THIN-FILM \( \Gamma \)-LIMIT OF THE MICROMAGNETIC FREE ENERGY

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Abstract. The asymptotic behavior of the micromagnetic free energy governing a ferromagnetic film is studied as its thickness gets smaller and smaller compared to its cross section. Here the static Maxwell equations are treated as a Murat’s constant rank PDE constraint on the energy functional. In contrast to previous work this approach allows to keep track of the induced magnetic field without solving the magnetostatic equations. In particular, the mathematical results of Gioia and James [Proc. R. Soc. Lond. A 453 (1997), pp. 213–223] regarding convergence of minimizers are extended by giving a full characterization of the corresponding \( \Gamma \)-limit.

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1. Introduction

Over the last twenty years there has been tremendous scientific progress in the research on thin-film devices pushing forward technology and leading to important industrial applications, first and foremost in the wide fields of semiconductor devices and optical coatings. By a thin film one understands a layer of material whose thickness ranges between a fractional amount of a nanometer and a couple of micrometers. When using films for computer data storage and solar cells, a deep understanding of the ferromagnetic properties of the thin material is of great importance [24]. A widely used mathematical procedure to achieve exactly that takes the theory of micromagnetics for bulk bodies [2, 4, 5, 16, 17, 26] as a starting point and derives a reduced theory capable of capturing the specific features of thin material layers by means of dimension reduction techniques.

It was in [14] that for the first time authors studied convergence of minimizers of the micromagnetic energy on a film whose thickness \( \varepsilon \) gets smaller and smaller compared to its cross section of length scale \( l \). More recent work within this scaling regime can be found for instance in [1,6], where properties of thin superconducting films and manifolds are investigated by deriving the dimension reducing \( \Gamma \)-limit of 3-d Ginzburg-Landau-type functionals under consideration of external magnetic fields.

In this paper we follow basically the approach of [14], but use an equivalent formulation in the sense of [8]. The latter indicates that micromagnetism is actually one of the examples, where the mathematical modeling of physical phenomena within a variational formulation leads to functionals that do not simply depend
on gradient fields. Instead one faces more intricate partial differential constraints which involve an interaction of divergence- and curl-free fields. More precisely, the magnetostatic equations read

\[ \text{div}(\vec{m} + \vec{h}) = 0 \quad \text{in } \mathbb{R}^3, \]
\[ \text{curl } \vec{h} = 0 \quad \text{in } \mathbb{R}^3, \]

where \( \Omega \subset \mathbb{R}^3 \) stands for the reference configuration of the ferromagnetic body, \( \vec{m} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3 \), which whenever necessary is identified with its trivial extension to the whole space by zero, denotes the magnetization and \( \vec{h} : \mathbb{R}^3 \to \mathbb{R}^3 \) is the induced magnetic field.

In the existing literature this mathematical difficulty is essentially tackled by explicitly solving (or proving existence of solutions to) the magnetostatic equations in their weak formulation at some point and expressing \( \vec{h} \) in terms of \( \vec{m} \). In the following we will give a rigorous \( \Gamma \)-convergence based proof of 3d–2d dimension reduction for the micromagnetic functional which works directly with the PDE constraint and allows us to keep track in the limit not only of the magnetization, but also of the magnetic field.

Let \( \Omega_\varepsilon = \omega \times (0, \varepsilon) \subset \mathbb{R}^3 \) with \( \varepsilon > 0 \) be the domain with cross section \( \omega \subset \mathbb{R}^2 \) occupied by a ferromagnetic body. The free energy that emerges in the theory of micromagnetism [4,20] is given by

\[ E_\varepsilon[\vec{m}, \vec{h}] = \begin{cases} \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \alpha |\nabla \vec{m}|^2 + \varphi(\vec{m}) \, dy + \frac{1}{2} \int_{\mathbb{R}^3} |\vec{h}|^2 \, dy, & \text{if } (\vec{m}, \vec{h}) \in \mathcal{V}_\varepsilon, \\ \infty, & \text{otherwise}. \end{cases} \]

Here \( \alpha > 0 \) is a material constant and \( \varphi : \mathbb{R}^3 \to [0, \infty) \) is a continuous, even function featuring crystallographic symmetry. Further,

\[ \mathcal{V}_\varepsilon = \{ (\vec{m}, \vec{h}) \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3) : \mathcal{A}^{\text{mag}} \begin{pmatrix} \vec{m} \\ \vec{h} \end{pmatrix} = 0 \text{ in } \mathbb{R}^3, |\vec{m}| = m_s \text{ in } \Omega_\varepsilon \}. \]

Let us remark that, depending on the regularity of \( \vec{m} \), the last equation may only be fulfilled pointwise almost everywhere in \( \Omega_\varepsilon \). We will adopt that notational convention throughout this work.

The first order PDE constraint in the definition of \( \mathcal{V}_\varepsilon \), namely

\[ \mathcal{A}^{\text{mag}} \begin{pmatrix} \vec{m} \\ \vec{h} \end{pmatrix} := \begin{pmatrix} \text{div} \\ \text{curl} \end{pmatrix} \begin{pmatrix} \vec{m} \\ \vec{h} \end{pmatrix} = \begin{pmatrix} \text{div}(\vec{m} + \vec{h}) \\ \text{curl } \vec{h} \end{pmatrix} = 0, \]

conveys the magnetostatic equations, which in terms of micromagnetics are frequently referred to as static Maxwell equations. The operator \( \text{curl} \) is supposed to be interpreted in the way \( \text{curl } = \nabla \times \), i.e.

\[ \text{curl } \vec{h} = (\partial_2 \vec{h}_3 - \partial_3 \vec{h}_2, \partial_3 \vec{h}_1 - \partial_1 \vec{h}_3, \partial_1 \vec{h}_2 - \partial_2 \vec{h}_1)^T. \]

Notice that here, and in what follows, all occurring differential operators and partial derivatives are to be understood in the sense of distributions, for example \( \text{div}(\vec{m} + \vec{h}) = 0 \) in \( \mathbb{R}^3 \) means \( \int_{\mathbb{R}^3} (\vec{m} + \vec{h}) \cdot \nabla \phi \, dy = 0 \) for all test functions \( \phi \in C^\infty(\mathbb{R}^3) \).

