Characterization of generalized Young measures in the $\mathcal{A}$-quasiconvexity context.

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January 24, 2013

Abstract

This work is devoted to the characterization of generalized Young measures generated by sequences of bounded Radon measures $\{\mu_n\} \subset \mathcal{M}(\Omega; \mathbb{R}^d)$ (with $\Omega \subset \mathbb{R}^N$ an open bounded set) such that $\{\mathcal{A}\mu_n\}$ converges to zero strongly in $W^{-1,q}$, for some $q \in \left(1, \frac{N}{N-1}\right)$, and $\mathcal{A}$ is a first order partial differential operator with constant rank.

KEYWORDS: Young Measures, Lower semicontinuity, $\mathcal{A}$-quasiconvexity.


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1 Introduction

Since introduced by L.C. Young [38, 39] in optimal control theory, classical Young measures (or parametrized measures) have been extensively used as a notion of generalized solution in many applications of the Calculus of Variations due to their capacity to capture the oscillations of minimizing sequences of nonlinear variational problems. We refer to Balder [7], Ball [8], Kinderlehrer and Pedregal [23], Müller [30], Pedregal [32], Roubíček [33], Tartar [36], Valadier [37] and references therein, for the general theory of Young measures, as well as for some applications. In particular, it is well known (see Pedregal [32]) that given a bounded sequence $\{u_j\} \subset L^p(\Omega; \mathbb{R}^d)$, $p \in [1, \infty)$, it is possible to associate to a subsequence $\{u_{j_k}\}$ a family of probability measures $\{\nu_x\}_{x \in \Omega}$ (the Young measure) such that for every Carathéodory function $f$, with $|f(x, \xi)| \leq C(1 + |\xi|^p)$, we have that

$$\int_{\Omega} f(x, u_{j_k}(x)) \, dx \to \int_{\Omega} \langle \nu_x, f(x, \cdot) \rangle \, dx$$

(1.1)

whenever $\{u_{j_k}\}$ is $p$-equi-integrable.
Nonlinear problems may exhibit, besides oscillation phenomena, also concentration effects which are neglected with this classical notion of Young measure. If we take them into account, (1.1) doesn’t hold anymore. Many extensions of the notion of Young measures have been developed to address problems of concentrations through the notions of weak-star defect measures (see Di Perna and Majda [10], Federer and Ziemer [16], Lions [28, 29]), reduced defect measures (see Di Perna and Majda [12]), generalized Young measures (see Di Perna and Majda [13], Alibert and Bouchité [2], Dal Maso, Desimone, Mora and Morini [14], Kristensen and Rindler [27], Béa and Santos [5]), varifold measures (Allard [1], Almgren [3], Fonseca, Pedregal and Müller [21]) and indicator measures (Fonseca [17] and Reshetnyak [34]).

The most interesting cases to be addressed, from the application point of view, are those where sequences are constrained to satisfy some PDE. The main example is that of sequences of gradients, i.e., \( u_j = Dv_j \), where \( v_j \) is a bounded sequence in \( W^{1,p} \), which have been characterized by Kinderleherer and Pedregal [23, 24] in the classical setting (under equi-integrability conditions), and by Kristensen and Rindler [27] for generalized Young measures with \( p = 1 \). More general constraints have been considered in the framework of \( \mathcal{A} \)-quasiconvexity by Fonseca and Müller [22], Santos [35] and Fonseca and Kruzik [18].

The objective of this work is to give a characterization of a generalized Young measure (in the sense described by Kristensen and Rindler [27]; see Section 2 for the main notations and concepts used throughout this work) generated by sequences of bounded Radon measures \( \{\mu_n\} \subset \mathcal{M}(\Omega; \mathbb{R}^d) \) verifying the condition

\[
\mathcal{A}\mu_n \to 0 \text{ in } W^{-1,q}(\Omega; \mathbb{R}^M),
\]

for some \( q \in \left(1, \frac{N}{N-1}\right) \), where \( \mathcal{A} \) is a first order partial differential operator of the form

\[
\mathcal{A} := \sum_{i=1}^N A^{(i)} \frac{\partial}{\partial x_i}, \quad A^{(i)} \in M^{M \times d}(\mathbb{R}), \quad M \in \mathbb{N},
\]

that we assume throughout to satisfy Murat’s condition of constant rank (see Murat [31]; Fonseca and Müller [22]) i.e., there exists \( c \in \mathbb{N} \) such that

\[
\text{rank} \left( \sum_{i=1}^N A^{(i)} \xi_i \right) = c \quad \text{for all } \xi = (\xi_1, \ldots, \xi_N) \in S^{N-1}.
\]

We will also assume throughout this work that the operator \( \mathcal{A} \) is such that:

(H) All \( \mathcal{A} \)-quasiconvex functions with linear growth are Lipschitz continuous.

We refer to Remark 2.16 for a sufficient condition for (H) to be satisfied and we note that (H) holds in the quasiconvexity context (\( \mathcal{A} = \text{curl} \)).

Our main result reads as follows.

**Theorem 1.1.** Let \( \nu \in Y(\Omega; \mathbb{R}^d) \) with \( \lambda_\nu(\partial \Omega) = 0 \) and \( \langle \langle \nu, \text{Id} \rangle \rangle = \mu \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d) \cap \text{Ker}\mathcal{A} \). Suppose further that

\[
|\mu^\alpha| - \text{a.e. } x_0 \limsup_{r \to 0} \frac{|\mu^\alpha|(B(x_0; r))}{r^{1+\alpha}} < \infty
\]

for some \( \alpha \in (0, 1) \). Then \( \nu \) is generated by a bounded sequence \( \{\mu_n\} \subset \mathcal{M}(\Omega; \mathbb{R}^d) \) such that \( \mu_n \rightharpoonup \mu \) and \( \mathcal{A}\mu_n \to 0 \) in \( W^{-1,q}(\Omega; \mathbb{R}^M) \), for \( q \leq \frac{N}{N-\alpha} \), if and only if the following conditions hold:

i) \( \int \langle \nu_x, \xi \rangle dx + \lambda_\nu(\Omega) < \infty \),

ii) \( f(\mu^\alpha(x)) \leq \langle \nu_x, f \rangle + \langle \nu_x^\infty, f^\infty \rangle \frac{dx_\nu}{dx_\nu^\infty}(x) \) for a.e. \( x \in \Omega \),

We refer to Remark 2.16 for a sufficient condition for (H) to be satisfied and we note that (H) holds in the quasiconvexity context (\( \mathcal{A} = \text{curl} \)).

Our main result reads as follows.
iii) $f^\infty \left( \frac{d\mu^s}{d\mu^s}(x) \right) |\mu^s| \leq \langle \nu_x^\infty, f^\infty \rangle \lambda^s$ as measures,

for every $\mathcal{A}$-quasiconvex function $f : \mathbb{R}^d \to \mathbb{R}$ with linear growth at infinity, i.e. such that there exist $C > 0$ in such a way that

$$|f(\xi)| \leq C(1 + |\xi|), \quad \xi \in \mathbb{R}^d,$$

where $f^\infty$ (the recession function of $f$) is defined as

$$f^\infty(\xi) := \limsup_{t \to \infty} \frac{f(t\xi)}{t}.$$  \hspace{1cm} (1.4)

The overall plan of this work in the ensuing sections will be as follows: in Section 2 we set up the notation, concepts and preliminary results that will be used throughout the paper. In Section 3 we prove some auxiliary results on generalized Young measures and $\mathcal{A}$-quasiconvexity required for the proof of Theorem 1.1 that can be found in Section 4.

2 Preliminaries

In this section we recall the main concepts and results used in our analysis.

2.1 General notations

Throughout the text we will use the following notations:

- $\Omega \subset \mathbb{R}^N$, $N \geq 1$, denotes an open bounded set with Lipschitz boundary;
- $\mathcal{L}^N$ is the $N$-dimensional Lebesgue measure. For the measure of a Lebesgue measurable set $A$ we will use $\mathcal{L}(A)$ or $|A|$;
- $S^{N-1}$ stands for the unit sphere in $\mathbb{R}^N$;
- $B$ stands for the unit open ball centered at the origin;
- $Q = \{x \in \mathbb{R}^N : |x_i| < 1/2, \ i = 1, ..., N\}$;
- $Q_\xi$ stands for any open unit cube centered at the origin with two of its faces normal to $\xi \in S^{N-1}$;
- $M^{M \times d}(\mathbb{R})$ stand for the set of $M \times d$ real matrices;
- $C^\infty_{\text{per}}(Q; \mathbb{R}^d)$ is the space of all $Q$-periodic functions in $C^\infty(\mathbb{R}^N; \mathbb{R}^d)$;
- $L^q_{\text{per}}(Q; \mathbb{R}^d)$ is the space of all $Q$-periodic functions in $L^q_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^d)$;
- $\mathcal{D}'(\Omega; \mathbb{R}^M)$ denotes the space of distributions in $\Omega$ with values in $\mathbb{R}^M$;
- $C$ represents a generic positive constant, which may vary from expression to expression;
- $\lim_{n,m} := \lim_{n \to \infty} \lim_{m \to \infty}$. 

3
2.2 Remarks on measure theory

In this section we recall some notations and well known results in Measure Theory (see e.g Ambrosio, Fusco & Pallara [4], Evans & Gariepy [15] and Fonseca & Leoni [19], as well as the bibliography therein).

Let $X$ be a locally compact metric space and let $C_c(X;\mathbb{R}^d)$, $d \geq 1$, denote the set of continuous functions with compact support on $X$. We denote by $C_0(X;\mathbb{R}^d)$ the completion of $C_c(X;\mathbb{R}^d)$ with respect to the supremum norm. Let $\mathcal{B}(X)$ be the Borel $\sigma$-algebra of $X$. By the Riesz-Representation Theorem the dual of the Banach space $C_0(X;\mathbb{R}^d)$, denoted by $\mathcal{M}(X;\mathbb{R}^d)$, is the space of finite $\mathbb{R}^d$-valued Radon measures $\mu : \mathcal{B}(X) \to \mathbb{R}^d$ under the pairing

$$<\mu, \varphi> := \int_X \varphi \, d\mu \equiv \sum_{i=1}^d \int_X \varphi_i \, d\mu_i$$

where $\varphi = (\varphi_1, \ldots, \varphi_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$. The space $\mathcal{M}(X;\mathbb{R}^d)$ will be endowed with the weak-$\ast$-topology deriving from this duality. In particular a sequence $\{\mu_n\} \subset \mathcal{M}(X;\mathbb{R}^d)$ is said to weak-$\ast$-converge to $\mu \in \mathcal{M}(X;\mathbb{R}^d)$ (indicated by $\mu_n \rightharpoonup \mu$) if for all $\varphi \in C_0(X;\mathbb{R}^d)$

$$\lim_{n \to \infty} \int_X \varphi \, d\mu_n = \int_X \varphi \, d\mu.$$

If $d = 1$ we write by simplicity $\mathcal{M}(X)$ and we denote by $\mathcal{M}^+(X)$ its subset of positive measures. Given $\mu \in \mathcal{M}(X;\mathbb{R}^d)$ let $|\mu|$ denote its total variation and let $\text{supp} \, \mu$ denote its support. We recall that a measure $\mu$ is said to be absolutely continuous with respect to a positive measure $\nu$, written $\mu << \nu$, if for every $E \in \mathcal{B}(X)$ the following implication holds:

$$\nu(E) = 0 \Rightarrow \mu(E) = 0.$$

Two positive measures $\mu$ and $\nu$ are said to be mutually singular, written $\mu \perp \nu$, if there exists $E \in \mathcal{B}(X)$ such that $\nu(E) = 0$ and $\mu(X \setminus E) = 0$. For general vector-valued measures $\mu$ and $\nu$ we say that $\mu \perp \nu$ if $|\mu| \perp |\nu|$.

