$$\hat{\gamma}^{(V)}(z,\omega) := \mathcal{F}_+\left\{\gamma^{(V)}(z,\circ)\right\}(\omega), \tag{71b}$$

(69a) and (69b) yield

$$\begin{aligned}\int_0^{+\infty} \hat{\sigma}_m^{(E)}(z,\omega)dz &= \int_0^{+\infty} k_E(z)\hat{\gamma}^{(E)}(z,\omega)dz \\ &= \frac{k_0\left(\tau_\alpha^{(E)}\ i\omega\right)^{\frac{\beta-1}{2}}}{\Gamma(2-2\beta)\Gamma(1-\beta)2^{-\beta}}\int_0^{+\infty} z^\beta \mathrm{K}_{1-\beta}\left(\frac{z}{\sqrt{\tau_\alpha^{(E)} i\omega}}\right) dz\ \hat{\gamma}(\omega)\end{aligned} \tag{72a}$$

$$\begin{aligned}\int_0^{+\infty} \hat{\sigma}_m^{(V)}(z,\omega)dz &= i\omega\int_0^{+\infty} c_V(z)\hat{\gamma}^{(V)}(z,\omega)dz \\ &= \frac{k_0\left(\tau_\alpha^{(V)}\ i\omega\right)^{\frac{\beta+2}{2}}}{\Gamma(2-2\beta)\Gamma(\beta)2^{\beta-1}}\int_0^{+\infty} z^{1-\beta} \mathrm{K}_{\beta}\left(z\sqrt{\tau_\alpha^{(V)} i\omega}\right) dz\ \hat{\gamma}(\omega).\end{aligned} \tag{72b}$$

It is not difficult to show (see Appendix B for the explicit calculations of the integrals in (72a) and (72b)) that the relationships above read as follows

$$\begin{aligned}\int_0^{+\infty} \hat{\sigma}_m^{(E)}(z,\omega)dz &= \frac{k_0\left(\tau_\alpha^{(E)}\right)^\beta}{2^{1-2\beta}}\frac{\Gamma(\beta)}{\Gamma(2-2\beta)\Gamma(1-\beta)}(i\omega)^{\beta-1}i\omega\hat{\gamma}(\omega) \\ &= \int_0^{+\infty} e^{-i\omega(\bar{t}+\tau)}\int_{-\infty}^{\bar{t}+\tau} G^{(E)}(\bar{t}+\tau-r)\dot{\gamma}(r)dr\ d\tau\end{aligned} \tag{73a}$$

$$\begin{aligned}\int_0^{+\infty} \hat{\sigma}_m^{(V)}(z,\omega)dz &= \frac{k_0\left(\tau_\alpha^{(V)}\right)^\beta}{2^{2\beta-1}}\frac{\Gamma(1-\beta)}{\Gamma(\beta)\Gamma(2-2\beta)}(i\omega)^{\beta-1}i\omega\hat{\gamma}(\omega) \\ &= \int_0^{+\infty} e^{-i\omega(\bar{t}+\tau)}\int_{-\infty}^{\bar{t}+\tau} G^{(V)}(\bar{t}+\tau-r)\dot{\gamma}(r)dr\ d\tau\end{aligned}. \tag{73b}$$

Henceforth, the inverse right Fourier transform of the relations above yield that (69a) and (69b) read as follows:

$$\int_0^{+\infty} \sigma_m^{(E)}(z,\bar{t}+\tau)dz = \int_{-\infty}^{\bar{t}+\tau} G^{(E)}(\bar{t}+\tau-r)\dot{\gamma}(r)dr \tag{74a}$$

$$\int_0^{+\infty} \sigma_m^{(V)}(z,\bar{t}+\tau)dz = \int_{-\infty}^{\bar{t}+\tau} G^{(V)}(\bar{t}+\tau-r)\dot{\gamma}(r)dr \tag{74b}$$

From the latter relations, we can conclude that no matter what rheological model is considered, their overall response, i.e. the resultant of the “microforces”, namely of the forces on the lateral springs (for the (E)-model) or dashpots (for the (V)-model), match exactly the force arising at the “macrolevel”, i.e. the force acting on the top plate.

Once more this is a confirmation of the fact that the detected “macroscopic” stress response, governed by a power-law relaxation function, is completely determined by constitutive properties of the submacroscopic structure which, in this case, is resembled by the rheological models (E) and (V).

