Spectra of Functionalized Operators arising from hypersurfaces

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Abstract. Functionalized energies, such as the Functionalized Cahn-Hilliard, model phase separation in amphiphilic systems, in which interface production is limited by competition for surfactant phase, which wets the interface. This is in contrast to classical phase-separating energies, such as the Cahn-Hilliard, in which interfacial area is energetically penalized. In binary amphiphilic mixtures, interfaces are characterized not by single-layers, which separate domains of phase A from those of phase B via a heteroclinic connection, but by bilayers, which divide the domain of the dominant phase, A, via thin layers of phase B formed by homoclinic connections. Evaluating the second variation of the Functionalized energy at a bilayer interface yields a functionalized operator. We characterize the center-unstable spectra of functionalized operators and obtain resolvent estimates to the operators associated with gradient flows of the Functionalized energies. This is an essential step to a rigorous reduction to a sharp-interface limit.

1. Introduction

Since its introduction in 1958, [7], the Cahn-Hilliard energy and associated gradient flows have been a fundamental model for diffusive interfaces and their dynamics in binary phase-separated systems which seek to minimize interfacial surface area. However, many systems of physical significance, including amphiphilic mixtures, [19], ionic membranes, [29], and membranes case within ionic liquids, [1], have the tendency to self-assemble interface, subject to the constraint imposed by limitation of surfactant. In binary interface-minimizing systems the interfacial structure is dominated by single-layers which connect phase A to phase B across a thin heteroclinic front. In contrast, interface assembling systems generate a variety of network type structures which the minority (surfactant) phase interpenetrates the majority phase. In some systems, such as lipid bilayers in solvent, the preferred network structure is dominated by thin, co-dimension one sheets of phase B which are surrounded on either side by phase A. In other regimes pore like networks or spherical, micellular inclusions are preferred.

Since a closed bilayer structure separates the physical domain into an inside and an outside, there is a tendency to view bilayer interfaces as equivalent to single-layer interfaces, with phase A representing the inside and phase B the outside of the bilayer. There are situations where this identification is reasonable, but there are also significant distinctions between single-layer and bilayers. First, a single-layer interfaces has no volume to conserve. For a bilayer interface, the surfactant phase lies principally upon the interface and the growth of interface is restricted by the availability of surfactant. Second, a single-layer interface cannot be pierced – it is defined by the two phases that lie on either side. Bilayer interfaces, which separate the same phase, can open up connections, just as the lipid bilayer forming a cell membrane can open a pore to connect the inside and outside of the cell. These distinctions are seen in the geometric flows derived as sharp interface limits of bilayer interfaces in amphiphilic models, [12], which belong in a different class of evolution problems than the Stefan and Mullin-Sekerka type flows derived for the single-layer evolution of the Cahn-Hilliard equation, [42].

The Cahn-Hilliard energy represents the free energy of a binary mixture in terms of the phase function $u : \Omega \subset \mathbf{R}^d \to \mathbf{R}$, via the functional

$$\mathcal{E}[u] = \int_\Omega \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\,dx, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^d$ is bounded, $\varepsilon \ll 1$ is an interfacial width, and W is a double-well potential which assigns energies to the mixture u of the two phases. It is well known that the critical points of the Cahn-Hilliard energy, that is the solutions of

$$\frac{\delta\mathcal{E}}{\delta u} := -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u) = 0, \tag{1.2}$$

are interfacial structures and that the Γ-limit of the Cahn-Hilliard energy is proportional to surface area, [39, 37, 50]. The Functionalized Cahn-Hilliard (FCH) energy, introduced in [45],

$$\mathcal{E}_F[u] = \int_\Omega \frac{1}{2}\left(-\epsilon\Delta u + \frac{1}{\varepsilon}W'(u)\right)^2 - \varepsilon\eta\left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right)dx, \tag{1.3}$$

balances the square of the variational derivative of the Cahn-Hilliard energy *against* a small multiple of the original energy. Energy formulations in the form (1.3) have been derived from experimental data for amphiphilic mixtures, see [19]. For $\varepsilon \ll 1$, the squared variational derivative term demands that minimizers of the FCH be close to *any* critical point of the Cahn-Hilliard energy, including saddle points. At the next order in ε, for $\eta > 0$ lowering the energy requires an *increase* in surface area. Indeed at this order the energy balances the η term against the residual of the first term, which for co-dimension one interfaces reduces to the square of the mean curvature of the interfacial surface. It has been shown, [46] that the FCH is bounded from below over reasonable function spaces, and possesses global minima which are distinct from those of the Cahn-Hilliard energy. Fourth order energies which resemble the FCH with $\eta < 0$ and an equal-depth well W, have been proposed, see, [35] and [54]. Indeed, the De Giorgi conjecture, which concerns the Γ-limit of the FCH energy for $\eta < 0$ with an equal-depth well has been established, [47]. Extensions of these models to address deformations of elastic vesicles subject to volume constraints, [14], and multicomponent models which incorporate a variable intrinsic curvature have been investigated, [36]. However, it is the single-layer interface which underpins all of the analysis for $\eta < 0$. In this work we address the linear structure of bilayer interfaces, which are dominant for $\eta > 0$.

We consider mass-preserving gradient flows of the form

$$u_t = -\varepsilon^2\mathcal{G}\frac{\delta\mathcal{E}_F}{\delta u} = -\mathcal{G}\left(-\varepsilon^2\Delta + W''(u) - \varepsilon^2\eta\right)\left(-\varepsilon^2\Delta u + W'(u)\right), \tag{1.4}$$

subject to various boundary conditions, where the gradient, $\mathcal{G}$, is a self-adjoint, non-negative operator with a simple kernel comprised of the constant functions. In [16] a formal sharp-interface reduction was obtained for the $\mathcal{G} := \Pi_0$, the $L^2(\Omega)$ projection off of the constant functions – the zero-mass projection over Ω. For the case of an equal-depth well W the authors derived the evolution of a bilayer dressed interface $\Gamma \subset \mathbf{R}^d$ in terms of its normal velocity

$$\mathbf{V}_n = \left(\Delta_s - \beta(|\Gamma|) + \frac{H^2}{2} - \mathrm{tr}(A^2)\right)H, \tag{1.5}$$

where H is the mean curvature of Γ, Δ_s is the surface diffusion or Laplace-Beltrami operator, A is the Weingarten map whose eigenvalues are the curvatures of Γ, and the function β couples the surface area of the bilayer interface to its geometric evolution. Indeed, as the interface grows in length, conservation of surfactant requires that the interface becomes thinner; thus β expresses the cost of surfactant scarcity, arresting further interfacial growth as the bilayer thins. Normal velocity relations were also derived for the single-layer interfaces which the FCH supports, however the single-layer geometric evolution generically leads to self-intersection while the bilayer interfaces support large families of stable equilibria. Somewhat surprisingly, the sharp-interface reductions for bilayer interfaces subject to the H^{-1} gradient flow, equivalent to (1.4) with $\mathcal{G} := -\Delta$, do not reproduce the familiar Stefan and Mullen-Sekerka for single-layer interfaces of Cahn-Hilliard, [42]. Rather, for a potential W

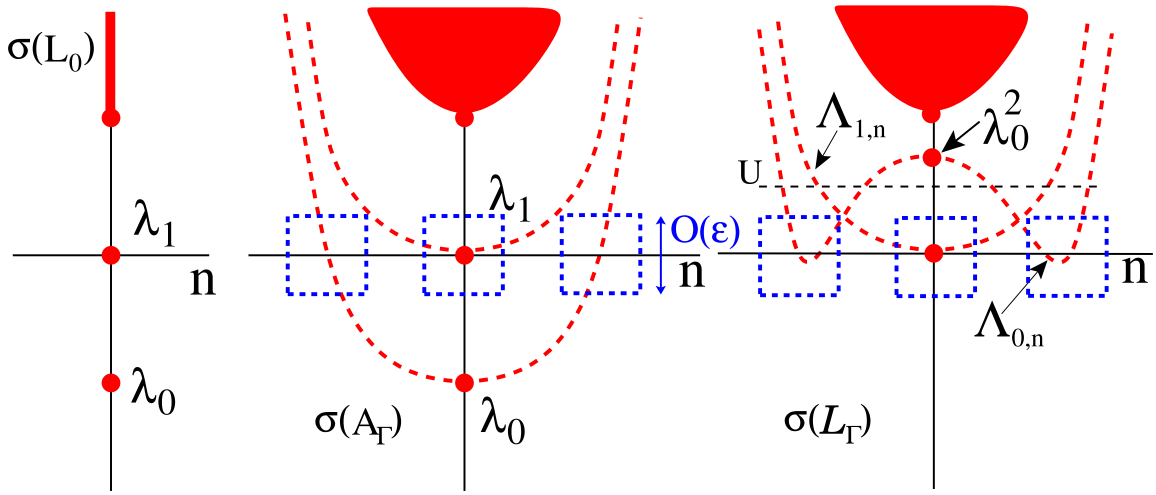

FIGURE 1. Caricature of spectrum verses the in-plane wave number j for the 1D operator, L_0, its extension, A_Γ, to $H^2(\mathbf{R}^d)$, and the linearization, $\mathcal{L}_\Gamma$, given in (1.10), about a bilayer dressed interface, Γ, for a well with unequal-depth minima. Eigenvalues $\Lambda_{i,j}$ denote the spectra of $\mathcal{L}_\Gamma$ associated to the i-th 1D eigenvalue λ_i and the j-th Laplace-Beltrami eigenvalue. The spectrum of A_Γ which is $O(\varepsilon)$ (blue boxes of center frame) are mapped onto the smallest eigenvalues of $\mathcal{L}_\Gamma \sim \mathcal{A}_\Gamma^2 + O(\varepsilon)$, (blue boxes of right frame). The center-unstable spectrum of $\mathcal{L}_\Gamma$ drives the geometric dynamics (near $j = 0$) and modulational or pearling instabilities (for $|j| \in [m_0, M_0]$) of bilayer interfaces.

with un-equal depth wells, one obtains a quenched mean-curvature driven flow on an intermediate time-scale, while the slow time-scale yields a high-order curvature and surface diffusion driven flow with normal velocity

$$\mathbf{V}_n = \sigma_b \Pi_\Gamma \left(\Delta_s + \frac{H^2}{2} - \mathrm{tr}(A^2) \right) H, \tag{1.6}$$

where Π_Γ is a curvature weighted projection associated to the interface Γ which insures that the evolution preserves total interfacial surface area, and $\sigma_b > 0$ depends only upon the well, W, [12]. Indeed, up to the value of the constant σ_b, this result also holds for $\mathcal{G} = \Pi_0$ for W with unequal depth wells. In this paper we consider only the gradient $\mathcal{G} = \Pi_0$.

An important step towards a rigorous sharp-interface reduction of the gradient flows (1.4) is to characterize the small-eigenvalue spectrum of the **functionalized operators** $\mathcal{L}_\Gamma$: the second variational derivative of the FCH energy evaluated at the bilayer solution associated to the admissible interface Γ, see Definition 2.1. This paper gives a rigorous justification to the heuristic images of the spectra of functionalized operators as depicted in Figures 1 and 2. The spectra of Cahn-Hilliard operators, the second variations of the Cahn-Hilliard energy about *single-layer* interfaces, was characterized first in two dimensions, [3], and subsequently in $\mathbf{R}^d$, [10]. These works showed that the eigenfunctions corresponding to small eigenvalues of the Cahn-Hilliard operators admit a leading-order separation of variables decomposition into normal and tangentially varying functions. This characterization played an instrumental role in the rigorous sharp-interface reduction, carried out in [2], of the Cahn-Hilliard equation to a Mullens-Sekerka flow on short time windows $[0, T]$ as $\varepsilon \to 0$.

While the FCH gradient flows engender mergings, buddings, and other morphological rearrangements, [17], they also support large classes of stable equilibria which are typically local minima of the energy landscape, [13]. Correspondingly, the gradient flows of the FCH energy support important classes of solutions which evolve without topological singularity for all time. The small-time restriction which appears in the Cahn-Hilliard reduction, seems particularly unnatural in this context. The removal of this restriction is complicated by the fact that the interface Γ evolves on a slow time-scale, rendering the full linearization, $\mathbf{L}_\Gamma := \mathcal{G}\mathcal{L}_\Gamma$, of the FCH gradient flow weakly time-dependent. The renormalization group techniques developed in [44, 6], yield uniform semi-group estimates for the time-dependent operators arising from the linearization about the slowly evolving structures. This can be achieved if, roughly speaking, each time-frozen operator is coercive off of an approximately invariant set, and these sets are sufficiently insensitive to the evolution of the underlying structure. While these statements have been made rigorous for families of multi-pulses, quantifying them for evolving interfaces remains future work. In this light it is natural to study not the evolution of individual eigenfunctions, but the collective evolution of the center-unstable space of $\mathcal{L}_\Gamma$ as Γ evolves. The main result, Theorem 5.2, shows that for each $U > 0$, sufficiently small but independent of ε, the associated functionalized operator, $\mathcal{L}_\Gamma$, possesses an approximately invariant, finite-dimensional center-unstable space $\mathcal{Z}_U$ such that $\mathbf{L}_\Gamma$ is coercive on $\mathcal{Z}_U^\perp$ with a bounded resolvent.

1.1. Overview of the Sharp Interface Reduction

For the FCH energy, we consider a smooth well $W : \mathbf{R} \to \mathbf{R}$ with two local minima, which may be taken at $u = \pm 1$, and assume that the $u = -1$ state is the majority phase, while the minority phase has the lower self-energy: $W(-1) = 0 > W(1)$. We consider functionalized operators arising from two classes of well W: the unequal-depth well case for which the two minima of $W(-1) - W(1) > 0$ is independent of ε, and the asymptotically equal-depth well case for which the two minima differ by $O(\varepsilon)$. Fixing an admissible hypersurface Γ, see Definition 2.1 we introduce the co-dimension one variables in terms of the scaled, signed distance $z = z(x)$ to Γ, see (2.1) and (6.37). For unequal wells the bilayer profile, $\phi(z)$, is the leading-order solution of the rescaling of (1.2),

$$\partial_z^2 \phi = W'(\phi), \tag{1.7}$$

which is homoclinic to the majority phase, $u = -1$. The existence of ϕ follows from a simple phase-plane analysis. As an operator on $H^2(\mathbb{R})$, the linearization, L_0, of (1.7) about ϕ

$$L_0 := -\partial_z^2 + W''(\phi), \tag{1.8}$$

has a translational eigenvalue, λ_1, at the origin, and an $O(1)$ ground state eigenvalue, $\lambda_0 < 0$, see the left panel of Figure 1. Extending $\phi = \phi(z(x))$ to Ω, see Definition 2.2, we introduce the second-order, scaled Cahn-Hilliard operator

$$\mathcal{A}_\Gamma := \varepsilon \frac{\delta^2 \mathcal{E}}{\delta u^2}(\phi) := -\varepsilon^2 \Delta + W''\big(\phi(x)\big). \tag{1.9}$$

Heuristically, the small-eigenvalue eigenfunctions of $\mathcal{A}_\Gamma$, acting on $H^2(\Omega)$, take the leading order separated-variables form $\psi_i(z)\Theta_j(s)$ where ψ_i is the i'th eigenfunction of L_0 and Θ_j is an eigenfunction of the Laplace-Beltrami operator, Δ_s, associated to Γ. The corresponding eigenvalues take the form $\lambda_i + \varepsilon^2 \mu_j$, where μ_j is the eigenvalue of Δ_s associated to Θ_j. This is the situation depicted in the center panel of Figure 1.