Physically speaking, the nonconvex constraint \( |\vec{m}| = m_s \) in \( \Omega_\varepsilon \) encodes the fundamental assumption that the body is locally saturated with saturation magnetization \( m_s > 0 \). The second term in the definition of \( E_\varepsilon \) is the anisotropy energy, which
penalizes magnetizations varying from special directions within the crystal lattice of the ferromagnet. The latter are commonly called directions of easy magnetization. An exchange energy contribution is conveyed by the first term of $E_\varepsilon$. It results from a force tending to align magnetic moments of neighboring atoms and therefore favors regions of constant magnetization. The third summand in $E_\varepsilon$ is an integral over the whole space $\mathbb{R}^3$ modeling the energy of the magnetic field $\vec{h}$ induced by $\vec{m}$. These three energy components impose competing requirements on the magnetization. As a result this gives rise to fine structures in form of Weiss domains, separated by thin Bloch walls. For more details on the physical motivation and interpretation of the nonlocal and nonconvex energy $E_\varepsilon$ see for example [2,4,5,8,14,16,17,20,26] and the references therein.

As originally stated in [25] and further discussed in [13], $A_{\text{mag}}$ is a first order differential operator meeting Murat’s constant rank property [23], i.e. the symbol $A_{\text{mag}}(\xi)$ satisfies

$$\text{rank} \ A_{\text{mag}}(\xi) = \text{const.} \quad \text{for all } \xi \in S^2. \quad (1.1)$$

Indeed, $A_{\text{mag}}(\xi) \in \text{Lin}(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{R} \times \mathbb{R}^3)$ and it holds

$$\ker A_{\text{mag}}(\xi) = \{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : \xi \cdot (x + y) = 0, \xi \times y = 0 \} = \{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : y = \lambda \xi, \xi \cdot x = -\lambda, \lambda \in \mathbb{R} \}.$$

This implies $\dim \ker A_{\text{mag}}(\xi) = 3$. Hence, the problem we are interested in can be studied in the more abstract context of dimension reduction for functionals on $A$-free vector fields, where $A$ is some constant rank operator. The work on variational problems within the $A$-free framework can be traced back to [7] and was pushed on by Fonseca and Müller [13], who came up with the notion of $A$-quasiconvexity (in its modern sense) and studied lower semicontinuity of functionals involving integrands with this property. Since then, a lot of papers investigating for instance relaxation, homogenization and Young measures in the $A$-free setting have emerged [3,11–13]. The first article to cover 3d–2d asymptotic analysis in such generality is [18] (see [19] for the case of functionals on solenoidal vector fields, meaning $A = \text{div}$). In fact, the latter provides the technical basis for the results presented in the following. Let us finally point out that there is a wide literature on the special case of thin-film limits of gradient dependent problems [10,21,22], which corresponds to $A = \text{curl}$.

In order to obtain a variational problem on the fixed domain $\Omega_1 = \omega \times (0, 1)$, we apply the standard parameter rescaling,

$$x = (x', x_d) = (y', \varepsilon^{-1} y_d) \quad \text{and} \quad m(x) = \bar{m}(x', \varepsilon x_d), \ h(x) = \bar{h}(x', \varepsilon x_d)$$

with $x' = (x_1, \ldots, x_{d-1})$, and $E_\varepsilon$ transforms into

$$E_\varepsilon(m, h) = \left\{ \begin{array}{ll}
\int_{\Omega_1} \alpha |\nabla m|^2 + \varphi(m) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h|^2 \, dx, & \text{if } (m, h) \in \mathcal{U}_\varepsilon, \\
\infty, & \text{otherwise,}
\end{array} \right.$$
The rescaled operators $\nabla_\varepsilon$ of $\nabla = (\nabla', \partial_3) = (\partial_1, \partial_2, \partial_3)$ and $A_{\varepsilon}^{\text{mag}}$ of $A^{\text{mag}}$ read

$$\nabla_\varepsilon = \left(\nabla', \frac{1}{\varepsilon} \partial_3\right)^T$$

and

$$A_{\varepsilon}^{\text{mag}} = A_{\varepsilon}^{\text{mag}}(\nabla_\varepsilon),$$

respectively. Accordingly, we define

$$A^0_{\text{mag}}(m) := \begin{pmatrix} \partial_3(m_3 + h_3) \\ -\partial_2 h_2 \\ \partial_3 h_1 \\ \partial_1 h_2 - \partial_2 h_1 \end{pmatrix} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

**Theorem 1.1.** The $\Gamma$-limit of $F_\varepsilon$ for $\varepsilon \to 0^+$ with respect to weak convergence in $W^{1,2}(\Omega_1; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$ exists and is represented as

$$F_0[m, h] = \begin{cases} \int_{\Omega_1} \alpha |\nabla' m|^2 + \varphi(m) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h|^2 \, dx, & \text{if } (m, h) \in \mathcal{U}_0, \\ \infty, & \text{otherwise}, \end{cases}$$

with

$$\mathcal{U}_0 = \left\{ (m, h) \in W^{1,2}(\Omega_1; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3) : A^0_{\text{mag}} \left( \begin{pmatrix} m \\ h \end{pmatrix} \right) = 0 \quad \text{in} \quad \mathbb{R}^3 \right\}.$$

and $A^0_{\text{mag}}$ defined through

$$A^0_{\text{mag}} \left( \begin{pmatrix} m \\ h \end{pmatrix} \right) := \begin{pmatrix} \varphi(m) \end{pmatrix} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