**Theorem 2.1** (Lebesgue-Radon-Nikodým Theorem). Let $\mu \in \mathcal{M}^+(X)$ and $\nu \in \mathcal{M}(X;\mathbb{R}^d)$. Then

(i) there exist two $\mathbb{R}^d$-valued measures $\nu^\mu$ and $\nu^s$ such that

$$\nu = \nu^\mu + \nu^s$$

(2.1)

with $\nu^\mu << \mu$ and $\nu^s \perp \mu$. Moreover, the decomposition (2.1) is unique, that is, if $\nu = \nu^\mu + \nu^\nu$ for some measures $\nu^\mu, \nu^\nu$, with $\nu^\mu << \mu$ and $\nu^\nu \perp \mu$, then $\nu^\mu = \nu^\nu$ and $\nu^\nu = \nu^\nu$;

(ii) there is a $\mu$-measurable function $u \in L^1(\Omega;\mathbb{R}^d)$ such that

$$\nu^\mu(E) = \int_E u \, d\mu$$

for every $E \in \mathcal{B}(\Omega)$. The function $u$ is unique up to a set of $\mu$ measure zero.

The decomposition $\nu = \nu^\mu + \nu^s$ is called the Lebesgue decomposition of $\nu$ with respect to $\mu$ (see [19, Theorem 1.115]) and the function $u$ is called the Radon-Nikodým derivative of $\nu$ with respect to $\mu$, denoted by $u = d\nu^\mu/d\mu$ (see [19, Theorem 1.101]).

We recall that $W_0^{1,p}(\Omega;\mathbb{R}^d) \subset C_0(\Omega;\mathbb{R}^d)$ for $p > N$. Then $\mathcal{M}(\Omega;\mathbb{R}^d)$ is compactly embedded in $W^{-1,q}(\Omega;\mathbb{R}^d)$, $1 < q < \frac{N}{N-1}$, where $W^{-1,q}(\Omega;\mathbb{R}^d)$ denotes the dual space of $W_0^{1,q}(\Omega;\mathbb{R}^d)$ with $q'$, the conjugate exponent of $q$, given by the relation $\frac{1}{q} + \frac{1}{q'} = 1$.

We finish this part by recalling the notion of parametrized measures and one of its important subclasses used throughout.
Definition 2.2 (Parametrized measures). Given $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^d$, open or closed, a parametrized measure is a map $\tau : E \to M(F)$. It is usual to identify $\tau \equiv (\tau_x)_{x \in E}$.

Definition 2.3 (The space $L^\infty(E, \mu; M(F))$). Given $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^d$, open or closed, $\mu \in M^+(E)$, we denote by $L^\infty(E, \mu; M(F))$ the set of parametrized measures $\tau : E \to M(F)$ that are weakly* $\mu$-measurable (meaning that for every Borel set $B \subset F$ the function $x \to \tau_x(B)$ is $\mu$-measurable in $E$) and $|\tau_x(F)| \leq L^\infty(E)$.

2.3 The notion of generalized Young measure and some class of integrands

We define here the class of Young measures and integrands considered throughout this work (see Kristensen and Rindler [27] for more details and properties).

Given a continuous function $f \in C(\Omega \times \mathbb{R}^d)$ we start by defining the linear mapping $T : C(\Omega \times \mathbb{R}^d) \to C(\Omega \times B)$ by

$$Tf(x, \xi) = (1 - |\xi|)f \left( x, \frac{\xi}{1 - |\xi|} \right) \quad (2.2)$$

and we introduce the set

$$E(\Omega \times \mathbb{R}^d) := \{ f \in C(\Omega \times \mathbb{R}^d) : Tf \text{ has a continuous extension to } C(\Omega \times B) \}.$$  

For $f \in E(\Omega \times \mathbb{R}^d)$, $f^\infty$ represents the continuous extension of $Tf$, precisely,

$$f^\infty(x, \xi) := \lim_{\xi' \to \xi \atop t \to \infty} \frac{f(x', t\xi')}{t} \quad (2.3)$$

Definition 2.4 (Generalized Young measures: the set $Y(\Omega; \mathbb{R}^d)$). The set of all generalized Young measures $Y(\Omega; \mathbb{R}^d)$ consists of those triples $\nu = (\nu_x, \lambda_x, \nu^\infty_x)$ where $(\nu_x)_{x \in \Omega} \in L^\infty_{w*}(\Omega; M(\mathbb{R}^d))$ with $\langle \nu_x, |.| \rangle \in L^1(\Omega)$ for $x \in \Omega$, and in addition $\lambda_x \in M^+(\Omega)$ and $(\nu^\infty_x)_{x \in \Omega} \in L^w_{w*}(\Omega, \lambda_x, M(S^{d-1}))$.

In what follows, for simplicity, we use the term Young measure to refer to a generalized Young measure.

Remark 2.5 (Duality). We have that $Y(\Omega; \mathbb{R}^d) \subset E'(\Omega \times \mathbb{R}^d)$ (dual of $E(\Omega \times \mathbb{R}^d)$) under the duality pairing

$$\langle \langle \nu, f \rangle \rangle := \int_\Omega \langle \nu_x, f(x, ...) \rangle \, d\lambda_x(x) + \int_\Omega \langle \nu^\infty_x, f^\infty(x, ...) \rangle \, d\lambda_x(x)$$

for all $\nu \in Y(\Omega; \mathbb{R}^d)$ and $f \in E(\Omega \times \mathbb{R}^d)$.

We note that the inclusion above is strict. Moreover given $\{\nu^k\} \subset Y(\Omega; \mathbb{R}^d)$ we say that $\nu^k$ weakly*-converges to $\nu \in Y(\Omega; \mathbb{R}^d)$ (and we denote it by $\nu^k \overset{*}{\rightharpoonup} \nu$) whenever

$$\langle \langle \nu^k, f \rangle \rangle \to \langle \langle \nu, f \rangle \rangle$$

holds for every $f \in E(\Omega \times \mathbb{R}^d)$.

Remark 2.6. To a given measure $\mu \in M(\Omega; \mathbb{R}^d)$ we associate the element of $Y(\Omega; \mathbb{R}^d)$

$$\delta_\mu := \left( \delta_{\mu^*(x)}, |\mu^*|, \delta_{\mu^*} \frac{\mu^*}{\mu^*}(x) \right).$$

The following theorem of existence of Young measures associated to subsequences of bounded Radon measures is a classical result (see Theorem 7 in Kristensen and Rindler [27] for its proof; see also [2]).

Theorem 2.7. Let $\{\mu_n\}$ be an uniformly bounded sequence in $M(\Omega; \mathbb{R}^d)$. Then there exists a subsequence (which we still denote by $\mu_n$) and a Young measure $\nu := (\nu_x, \lambda_x, \nu^\infty_x)$ such that for every $f \in E(\Omega \times \mathbb{R}^d)$ we have

$$\langle \langle \delta_{\mu_n}, f \rangle \rangle \to \langle \langle \nu, f \rangle \rangle. \quad (2.4)$$
Remark 2.8. Note that from (2.4) it follows that
\[ \mu_n \overset{\star}{\rightharpoonup} \mu := (\nu_x, \xi) \mathcal{L}^N + (\nu_x^\infty, \xi) \lambda, \]
and
\[ |\mu_n| \overset{\star}{\rightharpoonup} \Lambda := (\nu_x, |\xi|) \mathcal{L}^N + \lambda, \]
by considering \( f(x, \xi) = \xi \) and \( f(x, \xi) = |\xi| \), respectively.

Remark 2.9. We say that \( \nu = (\nu_x, \lambda, \nu_x^\infty) \) is the Young measure generated by the sequence \( \{\mu_n\} \) whenever (2.4) holds for every \( f \in E(\Omega \times \mathbb{R}^d) \). Indeed to check that a given Young measure \( \nu \) is generated by a sequence \( \mu_n \) it is enough to test (2.4) for a countable dense set of functions in \( f_k \in E(\Omega \times \mathbb{R}^d) \) of the form \( f_k(x, \xi) = \varphi_k(x) g_k(\xi) \), \( k \in \mathbb{N} \), with \( \varphi_k \in C(\Omega) \) and \( g_k : \mathbb{R}^d \to \mathbb{R} \) Lipschitz continuous (see Lemma 3 in [27] for more details).

The parametrized measure \((\nu_x)_{x \in \Omega}\) describes the oscillations of the generating sequence, \( \lambda \) gives the location and magnitude of its concentrations, and \( \nu_x^\infty \) gives the distribution of these concentrations by direction. We now present some simple 1-D examples that will clarify these ideas.

Example 2.10. (Oscillations, no concentrations) Given \( u \in L^1(0, 1) \) let us denote by \( \hat{u} \) its extension by periodicity to \( \mathbb{R} \). We now identify the Young measure associated with the sequence
\[ u_n(x) := \hat{u}(nx), \quad x \in (0, 1) \]
According to Remark 2.9 it is enough to consider functions in \( E((0, 1) \times \mathbb{R}) \) of the form \( f(x, \xi) = \varphi(x) g(\xi) \) (see 2.9). Then, by Riemman-Lebesgue Lemma
\[ \lim_{n \to \infty} \int_0^1 \varphi(x) g(\hat{u}(nx)) \, dx = \int_0^1 \varphi(y) \, dy \int_0^1 g(u(x)) \, dx \]
and thus
\[ < \nu_x, g >= \int_0^1 g(u(x)) \, dx, \quad \lambda = 0. \]

Example 2.11. (Concentrations at one point) Let \( x_0 \in (0, 1) \) and let \( u_n : (0, 1) \to \mathbb{R} \) be given by
\[ u_n(x) := \begin{cases} \frac{n}{2} & \text{if } x_0 - \frac{1}{n} < x < x_0 + \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases} \]
Then, for every \( f \in E((0, 1) \times \mathbb{R}) \) of the form \( f(x, \xi) = \varphi(x) g(\xi) \) we have that
\[ \lim_{n \to \infty} \int_0^1 \varphi(x) g(u_n(x)) \, dx = \int_0^1 \varphi(x) g(0) \, dx + \varphi(x_0) g^\infty(1), \]
and thus
\[ \nu_x = \delta_{x_0}, \quad \lambda = \delta_{x_0}, \quad \nu_x^\infty = \delta_1. \]

Example 2.12. (Distributed concentrations) Let \( v_n : (0, 1) \to \mathbb{R} \) be given by
\[ v_n(x) := \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise}, \end{cases} \]
which we extend by periodicity to all \( \mathbb{R} \), and define
\[ u_n(x) := v_n(nx), \quad x \in (0, 1) \]
Then
\[ \lim_{n \to \infty} \int_0^1 \varphi(x) g(v_n(nx)) \, dx = g(0) \int_0^1 \varphi(y) \, dy + g^\infty(1) \int_0^1 \varphi(y) \, dy. \]
Thus
\[ \nu_x = \delta_0, \quad \lambda = \mathcal{L}^1, \quad \nu_x^\infty = \delta_1. \]
2.4 Results on $\mathcal{A}$-quasiconvexity

We recall here the notion of $\mathcal{A}$-quasiconvexity introduced by Dacorogna [9], following the works of Murat and Tartar in compensated compactness (see [31] and [36]) and further developed by Fonseca & Müller [22]. We recall as well some of its main properties.

Let $\mathcal{A} : D'(\Omega; \mathbb{R}^d) \to D'(\Omega; \mathbb{R}^d)$ be the first order linear differential operator defined in (1.2).

**Definition 2.13.** ($\mathcal{A}$-quasiconvex function) A locally bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if

$$f(v) \leq \int_Q f(v + w(x)) \, dx$$

holds for all $v \in \mathbb{R}^d$ and for all $w \in C^\infty_{\text{per}}(Q; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ in $\mathbb{R}^N$ with $\int_Q w(x) \, dx = 0$.

**Remark 2.14.** If $f$ has $q$-growth, i.e. $|f(v)| \leq C(1 + |v|^q)$ for all $v \in \mathbb{R}^d$, then the space of test functions $C^\infty_{\text{per}}(Q; \mathbb{R}^d)$ in Definition 2.13 can be replaced by $L^q_{\text{per}}(Q; \mathbb{R}^d)$ (see Remark 3.3.2 in [22]).

