

MOTION OF ELASTIC THIN FILMS BY ANISOTROPIC SURFACE DIFFUSION WITH CURVATURE REGULARIZATION

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ABSTRACT. Short time existence, uniqueness, and regularity for a surface diffusion evolution equation with curvature regularization are proved in the context of epitaxially strained two-dimensional films. This is achieved by using the H^{-1} -gradient flow structure of the evolution law, via De Giorgi's minimizing movements. This seems to be the first short time existence result for a surface diffusion type geometric evolution equation in the presence of elasticity.

1. INTRODUCTION

In this paper we study the morphologic evolution of anisotropic epitaxially strained films, driven by stress and surface mass transport. This can be viewed as the evolutionary counterpart of the static theory developed in [10, 19, 21, 18].

We briefly recall the physical mechanism behind the evolution equation. The free interface is allowed to evolve via surface mass transport under the influence of a chemical potential μ . Mass transport in the bulk can be neglected, as it occurs at a much faster time scale (see [30]). According to the Einstein-Nernst relation, the surface flux of atoms is proportional to the tangential gradient of the chemical potential, whose divergence, in turn, equals the rate at which material is removed from or deposited on the interface, due to mass conservation. Thus, we get the *volume preserving* evolution law

$$V = C\Delta_\Gamma\mu, \tag{1.1}$$

where $C > 0$, V denotes the normal velocity of the evolving interface Γ , Δ_Γ stands for the tangential laplacian, and the chemical potential μ is given by the first variation of the free-energy functional.

In the case of three-dimensional epitaxially strained films with planar symmetries, the underlying model becomes two-dimensional and the free energy functional is given by

$$\int_{\Omega_h} Q(E(u)) dz + \int_{\Gamma_h} g(\theta) d\mathcal{H}^1, \tag{1.2}$$

where h is the function whose graph Γ_h describes the evolving profile of the film, Ω_h is the region occupied by the film, u is displacement of the material, which is assumed to be in (quasistatic) elastic equilibrium at each time, and \mathcal{H}^1 denotes the one-dimensional measure on Γ_h . Finally, g is an anisotropic surface energy density, evaluated at the angle θ that the outward surface normal ν forms with the x -axis. The first variation of (1.2) can be written as the sum of three contributions: A constant Lagrange multiplier related to mass conservation, the (anisotropic) curvature of the surface, and the elastic energy density evaluated at the displacement of the solid on the profile of the film. Hence, (1.1) takes the form (assuming $C = 1$)

$$V = \left((g_{\theta\theta} + g)k + Q(E(u)) \right)_{\sigma\sigma}, \tag{1.3}$$

where k is the curvature of Γ_h , $(\cdot)_\sigma$ stands for the tangential derivative along Γ_h , and $u(\cdot, t)$ is the elastic equilibrium in $\Omega_{h(\cdot, t)}$, i.e., the minimizer of the elastic energy under the prescribed

periodicity and boundary conditions (see (1.7) below). The corresponding evolution without surface diffusion, which reduces to

$$V = (g_{\theta\theta} + g)k + Q(E(u)), \quad (1.4)$$

has been studied by several authors under the assumption that $g_{\theta\theta} + g > 0$, i.e., the equation is parabolic (see [25] and the references therein). However, a highly anisotropic non-convex interfacial energy may lead to the existence of certain directions θ at which the coefficient $g_{\theta\theta} + g$ becomes negative. This is a situation often present in the materials science literature. In this case, equations (1.3) and (1.4) are backward parabolic and the corresponding initial value problem is ill-posed.

To overcome the ill-posedness of the evolution equation, a common approach found in the literature is to regularize higher order terms, i.e, to consider in (1.2) a curvature-dependent surface energy. This was first suggested by Herring ([26]) on physical grounds and later was adopted in [7] and [15], in the particular case of a surface energy of the form

$$g(\theta, k) = g(\theta) + \frac{\varepsilon}{2}k^2,$$

with ε a positive constant. The evolution without surface diffusion (1.4) is then replaced by

$$V = (g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon(k_{\sigma\sigma} + \frac{1}{2}k^3), \quad (1.5)$$

while in the context of surface diffusion, in view of (1.1), we have the *volume preserving* evolution law

$$V = \left((g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon(k_{\sigma\sigma} + \frac{1}{2}k^3) \right)_{\sigma\sigma}. \quad (1.6)$$

This equation was already proposed in [24] for the case without elasticity, and it was studied numerically in [32] for the evolution of voids in elastically stressed materials (see also [31, 11] and references therein). However, to the best of our knowledge no analytical results exist in the literature for equations (1.3), (1.5), and (1.6). Related analytical results concerning the diffuse interface version of such equations may be found, for instance, in [22, 23].

In this paper we prove short time existence, uniqueness, and regularity of a spatially periodic solution to (1.6) in the context of the epitaxially strained two dimensional elastic films over a rigid substrate. Precisely, for $b > 0$ we construct a local in time solution of the Cauchy problem

$$\begin{cases} \frac{1}{J} \frac{\partial h}{\partial t} = \left((g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon(k_{\sigma\sigma} + \frac{1}{2}k^3) \right)_{\sigma\sigma}, & \text{in } \mathbb{R} \times [0, T_0], \\ \operatorname{div} \mathbb{C}E(u) = 0 & \text{in } \Omega_h, \\ \mathbb{C}E(u)[\nu] = 0 & \text{on } \Gamma_h, \quad u(x, 0, t) = e_0(x, 0), \\ h(\cdot, t) \text{ and } \nabla u(\cdot, t) & \text{are } b\text{-periodic} \\ h(\cdot, 0) = h_0 \end{cases} \quad (1.7)$$

where e_0 is a nonzero constant that reflects the mismatch between the crystalline lattices of the film and the substrate and, we recall, $h : \mathbb{R} \times [0, T_0] \rightarrow (0, +\infty)$ denotes the one-dimensional function describing the profile Γ_h of the film,

$$J := \sqrt{1 + \left| \frac{\partial h}{\partial x} \right|^2}, \quad k := -\frac{\partial}{\partial x} \left(\frac{\frac{\partial h}{\partial x}}{\sqrt{1 + \left| \frac{\partial h}{\partial x} \right|^2}} \right),$$

$Q(E(u)) = \frac{1}{2} \mathbb{C}E(u) : E(u)$, and $h_0 \in H_{loc}^2(\mathbb{R})$ is a b -periodic function.

As observed by Cahn and Taylor in [12], this motion can be regarded as the H^{-1} -gradient flow for the total energy

$$G(h) := \int_{\Omega_h} Q(E(u_h)) dz + \int_{\Gamma_h} \left(g(\theta) + \frac{\varepsilon}{2}k^2 \right) d\mathcal{H}^1,$$

where $\Omega_h := \{(x, y) : 0 < x < b, 0 < y < h(x)\}$, Γ_h is the graph of h over the periodicity interval $(0, b)$, and u_h is the minimizer of the elastic energy in Ω_h under suitable boundary and periodicity conditions. Therefore, it is natural to adopt De Giorgi's minimizing movements approach (see [4]), which consists in constructing discrete time evolutions by solving iteratively suitable minimum incremental problems. Precisely, we start with a b -periodic initial datum $h_0 \in H_{loc}^2(\mathbb{R})$ and, given $T > 0$, $N \in \mathbb{N}$, for $i = 1, \dots, N$, we define inductively $h_{i,N}$ as the minimizer of

$$G(h) + \frac{1}{2\tau} d^2(h, h_{i-1,N}), \quad (1.8)$$

where $\tau := \frac{T}{N}$ and d is a suitable term measuring the H^{-1} -distance between h and $h_{i-1,N}$. We mention here that minimizing movements have been already successfully implemented to treat various mean curvature type flows without surface diffusion (see, e.g., [3, 13, 8]).

This paper is organized as follows. In Section 2 we set up the problem and introduce the discrete time evolutions. In Section 3 we show that they converge to a weak solution of (1.7) in $[0, T_0]$ for some $T_0 > 0$ (see Theorem 3.8). Precisely, a *b-periodic weak solution* of (1.7) is a function $h \in H^1(0, T_0; H_{loc}^{-1}(\mathbb{R})) \cap L^\infty(0, T_0; H_{loc}^2(\mathbb{R}))$, such that $h(\cdot, t)$ is b -periodic for all $t \in [0, T]$ and (h, u) satisfies (1.7) in the distributional sense (see Definition 3.1). We remark that Theorem 3.8 seems to be the first (short time) existence result for a surface diffusion type geometric evolution equation in the presence of elasticity. Moreover, also the use of minimizing movements instead of the more classical semigroup approach appears to be new in this context. We observe that in the case without elasticity and without curvature regularization, short time existence of a smooth solution was proved in [17], using semigroup techniques, for the motion of immersed hypersurfaces by surface diffusion. See also [16, 29, 9] and the work of Chen ([14]) for the Hele-Shaw equation.

A delicate point in the proof of our existence result is the choice of d in the penalization term in (1.8) (see Remark 2.2). A rather technical obstacle is overcome in Theorem 3.4, where it is proved that the solutions of the discrete time evolutions are equicontinuous in time with values in $C^{1,\alpha}$ for all $\alpha \in (0, \frac{1}{2})$, at least for an initial time interval $[0, T_0]$. This property is crucial to guarantee that the evolving graphs do not develop vertical parts in $[0, T_0]$. However, our variational procedure provides a global in time volume preserving evolution, which satisfies (1.7) until the vertical parts appear (see Theorem 3.3). The main existence result is established in Theorem 3.8.

In Section 4 we prove that the constructed solution h solves the equation in a much stronger sense. Namely, we show that $h \in L^2(0, T_0; H_{loc}^6(\mathbb{R}))$ and $\frac{\partial h}{\partial t} \in L^2(0, T_0; L_{loc}^2(\mathbb{R}))$, provided that the anisotropy ψ is sufficiently smooth (see Theorem 4.3). We note that this is obtained without requiring any further regularity on the initial datum, besides $H_{loc}^2(\mathbb{R})$. The presence of the elasticity term poses some serious technical difficulties in the proof of Theorem 4.3. To understand why, recall that by the classical elliptic theory if the profile $h \in C^{k,\alpha}$ for $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then the corresponding elastic equilibrium u_h is of class $C^{k,\alpha}$ up to Γ_h . However, in order to prove the desired regularity result we need to specify in a rather precise way how the constants in the elliptic estimates depend on h . This is achieved in Theorem 4.1, whose technical proof makes an essential use of the Airy functions associated to u_h .

Finally, in Section 5 we show that weak solutions to equation (1.7) are unique and thus coincide necessarily with the solution constructed via minimizing movements.

Future work will address other properties such as long time existence and asymptotic stability.

2. SETTING OF THE PROBLEM

In this section we introduce the precise mathematical setting needed to define the free-energy functional and the corresponding minimum incremental problems. Our formulation is similar to the one in [10] (see also [21]) to model the epitaxial growth of an elastic film over a rigid substrate

in the presence of a mismatch between the lattices of the two materials. Following the physical literature and as in [10], we work under periodicity conditions on the evolving profiles and on the corresponding elastic displacements. Given a positive b -periodic function $h : \mathbb{R} \rightarrow [0, +\infty)$, with locally finite pointwise variation, we set

$$\Omega_h := \{z = (x, y) : 0 < x < b, 0 < y < h(x)\}, \quad \Omega_h^\# := \{z = (x, y) : x \in \mathbb{R}, 0 < y < h(x)\},$$

$$\Gamma_h := \{z = (x, y) : x \in [0, b], h^-(x) \leq y \leq h^+(x)\},$$

where

$$h^-(x) := \min\{h(x-), h(x+)\} \quad \text{and} \quad h^+(x) := \max\{h(x-), h(x+)\},$$

with $h(x\pm)$ the right and left limit of h at x , respectively. We also consider the set

$$\Gamma_h^\# := \{z = (x, y) : x \in \mathbb{R}, h^-(x) \leq y \leq h^+(x)\}.$$

The set Ω_h represents the reference configuration of the film over the interval $(0, b)$ and Γ_h is the corresponding profile. We introduce the class of admissible profile functions AP , defined as

$$AP := \{h : \mathbb{R} \rightarrow [0, +\infty) : h \text{ } b\text{-periodic, lower semicontinuous, } \text{Var}(h; 0, b) < +\infty, \text{ and there exists } \gamma \in H_{\text{loc}}^2(\mathbb{R}; \mathbb{R}^2) \text{ such that } \gamma(t+1) = \gamma(t) + (b, 0), |\dot{\gamma}| \equiv \mathcal{H}^1(\Gamma_h), \text{ and } \gamma(\mathbb{R}) = \Gamma_h^\#\}.$$
 (2.1)

Given $h \in AP$, we denote

$$LD_\#(\Omega_h; \mathbb{R}^2) := \left\{ u \in L_{\text{loc}}^2(\Omega_h^\#; \mathbb{R}^2) : u(x, y) = u(x+b, y) \text{ for } (x, y) \in \Omega_h^\#, E(u)|_{\Omega_h} \in L^2(\Omega_h; \mathbb{R}^2) \right\},$$

where $E(u) := \frac{1}{2}(\nabla u + \nabla^T u)$, ∇u is the distributional gradient of u and $\nabla^T u$ is the transpose of ∇u . We work within the theory of small deformations, so that $E(u)$ represents the strain and u is the planar displacement. We also prescribe the Dirichlet boundary condition $u(x, 0) = e_0(x, 0)$ at the interface between film and substrate, which models the case of a film growing on an infinitely rigid substrate. This boundary condition forces the film to be strained, thus generating elastic energy. The positive constant e_0 measures the mismatch between the lattices of the two materials. Thus, we define

$$X_{e_0} := \left\{ (h, u) : h \in AP, u : \Omega_h^\# \rightarrow \mathbb{R}^2 \text{ s.t. } u(\cdot, \cdot) - e_0(\cdot, 0) \in LD_\#(\Omega_h; \mathbb{R}^2), \right. \\ \left. \text{and } u(x, 0) = (e_0 x, 0) \text{ for all } x \in \mathbb{R} \right\}.$$

The elastic energy density $Q : \mathbb{M}_{\text{sym}}^{2 \times 2} \rightarrow [0, +\infty)$ takes the form

$$Q(A) := \frac{1}{2} \mathbb{C} A : A$$

with \mathbb{C} a fourth-order tensor. We assume that $Q(A) > 0$ for all $A \in \mathbb{M}_{\text{sym}}^{2 \times 2} \setminus \{0\}$.

Let $\psi : \mathbb{R}^2 \rightarrow [0, +\infty)$ be a positively one-homogeneous function of class C^2 away from the origin. Note that, in particular,

$$c_1 |\xi| \leq \psi(\xi) \leq c_2 |\xi| \quad \text{for all } \xi \in \mathbb{R}^2, \quad (2.2)$$

for some $c_1, c_2 > 0$. We are ready to introduce the energy functional. For all $(h, u) \in X_{e_0}$ we set

$$F(h, u) := \int_{\Omega_h} Q(E(u)) \, dz + \int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{2} k^2 \right) d\mathcal{H}^1, \quad (2.3)$$

where k denotes the curvature of Γ_h , ν is the outer unit normal to Ω_h , and ε is a (small) positive constant.

Remark 2.1. Note that if γ is any constant speed parametrization of $\Gamma_h^\#$ as in (2.1), then by periodicity we have

$$\int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{2} k^2 \right) d\mathcal{H}^1 = \int_{\gamma(I)} \left(\psi(\nu) + \frac{\varepsilon}{2} k^2 \right) d\mathcal{H}^1$$

for every interval $I \subset \mathbb{R}$ of length one.

2.1. The incremental minimum problem. In this subsection we introduce the incremental minimum problems used to define the discrete time evolutions. As standing assumptions throughout this paper, we start from an initial configuration $(h_0, u_0) \in X_{e_0}$, such that

$$h_0 \in H_{\#}^2([0, b]), \quad h_0 > 0, \quad (2.4)$$

and u_0 minimizes the elastic energy in Ω_{h_0} among all u with $(h_0, u) \in X_{e_0}$. Here, and in what follows, we denote by $H_{\#}^k(0, l)$ the space of all functions in $H_{loc}^k(\mathbb{R})$ that are l -periodic, endowed with the norm of $H^k(0, l)$. A similar convention will be used also for other functional spaces. Also, $H_{\#}^{-1}(0, l)$ stands for the dual space of $H_{\#}^1(0, l)$.