The full linearization, $\mathbf{L}_\Gamma := \mathcal{G}\mathcal{L}_\Gamma$, arising from the linearization of (1.4) about the bilayer interface ϕ associated to Γ is expressed in terms of the functionalized operator,

$$\mathcal{L}_\Gamma := \left(\varepsilon^2 \Delta - W''(\phi) + \varepsilon^2 \eta\right)\left(\varepsilon^2 \Delta - W''(\phi)\right) - W'''(\phi)\left(\varepsilon^2 \Delta \phi - W'(\phi)\right). \tag{1.10}$$

However, the second term of $\mathcal{L}_\Gamma$ is small for ϕ solving (1.7), and formally the operator admits the expansion

$$\mathcal{L}_\Gamma = \mathcal{A}_\Gamma^2 + O(\varepsilon). \tag{1.11}$$

Since Γ is admissible, it is far from $\partial\Omega$ in the scaled distance, and the small-eigenvalue eigenfunctions of $\mathcal{A}_\Gamma$ are localized near the surface. This suggests that the spectrum of $\mathcal{L}_\Gamma$ is given, to $O(\varepsilon)$, by the square of the spectrum of $\mathcal{A}_\Gamma$. This is the situation depicted in the right panel of Figure 1. The coercivity of $\mathcal{A}_\Gamma$ on the set $\mathcal{Z}_U^\perp$ largely follows the construction in [10], we boot-strap this coercivity

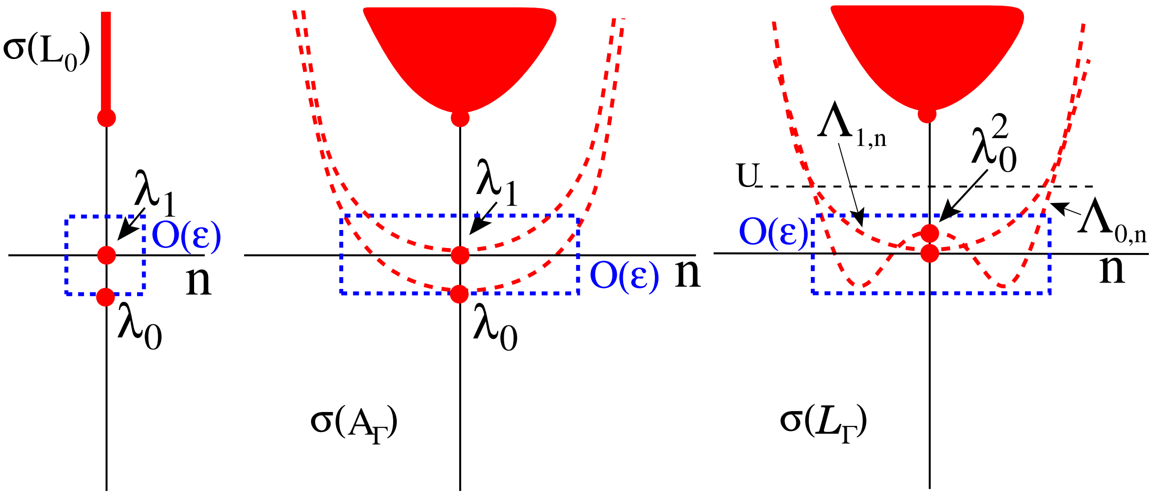

FIGURE 2. Caricature of the spectrum (real) verses the wave number n for the 1D operator, L_0, its Cahn-Hilliard extension, A_Γ, to $\mathbf{R}^d$, and the functionalized operator, $\mathcal{L}_\Gamma$, given in (1.10), about a bilayer dressed interface, Γ, in the case of an asymptotically equal-depth well. Eigenvalues $\Lambda_{i,n}$ indicate the spectrum of $\mathcal{L}_\Gamma$ arising from the i-th 1D eigenvalue λ_i. Theorem 2.7 characterizes the spectrum of $\mathcal{L}$ where U indicates the location of the cut-off of the spectrum in the definition of S_U. In the center and right frames, the blue boxes depict the spectrum which is $O(\varepsilon)$. In the right frame, the small spectrum of $\mathcal{L}$ arises from the square of the small spectrum of A_Γ. It is this spectrum which drives the geometric dynamics ($\{\Psi_{1,n}\}$) and the modulations to the width of the bilayer ($\{\Psi_{0,n}\}$).

to $\mathcal{L}_\Gamma$, which requires bounds on the differential structure of the underlying manifold Γ, and extend the coercivity to $\mathbf{L}_\Gamma$.

The sharp interface reduction follows from the decomposition, $u = \phi(x;\Gamma)+w$ where the interface $\Gamma = \Gamma(t)$ is parameterized by a large, but finite dimensional set of parameters $\vec{p} = \vec{p}(t)$ which are connected to w through the condition $w \in \mathcal{Z}_U^\perp(\Gamma)$. Formally Taylor expanding the gradient flow (1.4) in w yields the form

$$w_t + \nabla_{\vec{p}}\phi \cdot \vec{p}_t = R(\vec{p}) + \mathbf{L}_\Gamma w + Q(w), \tag{1.12}$$

where the residual R denotes the right-hand side of (1.4) evaluated at $\phi = \phi(\vec{p})$, and Q denotes terms which are formally quadratic or higher in w. Denoting the projection onto $\mathcal{Z}_U$, in an appropriate inner product, by Π_U, and its complement by $\widetilde{\Pi}_U := I - \Pi_U$ the gradient flow we project (1.12) with Π_U and $\widetilde{\Pi}_U$, obtaining the evolution

$$w_t = \widetilde{\Pi}_U(R - \nabla_{\vec{p}}\phi \cdot \vec{p}_t) + \widetilde{\Pi}_U \mathbf{L}_\Gamma \widetilde{\Pi}_U w + \widetilde{\Pi}_U Q(w), \tag{1.13}$$

$$\Pi_U \nabla_{\vec{p}}\phi \cdot \vec{p}_t = \Pi_U R + \Pi_U \mathbf{L}_\Gamma \widetilde{\Pi}_U w + \Pi_U Q(w). \tag{1.14}$$

The goal of this paper is to identify a set $\mathcal{Z}_U = \mathcal{Z}_U(\Gamma)$ and a norm for which $\widetilde{\Pi}_U \mathbf{L}_\Gamma \widetilde{\Pi}_U$ has a uniformly bounded resolvent which generates a contractive semi-group, and for which $\mathbf{L}_\Gamma$ is approximately invariant in the sense that $\|\Pi_U \mathbf{L}_\Gamma \widetilde{\Pi}_U\| \ll 1$. These two steps, which form the fulcrum of the sharp interface reduction, are presented in Theorem 5.2 for the gradient $\mathcal{G} = \Pi_0$.

For the case of asymptotically equal-depth wells, $W(-1) = 0 > W(1) = O(\varepsilon)$, the spectra of the functionalized operator are characterized in Figure 2. Following see [16], the bilayer structures are taken from a one-parameter family, $\phi(z;\tau)$, defined as the homoclinic solutions to

$$\partial_z^2 \phi = G'(\phi;\tau), \tag{1.15}$$

where the "tilted-well", G is obtained by adjusting value of G at the minority phase, $u=1$, according to,

$$G(u;\tau) := W(u) - \varepsilon\tau \int_{-1}^{u} \sqrt{W(s)}\, ds. \tag{1.16}$$

The linearization L_0 of (1.15) about $\phi(\cdot,\tau)$

$$L_0(\tau) := -\partial_z^2 + G''(\phi(z;\tau)), \tag{1.17}$$

has a translational eigenvalue, λ_1, at the origin, and an $O(\varepsilon)$ ground state eigenvalue, $\lambda_0(\tau)$, which is associated to the dynamic evolution of the spatially varying width of the bilayer structure. In particular, in the normal z and tangential coordinates, $s \in \mathbf{R}^{d-1}$, defined in a neighborhood of the interface Γ, it is natural to take the well-tilt parameter $\tau = \tau(s)$ to depend upon the tangential variable, and construct the leading order ansatz by dressing the interface Γ with $\phi(z;\tau(s))$. This corresponds to a bilayer structure with a variable width. In Cartesian coordinates this yields an anasatz of the form $\phi(x) = \phi\big(z(x);\tau(s(x))\big)$, where the geometry of the interface Γ is imbedded in the (s,z) coordinate system. The d dimensional, second-order linearized operator depends both upon the choice of interface Γ and upon the choice of dependence $\tau = \tau(x) = \tau(s(x))$ of the tilt upon position s along the interface

$$\mathcal{A}_\Gamma := -\varepsilon^2\Delta + G''(\phi(x;\Gamma,\tau(x))). \tag{1.18}$$

The full linearization $\mathbf{L}_\Gamma$ and the functionalized operator $\mathcal{L}_\Gamma$ are defined via (1.10), where $\phi = \phi(z,\tau(s))$ varies at an $O(1)$ rate in x over the surface of the interface Γ.

In Section 2 we present a general framework and precise statements of the main results, in particular Theorems 2.2 and 2.4 which yield the coercivity of the functionalized operators for equal and asymptotically unequal well depths. We also provide an overview of the proof of the main results and present the decomposition (2.39) which motivates the definition of the small-energy space $\mathcal{Z}_U$. In Section 3 we obtain upper bounds on the elements of the decomposition, proving Theorem 3.2 for the case of unequal depth wells. The proof of the main theorems is presented in Section 4, including the extension to the functionalized operators of the case of equal depth wells, and the extension of the coercivity results from the functionalized operator $\mathcal{L}_\Gamma$ to the full linearization, $\mathbf{L}_\Gamma$. Section 5 provides background material on the differential structure of the manifold Γ and derives estimates on the first and second fundamental forms.

2. Precise Statements of Results for $\mathcal{L}$

We fix a bounded domain $\Omega \subset \mathbb{R}^d$ which is simply connected with smooth boundary. Let $\varepsilon \ll 1$ be a small positive parameter. For a smooth, closed (compact and without boundary), oriented $d-1$ dimensional manifold Γ embedded in $\mathbb{R}^d$, the 'whiskered coordinates' are defined in a tubular neighborhood of Γ,

$$x = \varphi(s,z) := \gamma(s) + \varepsilon z \nu(s), \tag{2.1}$$

where $\gamma : \mathbf{S} \subset \mathbf{R}^{d-1} \to \Gamma$ is the local parametrization of Γ and $\nu(s)$ is the outward unit normal. The line segments $\big\{\gamma(s) \times [-t,t] \big| s \in \mathbf{S}\big\}$ are the *whiskers* of length t of Γ, and the pair (s,z) form the local whiskered coordinate system.

Definition 2.1. *For any $K,\ell > 0$ the family, $\mathcal{G}_{K,\ell}$, of "admissible interfaces" is comprised of closed (compact and without boundary), oriented $d-1$ dimensional manifolds Γ embedded in $\mathbb{R}^d$, which are far from self-intersection and with a smooth second fundamental form. More precisely,*

(i) The $W^{4,\infty}(\mathbf{S})$ norm of the 2nd Fundamental form of Γ and its principal curvatures are bounded by K.

(ii) Whiskers of length $3\ell < 1/K$, in the unscaled distance, neither intersect each-other nor $\partial\Omega$ (except when considering periodic boundary conditions).

(iii) The surface area, $|\Gamma|$, of Γ is bounded by K.

For such Γ the change of variables $x \to (s,z)$ is a C^4 (see Proposition 6.7) diffeomorphism on the neighborhood,

$$\Gamma(l) := \left\{ \varphi(s,z) \middle| s \in \mathbf{S}, -l/\varepsilon \le z \le l/\varepsilon \right\}, \tag{2.2}$$

for any $0 < l < 3\ell$.

Definition 2.2. *We say that a function f defined in $\Gamma(2\ell)$ converges to $f_+ \in \mathbf{R}$ at an $O(1)$ rate if*

$$\sup_s |f(s,z) - f_+| \le Ce^{-m|z|}, \tag{2.3}$$

for constants $C, m > 0$ that are independent of $\varepsilon > 0$. If f converges to zero at an $O(1)$ rate, then we say that f is localized on the interface. If f converges to f_+ at an $O(1)$ rate, then its extension to Ω is defined as the function equal to f_+ in $\Omega\backslash\Gamma(2\ell)$, and $(\eta_2(\varepsilon z))f_+ + \eta_1(\varepsilon z)f$ in $\Gamma(2\ell)$, where $\eta_1(\zeta) : \mathbf{R} \mapsto \mathbf{R}$ is a smooth cut-off function which is equal to 1 for $|\zeta| \le \ell$, 0 for $|\zeta| \ge 2\ell$ and monotone between and $\eta_2 = 1 - \eta_1$. We use f to denote both $f = f(s,z)$ on $\Gamma(2\ell)$ and its extension $f = f(x)$ to Ω.

Given $f, g \in L^2(\Gamma(2\ell))$, then for each $s \in \mathbf{S}$ the weighted inner product on the whisker at $\gamma(s)$ is defined

$$(f,g)_J = (f,g)_J(s) := \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} f(s,z)g(s,z)\tilde{J}(s,z)dz, \tag{2.4}$$

where $J(s,z)$ is the Jacobian of the map $x = \varphi(s,z)$ and $\tilde{J} = J/J_0$ where $J_0 = \sqrt{\det \mathbf{g}}$ is the square-root of the determinant of the first fundamental form of Γ, see (6.29). The associated norm is denoted $\|\cdot\|_J$. For admissible interfaces $\|J\|_{L^\infty(\Gamma(2\ell))}$ is $O(\varepsilon)$ and the $\|\cdot\|_J$ norm introduces a factor of $\varepsilon^{1/2}$. For each $s \in \mathbf{S}$ we also introduce the unscaled inner product

$$(f,g)_0 = (f,g)_0(s) := \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} f(s,z)g(s,z)dz, \tag{2.5}$$

For $f, g \in L^2(\Gamma)$ we have the inner product

$$\langle f, g\rangle_\Gamma := \int_\Gamma f(s)g(s)J_0(s)ds. \tag{2.6}$$

The Laplace-Beltrami eigenmodes are orthonormal in the $\langle\cdot,\cdot\rangle_\Gamma$ inner product. Moreover if the support of f, g is contained inside of $\Gamma(2\ell)$, then we may change to whiskered coordinates in the inner product

$$(f,g)_{L^2(\Omega)} = \int_\Gamma \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} f(s,z)g(s,z)J(s,z)\,dzds = \int_\Gamma (f,g)_J J_0(s)ds = \langle (f,g)_J, 1\rangle_\Gamma. \tag{2.7}$$

We consider two classes of operators which generalize those presented in the introduction.

2.1. Main Results for Unequal Depth Wells

The first class of operators generalizes the case of unequal-depth wells, and has potentials which do not depend upon s to leading order. This class is comprised of the Sturm-Liouville operator $L_0 : H^2(-2\ell/\varepsilon, 2\ell/\varepsilon) \to L^2(-2\ell/\varepsilon, 2\ell/\varepsilon)$, which acts on each whisker of Γ,

$$L_0 := -\partial_z^2 + q_0(z;\varepsilon), \tag{2.8}$$

where the potential q_0 converges exponentially to $2q_+$ at an $O(1)$ rate (see Definition 2.2), and has an extension $q_0 = q_0(x)$ defined on Ω. We define the "Cahn-Hilliard" extension of L_0 to $H^2(\Omega)$,

$$\mathcal{A} := -\varepsilon^2\Delta + q(x;\varepsilon) \tag{2.9}$$

and the functionalized operator

$$\mathcal{L} := \mathcal{A}^2 + \varepsilon\tilde{q}(x;\varepsilon) = \left(\varepsilon^2\Delta - q(x;\varepsilon)\right)^2 + \varepsilon\tilde{q}(x;\varepsilon), \tag{2.10}$$

in terms of the potential

$$q(x;\varepsilon) := q_0(x;\varepsilon) + \varepsilon q_1(x;\varepsilon), \tag{2.11}$$

where both q_1 and $\tilde{q}$ are localized on Γ. We use q to denote both $q = q(s,z)$ on $\Gamma(2\ell)$ and $q = q(x)$, its extension to Ω which equals $2q_+$ on $\Omega\backslash\Gamma(2\ell)$. We assume the potentials q_0 and q satisfy $q_0(x), q(x) > q^+$ when $x \in \Omega\backslash\Gamma(\ell)$. In addition, we assume that q_0, q_1 and $\tilde{q}$ satisfy

$$\sup_{m\le 4} \|\partial_z^m q_0\|_{L^\infty(\Gamma(2\ell))} + \sup_{|\alpha|,m\le 4} \|D_s^\alpha \partial_z^m q_1\|_{L^\infty(\Gamma(2\ell))} + \|\tilde{q}\|_{L^\infty(\Omega)} \le C, \tag{2.12}$$

where C is a constant independent of ε and $\Gamma \in \mathcal{G}_{K,l}$.

We consider the operator $\mathcal{L}$ acting on $H^4_n(\Omega)$, which is $H^4(\Omega)$ subject to natural boundary conditions, that is either zero flux conditions

$$\nabla_{\mathbf{n}} u = \nabla_{\mathbf{n}} \Delta u = 0, \qquad x \in \partial\Omega, \tag{2.13}$$

where $\mathbf{n}$ is the external normal to Ω, or periodic conditions with $\Omega = [0,L]^d$. For all $u, v \in H^2(\Omega)$, the bilinear form associated to $\mathcal{L}$ under natural boundary conditions is given by

$$B[u,v] := (\mathcal{A}u, \mathcal{A}v)_{L^2(\Omega)} + \varepsilon(\tilde{q}u, v)_{L^2(\Omega)}. \tag{2.14}$$

Our goal is to characterize a space, $\mathcal{Z}_U$, of asymptotically minimal dimension for which the bilinear form is coercive on the orthogonal compliment. Indeed, restricting the bilinear form to act on $\mathcal{Z}_U^\perp$ induces the constrained operator

$$\mathcal{L}_U := \Pi_U \mathcal{L}, \tag{2.15}$$

where Π_U is the L^2 orthogonal projection onto $\mathcal{Z}_U^\perp$. The constrained operator maps $\mathcal{Z}_U^\perp \subset H^4(\Omega)$ into $L^2(\Omega)$ and is self-adjoint. A key step is to show that its spectrum is uniformly bounded away from zero.

The following assumption on L_0 generalizes the case of unequal-depth wells considered in the introduction.

Assumption 2.3. (A1) *There exist $\nu_0, \underline{\lambda} > 0$ independent of ε, such that L_0, acting on $H^2([-\ell/\varepsilon, \ell/\varepsilon])$ subject to Neumann boundary conditions, has r eigenvalues satisfying $-\underline{\lambda} \le \lambda_i^0 \le \nu_0$ for $0 \le i \le r-1$, with the remainder of the spectrum of L_0 bounded from below by ν_0.*

Remark: Unadorned constants C will depend only upon the admissible interface parameters K, ℓ and the spectral bounds $\nu_0, \underline{\lambda}$.