The achieved result is consistent with what is encountered in micromechanics, where a displacement (in this case γ) is prescribed on the boundary (namely the top plate) of a representative volume element of heterogeneous material and a boundary value problem is solved in the interior of such an element. The latter is essentially provided by the procedure followed in [16]: there it was found that the response of the top plate, upon which a time varying displacement was prescribed, was precisely given by the convolution between the rate of change pf the imposed displacement and the relaxation function $G^{(E)}(\circ)$ or $G^{(V)}(\circ)$ for (E) or (V) respectively.

Nevertheless, (74a) and (74b) from (E) and (V), respectively, can be interpreted by invoking the balance of forces arising on each system. In particular, the free body diagrams displayed in Figure 6 are self explanatory about the fact that for each if the two models balance of forces yields that the following relation must hold

$$\sigma(t) = \int_0^\infty \sigma_m(z,t)dz \tag{75}$$

where $\sigma(t)$ is given by (61) and $\sigma_m(z,t)$ represents either (64a) and (64b). Since

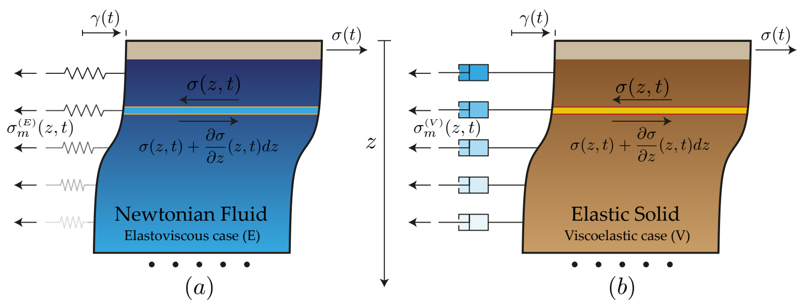


FIGURE 6. Elastoviscous (E) and Viscoelastic (V) deformed models and related external microstresses.

(61) is nothing but the representation in terms of (Caputo) fractional derivative of either $G^{(E)}(\circ)$ or $G^{(V)}(\circ)$, (75) reproduces exactly (74a) and (74b). Evidently, a third way to achieve the same result provided by the latter equation is that (59a) and (59b) deliver the balance of linear momentum of an arbitrary slab (of unitary width) in between the depths z and $z + dz$, i.e.

$$\frac{\partial \tau(z,t)}{\partial z} = \sigma_m(z,t) \tag{76}$$

where τ is the shear stress inside the column and the pair (τ, σ_m) for (E) and (V) agrees with $\left(c_{(E)}(z)\frac{\partial \dot\gamma(z,t)}{\partial z},\ \sigma_m^{(E)}\right)$, $\left(k_{(V)}(z)\frac{\partial \gamma(z,t)}{\partial z},\ \sigma_m^{(V)}\right)$ respectively. Henceforth, the simple integration of (76) across the depth of the column yields

$$\tau(0,t) = -\int_0^\infty \sigma_m(z,t)dz. \tag{77}$$

It is not difficult to check that a direct evaluation of the left-hand side for each of the two rheological models yield (74a) and (74b) back again.

Ultimately, the knowledge of the time history of the displacement of the top plates determines:

(i) the overall response of the system, no matter what rheological model is used;
(ii) the values of the displacement "at the microlevel" $\gamma^{(E)}(z,t)$, $\gamma^{(V)}(z,t)$ for (E) and (V) respectively (e.g. (65a) and (65b)) ultimately leading to the "microstresses" $\sigma_m^{(E)}(z,t)$ and $\sigma_m^{(V)}(z,t)$ through (64a) and (64b) respectively.

Physically it is unlikely to know the whole past strain history of a viscoelastic material, whereas residual stresses may be detectable. As remarked in Sect. 3 before defining $\mathcal{I}^{t_1}(t_2)$ with (45) (here t_1 and t_2 denote two arbitrary times), this variable represents such a stress at time t_2 for not necessarily known strain histories acting from the far past up to the initial observation time $t_1 < t_2$.

The approximation formulas (49) and (37) allow us to state that although (approximately) equivalent histories differ by a power-law strain, the knowledge of $\mathcal{I}^{t_1}(t_2)$ at any $t_1 < t_2$ allows for uniquely determining (i) and (ii).