Given $T > 0$, $N \in \mathbb{N}$, we set $\Delta T := \frac{T}{N}$. For $i = 1, \dots, N$ we define inductively $(h_{i,N}, u_{i,N})$ as the solution of the minimum problem

$$\min \left\{ F(h, u) + \frac{1}{2\Delta T} \int_{\Gamma_{h_{i-1,N}}} \left(\int_0^x (h(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 d\mathcal{H}^1(x, y) : (h, u) \in X_{e_0} \right. \\ \left. \int_0^b h dx = \int_0^b h_0 dx, \int_{\Gamma_{h_{i-1,N}}} \int_0^x (h(\zeta) - h_{i-1,N}(\zeta)) d\zeta d\mathcal{H}^1(x, y) = 0 \right\}. \quad (2.5)$$

Then for $x \in \mathbb{R}$ and $(i-1)\Delta T \leq t \leq i\Delta T$, $i = 1, \dots, N$, we define

$$h_N(x, t) := h_{i-1,N}(x) + \frac{1}{\Delta T} (t - (i-1)\Delta T) (h_{i,N}(x) - h_{i-1,N}(x)) \quad (2.6)$$

and we let $u_N(\cdot, t)$ be the *elastic equilibrium corresponding to $h_N(\cdot, t)$, i.e., the minimizer of the elastic energy in $\Omega_{h_N(\cdot, t)}$ among all u such that $(h_N(\cdot, t), u) \in X_{e_0}$* . We also denote by $\gamma_{i,N}$ and γ_0 admissible constant speed parametrizations of $\Gamma_{h_{i,N}}^\#$ and $\Gamma_{h_0}^\#$, respectively, as in (2.1) and oriented in such a way that $\dot{\gamma}_{i,N} \cdot e_1, \dot{\gamma}_0 \cdot e_1 \geq 0$.

Remark 2.2 (Interpretation of the penalization term). We remark that the penalization term in (2.5) coincides with the square of the $H^{-1}(\Gamma_{i-1,N})$ -norm of $f := \frac{h - h_{i-1,N}}{\sqrt{1 + h_{i-1,N}^2}}$, provided h satisfies the constraint in (2.5) and $h_{i-1,N}$ is of class C^1 . Indeed, writing Γ instead of $\Gamma_{i-1,N}$, the penalization and the constraints on h in (2.5) reduce to

$$\int_{\Gamma} \left(\int_{\Gamma(z_0, z)} f(w) d\mathcal{H}^1(w) \right)^2 d\mathcal{H}^1(z) \quad (2.7)$$

and

$$\int_{\Gamma} f d\mathcal{H}^1 = 0, \quad \int_{\Gamma} \int_{\Gamma(z_0, z)} f(w) d\mathcal{H}^1(w) d\mathcal{H}^1(z) = 0, \quad (2.8)$$

where $z_0 = (0, h_{i-1,N}(0))$ and $\Gamma(z_0, z)$ is the arc of Γ connecting z_0 and z . Note that if f is a function on Γ satisfying the constraints (2.8), then (2.7) reduces to $\|f\|_{H^{-1}(\Gamma)}^2$, once we define

$$\|\varphi\|_{H^1(\Gamma)}^2 := |\varphi(z_0)|^2 + \int_{\Gamma} \varphi_{\sigma}^2(z) d\mathcal{H}^1(z) \quad (2.9)$$

for every $\varphi \in H^1(\Gamma)$, where, we recall, $(\cdot)_{\sigma}$ stands for the tangential derivative along Γ . In fact, integrating by parts, we obtain

$$\|f\|_{H^{-1}(\Gamma)} = \sup_{\|\varphi\|_{H^1(\Gamma)}=1} \int_{\Gamma} f \varphi d\mathcal{H}^1 = \sup_{\|\varphi\|_{H^1(\Gamma)}=1} \left[- \int_{\Gamma} F \varphi_{\sigma} d\mathcal{H}^1 + \varphi(z_0) \int_{\Gamma} f d\mathcal{H}^1 \right],$$

where $F(z) := \int_{\Gamma(z_0, z)} f(w) d\mathcal{H}^1(w)$. Setting $\chi := \varphi_\sigma$ and recalling (2.9), we have

$$\begin{aligned} \|f\|_{H^{-1}(\Gamma)}^2 &= \sup_{0 \leq a \leq 1} \sup_{\substack{\|\chi\|_{L^2(\Gamma)} = \sqrt{1-a^2} \\ \int_{\Gamma} \chi d\mathcal{H}^1 = 0}} \left[- \int_{\Gamma} F \chi d\mathcal{H}^1 + a \int_{\Gamma} f d\mathcal{H}^1 \right]^2 \\ &= \sup_{0 \leq a \leq 1} \left[\sqrt{1-a^2} \left(\int_{\Gamma} \left(F(z) - \int_{\Gamma} F d\mathcal{H}^1 \right)^2 d\mathcal{H}^1(z) \right)^{\frac{1}{2}} + a \int_{\Gamma} f d\mathcal{H}^1 \right]^2 \\ &= \int_{\Gamma} \left(F(z) - \int_{\Gamma} F d\mathcal{H}^1 \right)^2 d\mathcal{H}^1(z) + \left(\int_{\Gamma} f d\mathcal{H}^1 \right)^2. \end{aligned} \quad (2.10)$$

Recalling (2.8), the right-hand side of the last identity reduces to (2.7).

Note that if h is a solution to the limiting evolution (1.7), then

$$\int_0^b h(x, t_1) dx = \int_0^b h(x, t_2) dx \quad \int_{\Gamma_{h(\cdot, t)}} \int_0^x \frac{\partial h}{\partial t}(\zeta, t) d\zeta d\mathcal{H}^1 = 0$$

for t, t_1 , and t_2 (see (3.4) for the first identity, the second one is proved similarly). This observation justifies the choice the constraints imposed in (2.5).

An alternative formulation of the incremental minimum problem would be

$$\min \left\{ F(h, u) + \frac{1}{2\Delta T} \left\| \frac{h - h_{i-1, N}}{J_{i-1, N}} \right\|_{H^{-1}(\Gamma_{i-1, N})}^2 : (h, u) \in X_{e_0} \right\},$$

where the $H^{-1}(\Gamma_{i-1, N})$ -norm is defined as in (2.10) and $J_{i-1, N}$ denotes the length factor of $\Gamma_{i-1, N}$. This should lead to the same limiting evolution equation, although we shall not pursue this approach here.

The remaining of this subsection is devoted to the proof of the existence of a minimizer for the minimum incremental problem (2.5). We start with a compactness lemma.

Lemma 2.3. *Let $\{h_n\} \subset AP$ be such that*

$$\sup_n \|\gamma_n\|_{H^2(0, 1; \mathbb{R}^2)} < +\infty, \quad (2.11)$$

where γ_n is a constant speed parametrization of $\Gamma_{h_n}^\#$ as in (2.1). Then, there exists $h \in AP$ such that, up to a subsequence,

- (i) $h_n \rightarrow h$ in $L^1(0, b)$,
- (ii) $\Gamma_{h_n}^\# \rightarrow \Gamma_h^\#$ in the Hausdorff metric;
- (iii) $\mathbb{R}_+^2 \setminus \Omega_{h_n}^\# \rightarrow \mathbb{R}_+^2 \setminus \Omega_h^\#$ in the Hausdorff metric;
- (iv) $\gamma_n \rightharpoonup \gamma$ weakly in $H^2(0, 1; \mathbb{R}^2)$, where γ is a constant speed parametrization of $\Gamma_h^\#$ as in (2.1).

Proof. For simplicity we set $\Gamma_n^\# := \Gamma_{h_n}^\#$. From (2.11) we have that $\sup_n \text{Var}(h_n; 0, b) < +\infty$. Hence, by the Helly theorem, and up to a (not relabeled) subsequence, we may conclude that there exists a b -periodic function h with locally finite pointwise variation such that $h_n \rightarrow h$ in $L_{loc}^1(\mathbb{R})$ and pointwise everywhere. Moreover, again by (2.11), we may also assume that $\gamma_n \rightharpoonup \gamma$ weakly in $H^2(0, 1; \mathbb{R}^2)$. Finally, by the Blaschke Compactness theorem (see [5, Theorem 6.1]), and using the periodicity of $\Gamma_n^\#$, we may also assume that there exists a closed set $\Gamma^\#$ such that $\Gamma_n^\# \rightarrow \Gamma^\#$ in the Hausdorff metric. Since $\dot{\gamma}_n \rightarrow \dot{\gamma}$ uniformly and $|\dot{\gamma}_n| \equiv \mathcal{H}^1(\gamma_n([0, 1]))$, we get that $|\dot{\gamma}| \equiv \mathcal{H}^1(\gamma([0, 1]))$. Moreover, since $\gamma_n \rightarrow \gamma$ uniformly, using the definition of Hausdorff convergence, we conclude that γ is a parametrization of $\Gamma^\#$.

We now show that $\Gamma^\# = \Gamma_h^\#$. Arguing as in the proof of [19, Lemma 2.5], we get that $\Gamma_h^\# \subset \Gamma^\#$ and the vertical section $\Gamma_x^\# := \{t \in \mathbb{R} : (x, t) \in \Gamma^\#\}$ is a closed interval for all $x \in \mathbb{R}$. Assume by contradiction that $\Gamma_x^\# \not\supseteq [h^-(x), h^+(x)]$ and, without loss of generality, that $\Gamma_x^\# = [y_1, y_2]$, with $y_1 \leq h^-(x)$ and $y_2 > h^+(x)$. Let $s \in (0, 1)$ such that $\gamma(s) = (x, y_2)$. We claim that there exists $\delta > 0$ such that $\gamma(t) \in \{x\} \times \Gamma_x^\#$ for all $t \in (s - \delta, s + \delta)$. Indeed, otherwise there would exist a sequence $t_n \rightarrow s$ such that $\gamma(t_n) \notin \{x\} \times \Gamma_x^\#$. Since $\gamma(t_n) = (x_n, y_n) \rightarrow (x, y_2)$ and h is lower semicontinuous, we would have $x_n \neq x$ and $y_n - h^+(x_n) > \frac{y_2 - h^+(x)}{2}$ for n large. This would imply that $\Gamma^\#$ contains infinitely many vertical segments of length greater than $\frac{y_2 - h^+(x)}{2}$ near (x, y_2) , which is in contradiction with the fact that $\Gamma^\#$ has locally finite length. Hence, writing $\gamma(t) = (x(t), y(t))$, we have that in the interval $(s - \delta, s + \delta)$, $x(t) = x$, and $y(s) = y_2 = \max_{t \in (s - \delta, s + \delta)} y(t)$. It follows that $\dot{\gamma}(s) = 0$, which is impossible. This shows that $\Gamma^\# = \Gamma_h^\#$ and that γ is a constant speed parametrization of $\Gamma_h^\#$ as in (2.1).

To conclude the proof, we recall that under our assumptions there exists (see [10], [21, Theorem 2.2]) a lower semicontinuous b -periodic function \tilde{h} , with locally finite pointwise variation, such that, up to a subsequence,

$$\mathbb{R}_+^2 \setminus \Omega_{h_n}^\# \rightarrow \mathbb{R}_+^2 \setminus \Omega_{\tilde{h}}^\# \quad \text{in the Hausdorff metric} \quad (2.12)$$

Moreover (see [19, Proposition 2.2]), for all $x \in \mathbb{R}$

$$\tilde{h}(x) = \inf \left\{ \liminf h_n(x_n) : x_n \rightarrow x \right\}$$

and (see [19, Lemma 2.5]) $\tilde{h} = h$ almost everywhere. Define

$$\tilde{\Gamma}_{\tilde{h}}^\# := \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{R}, \tilde{h}(x) \leq y \leq \tilde{h}^+(x) = h^+(x)\},$$

which is the union of the extended graph $\Gamma_h^\# = \Gamma_h^\#$ with the vertical segments of the type $\{x\} \times [\tilde{h}(x), \tilde{h}^-(x)]$.

From (2.12) it follows that $\tilde{\Gamma}_{\tilde{h}}^\#$ is contained in the Hausdorff limit of $\Gamma_{h_n}^\#$. Hence, by (ii) $\tilde{\Gamma}_{\tilde{h}}^\# \subset \Gamma_h^\#$. Since the opposite inclusion is obvious, we conclude that the two sets coincide and, in turn, $\Omega_{\tilde{h}} = \Omega_h$. This concludes the proof of the lemma. \square

Using this compactness lemma, we now show that the incremental minimum problem (2.5) admits a solution.

Theorem 2.4. *For every $i = 1, \dots, N$, the minimum problem (2.5) admits a solution $(h_{i,N}, u_{i,N}) \in X_{e_0}$.*

Proof. Let $(h_n, u_n) \in X_{e_0}$ be a minimizing sequence for (2.5) and let γ_n be an admissible constant speed parametrization of $\Gamma_{h_n}^\#$. Then,

$$\sup_n \left\{ F(h_n, u_n) + \frac{1}{2\Delta T} \int_{\Gamma_{h_{i-1,N}}} \left(\int_0^x (h_n(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 d\mathcal{H}^1(x, y) \right\} < +\infty. \quad (2.13)$$

Since by (2.5) and by (2.13) the functions h_n are bounded in $BV(0, b)$, we have

$$\sup_n \|h_n\|_{L^\infty(0,b)} \leq C. \quad (2.14)$$

Moreover, again by (2.13) and by (2.2), we have that

$$\sup_n \int_0^1 (|\dot{\gamma}_n|^2 + |\ddot{\gamma}_n|^2) dt < +\infty.$$

This, together with (2.14), yields (2.11). Thus we are in a position to apply Lemma 2.3 to conclude that there exists $h \in AP$ such that, for a not relabeled subsequence, (i)–(iv) of that lemma hold. In particular, using (i) and (iv), we have that

$$\int_{\Gamma_h} \left(\psi(\nu) + \frac{\varepsilon}{2} k^2 \right) d\mathcal{H}^1 \leq \liminf_n \int_{\Gamma_{h_n}} \left(\psi(\nu) + \frac{\varepsilon}{2} k^2 \right) d\mathcal{H}^1 \quad (2.15)$$

and

$$\int_{\Gamma_{h_{i-1,N}}} \left(\int_0^x (h(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 d\mathcal{H}^1 = \lim_n \int_{\Gamma_{h_{i-1,N}}} \left(\int_0^x (h_n(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 d\mathcal{H}^1. \quad (2.16)$$

Finally, since $\sup_n \int_{\Omega_{h_n}} |Eu_n|^2 dz < +\infty$, reasoning as in [19, Proposition 2.2], from (iii) and Korn's inequality we conclude that there exists $u \in H_{loc}^1(\Omega_h^\#; \mathbb{R}^2)$ such that $(h, u) \in X_{e_0}$ and, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_{loc}^1(\Omega_h^\#; \mathbb{R}^2)$. Therefore, we have that

$$\int_{\Omega_h} Q(E(u)) dz \leq \liminf_n \int_{\Omega_{h_n}} Q(E(u_n)) dz,$$

which, together with (2.15) and (2.16), allows us to conclude that (h, u) is a minimizer. \square

3. EXISTENCE OF THE EVOLUTION

In this section we show that solutions of the discrete time evolution problems defined in the previous section (see (2.6)) converge to a function $h = h(x, t)$ such that $\Gamma_{h(\cdot, t)}$ is a suitable weak solution (see Definition 3.1 below) of the following geometric evolution equation,

$$V = \left((g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon \left(k_{\sigma\sigma} + \frac{1}{2} k^3 \right) \right)_{\sigma\sigma}, \quad (3.1)$$

provided that the initial configuration $(h_0, u_0) \in X_{e_0}$ satisfies (2.4). Here V denotes the outer normal velocity of $\Gamma_{h(\cdot, t)}$, k is its curvature, $Q(E(u))$ is the trace of $Q(E(u(\cdot, t)))$ on $\Gamma_{h(\cdot, t)}$, with $u(\cdot, t)$ the elastic equilibrium in $\Omega_{h(\cdot, t)}$. Moreover, $g : [0, 2\pi] \rightarrow (0, +\infty)$ is defined as

$$g(\theta) = \psi(\cos \theta, \sin \theta) \quad (3.2)$$

and is evaluated at $\arg(\nu(\cdot, t))$, where $\nu(\cdot, t)$ is the outer normal to $\Gamma_{h(\cdot, t)}$.

As in Section 2, fix T , N , and define h_N as in (2.6). Throughout this section we will assume that the initial profile h_0 belongs to $H_\#^2(0, b)$ and is strictly positive. A function $f \in L^2(\Gamma_h)$ will be identified with the functional:

$$\varphi \in H_\#^1(0, b) \mapsto \langle f, \varphi \rangle := \int_0^b f \varphi dx.$$

We now introduce a suitable notion of weak solution of equation (3.1).