The Laplace-Beltrami operator $\Delta_s : H^2(\Gamma) \to L^2(\Gamma)$ of the underlying surface, Γ, is defined by

$$\Delta_s = \frac{1}{\sqrt{\det \mathbf{g}}} \sum_{i=1}^{d-1}\sum_{j=1}^{d-1} \frac{\partial}{\partial s_i} g^{ij} \sqrt{\det \mathbf{g}} \frac{\partial}{\partial s_j} = J_0^{-1} \nabla_s \cdot \left(\mathbf{g}^{-1} J_0 \nabla_s\right), \tag{2.16}$$

where $\mathbf{g}$ is the first fundamental form of Γ and g^{ij} are the elements of $\mathbf{g}^{-1}$, see Section 5. The eigenvalues $\{\beta_j\}_{j=0}^\infty$ of $-\Delta_s$ are non-negative, we denote the corresponding $L^2(\mathbf{S})$ orthonormalized eigenfunctions by $\{\Theta_j\}_{j\in\mathbf{N}_+}$, so that

$$-\Delta_s \Theta_j = \beta_j \Theta_j. \tag{2.17}$$

Fixing $U \ge 0$, then for each $k = 0, \cdots, r-1$ we denote by $m(k) \le M(k)$ the natural numbers for which

$$(\lambda_k^0 + \varepsilon^2 \beta_j)^2 \le U, \qquad \forall\, j \in [m(k), M(k)], \tag{2.18}$$

where λ_k^0 is the k-th eigenvalue of L_0. The wave numbers n in Figures 1 and 2 are related to j through the well ordering of the numbers $|n|^2$ for $n \in \mathbf{N}_+^{\mathbf{d-1}}$. We also introduce the spaces

$$S_U = \operatorname{span}\left\{\Theta_j \,\middle|\, j = m(k), \cdots, M(k),\ \text{and } k = 0, \cdots, r-1\right\}. \tag{2.19}$$

Definition 2.4. *For $k = 0, \cdots, r-1$, let $\psi_k = \eta_1 \tilde{J}^{-1/2} \psi_k^0$ denote the k^{th} eigenfunction, ψ_k^0, of L_0 rescaled by the reduced Jacobian $\tilde{J}$ and the cut-off function η_1, introduced in* (6.32) *and Definition* 2.2 *respectively. We define the basis elements*

$$Z_{jk} = \Theta_j(s)\psi_k(s,z), \tag{2.20}$$

and the slow space $\mathcal{Z}_U$, associated to the functionlized operator $\mathcal{L}$,

$$\mathcal{Z}_U := \text{span}\left\{ Z_{j,k} \middle| k = 0, \cdots, r-1 \quad \text{and} \quad j = m(k), \cdots, M(k) \right\}, \tag{2.21}$$

where $m(k)$ and $M(k)$ are defined in (2.18). The space $\mathcal{Z}_U^\perp$ is the orthogonal complement of $\mathcal{Z}_U$ in $L^2(\Omega)$.

The following theorem characterizes the spectra and coercivity of the Cahn-Hilliard and functionalized operators.

Theorem 2.5. *(Coercivity for Unequal Depth Wells.) Fix $\Omega \subset \mathbf{R}^d$ and $K, \ell > 0$. For any admissible interface $\Gamma \in \mathcal{G}_{K,\ell}$ and operators $\mathcal{A}$ and $\mathcal{L}$ as defined in (2.10), with potentials q and $\tilde{q}$ satisfying (2.12) and Assumption **A1**, there exists $\varepsilon_0, \underline{U}, U_0 > 0$ such that the following results hold for all $\varepsilon \in (0, \varepsilon_0)$ and $U \in (\varepsilon\underline{U}, U_0)$. The ground-state eigenvalue μ_0 of $\mathcal{L}$ acting on $H^4_n(\Omega)$ is bounded from below by*

$$\mu_0 \geq -C\varepsilon, \tag{2.22}$$

where $C > 0$ is independent of ε. Moreover, there exists $\rho > 0$ such that the constrained operator $\mathcal{L}_U$, defined in (2.15), induced by the slow space $\mathcal{Z}_U^\perp$, defined in (2.21), has no spectrum below ρU; in particular the following coercivity estimates hold for all $w \in \mathcal{Z}_U^\perp \cap H^4_n(\Omega)$,

$$B[w, w] \geq \rho^2 U^2 \|w\|^2_{L^2(\Omega)}, \tag{2.23}$$

$$\|\mathcal{A}w\|_{L^2(\Omega)} \geq \max\left\{ \rho U \|w\|_{L^2(\Omega)}, C\varepsilon^2 \|w\|_{H^2(\Omega)} \right\}, \tag{2.24}$$

$$\|\mathcal{A}^2 w\|_{L^2(\Omega)} \geq \max\left\{ \rho^2 U^2 \|w\|_{L^2(\Omega)}, \rho U \|\mathcal{A}w\|_{L^2(\Omega)} \right\}. \tag{2.25}$$

Here C is a positive constant independent of ε.

2.2. Main Results for Asymptotically Equal-Depth Wells

In the case of asymptotically equal-depth wells, the homoclinic profile $\phi(x) = \phi(z(x); \tau(s(x)))$, depends upon the tilt parameter which may vary with position s along the interface. The second class of operators we consider incorporates this case by allowing the leading-order term in the potential q to depend upon s. In particular, we consider a class of Sturm-Liouville operators $L_0[s] : H^2(-2\ell/\varepsilon, 2\ell/\varepsilon) \to L^2(-2\ell/\varepsilon, 2\ell/\varepsilon)$, which act on each whisker of Γ, with an s dependent potential q_0 converging to a common value $2q_+$ exponentially at $O(1)$ rate (see Definition 2.2),

$$L_0[s] := -\partial_z^2 + q_0(s, z; \varepsilon). \tag{2.26}$$

The Cahn-Hilliad extension of $L_0[s]$ to Ω,

$$\mathcal{A} := -\varepsilon^2 \Delta + q(x; \varepsilon) \tag{2.27}$$

and the functionalized operator

$$\mathcal{L} := \mathcal{A}^2 + \varepsilon\tilde{q}(x; \varepsilon) = \left(\varepsilon^2\Delta - q(x; \varepsilon)\right)^2 + \varepsilon\tilde{q}(x; \varepsilon), \tag{2.28}$$

depend upon the potential q which we assume takes the form

$$q(x; \varepsilon) = q(s, z; \varepsilon) = q_0(s, z; \varepsilon) + \varepsilon q_1(s, z; \varepsilon). \tag{2.29}$$

We use q_i to denote both $q_i = q_i(s, z)$ on $\Gamma(2\ell)$ and $q_i = q_i(x)$, the extension of q_i to Ω. The potentials $q_0 - 2q_+$, q_1 and $\tilde{q}$ are localized on Γ and satisfy

$$\sup_{|\alpha| \leq 4, m \leq 2} \|D_s^\alpha \partial_z^m q_0\|_{L^\infty(\Gamma(2\ell))} + \sup_{|\alpha|, m \leq 2} \|D_s^\alpha \partial_z^m q_1\|_{L^\infty(\Gamma(2\ell))} + \|\tilde{q}\|_{L^\infty(\Omega)} \leq C, \ i = 0, 1, \tag{2.30}$$

where C is a constant independent of ε and $\Gamma \in \mathcal{G}_{K,l}$. The Assumption **A1** from the unequal-depth wells case is replaced with

Assumption 2.6. **(A2)** *There exist $\nu_0, \underline{\lambda} > 0$ independent of ε and s, such that for ε sufficiently small, the operator $L_0[s]$, acting on $L^2([-\ell/2, \ell/2])$ subject to Neumann boundary conditions, has r eigenvalues satisfying $\sup_s |\lambda_i^0(s)| \leq \underline{\lambda}\varepsilon$ for $1 \leq i \leq r$, with the remainder of the spectrum of L_0 bounded from below by ν_0.*

Given $U \geq 0$, we denote by $M = M(U)$ the natural number for which

$$\varepsilon^4 \beta_j^2 \leq U, \qquad \forall j \in [1, M], \tag{2.31}$$

and $\varepsilon^4 \beta_{M+1}^2 > U$. We also define the space S_U (now independent of k) as

$$S_U = \operatorname{span}\{\Theta_j\}_{j=1}^M, \tag{2.32}$$

and the slow space $\mathcal{Z}_U$ as,

$$\mathcal{Z}_U := \operatorname{span}\left\{ Z_{j,k} \,\middle|\, k = 0, \cdots, r-1 \quad \text{and} \quad j = 1, \cdots, M \right\}, \tag{2.33}$$

where the basis elements $Z_{j,k}$ are defined in (2.20).

Theorem 2.7. *(Coercivity for Equal-Depth Wells.) Under the conditions of Theorem* 2.5 *with Assumption* **A1** *replaced with Assumption* **A2** *and the potentials q and $\tilde{q}$ subject to the conditions of this section, the conclusions of Theorem* 2.5 *hold for the operators $\mathcal{A}$ and $\mathcal{L}$ defined in* (2.27) *and* (2.28) *over orthogonal compliment to the slow space, $\mathcal{Z}_U^\perp$ defined in* (2.33).

2.3. Outline of the proof of Theorem 2.5

We consider the case of unequal-depth wells and remark that since $\mathcal{L} = \mathcal{A}^2 + O(\varepsilon)$, the coercivity estimates involving $\mathcal{A}$ are a consequence of the coercivity estimate on $B[w,w] = (\mathcal{L}w, w)$. For $\widetilde{U} > 0$ we introduce the set

$$X_{\widetilde{U}} := \operatorname{span}\left\{ \Psi_i \,\middle|\, i = 0, \cdots N_{\widetilde{U}} \right\},$$

comprised of the eigenfunctions of $\mathcal{L}$ corresponding to the eigenvalues smaller than $\widetilde{U}$. For two values of U and $\widetilde{U}$ a key step is to bound the angle between $\mathcal{Z}_U$ and $X_{\widetilde{U}}$. More precisely, we first establish the lower bound (2.22) on $\mathcal{L}$. Decomposing $w \in \mathcal{Z}_U^\perp$ as $w = w_X + w_{X^\perp}$ where $w_X \in X_{\widetilde{U}}$ and $w_{X^\perp} \in X_{\widetilde{U}}^\perp$, the $\mathcal{L}$ orthogonality of these spaces and the afore-mentioned lower bound yield the estimate

$$(\mathcal{L}w, w)_{L^2(\Omega)} = B[w,w] = B[w_X, w_X] + B[w_{X^\perp}, w_{X^\perp}] \geq \widetilde{U}\|w_{X^\perp}\|^2_{L^2(\Omega)} - \varepsilon C \|w_X\|^2_{L^2(\Omega)}. \tag{2.34}$$

If $X_{\widetilde{U}}^\perp$ lies within a cone of aperture $\bar{\alpha} > 0$, independent of ε, about $\mathcal{Z}_U^\perp$, that is if

$$\|w_{X^\perp}\|_{L^2(\Omega)} \geq \bar{\alpha}\|w_X\|_{L^2(\Omega)}, \tag{2.35}$$

for all $w \in \mathcal{Z}_U^\perp$, then we obtain the L^2 coercivity of $\mathcal{L}$ with the constant

$$C_\alpha := \frac{\widetilde{U}\bar{\alpha}^2 - \varepsilon C}{1 + \bar{\alpha}^2}. \tag{2.36}$$

To estimate $\bar{\alpha}$, in Proposition 4.1 we establish the bound

$$\|w_X\|_{L^2(\Omega)} \leq \alpha, \tag{2.37}$$

for all $w \in \mathcal{Z}_U^\perp$ with $\|w\|_{L^2(\Omega)} = 1$, where

$$\alpha := \sqrt{\frac{r(\widetilde{U} + \varepsilon C)}{U}} + \frac{\sqrt{\widetilde{U}}}{\nu_0} + \frac{\sqrt{2\widetilde{U}}}{q_+} + \varepsilon C, \tag{2.38}$$

see (4.4). From the $L^2(\Omega)$ orthogonality $\|w\|^2_{L^2(\Omega)} = \|w_X\|^2_{L^2(\Omega)} + \|w_{X^\perp}\|^2_{L^2(\Omega)}$, the estimate (2.37) yields (2.35) with constant $\bar{\alpha} = \sqrt{\alpha^{-2} - 1}$. In particular, taking $\widetilde{U}$ and U sufficiently larger than ε we may discard $O(\varepsilon)$ terms, and we have the lower bound $C_\alpha \geq \widetilde{U}(1 - \alpha^2)$. Subsequently, choosing $r\widetilde{U}/U$ sufficiently small and while also taking U sufficiently small, in terms of ν_0 and q_+, yields $\alpha < \frac{1}{2}$. That is, there exists a $\rho > 0$, independent of $\varepsilon < \varepsilon_0$, such that $C_\alpha > \rho U$ for $U \in (\varepsilon \underline{U}, U_0)$ with $\underline{U}$ sufficiently large and U_0 sufficiently small. This establishes (2.23).

The proof is encumbered by the differential structure associated to the manifold Γ, whose properties are outlined in the Appendix. We handle this structure term-by-term, decomposing $\Psi = w_X/\|w_X\|_{L^2(\Omega)}$ as $\Psi = \Psi_1 + \Psi_2$ where Ψ_1 is supported inside $\Gamma(2\ell)$, Ψ_2 is supported outside $\Gamma(\ell)$ and

$$\Psi_1(s,z) = \underbrace{\sum_{k=0}^{r-1} b_k(s)\psi_k(s,z)}_{\Psi_S} + \underbrace{\sum_{k=0}^{r-1} c_k(s)\psi_k(s,z)}_{\Psi_{S^\perp}} + \Psi^\perp(x), \tag{2.39}$$

where $b_k \in S_U$, $c_k \in S_U^\perp$, and $(\Psi^\perp, \psi_k)_J(s) = 0$ for $k = 0, \dots r-1$ and all $s \in \mathbf{S}$. In the context of Figure 1 (right), the $k = 0, 1$ parts of Ψ_S denote the projection of Ψ_1 onto the function-space corresponding to $\Lambda_{k,j}$ which lie respectively inside of the dashed-blue boxes, while the $k = 0, 1$ parts of $(\Psi_S^\perp)$ correspond to the projection on the spectra outside the dashed-blue box. In particular since $\Psi_S \in \mathcal{Z}_U$ we have $(\Psi_S, w) = 0$, while $\Psi_1 - \Psi_S \in \mathcal{Z}_U^\perp$. This decomposition is further elaborated in section 3. To obtain the cone property we have to bound the inner product of the remaining terms in the decomposition of Ψ with w. Since $w_X \in X_{\widetilde{U}}$, we have, $\Psi \in X_{\widetilde{U}}$ and hence $B[\Psi, \Psi] \le \widetilde{U}$. In Proposition 3.4 we show that $B[\Psi, \Psi] \le \widetilde{U}$ implies the mild, but necessary bounds

$$\varepsilon\|\Psi\|_{H^1(\Omega)} + \varepsilon^2\|\Psi\|_{H^2(\Omega)} \le C. \tag{2.40}$$

Using (2.39) to expand the bilinear form $B[\Psi, \Psi]$, we bound the mixed terms $B[\Psi_S, \Psi_{S^\perp}]$, $B[\Psi_S, \Psi^\perp]$, and $B[\Psi_{S^\perp}, \Psi^\perp]$, by exploiting an approximate invariance, see Lemma 3.6, of the corresponding spaces with respect to the operator $\mathcal{L} \sim (L_0 - \varepsilon^2\Delta_s)^2$. In Proposition 3.5, we obtain lower bounds on $B[\Psi_S, \Psi_S]$ and $B[\Psi_2, \Psi_2]$, establishing the inequality

$$B[\Psi^\perp, \Psi^\perp] + \|\mathcal{A}\Psi_{S^\perp}\|^2_{L^2(\Omega)} \le \widetilde{U} + C\varepsilon. \tag{2.41}$$

In Proposition 3.8 we control the second term on the left-hand side. Specifically, since $c_k \in S_U^\perp$, (2.41) the minimax principle applied to the Laplace Beltrami operator and $\mathcal{L}$ imply that

$$C\varepsilon + \widetilde{U} \ge \|\mathcal{A}\Psi_{S^\perp}\|^2_{L^2(\Omega} \sim \|(L_0 - \varepsilon^2\Delta_s)\Psi_{S^\perp}\|^2_{L^2(\Omega)} \sim \sum_{k=0}^{r-1}\int_\Gamma [(\lambda_k^0 - \varepsilon^2\Delta_s)c_k]^2 J_0 ds \ge U\sum_{k=0}^{r-1}\|c_k\|^2_{L^2(\Gamma)}, \tag{2.42}$$

which affords cone-control on $\Psi_{S^\perp}$ in (2.39). For $\Psi^\perp$, on each whisker it follows readily from the orthogonality of $\Psi^\perp$ to the small eigenvalue eigenfunctions of L_0 that

$$\nu_0^2\|\Psi^\perp\|_J \le (L_0^2\Psi^\perp, \Psi^\perp)_J + O(\varepsilon). \tag{2.43}$$

However obtaining an upper bound on $\|\Psi^\perp\|_{L^2(\Omega)}$ requires exploiting the structure of the 2nd fundamental form, this is achieved in Proposition 3.9, where we establish the bound

$$B[\Psi^\perp, \Psi^\perp] \sim \int_\Gamma (L_0^2\Psi^\perp, \Psi^\perp)_J J_0 ds. \tag{2.44}$$

3. The Small Energy-Space Decomposition

In this section we prove Theorem 3.3 which characterizes the small energy functions $\Psi \in H^2(\Omega)$ satisfying $B[\Psi, \Psi] \le \widetilde{U}$, in particular for those $\Psi \in X_{\widetilde{U}}$. We also establish the cone-condition (2.35) in Proposition 3.9.