Unlike (i), which has been treated in [16], (ii) needs to be clarified. If we set

$$
{}_a\varepsilon^*_b(r) := \begin{cases} {}_a\varepsilon^*(r) & a \leq r < b \\ 0 & b \leq r < \bar{t} + \tau \end{cases} \tag{78}
$$

then relation (48) delivering the corresponding residual stress at $\bar{t} + \tau$ reads as follows

$$
{}_a\mathcal{I}^b(\tau + \bar{t}) = \left({}_C\mathrm{D}^\beta_{a+}\ {}_a\varepsilon^*_b\right)(\tau + \bar{t}) + \frac{\varepsilon(b)}{(\bar{t} + \tau - b)^\beta} - \frac{\varepsilon(a)}{(\bar{t} + \tau - a)^\beta}. \tag{79}
$$

This allows for obtaining representatives of the equivalence class of histories by computing the fractional integrals I^β_{a+} of order β of both sides of such a relationship.

As far as the rheological models are concerned, we can state that, instead of prescribing the displacement history of the top plate, the knowledge of the associated residual stress ${}_a\mathcal{I}^b(\tau + \bar{t})$ allows for evaluating

$$
\left(\mathrm{I}^\beta_{a+}\ {}_a\mathcal{I}^b\right)(\tau + \bar{t}) = \frac{1}{\Gamma(\beta)} \int_{a^+}^{\tau + \bar{t}} {}_a\mathcal{I}^b(r)(t - r)^{\beta - 1} dr. \tag{80}
$$

This (besides the terms depending on $\varepsilon(b)$ and $\varepsilon(a)$, where the latter vanishes in the limit for $a \to -\infty$) divided by C_β permits to replace γ in (74a) and (74b) and hence allows for determining the overall response of the rheological models. In this sense, we may state that the "state at the microlevel", namely the knowledge of $\sigma^{(E)}_m(z,t)$ (or $\gamma^{(E)}(z,t)$), $\sigma^{(V)}_m(z,t)$ (or $\gamma^{(V)}(z,t)$) is fully determined through the knowledge of the "macroscopic" state.

6. **Conclusions.** The present study is strongly motivated by the revitalized interest on the hereditariness of biological and bioinspired substructured materials exhibiting macroscopic power law behavior. In this paper, we first provide a mathematical characterization of stress and strain response for such materials in the broader functional setting of strains in SBV_{loc}. In particular, we account for time jumps of such quantities by extending a result of fractional analysis dealing with the reconstruction of the strain for a given stress (see e.g. [38]). Prestressed FHM are then studied, given that power law hereditariness entails very "long tail" memory. This is done in this paper by exploring the notion of state based on [34] and taylored in [36, 37] for general linear viscoelastic materials. This leads to establish an equivalence class of past strain histories based on an integral equation, which here is recast in terms of Caputo fractional derivative. An approximation of this equation is considered and its solution is shown to generate a one parameter family of strains whose stresses converge in L^∞ to the original stress. In turn, the explicit expression of the state variable is singled out, namely the residual stress obtained by freezing the strain to zero for any arbitrary duration starting from a conventional observation time before which the past strain history is not known.

Finally the overall response of two new rheological models describing "functionally graded" viscoelastic submacroscopic structures and introduced in [16] is evaluated. The models are formed by an elastic/viscous continuum layer supported on a side by a bed of discrete viscous/elastic units and driven by an indefinite massless rigid plate adhering to the layer. The resultants of the associated "microstresses" are expliticly evaluated in this paper and they are shown to produce the same "macrostress" arising at the plate. This is expressed in terms of a Caputo fractional derivative of the imposed macroscopic time history which, in reality, is unlikely to be known. While the equality between the two macro-stresses is justified a-posteriori through the balance of linear momentum for each model, it is remarked that the unknown macroscopic past history can be replaced by the fractional integral of the state variable, ultimately owing to a relationship between the "macro-state" to the submacroscopic stresses.

Besides exploring the same issues from the point of view of creep [42], instead of just relaxation, further developments suggested by this new study address the issue of how hereditariness of the submacroscopic structure of more complex (possibly hierarchical) bioinspired or biological materials may determine their overall, and yet effective, state.