**Definition 2.15.** (Characteristic cone of $\mathcal{A}$) The characteristic cone of $\mathcal{A}$ is defined by

$$\mathcal{C} = \left\{ v \in \mathbb{R}^d : \exists w \in \mathbb{R}^N \setminus \{0\}, \left( \sum_{i=1}^N A^{(i)} w_i \right) v = 0 \right\}.$$

We note that the set $\mathcal{C}$ was introduced in the works of Murat and Tartar (see [31] and [36]) and that $\mathcal{A}$-quasiconvex functions are convex along directions of the characteristic cone (see [22]).

We introduce also a set that will be useful in the sequel. If $\lambda \in \mathcal{C}$ we define

$$\mathcal{V}_\lambda := \left\{ \omega \in \mathbb{R}^N : \left( \sum_{i=1}^N A^{(i)} w_i \right) \lambda = 0 \right\}.$$

Note that $\mathcal{V}_\lambda$ is a subspace of $\mathbb{R}^N$.

**Remark 2.16.** If $\text{Span}(\mathcal{C}) = \mathbb{R}^d$ then $\mathcal{A}$-quasiconvex functions with $q$-growth satisfy a growth condition of the type

$$|f(\xi_1) - f(\xi_2)| \leq C(1 + |\xi_1|^{q-1} + |\xi_2|^{q-1})|\xi_1 - \xi_2|$$

(see Fonseca, Leoni & Müller [20]). In [20] this result is stated for nonnegative integrands which turns out not to be restrictive since if $f(\zeta) \geq -C(1 + |\zeta|)$ then $g(\zeta) = f(\zeta) + C(1 + |\zeta|)$ is nonnegative. Thus if $g$ is Lipschitz, the same holds for $f$. We note that in particular from (2.5) it follows that $\mathcal{A}$-quasiconvex functions with linear growth are Lipchitz continuous, that is, condition (H) is satisfied.

We recall the notion of $\mathcal{A}$-quasiconvex envelope for functions that fail to be $\mathcal{A}$-quasiconvex.

**Definition 2.17.** ($\mathcal{A}$-quasiconvex envelope) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We define the $\mathcal{A}$-quasiconvex envelope of $f$, $\mathcal{Q}_\mathcal{A} f : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$, as

$$\mathcal{Q}_\mathcal{A} f(v) := \inf \left\{ \int_Q f(v + w(x)) \, dx : w \in C^\infty_{\text{per}}(Q; \mathbb{R}^d) \text{ such that } \mathcal{A}w = 0 \text{ in } \mathbb{R}^N \text{ and } \int_Q w(x) \, dx = 0 \right\}.$$

**Remark 2.18.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function.

i) If $\text{Span}(\mathcal{C}) = \mathbb{R}^d$ and $\mathcal{Q}_\mathcal{A} f(v_0) > -\infty$ for some $v_0 \in \mathbb{R}^d$, then $\mathcal{Q}_\mathcal{A} f(v)$ is finite for all $v \in \mathbb{R}^d$. This result, which can be found in [22], follows from the convexity of $\mathcal{Q}_\mathcal{A} f$ along directions of the cone in a similar way to was proved in [23] in the context of quasiconvexity. If in addition $f$ has linear growth at infinity, then $\mathcal{Q}_\mathcal{A} f$ also has linear growth at infinity (see Lemma 2.5 in [25] where the case of gradients is treated; the proof for our case is interely similar).
ii) If $f$ is Lipschitz continuous, then $QAf$ is also Lipschitz continuous. Indeed, given $\epsilon > 0$, for $v \in \mathbb{R}^d$

$$QAf(v) + \epsilon \geq \int_Q f(v + w(y)) \, dy$$

for some $w \in C_\text{per}^\infty(Q; \mathbb{R}^d)$. Thus, for $u \in \mathbb{R}^d$

$$QAf(u) - QAf(v) \leq \int_Q f(u + w(y)) \, dy - \int_Q f(v + w(y)) \, dy - \epsilon \leq L|u - v| - \epsilon,$$

from where letting $\epsilon \to 0$ and interchanging the role of $u$ and $v$ we get the result.

The next lemma gives an approximation result for $A$-quasiconvex functions with linear growth at infinity by functions in the space $E(\Omega \times \mathbb{R}^d)$. Its proof is similar to the one for the quasiconvexity case (see Lemma 5 in Kristensen and Rinder [27]).

**Lemma 2.19.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be an $A$-quasiconvex function such that

$$|f(\xi)| \leq C(1 + |\xi|)$$

for some $C > 0$. Then there exists a decreasing sequence $\{f_n\} \subset E(\Omega \times \mathbb{R}^d)$ (just dependent of the second variable, i.e., $f_n(x, \xi) = f_n(\xi)$) such that $f_n \to f, f_n^\infty \to f^\infty$ pointwise.

### 3 Auxiliary results on Young measures and $A$-quasiconvexity

**Proposition 3.1.** Let $\{u_n\} \subset L^1(\Omega; \mathbb{R}^d)$ with $u_n \rightharpoonup \mu$, $Au_n \to 0$ in $W^{-1,q}$ (for $1 < q < \frac{N}{N-1}$), and assume that this sequence generates a Young measure $\nu$. Let $m \in \mathbb{R}^d$. Then there exists a sequence $\{v_n\} \subset L^1(\Omega; \mathbb{R}^d)$ with $v_n = m$ in a neighborhood of $\partial \Omega$, $Av_n \to 0$ in $W^{-1,q}$ and such that $v_n$ generates the Young measure $\tilde{\nu} = (\tilde{\nu}_x, \lambda_{\tilde{\nu}}, \tilde{\nu}_x^\infty)$ where

$$\langle \tilde{\nu}_x, \varphi \rangle = \int_{\Omega} \langle \nu_x, \varphi \rangle \, dx, \varphi \in C_0(\mathbb{R}^d),$$

$$\lambda_{\tilde{\nu}} = \frac{\lambda_{\nu}(\Omega)}{|\Omega|} |\mathbb{S}^d|$$

$$\langle \tilde{\nu}_x^\infty, \varphi \rangle = \int_{\Omega} \langle \nu_x^\infty, \varphi \rangle \, d\lambda_{\nu}, \varphi \in C(S^{d-1}).$$

**Proof.** Consider a sequence of cut-off functions $\{\rho_l\} \subset C_\text{c}^\infty(\Omega; [0,1])$ such that

$$|\Omega \setminus \Omega_l| \to 0$$

as $l \to \infty$, where $\Omega_l := \{x \in \Omega : \rho_l(x) = 1\}$. Define the sequence

$$\tilde{u}_{l,n} := \rho_l(u_n - m) + m.$$

For each $k \in \mathbb{N}$, since $\Omega$ has a Lipschitz boundary, we can apply Vitali’s Covering Theorem and find a countable family of disjoint sets $\{a_i + r_i \Omega\} \subset \Omega$, with $i \in \mathbb{N}$ and $r_i < \frac{1}{k}$, such that

$$|\Omega \setminus (\cup_{i=1}^\infty a_i + r_i \Omega)| = 0.$$
Define the sequence

\[ v_{l,k,n}(x) := \begin{cases} \tilde{u}_{l,n}(\frac{x-a_i}{r_i}) & x \in a_i + r_i \Omega, \\ m, & \text{otherwise in } \Omega. \end{cases} \]

We have

\[ A v_{l,k,n} = \frac{1}{r_i} A \rho_i \left( \frac{x-a_i}{r_i} \right) \left( u_n \left( \frac{x-a_i}{r_i} \right) - m \right) + \frac{1}{r_i} \rho_i \left( \frac{x-a_i}{r_i} \right) A u_n \left( \frac{x-a_i}{r_i} \right) := A_1 + A_2 \]

for \( x \in a_i + r_i \Omega \). Since the second term above goes to 0 in \( W^{-1,q} \) as \( n \to \infty \), we just focus on the first. Let \( \phi \in C^\infty_0(\Omega; \mathbb{R}^d) \), we have

\[ \langle A_1, \phi \rangle = \sum_i r_i^{-1} \int_{a_i + r_i \Omega} A \rho_i \left( \frac{x-a_i}{r_i} \right) \left( u_n \left( \frac{x-a_i}{r_i} \right) - m \right) \phi(x) \, dx \]

\[ = \sum_i r_i^{N-1} \int_{\Omega} A \rho_i (u_n(y) - m) \phi(x + r_i y) \, dy. \]

Note that

\[ \max_{a_i + r_i \Omega} |\Phi| = |\Phi^j(\tilde{x})| \leq |\Phi^j(\tilde{y}) - |\Phi^j(\tilde{\bar{y}})\rangle| + |\Phi^j(\tilde{\bar{y}})| \leq r_i^{1 - \frac{N}{q}} \| \nabla \Phi^j \|_{L_q} + r_i^{-\frac{N}{q}} \| \Phi^j \|_{L_{q'}} \leq r_i^{-\frac{N}{q}} \| \Phi^j \|_{W^{1,q}}, \]

where \( \tilde{y} \) and \( \tilde{x} \) are the points where the absolute value of a generic component of \( \Phi \) (denoted by \( |\Phi^j| \) above) attains its minimum and maximum values, respectively, in \( a_i + r_i \Omega \).

Thus

\[ |\langle A_1, \phi \rangle| \leq \sum_i r_i^{N-1} \| A \rho_i \|_{L_\infty} \| u_n - m \|_{L_1} \| \phi (a_i + r_i y) \|_{L_\infty(\Omega)} \]

\[ \leq C \sum_i r_i^{N-1} \| A \rho_i \|_{L_\infty} \| u_n - m \|_{L_1} \| \phi \|_{W^{1,q'}(a_i + r_i \Omega)} \]

\[ \leq C \left( \frac{1}{k} \right)^{\frac{N}{q'} - 1} \| A \rho_i \|_{L_\infty} \| \phi \|_{W^{1,q'}(\Omega)}. \]

We conclude that

\[ A v_{l,k,n} \to 0 \quad \text{as } l, k, n \to \infty \] (3.5)

as \( l, k, n \to \infty \) (we have used that \( q < \frac{N}{N-1} \Rightarrow \frac{N}{q} - 1 > N - 2 \geq 0 \)).

Moreover

\[ \int_{\Omega} |v_{l,k,n}| = |\Omega| \int_{\Omega} |\tilde{u}_{l,n}| \leq C \quad (3.6) \]

and, for \( \varphi \in C(\overline{\Omega}) \),

\[ \int_{\Omega} |v_{l,k,n}| \varphi \, dx = \sum_i r_i^N \int_{\Omega} |\tilde{u}_{l,n}| \varphi (a_i + r_i y) \, dy \]

\[ = \sum_i r_i^N \varphi (a_i) \int_{\Omega} |\tilde{u}_{l,n}| \, dy + R_k(\varphi) \]

where

\[ |R_k(\varphi)| \leq C \sup_{a_i} |\varphi (a_i + r_i y) - \varphi (a_i)| \| \varphi \|_{L_\infty(\Omega)}. \]

Thus taking limits, first in \( n \) and then in \( k \) and \( l \), we get that

\[ \int_{\Omega} |v_{l,k,n}| \varphi \, dx \to \int_{\Omega} \varphi \, dy \left( \lambda_\nu(\Omega) + \int_{\Omega} \langle \nu_x, \zeta \rangle \, dx \right). \] (3.7)
In a similar way we can prove that
\[
\int_{\Omega} v_{l,k,n} \varphi \, dx \to \int_{\Omega} \varphi \, d(\langle \nu_x, |\xi| \rangle + \langle \nu_x, h \rangle d\lambda_\nu) = \int_{\Omega} \varphi \, d\mu(\Omega) / |\Omega| \quad (3.8)
\]

Using (3.5)-(3.8) we can find a new sequence \( v_l \equiv v_{l,k,n} \) such that
\[
v_l \sim \mu(\Omega) / |\Omega|, \quad |v_l| \sim \mathcal{L}^N(\Omega) \left( \lambda_\nu(\Omega) + \int_{\Omega} \langle \nu_x, |\xi| \rangle \, dx \right), \quad \mathcal{A}v_l \to 0. \quad (3.9)
\]

Moreover by extracting a subsequence, which we do not relabel, we can assume that \( v_l \) generates a Young measure \( \bar{\nu} = (\bar{\nu}_x, \lambda_\nu, \bar{\nu}_x^\infty) \). In the next steps we will identify each one of the measures in the triple \( \bar{\nu} \).