On the domain $\Gamma(\ell)$ the whiskered coordinates (6.37) and the form, (2.11), of the potential q allows us to write the operator $\mathcal{A}$ as

$$\mathcal{A} = -\partial_z^2 - \varepsilon\kappa(s,z)\partial_z + q_0(z) + \varepsilon q_1(s,z) - \varepsilon^2\Delta_G, \tag{3.1}$$

where Δ_G, given in (6.22) is the extension of the Laplace-Beltrami operator off of Γ. The leading-order terms of $\mathcal{A}$ are the $(\cdot,\cdot)_0$ self-adjoint form

$$L_0 := -\partial_z^2 + q_0(z), \tag{3.2}$$

Operator	Domain	Description	Eigenpairs
$L_0 = -\partial_z^2 + q_0(z)$	$I_\ell \subset \mathbb{R}$	1D Allen-Cahn Operator	(ψ_i^0, λ_i^0)
$L_J = -\partial_z^2 - \varepsilon\kappa(s,z)\partial_z + q(s,z)$	$I_\ell \subset \mathbb{R}$	1D Symmetrized Allen-Cahn	(ψ_i^a, λ_i^a)
$L = L_J^2 + \varepsilon\tilde{q}(s,z)$	$I_\ell \subset \mathbb{R}$	1D Functionalized	
$\mathcal{A} = -\varepsilon^2\Delta + q(x)$	$\Omega \subset \mathbb{R}^d$	Allen-Cahn Operator	
$\mathcal{L} = \mathcal{A}^2 + \varepsilon\tilde{q}(x)$	$\Omega \subset \mathbb{R}^d$	Functionalized Operator	(Ψ_i, μ_i)

Table 1. The operators, domains, and eigenpairs used in the text.

and the $(\cdot,\cdot)_J$ self-adjoint form, which includes terms to first order in ε,

$$L_J[s] := -\tilde{J}^{-1}\frac{\partial}{\partial z}\tilde{J}\frac{\partial}{\partial z} + q(s,z) = L_0 - \varepsilon\kappa(s,z)\partial_z + \varepsilon q_1(s,z), \tag{3.3}$$

where we recall $\partial_z\tilde{J} = \varepsilon\kappa\tilde{J}$. Similarly, we introduce the one-dimensional functionalized operator

$$L := L_J^2 + \varepsilon\tilde{q}(s,z) = (L_0 - \varepsilon\kappa\partial_z + \varepsilon q_1)^2 + \varepsilon\tilde{q}, \tag{3.4}$$

which contains the leading-order terms of $\mathcal{L}$. Indeed, in the whiskered coordinate system,

$$\mathcal{L} = \mathcal{A}^2 + \varepsilon\tilde{q} = L - \varepsilon^2\Delta_G L_J - \varepsilon^2 L_J\Delta_G + \varepsilon^4\Delta_G^2. \tag{3.5}$$

This proliferation of operators, each definded on different domains, is summarized in Tables 1 and 2. Here I_ℓ denotes the interval $(-2\ell/\varepsilon, 2\ell/\varepsilon)$, and the boundary conditions are natural for the appropriate bilinear form. The operators L_J and L depend upon $s \in \mathbf{S}$ as a parameter, and for each fixed whisker $w(s)$ act on $H^2(I_\ell)$. When viewed as operators on a fixed whisker, we will occasionally use $'$ to represent derivative with respect to the variable z.

Since the potential q_0 in L_0 converges to a constant $2q_+ > 0$ at an $O(1)$ exponential rate, its eigenfunctions satisfy the following classical estimates whose proof we omit.

Lemma 3.1. *There exists $m > 0$ such that the eigenfunctions $\{\psi_i^0\}_{i=0}^{r-1}$ of L_0 are uniformly bounded by C in $H^4(\ell)$ with respect to the $\|\cdot\|_0$ norm, and moreover*

$$\sup_{n\le 4}\partial_z^n\psi_i^0(\pm\ell/\varepsilon) = O(e^{-m/\varepsilon}). \tag{3.6}$$

Proposition 3.2. *(Coercivity of L_J) Fix $K,\ell > 0$ and let $\Gamma \in \mathcal{G}_{K,\ell}$. The set $\Sigma_J^n := \{\tilde{J}^{-1/2}\psi_k^0\}_{k=0}^n$ is an approximate basis for the first n eigenfunctions of L_J. That is, there exists a positive constant C independent of ε and $\Gamma \in \mathcal{G}_{K,l}$ such that for all $\psi \in H^1(I_\ell)$ satisfying Neumann boundary conditions and $\tilde{J}$-orthogonal to Σ_J^n we have the bound*

$$\frac{(L_J\psi,\psi)_J}{\|\psi\|_J^2} \ge \lambda_{n+1}^0 - C\varepsilon. \tag{3.7}$$

Proof. Inserting $\sqrt{\tilde{J}}\,u$ and $\sqrt{\tilde{J}}\,v$ into b_0 and integrating by parts yields

$$b_0[\sqrt{\tilde{J}}\,u, \sqrt{\tilde{J}}\,v] = b_a[u,v] + \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon}[\frac{1}{4}\tilde{J}^{-1}(\tilde{J}')^2 - \frac{1}{2}\tilde{J}'' - \varepsilon q_1]uv dz. \tag{3.8}$$

Fix $s \in \Gamma$ and let $\psi(s,z) \in H^1(I_\ell)$ and define $\psi^0 := \tilde{J}^{1/2}\psi$, so that by (3.8)

$$(L_J\psi,\psi)_J = (L_0\psi^0,\psi^0)_0 + \Big(\Big[(1/2)\tilde{J}^{-1}\tilde{J}'' - (1/4)\tilde{J}^{-2}(\tilde{J}')^2 + \varepsilon q_1\Big]\psi,\psi\Big)_J. \tag{3.9}$$

Operator	Inner Product	Bilinear Form
L_0	$\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} uvdz$	$b_0[u,v] = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon}(u'v' + q_0uv)dz$
L_J	$\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} uv\tilde{J}dz$	$b_a[u,v] = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon}(u'v' + quv)\tilde{J}dz$
L	$\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} uv\tilde{J}dz$	$b[u,v] = \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon}\left((L_Ju)(L_Jv) + \varepsilon\tilde{q}uv\right)\tilde{J}dz$
$\mathcal{A}$	$\int_\Omega uvdx$	$B_a[u,v] = \int_\Omega(\varepsilon^2\nabla u\cdot\nabla v + quv)dx$
$\mathcal{L}$	$\int_\Omega uvdx$	$B[u,v] := \int_\Omega((\mathcal{A}u)(\mathcal{A}v) + \varepsilon\tilde{q}uv)dx$

TABLE 2. The operators, inner products for which they are self-adjoint, and associated bilinear forms used in the text.

The expansion (6.32) and the estimates on the reduced Jacobian in (6.40) of Proposition 6.7 imply that there exists a constant $C > 0$ that depends only on K such that

$$\left\| -(1/4)\tilde{J}^{-2}(\tilde{J}')^2 + (1/2)\tilde{J}^{-1}\tilde{J}''\right\|_{L^\infty(\Gamma(2\ell))} \leq C\varepsilon^2, \tag{3.10}$$

while from (2.12), $\|q_1\|_{L^\infty(\Gamma(2\ell))} \leq C$. Using $(\psi^0,\psi_k^0)_0 = (\psi,\tilde{J}^{-1/2}\psi_k^0)_J = 0$ for $k = 0,\ldots,n$ together with the minimax characterization for eigenvalues of L_0 gives

$$(L_J\psi,\psi)_J \geq (L_0\psi^0,\psi^0)_0 - C\varepsilon\|\psi\|_J^2 \geq \lambda_{n+1}^0\|\psi_0\|_0^2 - C\varepsilon\|\psi\|_J^2 = (\lambda_{n+1}^0 - C\varepsilon)\|\psi\|_J^2, \tag{3.11}$$

where we used $\|\psi_0\|_0 = \|\tilde{J}^{1/2}\psi\|_0 = \|\psi\|_J$. □

We fix an admissible interface $\Gamma \in \mathcal{G}_{K,\ell}$ and recall the cut-off function η_1 introduced in Definition 2.2 and the decomposition of $\Psi \in H^2(\Omega)$ into localized and delocalized parts, $\Psi = \Psi_1 + \Psi_2$, where $\Psi_1 = \Psi\eta_1$ is localized on Γ and $\Psi_2 = (1-\eta_1)\Psi$. Recalling the definition (2.19) of the space S_U, we remark that if $a \in H^2(\Gamma)\cap S_U^\perp$, then for $k = 0,\cdots,r-1$ we have

$$\|(\lambda_k^0 - \varepsilon^2\Delta_s)a\|_{L^2(\Gamma)}^2 \geq U\|a\|_{L^2(\Gamma)}^2. \tag{3.12}$$

Recalling the $\tilde{J}$ normalized eigenfunction $\{\psi_k\}_{k=0}^{r-1}$ introduced in Definition 2.4, we introduce the s dependent coefficients $a_k(s) := (\Psi_1,\psi_k)_J = (\tilde{J}^{1/2}\Psi_1,\psi_k^0)_0$, and decompose Ψ on each whisker as

$$\Psi_1 = \sum_k a_k(s)\psi_k(s,z) + \Psi^\perp, \tag{3.13}$$

where $(\psi_k,\Psi^\perp)_J = 0$, for $k = 0,\ldots,r-1$ for each $s \in \mathbf{S}$. The normalization $\|\psi_k\|_J = 1$ implies,

$$\|a_k\|_{L^2(\Gamma)}^2 = \int_\Gamma (\Psi_1,\psi_k)_J^2\, J_0ds \leq \int_\Gamma \|\Psi_1\|_J^2\, J_0ds \leq \|\Psi\|_{L^2(\Omega)}. \tag{3.14}$$

Since the eigenfunctions ψ_i^0 do not depend on s the Jacobian bounds (6.40) yield the estimate

$$\|\Delta_s a_k\|_{L^2(\Gamma)}^2 \leq C\|\Psi_1\|_{H^2(\Omega)}^2. \tag{3.15}$$

Moreover, we further decompose each a_k into its projection onto S_U,

$$b_k = P_{S_U}a_k := \sum_{k=0}^{r-1}\sum_{i=m(k)}^{M(k)}(a_k,\Theta_i)_{L^2(\Gamma)}\Theta_i, \tag{3.16}$$

and the orthogonal complement

$$c_k = (I - P_{S_U})a_k. \tag{3.17}$$

Since $b_k \in S_U$ and $c_k \in S_U^\perp$ are orthogonal we have

$$\|b_k\|^2_{L^2(\Gamma)} + \|c_k\|^2_{L^2(\Gamma)} = \|a_k\|^2_{L^2(\Gamma)} \leq 1, \tag{3.18}$$

while from the bounds (3.14) and (3.15) we get the estimate,

$$\|b_k\|_{H^2(\Gamma)} + \|c_k\|_{H^2(\Gamma)} \leq C\|\Psi_1\|_{H^2(\Omega)}. \tag{3.19}$$

Defining Ψ_S and $\Psi_{S^\perp}$ as in (2.39) we have the decomposition

$$\Psi = \Psi_1 + \Psi_2 = \Psi_S + \Psi_{S^\perp} + \Psi^\perp + \Psi_2, \tag{3.20}$$

where $(\psi_k, \Psi^\perp)_J = 0$, for $k = 0, \ldots, r-1$. Moreover the components $\Psi_S, \Psi_{S^\perp}$, and $\Psi^\perp$ are mutually orthogonal and hence satisfy the estimate

$$\|\Psi_S\|^2_{L^2(\Omega)} + \|\Psi_{S^\perp}\|^2_{L^2(\Omega)} + \|\Psi^\perp\|^2_{L^2(\Omega)} = \|\Psi_1\|^2_{L^2(\Omega)} \leq \|\Psi\|^2_{L^2(\Omega)}. \tag{3.21}$$

In the next theorem we obtain bounds on c_k, $\Psi^\perp$, and Ψ_2 from an upper bound on the full bilinear form $B[\Psi, \Psi]$, associated with $\mathcal{L}$ of (2.10),

$$B[u, v] := (\mathcal{A}u, \mathcal{A}v) + \varepsilon(\tilde{q}u, v). \tag{3.22}$$

Theorem 3.3. *Let U_0 be a positive constant independent of ε. Assume $\Psi \in H^2(\Omega)$ satisfies $\|\Psi\|_{L^2(\Omega)} = 1$ and $\partial_\nu \Psi = 0$ or periodic boundary conditions on $\partial\Omega$, in addition to the bound*

$$B[\Psi, \Psi] \leq \widetilde{U}, \tag{3.23}$$

for the bilinear form B given in (3.22) and some $\widetilde{U} \in (0, U_0)$. Then for any spectral cutoff value $U \geq 0$, Ψ admits the decomposition (3.20) *where*

$$\nu_0^2 \|\Psi^\perp\|^2_{L^2(\Gamma(2\ell))} + q_+^2 \|\Psi_2\|^2_{L^2(\Omega)} \leq \widetilde{U} + C\varepsilon, \tag{3.24}$$

and

$$\sum_{k=0}^{r-1} \|c_k\|^2_{L^2(\Gamma)} \leq \frac{\widetilde{U} + C\varepsilon}{U}. \tag{3.25}$$

Remark: Recall that U represents the spectral cutoff bound in the construction of the spaces S_U and Z_U, while the coefficients c_k in the decomposition of Ψ represent the contribution to the small energy functions by Laplace-Beltrami modes in $S_U^\perp$. The further we cut the spectrum above $\widetilde{U}$, smaller the ratio $\frac{\widetilde{U}}{U}$, and consequently the smaller this contribution.

We present the proof in a series of propositions.

Proposition 3.4. *Under the conditions of Theorem 3.3, there exist $\varepsilon_0, C > 0$ such that for $\varepsilon < \varepsilon_0$*

$$\varepsilon\|\Psi\|_{H^1(\Omega)} + \varepsilon\|\Psi^\perp\|_{H^1(\Omega)} + \varepsilon^2\|\Psi\|_{H^2(\Omega)} + \varepsilon^2\|\Psi^\perp\|_{H^2(\Omega)} \leq C, \tag{3.26}$$

Moreover, $\Psi^\perp$ is supported on $\Gamma(2\ell)$ where it enjoys the estimates

$$\|\partial_z \Psi^\perp\|_{L^2(\Gamma(2\ell))} + \|\partial_{zz}\Psi^\perp\|_{L^2(\Gamma(2\ell))} + \varepsilon\|\nabla_s \Psi^\perp\|_{L^2(\Gamma(2\ell))} + \varepsilon\|\partial_z \nabla_s \Psi^\perp\|_{L^2(\Gamma(2\ell))} \leq C. \tag{3.27}$$

Proof. Using the Neumann boundary conditions, an integration by parts yields the equality

$$(\mathcal{A}\Psi, \Psi)_{L^2(\Omega)} = \int_\Omega [\varepsilon^2 |\nabla\Psi|^2 + q\Psi^2] dx, \tag{3.28}$$

while from (3.23) and the definition of the bilinear form B in (3.22) we have

$$\|\mathcal{A}\Psi\|^2_{L^2(\Omega)} = B[\Psi, \Psi] - \varepsilon(\tilde{q}\Psi, \Psi) \leq U_0, \tag{3.29}$$

for ε sufficiently small, since $\|\tilde{q}\|_{L^\infty(\Omega)} \leq C$ and $\|\Psi\|_{L^2(\Omega)} = 1$. Using (3.28) and (3.29) gives

$$\varepsilon^2 \|\nabla\Psi\|^2_{L^2(\Omega)} = (\mathcal{A}\Psi, \Psi)_{L^2(\Omega)} - (q\Psi, \Psi)_{L^2(\Omega)} \leq \sqrt{U_0} + C \leq C, \tag{3.30}$$

which yields the estimate on $\|\Psi\|_{H^1(\Omega)}$ in (3.26). To obtain the H^2 bound we observe that

$$\Delta\Psi = -\varepsilon^{-2}(\mathcal{A}\Psi + q\Psi^2), \tag{3.31}$$

and from classical elliptic regularity (see [15]), we have the bound

$$\|\Psi\|_{H^2(\Omega)} \leq C\Big(\|\Psi\|_{L^2(\Omega)} + \varepsilon^{-2}\|\mathcal{A}\Psi + q(x)\Psi^2\|_{L^2(\Omega)}\Big) \leq C\varepsilon^{-2}. \tag{3.32}$$