Acknowledgments. The authors gratefully acknowledges Prof. M. Di Paola, University of Palermo (Italy) for the useful conversations about fractional calculus and its applications in mechanics. Luca Deseri acknowledges the Department of Mathematical Sciences and the Center for Nonlinear Analysis, Carnegie Mellon University through the NSF Grant No. DMS-0635983 and the financial support from grant PIAP- GA-2011-286110-INTERCER2 "Modelling and optimal design of ceramic structures with defects and imperfect interfaces". Massimiliano Zingales is grateful to the grant Prin 2012 "Dynamics control and stability of flexible structures" from the Italian MIUR. Pietro Pollaci acknowledges the Doctoral School in Engineering of Civil and Mechanical Structural Systems, University of Trento for the financial support, and the Department of Civil and Environmental Engineering of Carnegie Mellon University for its hospitality.

Appendix A. **Notes on fractional calculus and Fourier transforms.** In this Appendix we address some basic notions about fractional calculus. The Euler-Gamma function $z \mapsto \Gamma(z)$ may be considered as the generalization of the factorial function since, as z assumes integer values as $\Gamma(z+1) = z!$ and it is defined as the result of the integral

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx. \tag{A1}$$

Riemann-Liouville fractional integrals and derivatives with $0 < \beta < 1$ of functions defined on the entire real axis $\mathbb{R}$ have the following forms:

$$\left(\mathrm{I}_+^\beta f\right)(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau \tag{A2a}$$

$$\left(\mathrm{D}_+^\beta f\right)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^{\beta}} d\tau, \tag{A2b}$$

whereas their counterparts defined over the whole real axis take the following forms:

$$\left(\mathrm{I}_a^\beta f\right)(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau \tag{A3a}$$

$$\left(\mathrm{D}_a^\beta f\right)(t) = \frac{f(a)}{\Gamma(1-\beta)(t-a)^\beta} + \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{\dot{f}(\tau)}{(t-\tau)^{\beta}} d\tau. \tag{A3b}$$

The relation (A3b) is a direct consequence of Corollary of Lemma 2.1 in [38] (p.32). Besides Riemann-Liouville fractional operators defined above, another class of fractional derivatives that is often used in the context of fractional viscoelasticity is represented by Caputo fractional derivatives defined as:

$$\left({}_C\mathrm{D}^{\beta}_{a^+}f\right)(t) := \mathrm{I}^{m-\beta}_{a^+}\left(\mathrm{D}^{m}_{a^+}f\right)(t) \quad m-1<\beta<m \tag{A4}$$

and whenever $0<\beta<1$ it reads as follows:

$$\left({}_C\mathrm{D}^{\beta}_{a^+}f\right)(t) = \frac{1}{\Gamma(1-\beta)}\int_a^t \frac{\dot{f}(\tau)}{(t-\tau)^{\beta}}d\tau \tag{A5}$$

A closer analysis of (A3b) and (A5) shows that Caputo fractional derivative coincides with the integral part of the Riemann-Liouville fractional derivative in bounded domain. Moreover, the definition in (A4) implies that the function $f(t)$ has to be absolutely integrable of order m (in (A5) $m=1$). Whenever $f(a)=0$ Caputo and Riemann-Liouville fractional derivatives coalesce.

Similar considerations hold true also for Caputo and Riemann-Liouville fractional derivatives defined on the entire real axis. Caputo fractional derivatives may be considered as the interpolation among the well-known, integer-order derivatives, operating over functions $f(\circ)$ that belong to the class of Lebesgue integrable functions. As a consequence, they are very useful in the mathematical description of complex systems evolution.

The right-Fourier Transform of Caputo fractional derivative and Riemann-Liouville fractional integral read as follows:

$$\mathcal{F}_+\left\{\left(\mathrm{I}^{\beta}_{+}f\right)(\circ)\right\}(\omega) = (-i\omega)^{-\beta}\hat{f}_+(\omega) \tag{A6a}$$

$$\mathcal{F}_+\left\{\left({}_C\mathrm{D}^{\beta}_{+}f\right)(\circ)\right\}(\omega) = (-i\omega)^{\beta}\hat{f}_+(\omega) \tag{A6b}$$

where

$$\mathcal{F}_+\left\{f(\circ)\right\}(\omega) := \hat{f}_+(\omega) := \int_0^{\infty} f(t)\mathrm{e}^{-i\omega t}dt. \tag{A7}$$