Definition 3.1. We say that $h \in H^1(0, T_0; H_\#^{-1}(0, b)) \cap L^\infty(0, T_0; H_\#^2(0, b))$ is a *weak solution* of (3.1) in $[0, T_0]$ if:

- (i) for almost every $t \in [0, T_0]$ the function $h(\cdot, t) \in AP \cap H_\#^4(0, b)$;
- (ii) $(g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon \left(k_{\sigma\sigma} + \frac{1}{2} k^3 \right) \in L^2(0, T_0; H_\#^1(0, b))$, where $Q(E(u))$ is the trace of $Q(E(u(\cdot, t)))$ on $\Gamma_{h(\cdot, t)}$, and $u(\cdot, t)$ is the elastic equilibrium in $\Omega_{h(\cdot, t)}$;
- (iii) for almost every $t \in [0, T_0]$

$$\frac{\partial h}{\partial t} = J \left((g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon \left(k_{\sigma\sigma} + \frac{1}{2} k^3 \right) \right)_{\sigma\sigma} \quad \text{in } H_\#^{-1}(0, b).$$

Note that in context of the Definition 3.1, $J = \sqrt{1 + h_x^2}$, $(\cdot)_\sigma = \frac{1}{J}(\cdot)_x$, and the outer normal velocity V coincides with $\frac{1}{J} \frac{\partial h}{\partial t}$.

Remark 3.2. Concerning the definition above we observe that

(i) from Lemmas 6.6 and 6.7 it follows that

$$(g_{\theta\theta} + g)k - \varepsilon(k_{\sigma\sigma} + \frac{1}{2}k^3) = \varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x + (\psi_x(-h_x, 1))_x,$$

and so

$$\frac{\partial h}{\partial t} = \left[\frac{1}{J} \left(\varepsilon\left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5\varepsilon}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x + (\psi_x(-h_x, 1))_x \right) \right]_x. \quad (3.3)$$

(ii) Another immediate consequence of the above definition is that the evolution is volume preserving, that is, $\int_0^b h(x, t) dx = \int_0^b h_0(x) dx$ for all $t \in [0, T_0]$. Indeed, for all $t_1, t_2 \in [0, T_0]$ and for $\varphi \in H_{\#}^1(0, b)$ we have

$$\begin{aligned} \int_0^b [h(x, t_2) - h(x, t_1)] \varphi(x) dx &= \int_{t_1}^{t_2} \left\langle \frac{\partial h}{\partial t}(\cdot, t), \varphi \right\rangle dt \\ &= \int_{t_1}^{t_2} \left\langle J \left((g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon(k_{\sigma\sigma} + \frac{1}{2}k^3) \right)_{\sigma\sigma}, \varphi \right\rangle dt \\ &= - \int_{t_1}^{t_2} \int_0^b \left((g_{\theta\theta} + g)k + Q(E(u)) - \varepsilon(k_{\sigma\sigma} + \frac{1}{2}k^3) \right)_{\sigma} \varphi' dx dt. \end{aligned} \quad (3.4)$$

Choosing $\varphi = 1$ we conclude that

$$\int_0^b h(x, t_2) dx = \int_0^b h(x, t_1) dx.$$

The remaining part of this section is devoted to showing that the functions h_N constructed in Section 2 converge to a weak solution of (3.1). We start by proving that the functions h_N are uniformly bounded in $H^1((0, T); H_{\#}^{-1}(0, b))$. Precisely, we have

Theorem 3.3. *For all $N, i = 1, \dots, N$, and T we have*

$$\int_0^T \int_0^b \left(\int_0^x \frac{\partial h_N}{\partial t}(\zeta, t) d\zeta \right)^2 dx dt \leq 2F(h_0, u_0), \quad (3.5)$$

$$F(h_{i,N}, u_{i,N}) \leq F(h_0, u_0), \quad (3.6)$$

and

$$\sup_{i,N} \|\gamma_{i,N}\|_{H^2(0,1;\mathbb{R}^2)} < +\infty. \quad (3.7)$$

Moreover, up to a subsequence,

$$h_N \rightarrow h \text{ in } C^{0,\alpha}([0, T]; L^2(0, b)) \text{ for all } \alpha \in (0, \frac{1}{8}), \quad h_N \rightharpoonup h \text{ weakly in } H^1(0, T; H_{\#}^{-1}(0, b)), \quad (3.8)$$

for some function h such that $h(\cdot, t) \in AP$ for every $t \in [0, T]$.

Proof. By the minimality of $(h_{i,N}, u_{i,N})$ we have that

$$F(h_{i,N}, u_{i,N}) + \frac{1}{2\Delta T} \int_{\Gamma_{h_{i-1,N}}} \left(\int_0^x (h_{i,N}(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 d\mathcal{H}^1 \leq F(h_{i-1,N}, u_{i-1,N}) \quad (3.9)$$

for all $i = 1, \dots, N$. Hence,

$$\frac{1}{2\Delta T} \int_0^b \left(\int_0^x (h_{i,N}(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 dx \leq F(h_{i-1,N}, u_{i-1,N}) - F(h_{i,N}, u_{i,N}).$$

Summing over $i = 1, \dots, N$, we obtain

$$\sum_{i=1}^N \frac{1}{2\Delta T} \int_0^b \left(\int_0^x (h_{i,N}(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 dx \leq F(h_0, u_0),$$

from which we deduce the estimate (3.5) (see (2.6)).

Moreover, by iterating (3.9) we obtain (3.6) and, in particular, recalling (2.2)

$$\mathcal{H}^1(\Gamma_{h_{i,N}}) \leq \frac{1}{c_1} F(h_0, u_0).$$

Hence,

$$\sup_{N,t} |Dh_N(\cdot, t)|(0, b) \leq C \quad (3.10)$$

for some positive constant $C > 0$. By (3.5) we have for $t_2 > t_1$

$$\begin{aligned} \|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{H^{-1}} &\leq \int_{t_1}^{t_2} \left\| \frac{\partial h_N(\cdot, t)}{\partial t} \right\|_{H^{-1}} dt \\ &\leq (t_2 - t_1)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \left\| \frac{\partial h_N(\cdot, t)}{\partial t} \right\|_{H^{-1}}^2 dt \right)^{\frac{1}{2}} \leq C(t_2 - t_1)^{\frac{1}{2}}. \end{aligned} \quad (3.11)$$

Applying Remark 6.5 to the $f(x) = \int_0^x (h_N(\zeta, t_1) - h_N(\zeta, t_2)) d\zeta$, which vanishes at 0 and b (see (2.5)), and recalling (3.10), from (3.11) we get

$$\begin{aligned} \|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{L^2} &\leq C \left(|D(h_N(\cdot, t_2) - h_N(\cdot, t_1))|(0, b) \right)^{\frac{3}{4}} \|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{H^{-1}}^{\frac{1}{4}} \\ &\leq C(t_2 - t_1)^{\frac{1}{8}}. \end{aligned} \quad (3.12)$$

By the Ascoli-Arzelà theorem (see e.g. [6, Proposition 3.3.1]) we find a subsequence (not relabeled) and a function h such that

$$h_N \rightarrow h \quad \text{in } C^{0,\alpha}(0, T; L^2(0, b)) \text{ for all } \alpha \in (0, \frac{1}{8}).$$

Finally, observe that from (3.6) we have (3.7). Hence, since for every $t \in [0, T]$ we may find a sequence (i_n, N_n) such that $h_{i_n, N_n} \rightarrow h(\cdot, t)$ in $L^2(0, b)$, by Lemma 2.3 we conclude that $h(\cdot, t) \in AP$. The weak convergence of h_N to h in $H^1(0, T; H_{\#}^{-1}(0, b))$ follows from (3.5). This concludes the proof of the theorem. \square

In the following, h_N and h will denote the subsequence and the function provided by Theorem 3.3, respectively. The next result shows that the convergence of h_N to h can be significantly improved for short time.

Theorem 3.4. *There exist $T_0 \in (0, T]$ and $C > 0$ depending only (h_0, u_0) such that:*

- (i) $h_N \rightarrow h$ in $C^{0,\beta}([0, T_0]; C^{1,\alpha}([0, b]))$ for every $\alpha \in (0, \frac{1}{2})$, and $\beta \in (0, (1 - 2\alpha)/16)$;
- (ii) $\sup_{t \in [0, T_0]} \|h_N(\cdot, t)\|_{H^2(0, b)} \leq C$;
- (iii) $\sup_{t \in [0, T_0]} \|\nabla u_N(\cdot, t)\|_{C^{0, \frac{1}{2}}(\bar{\Omega}_{h_N(\cdot, t)})} \leq C$;
- (iv) $E(u_N(\cdot, h_N)) \rightarrow E(u(\cdot, h))$ in $C^{0,\beta}([0, T_0]; C^{1,\alpha}([0, b]))$ for every $\alpha \in (0, \frac{1}{2})$, and $0 \leq \beta < (1 - 2\alpha)/16$, where $u(\cdot, t)$ is the elastic equilibrium in $\Omega_{h(\cdot, t)}$.

In particular, $h(\cdot, t) \rightarrow h_0$ in $C^{1,\alpha}([0, b])$ as $t \rightarrow 0^+$ and $h_N, h \geq C_0 > 0$ for some positive constant C_0 , provided that N is sufficiently large and T_0 is small enough.

Proof. We claim that for every $\eta \in (0, 1)$ there exists $\delta_\eta > 0$ such that if $\frac{iT}{N} \leq \delta_\eta$, then

$$\inf_{0 \leq \tau \leq 1} \|\gamma_{i,N} - \gamma_0(\cdot + \tau)\|_{L^2(0,1)} < \eta, \quad (3.13)$$

where, we recall, $\gamma_0, \gamma_{i,N}$ are admissible constant speed parametrization of $\Gamma_{h_0}^\#, \Gamma_{h_{i,N}}^\#$, respectively, such that $\dot{\gamma}_0 \cdot e_1, \dot{\gamma}_{i,N} \cdot e_1 \geq 0$. Indeed, if not there exist $\eta_0 > 0$ and i_n, N_n such that $\frac{i_n T}{N_n} \leq \frac{1}{n}$ and

$$\inf_{0 \leq \tau \leq 1} \|\gamma_{i_n, N_n} - \gamma_0(\cdot + \tau)\|_{L^2(0,1)} \geq \eta_0 \quad (3.14)$$

for all $n \in \mathbb{N}$. Since $h_{i_n, N_n} = h_{N_n}(\cdot, \frac{i_n T}{N_n})$ (see (2.6)), by Theorem 3.3 we have that h_{i_n, N_n} converge in $L^2(0, b)$ to h_0 . Recalling (3.7), by Lemma 2.3 we have that, up to a further (not relabeled) subsequence, $\gamma_{i_n, N_n} \rightharpoonup \gamma_1$ weakly in $H_{\#}^2(0, 1; \mathbb{R}^2)$, where γ_1 is an admissible constant speed parametrization of $\Gamma_{h_0}^{\#}$. Hence, $\gamma_1 = \gamma_0(\cdot + \tau_0)$ for some $0 \leq \tau_0 \leq 1$. We deduce that

$$\|\gamma_{i_n, N_n} - \gamma_0(\cdot + \tau_0)\|_{L^2(0, 1)} \rightarrow 0,$$

which is in contradiction with (3.14). This proves the claim.

Let $\tau_{i, N} \in [0, 1]$ be such that

$$\|\gamma_{i, N} - \gamma_0(\cdot + \tau_{i, N})\|_{L^2(0, 1)} < \eta$$

for $\frac{iT}{N} \leq \delta_\eta$. We now apply Theorem 6.4 to $f := \gamma_{i, N} - \gamma_0(\cdot + \tau_{i, N})$, to obtain

$$\begin{aligned} \|\dot{\gamma}_{i, N} - \dot{\gamma}_0(\cdot + \tau_{i, N})\|_{L^\infty} &\leq C \|\ddot{\gamma}_{i, N} - \ddot{\gamma}_0(\cdot + \tau_{i, N})\|_{L^2}^{\frac{3}{4}} \|\gamma_{i, N} - \gamma_0(\cdot + \tau_{i, N})\|_{L^2}^{\frac{1}{4}} \\ &\quad + C \|\gamma_{i, N} - \gamma_0(\cdot + \tau_{i, N})\|_{L^2} \leq C \eta^{\frac{1}{4}}, \end{aligned}$$

where the last inequality follows from (3.7) and (3.13), provided that $\frac{iT}{N} \leq \delta_\eta$.

Since $\gamma_0(s) = (x_0(s), y_0(s))$ is a constant speed parametrization of a C^1 graph, we have that $\dot{x}_0(s) \geq c_0$ for some $c_0 > 0$. Therefore, if η is sufficiently small, and writing $\gamma_{i, N}(s) = (x_{i, N}(s), y_{i, N}(s))$, we have that $\dot{x}_{i, N}(s) \geq \frac{c_0}{2}$. This fact implies in particular that $h_{i, N} \in C^1([0, b])$ and

$$\left\| \frac{h'_{i, N}}{\sqrt{1 + (h'_{i, N})^2}} - \frac{h'_0}{\sqrt{1 + (h'_0)^2}} \right\|_{L^\infty(0, b)} \leq \omega(\eta), \quad (3.15)$$

where $\omega(\eta) \rightarrow 0$ as $\eta \rightarrow 0^+$. Let

$$M := \max_{x \in [0, b]} \left| \frac{h'_0(x)}{\sqrt{1 + (h'_0(x))^2}} \right| < 1.$$

By taking η in (3.15) so small that $M + \omega(\eta) < 1$, we obtain

$$\left\| \frac{h'_{i, N}}{\sqrt{1 + (h'_{i, N})^2}} \right\|_{L^\infty(0, b)} \leq M + \omega(\eta) < 1,$$

for $\frac{iT}{N} \leq \delta_\eta =: T_0$. Hence, $\|h'_{i, N}\|_{L^\infty(0, b)} \leq C$ and, in turn, by (3.7) we get that

$$\|h''_{i, N}\|_{L^2(0, b)} \leq C,$$

for $\frac{iT}{N} \leq T_0$ and for some $C > 0$. From this estimate, recalling (2.6), assertion (ii) of the statement follows.

To prove assertion (i), we start by observing that by Theorem 6.4, the property (ii) just proved, and (3.12) we have

$$\begin{aligned} \left\| \frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right\|_{L^\infty} &\leq C \left\| \frac{\partial^2 h_N}{\partial x^2}(\cdot, t_2) - \frac{\partial^2 h_N}{\partial x^2}(\cdot, t_1) \right\|_{L^2}^{\frac{1}{2}} \|h_N(\cdot, t_2) - h_N(\cdot, t_1)\|_{L^2}^{\frac{1}{2}} \\ &\leq C |t_2 - t_1|^{\frac{1}{16}} \end{aligned} \quad (3.16)$$

for all $t_1, t_2 \in [0, T_0]$. Notice that from property (ii) we have

$$\sup_{N, t \in [0, T_0]} \|h_N(\cdot, t)\|_{C^{1, \frac{1}{2}}([0, b])} < +\infty. \quad (3.17)$$

Take $\alpha \in (0, \frac{1}{2})$ and observe that,

$$\left| \frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right|_\alpha \leq \left| \frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right|_{\frac{1}{2}}^{2\alpha} \left[\operatorname{osc}_{[0,b]} \left(\frac{\partial h_N}{\partial x}(\cdot, t_2) - \frac{\partial h_N}{\partial x}(\cdot, t_1) \right) \right]^{1-2\alpha},$$

where $|\cdot|_\beta$ denotes the β -Hölder seminorm. From this inequality, (3.16), and (3.17) assertion (i) follows.

Standard elliptic estimates (see [21, Proposition 8.9]) ensure that if $h_N(\cdot, t) \in C^{1,\alpha}([0, b])$ for some $\alpha \in (0, 1)$, then $\nabla u_N(\cdot, t)$ can be estimated in $C^{0,\alpha}(\overline{\Omega}_{h_N(\cdot, t)})$ with a constant depending only on the $C^{1,\alpha}$ -norm of $h_N(\cdot, t)$. Hence, assertion (iii) follows from (3.17). Assertion (iv) is an immediate consequence of (i) and Lemma 6.10. \square

In what follows $J_{i,N}$ stands for

$$J_{i,N} := \sqrt{1 + (h'_{i,N})^2}.$$

Theorem 3.5. *Let T_0 and h_N be as in Theorem 3.4. Then there exists $C > 0$ such that*

$$\int_0^{T_0} \int_0^b \left| \frac{\partial^4 h_N}{\partial x^4}(x, t) \right|^2 dx dt \leq C \quad (3.18)$$

for $N \in \mathbb{N}$.