Note that the result of Theorem 1.1 is in complete agreement with the limiting micromagnetic energy derived in [14]. To demonstrate this we will prove that the $\Gamma$-limit of $F_\varepsilon$ can be equivalently expressed in the form

$$\tilde{F}_0[\tilde{m}, \tilde{h}] = \begin{cases} \int_\omega \alpha |\nabla' \tilde{m}|^2 + \varphi(\tilde{m}) + \frac{1}{2} \tilde{m}_3^2 \, dx', & \text{if } (\tilde{m}, \tilde{h}) \in \tilde{\mathcal{U}}_0, \\ \infty, & \text{otherwise}, \end{cases}$$

where

$$\tilde{\mathcal{U}}_0 = \left\{ (\tilde{m}, \tilde{h}) \in W^{1,2}(\omega; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3) : |\tilde{m}| = m_s \text{ in } \omega, \ \tilde{h} = -(0, 0, \tilde{m}_3)^T \right\}.$$

Indeed, if we identify $\tilde{m}$ and $\tilde{h}$ with their constant extensions in space direction $x_3$ cut off outside $\Omega_1$, it is sufficient to show that

$$\mathcal{U}_0 = \tilde{\mathcal{U}}_0.$$
The reverse inclusion follows simply from the above mentioned identification and a straightforward calculation.

In view of its representation $\tilde{F}_0$, the $\Gamma$-limit of $F_\varepsilon$ is purely two-dimensional. The third additive term in the density of $\tilde{F}_0$ indicates that magnetizations pointing out of plane are penalized and hence less favorable when it comes to energy minimization. Interestingly, we also observe that the limit functional $\tilde{F}_0$ is local. In fact, Maxwell’s equations disappear as $\varepsilon$ tends to zero, so that the asymptotic problem is free of any magnetostatic constraints of this form. Instead, the relation between $\tilde{m}$ and $\tilde{h}$ is ruled by the pointwise equation $\tilde{h} = -(0,0,\tilde{m}_3)^T$. For a detailed interpretation of our main result with respect to physical and engineering applications we refer to [14, Section 5]. There, Gioia and James analyze the qualitative effect of external fields and the practical scope of the theory regarding the thickness of films by giving effective estimates for some relevant materials.

Compared to [14, Theorem 4.1], where the authors focus exclusively on the convergence of minimizers of $F_\varepsilon$, the main finding of this paper provides a mathematically stronger statement. Another advantage of our approach is the convenient access to complete information about the induced field $\tilde{h}$, which is naturally included in $\tilde{F}_0$. In contrast, the reasoning in [14] requires explicit solving of the magnetostatic equations together with a limit analysis for the solutions as $\varepsilon \to 0^+$ to gain the same insight.

The choice of the appropriate scaling for the micromagnetic energy in this work rests fundamentally upon the assumption that the material parameter $\alpha$ does not depend on the thickness $\varepsilon$. We refer to [9] for a paper investigating a different reduction regime. Here an additional characteristic length scale $d$ of the magnetic material is introduced and related to $\varepsilon$ and $l$, the length scale of the cross section, in the sense that the $\Gamma$-limit for $d^2/|l\varepsilon| \to 0$ and $\varepsilon/l \to 0$ is determined. A detailed comparison of these two scaling regimes considering the particular benefits and drawbacks concerning their applicability can be found in [14, Section 5] and [9, Section 3.2]. Let us mention that, in particular, the results obtained here are physically relevant only for thin-film samples of sufficiently small lateral expansion.

In [18] dimension reduction within the general $A$-free setting is investigated. One of the purposes of this work is to illustrate the power of the concepts and tools developed there by means of another example of a physically relevant situation. An immediate first application to the bending of elastic thin films is studied in [18, Section 5].

Since the functional $F_\varepsilon$ contains first order derivatives of $m$ and a nonconvex constraint, it does not exactly fit into the framework of [18]. (Notice that it is not clear, in particular, how to extend that theory to mixed order differential operators.) Therefore, we consider $F_\varepsilon$ as split into a part which is convex in the higher derivatives of $m$ and one that is in line with [18]. It is due to the coercivity of the first term that relaxation effects with respect to $m$ are prevented. When it comes to the construction of a recovery sequence, the crucial step is to exploit a tool introduced in [18], which yields convergence of the symbols of $A_*^{\text{mag}}$ for $\varepsilon \to 0^+$. That provides a first candidate for the recovery sequence which, however, lacks the necessary regularity and fails to meet the nonconvex constraint imposed by the requirement of local saturation. To handle this issue, we select the constant sequence with respect to magnetization and adjust the exterior fields with the help of projection operators onto curl-$\varepsilon$-free fields.
In the remaining two sections we give the detailed proof of Theorem 1.1 by showing separately the required upper and lower bounds. Notice that throughout this work we are using generalized sequences with index \( \varepsilon > 0 \), like \((u_\varepsilon)_\varepsilon\), by which we refer to any sequence \((u_\varepsilon)_j\) with \( \varepsilon_j \to 0^+ \) as \( j \to \infty \).

2. PROOF OF COMPACTNESS AND THE LOWER BOUND

We begin by proving the following compactness result, which is essentially based on the coercivity of the micromagnetic energy. Notice that extracted subsequences are not relabeled in the sequel.