Let \( f(x, \xi) = \varphi(x)g(\xi) \) be a function in \( E(\Omega \times \mathbb{R}^d) \). Using an argument similar to the previous step we get that
\[
\lim_{k \to \infty} \int_{\Omega} \varphi(x)g(v_l(x)) \, dx = \int_{\Omega} \varphi \, dx \left( \int_{\Omega} \langle \nu_x, g \rangle \, dx + \int_{\Omega} \langle \nu_x^\infty, g \rangle \, d\lambda_\nu \right). \quad (3.10)
\]

If we consider \( g \in C_0(\mathbb{R}^d) \) in (3.10) we get (3.1). From (3.1) and (3.9) we get (3.2). In order to get (3.3) consider \( h \in C(S^{d-1}) \) and test the inequality above with the class of functions \( g_M(\xi) = \Psi_M(\xi) |h(\xi) / M| \), where \( \Psi \in C^\infty([0, \infty); [0, 1]) \) with \( \Psi(t) = 0 \) for \( t < M \) and \( \Psi(t) = 1 \) for \( t > M \). Indeed, we get
\[
\int_{\Omega} \varphi(\bar{\nu}_x, g_M) + \int_{\Omega} \varphi(\bar{\nu}_x^\infty, h) \, d\lambda_\nu = \int_{\Omega} \varphi \, dx \left( \int_{\Omega} \langle \nu_x, g_M \rangle + \int_{\Omega} \langle \nu_x^\infty, h \rangle \, d\lambda_\nu \right)
\]
and now the result follows by letting \( M \to \infty \) and using (3.2).

**Lemma 3.2.** Let \( \nu^i \in Y(\Omega; \mathbb{R}^d) \), \( i = 1, 2 \), satisfying \( \lambda_\nu^i(\partial \Omega) = 0 \) generated respectively by \( \{u_{n_i}\} \subset L^1(\Omega; \mathbb{R}^d) \) and \( \{v_{n_i}\} \subset L^1(\Omega; \mathbb{R}^d) \), such that \( u_n \rightharpoonup^* \mu, v_n \rightharpoonup^* \mu \). Suppose further that \( \mathcal{A}u_n \to 0 \) in \( W^{-1,q} \), \( \mathcal{A}v_n \to 0 \) in \( W^{-1,q} \), for \( 1 < q < \frac{N}{N-1} \). Let \( m \in \mathbb{R}^d \). Then, for each \( \theta \in (0, 1) \) there exists a sequence \( \{w_{n_i}\} \subset L^1(\Omega; \mathbb{R}^d) \) that agrees with \( m \) on a neighborhood of \( \partial \Omega \), \( \mathcal{A}w_n \to 0 \) in \( W^{-1,q} \) and such that \( w_n \) generates the Young measure \( \theta \nu^1 + (1-\theta)\nu^2 \).

**Proof.** Consider a partition of \( \Omega \) into two disjoint open sets \( \Omega_i, i = 1, 2 \), such that \( |\Omega_i| = \theta|\Omega| \) and \( |\Omega_2| = (1-\theta)|\Omega| \).

Let \( \pi_n \) and \( \bar{\pi}_n \) be the sequences constructed by the application of Proposition 3.1 in \( \Omega \), and note that the construction can be made in such a way that both sequences take the value \( m \) in a neighborhood of \( \partial \Omega_1 \) and \( \partial \Omega_2 \). We denote the sequences by \( \bar{u}_n \) and \( \bar{v}_n \), respectively. Notice that \( \bar{u}_n \) generates the Young measure \( \bar{\nu}^1 \) and \( \bar{v}_n \) generates the Young measure \( \bar{\nu}^2 \). Now we proceed in a similar way to the proof of Proposition 3.1. We consider Vitali coverings of \( \Omega \), \( \{a_i + r_i \Omega \} \) with \( r_i < \frac{1}{j} \), and define the sequence
\[
w_{j,n}(x) := \begin{cases} \bar{u}_n(x - a_i r_i) & x \in a_i + r_i \Omega_1, \\ \bar{v}_n(x - a_i r_i) & x \in a_i + r_i \Omega_2. \end{cases}
\]

By a change of variables we get, by a similar argument than before,
\[
\int_{\Omega} |w_j| \, dx = \sum_i r_i^{N} \left( \int_{\Omega_1} |\bar{u}_n| \, dy + \int_{\Omega_2} |\bar{v}_n| \, dy \right) \leq C.
\]

With an appropriate diagonalization argument we obtain a sequence \( w_j \equiv w_{j,n_j} \) in such a way that
\[
\mathcal{A}w_j \to 0,
\]
and \( w_j \) generates a Young measure \( \tau \). We now prove that
\[
\tau = \theta \nu^1 + (1 - \theta) \nu^2.
\] (3.11)

Let \( f(x, \xi) = \varphi(x)g(\xi) \) be a function in \( E(\Omega \times \mathbb{R}^d) \). We then have
\[
\int_{\Omega} \varphi g(w_j) \, dx = \sum_i r_i^N \int_{\Omega_1} \varphi(a_i + r_i y) g(\tilde{u}_n) + \sum_i r_i^N \int_{\Omega_2} \varphi(a_i + r_i y) g(\tilde{v}_n)
\]
where \( R_j \to 0 \) as \( j \to \infty \). Taking limits we get
\[
\lim_j \int_{\Omega} \varphi(x) g(w_j(x)) \, dx = \int_{\Omega} \varphi \left[ \theta \left( \int_{\Omega} \langle \nu^1, g \rangle + \int_{\Omega} \langle \nu^1, g \rangle d\lambda_{\nu^1} \right) + (1 - \theta) \left( \int_{\Omega} \langle \nu^2, g \rangle + \int_{\Omega} \langle \nu^2, g \rangle d\lambda_{\nu^2} \right) \right]
\]
and (3.11) follows.

4 Proof of the Characterization Theorem

In this section we prove Theorem 1.1.

Remark 4.1. Any Young measure \( \nu \) (with \( \lambda_{\nu}(\partial \Omega) = 0 \)) generated by a bounded sequence \( \{\mu_n\} \subset \mathcal{M}(\Omega; \mathbb{R}^d) \) with \( A_{\mu_n} \to 0 \) in \( W^{-1,q} \), for some \( q \in \left( 1, \frac{N}{N-1} \right) \), is also generated by a sequence of \( C^\infty \)-functions \( \{u_n\} \subset L^1(\Omega; \mathbb{R}^d) \) uniformly bounded such that \( A_{u_n} \to 0 \). The result follows from a regularization argument together with Reshetnyak’s Theorem (see [6]).

Before proving Theorem 1.1 we need three auxiliary results. We start with the following definition.

Definition 4.2.
\[ AY_{\mu} := \{ \nu \in Y(\Omega; \mathbb{R}^d) : \nu \text{ is generated by a sequence } u_n \in L^1(\Omega; \mathbb{R}^d), A_{u_n} \to 0, \ll \nu, \text{Id} \gg = \mu \} \]

Lemma 4.3. The set of Young measures \( AY_{\mu} \) is weak*-closed.

Proof. Let \( \nu \) be a Young measure in the weakly*-closure of \( AY_{\mu} \), i.e., there exist \( \nu^k \in AY_{\mu} \) such that \( \nu^k \rightharpoonup \nu \) in \( E(\Omega \times \mathbb{R}^d) \). For each \( k \) we can find a generating sequence \( \{u^k_j\} \subset L^1(\Omega; \mathbb{R}^d) \) for \( \nu^k \) such that
\[
u^k \rightharpoonup \nu, \quad A_{u^k_j} \to 0
\]
as \( j \to \infty \). For \( f \in E(\Omega \times \mathbb{R}^d) \) we have
\[
\lim_k \lim_j \int_{\Omega} f(x, u^k_j) \, dx = \langle \nu, f \rangle
\]
Thus by an appropriate diagonalization procedure (note that it is enough to check the equality above for countably many functions in \( E(\Omega \times \mathbb{R}^d) \)) we can find a sequence \( v_k := u^k_j \) such that
\[
v_k \rightharpoonup \nu, \quad A_{v_k} \to 0
\]
and that generates \( \nu \). Thus \( \nu \in AY_{\mu} \).
We recall here some results from [6] on the blow-up of a measure $\mu$, with $A\mu = 0$, around absolutely continuous and singular points, that will be needed in the proof of the next proposition.

At almost every point $x_0 \in \Omega$ we have that
\[
\frac{T_k \mu}{r_k} \rightharpoonup \mu^a(x_0) \text{ in } \mathcal{M}(Q; \mathbb{R}^d),
\]
where
\[
<T_k \mu, \phi> = \int_{Q(x_0, r_k)} \phi \left( \frac{x - x_0}{r_k} \right) d\mu,
\]
and $r_k \to 0$.

At $|\mu^s|$-almost every $x_0 \in \Omega$ where $v_{x_0} := \frac{d\mu^s}{d|\mu^s|}(x_0) \in C$ (see definition 2.15) we have that
\[
\frac{T_\omega k \mu}{|\mu^s|(Q_\omega(x_0, r_k))} \rightharpoonup \tau \text{ in } \mathcal{M}(Q_\omega; \mathbb{R}^d),
\]
where $Q_\omega$ is an unitary cube with faces orthogonal to $\{\omega^1, ..., \omega^l\}$, an orthonormal basis for $V_{x_0}$,
\[
<T_\omega k \mu, \phi> = \int_{Q_\omega(x_0, r_k)} \phi \left( \frac{x - x_0}{r_k} \right) d\mu,
\]
and the tangent measure $\tau \in \mathcal{M}(Q_\omega; \mathbb{R}^d)$ is such that
\[
|\tau| = v_{x_0} \tau, \quad |\tau|(\partial Q_\omega) = 0,
\]
if the sequence $r_k \to 0$ is choosen appropriately. Moreover we have $\tau = \tau(x, \omega^1, ..., x, \omega^l)$, that is, the measure $\tau$ is invariant for translations in directions orthogonal to $\text{Span} \{\omega^1, ..., \omega^l\}$.

The singular points where $v_{x_0} \notin C$ the blow-up is a constant like in the absolutely continuous part.

**Proposition 4.4.** The set of Young measures $AY_{\mu, 0} := \{\nu \in AY_{\mu} : \lambda_\nu(\partial \Omega) = 0\}$ is convex.

**Proof.** Step 1: We start by constructing piecewise constant approximations (in the $W^{-1,q}$ sense) to the blow-up of a measure $\mu$ around absolutely continuous and singular points. Fix $\epsilon > 0$. Let $x_0 \in \text{supp } \mu^a$ and consider a cube $Q_k = Q(x_0; r_k)$. Let
\[
a_k = \frac{\mu(Q_k)}{r_k^d} \quad (4.4)
\]
and note that $a_k \to \mu^a(x_0)$ at a.e. $x_0$. We have that
\[
\frac{T_k \mu}{r_k^d} - a_k \rightharpoonup 0 \text{ in } \mathcal{M}(Q; \mathbb{R}^d),
\]
where
\[
<T_k \mu, \phi> = \int_{Q_k} \phi \left( \frac{x - x_0}{r_k} \right) d\mu.
\]
From (4.5) it follows that we can find $r_0 > 0$ such that, for $r_k < r_0$, we have
\[
\left\| \frac{T_k \mu}{r_k^d} - a_k \right\|_{W^{-1,q}} < \epsilon.
\]

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If \( x_0 \in \text{supp } |\mu^s| \), consider a cube (eventually rotated) \( Q_k = Q(x_0, r_k) \). Let

\[ v_{x_0} := \frac{d\mu^s}{d|\mu^s|}(x_0) \in C \]

(see definition 2.15) and suppose that

\[ \mathcal{V}_{v_{x_0}} = \text{span}\{ w^1, \ldots, w^l \} \].