The bounds on $\Psi^\perp$ follow from applying (3.19) to $\Psi^\perp = \Psi_1 - \Psi_S - \Psi_{S^\perp}$. The estimates (3.27) are immediate consequences of (6.45). ☐

Proposition 3.5. *Under the conditions of Theorem* 3.3*, there exist* $\varepsilon_0, C, C_1 > 0$ *such that for* $\varepsilon < \varepsilon_0$*,*

$$B[\Psi^\perp,\Psi^\perp] + \|\mathcal{A}\Psi_{S^\perp}\|^2 + \|\mathcal{A}\Psi_S\|^2_{L^2(\Omega)} + C_1\varepsilon^4\|\Psi_2\|^2_{H^2(\Omega)} + q_+2\varepsilon^2\|\Psi_2\|^2_{H^1(\Omega)} + q_+^2\|\Psi_2\|^2_{L^2(\Omega)} \leq \widetilde{U} + C\varepsilon. \tag{3.33}$$

Proof. Using the decomposition (2.39), we expand the bi-linear form B as

$$\begin{aligned} B[\Psi,\Psi] \quad &= \quad B[\Psi_S,\Psi_S] + B[\Psi_{S^\perp},\Psi_{S^\perp}] + B[\Psi^\perp,\Psi^\perp] + B[\Psi_2,\Psi_2] + \\ &\quad\quad 2B[\Psi_S,\Psi_S^\perp] + 2B[\Psi_S,\Psi^\perp] + 2B[\Psi_{S^\perp},\Psi^\perp] + 2B[\Psi_1,\Psi_2]. \end{aligned} \tag{3.34}$$

The potentials q and $\tilde{q}$ are bounded in $L^\infty(\Omega)$, so (3.22) and (3.21) immediately imply,

$$B[\Psi_{S^\perp},\Psi_{S^\perp}] + B[\Psi_S,\Psi_S] \geq \|\mathcal{A}\Psi_{S^\perp}\|^2_{L^2(\Omega)} + \|\mathcal{A}\Psi_S\|^2_{L^2(\Omega)} - C\varepsilon. \tag{3.35}$$

To address the term $B[\Psi_1,\Psi_2]$, we expand the Bilinear form

$$B[\Psi_1,\Psi_2] = \int_\Omega (-\varepsilon^2\Delta(\eta_1\Psi) + q\eta_1\Psi)(-\varepsilon^2\Delta(\eta_2\Psi) + q\eta_2\Psi)dx + \varepsilon\int_{\Gamma_{01}} \tilde{q}\eta_1\eta_2\Psi^2 dx. \tag{3.36}$$

The cut-off functions $\eta_i = \eta_i(z)$ are smooth with $O(1)$ derivatives in x. We distribute the derivatives, and from the bounds (3.26) and (3.21) it is easy to see that

$$\varepsilon^2\|\eta_i\Delta\Psi\|_{L^2(\Omega)} + \varepsilon\|\nabla\eta_i\cdot\nabla\Psi\|_{L^2(\Omega)} + \|(\Delta\eta_i)\Psi)\|_{L^2(\Omega)} + \|q\eta_i\Psi\|_{L^2(\Omega)} \leq C,$$

for $i = 1,2$, while the remaining term, with all derivatives on Ψ, is positive since $\eta_1\eta_2 \geq 0$. This yields the lower bound

$$B[\Psi_1,\Psi_2] = \int_\Omega \eta_1\eta_2(-\varepsilon^2\Delta\Psi + q\Psi)^2 dx + O(\varepsilon) \geq -C\varepsilon. \tag{3.37}$$

☐

We complete the proof of Proposition 3.5 via Lemmas 3.6 and 3.7 below. The first of these bounds the cross-terms within Ψ_1.

Lemma 3.6. *Under the conditions of Theorem* 3.3*, there exist* $\varepsilon_0, C > 0$ *such that for* $\varepsilon < \varepsilon_0$*,*

$$\Big|B[\Psi_S,\Psi_{S^\perp}]\Big| + \Big|B[\Psi_S,\Psi^\perp]\Big| + \Big|B[\Psi_{S^\perp},\Psi^\perp]\Big| \leq C\varepsilon, \tag{3.38}$$

Proof. We bound only the first term, the other two are handled with similar arguments. We recall the definition (2.39) of Ψ_S, where b_k are defined in (3.16). From (3.26), (3.18) and (3.19) we have the estimate

$$\sup_{m=0,\cdots,2} \|\varepsilon^m b_k\|_{H^m(\Gamma)} \leq C. \tag{3.39}$$

The $\tilde{J}$-normalized eigenfunctions of L_0 satisfy,

$$L_0\psi_k = L_0(\tilde{J}^{-1/2}\psi_k^0) = \lambda_k^0\psi_k - \frac{3}{4}\tilde{J}^{-5/2}(\partial_z\tilde{J})^2\psi_k^0 + \frac{1}{2}\tilde{J}^{-3/2}(\partial_z^2\tilde{J})\psi_k^0 + \tilde{J}^{-3/2}(\partial_z\tilde{J})(\partial_z\psi_k^0). \tag{3.40}$$

From the definition, (3.3), of L_J, the bounds on κ and $\tilde{J}$ in Proposition 6.7, the bounds (2.12) on q, and the bounds (3.6) on ψ_k^0, we write

$$L_J\psi_k = L_0\psi_k - \varepsilon\kappa\partial_z\psi_k + \varepsilon q_1\psi_k = \lambda_k^0\psi_k + r_1(s,z), \tag{3.41}$$

where the residual r_1 enjoys the bounds

$$\sup_{|\alpha|\le 4, i=0,1} \sup_s \|z^i D_s^\alpha r_1(s,z)\|_J \le C\varepsilon. \tag{3.42}$$

Similarly we may write the action of the functionalized 1D operator, L, defined in (3.4), on ψ_k in the form

$$L\psi_k = L_J^2\psi_k + \varepsilon\tilde{q}\psi_k = (L_0 - \varepsilon\kappa\partial_z + \varepsilon q_1)L_J\psi_k + \varepsilon\tilde{q}\psi_k = (\lambda_k^0)^2\psi_k + r_2(s,z), \tag{3.43}$$

where the residual r_2 enjoys the same bounds as r_1. From the expression (3.5) for $\mathcal{L}$, we write its action on $b_k(s)\psi_k(z,s)$ in the form

$$\mathcal{L}(b_k\psi_k) = L(b_k\psi_k) - \varepsilon^2\Delta_G L_J(b_k\psi_k) - \varepsilon^2 L_J\Delta_G(b_k\psi_k) + \varepsilon^4\Delta_G^2(b_k\psi_k). \tag{3.44}$$

Recalling the form of Ψ_S from (2.39) and that the bilinear form B is induced by $\mathcal{L}$, we have the expansion

$$\begin{aligned} B[\Psi_S, \Psi_{S^\perp}] \;=\; & \sum_{k=0}^{r-1}\Big(\overbrace{(L(b_k\psi_k), \Psi_{S^\perp})_{L^2(\Omega)}}^{I_{1k}} - \varepsilon^2\overbrace{(\Delta_G L_J(b_k\psi_k), \Psi_{S^\perp})_{L^2(\Omega)}}^{I_{2k}} + \\ & -\varepsilon^2\overbrace{(L_J\Delta_G(b_k\psi_k), \Psi_{S^\perp})_{L^2(\Omega)}}^{I_{3k}} + \varepsilon^4\overbrace{(\Delta_G^2(b_k\psi_k), \Psi_{S^\perp})_{L^2(\Omega)}}^{I_{4k}}\Big) \end{aligned} \tag{3.45}$$

We estimate each term individually. For I_1 we observe that $L(b_k\psi_k) = b_kL\psi_k$ which from (3.43) has leading order term proportional to $b_k\psi_k$ which is orthogonal to $\Psi_{S^\perp}$; thus the leading order term in I_{1k} arises from the inner product of r_2 with $\Psi_{S^\perp}$, which is $O(\varepsilon)$. For I_2, we use (6.23) of Proposition 6.6 to expand Δ_G as

$$\Delta_G L_J(b_k\psi_k) = \Delta_s L_J(b_k\psi_k) + \varepsilon z D_{s,2}(b_kL_J\psi_k), \tag{3.46}$$

where $D_{s,2}$ is the second order differential operator defined in (6.24) whose coefficients satisfy the bounds (6.25). Combining (3.41) and (3.46) we find the leading order expression for I_2,

$$I_{2k} = \lambda_k^0\,(\psi_k\Delta_s b_k, \Psi_{S^\perp})_{L^2(\Omega)} = \lambda_k^0\int_\Gamma (\Delta_s b_k)c_k\, J_0 ds, \tag{3.47}$$

where we used (2.7) in conjunction with the ψ_j-ψ_k orthonormality in the $\tilde{J}$ inner-product. However $b_k \in S_U$ which is invariant under Δ_s, so that $\Delta_s b_k \in S_U$ which is s-orthogonal to $\Psi_{S^\perp}$, that is

$$(\Delta_s b_k, c_j)_\Gamma = 0, \tag{3.48}$$

for all $j,k = 0,\cdots,r-1$. The remaining terms in I_2 are all one order of ε lower, and from the bounds (3.39) as well as the estimates on Δ_G, we find that $I_{2k} = O(\varepsilon)$. A similar estimate on I_{3k} follows since Δ_G commutes with L_J at leading order, while the estimates on I_{4k} follow from (6.23) and the bounds (3.39) on b_k. Returning these estimates to (3.45) gives the first bound of (3.38). □

Lemma 3.7. *Under the conditions of Theorem* 3.3, *there exists* $\varepsilon_0, C_1 > 0$ *such that for* $\varepsilon < \varepsilon_0$

$$B[\Psi_2, \Psi_2] \ge C_1\varepsilon^4\|\Psi_2\|_{H^2(\Omega)}^2 + 2q_+\varepsilon^2\|\Psi_2\|_{H^1(\Omega)}^2 + q_+^2\|\Psi_2\|_{L^2(\Omega)}^2. \tag{3.49}$$

Proof. We first observe that Ψ_2 is supported inside $\Omega\backslash\Gamma(\ell)$. We distribute the derivatives in the bilinear form and use Neumann boundary conditions to integrate by parts yielding

$$B[\Psi_2, \Psi_2] = \int_{\Omega\backslash\Gamma(\ell)} \left(\varepsilon^4|\Delta\Psi_2|^2dx + 2\varepsilon^2 q|\nabla\Psi_2|^2dx + 2\varepsilon^2(\nabla q\cdot\nabla\Psi_2)\Psi_2dx + q^2\Psi_2^2dx + \varepsilon\tilde{q}\Psi_2^2\right)dx. \tag{3.50}$$

In $\Omega\backslash\Gamma(\ell)$, $q > q_+$, and using Young's Inequality we obtain,

$$\begin{aligned} B[\Psi_2, \Psi_2] \;\ge\; & \varepsilon^4\|\Delta\Psi_2\|_{L^2(\Omega)}^2 + \varepsilon^2\Big(2q_+ - \varepsilon\|\nabla q\|_{L^\infty(\Omega\backslash\Gamma(\ell))}\Big)\|\nabla\Psi_2\|_{L^2(\Omega)}^2 \\ + \; & \Big(q_+^2 - \varepsilon\|\nabla q\|_{L^\infty(\Omega\backslash\Gamma(\ell))} - \varepsilon\|\tilde{q}\|_{L^\infty(\Omega\backslash\Gamma(\ell))}\Big)\|\Psi_2\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.51}$$

Elliptic regularity theory (see [15]) guarantees the existence of a constant C_1 independent of ε such that

$$\|\Delta\Psi_2\|^2_{L^2(\Omega)} \geq C_1\|\Psi_2\|^2_{H^2(\Omega)} - \|\Psi_2\|^2_{L^2(\Omega)}. \tag{3.52}$$

Returning this estimate to (3.51) and using (2.12) to bound q and $\tilde{q}$ yields (3.49) for ε sufficiently small. □

Proposition 3.8. *Under the conditions of Theorem* 3.3*, there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$ *and* $U > 0$*,*

$$\sum_{k=0}^{r-1} \|c_k\|^2_{L^2(\Gamma)} \leq \frac{\widetilde{U} + \varepsilon C}{U}, \tag{3.53}$$

where recall that $c_k \in S_U^\perp$*.*

Proof. Following Lemma 3.6, from (3.41) and Proposition 6.6 we see at leading order that

$$\mathcal{A}\Psi_{S^\perp} = \sum_{k=0}^{r-1}[(\lambda_k^0 - \varepsilon^2\Delta_s)c_k]\psi_k, \tag{3.54}$$

where $\Psi_{S^\perp}$ is as given in (2.39). Taking the $L^2(\Omega)$ norm of both sides, recalling the bound (3.33), changing the integral to the whiskered coordinates and using the $\|\cdot\|_J$ orthonormality of the $\{\psi_k\}$ we obtain

$$C\varepsilon + \widetilde{U} \geq \|\mathcal{A}\Psi_{S^\perp}\|^2_{L^2(\Omega} \geq \sum_{k=0}^{r-1}\int_\Gamma [(\lambda_k^0 - \varepsilon^2\Delta_s)c_k]^2\, J_0 ds - \varepsilon C \geq U\sum_{k=0}^{r-1}\|c_k\|^2_{L^2(\Gamma)} - \varepsilon C, \tag{3.55}$$

where in the last inequality we employed the lower bound (3.12) to $c_k \in S_U^\perp$. □

Proposition 3.9. *Under the conditions of Theorem* 3.3*, there exist* $\varepsilon_0, C > 0$ *such that for* $\varepsilon < \varepsilon_0$

$$\widetilde{U} + C\varepsilon \geq B[\Psi^\perp, \Psi^\perp] \geq \int_\Gamma (L_J\Psi^\perp, L_J\Psi^\perp)_J\, J_0 ds + \varepsilon^4 \int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_G\Psi^\perp)^2 J dz ds - C\varepsilon. \tag{3.56}$$

Proof. Since the support of $\Psi^\perp$ is contained in $\Gamma(2\ell)$ we use the whiskered coordinate form (3.1) of $\mathcal{A}$ to expand the bilinear form (3.22) as

$$B[\Psi^\perp, \Psi^\perp] = \int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left((L_J\Psi^\perp)^2 + \varepsilon\tilde{q}(\Psi^\perp)^2 + 2\varepsilon^2\left(-L_J\Psi^\perp\right)\Delta_G\Psi^\perp + \varepsilon^4(\Delta_G\Psi)^2\right) J dz ds. \tag{3.57}$$

We bound the second and third terms from below. The uniform bound on the potential $\tilde{q}$ readily imply

$$\int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \varepsilon\tilde{q}(\Psi^\perp)^2 J dz ds \geq -C\varepsilon. \tag{3.58}$$

Addressing the third term on the right-hand side we use (3.3) and (6.22) to expand L_J and Δ_G

$$\int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left(-L_J\Psi^\perp\right)\Delta_G\Psi^\perp J dz ds = \int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon}\sum_{i,j=1}^{d}\left(\left(\frac{\partial}{\partial z}\tilde{J}\frac{\partial\Psi^\perp}{\partial z}\right)\tilde{J}^{-1} - q\Psi^\perp\right)\frac{\partial}{\partial s_i}\left(G^{ij}J\frac{\partial\Psi^\perp}{\partial s_j}\right)dz ds, \tag{3.59}$$

where G^{ij} are the components of the inverse of the metric tensor $\mathbf{G}$. We recall that $\Psi^\perp$ is localized on Γ and Γ is closed, we integrate by parts in z and s, obtaining up to exponentially small terms

$$\begin{aligned}\int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left(-L_J\Psi^\perp\right)\Delta_G\Psi^\perp J dz ds \quad &= \quad B_G[\Psi^\perp, \Psi^\perp] + \int_\Gamma\int_{-2\ell/\varepsilon}^{2\ell/\varepsilon}\sum_{i,j}^{d-1}\Big[(\partial_z G^{ij})\frac{\partial^2\Psi^\perp}{\partial z\partial s_i}\frac{\partial\Psi^\perp}{\partial s_j} + \\ &\qquad -\partial_{s_i}\left(\frac{J_z}{J}\right)G^{ij}\frac{\partial\Psi^\perp}{\partial z}\frac{\partial\Psi^\perp}{\partial s_j} + \frac{\partial q}{\partial s_i}\Psi^\perp\frac{\partial\Psi^\perp}{\partial s_j}G^{ij}\Big] J dz ds \\ &,\end{aligned} \tag{3.60}$$

where the B_G bilinear form is given by

$$B_G[u,v] := \int_\Gamma \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \sum_{i,j} \Big(\frac{\partial^2 u}{\partial z \partial s_i} \frac{\partial^2 v}{\partial z \partial s_j} + q \frac{\partial u}{\partial s_j} \frac{\partial v}{\partial s_i} \Big) G^{ij} J dz ds. \tag{3.61}$$

The estimates (6.40) on the metric tensor $\mathbf{G}$, the Jacobian J, and (3.26) on $\Psi^\perp$ show that only the B_G term is leading order, in particular

$$2\varepsilon^2 \int_\Gamma \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left(-L_J \Psi^\perp\right) \Delta_G \Psi^\perp J dz ds \geq 2\varepsilon^2 B_G[\Psi^\perp, \Psi^\perp] - C\varepsilon. \tag{3.62}$$