**Proposition 2.1 (Compactness).** Let \( \varepsilon_j \to 0^+ \) for \( j \to \infty \). Further assume \((m_\varepsilon_j, h_\varepsilon_j)_j \subset W^{1,2}(\Omega_1;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^3)\) to be a bounded energy sequence, i.e.

\[
F_{\varepsilon_j}[m_\varepsilon_j, h_\varepsilon_j] \leq C < \infty \quad \text{for all } j \in \mathbb{N}.
\]

Then there is a subsequence \((m_\varepsilon_j, h_\varepsilon_j)_j\) and \((m, h) \in W^{1,2}(\Omega_1;\mathbb{R}^3) \times L^2(\mathbb{R}^3;\mathbb{R}^3)\) such that

\[
m_\varepsilon_j \rightharpoonup m \quad \text{in } W^{1,2}(\Omega_1;\mathbb{R}^3),

h_\varepsilon_j \to h \quad \text{in } L^2(\mathbb{R}^3;\mathbb{R}^3)
\]

for \( j \to \infty \). Moreover, it holds \((m, h) \in \mathcal{U}_0\).

**Proof.** In view of the constraint \( |m_\varepsilon_j| = m_s \) in \( \Omega_1 \) and the fact that

\[
\|\nabla_\varepsilon_j m_\varepsilon_j\|_{L^2(\Omega_1;\mathbb{R}^{3 \times 3})} \leq C < \infty \quad \text{for all } j \in \mathbb{N}, \quad (2.1)
\]

one infers that \( \|m_\varepsilon_j\|_{W^{1,2}(\Omega_1;\mathbb{R}^{3 \times 3})} \) is bounded uniformly with respect to \( j \), which implies the existence of a subsequence \((m_\varepsilon_j)_j\) and a function \( m \in W^{1,2}(\Omega_1;\mathbb{R}^3)\) such that \( m_\varepsilon_j \to m \) in \( W^{1,2}(\Omega_1;\mathbb{R}^3)\). By compact embedding we find (after passing to a subsequence) that \( m_\varepsilon_j \to m \) pointwise a.e. in \( \Omega_1 \) as \( j \to \infty \), so that \( |m| = m_s \) in \( \Omega_1 \). Recalling the definition of \( \nabla_\varepsilon \) in (1.2) we conclude from (2.1) that \( \partial_3 m_\varepsilon_j \to 0 \) in \( L^2(\Omega_1;\mathbb{R}^3) \). Thus, \( \partial_3 m = 0 \) in \( \Omega_1 \) by uniqueness of the limit. Since the induced energy contribution of \( F_{\varepsilon_j}[m_\varepsilon_j, h_\varepsilon_j] \) is bounded, one can extract a subsequence of \((h_\varepsilon_j)_j\) satisfying

\[
h_\varepsilon_j \to h \quad \text{in } L^2(\Omega_1;\mathbb{R}^3)
\]

for some \( h \in L^2(\Omega_1;\mathbb{R}^3) \). The expression \( \text{curl}_\varepsilon_j h_\varepsilon_j = 0 \) in \( \mathbb{R}^3 \) for all \( j \in \mathbb{N} \) is equivalent to

\[
\partial_3(h_\varepsilon_j)_3 - 1/\varepsilon_j \partial_3(h_\varepsilon_j)_2 = 0 \quad \text{in } \mathbb{R}^3,

1/\varepsilon_j \partial_3(h_\varepsilon_j)_1 - \partial_1(h_\varepsilon_j)_3 = 0 \quad \text{in } \mathbb{R}^3,

\partial_1(h_\varepsilon_j)_2 - \partial_2(h_\varepsilon_j)_1 = 0 \quad \text{in } \mathbb{R}^3.
\]

When passing to the limit \( j \to \infty \), it follows \( \partial_3 h_2 = \partial_3 h_1 = 0 \) and \( \partial_1 h_2 = \partial_2 h_1 \) in \( \mathbb{R}^3 \). Finally, we exploit \( \text{div}_\varepsilon_j(m_\varepsilon_j + h_\varepsilon_j) = 0 \) in \( \mathbb{R}^3 \) to derive

\[
\partial_3(m_3 + h_3) = 0 \quad \text{in } \mathbb{R}^3.
\]

This shows \((m, h) \in \mathcal{U}_0\). □
Let $m_\varepsilon \to m$ in $W^{1,2}(\Omega_1; \mathbb{R}^3)$ and $h_\varepsilon \to h$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ as $\varepsilon$ tends to zero. Consider any $\varepsilon_j \to 0^+$ for $j \to \infty$ and assume that
\[
\lim_{j \to \infty} F_{\varepsilon_j}[m_{\varepsilon_j}, h_{\varepsilon_j}] = \lim_{j \to \infty} F_{\varepsilon_j}[m_{\varepsilon_j}, h_{\varepsilon_j}] < \infty,
\]
otherwise the corresponding liminf-inequality is immediate. Then, up to a subsequence, $(m_{\varepsilon_j}, h_{\varepsilon_j}) \subset U_{\varepsilon_j}$ is of bounded energy and $(m, h) \in U_0$ by Proposition 2.1.