For simplicity suppose that \( w^1 = e_1, \ldots, w^l = e_l \) and split the cube \( Q_k \) into rectangles \( R_{i,k} = R(c_{i,k}, r_{i,k}) \) where

\[ R(c_{i,k}, r_{i,k}) := Q_k \cap \{ x \in \mathbb{R}^N : (x - c_{i,k}).e_j < r_{i,k}, j = 1, \ldots, l \} \].

Note that we can find a partition of the unit cube \( \tilde{Q} \) (eventually rotated) into rectangles \( \tilde{R}_{i,k} \) in such a way that \( R_{i,k} = x_0 + r_k \tilde{R}_{i,k} \). Set

\[ a_{i,k} := \frac{\mu(R_{i,k})}{|R_{i,k}|} = \frac{\mu(R_{i,k})}{r_{i,k}^N} \].

We claim that for \( \phi \in C_0(Q) \), \( \phi(x) = \tilde{\phi}(x.e_1, \ldots, x.e_l) \), we have

\[ \frac{1}{t_k} \sum_i a_{i,k} \int_{\tilde{R}_{i,k}} \phi(y) \, dy \to \frac{1}{|\mu^s|(Q_k)} \int_{Q_k} \phi \left( \frac{x - x_0}{r_k} \right) \, d\mu \quad \text{(4.7)} \]

as \( i, k \to \infty \), where \( t_k := \frac{|\mu^s|(Q_k)}{r_k} \). Indeed, given \( \delta > 0 \), we can find a partition \( \tilde{R}_{i,k} \) of the unit cube \( Q \) (and thus a corresponding partition \( R_{i,k} \) of \( Q_k \)) in such a way that

\[ |\phi(y) - \phi(\tilde{c}_{i,k})| < \delta, \]

for \( y \in \tilde{R}_{i,k} \) where \( \tilde{c}_{i,k} \) denotes the center of the rectangle \( \tilde{R}_{i,k} \). We then have

\[
\frac{1}{t_k} \sum_i a_{i,k} \int_{R_{i,k}} \phi(y) \, dy = \frac{1}{t_k} \sum_i a_{i,k} \phi(\tilde{c}_{i,k}) \left( \frac{r_{i,k}}{l_{i,k}} \right)^l + \frac{1}{t_k} \sum_i a_{i,k} \int_{R_{i,k}} (\phi(y) - \phi(\tilde{c}_{i,k})) \, dy
\]

\[ = \frac{1}{t_k} \sum_i a_{i,k} \phi(\tilde{c}_{i,k}) \left( \frac{r_{i,k}}{l_{i,k}} \right)^l + E_1 \]

\[ = \frac{1}{|\mu^s|(Q_k)} \int_{Q_k} \phi \left( \frac{x - x_0}{r_k} \right) \, d\mu \]

\[ + \frac{1}{|\mu^s|(Q_k)} \sum_i \int_{R_{i,k}} \left( \phi(\tilde{c}_{i,k}) - \phi \left( \frac{x - x_0}{r_k} \right) \right) \, d\mu + E_1
\]

\[ = \frac{1}{|\mu^s|(Q_k)} \int_{Q_k} \phi \left( \frac{x - x_0}{r_k} \right) \, d\mu + E_2 + E_1 \]

where \( |E_i| \leq \frac{\delta |\mu^s|(Q_k)}{|\mu^s|(Q_k)} \), \( i = 1, 2 \). Thus we have (see (4.2))

\[ \frac{1}{t_k} \sum a_{i,k} \int_{\tilde{R}_{i,k}} \phi(y) \, dy \to \langle \tau, \phi \rangle \]
as $i, k \to \infty$. Using an appropriate diagonalization argument we can find a sequence $r_k$ (of radii of the cube $Q_k$), and a corresponding partition of $Q_k$ in rectangles $R_{i,k}$, $i = 1, \ldots, k$, in such a way that

$$
\frac{1}{t_k} \sum_{i=1}^{t_k} a_{i,k} 1_{R_{i,k}} \overset{a.e.}{\rightharpoonup} \tau.
$$

Thus we have that

$$
\frac{T_k \mu}{\mu^s(Q_k)} - \frac{1}{t_k} \sum_{i=1}^{t_k} a_{i,k} 1_{R_{i,k}} \overset{a.e.}{\rightharpoonup} 0,
$$

around points $x_0 \in \text{supp}\mu^s$. Therefore we can find $r_0 > 0$ such that for $r_k < r_0$,

$$
\left\| \frac{T_k \mu}{\mu^s(Q_k)} - \frac{1}{t_k} \sum_{i=1}^{t_k} a_{i,k} 1_{R_{i,k}} \right\|_{W^{-1,q}} < \epsilon. \tag{4.8}
$$

Finally, in the singular points where $v_{x_0} \notin \mathcal{C}$, we get (4.8) with a constant $a_k = \frac{\mu(Q_k)}{r_k}$ in the place of the piecewise constant function $\sum a_{i,k} 1_{R_{i,k}}$

In this way we can find (for each fixed $\epsilon$) a partition of $\Omega$ into disjoint cubes (up to a set of $\mu$-measure zero) and verifying conditions (4.6) and (4.8). This will be the basis for constructing a piecewise constant approximation to the measure $\mu$ with some control on the $W^{-1,q}$ norm of the operator $A$.

**Step 2:**

Let $\nu^1$ and $\nu^2$ be Young measures in $\mathcal{AY}_{\mu,0}$ and $0 < \theta < 1$. Our goal is to prove that $\theta \nu^1 + (1-\theta)\nu^2 \in \mathcal{AY}_{\mu,0}$. Fix $l > 0$ and consider $\rho_l \in C^\infty_c(Q)$ a cut-off function, such that

$$
|\{x \in Q : \rho_l \neq 1\}| < \frac{1}{l}. \tag{4.9}
$$

For cubes around singular points we will need to adjust the cut-off function to the tangent measure as follows: at a point $x_k$ choose $\rho_{k,l} \in C^\infty_c(Q)$ such that

$$
|\tau_k|\{|x \in Q : \rho_{k,l} \neq 1\}| < \frac{1}{l}, \tag{4.10}
$$

where $\tau_k$ is a tangent measure at $x_k$ verifying (4.3).

Now consider cubes $Q(x_k, r_k)$ such chosen in such a way that (cf. Step 1)

$$
\left\| \frac{T_k \mu}{r_k^N} - a_k \right\|_{W^{-1,q}} < \frac{1}{l}, \quad |\mu^s| \left( Q(x_k, r_k) \setminus \left\{ x : \rho_l \left( \frac{x - x_k}{r_k} \right) = 1 \right\} \right) \leq \frac{r_k^N}{l} \tag{4.11}
$$

if the center of the cube belongs to the support of $\mu^s$, where the second estimate follows from $\frac{d|\mu^s|}{dx} = 0$ a.e., or

$$
C \left\| \frac{T_k \mu}{\mu^s(Q_k)} - \frac{1}{t_k} \sum_{i=1}^{t_k} a_{i,k} 1_{R_{i,k}} \right\|_{W^{-1,q}} < \frac{1}{l}, \quad |\mu^s| \left( Q(x_k, r_k) \setminus \left\{ x : \rho_{k,l} \left( \frac{x - x_k}{r_k} \right) = 1 \right\} \right) \leq \frac{|\mu^s|(Q(x_k, r_k))}{l} \tag{4.12}
$$

if the center of the cube belongs to the support of $|\mu^s|$, where where the second estimate follows from (4.10), and

$$
C := C_0 ||A \rho_{k,l}||_\infty,
$$

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where $C_0$ is a constant that follows from (1.3). Note that for each $x_k$ the conditions above hold for all $r_k$ small enough (choosing appropriate sequences), thus we can use Morse covering theorem to select a disjoint cover of $\Omega$, made of cubes $Q_k^l$, up to a set of zero $\mu$ measure. Moreover, we can assume that $(\lambda_{\nu} + \lambda_{\omega})(\partial Q_k^l) = 0$, $\forall Q_k^l$; indeed, for cubes with centers around the support of $\mu^s$ it is clear (since the conditions in (4.11) hold for all $r_k$ small enough); for cubes with centers around singular points we can choose an appropriate sequence associated to some fixed tangent measure verifying (4.3) (see Remark 2.6 ii) in [6] for details).

We can now apply Lemma 3.2 in each cube $Q_k^l$, with $m = a_k$ for cubes around absolutely continuous points, and around singular points we apply the Lemma to each rectangle $R_i^i$ of an appropriate subpartition, with $m = \frac{\tau(R_i^i)}{|R_i^i|}$. Note that in both cases we have $Am = 0$ in $Q_k^l$. Thus we get a sequence $w_n^l$ that generates the measure $(\theta \nu^T + (1 - \theta)\nu^T)|Q_k^l$. Now set

$$\tilde{w}_n^l = \begin{cases} \sum_k \rho_{k,l} \left( \frac{x - x_k}{r_k} \right) (w_n^l - \mu) & x \in \bigcup Q_k^l, \\ \rho_l & \text{otherwise}, \end{cases}$$

note that around absolutely continuous points we have $\rho_{k,l} = \rho_l$ as in (4.9).

We claim that

$$||A\tilde{w}_n^l||_{W^{-1,q}} = O \left( \frac{1}{l} \right).$$

In fact, since

$$A\tilde{w}_n^l = \sum_k \frac{1}{r_k} A\rho_{k,l} \left( \frac{x - x_k}{r_k} \right) (w_n^l - \mu) + \sum_k \rho_{k,l} \left( \frac{x - x_k}{r_k} \right) Aw_n^l,$$

and the second term goes to zero as $n \to \infty$, we just need to control

$$\left| \left( \sum_k \frac{1}{r_k} A\rho_{k,l} \left( \frac{x - x_k}{r_k} \right) (w_n^l - \mu), \phi \right) \right|,$$

for $\phi \in W_0^{1,q}(\Omega; \mathbb{R}^d)$. We split the sum into sum around points $x_k \in \text{supp} \mu^a$ (which we denote by $k_{ab}$) and around points $x_k \in \text{supp} \mu^s$ (which we denote by $k_{sing}$). In the first case we have:

$$\left| \left( \sum_k \frac{1}{r_k} A\rho_{l} \left( \frac{x - x_k}{r_k} \right) (w_n^l - \mu), \phi \right) \right|$$

$$= \left| \frac{1}{r_k} \int_{Q_k} A\rho_{l} \left( \frac{x - x_k}{r_k} \right) w_n^l(x) \phi(x) \, dx - \int_{Q_k} \sum_k \frac{1}{r_k} A\rho_{l} \left( \frac{x - x_k}{r_k} \right) \phi(x) \, d\mu \right|$$

$$= \left| \sum_k \frac{1}{r_k} \int_{Q_k} A\rho_{l}(y) \phi(x + r_k y) w_n^l(x + r_k y) \, dy - \sum_k \frac{1}{r_k} \int_{Q_k} A\rho_{l}(y) \phi(x + r_k y) \, d(T_k \mu) \right|$$

$$= \left| \sum_k \frac{1}{r_k} \int_{Q_k} A\rho_{l}(y) \phi(x + r_k y) \left( w_n^l(x + r_k y) \, dy - d \left( \frac{T_k \mu}{r_k} \right) \right) \right|$$

$$\leq C \sum_k \frac{1}{r_k} \|A\rho_{l}\|_{\infty} \left\| w_n^l(x + r_k y) - \frac{T_k \mu}{r_k} \right\|_{-1,q} \|\phi\|_{W^{1,q}(Q_k)}$$
As \( w_n^I(x_k + r_k y) \xrightarrow{\ast} \alpha_k \) by (4.11) we have the desired control. Similarly, we have that

\[
\left| \sum_{k\in\text{sing}} \frac{1}{r_k} \mathcal{A}_{\rho_k,l} \left( \frac{x - x_0}{r_k} \right) (w_n^I - \mu), \phi \right| \leq C \sum_{k\in\text{sing}} |\mu^s(Q_k)| ||\mathcal{A}_{\rho_k,l}|| \left| \frac{w_n^I(x_0 + r_k y)}{r_k} - \frac{T_k \mu}{|\mu^s(Q_k)|} \right|_{-1,q} \cdot ||\phi||_{W^{1,q}(Q_k)}
\]

and as \( \frac{w_n^I(x_0 + r_k y)}{r_k} \xrightarrow{\ast} \frac{1}{t_k} \sum_{i=1}^{l_k} a_i^k 1_{R_{i-k}} \) as \( n \to \infty \), the result follows from (4.12) and (1.3).