Proof. For every $N, i = 1, \dots, N$, the Euler-Lagrange equation satisfied by $h_{i,N}$ is

$$\int_0^b \left[\varepsilon \frac{h''_{i,N}}{J_{i,N}^5} \varphi'' - \frac{5\varepsilon (h''_{i,N})^2}{2 J_{i,N}^7} h'_{i,N} \varphi' - \psi_x(-h'_{i,N}, 1) \varphi' \right] dx + \int_0^b (Q_{i,N} - H_{i,N}) \varphi dx = 0 \quad (3.19)$$

for every $\varphi \in H_{\#}^2(0, b)$ such that $\int_0^b \varphi dx = 0$ and $\int_{\Gamma_{h_{i-1,N}}} \int_0^x \varphi d\zeta d\mathcal{H}^1 = 0$, where

$$H_{i,N}(x) := \int_0^x \left(\int_0^\zeta \frac{h_{i,N}(r) - h_{i-1,N}(r)}{\Delta T} dr \right) J_{i-1,N}(\zeta) d\zeta, \quad (3.20)$$

$$Q_{i,N}(x) := Q(E(u_{i,N}(x, h_{i,N}(x)))) .$$

We start by showing that $h_{i,N} \in H_{\#}^4(0, b)$ and that its fourth derivative, $h_{i,N}^{(\text{iv})}$, is continuous and thus periodic. To see this, note that (3.19) implies that

$$\int_0^b f_{i,N} \varphi'' dx = 0$$

for all $\varphi \in C_c^2(0, b)$ with $\int_0^b \varphi dx = 0$ and $\int_{\Gamma_{h_{i-1,N}}} \int_0^x \varphi d\zeta d\mathcal{H}^1 = 0$, where

$$f_{i,N}(x) := \varepsilon \frac{h''_{i,N}}{J_{i,N}^5} + \int_0^x \left(\frac{5\varepsilon (h''_{i,N})^2}{2 J_{i,N}^7} h'_{i,N} + \psi_x(-h'_{i,N}, 1) \right) dr + \int_0^x \int_0^r [Q_{i,N} - H_{i,N}] d\zeta dr \quad (3.21)$$

is an L^2 -function. Hence, by Lemma 6.8 we get that

$$f_{i,N}(x) = a_{i,N} \int_0^x \int_0^{x_2} \int_0^{x_1} J_{i-1,N} dr dx_1 dx_2 + c_{i,N} x^2 + d_{i,N} x + e_{i,N} \quad (3.22)$$

for some constants $a_{i,N}$, $c_{i,N}$, $d_{i,N}$, and $e_{i,N}$. Using (3.21) and (3.22), we may conclude that $h''_{i,N} \in W^{1,1}(0, b)$. By a bootstrap argument, using again (3.21), (3.22), standard elliptic estimates, and the fact $\psi \in C^2(\mathbb{R}^2 \setminus \{0\})$, we get that $h_{i,N} \in H^4(0, b)$ and $h_{i,N}^{(\text{iv})}$ is continuous in $(0, b)$.

It remains to prove periodicity. To this purpose, note that, given any $a \in \mathbb{R}$, for every $i = 1, \dots, N$, $h_{i,N}$ is also a minimizer of the functional

$$\begin{aligned} \int_a^{a+b} \int_0^{h(x)} Q(E(u)) dy dx + \int_a^{a+b} \left(\psi(\nu) + \frac{\varepsilon}{2} k^2 \right) J dx \\ + \frac{1}{2\Delta T} \int_a^{a+b} \left(\int_0^x (h(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 J_{i-1,N} dx \end{aligned}$$

among all $(h, u) \in X_{e_0}$ such that $h \in H_{\#}^2(0, b)$, $\int_0^b h dx = \int_0^b h_0 dx$, and $\int_{\Gamma_{h_{i-1,N}}} \int_0^x (h(\zeta) - h_{i-1,N}(\zeta)) d\zeta d\mathcal{H}^1 = 0$. This is a consequence of the translation invariance of the energy, which in turn follows from the b -periodicity of h , $h_{i-1,N}$, and $x \mapsto \int_0^x (h - h_{i-1,N}) d\zeta$ (see (2.5)). Writing the Euler-Lagrange equation satisfied by $h_{i,N}$ in the interval $(a, a+b)$, we get

$$\int_a^{a+b} \left[\varepsilon \frac{h''_{i,N}}{J_{i,N}^5} \varphi'' - \frac{5\varepsilon}{2} \frac{(h''_{i,N})^2}{J_{i,N}^7} h'_{i,N} \varphi' - \psi_x(-h'_{i,N}, 1) \varphi' \right] dx + \int_a^{a+b} (Q_{i,N} - H_{i,N}) \varphi dx = 0, \quad (3.23)$$

where we have used the fact that

$$\begin{aligned} & \frac{d}{d\eta} \left[\frac{1}{2\Delta T} \int_a^{a+b} \left(\int_0^x (h_{i,N}(\zeta) + \eta \varphi(\zeta) - h_{i-1,N}(\zeta)) d\zeta \right)^2 J_{i-1,N} dx \right]_{|\eta=0} \\ &= \int_a^{a+b} \left[\int_0^x \varphi d\zeta \int_0^x \frac{h_{i,N} - h_{i-1,N}}{\Delta T} d\zeta \right] J_{i-1,N} dx \\ &= H_{i,N}(a+b) \int_0^{a+b} \varphi d\zeta - H_{i,N}(a) \int_0^a \varphi d\zeta - \int_a^{a+b} \varphi H_{i,N} dx \\ &= (H_{i,N}(a+b) - H_{i,N}(a)) \int_0^a \varphi d\zeta - \int_a^{a+b} \varphi H_{i,N} dx \\ &= - \int_a^{a+b} \varphi H_{i,N} dx. \end{aligned}$$

Observe that the second equality follows by integrating by parts, while the last two follow from the fact that $\int_a^{a+b} \varphi d\zeta = 0$ and the periodicity of $H_{i,N}$, respectively.

Now arguing as before in any interval of the form $(a, a+b)$ and using equation (3.23), we deduce that $h_{i,N} \in H_{loc}^4(\mathbb{R})$ and $h_{i,N}^{(iv)}$ is continuous. Hence, since $h_{i,N}$ is b -periodic, $h_{i,N} \in H_{\#}^4(0, b)$ and $h_{i,N}^{(iv)}$ is b -periodic too.

Differentiating (3.22), we obtain

$$f''_{i,N}(x) = a_{i,N} \int_0^x J_{i-1,N} d\zeta + 2c_{i,N}.$$

By (3.21), the b -periodicity of $h_{i,N}$, its derivatives up to order 4, $Q_{i,N}$, and $H_{i,N}$, we conclude that $f''_{i,N}$ is b -periodic and so $a_{i,N} = 0$.

Differentiating twice (3.21), we obtain

$$\varepsilon \left(\frac{h''_{i,N}}{J_{i,N}^5} \right)'' + \left(\frac{5\varepsilon}{2} \frac{(h''_{i,N})^2}{J_{i,N}^7} h'_{i,N} + \psi_x(-h'_{i,N}, 1) \right)' + Q_{i,N} - H_{i,N} = 2c_{i,N}. \quad (3.24)$$

Integrating both sides of the above equality in $(0, b)$ and using periodicity, we deduce

$$2bc_{i,N} = \int_0^b (Q_{i,N} - H_{i,N}) dx.$$

From (3.24), using Theorem 3.4 and Young inequality to estimate $h'_{i,N}$, $J_{i,N}$, and $Q_{i,N}$, we have

$$\begin{aligned} \int_0^b |h_{i,N}^{(\text{iv})}|^2 dx &\leq C \int_0^b (1 + |h''_{i,N}|^6 + |h''_{i,N}|^2 |h'''_{i,N}|^2 + H_{i,N}^2) dx \\ &\leq C \int_0^b |h'''_{i,N}|^3 dx + C \int_0^b |h''_{i,N}|^6 dx + C \int_0^b (1 + H_{i,N}^2) dx. \end{aligned} \quad (3.25)$$

Using Theorem 6.4 to estimate $\|h''_{i,N}\|_{L^6}$ and $\|h'''_{i,N}\|_{L^3}$, we obtain from (3.25)

$$\begin{aligned} \|h_{i,N}^{(\text{iv})}\|_{L^2}^2 &\leq C \|h''_{i,N}\|_{L^2}^{\frac{5}{4}} \|h_{i,N}^{(\text{iv})}\|_{L^2}^{\frac{7}{4}} + C \|h''_{i,N}\|_{L^2}^5 \|h_{i,N}^{(\text{iv})}\|_{L^2} + C \int_0^b (1 + H_{i,N}^2) dx \\ &\leq \eta \|h_{i,N}^{(\text{iv})}\|_{L^2}^2 + C_\eta \int_0^b (1 + H_{i,N}^2) dx, \end{aligned}$$

where in the last inequality we used Theorem 3.4(ii). Choosing η sufficiently small and summing over all i such that $i \frac{T}{N} \leq T_0$, and recalling (3.5), we finally conclude that

$$\int_0^{T_0} \int_0^b \left| \frac{\partial^4 h_N}{\partial x^4}(x, t) \right|^2 dx dt \leq C \int_0^{T_0} \int_0^b (1 + H_N^2) dx dt \leq C, \quad (3.26)$$

thus proving (3.18). \square

Remark 3.6. Denote by \tilde{h}_N the piecewise interpolation function

$$\tilde{h}_N(x, t) := \sum_{i=1}^N \chi_{[\frac{i-1}{N}T, \frac{i}{N}T)}(t) h_{i,N}(x).$$

The argument used in the proof of Theorem 3.5 allows us to write in place of (3.26) the following inequality

$$\int_0^{T_0} \int_0^b \left| \frac{\partial^4 \tilde{h}_N}{\partial x^4}(x, t) \right|^2 dx dt \leq C.$$

Note also that property (i) in Theorem 3.4 implies that $\tilde{h}_N \rightarrow h$ in $L^\infty(0, T_0; C^{1,\alpha}([0, b]))$ for all $\alpha \in (0, \frac{1}{2})$.

As a consequence of estimate (3.18) we get that the discrete time evolutions h_N converge to h in a much stronger sense.

Corollary 3.7. *Let $T_0 > 0$ be as in Theorem 3.4. Then*

$$h_N, \tilde{h}_N \rightarrow h \quad \text{in } L^{\frac{12}{5}}(0, T_0; C_{\#}^{2,1}([0, b])) \cap L^{\frac{24}{5}}(0, T_0; C_{\#}^{1,1}([0, b])).$$

Proof. Fix $N > M$. By applying Theorem 6.4, we get that

$$\begin{aligned} &\left\| \frac{\partial^3 h_N}{\partial x^3}(\cdot, t) - \frac{\partial^3 h_M}{\partial x^3}(\cdot, t) \right\|_{L^\infty(0, b)} \\ &\leq C \left(\int_0^b \left| \frac{\partial^4 h_N}{\partial x^4}(x, t) - \frac{\partial^4 h_M}{\partial x^4}(x, t) \right|^2 dx \right)^{\frac{5}{12}} \left(\int_0^b \left| \frac{\partial h_N}{\partial x}(x, t) - \frac{\partial h_M}{\partial x}(x, t) \right|^2 dx \right)^{\frac{1}{12}}. \end{aligned}$$

Raising both sides to the power $\frac{12}{5}$, integrating with respect to time, and recalling (3.18) we have that

$$\int_0^{T_0} \left\| \frac{\partial^3 h_N}{\partial x^3}(\cdot, t) - \frac{\partial^3 h_M}{\partial x^3}(\cdot, t) \right\|_{L^\infty(0, b)}^{\frac{12}{5}} dt \leq C \int_0^{T_0} \left\| \frac{\partial h_N}{\partial x}(\cdot, t) - \frac{\partial h_M}{\partial x}(\cdot, t) \right\|_{L^\infty(0, b)}^{\frac{2}{5}} dt.$$

The conclusion then follows from property (i) in Theorem 3.4. This proves the convergence of h_N to h in $L^{\frac{12}{5}}(0, T_0; C_{\#}^{2,1}([0, b]))$. The convergence of h_N to h in $L^{\frac{24}{5}}(0, T_0; C_{\#}^{1,1}([0, b]))$ is obtained similarly by observing that from Theorem 6.4

$$\left\| \frac{\partial^2 h_N}{\partial x^2}(\cdot, t) - \frac{\partial^2 h_M}{\partial x^2}(\cdot, t) \right\|_{L^\infty(0, b)} \leq C \left\| \frac{\partial^3 h_N}{\partial x^3}(\cdot, t) - \frac{\partial^3 h_M}{\partial x^3}(\cdot, t) \right\|_{L^\infty(0, b)}^{\frac{1}{2}} \left\| \frac{\partial h_N}{\partial x}(\cdot, t) - \frac{\partial h_M}{\partial x}(\cdot, t) \right\|_{L^\infty(0, b)}^{\frac{1}{2}}$$

for almost every t . Finally, the corresponding convergence for \tilde{h}_N are proved analogously in view of Remark 3.6. \square

The next theorem establishes the existence of a weak solution to (3.1) for all $h_0 \in H_{\#}^2(0, b)$ strictly positive.

Theorem 3.8. *Let T_0 be as in Theorem 3.4. Then equation (3.1) admits a weak solution in $[0, T_0]$ in the sense of Definition 3.1 with initial datum h_0 . Moreover, if $\psi \in C^3(\mathbb{R}^2 \setminus \{0\})$ then $h(\cdot, t) \in H_{\#}^5(0, b)$ for almost every $t \in [0, T_0]$.*

Proof. Define

$$\tilde{H}_N(x, t) := H_{i,N}(x), \quad \tilde{Q}_N(x, t) := Q_{i,N}(x), \quad \tilde{J}_N(x, t) := J_{i,N}(x) \quad \text{if } (i-1)\frac{T}{N} \leq t < i\frac{T}{N},$$

where $H_{i,N}$ and $Q_{i,N}$ have been introduced in (3.20). Fix $t \in (0, T_0)$ and a sequence (j_n, N_n) such that $\frac{j_n}{N_n}T \rightarrow t$. Fix also $\chi \in C_c^\infty(0, b)$ and $\varphi_{i,N_n} := (\frac{\chi'}{J_{i-1,N_n}})'$. Define also $\tilde{\varphi}_n(x, t) := \varphi_{i,N_n}(x)$ if $(i-1)\frac{T}{N} \leq t < i\frac{T}{N}$. Inserting φ_{i,N_n} in (3.19) and summing up the resulting equations from $i = 1$ to $i = j_n$, we get

$$\begin{aligned} \int_0^t \int_0^b \left[\varepsilon \frac{(\tilde{h}_{N_n})_{xx}}{\tilde{J}_{N_n}^5} (\tilde{\varphi}_n)_{xx} - \frac{5\varepsilon}{2} \frac{(\tilde{h}_{N_n})_{xx}^2}{\tilde{J}_{N_n}^7} (\tilde{h}_{N_n})_x (\tilde{\varphi}_n)_x - \psi_x(-(\tilde{h}_{N_n})_x, 1) (\tilde{\varphi}_n)_x \right] dx d\tau \\ + \int_0^t \int_0^b (\tilde{Q}_{N_n} - \tilde{H}_{N_n}) \tilde{\varphi}_n dx d\tau = 0. \end{aligned}$$

Integrating by parts the first term in the integral, we obtain

$$\begin{aligned} \int_0^t \int_0^b \left[-\varepsilon \left(\frac{(\tilde{h}_{N_n})_{xx}}{\tilde{J}_{N_n}^5} \right)_x (\tilde{\varphi}_n)_x - \frac{5\varepsilon}{2} \frac{(\tilde{h}_{N_n})_{xx}^2}{\tilde{J}_{N_n}^7} (\tilde{h}_{N_n})_x (\tilde{\varphi}_n)_x - \psi_x(-(\tilde{h}_{N_n})_x, 1) (\tilde{\varphi}_n)_x \right] dx d\tau \\ + \int_0^t \int_0^b (\tilde{Q}_{N_n} - \tilde{H}_{N_n}) \tilde{\varphi}_n dx d\tau = 0. \end{aligned}$$

To pass to the limit in this equation we recall the following facts:

- (i) by Corollary 3.7 we have that $(\frac{(\tilde{h}_{N_n})_{xx}}{\tilde{J}_{N_n}^5})_x \rightarrow (\frac{h_{xx}}{J^5})_x$ in $L^{\frac{12}{5}}(0, T_0; L^\infty(0, b))$;
- (ii) by (3.8) \tilde{H}_N converges weakly in $L^2(0, T_0; H_{\#}^1(0, b))$ to the unique function H such that $H(0, t) = H(b, t) = 0$ and

$$\left(\frac{H_x}{J} \right)_x = \frac{\partial h}{\partial t} \text{ in } H_{\#}^{-1}(0, b) \quad (3.27)$$

for almost every t . Indeed, from (3.20) we have that

$$\left(\frac{H_{i,N}}{J_{i-1,N}} \right)_x = \frac{\partial h_N}{\partial t} \quad \text{for } t \in \left[\frac{i-1}{N}T, \frac{i}{N}T \right)$$

and (3.27) easily follows by (3.8) and property (i) in Theorem 3.4.