Appendix B. **Direct evaluation of the microstresses in models (E) and (V).** The displacement functions along the depth of the continuum column of the mechanical models displayed in Figure 4 (see [16] for more details) analyzed in the right Fourier domain assume the following forms

$$\hat{\gamma}^{(E)}(z,\omega) = \hat{\gamma}(\omega)\frac{(\tau_\alpha^{(E)}i\omega)^{-\frac{\overline{\beta}}{2}}}{\Gamma(\overline{\beta})2^{\overline{\beta}-1}}z^{\overline{\beta}}\mathrm{K}_{\overline{\beta}}\left(\frac{z}{\sqrt{\tau_\alpha^{(E)}i\omega}}\right) \tag{B1a}$$

$$\hat{\gamma}^{(V)}(z,\omega) = \hat{\gamma}(\omega)\frac{(\tau_\alpha^{(V)}i\omega)^{\frac{\beta}{2}}}{\Gamma(\beta)2^{\beta-1}}z^{\beta}\mathrm{K}_{\beta}\left(z\sqrt{\tau_\alpha^{(V)}i\omega}\right), \tag{B1b}$$

for the (E) and (V) case respectively, where $\hat{\gamma}(\omega)$ is the right Fourier transform of the imposed displacement at the top plate, $\mathrm{K}_\nu(\circ)$ is the modified Bessel function of the second kind of order ν, $\tau_\alpha^{(E)}$ and $\tau_\alpha^{(V)}$ have been defined in (62b) and (63b) respectively and $\overline{\beta}=1-\beta$. The relationships above are preparatory to enable us evaluating $\hat{\sigma}_m^{(E)}$ and $\hat{\sigma}_m^{(V)}$, namely the time-(right)-Fourier transforms of the microstress, arising in the external devices for both models.

Bearing in mind that $\overline{\alpha} = 2\overline{\beta} - 1$, the right-Fourier Transform of the microstress related to the external springs for the (E) case may be written as follows:

$$\begin{aligned}
\hat{\sigma}_m^{(E)}(z,\omega) &= \frac{1}{A} k_E(z)\hat{\gamma}^{(E)}(z,\omega) \\
&= \frac{G_0}{\Gamma(1+\overline{\alpha})} z^{-\overline{\alpha}}\ \hat{\gamma}(\omega) \frac{(\tau_\alpha^{(E)} i\omega)^{-\frac{\overline{\beta}}{2}}}{\Gamma(\overline{\beta})2^{\overline{\beta}-1}} z^{\overline{\beta}} \mathrm{K}_{\overline{\beta}} \left(\frac{z}{\sqrt{\tau_\alpha^{(E)} i\omega}} \right) \\
&= \hat{\gamma}(\omega) \frac{G_0 \left(\tau_\alpha^{(E)} i\omega\right)^{\frac{\beta-1}{2}}}{\Gamma(2-2\beta)\Gamma(1-\beta)2^{-\beta}} z^{\beta} \mathrm{K}_{1-\beta} \left(\frac{z}{\sqrt{\tau_\alpha^{(E)} i\omega}} \right).
\end{aligned} \tag{B2}$$

The resultant of such quantities can be computed as the integral across the external devices (the springs for the (E) case). In order to evaluate this quantity, we recall that following result holds for a modified Bessel function of second kind integral [43]:

$$\int_0^\infty z^{\mu-1} \mathrm{K}_\nu (Az)\, dz = 2^{\mu-2} A^{-\mu} \Gamma\left(\frac{\mu-\nu}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}\right). \tag{B3}$$

By assuming $\mu - 1 = \beta$ and $\nu = 1 - \beta$ in (B3) and with the aid of (B2), the resultant of the right-Fourier Transform of the microstress for the (E) case arises in the following form:

$$\begin{aligned}
\int_0^\infty \hat{\sigma}_m^{(E)}(z,\omega) dz &= \hat{\gamma}(\omega) \frac{G_0 \left(\tau_\alpha^{(E)} i\omega\right)^{\frac{\beta-1}{2}}}{\Gamma(2-2\beta)\Gamma(1-\beta)2^{-\beta}} \int_0^\infty z^{\beta} \mathrm{K}_{1-\beta} \left(\frac{z}{\sqrt{\tau_\alpha^{(E)} i\omega}} \right) dz \\
&= \hat{\gamma}(\omega) \frac{G_0 \left(\tau_\alpha^{(E)} i\omega\right)^{\frac{\beta-1}{2}}}{\Gamma(2-2\beta)\Gamma(1-\beta)2^{-\beta}}\ 2^{\beta-1} \left(\frac{1}{\tau_\alpha^{(E)} i\omega} \right)^{-\frac{(\beta+1)}{2}} \Gamma(\beta) \\
&= \hat{\gamma}(\omega) \underbrace{\frac{G_0 \left(\tau_\alpha^{(E)}\right)^{\beta} \Gamma(\beta)}{\Gamma(2-2\beta)\Gamma(1-\beta)2^{1-2\beta}}}_{C_\beta^E} (i\omega)^{\beta} \left(\frac{i\omega}{i\omega} \right) \\
&= (i\omega)\hat{\gamma}(\omega)\ C_\beta^E (i\omega)^{\beta-1}.
\end{aligned} \tag{B4}$$