In view of the compact embedding $W^{1,2}(\Omega_1; \mathbb{R}^3) \hookrightarrow L^2(\Omega_1; \mathbb{R}^3)$ we obtain a subsequence with $m_{\varepsilon_j} \to m$ pointwise a.e. in $\Omega_1$, so that by the lower semicontinuity of the $L^2$-norm and Fatou’s lemma,
\[
\lim_{j \to \infty} F_{\varepsilon_j}[m_{\varepsilon_j}, h_{\varepsilon_j}] \geq \lim_{j \to \infty} \int_{\Omega_1} \alpha |\nabla' m_{\varepsilon_j}|^2 + \varphi(m_{\varepsilon_j}) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h_{\varepsilon_j}|^2 \, dx.
\]
Hence,
\[
\liminf_{\varepsilon \to 0^+} F_{\varepsilon}[m_\varepsilon, h_\varepsilon] \geq F_0[m, h],
\]
which is the required liminf-inequality. This concludes the proof of the lower bound.

3. Construction of a recovery sequence

The upcoming section is based on arguments involving operators of the following form: For given matrices $A^{(1)}, \ldots, A^{(d)} \in \mathbb{R}^{l \times n}$, we define a linear partial differential operator of first order $A$ through
\[
Au := \sum_{k=1}^d A^{(k)} \partial_k u, \quad L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^n).
\]
(3.1)
The symbol $A_A$ of $A$ is given by
\[
A_A(\xi) := \sum_{k=1}^d A^{(k)} \xi_k, \quad \xi \in \mathbb{R}^d.
\]
The essential assumption on $A$ is Murat’s constant rank condition [13, 23], precisely
\[
\text{rank } A_A(\xi) = \text{const.} \quad \text{for all } \xi \in S^{d-1}.
\]
By [18, Lemma 2.2] the operators $A_\varepsilon := A_A(\nabla_\varepsilon)$ are of constant rank for all $\varepsilon > 0$ provided $A$ has the same property.

As established in the introduction (see (1.1)) $A^{\text{mag}}$ fits into the framework described above with
\[
d = 3, \quad n = 6, \quad l = 4, \quad u = \begin{pmatrix} m \\ h \end{pmatrix}.
\]

So, after having proved some technical tools for general constant rank operators we will be able to apply these findings to the context of micromagnetics.

The next theorem is a modification of [18, Theorem 2.7], which is formulated on the torus, for $L^2$-functions on the whole space.

**Theorem 3.1 (Projection onto $A_\varepsilon$-free fields).** Suppose $A$ is a constant rank operator as defined in (3.1) and $\varepsilon > 0$. Then there exist bounded operators $P_{A_\varepsilon} : L^2(\mathbb{R}^d; \mathbb{R}^n) \to L^2(\mathbb{R}^d; \mathbb{R}^n)$ featuring the following properties:
(i) \((P_{A_e} \circ P_{A_e})_u = P_{A_e} u\) for all \(u \in L^2(\mathbb{R}^d; \mathbb{R}^n)\).
(ii) \((A_e \circ P_{A_e})_u = 0\) in \(\mathbb{R}^d\) for all \(u \in L^2(\mathbb{R}^d; \mathbb{R}^n)\).
(iii) The operators \(P_{A_e}\) are uniformly bounded with respect to \(\varepsilon\), i.e.
\[
\left\| P_{A_e} u \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \leq C \left\| u \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}
\]
for all \(u \in L^2(\mathbb{R}^d; \mathbb{R}^n)\) with a constant \(C > 0\) independent of \(\varepsilon\).
(iv) There exists a constant \(C > 0\) such that
\[
\left\| u - P_{A_e} u \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \leq C \left\| A_e u \right\|_{W^{-1,2}(\mathbb{R}^d; \mathbb{R}^l)}
\]
for all \(u \in L^2(\mathbb{R}^d; \mathbb{R}^n)\).

\textbf{Remark 3.2.} For \(1 < p < \infty\) the statement of Theorem 3.1 is still true. However, the line of reasoning is more involved due to the fact that the Fourier inversion formula does not hold in general. This difficulty can be overcome by using approximation arguments via smooth functions similar to those in [18].

\textbf{Proof.} We basically adapt the proof of the projection theorem in [18] replacing Fourier series with Fourier transforms. In the sequel we employ the common notation \(\mathcal{F}\) to refer to the Fourier transform and use \(\mathcal{F}^{-1}\) for its inversion.

For \(\xi \in \mathbb{R}^d \setminus \{0\}\) let \(P_{A_e}(\xi)\) be the orthogonal projector onto \(\ker A_{A_e}(\xi)\). The operator \(P_{A_e} : \mathbb{R}^d \setminus \{0\} \to \text{Lin}(\mathbb{R}^n; \mathbb{R}^n)\) is 0-homogeneous, smooth and uniformly bounded with respect to \(\varepsilon\), for details see [18, Lemma 2.6]. Then the application of the Mihlin Multiplier Theorem [15, Theorem 5.2.7] provides that \(P_{A_e}\) is a Fourier multiplier. This renders \(P_{A_e}\), defined by
\[
P_{A_e} u = \mathcal{F}^{-1}(P_{A_e}(\cdot)\mathcal{F} u), \quad u \in L^2(\mathbb{R}^d; \mathbb{R}^n)
\]
a Fourier multiplier operator satisfying the estimate
\[
\left\| P_{A_e} u \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \leq C \left\| u \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}
\]
for all \(u \in L^2(\mathbb{R}^d; \mathbb{R}^n)\) with a constant \(C > 0\) independent of \(\varepsilon\). This proves (iii). Properties (i)-(ii) are an immediate consequence of the structure of \(P_{A_e}\) together with (3.2).