- (iii) by property (iv) of Theorem 3.4 we have that $\tilde{Q}_N \rightarrow Q(E(u(\cdot, h(\cdot, \tau)))) =: Q(\cdot, \tau)$ in $L^\infty(0, T_0; C^{1,\alpha}([0, b]))$ for all $\alpha \in (0, \frac{1}{2})$;
- (iv) by Corollary 3.7 $(\tilde{\varphi}_n)_x \rightarrow \varphi_x$ in $L^{\frac{12}{5}}(0, T_0; L^\infty(0, b))$, where $\varphi(x, t) := (\frac{\chi'(x)}{J(x,t)})_x$.

Therefore, passing to the limit in n , we get

$$\int_0^t \int_0^b \left[-\varepsilon \left(\frac{h_{xx}}{J^5} \right)_x - \frac{5\varepsilon}{2} \frac{(h_{xx})^2}{J^7} h_x - \psi_x(-h_x, 1) \right] \varphi_x dx d\tau + \int_0^t \int_0^b (Q - H) \varphi dx d\tau = 0. \quad (3.28)$$

From (3.28) it follows that for every $t \in (0, T_0) \setminus N_\chi$, with $\mathcal{L}^1(N_\chi) = 0$,

$$\int_0^b \left[-\varepsilon \left(\frac{h_{xx}}{J^5} \right)_x - \frac{5\varepsilon}{2} \frac{(h_{xx})^2}{J^7} h_x - \psi_x(-h_x, 1) \right] \varphi_x dx + \int_0^b (Q - H) \varphi dx = 0. \quad (3.29)$$

Letting χ vary in a countable dense subset of $C_c^\infty(0, b)$ we conclude that for almost every $t \in [0, T_0]$ equation (3.29) holds for all $\chi \in C_c^\infty(0, b)$ and thus

$$\int_0^b \left[-\varepsilon \left(\frac{h_{xx}}{J^5} \right)_x - \frac{5\varepsilon}{2} \frac{(h_{xx})^2}{J^7} h_x - \psi_x(-h_x, 1) \right] \eta' dx + \int_0^b (Q - H) \eta dx = 0 \quad (3.30)$$

for all $\eta \in C_c^2([0, b])$ such that $\int_0^b \eta dx = 0$ and $\int_{\Gamma_h} \int_0^x \eta d\zeta d\mathcal{H}^1 = 0$.

We claim that from (3.30) it follows that $h(\cdot, t) \in C_\#^4([0, b])$ and

$$\varepsilon \left(\frac{h_{xx}}{J^5} \right)_{xx} + \frac{5\varepsilon}{2} \left(\frac{h_{xx}^2}{J^7} h_x \right)_x + (\psi_x(-h_x, 1))_x + Q - H = \alpha\sigma + \beta. \quad (3.31)$$

for suitable time-dependent functions $\alpha = \alpha(t)$ and $\beta = \beta(t)$. Here, $\sigma(x, t) := \int_0^x J(x, \tau) d\tau$. To prove the claim note that since by Theorem 3.5 $h \in L^2(0, T_0; H_\#^4(0, b))$, (3.30) can be written as

$$\int_0^b f \eta dx = 0 \quad \text{for all } \eta \text{ as above,}$$

where

$$f(x) := \varepsilon \left(\frac{h_{xx}}{J^5} \right)_{xx} + \frac{5\varepsilon}{2} \left(\frac{h_{xx}^2}{J^7} h_x \right)_x + (\psi_x(-h_x, 1))_x + Q - H.$$

By Lemma 6.8, (3.31) follows. Note that from (3.31), recalling that $\psi \in C^2(\mathbb{R}^2 \setminus \{0\})$, we obtain that $h(\cdot, t) \in C^4(0, b)$. Now using equation (3.23) in any interval of the form $(a, a+b)$ and arguing as before, we deduce that $h(\cdot, t) \in C^4(\mathbb{R})$. In particular, $h(\cdot, t) \in C_\#^4([0, b])$. In turn, the left-hand side of (3.31) is b -periodic, and since H is also b -periodic we conclude that α must be zero. Note that since $h(\cdot, t) \in C_\#^4([0, b])$, by elliptic regularity we have that $u(\cdot, t) \in C^3(\overline{\Omega}_{h(\cdot, t)}; \mathbb{R}^2)$. Therefore, since $H(\cdot, t) \in H_\#^1(0, b)$, from equation (3.31) we deduce that $h(\cdot, t) \in H_\#^5(0, b)$ if $\psi \in C^3(\mathbb{R}^2 \setminus \{0\})$.

Integrating (3.31) with respect to x and recalling that $\alpha = 0$, we get that

$$\beta(t) = \frac{1}{b} \int_0^b (Q(x, t) - H(x, t)) dx.$$

Therefore, since $H \in L^2(0, T_0; H_\#^1(0, b))$ we deduce that

$$\varepsilon \left(\frac{h_{xx}}{J^5} \right)_{xx} + \frac{5\varepsilon}{2} \left(\frac{h_{xx}^2}{J^7} h_x \right)_x + (\psi_x(-h_x, 1))_x + Q \in L^2(0, T_0; H_\#^1(0, b))$$

thus proving condition (ii) in Definition 3.1, thanks to part (i) of Remark 3.2.

Finally, in view of part (i) of Remark 3.2, condition (iii) in Definition 3.1 follows by combining (3.27) with (3.31), recalling that $\alpha = 0$.

□

4. REGULARITY

In this section we show that if ψ is of class C^4 away from the origin and $h_0 \in H_{\#}^3(0, b)$, $h_0 > 0$, then the solution constructed in the previous section satisfies (3.1) in a stronger sense, precisely, $h \in L^2(0, T_0; H_{\#}^6(0, b)) \cap H^1(0, T_0; L^2(0, b))$.

Next we need a delicate and crucial elliptic estimate on the trace of the $E(u)$ on Γ_h , where u is the elastic equilibrium in Ω_h .

Theorem 4.1. *Let $(h, u) \in X_{e_0}$ be such that $h \in H_{\#}^5(0, b)$, $h \geq c_0 > 0$, and u is the corresponding elastic equilibrium. Then, there exists a constant C depending only on $\|h\|_{H^2(0, b)}$, c_0 , and $\|E(u)\|_{L^\infty(\Omega_h)}$ such that the following estimates hold:*

$$\int_{\Gamma_h} |DE(u)|^2 d\mathcal{H}^1 \leq C \int_0^b (1 + |h^{(\text{iv})}|^2) dx \quad (4.1)$$

and

$$\int_{\Gamma_h} |D^2 E(u)|^2 d\mathcal{H}^1 + \int_{\Gamma_h} |D_\sigma(E(u))|^4 d\mathcal{H}^1 \leq C \int_0^b (1 + |h^{(\text{v})}|^2) dx, \quad (4.2)$$

where D_σ denotes the tangential derivative along Γ_h .

Proof. We split the proof into several steps.

Step 1. (*Airy Functions*) Since $h \in H_{\#}^5(0, b)$, by elliptic regularity we have that $u \in C^{4, \frac{1}{2}}(\bar{\Omega}_h^\#)$. We now introduce the Airy function w associated to u , which satisfies

$$\begin{pmatrix} w_{xx} & w_{xy} \\ w_{xy} & w_{yy} \end{pmatrix} = \begin{pmatrix} (\mathbb{C}E(u))_{22} & -(\mathbb{C}E(u))_{12} \\ -(\mathbb{C}E(u))_{12} & (\mathbb{C}E(u))_{11} \end{pmatrix}. \quad (4.3)$$

It is well known that w is biharmonic (see for instance [20, Ch. 12]). Note that we can choose w to satisfy the additional conditions

$$w = 0 \quad \text{and} \quad \nabla w = 0 \quad \text{on } \Gamma_h^\#. \quad (4.4)$$

Indeed, since $\mathbb{C}E(u)[\nu] = 0$ on $\Gamma_h^\#$, it follows from (4.3) that $D_\sigma(\nabla w) = 0$. Hence, by subtracting a suitable affine function, if necessary, we may impose that (4.4) holds. Moreover, from the regularity of u we have that $w \in C^{5, \frac{1}{2}}(\Omega_h^\#)$.

Step 2. (*Straightening the boundary*) We denote by $Q_h^\#$ the image of $\Omega_h^\#$ under the diffeomorphism $\Phi_h(x, y) := (x, y - h(x))$. Note that $\Phi_h(\Gamma_h^\#) = \{y = 0\}$. Set $v(x, y) := w(x, y + h(x))$ for all $(x, y) \in Q_h^\#$. Note that, since

$$v_x = w_x + h'w_y, \quad v_y = w_y,$$

and

$$v_{xx} = w_{xx} + 2h'w_{xy} + h'^2w_{yy} + h''w_y, \quad v_{xy} = w_{xy} + h'w_{yy}, \quad v_{yy} = w_{yy}, \quad (4.5)$$

from the assumptions on h and u we conclude that

$$|Dv| \leq C \quad \text{and} \quad |D^2v| \leq C(1 + |h''|) \quad (4.6)$$

for some constant $C > 0$ depending only on $\|h'\|_{L^\infty}$ and $\|E(u)\|_{L^\infty}$. Since $w(x, y) = v(x, y - h(x))$ and $\Delta^2 w = 0$, a lengthy but straightforward calculation shows that v satisfies the 4-th order elliptic equation $L(v) = f$ in $Q_h^\#$, where

$$f := 6h''v_{xxy} - 12h'h''v_{xyy} + (2 + 6h'^2)h''v_{yyy} + 4h'''v_{xy} - (3h''^2 + 4h'h''')v_{yy} + h^{(\text{iv})}v_y, \quad (4.7)$$

and

$$L(v) := v_{xxxx} - 4h'v_{xxxy} + (2 + 6h'^2)v_{xxyy} - (4h' + 4h'^3)v_{xyyy} + (1 + h'^2)^2v_{yyyy}. \quad (4.8)$$

Moreover, v satisfies the boundary conditions (see (4.4))

$$v = 0 \quad \text{and} \quad \nabla v = 0 \quad \text{on } \{y = 0\}.$$

Set $S := [0, b] \times \{0\}$, fix two bounded open sets U' and V' such that $S \subset U' \subset\subset V'$, with $\overline{V'} \cap \Phi_h(\{y = 0\}) = \emptyset$, and denote $U := U' \cap Q_h^\#$ and $V := V' \cap Q_h^\#$. From well-known elliptic estimates (see [2]) we have that

$$\int_U |D^4 v|^2 dz \leq C \int_V (|v|^2 + |f|^2) dz. \quad (4.9)$$

Step 3. (*Estimate of $D^4 v$*) From (4.8) and (4.9), recalling (4.6) and using Young's inequality, we get

$$\begin{aligned} \int_U |D^4 v|^2 dz &\leq C \int_V [1 + |h^{(\text{iv})}|^2 |Dv|^2 + (h''^4 + h'''^2) |D^2 v|^2 + h''^2 |D^3 v|^2] dz \\ &\leq C_\eta \int_0^b (1 + h''^6 + h'''^2 + h''^2 h'''^2 + |h^{(\text{iv})}|^2) dx + \eta \int_V |D^3 v|^3 dz \\ &\leq C_\eta \int_0^b (1 + h''^6 + h'''^3 + |h^{(\text{iv})}|^2) dx + \eta \int_V |D^3 v|^3 dz. \end{aligned} \quad (4.10)$$

We now use Theorem 6.4 to estimate

$$\int_0^b (h''^6 + h'''^3) dx \leq C \left(\|h^{(\text{iv})}\|_{L^2} \|h''\|_{L^2}^5 + \|h^{(\text{iv})}\|_{L^2}^{\frac{7}{2}} \|h''\|_{L^2}^{\frac{5}{2}} \right) \leq C \int_0^b (1 + |h^{(\text{iv})}|^2) dx \quad (4.11)$$

for a suitable constant $C > 0$ depending only on $\|h''\|_{L^2}$. Using Theorem 6.4 and (4.6), we have

$$\begin{aligned} \int_V |D^3 v|^3 dz &\leq C \int_V |D^4 v|^2 dz \left(\int_V |D^2 v|^2 dz \right)^{\frac{1}{2}} + \left(\int_V |D^2 v|^2 dz \right)^{\frac{3}{2}} \\ &\leq C \int_V |D^4 v|^2 dz \left(\int_0^b (1 + h''^2) dx \right)^{\frac{1}{2}} + C \left(\int_0^b (1 + h''^2) dx \right)^{\frac{3}{2}} \leq C \int_V (1 + |D^4 v|^2) dz \end{aligned}$$

for some constant $C > 0$ depending only on $\|E(u)\|_{L^\infty}$ and $\|h''\|_{L^2}$. Inserting this estimate in (4.10), recalling (4.11), and taking η sufficiently small, we conclude that

$$\int_U |D^4 v|^2 dz \leq C \int_0^b (1 + |h^{(\text{iv})}|^2) dx + C \int_{V \setminus U} |D^4 v|^2 dz. \quad (4.12)$$

Step 4. (*Estimate of $D^5 v$*) Differentiating equation (4.7) with respect to x , since $v_x = 0$ and $\nabla v_x = 0$ on $\{y = 0\}$, as before from elliptic regularity we have

$$\int_U |D^4 v_x|^2 dz \leq C \int_V (1 + |v_x|^2 + |f_x|^2 + h''^2 |D^4 v|^2) dz. \quad (4.13)$$

From (4.7), recalling (4.6) and using Young's inequality, we get

$$\begin{aligned} \int_V (|f_x|^2 + h''^2 |D^4 v|^2) dz &\leq C \int_V [|h^{(\text{v})}|^2 |Dv|^2 + (h''^2 h'''^2 + |h^{(\text{iv})}|^2) |D^2 v|^2] dz \\ &\quad + C \int_V [(1 + h''^4 + h'''^2) |D^3 v|^2 + h''^2 |D^4 v|^2] dz \\ &\leq \eta \int_V (|D^3 v|^4 + |D^4 v|^{\frac{8}{3}}) dz + C_\eta \int_0^b (1 + h''^8 + h'''^4 + |h^{(\text{iv})}|^{\frac{8}{3}} + |h^{(\text{v})}|^2) dx. \end{aligned} \quad (4.14)$$

Using repeatedly Theorem 6.4, we get

$$\begin{aligned} \int_0^b (h''^8 + h''^4 + |h^{(\text{iv})}|^{\frac{8}{3}}) dx &\leq C \left(\|h^{(\text{v})}\|_{L^2} \|h''\|_{L^2}^7 + \|h^{(\text{v})}\|_{L^2}^{\frac{5}{3}} \|h''\|_{L^2}^{\frac{7}{3}} + \|h^{(\text{v})}\|_{L^2}^{\frac{17}{9}} \|h''\|_{L^2}^{\frac{7}{9}} \right) \\ &\leq C \int_0^b (1 + |h^{(\text{v})}|^2) dx \end{aligned} \quad (4.15)$$

for some constant C depending only on $\|h\|_{H^2}$. Similarly, using Theorem 6.4 and again (4.6),

$$\begin{aligned} \int_V (|D^3 v|^4 + |D^4 v|^{\frac{8}{3}}) dz &\leq C \left(\|D^5 v\|_{L^2(V)}^2 \|D^2 v\|_{L^2(V)}^2 + \|D^2 v\|_{L^2(V)}^4 + \|D^5 v\|_{L^2(V)}^2 \|D^2 v\|_{L^2(V)}^{\frac{2}{3}} + \|D^2 v\|_{L^2(V)}^{\frac{8}{3}} \right) \\ &\leq C \left(1 + \|D^5 v\|_{L^2(V)}^2 \right). \end{aligned}$$

Inserting the last estimate and (4.15) in (4.14), and recalling (4.13), we have

$$\int_U |D^4 v_x|^2 dz \leq C\eta \int_V |D^5 v|^2 dz + C_\eta \int_0^b (1 + |h^{(\text{v})}|^2) dx. \quad (4.16)$$