Hence, by taking the inverse right-Fourier transform of both sides and by using Fubini's Theorem on the left-hand side, we have:

$$\int_0^\infty \sigma_m^{(E)}(z,t) dz = \int_{-\infty}^t G(t-\tau)\dot{\gamma}(\tau) d\tau, \tag{B5}$$

since the right Fourier transform of the assumed relaxation function in (4) takes the form:

$$\mathcal{F}_+ \left\{ \frac{C_\beta}{\Gamma(1-\beta)} t^{-\beta} \right\} (\omega) = C_\beta (i\omega)^{\beta-1}. \tag{B6}$$

Similarly, whenever it is assumed that $\alpha = 2\beta - 1$ the microstress for the external dashpots for the (V) case can be evaluated as follows:

$$\begin{aligned}
\hat{\sigma}_m^{(V)}(z,\omega) &= \frac{1}{A} c_V(z)(i\omega)\hat{\gamma}^{(V)}(z,\omega) \\
&= \frac{\eta_0}{\Gamma(1+\alpha)} z^{-\alpha}\ (i\omega)\hat{\gamma}(\omega) \frac{\left(\tau_\alpha^{(V)} i\omega\right)^{\frac{\beta}{2}}}{\Gamma(\beta)2^{\beta-1}} z^\beta \mathrm{K}_\beta\left(z\sqrt{\tau_\alpha^{(V)} i\omega}\right) \\
&= \frac{G_0 \tau_\alpha^{(V)}}{\Gamma(1+\alpha)} \frac{\Gamma(1+\alpha)}{\Gamma(2-2\beta)}\ (i\omega)\hat{\gamma}(\omega) \frac{\left(\tau_\alpha^{(V)} i\omega\right)^{\frac{\beta}{2}}}{\Gamma(\beta)2^{\beta-1}} z^{1-\beta} \mathrm{K}_\beta\left(z\sqrt{\tau_\alpha^{(V)} i\omega}\right) \\
&= \hat{\gamma}(\omega) \frac{G_0\left(\tau_\alpha^{(V)} i\omega\right)^{\frac{\beta+2}{2}}}{\Gamma(\beta)\Gamma(2-2\beta)2^{\beta-1}} z^{1-\beta} \mathrm{K}_\beta\left(z\sqrt{\tau_\alpha^{(V)} i\omega}\right)
\end{aligned} \tag{B7}$$

where equation (63b) has been used in the following form:

$$\tau_\alpha^{(V)} = \frac{\eta_0}{G_0}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \quad \Longrightarrow \quad \eta_0 = \tau_\alpha^{(V)} G_0 \frac{\Gamma(1+\alpha)}{\Gamma(2-2\beta)}. \tag{B8}$$

The use of (B7) and the assumptions $\mu - 1 = 1 - \beta$ and $\nu = \beta$ in (B3) allow for writing the overall microstress for the (V) case in the following form:

$$\begin{aligned}
\int_0^\infty \hat{\sigma}_m^{(V)}(z,\omega)dz &= \hat{\gamma}(\omega) \frac{G_0\left(\tau_\alpha^{(V)} i\omega\right)^{\frac{\beta+2}{2}}}{\Gamma(\beta)\Gamma(2-2\beta)2^{\beta-1}} \int_0^\infty z^{1-\beta}\mathrm{K}_\beta\left(z\sqrt{\tau_\alpha^{(E)} i\omega}\right)dz \\
&= \hat{\gamma}(\omega) \frac{G_0\left(\tau_\alpha^{(V)} i\omega\right)^{\frac{\beta+2}{2}}}{\Gamma(\beta)\Gamma(2-2\beta)2^{\beta-1}}\ 2^{-\beta}\left(\tau_\alpha^{(V)}\right)^{\frac{\beta-2}{2}}\Gamma(1-\beta) \\
&= \hat{\gamma}(\omega) \underbrace{\frac{G_0\left(\tau_\alpha^{(V)}\right)^\beta \Gamma(1-\beta)}{\Gamma(\beta)\Gamma(2-2\beta)2^{2\beta-1}}}_{C_\beta^V} (i\omega)^\beta \left(\frac{i\omega}{i\omega}\right) \\
&= (i\omega)\hat{\gamma}(\omega)\ C_\beta^V (i\omega)^{\beta-1}.
\end{aligned} \tag{B9}$$