Set \( Q^I_k := Q_k \cap \{ x : \rho_l \left( \frac{x - x_k}{r_k} \right) = 1 \} \). Then, for \( f(x, \zeta) = \phi(x)g(\zeta) \in E(\Omega; \mathbb{R}^d) \),

\[
\langle \langle \tilde{w}_n^I, f \rangle \rangle = \sum_k \int_{Q^I_k} \phi(x)g((\tilde{w}_n^I)^s) \, dx
\]

\[
= \sum_k \int_{Q^I_k} \phi(x)g(w_n^I) \, dx
\]

\[
+ \int_{Q^I_k \setminus Q^I_k} \phi(x)g \left( \rho_{k,l} \left( \frac{x - x_k}{r_k} \right) (w_n^I - \mu^s) + \mu^s \right) \, dx
\]

\[
+ \int_{Q^I_k \setminus Q^I_k} \phi(x) \left( 1 - \rho_{k,l} \left( \frac{x - x_k}{r_k} \right) \right) \mu^s \left( \frac{d\mu^s}{d|\mu^s|} \right) \, d|\mu^s|
\]

\[
= \sum_k \int_{Q^I_k} \phi(x)g(w_n^I) \, dx + A_1 + A_2
\]

where, by the choices of \( \rho_l \) and \( \rho_{k,l} \) (see (4.11)-(4.12)) and the linear growth of \( f = \phi g \), we have that \( |A_i| \leq C_i^I, i = 1, 2 \).

On the other hand, by Lemma 3.2, we have as \( n \to \infty \)

\[
\lim_{n \to \infty} \sum_k \int_{Q^I_k} \phi(x)g(w_n^I) \, dx = \sum_{k\in\text{sing}} \int_{Q^I_k} \phi \left( \int_{Q^I_k} \langle \nu_1^I, g \rangle + \int_{Q^I_k} \langle \nu_1^I, g^\infty \rangle d\lambda_{\nu_1} \right)
\]

\[
+ \int_{Q^I_k} \phi \left( (1 - \theta) \left( \int_{Q^I_k} \langle \nu_2^I, g \rangle + \int_{Q^I_k} \langle \nu_2^I, g^\infty \rangle d\lambda_{\nu_2} \right) \right)
\]

\[
+ \sum_{k\in\text{sing}} \sum_i \int_{R_{i-k}^I} \phi \left( \int_{R_{i-k}^I} \langle \nu_1^I, g \rangle + \int_{R_{i-k}^I} \langle \nu_1^I, g^\infty \rangle d\lambda_{\nu_1} \right)
\]

\[
+ \int_{R_{i-k}^I} \phi \left( (1 - \theta) \left( \int_{R_{i-k}^I} \langle \nu_2^I, g \rangle + \int_{R_{i-k}^I} \langle \nu_2^I, g^\infty \rangle d\lambda_{\nu_2} \right) \right)
\]

\[
= \langle \langle \tilde{w}_I^I, f \rangle \rangle = \langle \theta I^I + (1 - \theta) \tilde{w}_I^I \rangle.
\]
where the superscript \( l \) indicates that we are averaging the measure over sets whose centers are changing with dependence in \( l \).

We claim that
\[
\tilde{\nu}^{1,l} \overset{*}{\rightharpoonup} \nu^1 \quad \text{and} \quad \tilde{\nu}^{2,l} \overset{*}{\rightharpoonup} \nu^2 \quad \text{in} \quad E'(\Omega \times \mathbb{R}^d). \tag{4.13}
\]
Indeed, we have
\[
\langle \langle \tilde{\nu}^{1,l}, \varphi g \rangle \rangle = \sum_{k_{ab}} \int_{Q_k^e} \phi \left( \int_{Q_k^e} \langle \nu^1_{x,z}, g \rangle + \int_{Q_k^e} \langle \nu^1_{x,\infty}, g^\infty \rangle d\lambda_{\nu^1} \right) + \sum_{k_{xing}} \sum_i \int_{R_{i,k}^1} \phi \left( \int_{R_{i,k}^1} \langle \nu^1_{x,z}, g \rangle + \int_{R_{i,k}^1} \langle \nu^1_{x,\infty}, g^\infty \rangle d\lambda_{\nu^1} \right) \nonumber
\]
\[
= \sum_{k_{ab}} \phi(x_k) \left( \int_{Q_k^e} \langle \nu^1_{x,z}, g \rangle + \int_{Q_k^e} \langle \nu^1_{x,\infty}, g^\infty \rangle d\lambda_{\nu^1} \right) + \sum_{k_{xing}} \sum_i \phi(c_{i,k}) \left( \int_{R_{i,k}^1} \langle \nu^1_{x,z}, g \rangle + \int_{R_{i,k}^1} \langle \nu^1_{x,\infty}, g^\infty \rangle d\lambda_{\nu^1} \right) + R_1^1
\]
\[
= \left( \int_{\Omega} \phi \langle \nu^1_{x,z}, g \rangle + \int_{\Omega} \phi \langle \nu^1_{x,\infty}, g^\infty \rangle d\lambda_{\nu^1} \right) + R_1^1 + R_2^1
\]
where the terms \( R_1^1 \) and \( R_2^1 \) converge to zero as \( l \to \infty \) by the uniform continuity of \( \phi \). Thus, the claim follows by taking the limit in \( l \). A similar argument holds for \( \tilde{\nu}^{2,l} \).

Finally, with an appropriate diagonalization argument one can construct a sequence \( \tilde{\nu}_l = \tilde{\nu}^{l}_{n(l)} \) such that
\[
\tilde{\nu}_l \overset{*}{\rightharpoonup} \mu, \quad A\tilde{\nu}_l \to 0
\]
and with \( \tilde{\nu}_l \) generating the Young measure \( \theta \nu^1 + (1-\theta)\nu^2 \).

\[\square\]

**Proof of Theorem 1.1**

**STEP 1. (Necessary Condition)** Let \( \{f_n\} \subset E(\Omega \times \mathbb{R}^d) \) be an approximate decreasing sequence for \( f \) (see Lemma 2.19). Let \( \nu \) be a Young measure generated by a bounded sequence \( \{u_n\} \subset L^1(\Omega; \mathbb{R}^d) \) such that (see Remark 4.1)
\[
u_n \overset{*}{\rightharpoonup} \mu, \quad A\nu_n \to 0,
\]
for some \( \mu \in \mathcal{M}(\Omega; \mathbb{R}^d) \). As
\[
|u_n| \overset{*}{\rightharpoonup} \langle \nu_x, |.| \rangle + \lambda_{\nu}
\]
then condition i) follows. Let \( A \subset \Omega \) be a Borel set such that \( (\mathcal{L}^N + \lambda_{\nu})(\partial A) = 0 \). Then we have
\[
\int_A f_n(u_n) \to \int_A \langle \nu_x, f_n \rangle + \int_A \langle \nu^\infty_{x}, f_{n}^\infty \rangle d\lambda_{\nu} \tag{4.14}
\]
(see Proposition 2 in [27]). On the other hand, by Theorem 1.1 in [6], for an open ball \( A \) with \( \lambda_{\nu}(\partial A) = 0 \) (note that by hypothesis (H) the function \( f \) is Lipschitz) we have
\[
\lim_{n \to \infty} \int_A f_n(u_n) \geq \liminf_{n \to \infty} \int_A f(u_n) \geq \int_A f(\mu^a(x)) \, dx + \int_A f^\infty \left( \frac{d\mu^a}{d|\mu^a|}(x) \right) \, d|\mu^a|. \tag{4.15}
\]
Using (4.14) and (4.15), and taking the infimum in \( h \) we get the inequality

\[
\int_A \langle \nu_x, f \rangle + \int_A \langle \nu_x^\infty, f^\infty \rangle d\lambda \nu \geq \int_A f(\mu^a(x)) \, dx + \int_A f \left( \frac{d\mu^a}{d|\mu^a|}(x) \right) \, d|\mu^a|.
\]

(4.16)

Now ii) and iii) follows by differentiation with respect to Lebesgue measure (see Theorem 2.1).

**STEP 2. (Sufficient Condition)**

Let \( \tilde{\nu} \) be a Young measure satisfying \( \lambda_{\tilde{\nu}}(\partial \Omega) = 0 \), \( \langle \tilde{\nu}, Id \rangle = \mu \) and the conditions i)-iii) in the statement of Theorem 1.1. We know that the set of Young measures \( \nu \) in \( AY_{\mu,0} \) is convex (see Proposition 4.4). If we assume that \( \tilde{\nu} \) does not belong to \( AY_{\mu,0} \) (which contains \( AY_{\mu,0} \) by Lemma 4.3) we will arrive to a contradiction. Indeed in that case by Hahn-Banach Theorem we can separate the weakly*-closed and convex set \( AY_{\mu,0} \) from the point \( \tilde{\nu} \) by an hyperplane defined as follows

\[
H = \{ \nu \in E'(\Omega \times \mathbb{R}^d) : \langle \nu, f_H \rangle = 0 \}
\]

for some \( f_H \in E(\Omega \times \mathbb{R}^d) \). Therefore we have that:

\[
\langle \langle \tilde{\nu}, f_H \rangle \rangle < 0 \text{ and } \langle \langle \nu, f_H \rangle \rangle > 0 \text{ for all } \nu \in AY_{\mu,0}.
\]

(4.17)

Let

\[
f_\gamma := f_H + \gamma |\xi|,
\]

defined for \( \gamma > 0 \). Let \( \delta > 0 \) and consider a partition of \( \Omega \) into a countable number of cubes \( C_k \), up to a set of measure zero, such that

1) \( \lambda_{\tilde{\nu}}(\partial C_k) = 0 \), \( \forall k \);

2) \( |Tf_H(x, \zeta) - Tf_H(y, \zeta)| \leq \delta \), \( x, y \in C_k \);

3) \(- \sum_k Q_Af_H(z_k, 0)|C_k| \leq - \int_{\Omega} Q_Af_H(x, 0) \, dx + 1\);

where \( z_k \) denotes a point in the cube \( C_k \) where 3) holds and \( Q_Af_H(z_k, \cdot) \) is finite with linear growth (see Lemma 5.3). Thus,

\[
\langle \langle \tilde{\nu}, f_\gamma \rangle \rangle = \sum_k \int_{C_k} \langle \tilde{\nu}_x, f_\gamma(x, \cdot) \rangle \, dx + \sum_k \int_{C_k} \langle \tilde{\nu}_x^\infty, f_\gamma^\infty(x, \cdot) \rangle \, d\lambda_{\tilde{\nu}}
\]

\[
= \sum_k \int_{C_k} \langle \tilde{\nu}_x, f_\gamma(z_k, \cdot) \rangle \, dx + \sum_k \int_{C_k} \langle \tilde{\nu}_x^\infty, f_\gamma^\infty(z_k, \cdot) \rangle \, d\lambda_{\tilde{\nu}} + R_3,
\]

where \( \lim_{\delta \to 0} R_3 = 0 \) (by condition 2). Now, by hypothesis ii) - iii), we have that

\[
0 > \langle \langle \tilde{\nu}, f_\gamma \rangle \rangle \geq \sum_k \int_{C_k} Q_Af_\gamma(z_k, \mu^a(x)) \, dx + \sum_k \int_{C_k} Q_Af_\gamma^\infty \left( z_k, \frac{d\mu^a}{d|\mu^a|}(x) \right) \, d|\mu^a| + R_3.
\]