Differentiating (4.7) with respect to y , we easily get

$$\|v_{yyyyyy}\|_{L^2(U)}^2 \leq C \int_U |D^4 v_x|^2 dz + \int_U [h''^2 |D^4 v|^2 + (h''^4 + h''^2) |D^3 v|^2 + |h^{(\text{iv})}|^2 |D^2 v|^2] dz.$$

Estimating the second integral in the previous formula as we did for the right-hand side of (4.14) and recalling (4.16), we have

$$\int_U |D^5 v|^2 dz \leq C\eta \int_V |D^5 v|^2 dz + C_\eta \int_0^b (1 + |h^{(\text{v})}|^2) dx.$$

Choosing η small enough, we conclude that

$$\int_U |D^5 v|^2 dz \leq C \int_0^b (1 + |h^{(\text{v})}|^2) dx + C \int_{V \setminus U} |D^5 v|^2 dz. \quad (4.17)$$

Step 5. (*Proof of (4.1)*) Let W' be an open set such that $\Phi_h^{-1}(V \setminus U) \subset \subset W'$ and set $W := W' \cap \Omega_h^\#$. We first observe that since $u \in e_0(\cdot, 0) + LD_\#(\Omega_h; \mathbb{R}^2)$ solves the linear system

$$\begin{cases} \int_{\Omega_h} \mathbb{C}E(u) : E(\varphi) = 0 & \text{for all } \varphi \in LD_\#(\Omega_h; \mathbb{R}^2) \text{ s.t. } \varphi(x, 0) = 0, \\ u(x, 0) = e_0(x, 0), \end{cases}$$

then standard elliptic regularity implies that for all $n \in \mathbb{N}$

$$\int_{\Phi_h^{-1}(V \setminus U)} |D^n u|^2 dz \leq C \int_W (1 + |u|^2) dz \leq C$$

for some positive constant C depending only on n and e_0 . Therefore, if $n \geq 2$ from (4.3) it follows that

$$\int_{\Phi_h^{-1}(V \setminus U)} |D^n w|^2 dz \leq C \int_{\Phi_h^{-1}(V \setminus U)} |D^{n-1} u|^2 dz \leq C \quad (4.18)$$

for some constant depending only on n and e_0 . Observe that similarly to (4.5) we have

$$w_{xx} = v_{xx} - 2h'v_{xy} + h'^2v_{yy} - h''v_y, \quad w_{xy} = v_{xy} - h'v_{yy}, \quad \text{and} \quad w_{yy} = v_{yy}. \quad (4.19)$$

Differentiating these inequalities with respect to x and y and using (4.3) and (4.6), we get

$$\begin{aligned}
\int_{\Gamma_h} |DE(u)|^2 d\mathcal{H}^1 &\leq C \int_{\Gamma_h} |D^3 w|^2 d\mathcal{H}^1 \leq C \int_S (|D^3 v|^2 + |D^2 v|^2 h''^2 + |Dv|^2 h'''^2) dx \\
&\leq C \int_U (|D^3 v|^2 + |D^4 v|^2) dz + C \int_0^b (1 + h''^4 + h'''^2) dx \\
&\leq C \int_U (|D^2 v|^2 + |D^4 v|^2) dz + C \int_0^b (1 + h''^4 + h'''^2) dx \\
&\leq C \int_U |D^4 v|^2 dz + C \int_0^b (1 + h''^4 + h'''^2) dx
\end{aligned}$$

where we have used the trace theorem to control $\|D^3 v\|_{L^2(S)}$ with $\|D^3 v\|_{H^1(U)}$. Estimating the last integral as we did in Step 3 and recalling (4.12), we obtain

$$\int_{\Gamma_h} |DE(u)|^2 d\mathcal{H}^1 \leq C \int_0^b (1 + |h^{(iv)}|^2) dx + C \int_{V \setminus U} |D^4 v|^2 dz. \quad (4.20)$$

Concerning the last integral, we have arguing as in (4.10)

$$\begin{aligned}
\int_{V \setminus U} |D^4 v|^2 dz &\leq C \int_{\Phi_h^{-1}(V \setminus U)} [1 + |h^{(iv)}|^2 |Dw|^2 + (h''^4 + h'''^2) |D^2 w|^2 + h''^2 |D^3 w|^2] dz \\
&\leq C_\eta \int_0^b (1 + h''^6 + h'''^3 + |h^{(iv)}|^2) dx + \eta \int_{\Phi_h^{-1}(V \setminus U)} |D^3 w|^3 dx \\
&\leq C_\eta \int_0^b (1 + h''^6 + h'''^3 + |h^{(iv)}|^2) dx + C,
\end{aligned}$$

where we used (4.18). Controlling the last integral by $\int_0^b (1 + |h^{(iv)}|^2) dx$ as we did in Step 3 (see (4.11)), and inserting the resulting estimate in (4.20), we conclude the proof of (4.1).

Step 6. (*Proof of (4.2)*) Differentiating (4.19) twice and using (4.3) and (4.6), we have

$$\begin{aligned}
\int_{\Gamma_h} |D^2 E(u)|^2 d\mathcal{H}^1 &\leq C \int_{\Gamma_h} |D^4 w|^2 d\mathcal{H}^1 \\
&\leq C \int_S [|D^4 v|^2 + h''^2 |D^3 v|^2 + (h''^4 + h'''^2) |D^2 v|^2 + |h^{(iv)}|^2 |Dv|^2] dx \\
&\leq C \int_S (|D^3 v|^{\frac{8}{3}} + |D^4 v|^2) dx + C \int_0^b (1 + h''^8 + h'''^4 + |h^{(iv)}|^2) dx \\
&\leq C \int_S (|D^3 v|^{\frac{8}{3}} + |D^4 v|^2) dx + C \int_0^b (1 + |h^{(v)}|^2) dx, \quad (4.21)
\end{aligned}$$

where the last inequality follows from (4.15). We now estimate, using Theorem 6.9 and Theorem 6.4

$$\begin{aligned}
\int_S (|D^3 v|^{\frac{8}{3}} + |D^4 v|^2) dx &\leq C \left(\|D^3 v\|_{L^{\frac{16}{11}}(U)}^{\frac{8}{3}} + \|D^4 v\|_{L^{\frac{16}{11}}(U)}^{\frac{8}{3}} + \|D^4 v\|_{L^2(U)}^2 + \|D^5 v\|_{L^2(U)}^2 \right) \\
&\leq C \left(\|D^5 v\|_{L^2(U)}^{\frac{5}{9}} \|D^2 v\|_{L^2(U)}^{\frac{19}{9}} + \|D^5 v\|_{L^2(U)}^{\frac{13}{9}} \|D^2 v\|_{L^2(U)}^{\frac{11}{9}} \right. \\
&\quad \left. + \|D^2 v\|_{L^2(U)}^{\frac{8}{3}} + \|D^4 v\|_{L^2(U)}^2 + \|D^5 v\|_{L^2(U)}^2 \right) \\
&\leq C \int_U (1 + |D^4 v|^2 + |D^5 v|^2) dz \leq C \int_U (1 + |D^5 v|^2) dz.
\end{aligned}$$

From this inequality, (4.17), and (4.21), we have

$$\int_{\Gamma_h} |D^2 E(u)|^2 d\mathcal{H}^1 \leq C \int_0^b (1 + |h^{(v)}|^2) dx + C \int_{V \setminus U} |D^5 v|^2 dz. \quad (4.22)$$

To estimate the tangential derivative of $E(u)$, we observe that by Theorem 6.4

$$\|D_\sigma E(u)\|_{L^4(\Gamma_h)} \leq C \|D_{\sigma\sigma}^2 E(u)\|_{L^4(\Gamma_h)}^{\frac{1}{2}} \|E(u)\|_{L^4(\Gamma_h)}^{\frac{1}{2}} \leq C \|D_{\sigma\sigma}^2 E(u)\|_{L^4(\Gamma_h)}^{\frac{1}{2}}.$$

Hence, differentiating twice the identities in (4.19), recalling (4.6)₂, and using Young's inequality, we have after some straightforward calculations,

$$\begin{aligned} \int_{\Gamma_h} |D_\sigma E(u)|^4 d\mathcal{H}^1 &\leq C \left(\int_S \left[|D^4 v|^4 + h''^4 |D^3 v|^4 + (h''^8 + h'''^4) |D^2 v|^4 + |h^{(\text{iv})}|^4 |Dv|^4 \right] dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_S \left(|D^3 v|^{\frac{24}{5}} + |D^4 v|^4 \right) dx \right)^{\frac{1}{2}} + C \left(\int_0^b \left(1 + h''^{24} + |h'''|^{\frac{24}{5}} + |h^{(\text{iv})}|^4 \right) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (4.23)$$

Using Theorem 6.4, we have

$$\begin{aligned} \int_0^b \left(1 + h''^{24} + |h'''|^{\frac{24}{5}} + |h^{(\text{iv})}|^4 \right) dx &\leq C \left(\|h^{(\text{v})}\|_{L^2}^{\frac{11}{3}} \|h''\|_{L^2}^{\frac{61}{3}} + \|h^{(\text{v})}\|_{L^2}^{\frac{31}{15}} \|h''\|_{L^2}^{\frac{41}{15}} + \|h^{(\text{v})}\|_{L^2}^3 \|h''\|_{L^2} \right) \\ &\leq C \left(1 + \|h^{(\text{v})}\|_{L^2}^{\frac{11}{3}} \right) \leq C \left(\int_0^b (1 + |h^{(\text{v})}|^2) dx \right)^2. \end{aligned} \quad (4.24)$$

On the other hand, using Theorem 6.9 and (4.6), we can estimate

$$\begin{aligned} \int_S \left(|D^3 v|^{\frac{24}{5}} + |D^4 v|^4 \right) dx &\leq C \left(\|D^3 v\|_{W^{1, \frac{48}{29}}(U)}^{\frac{24}{5}} + \|D^4 v\|_{W^{1, \frac{8}{5}}(U)}^4 \right) \\ &\leq C \left(\|D^4 v\|_{L^{\frac{48}{29}}(U)}^{\frac{24}{5}} + \|D^3 v\|_{L^{\frac{48}{29}}(U)}^{\frac{24}{5}} + \|D^5 v\|_{L^{\frac{8}{5}}(U)}^4 + \|D^4 v\|_{L^{\frac{8}{5}}(U)}^4 \right) \\ &\leq C \left(\|D^4 v\|_{L^{\frac{48}{29}}(U)}^{\frac{24}{5}} + \|D^2 v\|_{L^{\frac{48}{29}}(U)}^{\frac{24}{5}} + \|D^5 v\|_{L^{\frac{8}{5}}(U)}^4 + \|D^2 v\|_{L^{\frac{8}{5}}(U)}^4 \right) \\ &\leq C \left(1 + \|D^4 v\|_{L^{\frac{48}{29}}(U)}^{\frac{24}{5}} + \|D^5 v\|_{L^{\frac{8}{5}}(U)}^4 \right). \end{aligned}$$

Combining this estimate with (4.23) and (4.24), using Theorem 6.4, and recalling (4.6), we have

$$\begin{aligned} \int_{\Gamma_h} |D_\sigma E(u)|^4 d\mathcal{H}^1 &\leq C \left(\|D^4 v\|_{L^{\frac{48}{29}}(U)}^{\frac{12}{5}} + \|D^5 v\|_{L^{\frac{8}{5}}(U)}^2 \right) + C \int_0^b (1 + |h^{(\text{v})}|^2) dx \\ &\leq C \left(\|D^5 v\|_{L^2(U)}^{\frac{43}{30}} \|D^2 v\|_{L^2(U)}^{\frac{29}{30}} + \|D^2 v\|_{L^2(U)}^{\frac{12}{5}} + \|D^5 v\|_{L^{\frac{8}{5}}(U)}^2 \right) + C \int_0^b (1 + |h^{(\text{v})}|^2) dx \\ &\leq C \|D^5 v\|_{L^2}^2 + C \int_0^b (1 + |h^{(\text{v})}|^2) dx. \end{aligned}$$

The last estimate combined with (4.17) yields

$$\int_{\Gamma_h} |D_\sigma E(u)|^4 d\mathcal{H}^1 \leq C \int_0^b (1 + |h^{(\text{v})}|^2) dx + C \int_{V \setminus U} |D^5 v|^2 dz. \quad (4.25)$$

Concerning the last integral, arguing as in (4.14) and (4.15), and using (4.18), we have

$$\begin{aligned} \int_{V \setminus U} |D^5 v|^2 dz &\leq C \int_{\Phi_h^{-1}(V \setminus U)} \left[|h^{(\text{v})}|^2 |Dw|^2 + (h''^2 h'''^2 + |h^{(\text{iv})}|^2) |D^2 w|^2 \right] dz \\ &\quad + C \int_{\Phi_h^{-1}(V \setminus U)} \left[(1 + h''^4 + h'''^2) |D^3 w|^2 + h''^2 |D^4 w|^2 \right] dz + C \int_{\Phi_h^{-1}(V \setminus U)} |D^5 w|^2 dz \\ &\leq C \int_{\Phi_h^{-1}(V \setminus U)} \left(|D^3 w|^4 + |D^4 w|^{\frac{8}{3}} \right) dz + C \int_0^b (1 + h''^8 + h'''^4 + |h^{(\text{iv})}|^2 + |h^{(\text{v})}|^2) dx \\ &\leq C \int_0^b (1 + |h^{(\text{v})}|^2) dx. \end{aligned}$$

Hence, (4.2) follows by inserting the last estimate in (4.22) and (4.25). This concludes the proof of the theorem. \square

We now state a regularity result which applies to any weak solution to (3.1).

Theorem 4.2 (Properties of weak solutions). *Let h be a weak solution of (3.1) in $[0, T_0]$ in the sense of Definition 3.1. Then, the following properties hold:*

- (i) $h \in C^{0,\beta}([0, T_0]; C^{1,\alpha}([0, b]))$ for all $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{16})$;
- (ii) $h \in L^2(0, T_0; H_{\#}^4(0, b))$;
- (iii) if $\psi \in C^3(\mathbb{R}^2 \setminus \{0\})$ then $h \in L^2(0, T_0; H_{\#}^5(0, b))$.

Proof. Property (i) is a consequence of the fact that $H^1(0, T; H_{\#}^{-1}(0, b)) \cap L^{\infty}(0, T; H_{\#}^2(0, b))$ is continuously embedded in $C^{0,\beta}([0, T]; C^{1,\alpha}([0, b]))$ for all $\alpha \in (0, \frac{1}{2})$ and $\beta \in (0, \frac{1-2\alpha}{16})$, as shown in the proof of property (i) of Theorem 3.4.

Property (ii) can be proved arguing exactly as in the proof of Theorem 3.5, replacing (3.24) by (3.31), with $\alpha = 0$.

Finally, we establish (iii). To this aim, we note that if $\psi \in C^3(\mathbb{R}^2 \setminus \{0\})$ then $h(\cdot, t) \in H_{\#}^5(0, b)$ for almost every t . Indeed, this can be proved as in the final part of the proof of Theorem 3.8.

To simplify the notation we use $(\cdot)'$ to denote differentiation with respect to x . Differentiating equation (3.31) with respect to x , estimating Q_x by (4.1), one gets that for almost every $t \in (0, T_0)$

$$\begin{aligned} \int_0^b |h^{(\text{iv})}|^2 dx &\leq C \int_0^b (1 + h''^8 + h''^4 h'''^2 + h'''^4 + h''^2 |h^{(\text{iv})}|^2 + |h^{(\text{iv})}|^2 + H'^2) dx \\ &\leq C \int_0^b (1 + h''^8 + h'''^4 + |h^{(\text{iv})}|^{\frac{8}{3}} + H'^2) dx. \end{aligned}$$

Estimating

$$\int_0^b (h''^8 + h'''^4 + |h^{(\text{iv})}|^{\frac{8}{3}}) dx$$

as in (4.15), and using Young's inequality to estimate the products of the L^2 -norms of h'' and $h^{(\text{iv})}$, we arrive at

$$\int_0^b |h^{(\text{iv})}|^2 dx \leq \eta \int_0^b |h^{(\text{iv})}|^2 dx + C_{\eta} \int_0^b (1 + H'^2) dx.$$

By this inequality, taking η sufficiently small and integrating in time, property (iii) follows. \square

We now come to main result of the section.