In order to show (iv) let \(Q_{A_e}(\xi) \in \text{Lin}(\mathbb{R}^l; \mathbb{R}^n)\) with \(\xi \in \mathbb{R}^d \setminus \{0\}\) be given by
\[
Q_{A_e}(\xi) v = \begin{cases} 
z - [P_{A_e}(\xi)z] & \text{for } v \in \text{range } A_{A_e}(\xi) \text{ with } v = A_{A_e}(\xi)z, z \in \mathbb{R}^n, \\
0 & \text{for } v \in (\text{range } A_{A_e}(\xi))^\perp,
\end{cases}
\]
as in [18, Section 2.7]. Notice that by [18, Lemma 2.6], \(Q_{A_e}\) is homogeneous of degree \(-1\), smooth and \(Q_{A_e}(\cdot/\cdot|\cdot)\) is bounded in the operator norm (uniformly in \(\varepsilon\)). Hence, \(Q_{A_e}(\cdot/\cdot|\cdot)\) is a Fourier multiplier, so that it holds for \(w_{\varepsilon} \in L^2(\mathbb{R}^d; \mathbb{R}^l)\) given through \(\mathcal{F}w_{\varepsilon} = \lVert \cdot \rVert^{-1}A_{A_e}(\cdot)\mathcal{F}u\) that
\[
\left\| \mathcal{F}^{-1}(Q_{A_e}(\cdot/\cdot|\cdot)\mathcal{F} w_{\varepsilon}) \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)} \leq C \left\| w_{\varepsilon} \right\|_{L^2(\mathbb{R}^d; \mathbb{R}^l)}
\]
\[
\leq C \left\| A_{\varepsilon} u \right\|_{\text{W}^{-1,2}(\mathbb{R}^d; \mathbb{R}^l)}.
\]
The last inequality holds by the definition of the \(W^{-1,2}\)-norm and another application of Mihlin’s theorem (compare [18, proof of Theorem 2.7]). Using the properties
of $Q_{A_0}, P_{A_0}$ and $\hat{A}_{A_0}$ yields
\[
Q_{A_0}(\cdot/|\cdot)|Fw_{\varepsilon} = Q_{A_0}(\cdot/|\cdot)|\hat{A}_{A_0}(\cdot/|\cdot)|Fu
= Fu - P_{A_0}(\cdot/|\cdot)|Fu = Fu - P_{A_0}(\cdot)Fu.
\]
In view of (3.3), (3.2) and the Fourier inversion formula for $L^2$-functions this finally proves (iv).

As an essential tool towards the construction of a recovery sequence, we show the following analog of [18, Proposition 4.1] within the setting of functions defined on the whole space. The proof follows closely along the lines of [18], but is substantially easier, since one of the two relevant terms is forced to vanish here.

**Proposition 3.3.** Let $A$ be a constant rank operator as in (3.1) such that the number of non-zero rows of the matrix $A^{(d)}$ is equal to the rank of $A^{(d)}$. Further suppose $u \in L^2(\mathbb{R}^d; \mathbb{R}^n)$ satisfies $A_0u = 0$ in $\mathbb{R}^d$, where
\[
A_0 := \left\{ \begin{array}{ll}
[A^{(d)}]^i \partial_d, & \text{if } [A^{(d)}]^i \neq 0, \\
\sum_{k=1}^{d-1} [A^{(k)}]^i \partial_k, & \text{if } [A^{(d)}]^i = 0
\end{array} \right\}_{i=1, \ldots, l}
\]
and $[A^{(k)}]^i$ the $i$th row of $A^{(k)}$. Then there exists $(u_\varepsilon)_\varepsilon \subset L^2(\mathbb{R}^d; \mathbb{R}^n)$ with $A_\varepsilon u_\varepsilon = 0$ in $\mathbb{R}^d$ and $u_\varepsilon \to u$ in $L^2(\mathbb{R}^d; \mathbb{R}^n)$ for $\varepsilon \to 0^+$.

For the proof of this proposition the following auxiliary “symbol” will be needed,
\[
\hat{A}_0(\xi) := \left\{ \begin{array}{ll}
[A^{(d)}]^i \xi_d, & \text{if } [A^{(d)}]^i \xi_d \neq 0, \\
\sum_{k=1}^{d-1} [A^{(k)}]^i \xi_k, & \text{if } [A^{(d)}]^i \xi_d = 0
\end{array} \right\}_{i=1, \ldots, l}, \quad \xi \in \mathbb{R}^d.
\]
By $\hat{P}_0(\xi)$ we denote the orthogonal projection onto ker $\hat{A}_0(\xi)$. The symbol $A_{A_0}$ coincides with $\hat{A}_0$ outside the hyperplane where $\xi_d = 0$, in formulas
\[
\hat{A}_0(\xi) = A_{A_0}(\xi) \quad \text{for all } \xi \in \mathbb{R}^d \text{ with } \xi_d \neq 0. \tag{3.4}
\]
Notice that in contrast to $A_{A_0}$, the expression $\hat{A}_0$ is non-polynomial and hence cannot be the symbol of a constant-coefficient operator (see [18, Remark 4.2]).

Actually, $\hat{A}_0$ turns out to characterize the limit behavior of the symbols $A_{A_0}$. The exact result is formulated in the next lemma, which was proven in [18] and is repeated here for the readers’ convenience.

**Lemma 3.4** ([18, Lemma 4.3]). If $A$ meets the assumptions of Proposition 3.3, the symbols $A_{A_0}$ converge to the symbol $\hat{A}_0$ for $\varepsilon \to 0$ in the sense that $P_{A_0}(\xi)v \to \hat{P}_0(\xi)v$ for all $\xi \in \mathbb{R}^d$ and all $v \in \mathbb{R}^n$.