Condition 1) implies that \( |\mu|(+\partial C_k) = 0 \) and we can apply the relaxation result Theorem 5.1 (see
Remark 5.2) in each cube $C_k$. We have that there exist sequences $\mu_{k,n} \in \mathcal{M}(C_k; \mathbb{R}^d)$ such that

$$
\langle \langle \hat{\nu}, f_\gamma \rangle \rangle \geq \sum_k \int_{C_k} QA f_\gamma (z_k, \mu^\alpha (x)) \, dx + \sum_k \int_{C_k} QA f_\gamma^\infty \left( z_k, \frac{d\mu^s}{d|\mu^s|} (x) \right) \, d|\mu^s| + R_\delta
$$

$$
= \sum_k \lim_{n \to \infty} \left( \int_{C_k} f_\gamma (z_k, \mu^\alpha_{k,n} (x)) \, dx + \int_{C_k} f_\gamma^\infty \left( z_k, \frac{d\mu^s_{k,n}}{d|\mu^s_{k,n}|} \right) \, d|\mu^s_{k,n}| \right) + R_\delta
$$

$$
= \sum_k \lim_{n \to \infty} \left( \int_{C_k} f_\gamma (x, \mu^\alpha_{k,n} (x)) \, dx + \int_{C_k} f_\gamma^\infty \left( x, \frac{d\mu^s_{k,n}}{d|\mu^s_{k,n}|} \right) \, d|\mu^s_{k,n}| \right) + \hat{R}_\delta + E_{\delta,n}
$$

where $\nu^k$ denotes a Young measure in $C_k$ generated by $\mu_{k,n}$ in $C_k$.

We note that $|E_{n,\delta}| \leq C\delta$ (4.18)

for some $C$ is independent of $n$ and of the partition of $\Omega$. Indeed, for $n$ large enough, we have that

$$
\gamma|\mu_{k,n}|(C_k) \leq \int_{C_k} QA f_\gamma (z_k, \mu^\alpha (x)) \, dx + \int_{C_k} QA f_\gamma^\infty \left( z_k, \frac{d\mu^s}{d|\mu^s|} (x) \right) \, d|\mu^s|
$$

$$
- \left( \int_{C_k} f_H (z_k, \mu^\alpha_{k,n} (x)) \, dx + \int_{C_k} f_H^\infty \left( z_k, a_{k,n} (x) \frac{d\mu^s}{d|\mu^s|} \right) \, d|\mu^s| \right) + |C_k|
$$

$$
\leq (CH + \gamma) (|C_k| + |\mu|(C_k)) - \left( \int_{C_k} f_H (z_k, \mu^\alpha_{k,n} (x) - \mu^\alpha (x)) \, dx + \int_{C_k} f_H^\infty \left( z_k, (a_{k,n} (x) - 1) \frac{d\mu^s}{d|\mu^s|} \right) \, d|\mu^s| \right)
$$

$$
+ L|\mu|(C_k) + |C_k|
$$

$$
\leq C(|\mu|(C_k) + |C_k|) - QA f_H (z_k, 0) |C_k|,
$$

where $\mu^\alpha_{k,n} = a_{k,n} n(x) \mu^s$ (see the construction in Remark 5.2), $C_H$ denotes the constant in the upper bound of $f_H$, i.e.,

$$
f_H (x, \xi) \leq C_H (1 + |\xi|),
$$

and $L$ denotes the Lipschitz constant of $QA f_H (x, \cdot)$ which can be taken independent of $x$ (it is bounded my a multiple of $C_H$, see Lemma 2.5 in [25] and page 32 in [9] for details in the gradient case, which is entirely similar to our setting). We thus have (using condition 3))

$$
|\mu_{k,n}|(\Omega) \leq \frac{1}{\gamma} \left( C(|\mu|(\Omega) + |\Omega|) - \int_{\Omega} QA f_H (x, 0) \, dx \right),
$$

and (4.18) follows from condition 2).

Set $\mu_n := \mu_{k,n}$ in $C_k$ and denote by $\nu$ the Young measure generated by $\mu_n$. Then, we arrive at

$$
\langle \langle \hat{\nu}, f_\gamma \rangle \rangle \geq \sum_k \langle \langle \nu^k, f_\gamma \rangle \rangle + \hat{R}_\delta = \lim_{n \to \infty} \int_{\Omega} f_\gamma (x, \mu^\alpha_n (x)) \, dx + \int_{\Omega} f_\gamma^\infty \left( x, \frac{d\mu^s_n}{d|\mu^s_n|} \right) \, d|\mu^s_n| + \hat{R}_\delta
$$

$$
= \langle \langle \nu, f_\gamma \rangle \rangle + \hat{R}_\delta.
$$

Since $\nu \in \mathcal{AY}_{\mu_0}$, letting first $\delta \to 0$ and then $\gamma \to 0$, we get

$$
\langle \langle \hat{\nu}, f_H \rangle \rangle \geq 0
$$

which contradicts (4.17).
5 A relaxation result

In this section we extend Theorem 1.2 of [6] to continuous integrands. Precisely, we prove that

**Theorem 5.1.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be a continuous function with linear growth, and assume that \( \text{Span}(\mathcal{C}) = \mathbb{R}^d \). We then have that

\[
\mathcal{F}(\mu) := \inf \left\{ \liminf_{\Omega} \int_{\Omega} f(\mu_n^*) \, dx + \int_{\Omega} f^\infty \left( \frac{d\mu_n^*}{d\mu^*} \right) d|\mu^*| : \mu_n \rightharpoonup \mu; A\mu_n \to 0, |u_n| \rightharpoonup \Lambda, A(\partial \Omega) = 0 \right\}
\]

is given by

\[
\mathcal{F}(\mu) = \begin{cases} \int_{\Omega} Q_Af(\mu^a(x)) \, dx + \int_{\Omega} (Q_Af)^\infty \left( \frac{d\mu^a}{d\mu^*} \right) d|\mu^*| & \text{if } \exists \zeta_0 \in \mathbb{R}^d : Q_Af(\zeta_0) > -\infty, \\ -\infty & \text{otherwise}. \end{cases}
\]

**Proof.** Suppose first that \( \exists \zeta_0 \in \mathbb{R}^d : Q_Af(\zeta_0) > -\infty \). Then (see Remark 2.18) we have that \( Q_Af \) is finite with linear growth.

Let \( \gamma > 0 \) and define

\[
f_\gamma(\xi) := f(\xi) + \gamma |\xi|.
\]

Our goal is to prove that

\[
\mathcal{F}(\mu) \leq \int_{\Omega} Q_Af_\gamma(\mu^a(x)) \, dx + \int_{\Omega} (Q_Af_\gamma)^\infty \left( \frac{d\mu^a}{d\mu^*} \right) d|\mu^*|,
\]

and the upper bound follows by letting \( \gamma \to 0 \).

Let \( \{u_n\} \subset C^\infty(\mathbb{R}^N; \mathbb{R}^d) \) be such that \( u_n \rightharpoonup \mu \), \( (u_n)(\Omega) \to (\mu)(\Omega) \), \( A\mu_n \to 0 \) in \( W^{-1,q}_{loc}(\Omega; \mathbb{R}^M) \) (such a sequence can be constructed by regularization of \( \mu \), see Lemma 2.22 in [6]). Fix \( n \) and let \( x_0 \in \Omega \). Consider cubes \( Q(x_0, r) \) with \( r > 0 \) small enough such that

\[
\left| u_n(x) - \frac{1}{Q(x_0, r)} \right| < \frac{1}{n} \quad (5.1)
\]

for every \( x \in Q(x_0, r) \). We can consider for each \( n \) a finite disjoint set of cubes \( Q(x_k, r_k) \subset \subset \Omega \) verifying the condition above and such that

\[
|\Omega \setminus \bigcup Q(x_k, r_k)| < \frac{1}{n}, \quad \int_{\Omega \setminus \bigcup Q(x_k, r_k)} |u_n| \, dx < \frac{1}{n} \quad (5.2)
\]

moreover we assume that \( r_k < \frac{1}{n} \). Define

\[
\tilde{u}_n(x) = \begin{cases} u_n + \Phi_n \left( \frac{x-x_k}{r_k} \right) (a_k^n - u_n), & \text{if } x \in Q(x_k, r_k) \\
_n, & \text{otherwise in } \Omega.
\end{cases}
\]

where \( a_k^n : = \int_{Q(x_k, r_k)} u_n \) and \( \Phi_n \in C^\infty_c(Q; [0, 1]) \) is chosen in such a way that

\[
\sum_{Q(x_k, r_k) \cap \Phi_n \left( \frac{x-x_k}{r_k} \right) < 1} |u_n| \, dx < \frac{1}{n} \quad (5.3)
\]

It is easy to check that

\[
||\tilde{u}_n - u_n||_{L^\infty} \to 0.
\]
thus it follows that \( \tilde{u}_n \overset{\ast}{\rightharpoonup} \mu, \langle \tilde{u}_n \rangle(\Omega) \to \langle \mu \rangle(\Omega) \), \( A\tilde{u}_n \to 0 \) in \( W^{-1,q}_{\text{loc}}(\Omega; \mathbb{R}^M) \).

For each \( k, n \) we can find \( \omega_{k,n} \in C^\infty_c(Q; \mathbb{R}^d) \) with \( A\omega_{k,n} = 0 \) and \( \int_Q \omega_{k,n} = 0 \) such that

\[
\int_Q f_\gamma(a^n_k + \omega_{k,n}) \leq QA f_\gamma(a^n_k) + \frac{1}{n}. \tag{5.4}
\]

Note that there exists a constant \( K_n \) such that

\[
||\omega_{k,n}||_{L^\infty} < K_n
\]

for all \( k \). By modifying the function \( \Phi_n \in C^\infty_c(Q; [0,1]) \), if necessary, we may assume that \( \Phi_n \equiv 1 \) on \( Q(0, \tau_n) \), where \( \tau_n \) is chosen in such a way that

\[
K_n|\Omega|(1 - \tau_n^N) \leq \frac{1}{n}. \tag{5.5}
\]

Define

\[
v_{m,n} = \begin{cases} \tilde{u}_n + \Phi_n \left( \frac{x - x_k}{r_k} \right) \omega_{k,n} \left( m \left( \frac{x - x_k}{r_k} \right) \right), & \text{if } x \in Q(x_k, r_k) \\ u_n, & \text{otherwise in } \Omega. \end{cases}
\]

We claim that

\[
\int_{\Omega} |v_{m,n}| \, dx \leq C \tag{5.6}
\]

Since \( \tilde{u}_n \) is bounded in \( L^1 \) and \( ||\Phi_n||_{\infty} = 1 \) to see (5.6) it is enough to prove that

\[
\sum r_N^k \int_{Q(x_k, r_k)} |\omega_{k,n} \left( m \left( \frac{x - x_k}{r_k} \right) \right)| \leq C.
\]

By a change of variables we get

\[
\sum r_N^k \int_{Q(x_k, r_k)} |\omega_{k,n} \left( m \left( \frac{x - x_k}{r_k} \right) \right)| = \sum r_N^k \int_Q |\omega_{k,n}(y)| \tag{5.7}
\]

Now we use (5.4) to bound (5.7). Indeed we have

\[
\sum r_N^k QA f_\gamma(a^n_k) \leq \int_{\Omega} QA f_\gamma(\tilde{u}_n) + \frac{C}{n} \leq C
\]

so that

\[
\sum r_N^k \int_Q f_\gamma(a^n_k + \omega_{k,n}) \leq C \tag{5.8}
\]

On the other hand

\[
\sum r_N^k \int_Q f(a^n_k + \omega_{k,n}) \geq \sum r_N^k QA f(a^n_k) \geq \int_{\Omega} QA f(\tilde{u}_n) - \frac{C}{n} \geq -C
\]

which together with (5.8) implies

\[
\sum r_N^k \int_Q |a^n_k + \omega_{k,n}| \leq C
\]

Therefore (5.7) follows, and thus (5.6).