Theorem 4.3. *Let $h_0 \in H_{\#}^3(0, b)$, $h_0 > 0$, let $\psi \in C^4(\mathbb{R}^2 \setminus \{0\})$, and let T_0 be as in Theorem 3.4. Then the solution to (3.1) constructed in Theorem 3.8 belongs to $H^1(0, T_0; L^2(0, b)) \cap L^2(0, T_0; H^6(0, b))$.*

Proof. We start by observing that (3.24) gives

$$\begin{aligned} \frac{h_{i,N}^{(\text{iv})}}{J_{i,N}^5} &= 10 \frac{h'_{i,N} h''_{i,N} h'''_{i,N}}{J_{i,N}^7} + \frac{5}{2} \frac{h_{i,N}''^3}{J_{i,N}^7} - \frac{35}{2} \frac{h_{i,N}'^2 h_{i,N}''^3}{J_{i,N}^9} + \psi_{xx}(-h'_{i,N}, 1) \frac{h_{i,N}''}{\varepsilon} \\ &\quad + \frac{1}{\varepsilon} H_{i,N} + \frac{2}{\varepsilon} c_{i,N} - \frac{1}{\varepsilon} Q_{i,N} \\ &= M_{i,N} + \frac{1}{\varepsilon} H_{i,N} + \frac{2}{\varepsilon} c_{i,N} - \frac{1}{\varepsilon} Q_{i,N}, \end{aligned} \tag{4.26}$$

where we set

$$M_{i,N} := 10 \frac{h'_{i,N} h''_{i,N} h'''_{i,N}}{J_{i,N}^7} + \frac{5}{2} \frac{h_{i,N}''^3}{J_{i,N}^7} - \frac{35}{2} \frac{h_{i,N}'^2 h_{i,N}''^3}{J_{i,N}^9} + \psi_{xx}(-h'_{i,N}, 1) \frac{h_{i,N}''}{\varepsilon}. \tag{4.27}$$

Differentiating (4.26) with respect to x , recalling (3.20), and setting $\frac{d}{d\sigma_{i,N}} := \frac{1}{J_{i,N}} \frac{d}{dx}$, we get

$$\left(\frac{h_{i,N}^{(\text{iv})}}{J_{i,N}^5} \right)' = M'_{i,N} + \frac{J_{i-1,N}}{\varepsilon} \int_0^x \frac{h_{i,N} - h_{i-1,N}}{\Delta T} d\zeta - \frac{J_{i,N}}{\varepsilon} \mathbb{C}E(u_{i,N}) : \frac{d}{d\sigma_{i,N}} E(u_{i,N}).$$

Dividing both sides of the last identity by $J_{i-1,N}$ and differentiating again with respect to x , we get

$$\left[\frac{1}{J_{i-1,N}} \left(\frac{h_{i,N}^{(\text{iv})}}{J_{i,N}^5} \right)' \right]' = \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' + \frac{1}{\varepsilon} \frac{h_{i,N} - h_{i-1,N}}{\Delta T} - \left(\frac{J_{i,N}}{\varepsilon J_{i-1,N}} \mathbb{C}E(u_{i,N}) : \frac{d}{d\sigma_{i,N}} E(u_{i,N}) \right)'.$$

Expanding both sides of the above equation, we obtain

$$\begin{aligned} & \frac{h_{i,N}^{(\text{vi})}}{J_{i,N}^5 J_{i-1,N}} - \frac{10h'_{i,N} h''_{i,N} h_{i,N}^{(\text{v})}}{J_{i,N}^7 J_{i-1,N}} - \frac{5h''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^7 J_{i-1,N}} - \frac{5h'_{i,N} h'''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^7 J_{i-1,N}} + \frac{35h_{i,N}^{\prime 2} h''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^9 J_{i-1,N}} \\ & - \frac{h'_{i-1,N} h''_{i-1,N}}{J_{i-1,N}^3} \left(\frac{h_{i,N}^{(\text{v})}}{J_{i,N}^5} - \frac{5h'_{i,N} h''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^7} \right) = \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' + \frac{1}{\varepsilon} \frac{h_{i,N} - h_{i-1,N}}{\Delta T} \\ & - \frac{1}{\varepsilon} \left[\frac{h'_{i,N} h''_{i,N}}{J_{i,N} J_{i-1,N}} - \frac{h'_{i-1,N} h''_{i-1,N} J_{i,N}}{J_{i-1,N}^3} \right] \left(\mathbb{C}E(u_{i,N}) : \frac{d}{d\sigma_{i,N}} E(u_{i,N}) \right) \\ & - \frac{J_{i,N}}{\varepsilon J_{i-1,N}} \left[J_{i,N} \mathbb{C}E(u_{i,N}) : \frac{d^2}{d\sigma_{i,N}^2} E(u_{i,N}) + J_{i,N} \mathbb{C} \frac{d}{d\sigma_{i,N}} E(u_{i,N}) : \frac{d}{d\sigma_{i,N}} E(u_{i,N}) \right]. \end{aligned} \quad (4.28)$$

Let us now set

$$\begin{aligned} R_{i,N} := & \frac{10h'_{i,N} h''_{i,N} h_{i,N}^{(\text{v})}}{J_{i,N}^7 J_{i-1,N}} + \frac{5h''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^7 J_{i-1,N}} + \frac{5h'_{i,N} h'''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^7 J_{i-1,N}} - \frac{35h_{i,N}^{\prime 2} h''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^9 J_{i-1,N}} \\ & + \frac{h'_{i-1,N} h''_{i-1,N}}{J_{i-1,N}^3} \left(\frac{h_{i,N}^{(\text{v})}}{J_{i,N}^5} + \frac{5h'_{i,N} h''_{i,N} h_{i,N}^{(\text{iv})}}{J_{i,N}^7} \right). \end{aligned} \quad (4.29)$$

Multiplying both sides of (4.28) by $h_{i,N}^{(\text{vi})}$, integrating with respect to x , and recalling that all the $J_{i,N}$ are equibounded in L^∞ , after three integration by parts in the second integral, we have

$$\begin{aligned} & C_0 \int_0^b |h_{i,N}^{(\text{vi})}|^2 dx + \frac{1}{\varepsilon} \int_0^b \frac{h_{i,N}^{\prime \prime 2} - h_{i-1,N}^{\prime \prime 2} h_{i,N}^{\prime \prime \prime}}{\Delta T} dx \leq C \int_0^b |h_{i,N}^{(\text{vi})}| \left[|R_{i,N}| + \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right| \right] dx \\ & + C \int_0^b (|h_{i,N}''| + |h_{i-1,N}''|) |h_{i,N}^{(\text{vi})}| |D_{\sigma_{i,N}}(E(u_{i,N}))| dx + C \int_0^b |h_{i,N}^{(\text{vi})}| |D_{\sigma_{i,N}}(E(u_{i,N}))|^2 dx \\ & + C \int_0^b |h_{i,N}^{(\text{vi})}| |D_{\sigma_{i,N} \sigma_{i,N}}^2(E(u_{i,N}))| dx \\ & \leq \frac{C_0}{2} \int_0^b |h_{i,N}^{(\text{vi})}|^2 dx + C \int_0^b \left[|R_{i,N}|^2 + \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right|^2 \right] dx + C \int_0^b (|h_{i,N}''|^4 + |h_{i-1,N}''|^4) dx \\ & + C \int_0^b (|D_{\sigma_{i,N}}(E(u_{i,N}))|^4 + |D_{\sigma_{i,N} \sigma_{i,N}}^2(E(u_{i,N}))|^2) dx. \end{aligned}$$

Observing that $h_{i,N}''' - h_{i-1,N}''' \geq \frac{1}{2}(h_{i,N}''' - h_{i-1,N}''')$, recalling (4.2), and using Theorem 6.4, we get

$$\begin{aligned} \frac{C_0}{2} \int_0^b |h_{i,N}^{(vi)}|^2 dx + \frac{1}{2\varepsilon} \int_0^b \frac{(h_{i,N}''' - h_{i-1,N}''')}{\Delta T} dx &\leq C \int_0^b \left[|R_{i,N}| + \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right|^2 \right] dx \\ &\quad + C \int_0^b (1 + |h_{i,N}^{(v)}|^2 + |h_{i-1,N}^{(v)}|^2) dx \\ &\leq \eta \int_0^b (|h_{i,N}^{(vi)}|^2 + |h_{i-1,N}^{(vi)}|^2) dx + C_\eta \int_0^b (1 + |h_{i,N}^{(v)}|^2 + |h_{i-1,N}^{(v)}|^2) dx, \end{aligned}$$

where in the last inequality we used Lemma 4.4 below. Summing up over i , we have

$$\begin{aligned} \frac{C_0}{2} \int_0^{T_0} \int_0^b \left| \frac{\partial^6 h_N}{\partial x^6} \right|^2 dx dt + \frac{1}{2\varepsilon} \int_0^b (h_{N,N}''' - h_0''') dx \\ \leq 3\eta \int_0^{T_0} \int_0^b \left| \frac{\partial^6 h_N}{\partial x^6} \right|^2 dx dt + C_\eta \int_0^{T_0} \int_0^b \left(1 + \left| \frac{\partial^4 h_N}{\partial x^4} \right|^2 \right) dx. \end{aligned}$$

Choosing η small enough and recalling (3.18), we conclude that

$$\int_0^{T_0} \int_0^b \left| \frac{\partial^6 h_N}{\partial x^6} \right|^2 dx dt \leq C \quad (4.30)$$

with $C > 0$ independent of N . This shows that h_N is bounded in $L^2(0, T_0; H_{\#}^6(0, b))$, therefore $h \in L^2(0, T_0; H_{\#}^6(0, b))$.

To conclude the proof, observe that from (4.28), and arguing as above, we have

$$\begin{aligned} \int_0^b \left| \frac{h_{i,N} - h_{i-1,N}}{\Delta T} \right|^2 dx &\leq C \int_0^b |h_{i,N}^{(vi)}|^2 dx + C \int_0^b \left[|R_{i,N}|^2 + \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right|^2 \right] dx \\ &\quad + C \int_0^b (|h_{i,N}''|^2 + |h_{i-1,N}''|^2) |D_{\sigma_{i,N}}(E(u_{i,N}))|^2 dx + C \int_0^b |D_{\sigma_{i,N}}(E(u_{i,N}))|^4 dx \\ &\quad + C \int_0^b |D_{\sigma_{i,N}\sigma_{i,N}}^2(E(u_{i,N}))|^2 dx. \end{aligned}$$

Using (4.30), the right-hand side of this inequality and thus we conclude that

$$\int_0^{T_0} \int_0^b \left| \frac{\partial h_N}{\partial t} \right|^2 dx dt \leq C$$

with $C > 0$ independent of N . Hence, $h \in H^1(0, T_0; L^2(0, b))$. \square

Lemma 4.4. *Let $M_{i,N}$ and $R_{i,N}$ be defined as in (4.27) and (4.29). Then for any $\eta > 0$*

$$\int_0^b \left[|R_{i,N}|^2 + \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right|^2 \right] dx \leq \eta \int_0^b (|h_{i,N}^{(vi)}|^2 + |h_{i-1,N}^{(vi)}|^2) dx + C_\eta.$$

Proof. Recalling Theorem 3.4-(i), we have

$$|R_{i,N}|^2 \leq C \left[|h_{i,N}^{(v)}|^2 (h_{i,N}'' + h_{i-1,N}'') + |h_{i,N}^{(iv)}|^2 (h_{i,N}'''' + h_{i-1,N}'''' + h_{i,N}''') \right]. \quad (4.31)$$

Moreover, a lengthy but straightforward calculation together with property (i) of Theorem 3.4 leads to the estimate

$$\begin{aligned} \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right|^2 &\leq C \left[|h_{i,N}^{(v)}|^2 h_{i,N}'' + |h_{i,N}^{(iv)}|^2 (h_{i,N}'''' + h_{i-1,N}'''' + h_{i,N}''') \right. \\ &\quad \left. + h_{i,N}'''' (h_{i,N}'' + h_{i-1,N}'') + h_{i,N}'''' (h_{i,N}'''' + h_{i-1,N}''') + (h_{i,N}'''' + h_{i-1,N}''') \right]. \end{aligned}$$

Using (4.31) and the above estimate, we get by Young's inequality

$$\int_0^b \left[|R_{i,N}|^2 + \left| \left(\frac{M'_{i,N}}{J_{i-1,N}} \right)' \right|^2 \right] dx \leq \int_0^b \left[|h_{i,N}^{(v)}|^{\frac{5}{2}} + |h_{i,N}^{(iv)}|^{\frac{10}{3}} + |h_{i,N}'''|^5 + h_{i,N}''^{10} + h_{i-1,N}''^{10} \right] dx. \quad (4.32)$$

By repeatedly using Theorem 6.4 and recalling property (ii) of Theorem 3.4, we get

$$\begin{aligned} \int_0^b & \left[|h_{i,N}^{(v)}|^{\frac{5}{2}} + |h_{i,N}^{(iv)}|^{\frac{10}{3}} + |h_{i,N}'''|^5 + h_{i,N}''^{10} + h_{i-1,N}''^{10} \right] dx \leq \left[\|h_{i,N}^{(vi)}\|_{L^2}^{\frac{31}{16}} \|h_{i,N}''\|_{L^2}^{\frac{9}{16}} + \|h_{i,N}^{(vi)}\|_{L^2}^{\frac{11}{6}} \|h_{i,N}''\|_{L^2}^{\frac{3}{2}} \right. \\ & \quad \left. + \|h_{i,N}^{(vi)}\|_{L^2}^{\frac{13}{8}} \|h_{i,N}''\|_{L^2}^{\frac{27}{8}} + \|h_{i,N}^{(vi)}\|_{L^2} \|h_{i,N}''\|_{L^2}^9 + \|h_{i-1,N}^{(vi)}\|_{L^2} \|h_{i-1,N}''\|_{L^2}^9 \right] \\ & \leq \eta \int_0^b (|h_{i,N}^{(vi)}|^2 + |h_{i-1,N}^{(vi)}|^2) dx + C_\eta. \end{aligned}$$

Combining the last estimate with (4.32) we obtain the thesis. \square

5. UNIQUENESS

In this section we show that if ψ is of class C^3 away from the origin, then local weak solutions are unique. As before, h_0 satisfies (2.4).

Theorem 5.1 (Uniqueness). *Let $\psi \in C^3(\mathbb{R}^2 \setminus \{0\})$, and let h_1, h_2 be two weak solutions of (3.1) in $[0, T_0]$ with initial datum h_0 . Then $h_1 = h_2$.*

Proof. To simplify the notation, throughout the proof we use $(\cdot)'$ to denote differentiation with respect to x . First we observe that by Theorem 4.2 and an approximation argument, it follows that if h is a weak solution of (3.1), then the function $t \mapsto \int_0^b h'^2 dx$ has a weak derivative, which coincides for almost every t with $-2 \int_0^b \langle \frac{\partial h}{\partial t}, h'' \rangle dx$. Therefore, multiplying equations (3.3) for h_1 and h_2 by $h_2'' - h_1''$ and integrating by parts, we get for almost every t

$$\begin{aligned} & -\frac{1}{2} \frac{\partial}{\partial t} \int_0^b |h_2' - h_1'|^2 dx \\ & = \int_0^b \left[\varepsilon \left(\frac{h_2''}{J_2^5} \right)'' + \frac{5\varepsilon}{2} \left(\frac{h_2''^2}{J_2^7} h_2' \right)' + (\psi_x(-h_2', 1))' + Q_2 \right] \left[\frac{h_2^{(iv)} - h_1^{(iv)}}{J_2} - \frac{(h_2''' - h_1''')h_2''h_2'}{J_2^3} \right] dx \\ & \quad - \int_0^b \left[\varepsilon \left(\frac{h_1''}{J_1^5} \right)'' + \frac{5\varepsilon}{2} \left(\frac{h_1''^2}{J_1^7} h_1' \right)' + (\psi_x(-h_1', 1))' + Q_1 \right] \left[\frac{h_2^{(iv)} - h_1^{(iv)}}{J_1} - \frac{(h_2''' - h_1''')h_1''h_1'}{J_1^3} \right] dx, \end{aligned}$$

where the subscripts i refer to the functions h_i , $i = 1, 2$. After repeated use of Young's inequality, using the fact that h_i' is bounded, and that the function $t \mapsto (1 + t^2)^{-\frac{m}{2}}$, $m > 0$, is Lipschitz continuous, we get

$$\begin{aligned} & \int_0^b |h_2^{(iv)} - h_1^{(iv)}|^2 dx + \frac{\partial}{\partial t} \int_0^b |h_2' - h_1'|^2 dx \\ & \leq C \int_0^b \left[|h_1^{(iv)}|^2 + |h_2^{(iv)}|^2 + h_1''^2 h_1'''^2 + h_2''^2 h_2'''^2 + h_1''^6 + h_2''^6 + 1 \right] |h_2' - h_1'|^2 dx \\ & \quad + C \int_0^b \left[h_1'''^2 + h_2'''^2 + h_1''^4 + h_2''^4 \right] |h_2'' - h_1''|^2 dx + C \int_0^b \left[h_1''^2 + h_2''^2 \right] |h_2''' - h_1'''|^2 dx \\ & \quad + C \int_0^b |Q_2 - Q_1|^2 dx =: I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (5.1)$$