To continue, from (6.41) we expand G^{ij} in terms of the entries ϑ_{ij} of the inverse Jacobian, and rewrite B_G as

$$B_G[\Psi^\perp] = \sum_m^{d-1} \int_\Gamma \Big[(L_J R_m, R_m)_J - \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \left(\left(\sum_{i=1}^{d-1} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi^\perp}{\partial s_i} \right)^2 - 2 \sum_{i,j=1}^{d-1} \vartheta_{mj} \frac{\partial \vartheta_{mi}}{\partial z} \frac{\partial \Psi^\perp}{\partial s_i} \frac{\partial^2 \Psi^\perp}{\partial z \partial s_j} \right) \tilde{J} dz \Big] J_0 ds, \tag{3.63}$$

where we have introduced

$$R_m(s,z) := \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi^\perp}{\partial s_i}. \tag{3.64}$$

From the L^∞ estimates (6.42) on ϑ and the L^2 estimates (3.27) on $\Psi^\perp$ we see that the middle term is $O(1)$, while the third term is $O(\varepsilon^{-1})$. For the first term we decompose R_m as

$$R_m = \sum_{k=0}^{r-1} b_{mk}(s) \psi_k(z) + R_m^\perp, \tag{3.65}$$

where $R_m^\perp$ is $\tilde{J}$-orthogonal to Σ_J^r, as defined in Proposition 3.2, and $b_{km}(s)$ is the projection of $R_m(s,z)$ onto ψ_k. We observe from (3.7) that

$$\int_\Gamma (L_J R_m, R_m)_J \, J_0 ds \geq \sum_{k=0}^{r-1} \int_\Gamma (\lambda_0^k - C\varepsilon) b_{mk}^2 \, J_0 ds + \nu_0 \|R_m^\perp\|_{L^2(\Gamma(2\ell)}^2 \geq -(\underline{\lambda} + C\varepsilon) \sum_{k=0}^{r-1} ||b_{mk}||_{L^2(\Gamma)}^2, \tag{3.66}$$

where $\nu_0 > 1$ from Assumption **A1** is the lower bound on the spectrum of L_0 above λ_r^0 and $-\underline{\lambda} < 0$ is the $O(1)$ lower bound on the spectrum of L_0. It remains to bound b_{mk},

$$\begin{aligned} ||b_{mk}||_{L^2(\Gamma)}^2 &= \int_\Gamma (R_m, \psi_k)_J^2 \, J_0 ds = \int_\Gamma \left(\sum_{i=1}^{d-1} \left(\vartheta_{mi} \frac{\partial \Psi^\perp}{\partial s_i}, \psi_k \right)_J \right)^2 J_0 ds \\ &\leq C \int_\Gamma \sum_{i=1}^{d-1} (\vartheta_{mi}^0(s))^2 \left(\frac{\partial \Psi^\perp}{\partial s_i}, \psi_k \right)_J^2 J_0 ds + O(1), \end{aligned} \tag{3.67}$$

where we used (6.42) and the exponential decay of ψ_k to bound the ϑ_{mi}^1 integral. Substituting the normalization $\psi_k = \tilde{J}^{-1/2}(s,z)\psi_k^0(z)$ into the equality $(\Psi^\perp, \psi_k)_J = 0$, and taking the s_i derivative yields the expression

$$\left(\frac{\partial \Psi^\perp}{\partial s_i}, \psi_k \right)_J = -\frac{1}{2} \left(\Psi^\perp, \psi_k \frac{\tilde{J}_{s_i}}{\tilde{J}} \right)_J. \tag{3.68}$$

However from (6.29) $\tilde{J}_{s_i}$ is $O(\varepsilon)$ compared to $\tilde{J}$, and we deduce that

$$||b_{km}||_{L^2(\Gamma)}^2 \leq C \|\Psi^\perp\|_{L^2(\Omega)}^2 \sum_{i=1}^{d-1} \int_\Gamma \|\vartheta_{mi}^0\|_{L^\infty}^2 \left\| \psi_k \frac{J_{s_i}}{J} \right\|_J^2 J_0 ds + O(1) \leq O(1). \tag{3.69}$$

Returning this estimate to (3.66) shows that

$$\int_\Gamma (L_J R, R)_J ds \geq -C, \tag{3.70}$$

and hence

$$B_G[\Psi^\perp, \Psi^\perp] \geq -C\varepsilon^{-1}. \tag{3.71}$$

Combining (3.71) with (3.62) and (3.58), permits us to bound each of the possibly negative terms of (3.57) from below, yielding the second inequality of (3.56). The first inequality of (3.56) is a consequence of (3.33). □

Proof of Theorem 3.3. We note that (3.33) of Proposition 3.5 implies the bound (3.24) on Ψ_2. Next, observe that $\Psi^\perp$ is $\tilde{J}$-orthogonal to Σ_J^r, as defined in Proposition 3.2. Applying Proposition 3.2 to $\Psi^\perp$ yields

$$(L_J\Psi^\perp, \Psi^\perp)_J \geq (\nu_0 - C\varepsilon)\|\Psi^\perp\|_J^2, \tag{3.72}$$

where $\nu_0 > 1$ from Assumption **A1** is the lower bound on the spectrum of L_0 above λ_r^0. Applying the Cauchy inequality to the right-hand side of (3.72) and dividing both sides by $\|\Psi^\perp\|_J$, gives

$$\|\Psi^\perp\|_J \leq \frac{\|L_J\Psi^\perp\|_J}{\nu_0 - C\varepsilon}, \tag{3.73}$$

and the bound on $\Psi^\perp$ in (3.24) follows by squaring both sides of (3.73), integrating over Γ and using the bound (3.56) on $(L_J\Psi^\perp, L_J\Psi^\perp)_J$,

$$\|\Psi^\perp\|^2_{L^2(\Gamma(2\ell))} = \int_\Gamma \|\Psi^\perp\|_J^2 \, J_0 ds \leq \frac{1}{(\nu_0 - C\varepsilon)^2}\int_\Gamma (L_J\Psi^\perp, L_J\Psi^\perp)_J \, J_0 ds \leq \frac{\widetilde{U} + C\varepsilon}{(\nu_0 - C\varepsilon)^2}. \tag{3.74}$$

□

4. Coercivity of the Bilinear Form

We establish the coercivity results of Theorem 2.2 and 2.4 for the unequal-depth and asymptotically equal-depth wells.

4.1. The case of unequal depth wells

As described in section 2.3, decomposing $w \in \mathcal{Z}_U^\perp$ as $w = w_X + w_{X^\perp}$, where $w_X \in X_{\widetilde{U}}$ and $w_{X^\perp} \in X_{\widetilde{U}}^\perp$, we first characterize the angle between $\mathcal{Z}_U$ and $X_{\widetilde{U}}$ via the value of $\bar{\alpha}$ in the following proposition.

Proposition 4.1. *Fix $U_0 > 0$ independent of ε and let $U \in (0, U_0)$. Then, there exist $\widetilde{U} > 0$, independent of $\varepsilon \in (0, \varepsilon_0)$, such that $X_{\widetilde{U}}^\perp$ lies within a cone of aperture $\bar{\alpha} \in (0,1)$, independent of ε, about $\mathcal{Z}_U^\perp$. That is*

$$\|w_{X^\perp}\|_{L^2(\Omega)} \geq \bar{\alpha}\|w_X\|_{L^2(\Omega)}, \tag{4.1}$$

for all $w \in \mathcal{Z}_U^\perp$.

Proof. As discussed in section 2.3, it is equivalent to show (2.37) for $w \in Z_U^\perp$ satisfying $\|w\|_{L^2(\Omega)} = 1$. Setting $\Psi := w_X/\|w_X\|_{L^2(\Omega)}$, we use the decomposition of Theorem 3.3 together with the identities $\|w_X\|_{L^2(\Omega)} = (w, \Psi)_{L^2(\Omega)}$ and $\Psi = \Psi_1 + \Psi_2 = \eta_1\Psi_1 + (1+\eta_1)\Psi_2$, where η_1 is the cutoff function of Definition 2.2, to obtain

$$\|w_X\|_{L^2(\Omega)} = \Big(w, \sum_{k=0}^{r-1}\eta_1 b_k\psi_k\Big)_{L^2(\Gamma(2\ell))} + \Big(w, \sum_{k=0}^{r-1}\eta_1 c_k\psi_k + \eta_1\Psi^\perp + (1+\eta_1)\Psi_2\Big)_{L^2(\Omega)}, \tag{4.2}$$

Since $w \in \mathcal{Z}_U$ we may eliminate the first term we obtain

$$\|w_X\|_{L^2(\Omega)} = \Big(w, \sum_{k=0}^{r-1}\eta_1 c_k\psi_k\Big)_{L^2(\Gamma(2\ell))} + \Big(w, \eta_1\Psi^\perp\Big)_{L^2(\Gamma(2\ell))} + \Big(w, \Psi_2\Big)_{L^2(\Omega)} + O(\varepsilon). \tag{4.3}$$

Estimates (3.24) and (3.25) on c_k, $\Psi^\perp$, and Ψ_2 give

$$\|w_X\|_{L^2(\Omega)} \leq \sqrt{\frac{r(\widetilde{U}+\varepsilon C)}{U}} + \frac{\sqrt{\widetilde{U}}}{\nu_0} + \frac{\sqrt{2\widetilde{U}}}{q_+} + C\varepsilon =: \alpha, \tag{4.4}$$

and the result of the proposition follows for $\widetilde{U}$ sufficiently small compared to ν_0 and q_+ and U sufficiently large. □

Proof of Theorem 2.5**.** The lower bound on the eigenvalue μ_0 follows readily from

$$\mu_0 = \inf_{\|u\|_{L^2(\Omega)}=1} B[u,u] = \inf_{\|u\|_{L^2(\Omega)}=1} [(\mathcal{A}u,\mathcal{A}u) + \varepsilon(\tilde{q}u,u)] \geq -\varepsilon\|\tilde{q}\|_{L^\infty(\Omega)} \geq -C\varepsilon. \tag{4.5}$$

Proposition 4.1 gives the cone condition (2.35), which used in (2.34) gives (2.23) for ε sufficiently small. To obtain the remaining coercivity estimates note that using the definition of B in (3.22)

$$\|\mathcal{A}w\|^2_{L^2(\Omega)} = B[w,w] - \varepsilon(\tilde{q}w,w), \tag{4.6}$$

and the L^2 bound in (2.24) follows from (2.23) and $\|\tilde{q}\|_{L^\infty(\Omega)} = O(1)$. The inequality

$$\|\mathcal{A}^2 w\|_{L^2(\Omega)}\|w\|_{L^2(\Omega)} \geq (\mathcal{A}^2 w, w) = \|\mathcal{A}w\|^2_{L^2(\Omega)}, \tag{4.7}$$

implies (2.25). To obtain the H^2 coercivity in (2.25) we write Δw in terms of $\mathcal{A}w$

$$\Delta w = -\frac{1}{\varepsilon^2}\mathcal{A}w + \frac{1}{\varepsilon^2}qw, \tag{4.8}$$

and use elliptic regularity (see [15]) to obtain

$$\|w\|_{H^2(\Omega)} \leq C\left(\|w\|_{L^2(\Omega)} + \|\Delta w\|_{L^2(\Omega)}\right) \leq C\Big(\|w\|_{L^2(\Omega)} + \varepsilon^{-2}\|\mathcal{A}w\| + \varepsilon^{-2}\|q\|_\infty\|w\|_{L^2(\Omega)}\Big). \tag{4.9}$$

The H^2 coercivity in (2.24) now follows from the L^2 coercivity in (2.24). □

4.2. Modifications for the case of Asymptotically Equal-Depth Wells

For asymptotically equal-depth wells, the potential q_0 in the definition of the 1D operator L_0 (see (2.8)) depends on the position, s, along the interface. However the negative eigenvalues of the operator L_0 can only be asymptotically small. Indeed, below an $O(1)$ cut-off, ν_0, the each operator has r point eigenvalues: $\lambda_0(s), \ldots, \lambda_{r-1}(s)$ that are $O(\varepsilon)$ uniformly in s.

The tighter bounds on the small spectrum of L_0 are required to control the $B[\Psi_S, \Psi_{S^\perp}]$ term in Lemma 3.6, however they are also natural in light of the motivating example. In particular, due to additional s dependence of eigenfunctions ψ_k of L_0, we can no longer bound the I_{2k} of (3.47) by $O(\varepsilon)$. However the additional assumptions open a more direct approach which eliminates the need for Lemma 3.6 altogether. Indeed, we may initially replace the decomposition of $\Psi = w_X/\|w_X\|_{L^2(\Omega)}$ in (2.39) with a simpler one. Writing $\Psi = \Psi_1 + \Psi_2$ where Ψ_1 is supported inside $\Gamma(2\ell)$, Ψ_2 is supported outside $\Gamma(\ell)$ we merely break Ψ_1 into two terms

$$\Psi_1(s,z) = \sum_{k=0}^{r-1} a_k(s)\psi_k(s,z) + \Psi^\perp(x), \tag{4.10}$$

where $a_k \in H^2(\Gamma)$ is the full projection of Ψ_1 onto ψ_k, i.e.

$$a_k := (\Psi_1, \psi_k)_J, \tag{4.11}$$

in particular its Laplace-Beltrami modes are not limited. Thus we have $(\Psi^\perp, \psi_k)_J(s) = 0$ for $k = 0, \ldots r-1$ and all $s \in \mathbf{S}$. In the context of Figure 2, we do not yet project Ψ_1 onto the function-spaces corresponding to $\Lambda_{k,j}$ which lie inside and outside of the dashed-blue boxes – both can be treated together.

In this case, Proposition 3.9 holds with $\Psi^\perp$ replaced with Ψ_1. The lower bound on $(L_J R_m, R_m)_J$ in (3.63) (for which $R_m(s,z) := \sum_{i=1}^{d-1} \vartheta_{mi} \frac{\partial \Psi_1}{\partial s_i}$) follows immediately from Proposition 3.2 since the spectrum of L_0 is bounded from below by $O(\varepsilon)$ in Assumption **A2**. As a result of Proposition 3.9, we obtain,

$$\int_\Gamma (L_J\Psi_1, \Psi_1)_J\, J_0 ds + \varepsilon^4 \int_\Gamma \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} (\Delta_G \Psi_1)^2 J dz ds \leq \widetilde{U} + C\varepsilon. \tag{4.12}$$

To control the component of Ψ_1 belonging to $Z_U^\perp$, at this point, we further decompose Ψ_1 into its projection onto Z_U and the orthogonal complement as in (2.39), yielding

$$\sum_{k=0}^{r-1} \int_\Gamma \lambda_k^2(s) b_k^2(s)\, J_0 ds + \sum_{k=0}^{r-1} \int_\Gamma \lambda_k^2(s) c_k^2(s)\, J_0 ds + \int_\Gamma (L_J \Psi^\perp, \Psi^\perp)_J\, J_0 ds \leq \widetilde{U} + C\varepsilon. \tag{4.13}$$

Following the lines of argument in the proof of Theorem 3.3 gives the bounds on $\Psi^\perp$ in (3.24). The control of c_k follows from the spectral bound in (2.31) as for $c_k \in S_U^\perp$

$$\|c_k\|_{L^2(\Gamma)}^2 \leq \frac{1}{U}\varepsilon^4 \|\Delta_s c_k\|^2 \leq \frac{\varepsilon^4}{U}\|\Delta_G \Psi_1\|_{L^2}^2 + O(\varepsilon) \leq \frac{\widetilde{U} + C\varepsilon}{U}. \tag{4.14}$$

This completes the proof of Proposition 3.9. Proposition 4.1 and Theorem 2.5 follow from Proposition 3.9 without any modifications in the proofs.

5. Approximately Invariant Spaces and Resolvent Estimates for L

For an admissible Γ, we have shown that the functionalized operator, $\mathcal{L}$ is coercive off of the space $\mathcal{Z}_U$. We take the gradient operator $\mathcal{G} = \Pi_0$ for which the full operator $\mathbf{L} = \Pi_0 \mathcal{L}$ is not self-adjoint; however, we can symmetrize $\mathbf{L}$. We define the space

$$\mathcal{Y}_U := \text{span}\{\mathcal{Z}_U, Y_0\}, \tag{5.1}$$

where we introduce the element

$$Y_0 := n_0 \mathcal{L} 1 = n_0 \left(q^2 - 2\varepsilon^2 \Delta q + \varepsilon \tilde{q}\right), \tag{5.2}$$

whose normalization $n_0 \in \mathbb{R}$ is chosen to render Y_0 of unit norm in $L^2(\Omega)$. In particular from the form (2.28) of $\mathcal{L}$ we see that up to exponentially small terms $\mathcal{L}1 = 4q_+^2$ off of Γ_ε and hence $n_0 = 1/(4q_+^2\sqrt{|\Omega|}) + O(\varepsilon)$ and

$$Y_0 = \frac{1}{\sqrt{|\Omega|}} + O(\varepsilon|\Gamma|), \tag{5.3}$$

in $L^2(\Omega)$. We introduce Π_U, the L^2-orthogonal projection onto $\mathcal{Y}_U$, and its compliment, $\widetilde{\Pi}_U := I - \Pi_U$. The key observation is that $\mathbf{L}\widetilde{\Pi}_U = \Pi_0 \mathcal{L} \widetilde{\Pi}_U = \mathcal{L}\widetilde{\Pi}_U$. Indeed,

$$\Pi_0 \mathcal{L}\widetilde{\Pi}_U w = \mathcal{L}\widetilde{\Pi}_U w - |\Omega|^{-1}\left(\mathcal{L}\widetilde{\Pi}_U w, 1\right)_{L^2} = \mathcal{L}\widetilde{\Pi}_U w - n_0^{-1}|\Omega|^{-1}\left(w, \widetilde{\Pi}_U Y_0\right)_{L^2} = \mathcal{L}\widetilde{\Pi}_U w, \tag{5.4}$$

since $Y_0 \in \ker(\widetilde{\Pi}_U)$. In particular the operator $\widetilde{\Pi}_U \mathbf{L} \widetilde{\Pi}_U = \widetilde{\Pi}_U \mathcal{L} \widetilde{\Pi}_U$ is self adjoint.