**Proof of Proposition 3.3.** The fact that $u$ is $A_0$-free implies
\[
A_{A_0}(\xi)(Fu)(\xi) = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^d. \tag{3.5}
\]
Now we split $F u$ into

$$(F u)^{(1)}(\xi) := \begin{cases} (F u)(\xi) & \text{if } \xi_d \neq 0, \\ 0 & \text{if } \xi_d = 0, \end{cases} \quad \text{and} \quad (F u)^{(2)}(\xi) := \begin{cases} 0 & \text{if } \xi_d \neq 0, \\ (F u)(\xi) & \text{if } \xi_d = 0, \end{cases}$$

so that by means of Fourier inversion,

$$u = F^{-1}(F u)^{(1)} + F^{-1}(F u)^{(2)} = u^{(1)} + u^{(2)}.$$

With this definition, $u^{(1)}$ and $u^{(2)}$ are $L^2(\mathbb{R}^d; \mathbb{R}^n)$-functions satisfying $F u^{(i)} = (F u)^{(i)}$, $i = 1, 2$, and we may conclude from (3.5) that

$$A_0 u^{(1)} = A_0 u^{(2)} = 0 \quad \text{in } \mathbb{R}^d.$$

Turning to $u^{(2)}$, we notice that by construction $\partial_d u^{(2)} = 0$. In order to have the quadratic integrability of $u^{(2)}$ on $\mathbb{R}^d$ preserved it needs to hold that $u^{(2)} \equiv 0$. Hence, for the proof of this proposition it will be enough to show the existence of an $A_0$-free sequence $(u_\varepsilon)_{\varepsilon}$ with $u_\varepsilon \rightharpoonup u^{(1)}$ in $L^2(\mathbb{R}^d; \mathbb{R}^n)$ as $\varepsilon \to 0^+$.

To this end set $u_\varepsilon := P_{A_0} u^{(1)}$ or speaking in terms of Fourier transforms

$$\mathcal{F} u_\varepsilon := P_{A_0} \mathcal{F} u^{(1)}$$

with $P_{A_0}$ and $P_{A_0}$ as in Theorem 3.1. In view of (3.4) and (3.5) one finds for a.e. $\xi \in \mathbb{R}^d$ that

$$\hat{\alpha}_0(\xi) (F u^{(1)})(\xi) = 0,$$

which implies

$$\hat{\alpha}_0(\xi) (F u^{(1)})(\xi) = (F u^{(1)})(\xi).$$

Recalling that $F$ (and $F^{-1}$) are $L^2$-isometries we may calculate

$$\|u_\varepsilon - u^{(1)}\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}^2 = \|F u_\varepsilon - F u^{(1)}\|_{L^2(\mathbb{R}^d; \mathbb{R}^n)}^2$$

$$= \int_{\mathbb{R}^d} \left| \left( P_{A_0}(\xi) - \hat{\alpha}_0(\xi) \right) (F u^{(1)})(\xi) \right|^2 d\xi. \quad (3.6)$$

At this point we apply Lemma 3.4. This, together with the uniform boundedness of the projection operators $P_{A_0}$, allows us to use Lebesgue’s dominated convergence theorem and we conclude that the right-hand side in (3.6) tends to zero as $\varepsilon \to 0^+$. \qed

As discussed at the beginning of this section, $A_{\text{mag}}$ is a first order differential operator of the form (3.1) meeting the constant rank condition and one can easily check that both the rank of $(A_{\text{mag}})^{\text{(3)}}$ and its number of non-zero rows is three. Besides, a straightforward calculation shows that $A_{\text{mag}}^0$ as in the statement of Theorem 1.1 corresponds to $A_{\text{mag}}$ in the sense of Proposition 3.3 (also see [18, Example 2.3] for a detailed discussion of the constant rank operators $\text{div}$ and $\text{curl}$). Hence, the next corollary is an immediate consequence of the previous proposition.

**Corollary 3.5.** For every $(m, h) \in U_0$ there exists a sequence $(\hat{m}_\varepsilon, \hat{h}_\varepsilon)_{\varepsilon} \subset L^2(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$ such that $(\hat{m}_\varepsilon, \hat{h}_\varepsilon) \rightharpoonup (m, h)$ in $L^2(\mathbb{R}^3; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$ as $\varepsilon \to 0^+$ and

$$A_{\text{mag}} \left( \frac{\hat{m}_\varepsilon}{\hat{h}_\varepsilon} \right) = 0 \quad \text{in } \mathbb{R}^3 \text{ for all } \varepsilon > 0.$$
Before we prove the upper bound, we establish the following relation between \( \text{curl} \)- and \( \text{div} \)-free fields.

**Lemma 3.6.** It holds that

\[
\mathcal{I} - \mathcal{P}_{\text{curl}_\varepsilon} = \mathcal{P}_{\text{div}_\varepsilon},
\]

where \( \mathcal{P}_{\text{curl}_\varepsilon}, \mathcal{P}_{\text{div}_\varepsilon} : L^2(\mathbb{R}^3; \mathbb{R}^3) \to L^2(\mathbb{R}^3; \mathbb{R}^3), \varepsilon > 0, \) are projection operators as defined in Theorem 3.1 and \( \mathcal{I} \) is the identity map on \( L^2(\mathbb{R}^3; \mathbb{R}^3) \).

**Proof.** Accounting for (3.2), the claim holds true if

\[
\mathcal{I} - \mathcal{P}_{\text{curl}_\varepsilon}(\xi) = \mathcal{P}_{\text{div}_\varepsilon}(\xi) \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\}.
\]

Here \( \mathcal{I} : \mathbb{R}^3 \to \mathbb{R}^3 \) is the identity function, and in the sequel we want to use the notation \( \xi_\varepsilon = (\xi_1, \xi_2, 1/\varepsilon \xi_3)^T, \xi \in \mathbb{R}^3 \).