Now, proceeding like in (B5) we get:

$$\int_0^\infty \sigma_m^{(V)}(z,t)dz = \int_{-\infty}^t G(t-\tau)\dot{\gamma}(\tau)d\tau \tag{B10}$$

The results addressed by (B4)-(B5) and (B9)-(B10) show that it is possible to compute the resultant of the microstresses in both models without knowing explicitly the transfer function $H(z,t)$ in the time domain.

REFERENCES

[1] P. Nutting *A new general law of deformation*, Journal of The Franklin Institute, **191** (1921), 679–685.

[2] G. Scott Blair *The role of psychophysics in rheology*, Journal of Colloid Science, **2** (1947), 21–32.

[3] A. Jaishankar and G. H. McKinley, *Power-law rheology in the bulk and at the interface: quasi-properties and fractional constitutive equations*, Proceedings of the Royal Society A, **469** (2013), doi:10.1098/rspa.2012.0284

[4] S. Nawaz, P. Sanchez, K. Bodensiek, S. Li, M. Simons and I. A. T. Schaap, *Cell viscoelasticity measured with AFM and vertical optical trapping at sub-micrometer deformations*, PLoS ONE, **7**(9):e45297 (2012), doi:10.1371/journal.pone.0045297

[5] V. Sharma, A. Jaishankar, Y. C. Wang and G. H. McKinley, *Rheology of globular proteins: apparent yield stress, high shear rate viscosity and interfacial viscoelasticity of bovine serum albumin solutions*, Soft Matter, **7** (2011), 5150–5160, doi:10.1039/c0sm01312a

[6] R. L. Magin and T. J. Royston, *Fractional-order elastic models of cartilage: A multi-scale approach*, Communications in Nonlinear Science and Numerical Simulation, **15** (2010), 657–664.

[7] N. Sarkar and A. Lahiri, *Effect of fractional parameter on plane waves in a rotating elastic medium under fractional order generalized thermoelasticity*, International Journal of Applied Mechanics, **4** (2012), doi:10.1142/S1758825112500305

[8] L. Deseri and G. Zurlo, *The stretching elasticity of biomembranes determines their line tension and bending rigidity*, Biomechanics and Modeling in Mechanobiology, (2013), doi:10.1007/S10237-013-0478-Z

[9] L. Deseri, M. D. Piccioni and G. Zurlo, *Derivation of a new free energy for biological membranes*, Continuum Mechanics and Thermodynamics, **20** (2008), 255–273, doi:10.1007/s00161-008-0081-1

[10] L. Deseri, M. Di Paola, M., Zingales and Pollaci, *A hierarchic model of bone tissue hereditariness*, submitted.

[11] M. Baumgaertel and H. H. Winter, *Interrelation between continuous and discrete relaxation time spectra*, Journal of Non-Newtonian Fluid Mechanics, **44** (1992), 15–36.

[12] R. Guedes, J. Simoes and J. Morais, *Viscoelastic behaviour and failure of bovine cancellous bone under constant strain rate*, Journal of Biomechanics, **39** (2006), 49–60.

[13] B. D. Coleman and D. C. Newman, *On the rheology of cold drawing II. Viscoelastic materials*, Journal of Polymer Science: Part B: Polymer Physics, **30** (1992), 25–47.

[14] A. Gemant, *A method of analyzing experimental results obtained from elasto-viscous bodies*, Physics, **7** (1936), 311–317.

[15] M. Di Paola, F. P. Pinnola and M. Zingales, *A discrete mechanical model of fractional hereditary materials*, Meccanica, (2013), doi:0.1007/s11012-012-9685-4

[16] M. Di Paola and M. and Zingales, *Exact mechanical models of fractional hereditary materials*, Journal of Rheology, **56** (2012), 983–1004, http://dx.doi.org/10.1122/1.4717492

[17] V. Quaglini, V. La Russa and S. Corneo, *Nonlinear stress relaxation of trabecular bone*, Mechanics Research Communications, **36** (2009), 275–283.