The projection of $w \in L^2(\Omega)$ onto $\mathcal{Y}_U$ can be written as

$$\Pi_U w := \alpha_0 Y_0 + \sum_{jk} \alpha_{jk} Z_{jk}, \tag{5.5}$$

where the coefficients are chosen so that $\widetilde{\Pi}_U w$ is orthogonal to $\mathcal{Y}_U$. This amounts to satisfying the equations

$$(w, Z_{j'k'})_{L^2(\Omega)} = \alpha_0 (Y_0, Z_{j'k'})_{L^2(\Omega)} + \sum_{jk} \alpha_{jk} (Z_{jk}, Z_{j'k'})_{L^2(\Omega)}, \tag{5.6}$$

for $j' = m(k), \cdots, M(k)$ and $k' = 0, \cdots, r-1$, in addition to

$$(w, Y_0)_{L^2(\Omega)} = \alpha_0 (Y_0, Y_0)_{L^2(\Omega)} + \sum_{jk} \alpha_{jk} (Z_{jk}, Y_0^\dagger)_{L^2(\Omega)}. \tag{5.7}$$

Lemma 5.1. *If $\Gamma \subset \mathbb{R}^d$ is admissible and $U_0 > 0$ is given by Theorem 2.5 (or Theorem 2.7), then for all $U < U_0$ the projection Π_U defined in (5.5) is well posed and there exist $C_\pm > 0$, independent of ε, such that the coefficients satisfy*

$$C_- \|\Pi_U w\|^2_{L^2(\Omega)} \le |\alpha_0|^2 + \sum_{k=0}^{r-1} \sum_{j=m(k)}^{M(k)} |\alpha_{jk}|^2 \le C_+ \|\Pi_U w\|^2_{L^2(\Omega)}. \tag{5.8}$$

Proof. The well-posedness of the decomposition (5.5) is equivalent to the invertibility of the symmetric matrix $\mathbf{M}$ with entries

$$\mathbf{M}_{n,n'} := (Z_{jk}, Z_{j'k'})_{L^2(\Omega)} = \int_\Gamma \Theta_j \Theta_{j'}(s) J_0(s) ds \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \psi_k^0(z) \psi_{k'}^0(z)\, dz + O(e^{-\mu\ell/\varepsilon}), \tag{5.9}$$

$$\mathbf{M}_{n,0} = \mathbf{M}_{0,n} := (Z_{jk}, Y_0)_{L^2(\Omega)} = n_0 \sqrt{\varepsilon} \int_\Gamma \Theta_j J_0(s) ds \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \mathcal{L}1(s,z) \psi_k^0(z)\, dz + O(e^{3/2}), \tag{5.10}$$

and $\mathbf{M}_{0,0} := (Y_0, Y_0)_{L^2(\Omega)} = 1$. Here the indices $n = n(j,k)$ and $n' = n'(j',k')$ denote the enumeration of Z_{jk} and $Z_{j'k'}$ in a linear ordering of the basis of $\mathcal{Y}_U$ which start with Y_0 and run over $k = 0, \cdots, r-1$ and $j = m(k), \cdots, M(k)$. In particular, since the Laplace-Beltrami eigenmodes are J_0-orthonormal while the eigenfunctions $\{\psi_k^0\}_{k=0}^{r-1}$ are $L^2(\mathbb{R})$ orthonormal we see that $\mathbf{M}_{n,n'} = \delta_{n,n'} + O(e^{-\mu\ell/\varepsilon})$. In particular, up to exponentially small terms, $\det \mathbf{M} = 1 - |\mathbf{M}_0|^2$ where $\mathbf{M}_0 = (\mathbf{M}_{1,0}, \cdots, \mathbf{M}_{N,0})^t$ comprises the first row of $\mathbf{M}$, less the $\mathbf{M}_{0,0}$ entry. For $k = 0, \cdots, r-1$ we introduce

$$\rho_k(s) := \int_{-2\ell/\varepsilon}^{2\ell/\varepsilon} \mathcal{L}1(s,z) \psi_k^0(z)\, dz,$$

and observe that

$$M_{n(jk),0} = n_0 \sqrt{\varepsilon} \langle \Theta_j, \rho_k \rangle_\Gamma.$$

Since $\{\Theta_j\}_{j=0}^\infty$ is an orthonormal basis for $L^2(\Gamma)$ with this inner product, it follows that

$$\sum_{j=m(k)}^{M(k)} M^2_{n(jk),0} \le \sum_{j=0}^{\infty} M^2_{n(jk),0} = n_0^2 \varepsilon \|\rho_k\|^2_{L^2(\Gamma)}.$$

In particular

$$\det \mathbf{M} = 1 - n_0^2 \varepsilon \sum_{k=0}^{r-1} \|\rho_k\|^2_{L^2(\Gamma)} = 1 + O(\varepsilon).$$

Indeed, $\mathbf{M}$ is a diagonally dominant perturbation of the $N \times N$ identity, and is bounded and boundedly invertible, uniformly in $\varepsilon < \varepsilon_0$; this establishes (5.8). □

The utility of the space $\mathcal{Y}_U$ as a decomposition of the flow of the full PDE is that it is approximately invariant under the action of $\mathbf{L} = \Pi_0 \mathcal{L}$, whose bilinear form is uniformly coercive when restricted to $\mathcal{Y}^\perp$. The following results extend the coercivity of $\mathcal{L}$ to $\mathbf{L}$ in a meaningful way.

Theorem 5.2. *Consider the operator $\mathbf{L} := \Pi_0 \mathcal{L}$ where Γ is admissible and U_0 is as given in Thoerem 2.5 (or Theorem 2.7), then for all $U < U_0$ there exists $C > 0$, independent of ε, such that for all $w \in \mathcal{Y}_U$, $w^\perp \in \mathcal{Y}_U^\perp$ we have the bounds*

$$\|\widetilde{\Pi}_U \mathbf{L} w\|_{L^2(\Omega)} \le C\varepsilon \|w\|_{L^2(\Omega)}, \tag{5.11}$$

$$\|\Pi_U \mathbf{L} w^\perp\|_{L^2(\Omega)} \le C\varepsilon \|w^\perp\|_{L^2(\Omega)}. \tag{5.12}$$

Moreover given $C_\alpha \in (0, U_0/2)$ then for all $\varepsilon < \varepsilon_0$ and all $U \in (C_\alpha, U_0)$ the operator $\mathbf{L}_U := \widetilde{\Pi}_U \mathbf{L} \widetilde{\Pi}_U : \mathcal{Y}_U^\perp \mapsto \mathcal{Y}_U^\perp$ has no spectrum in the set $\{\Re\lambda < C_\alpha\}$, and for λ on this set the resolvent satisfies the bound

$$\|(\mathbf{L}_U - \lambda)^{-1} w^\perp\|_{L^2(\Omega)} \le \frac{C}{C_\alpha - \Re\lambda} \|w^\perp\|_{L^2(\Omega)}, \tag{5.13}$$

for all $w^\perp \in \mathcal{Y}_U^\perp$, where C is independent of the choice of ε and U.

Proof. To establish (5.11) we remark that any $w \in \mathcal{Y}_U$ can be represented as in (5.5) where the coefficients satisfy (5.8). We first observe that $\widetilde{\Pi}_U u$ is $L^2(\Omega)$ orthogonal to Y_0 for any $u \in L^2(\Omega)$. Since $\|Y_0 - 4q_+^2\|_{L^2(\Omega)} = O(\varepsilon)$, there exists $C > 0$ such that

$$\|\widetilde{\Pi}_U (I - \Pi_0)\mathcal{L}u\|_{L^2(\Omega)} \leq C\varepsilon \left|(\mathcal{L}u, 1)_{L^2(\Omega)}\right| \leq C\varepsilon\|u\|_{L^2(\Omega)}. \tag{5.14}$$

It is thus sufficient to establish (5.11) for $\widetilde{\Pi}_U\mathcal{L}$. To this end we follow the proof of Lemma 3.6; using the expression (3.5) for $\mathcal{L}$ we write its action on $Z_{jk} = \Theta_j\psi_k$ as

$$\mathcal{L}Z_{jk} = L\Theta_j\psi_k - \varepsilon^2\Delta_G L_J\Theta_j\psi_k - \varepsilon^2 L_J\Delta_G\Theta_j\psi_k + \varepsilon^4\Delta_G^2\Theta_j\psi_k. \tag{5.15}$$

However $\tilde{J} = \varepsilon + O(\varepsilon^2)$ over the support of ψ_k so that $\psi_k = \varepsilon^{-1/2}\psi_k^0 + O(\varepsilon)$ in $H^4(\mathbf{R})$. In particular from (3.4) we have

$$L\Theta_j\psi_k = \lambda_k^2\Theta_j\psi_k + O(\varepsilon), \tag{5.16}$$

and

$$L_J\Theta_j\psi_k = \lambda_k\Theta_j\psi_k + O(\varepsilon). \tag{5.17}$$

Similarly from (6.23) we see that

$$\Delta_G\Theta_j\psi_k = \beta_j\Theta_j\psi_k + O(\varepsilon\|\Theta_j\|_{H^2(\Gamma)}). \tag{5.18}$$

The projection $\widetilde{\Pi}_U$ eliminates the leading order terms in (5.16)-(5.18) and we deduce that

$$\|\widetilde{\Pi}_U\mathcal{L}Z_{jk}\|_{L^2(\Omega)} \leq C\left(\varepsilon + \varepsilon^3\|\Theta_j\|_{H^2(\Gamma)} + \varepsilon^5\|\Theta_j\|_{H^4(\Gamma)}\right). \tag{5.19}$$

From the Laplace-Beltrami eigenvalue equation for Θ_j, we deduce that

$$\int_\Gamma \left((\nabla_s\Theta_j)^t\mathbf{g}^{-1}\nabla_s\Theta_j\right) J_0(s)ds = \beta_j\int_\Gamma \Theta_j^2 J_0(s)ds = \beta_j.$$

Since $\mathbf{g}^{-1}$ is uniformly elliptic, and $J_0 > 0$ is bounded from below we have the estimate

$$\|\Theta_j\|_{H^1(\Gamma)} \leq C\sqrt{\beta_j}.$$

From the Weyl asymptotics for eigenvalues of the Laplace Beltrami operator, [9], there exists $c_\pm > 0$ such that

$$c_+ j^{2/(d-1)} \geq \beta_j \geq c_- j^{2/(d-1)}. \tag{5.20}$$

In particular we see that $j \leq M(k) \leq CU^{(d-1)/4}\varepsilon^{-(d-1)}$ so that $\beta_j \leq C\sqrt{U}\varepsilon^{-2}$. This argument, and similar ones for higher order derivatives show that $\|\Theta_j\|_{H^l(\Gamma)} \leq C\varepsilon^l$, for $l = 1, \cdots, 4$. In conjunction with (5.19) these results show that

$$\|\widetilde{\Pi}_U\mathcal{L}Z_{jk}\|_{L^2(\Omega)} \leq C\varepsilon. \tag{5.21}$$

It remains to bound $\widetilde{\Pi}_U\mathcal{L}Y_0 = n_0\widetilde{\Pi}_U\mathcal{L}^2 1$. However from the bounds (2.12) we see that $\mathcal{L}^2 1 = 16q_+^4 + O(\varepsilon)$ in $L^2(\Omega)$ and since Y_0 is also constant to $O(\varepsilon)$ we have

$$||\widetilde{\Pi}_U\mathcal{L}Y_0||_{L^2(\Omega)} \leq C\varepsilon. \tag{5.22}$$

Together with (5.8), the bounds (5.21)-(5.22) yield (5.11). To establish (5.12) we recall that $\Pi_0\mathcal{L}w^\perp = \mathcal{L}w^\perp$ and remark that

$$\|\Pi_U\mathcal{L}w^\perp\| = \sup_{w\in\mathcal{Y}_U}\left(w^\perp, \widetilde{\Pi}_U\mathcal{L}w\right)_2/\|w\|_{L^2} \leq C\varepsilon\|w^\perp\|_{L^2(\Omega)}, \tag{5.23}$$

where we used (5.11) to bound the last term.

Finally, since $\mathbf{L}_U = \widetilde{\Pi}_U\mathbf{L}\widetilde{\Pi}_U = \widetilde{\Pi}_U\mathcal{L}\widetilde{\Pi}_U$ we see that $\mathbf{L}_U$ is self-adjoint, and deduce from (2.23) of Theorem 2.5 (or Theorem 2.7) that $\sigma(\mathbf{L}_U)$ lies to the right of C_α. Since the resolvent is bounded by the inverse distance to the spectrum, the estimate (5.13) follows. □

6. Appendix: Whiskered Coordinates and Bounds on the Fundamental Forms

This section lays out the necessary framework on the differential geometry of co-dimension one interfaces in $\mathbb{R}^d$ required for the proof of main results.

6.1. Weingarten Map and Fundamental Forms in Local Coordinates

To make the presentation self-contained we first summarize some definitions from differential geometry, further background can be found in [31], for example. Let Γ be a $d-1$ dimensional smooth manifold embedded in $\mathbb{R}^d$ with a chosen orientation. Let $\mathbf{S}$ be an open set in $\mathbb{R}^{d-1}$ and $\gamma : \mathbf{S} \to \Gamma$ be a local parametrization. Then the tangent space $T_{\gamma(s)}\Gamma$ is defined as the image of $T_s\mathbf{S} \cong \mathbb{R}^{d-1}$ under the map $D\gamma|_s$. Denote the unit sphere, $\|x\| = 1$, in $\mathbb{R}^d$ by $\mathbf{S}^{d-1}$ and let $\{e_i\}_{i=1}^d$ denote the canonical basis of $\mathbf{R}^d$.