Let \( \xi \in \mathbb{R}^3 \setminus \{0\} \) be fixed. Then \( \ker \mathcal{A}_{\text{div}_\varepsilon}(\xi) = \{ v \in \mathbb{R}^3 : \xi_\varepsilon \cdot v = 0 \} = \text{span}\{\xi_\varepsilon \perp 1, \xi_\varepsilon \perp 2\} \) with orthonormal vectors \( \xi_\varepsilon \perp 1 \) and \( \xi_\varepsilon \perp 2 \), while

\[
\ker \mathcal{A}_{\text{curl}_\varepsilon}(\xi) = \{ v \in \mathbb{R}^3 : \xi_\varepsilon \times v = 0 \} = \text{span}\{\xi_\varepsilon / |\xi_\varepsilon|\}.
\]

Thus, for \( v \in \mathbb{R}^3 \) it follows

\[
(\mathcal{I} - \mathcal{P}_{\text{curl}_\varepsilon}(\xi))v = v - (v \cdot \xi_\varepsilon)\xi_\varepsilon / |\xi_\varepsilon|^2 = (v \cdot \xi_\varepsilon \perp 1)\xi_\varepsilon \perp 1 + (v \cdot \xi_\varepsilon \perp 2)\xi_\varepsilon \perp 2 = \mathcal{P}_{\text{div}_\varepsilon}(\xi)v
\]

and the proof is complete. \( \Box \)

Observe that the functions \( \hat{m}_\varepsilon \) in Corollary 3.5 do not have \( W^{1,2} \)-regularity nor do they fulfill the required nonconvex constraint of local saturation. Consequently, \( (\hat{m}_\varepsilon, \hat{h}_\varepsilon)_\varepsilon \) fails to be a correct recovery sequence. In order to overcome this problem we make a construction based on the use of appropriate projection operators and prove the following proposition.

**Proposition 3.7 (Recovery sequence).** For every \( (m, h) \in \mathcal{U}_0 \) there exists a sequence \( (m_\varepsilon, h_\varepsilon)_\varepsilon \subset \mathcal{U}_\varepsilon \) with \( (m_\varepsilon, h_\varepsilon) \rightharpoonup (m, h) \) in \( W^{1,2}(\Omega_1; \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3) \) satisfying

\[
\lim_{\varepsilon \to 0^+} F_\varepsilon[m_\varepsilon, h_\varepsilon] = F_0[m, h].
\]

**Proof.** For given \( (m, h) \in \mathcal{U}_0 \) let \( (\hat{m}_\varepsilon, \hat{h}_\varepsilon)_\varepsilon \) be as in Corollary 3.5. We set for \( \varepsilon > 0 \),

\[
m_\varepsilon = m, \quad h_\varepsilon = \mathcal{P}_{\text{curl}_\varepsilon}(\hat{h}_\varepsilon - m + \hat{m}_\varepsilon).
\]

The claim is now that this definition of \( (m_\varepsilon, h_\varepsilon)_\varepsilon \) provides a recovery sequence for \( (m, h) \).

Indeed, it holds \( \mathcal{A}_\varepsilon^\text{max}(m_\varepsilon, h_\varepsilon) = 0 \) in \( \mathbb{R}^3 \), since \( \text{curl}_\varepsilon h_\varepsilon = 0 \) by Theorem 3.1 (ii) and

\[
\text{div}(m_\varepsilon + h_\varepsilon) = \text{div}(m + \mathcal{P}_{\text{curl}_\varepsilon}(\hat{h}_\varepsilon - m + \hat{m}_\varepsilon)) = \text{div}(\hat{m}_\varepsilon + \hat{h}_\varepsilon) + \text{div}(\mathcal{P}_{\text{curl}_\varepsilon} - \mathcal{I})(\hat{h}_\varepsilon - m + \hat{m}_\varepsilon) = 0
\]
in view of Lemma 3.6 and Corollary 3.5. Trivially, \( m_\varepsilon \) satisfies \(|m_\varepsilon| = |m| = m\) in \( \Omega_1 \) for all \( \varepsilon > 0 \). Moreover, applying Theorem 3.1 (iii)-(iv) yields
\[
\|h_\varepsilon - h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq \|\hat{h}_\varepsilon - \hat{h}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|h_\varepsilon - h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
\leq \|\text{curl}_\varepsilon \hat{h}_\varepsilon - \hat{h}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\text{curl}_\varepsilon (\hat{m}_\varepsilon - m)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\hat{h}_\varepsilon - h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
\leq C (\|\text{curl}_\varepsilon \hat{h}_\varepsilon\|_{W^{1,2}(\mathbb{R}^3;\mathbb{R}^3)} + \|\hat{m}_\varepsilon - m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\hat{h}_\varepsilon - h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}) \\
= C (\|\hat{m}_\varepsilon - m\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \|\hat{h}_\varepsilon - h\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}).
\]

This expression tends to zero by Corollary 3.5.

Summarizing, we find \((m_\varepsilon, h_\varepsilon) \in U_\varepsilon\) for all \( \varepsilon > 0 \) and \( h_\varepsilon \to h \in L^2(\mathbb{R}^3;\mathbb{R}^3) \) as \( \varepsilon \to 0^+ \). Then,
\[
\lim_{\varepsilon \to 0^+} \int_{\Omega_1} \alpha |\nabla m_\varepsilon|^2 + \varphi(m_\varepsilon) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h_\varepsilon|^2 \, dx \\
= \int_{\Omega_1} \alpha |\nabla m|^2 + \varphi(m) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |h|^2 \, dx.
\]

\( \Box \)

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References


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