[18] M. Fabrizio and A. Morro, *Mathematical problems in linear viscoelasticity*, SIAM Philadelphia (1992).

[19] M. Fabrizio and M. J. Golden, *Minimum free energies for materials with finite memory*, Journal of Elasticity, **72** (2003), 121–143

[20] G. Amendola, M. Fabrizio and M. J. Golden, *Thermodynamics of materials with memory: Theory and applications*, Springer (2012), ISBN: 978-1-4614-1691-3.

[21] L. Deseri, G. Gentili and M. J. Golden, *An expression for the minimal free energy in linear viscoelasticity* Journal of Elasticity, *54* (1999), 141–185

[22] L. Deseri, M. Fabrizio and M. J. Golden, *The concept of a minimal state in viscoelasticity: New free energies and applications to PDEs*, Archive for Rational Mechanics and Analysis, **181**(2006), 43–96, doi:10.1007/s00205-005-0406-1

[23] L. Deseri and M. J. Golden, *The minimum free energy for continuous spectrum materials*, SIAM Journal on Applied Mathematics, **67** (2007) 869–892

[24] M. Fabrizio, G. Gentili and M. J. Golden, *The minimum free energy for a class of compressible viscoelastic fluids*, Differential and Integral Equations, **7** (2002), 319–342.

[25] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, in "Fractals and Fractional Calculus in Continuum Mechanics", Springer Verlag, Wien (1997), 291–348, arXiv:1201.0863

[26] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity*, Imperial College Press, London (2010).

[27] F. Mainardi and G. Spada, *Creep, relaxation and viscosity properties for basic fractional models in rheology*, The European Physical Journal, **193** (2011), 133–160, arXiv:1110.3400

[28] R. L. Bagley and P. J. Torvik, *Fractional calculus - A different approach to analysis of viscoelastically damped structures*, The American Institute of Aeronautics and Astronautics, **21** (1983), 741–748.

[29] R. L. Bagley and P. J. Torvik, *On the fractional calculus model of viscoelastic behavior*, Journal of Rheology, **30** (1986), 133–155.

[30] F. Riewe, *Mechanics with fractional derivatives*, Physical Review E , **55** (1997), 3581-3592.

[31] S. Saha Ray, B. P. Poddar and R. K. Bera, *Analytical solution of a dynamic system containing fractional derivative of order one-half by Adomian decomposition method*, Journal of Applied Mechanics, 2005, **72**(2), 290–295, doi:10.1115/1.1839184

[32] M. R. Sunny, R. K. Kapania, R. D. Moffitt, A. Mishra and N. Goulbourne, *A modified fractional calculus approach to model hysteresis*, Journal of Applied Mechanics, **77** (2010), doi:10.1115/1.4000413.

[33] M. K. Suchorsky and R. H. Rand, *A pair of van der Pol oscillators coupled by fractional derivatives*, Nonlinear Dynamics, **69** (2012), 313–324, 10.1007/s11071-011-0266-1

[34] W. Noll, *A new mathematical theory of simple materials*, Archive for Rational Mechanics and Analysis, **48** (1972), 1–50.

[35] D. Graffi and M. Fabrizio, *Sulla nozione di stato per materiali viscoelastici di tipo "rate"*, Atti Accademia dei Lincei, Rendiconti di Fisica, **83** (1989), 201–208.

[36] G. Del Piero and L. Deseri, *On the Concepts of state and free energy in linear viscoelasticity*, Archive for Rational Mechanics and Analysis, **138** (1997), 1–35.

[37] G. Gentili, *Maximum recoverable work, minimum free energy and state space in linear viscoelasticity*, Quarterly of Applied Mathematics, **60** (2002), 153–182

[38] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integral and derivatives. Theory and applications*, Gordon and Breach Science Publishers (1993).

[39] A. N. Kolmogorov and S. V. Fomin, *Introductory Real Analysis*, Prentice Hall (1970).

[40] R. Lakes, *Viscoelastic materials*, Cambridge University Press (2009).

[41] C. Giorgi and A. Morro, *Viscoelastic solids with unbounded relaxation function*, Continuum Mechanics and Thermodynamics, **4** (1992), 151–165.

[42] L. Deseri, M. Di Paola, P. Pollaci and M. Zingales, *Hereditary creep: state and free energy*, In preparation.

[43] http://functions.wolfram.com/Bessel-TypeFunctions/BesselK/21/02/01/0002/

Received xxxx 20xx; revised xxxx 20xx.

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