Definition 6.1. *The Gauss map,* $\nu : \mathbf{S} \to \mathbf{S}^{d-1}$ *maps points of* $\mathbf{S}$ *into unit normal vectors* $\nu(s)$ *orthogonal to* $T_{\gamma(s)}\Gamma$*.*

Definition 6.2. *The Weingarten map,* $A : T_{\gamma(s)}\Gamma \to T_{\gamma(s)}\Gamma$ *is defined by*

$$A = -D\nu \circ (D\gamma)^{-1}. \tag{6.1}$$

Remark 6.3. *The map* $D\nu : T_s\mathbf{S} \cong \mathbb{R}^{d-1} \to T_{\gamma(s)}\Gamma$ *since* $\frac{\partial \nu}{\partial s_i} = (D\nu)e_i$ *belongs to the tangent space* $T_{\gamma(s)}\Gamma$*, as can be verified by differentiating* $\nu(s) \cdot \nu(s) = 1$ *with respect to* s_i*.*

Definition 6.4. *For* $X, Y \in T_{\gamma(s)}\Gamma$ *the k-th fundamental for of* Γ *is defined by*

$$\left\langle A^{k-1}X, Y \right\rangle, \tag{6.2}$$

where the brackets represent the euclidean inner product in $\mathbb{R}^d$*.*

Definition 6.5. *The principal curvatures* $k_1, \ldots, k_{d-1}$ *of* Γ *are defined as the eigenvalues of the Weingarten map* A*.*

From the definitions of $D\gamma$ and $D\nu$ we have

$$(D\gamma)e_i = \frac{\partial \gamma}{\partial s_i}, \tag{6.3}$$

and

$$(D\nu)e_i = \frac{\partial \nu}{\partial s_i}. \tag{6.4}$$

It follows from the definition of Γ that $D\gamma$ is full-rank and so the vectors $\left\{ \frac{\partial \gamma}{\partial s_1}, \ldots \frac{\partial \gamma}{\partial s_{d-1}} \right\}$ form a basis for the tangent space $T_{\gamma(s)}\Gamma$. Assume that $X, Y \in T_{\gamma(s)}\Gamma$ are given by

$$X = \sum_{i=1}^{d-1} \xi^i \frac{\partial \gamma}{\partial s_i}, \; Y = \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j}. \tag{6.5}$$

Then,

$$\langle X, Y \rangle = \left\langle \sum_{i=1}^{d-1} \xi^i \frac{\partial \gamma}{\partial s_i}, \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j} \right\rangle = \sum_{i,j=1}^{d-1} \xi^i \eta^j g_{ij}, \tag{6.6}$$

where $g_{ij} := \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle$, is the representation of the first fundamental form in local coordinates. In addition, by definition, the Weingarten map $A : \frac{\partial \gamma}{\partial s_i} \to -\frac{\partial \nu}{\partial s_i}$, so that

$$\langle AX, Y \rangle = \left\langle \sum_{i=1}^{d-1} \xi^i A \frac{\partial \gamma}{\partial s_i}, \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j} \right\rangle = -\left\langle \sum_{i=1}^{d-1} \xi^i \frac{\partial \nu}{\partial s_i}, \sum_{j=1}^{d-1} \eta^j \frac{\partial \gamma}{\partial s_j} \right\rangle = \sum_{i,j=1}^{d-1} \xi^i \eta^j h_{ij}, \tag{6.7}$$

where

$$h_{ij} := -\left\langle \frac{\partial \nu}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle = \left\langle \nu, \frac{\partial^2 \gamma}{\partial s_i \partial s_j} \right\rangle, \tag{6.8}$$

is the representation of the second fundamental form in local coordinates. The last equality in (6.8) results from differentiating the equality $\left\langle \nu, \frac{\partial\gamma}{\partial s_i}\right\rangle = 0$ with respect to s_j. Similarly,

$$\langle A^2X, Y\rangle = \sum_{i,j=1}^{d-1} \xi^i \eta^j e_{ij}, \tag{6.9}$$

where $e_{ij} = \left\langle \frac{\partial\nu}{\partial s_i}, \frac{\partial\nu}{\partial s_j}\right\rangle = -\left\langle \nu, \frac{\partial^2\nu}{\partial s_i\partial s_j}\right\rangle$, is the representation of the third fundamental form. In addition if we write $\frac{\partial\nu}{\partial s_i}$ in the tangent basis as

$$\frac{\partial\nu}{\partial s_i} = -\sum_{j=1}^{d-1} \frac{\partial\gamma}{\partial s_j} h_i^j, \tag{6.10}$$

then referring to (6.8) we see that

$$h_{ij} = \sum_{j=1}^{d-1} \left\langle \frac{\partial\gamma}{\partial s_j} h_i^j, \frac{\partial\gamma}{\partial s_j}\right\rangle = \sum_{j=1}^{d-1} h_i^j g_{ij}, \text{ and } h_i^j = \sum_k h_{ik} g^{kj}, \tag{6.11}$$

where g^{kj} represents the elements of the inverse matrix of g_{ij}. The key relation (6.10) shows that h_i^j are the coefficients that express the linear variation of the normal vector in terms of the tangent basis. Moreover,

$$AX = \sum_{i=1}^{d-1} \xi^i A \frac{\partial\gamma}{\partial s_i} = -\sum_{i=1}^{d-1} \xi^i \frac{\partial\nu}{\partial s_i} = \sum_{i=1}^{d-1} \xi^i h_i^j \frac{\partial\gamma}{\partial s_i}, \tag{6.12}$$

so that $\{h_i^j\}$ is the matrix representation of the linear operator A in the basis $\left\{\frac{\partial\gamma}{\partial s_1}, \dots \frac{\partial\gamma}{\partial s_{d-1}}\right\}$, and k_i is the i^{th} eigenvalue of the matrix $\{h_i^j\}$. The matrix operator norm of h_i^j is then

$$\|\{h_i^j\}\| = \max_{1\le i\le d-1} k_i. \tag{6.13}$$

Finally, defining $e_i^j := \sum_k e_{ik} g^{kj}$, we use (6.10) to express e_i^j in terms of the Weingarten matrix h_i^j

$$\begin{aligned} e_i^j &= \sum_k e_{ik} g^{kj} = \sum_k \left\langle \frac{\partial\nu}{\partial s_i}, \frac{\partial\nu}{\partial s_k}\right\rangle g^{kj} = \sum_{k,l,m} h_i^l h_k^m \left\langle \frac{\partial\gamma}{\partial s_l}, \frac{\partial\gamma}{\partial s_m}\right\rangle g^{kj} \\ &= \sum_{k,l,m} h_i^l h_k^m g_{lm} g^{kj} = \sum_{k,l} h_i^l h_{kl} g^{kj} = \sum_l h_i^l h_l^j. \end{aligned} \tag{6.14}$$

That is $\{e_i^j\} = \{h_i^j\}^2$ and we bound the operator norm of $\{e_i^j\}$ by

$$\|\{e_i^j\}\| = \max_{1\le i\le d-1} k_i^2. \tag{6.15}$$

6.2. Whiskered Coordinates

We recall the whiskered coordinate system introduced in (2.1). We introduce the variables $y = (s_1, \dots, s_{d-1}, z)$ then $x = \varphi(y)$ and φ^{-1} defines a chart for $\Gamma(\ell)$. The gradient and the Laplace operator in $\mathbb{R}^d$ can be written in the y- coordinates as

$$\nabla_x^i = \sum_{j=1}^{d} G^{ij} \frac{\partial}{\partial y_j}, \ \ i = 1, \dots d \tag{6.16}$$

and

$$\Delta_x = \frac{1}{\sqrt{\det(\mathbf{G})}} \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial}{\partial y_i} G^{ij} \sqrt{\det(\mathbf{G})} \frac{\partial}{\partial y_j}, \tag{6.17}$$

where $\mathbf{G}$ is the metric tensor

$$G_{ij} = \left\langle \frac{\partial x}{\partial y_i}, \frac{\partial x}{\partial y_j}\right\rangle_{\mathbb{R}^d}, \tag{6.18}$$

and G^{ij} is the ij component of the inverse of the $\mathbf{G}$. Letting $\mathbf{J}$ denote the Jacobian matrix for $\varphi = \varphi(s,z)$, we have,

$$\mathbf{G} = \mathbf{J}^T\mathbf{J}, \tag{6.19}$$

and consequently, $\det(\mathbf{G}) = J^2$, where $J = J(s,z)$ is the associated Jacobian.

With this $\mathbf{G}$, in the whiskered variables the gradient and the Laplacian take the form,

$$\nabla_x^i = \sum_{j=1}^{d-1} G^{ij}\frac{\partial}{\partial s_j} \text{ for } i = 1,\dots,d-1, \quad \nabla_x^d = \varepsilon^{-2}\frac{\partial}{\partial z}, \tag{6.20}$$

and

$$\Delta_x = \varepsilon^{-2}J^{-1}\frac{\partial}{\partial z}J\frac{\partial}{\partial z} + \Delta_G, \tag{6.21}$$

where Δ_G is

$$\Delta_G := \frac{1}{\sqrt{\det(\mathbf{G})}}\sum_{i,j=1}^{d-1}\frac{\partial}{\partial s_i}G^{ij}\sqrt{\det(\mathbf{G})}\frac{\partial}{\partial s_j} = J^{-1}\sum_{i,j=1}^{d-1}\frac{\partial}{\partial s_i}G^{ij}J\frac{\partial}{\partial s_j}. \tag{6.22}$$

On the interface Γ, where $z = 0$, Δ_G reduces to the Laplace-Beltrami operator on Γ, defined in (2.16).

Proposition 6.6. *Fix $K > 0$ and $\Gamma \in \mathcal{G}_{K,l}$. Let G be the metric tensor for the whiskered coordinates defined in (6.18). Let Δ_G be the laplacian in whiskered coordinates given in (6.22) and let Δ_s be the Laplace-Beltrami operator on the interface Γ. Then the following relationship holds between the operators Δ_G and Δ_s*

$$\Delta_G = \Delta_s + \varepsilon z D_{s,2}, \tag{6.23}$$

where

$$D_{s,2} := \sum_{i,j=1}^{d-1} d_{ij}(s,z)\frac{\partial^2}{\partial s_i \partial s_j} + \sum_{j=1}^{d-1} d_j(s,z)\frac{\partial}{\partial s_j}, \tag{6.24}$$

and

$$\max_{ij}\left(\|\partial_z^m d_{ij}\|_{L^\infty(\Gamma(2\ell))}, \|\partial_z^m d_j\|_{L^\infty(\Gamma(2\ell))}\right) \le C\varepsilon^m, \tag{6.25}$$

where the constant C is independent of ε and $\Gamma \in \mathcal{G}_{K,l}$.

Sketch of the Proof: Substituting the expansion

$$G_{ij} = \left\langle \frac{\partial\gamma(s)}{\partial s_i} + \varepsilon z\frac{\partial\nu(s)}{\partial s_i}, \frac{\partial\gamma(s)}{\partial s_j} + \varepsilon z\frac{\partial\nu(s)}{\partial s_j}\right\rangle = g_{ij} - 2\varepsilon z h_{ij} + \varepsilon^2 z^2 e_{ij}, \tag{6.26}$$

into (6.22) and using the bounds on the first and second fundamental forms afforded by $\Gamma \in \mathcal{G}_{K,\ell}$ yields the result. □

To simplify the z derivatives in (6.21) we derive an expression for J in terms of the principal curvatures $\{k_i\}_{i=1}^{d-1}$ of Γ. We first observe that identity (6.10) implies,

$$\frac{\partial\varphi}{\partial s_i} = \frac{\partial\gamma}{\partial s_i} + \varepsilon z\frac{\partial\nu}{\partial s_i} = \frac{\partial\gamma}{\partial s_i} - \varepsilon z\sum_{j=1}^{d-1}\frac{\partial\gamma}{\partial s_j}h_i^j. \tag{6.27}$$

From (6.27) we see that the Jacobian matrix takes the form,

$$\mathbf{J} = \begin{pmatrix} \frac{\partial\gamma}{\partial s_1} & \frac{\partial\gamma}{\partial s_2} & \cdots & \frac{\partial\gamma}{\partial s_{d-1}} & \nu \end{pmatrix}\begin{pmatrix} I_{\mathrm{d}-1} - \varepsilon z h_i^j & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{6.28}$$

where $I_{\mathrm{d}-1}$ is the $(d-1)\times(d-1)$ identity matrix. The determinant, J, of the Jacobian matrix, $\mathbf{J}$, satifies

$$J(s,z) = \varepsilon J_0(s)\det(I_{d-1} - \varepsilon z h_i^j) = \varepsilon J_0(s)\prod_{i=1}^{d-1}(1 - \varepsilon z k_i) = J_0(s)\sum_{j=0}^{d}\varepsilon^{j+1}K_j z^j, \tag{6.29}$$

where

$$K_0 = 1, \quad K_i := (-1)^i \sum_{j_1 < \cdots < j_i} k_{j_1} \cdots k_{j_i}, \tag{6.30}$$

and $J_0 = \det\left(\frac{\partial\gamma}{\partial s_1} \cdots \frac{\partial\gamma}{\partial s_{d-1}}\, \nu\right)$. However, since $\frac{\partial\gamma}{\partial s_i} \in T_{\gamma(s)}$ is orthogonal to the normal ν we find

$$J_0 = \left(\det\left[\left(\frac{\partial\gamma}{\partial s_1} \cdots \frac{\partial\gamma}{\partial s_{d-1}}\right)\left(\frac{\partial\gamma}{\partial s_1} \cdots \frac{\partial\gamma}{\partial s_{d-1}}\right)^T\right]\right)^{1/2} = \sqrt{\det(\mathbf{g})}, \tag{6.31}$$

where $\mathbf{g}_{ij} = \langle \frac{\partial\gamma}{\partial s_i}, \frac{\partial\gamma}{\partial s_j}\rangle$ is the metric tensor for Γ. In particular the reduced Jacobian, $\tilde{J} := J/J_0$, has the expansion

$$\tilde{J} = \varepsilon \sum_{j=0}^{d} \varepsilon^j K_j z^j. \tag{6.32}$$

Taking the z derivative of the product form of the Jacobian expression in (6.29) we obtain the identity,

$$\partial_z J = -\varepsilon^2 J_0(s) \sum_{i=1}^{d-1} k_i \prod_{j \neq i} (1 - \varepsilon z k_j) = \varepsilon \kappa J, \tag{6.33}$$

where the extended curvature

$$\kappa(s,z) := \partial_z J/(\varepsilon J) = -\sum_{i=1}^{d-1} \frac{k_i}{1 - \varepsilon z k_i} = \sum_{j=0}^{\infty} \kappa_j \varepsilon^j z^j, \tag{6.34}$$

is expressed in terms of the coefficients

$$\kappa_j(s) = -\sum_{i=1}^{d-1} k_i^{j+1}(s) = -tr(A^{j+1}), \tag{6.35}$$

where A is the Weingarten map. In particular,

$$H = \sum_{i=1}^{d-1} k_i = -\kappa_0, \tag{6.36}$$

is the mean curvature.

We remark that the Jacobian remains smooth, in fact it is a polynomial of degree at most d in z. However the extended curvature becomes singular when the whiskers intersect. Distributing the z derivative in (6.21) and using the identity (6.34) yields the central result of this subsection, the local decomposition of the Laplacian in the whiskered coordinates,

$$\Delta_x = \varepsilon^{-2}\partial_z^2 + \varepsilon^{-1}\kappa(s,z)\partial_z + \Delta_G. \tag{6.37}$$

We call κ the extended mean curvature since $\kappa = \nabla_x \cdot \nu$ is the Cartesian divergence of the normal to Γ when the normal is extended off of Γ as a constant along whiskers.

We conclude the section with estimates on G, J, and κ.

Proposition 6.7.

$$\mathbf{G}(s,z) = \begin{pmatrix} \mathbf{G_0}(s,z) & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \tag{6.38}$$

and

$$\mathbf{G_0}(s,0) = \mathbf{g}(s), \tag{6.39}$$

where $\mathbf{g} = \{g_{ij}\}$ *is the first fundamental form (metric tensor) on* Γ.

In addition, if we fix $K > 0$ and $\ell \le \frac{1}{4K}$, then, for all $m \in \mathbb{N}$ there exists $C > 0$ such that for all $\Gamma \in \mathcal{G}_{K,l}$ and ε sufficiently small the following estimates hold for the metric tensor $\mathbf{G}$, the Jacobian J, and the extended mean curvature κ

$$\sup_{\substack{i,j \le d-1 \\ m,|\alpha| \le 2}} \Big(\|\partial_z^m G^{ij}\|_{L^\infty(\Gamma(2\ell))} + \|\partial_z^m G_{ij}\|_{L^\infty(\Gamma(2\ell))} + \|D_s^\alpha \partial_z^m \kappa\|_{L^\infty(\Gamma(2\ell))} + \varepsilon^{-1} \|D_s^\alpha \partial_z^m \tilde{J}\|_{L^\infty(\Gamma(2\ell))} \Big) \le C\varepsilon^m. \tag{6.40}$$

Moreover, G^{ij} has the following expansion

$$G^{ij} = \sum_{m=1}^{d-1} \vartheta_{mi} \vartheta_{mj}, \tag{6.41}$$

where $\vartheta_{ij}(s,z) = \vartheta_{ij}^0(s) + \varepsilon z \vartheta_{ij}^1(s,z)$ are the entries of the inverse Jacobian, $\vartheta = (\mathbf{J}^T)^{-1}$ and

$$\sup_{i,j \le d-1} \big(\|\vartheta_{ij}^0\|_{L^\infty(\Gamma(2\ell))} + \varepsilon^{-m} \|\partial_z^m \vartheta_{ij}^1\|_{L^\infty(\Gamma(2\ell))} \big) \le C. \tag{6.42}$$

The constants in these estimates depend on K only.

Proof: The expression (6.38) comes from differentiating (2.1) with respect to z and s_i

$$\frac{\partial \varphi}{\partial z} = \varepsilon \nu, \quad \frac{\partial \varphi}{\partial s_i} = \frac{\partial \gamma}{\partial s_i} + \varepsilon z \frac{\partial \nu}{\partial s_i} \in T_{\gamma(s)}\Gamma, \tag{6.43}$$

(see Remark 6.3). To obtain (6.39) we observe that for $i, j = 1, \dots, d-1$,

$$(G_0)_{ij}\Big|_{z=0} = G_{ij}\Big|_{z=0} = \left\langle \frac{\partial \varphi}{\partial s_i}, \frac{\partial \varphi}{\partial s_j} \right\rangle \Big|_{z=0} = \left\langle \frac{\partial \gamma}{\partial s_i}, \frac{\partial \gamma}{\partial s_j} \right\rangle = g_{ij}. \tag{6.44}$$

The remaining estimates follow from (6.29) and (6.34). □

In particular $\det \mathbf{g} = \varepsilon^2 \det \mathbf{G_0}$. As a consequence of (2.1) we also have the following inequalities.

Proposition 6.8. *For any function $f(x) = f(s,z)$ and $m = 1, 2$,*

$$\|\partial_z^m f\|_{L^2(\Gamma(2\ell))} \le \varepsilon^m \|f\|_{H^m(\Gamma(2\ell))} \text{ and } \|\partial_{s_i} f\|_{L^2(\Gamma(2\ell))} \le C\|f\|_{H^1(\Gamma(2\ell))}. \tag{6.45}$$

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