Note that as \( \int_Q \omega_{k,n} = 0 \) we have by the Riemann-Lebesgue Lemma that

\[
\omega_{k,n} \left( m \left( \frac{x - x_k}{r_k} \right) \right) \overset{\ast}{\rightharpoonup} 0 \tag{5.9}
\]
as $m \to \infty$. In addition
\[ \omega_{k,n} \left( m \left( \frac{x - x_k}{r_k} \right) \right) \to 0 \]
and so $A v_{m,n} \to 0$ in $W^{-1,q}_{\text{loc}}(Q; \mathbb{R}^M)$ as $n, m \to \infty$. Thus we can find a sequence $v_{m,n}$ such that $v_{m,n} \to \mu$ and $A v_{m,n} \to 0$ in $W^{-1,q}_{\text{loc}}(Q; \mathbb{R}^M)$.

Let $|\tilde{u}_n| = \Lambda$, $\sum \omega_{k,n} \left( m \left( \frac{x - x_k}{r_k} \right) \right) 1_{Q(x_k, r_k)} \to \Lambda_1$ and $\Omega' \subset \subset \Omega$ be an open set such that
\[ \Lambda(\partial \Omega') = \Lambda_1(\partial \Omega') = 0 \] (5.10)

Let $\Psi \in C_c^\infty(Q'; [0,1])$ and
\[ \tilde{v}_n := \Psi(v_{m,n}) + (1 - \Psi)\tilde{u}_n \]

We then have using (5.1)-(5.3), (5.5)
\[ \int_{\Omega} f_\gamma(\tilde{v}_n) \, dx \leq \int_{\Omega'} f_\gamma(v_{m,n}) + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |v_{m,n}|) + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |\tilde{u}_n|) \]
\[ \leq \sum Q(x_k, r_k) f_\gamma \left( a^n_k + \omega_{k,n} \left( m_n \left( \frac{x - x_k}{r_k} \right) \right) \right) \]
\[ + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |v_{m,n}|) + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |\tilde{u}_n|) + \frac{C}{n} \]
\[ \leq \sum Q(x_k, r_k) f_\gamma(\tilde{u}_n) + C \int_{\Omega' \setminus \{\Psi < 1\}} \left( 1 + |\tilde{u}_n| + \omega_{k,n} \left( m_n \left( \frac{x - x_k}{r_k} \right) \right) \right) \]
\[ + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |v_{m,n}|) + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |\tilde{u}_n|) + \frac{C}{n} \]
\[ \leq \int_{\Omega} Q \lambda f_\gamma(\tilde{u}_n) + C \int_{\Omega' \setminus \{\Psi < 1\}} \left( 1 + |\tilde{u}_n| + \omega_{k,n} \left( m_n \left( \frac{x - x_k}{r_k} \right) \right) \right) \]
\[ + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |v_{m,n}|) + C \int_{\Omega' \setminus \{\Psi < 1\}} (1 + |\tilde{u}_n|) + \frac{C}{n} \]

where the sum above is taken over the cubes $Q(x_k, r_k)$ that intersect $\Omega'$. Now the result follows by letting $n \to \infty$, $\Psi \to 1_{\Omega'}$, (5.10) and from the Corollary 2.11 (to the Reshetnyak Theorem) in [6]. Finally let $\Omega' \to \Omega$.

The lower bound follows from Theorem 1.1 in [6].

The proof in the case $Q_a f(\zeta_0) = -\infty$ for some $\zeta_0 \in \mathbb{R}^d$ (and hence $Q \lambda f(\zeta) = -\infty$, $\forall \zeta \in \mathbb{R}^d$ (cf. Remark 2.18)) follows the lines of the proof of Lemma 5.3 below. $\Box$

**Remark 5.2.** It follows from the theorem above that for every $\mu$ and every open set $Q \subset \Omega$ with $|\mu(\partial Q)| = 0$ there exists a sequence of measures $(\mu_n) \subset \mathcal{M}(Q; \mathbb{R}^d)$ with $\mu_n = \mu$ on a neighborhood of $\partial Q$, $\mu_n \to \mu$, $A(\mu_n) \to 0$ such that
\[ \int_Q f_\gamma(\mu_n(x)) \, dx + \int_Q f_\gamma(\mu_n(x)) \, d|\mu_n| \to \int_Q Q \lambda f_\gamma(\mu(x)) \, dx + \int_Q Q \lambda f_\gamma(\mu(x)) \, d|\mu|, \]
where $f_\gamma := f + \gamma |\xi|$ and $\gamma > 0$.
Proof. First we prove that there exists a sequence \((u_n) \subset M(Q; \mathbb{R}^d)\), with \(u_n \rightharpoonup^* \mu, A(u_n) \to 0\) in \(W_{\text{loc}}^{-1,q}(\Omega; \mathbb{R}^M)\), such that
\[
\int_Q f_\gamma(u_n) \, dx \to \int_Q Q_A f_\gamma(\mu^a(x)) \, dx + \int_Q (Q_A f_\gamma)^\infty \left( \frac{d\mu^s}{d|\mu^s|} \right) \, d|\mu^s|.
\]
Indeed let \((\varphi_i) \subset C(\overline{\Omega})\) be a countable dense subset and for every \(n\) choose \(u_n\) such that
\[
\int_Q Q_A f_\gamma(\mu^a(x)) \, dx + \int_Q (Q_A f_\gamma)^\infty \left( \frac{d\mu^s}{d|\mu^s|} \right) \, d|\mu^s| + \frac{1}{n} \geq \int_Q f_\gamma(u_n) \, dx \tag{5.11}
\]
and
\[
\left| \int_Q u_n \varphi_i - \int_Q \varphi_i \, d\mu \right| < \frac{1}{n}
\]
for \(i = 1, \ldots, n\). Moreover we may assume that
\[
||A u_n||_{W^{-1,q}(\Omega_n)} < \frac{1}{n},
\]
where \(\Omega_n := \{x \in \Omega : \text{dist}(x, \partial \Omega) < \frac{1}{n}\}\). We have that
\[
||u_n||_{L^1} \leq C
\]
which follows from (5.11), because
\[
\gamma ||u_n|| \leq C - \int_Q f(u_n) \leq C
\]
Thus by density we have that \(u_n \rightharpoonup^* \mu\).

Then it is enough to modify the sequence to get \(\mu\) on a neighborhood of the boundary. \(\square\)

**Lemma 5.3.** Let \(f_H \in E(\Omega \times \mathbb{R}^d)\) be such that
\[
\langle \langle \nu, f_H \rangle \rangle > 0
\]
for every \(\nu \in AY_{\mu,0}\). Then there exits a dense subset \(D \subset \Omega\) such that for every \(x \in D\) the relaxation \(Q_A f_H(x,.)\) is finite with linear growth.

**Proof.** Suppose there exists an open set \(A \subset \Omega\) such that
\[
Q_A f_H(x,.) \equiv -\infty \tag{5.12}
\]
for every \(x \in A\). Fix \(\delta > 0\) and choose a compact set \(B \subset A\) such that \(\mu^a\) is continuous and
\[
|A \setminus B| < \delta, \quad \int_{A \setminus B} |\mu^a| \, dx < \delta \tag{5.13}
\]
Fix a large positive number \(L\). For every \(z \in B\) we can find a function \(w_z \in C^\infty_{\text{per}}(Q; \mathbb{R}^d)\), with \(A w_z = 0\) and \(\int_Q w_z = 0\), such that
\[
\int_Q f_H(z, \mu^a(z) + w_z(x)) \, dx < -L. \tag{5.14}
\]
Choose cubes \(Q(z, r)\), with \(r < r_z\) small enough, such that
\[
|T g_H(x, \xi) - T g_H(z, \xi)| < \frac{1}{1 + \int_Q |w_z|} \tag{5.15}
\]

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for every \( x \in B \cap Q(z, r) \), where \( g_H(x, \xi) = f_H(x, \mu^a(x) + \xi) \). Moreover, if \( z \) is a Lebesgue point for \( 1_B \), we may also assume that
\[
\frac{1}{r^n} \int_{Q(z,r) \setminus B} < \frac{1}{||w_z||}\infty.
\]
The cubes described above set up a fine covering for the compact set \( B \), from which we can extract a finite subcovering \( Q_k = Q(x_k, r_k) \).

Let \( \varphi_l \in C_c^\infty(Q) \) such that \( |\{\varphi_l \neq 1\}| \to 0 \). Define
\[
\mu_{l,n} = \begin{cases}
  \mu + \varphi_l \left( \frac{x-x_k}{r_k} \right) \omega_k \left( n \left( \frac{x-x_k}{r_k} \right) \right), & \text{if } x \in Q_k \\
  \mu, & \text{otherwise in } \Omega.
\end{cases}
\]
Then we can find a sequence \( \mu_{l,n} \) such that
\[
\mu_{l,n} \rightharpoonup \mu, \quad \mathcal{A} \mu_{l,n} \to 0
\]
and passing to a subsequence, not relabeled, we may assume that generates a Young measure \( \nu \in \mathcal{A}_{\mu,0} \).

On the other hand we have
\[
\langle \delta_{\mu_{l,n}}, f_H \rangle = \int_{Q \cup Q_k} f_H(\mu^a(x)) \, dx + \int_{Q} f_H \left( \frac{d\mu^a}{d|\mu^a|} \right) \, d|\mu^a| \\
+ \sum_k \int f_H \left( x, \mu^a(x) + \varphi_l \left( \frac{x-x_k}{r_k} \right) \omega_k \left( n \left( \frac{x-x_k}{r_k} \right) \right) \right) \, dx
\]
\[
= F + \sum_k \int f_H \left( x, \mu^a(x) + \varphi_l \left( \frac{x-x_k}{r_k} \right) \omega_k \left( n \left( \frac{x-x_k}{r_k} \right) \right) \right) \, dx + E_1
\]
\[
= F + \sum_k \int f_H \left( x, \mu^a(x) + \omega_k \left( n \left( \frac{x-x_k}{r_k} \right) \right) \right) \, dx + E_2 + E_1
\]
where \( |E_1| \to 0 \) as \( l \to \infty \), and by (5.13)-(5.16) the estimation of \( E_2 \) is as follows:
\[
|E_2| = \left| \sum_k \int_{Q_k} f_H \left( x, \mu^a(x) + \omega_k \left( n \left( \frac{x-x_k}{r_k} \right) \right) \right) \, dx \right| \\
\leq \left| \sum_k \int_{Q_k \cap B} f_H \left( x, \mu^a(x) + \omega_k \left( n \left( \frac{x-x_k}{r_k} \right) \right) \right) \, dx \right| \\
+ C \sum_k \left( \int_{Q_k \setminus B} (1 + |\mu_a|) \, dx + r_k^N \right)
\]
\[
\leq |A| + C(2\delta + |A|) = |A|(C + 1) + 2\delta C
\]
Thus we have
\[
\lim \langle \delta_{\mu_{l,n}}, f_H \rangle \leq F - L|B| + |A|(C + 1) + 2\delta C < 0,
\]
for appropriate \( L > 0 \). On the other we have
\[
\langle \delta_{\mu_{l,n}}, f_H \rangle \to \langle \nu, f_H \rangle > 0
\]
from which a contradiction follows.

Thus we cannot have (5.12), and as the characteristic cone of \( \mathcal{A} \) generates all the space \( \mathbb{R}^d \) we have that \( Q_{A} f(x,.) \) is finite, with linear growth, for \( x \) in a dense subset \( D \) of \( \Omega \).

\( \square \)

Acknowledgements: This work was partially supported by the Fundação para a Ciência e a Tecnologia (FCT/Portugal) and by the project UTA-CMU/MAT/0005/2009.
References