Setting $B(t) := \|h_1^{(v)}\|_2^2 + \|h_2^{(v)}\|_2^2$, using Theorem 6.4 to estimate $\|h_i''\|_\infty$, $\|h_i'''\|_\infty$, $\|h_i^{(iv)}\|_\infty$ with $\|h_i^{(v)}\|_2$, and recalling that $\|h_i''\|_2$ is bounded, we have

$$I_1 \leq C(1 + B(t)) \int_0^b |h_2' - h_1'|^2 dx. \quad (5.2)$$

Estimating $\|h_i''\|_\infty$, $\|h_i'''\|_\infty$, with $\|h_i^{(v)}\|_2$ and using Theorem 6.4 again, we get by Young's inequality

$$I_2 + I_3 \leq \eta \int_0^b |h_2^{(iv)} - h_1^{(iv)}|^2 dx + C_\eta(1 + B(t)) \int_0^b |h_2' - h_1'|^2 dx. \quad (5.3)$$

By Lemma 6.10 we get

$$I_4 \leq \|h_2 - h_1\|_{C^{1,\alpha}([0,b])}^2 \leq \int_0^b |h_2'' - h_1''|^2 dx \leq \eta \int_0^b |h_2^{(iv)} - h_1^{(iv)}|^2 dx + C_\eta \int_0^b |h_2' - h_1'|^2 dx.$$

From this inequality, (5.1), (5.2), and (5.3), and choosing η sufficiently small, we have

$$\frac{\partial}{\partial t} \int_0^b |h_2' - h_1'|^2 dx \leq C(1 + B(t)) \int_0^b |h_2' - h_1'|^2 dx.$$

Since $B \in L^1(0, T)$ by Theorem 4.2-(iii), using Gronwall's lemma we conclude that $\int_0^b |h_2' - h_1'|^2 dx = 0$ for every t . Hence, the conclusion follows. \square

Remark 5.2. From Theorem 5.1 we deduce that if $\psi \in C^3(\mathbb{R}^2 \setminus \{0\})$, then the solutions of the discrete time evolution problem constructed in Section 2 converge to the unique solution of (3.1). If, in addition, ψ is of class C^4 away from the origin, then, in view of Theorem 4.3 we conclude that the unique weak solution of (3.1) belongs to $H^1(0, T; L^2(0, b)) \cap L^2(0, T; H_\#^6(0, b))$.

Remark 5.3. Combining Corollary 3.7 and Theorem 4.3, by interpolation we conclude that whole sequence h_N constructed in (2.6) satisfies

$$h_N \rightarrow h \quad \text{in } L^2(0, T; H_\#^5(0, b)).$$

6. APPENDIX

Here we collect some interpolation inequalities and other auxiliary results that are used throughout the paper.

The next two theorems are particular cases of more general statements proved in [1]. Precisely, from [1, Theorem 5.2] we have

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let $1 \leq p \leq \infty$ and j, m be two integers such that $0 \leq j \leq m$ and $m \geq 1$. Then there exists $C > 0$ such that*

$$\|D^j f\|_{L^p(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^{\frac{j}{m}} \|f\|_{L^p(\Omega)}^{\frac{m-j}{m}} + \|f\|_{L^p(\Omega)}) \quad (6.1)$$

for all $f \in W^{m,p}(\Omega)$. Moreover, if $n = 1$, Ω is an interval, and $f \in W_\#^{m,p}(\Omega)$, then (6.1) holds in the stronger form

$$\|D^j f\|_{L^p(\Omega)} \leq C\|D^m f\|_{L^p(\Omega)}^{\frac{j}{m}} \|f\|_{L^p(\Omega)}^{\frac{m-j}{m}}. \quad (6.2)$$

Proof. Inequality (6.1) follows by combining inequalities (3) and (1) in [1, Theorem 5.2].

If f is periodic in the interval Ω , the proof of (6.2) is contained in [28]. \square

Remark 6.2. If $p = 1$ the conclusion of the above theorem holds also for $f \in W^{m-1,1}(\Omega)$ such that $D^{m-1}f \in BV(\Omega)$, provided that in (6.1) we replace $\|D^m f\|_{L^1(\Omega)}$ by the total variation $|D(D^{m-1}f)|(\Omega)$. This can be easily seen by a standard approximation argument.

The next interpolation result is essentially contained in [1, Theorem 5.8].

Theorem 6.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let $1 \leq p \leq q < \infty$ if $mp \geq n$, and let $1 \leq p \leq q \leq \infty$ if $mp > n$. Then there exists $C > 0$ such that*

$$\|f\|_{L^q(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^\theta \|f\|_{L^p(\Omega)}^{1-\theta} + \|f\|_{L^p(\Omega)}), \quad (6.3)$$

for all $f \in W^{m,p}(\Omega)$, where $\theta := \frac{n}{mp} - \frac{n}{mq}$. Moreover, if $n = 1$, Ω is an interval, $f \in W_{\#}^{m,p}(\Omega)$, and either f vanishes at the boundary or $\int_{\Omega} f dx = 0$, (6.3) holds in the stronger form

$$\|f\|_{L^q(\Omega)} \leq C\|D^m f\|_{L^p(\Omega)}^\theta \|f\|_{L^p(\Omega)}^{1-\theta}. \quad (6.4)$$

Proof. Inequality (6.3) follows by combining inequality (10) in [1, Theorem 5.8] with (1) in [1, Theorem 5.2].

If f is periodic in the interval Ω and either f vanishes at the boundary or $\int_{\Omega} f dx = 0$, (6.4) follows from inequality (10) [1, Theorem 5.8], observing that

$$\|f\|_{W^{m,p}(\Omega)} \leq C\|D^m f\|_{L^p(\Omega)},$$

as a straightforward application of the Poincaré inequality. \square

Combining Theorems 6.1 and 6.3 we have the following theorem.

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let s, j , and m be integers such that $0 \leq s \leq j \leq m$. Let $1 \leq p \leq q < \infty$ if $(m-j)p \geq n$, and let $1 \leq p \leq q \leq \infty$ if $(m-j)p > n$. Then, there exists $C > 0$ such that*

$$\|D^j f\|_{L^q(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^\theta \|D^s f\|_{L^p(\Omega)}^{1-\theta} + \|D^s f\|_{L^p(\Omega)}) \quad (6.5)$$

for all $f \in W^{m,p}(\Omega)$, where

$$\theta := \frac{1}{m-s} \left(\frac{n}{p} - \frac{n}{q} + j - s \right).$$

Moreover, if $n = 1$, Ω is an interval, $f \in W_{\#}^{m,p}(\Omega)$, and either f vanishes at the boundary or $\int_{\Omega} f dx = 0$, (6.5) holds in the stronger form

$$\|D^j f\|_{L^q(\Omega)} \leq C\|D^m f\|_{L^p(\Omega)}^\theta \|D^s f\|_{L^p(\Omega)}^{1-\theta}. \quad (6.6)$$

Proof. Since $(m-j)p \geq n$, we may apply Theorem 6.3 to $D^j f$ to obtain

$$\|D^j f\|_{L^q(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^{\tilde{\theta}} \|D^j f\|_{L^p(\Omega)}^{1-\tilde{\theta}} + \|D^j f\|_{L^p(\Omega)}) \quad (6.7)$$

where $\tilde{\theta} = \frac{n}{m-j} \left(\frac{1}{p} - \frac{1}{q} \right)$. On the other hand, by applying Theorem 6.1 to $D^s f$, we get

$$\|D^j f\|_{L^p(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^{\frac{j-s}{m-s}} \|D^s f\|_{L^p(\Omega)}^{\frac{m-j}{m-s}} + \|D^s f\|_{L^p(\Omega)}). \quad (6.8)$$

Inserting this inequality in the right-hand side of (6.7) we get

$$\|D^j f\|_{L^q(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^\theta \|D^s f\|_{L^p(\Omega)}^{1-\theta} + \|D^m f\|_{L^p(\Omega)}^{\tilde{\theta}} \|D^s f\|_{L^p(\Omega)}^{1-\tilde{\theta}} + \|D^j f\|_{L^p(\Omega)}). \quad (6.9)$$

Since $\tilde{\theta} \leq \theta$, a simple application of the Young inequality shows that

$$\|D^m f\|_{L^p(\Omega)}^{\tilde{\theta}} \|D^s f\|_{L^p(\Omega)}^{1-\tilde{\theta}} \leq C(\|D^m f\|_{L^p(\Omega)}^\theta \|D^s f\|_{L^p(\Omega)}^{1-\theta} + \|D^s f\|_{L^p(\Omega)}).$$

Therefore, inserting this inequality in (6.9) and using (6.8) again, we get

$$\|D^j f\|_{L^q(\Omega)} \leq C(\|D^m f\|_{L^p(\Omega)}^\theta \|D^s f\|_{L^p(\Omega)}^{1-\theta} + \|D^m f\|_{L^p(\Omega)}^{\frac{j-s}{m-s}} \|D^s f\|_{L^p(\Omega)}^{\frac{m-j}{m-s}} + \|D^s f\|_{L^p(\Omega)}).$$

Oberving that $\frac{j-s}{m-s} \leq \theta$, another application of the Young inequality leads to

$$\|D^m f\|_{L^p(\Omega)}^{\frac{j-s}{m-s}} \|D^s f\|_{L^p(\Omega)}^{\frac{m-j}{m-s}} \leq C(\|D^m f\|_{L^p(\Omega)}^\theta \|D^s f\|_{L^p(\Omega)}^{1-\theta} + \|D^s f\|_{L^p(\Omega)}).$$

Hence, (6.5) follows. The proof of (6.6) is obtained in a similar (but simpler) way by combining (6.2) and (6.4). \square

Remark 6.5. As in Remark 6.2, the conclusion of Theorem 6.4 holds for a function $f \in W^{m-1,1}(\Omega)$ such that $D^{m-1}f \in BV(\Omega)$, provided that in (6.5) we replace $\|D^m f\|_{L^1(\Omega)}$ by $|D(D^{m-1}f)|(\Omega)$.

Nest we prove some identities, which are used in the derivation of the Euler-Lagrange equation, for the energy functional (2.3).

Lemma 6.6. *Let g be the function introduced in (3.2). Then, for every $\theta \in (0, 2\pi) \setminus \{\pi\}$*

$$g(\theta) + g_{\theta\theta}(\theta) = \frac{\psi_{xx}(\cos \theta, \sin \theta)}{\sin^2 \theta}.$$

Proof. Since ψ is positively one-homogeneous, we have that $\psi(x, y) = x\psi_x(x, y) + y\psi_y(x, y)$. Therefore we have

$$\begin{aligned} g(\theta) + g_{\theta\theta}(\theta) &= \psi(\cos \theta, \sin \theta) + \psi_{xx}(\cos \theta, \sin \theta) \sin^2 \theta - 2\psi_{xy}(\cos \theta, \sin \theta) \sin \theta \cos \theta \\ &\quad + \psi_{yy}(\cos \theta, \sin \theta) \cos^2 \theta - \psi_x(\cos \theta, \sin \theta) \cos \theta - \psi_y(\cos \theta, \sin \theta) \sin \theta \\ &= \psi_{xx}(\cos \theta, \sin \theta) \sin^2 \theta - 2\psi_{xy}(\cos \theta, \sin \theta) \sin \theta \cos \theta + \psi_{yy}(\cos \theta, \sin \theta) \cos^2 \theta. \end{aligned}$$

Since ψ_x and ψ_y are positively zero-homogeneous, we have

$$x\psi_{xx}(x, y) + y\psi_{xy}(x, y) = 0 \quad \text{and} \quad x\psi_{xy}(x, y) + y\psi_{yy}(x, y) = 0. \quad (6.10)$$

Inserting these identities in the formula above, we get

$$(g + g_{\theta\theta})(\theta) = \psi_{xx}(\cos \theta, \sin \theta) + \psi_{yy}(\cos \theta, \sin \theta).$$

Again from (6.10) we have that $\psi_{xx}(\cos \theta, \sin \theta) \cos^2 \theta = \psi_{yy}(\cos \theta, \sin \theta) \sin^2 \theta$. Therefore,

$$\psi_{xx}(\cos \theta, \sin \theta) + \psi_{yy}(\cos \theta, \sin \theta) = \frac{\psi_{xx}(\cos \theta, \sin \theta)}{\sin^2 \theta}.$$

\square

Lemma 6.7. *The following identity holds*

$$k_{\sigma\sigma} + \frac{1}{2}k^3 = -\left(\frac{h_{xx}}{J^5}\right)_{xx} - \frac{5}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x$$

for h sufficiently smooth, where $J := \sqrt{1 + h_x^2}$.

Proof. Since $\frac{\partial}{\partial x} = J \frac{\partial}{\partial \sigma}$,

$$\begin{aligned} \left(\frac{h_{xx}}{J^5}\right)_{xx} + \frac{5}{2}\left(\frac{h_{xx}^2}{J^7}h_x\right)_x &= -\left(\frac{k}{J^2}\right)_{xx} + \frac{5}{2}\left(\frac{k^2}{J}h_x\right)_x \\ &= \left(-\frac{k_{\sigma}}{J} - 2\frac{k^2}{J}h_x\right)_x + \frac{5}{2}\left((k^2)_x \frac{h_x}{J} - k^3\right) \\ &= -k_{\sigma\sigma} - \frac{1}{2}k^3. \end{aligned}$$

\square

The following lemma has been used in the proof of the existence theorem.

Lemma 6.8. *Let $k \in \mathbb{N} \cup \{0\}$, let $h \in H_{\#}^2(0, b)$, and let $f \in L^2(0, b)$ be such that $\int_0^b f \varphi^{(k)} dx = 0$ for all $\varphi \in C_c^k(0, b)$ with $\int_0^b \varphi dx = 0$ and $\int_{\Gamma_h} \varphi d\zeta d\mathcal{H}^1 = 0$. Then if $k \geq 1$*

$$f(x) = a \int_0^x \int_0^{x_k} \cdots \int_0^{x_1} \sqrt{1 + (h'(r))^2} dr \dots dx_{k-1} dx_k + P_k(x)$$

for some $a \in \mathbb{R}$ and some polynomial P_k of degree k , and if $k = 0$

$$f(x) = a \int_0^x \sqrt{1 + (h'(r))^2} dr + a_0$$

for some $a_0, a \in \mathbb{R}$.

Proof. Let $\chi \in C_c^{k+2}(0, b)$ and set $\varphi_\varepsilon(x) := \left(\frac{\chi'(x)}{J_\varepsilon(x)}\right)'$, where $J_\varepsilon(x) := \sqrt{1 + h_\varepsilon'^2(x)}$ and h_ε is the standard mollification of h .

Let $\psi \in C_c^\infty(0, b)$ be such that

$$\int_0^b \psi dx = 0, \quad \int_{\Gamma_h} \int_0^x \psi d\zeta d\mathcal{H}^1 = 1$$

and define

$$\psi_\varepsilon := \varphi_\varepsilon - c_\varepsilon \psi,$$

where $c_\varepsilon := \int_{\Gamma_h} \int_0^x \varphi_\varepsilon d\zeta d\mathcal{H}^1$. Note that by construction $\psi_\varepsilon \in C_c^k(0, b)$ and satisfies $\int_0^b \psi_\varepsilon dx = 0$ and $\int_{\Gamma_h} \int_0^x \psi_\varepsilon d\zeta d\mathcal{H}^1 = 0$. Hence,

$$\int_0^b \left(\frac{\chi'}{J_\varepsilon}\right)^{(k+1)} f dx = c_\varepsilon \int_0^b f \psi^{(k)} dx.$$

If f is smooth then the previous identity is equivalent to

$$(-1)^{k+2} \int_0^b \left(\frac{f^{(k+1)}}{J_\varepsilon}\right)' \chi dx = c_\varepsilon \int_0^b f \psi^{(k)} dx,$$

and letting $\varepsilon \rightarrow 0$, we obtain

$$\int_0^b \left(\frac{f^{(k+1)}}{J}\right)' \chi dx = 0$$

for all $\chi \in C_c^{k+2}(0, b)$ and the conclusion follows. The general case is obtained via a standard approximation argument. \square

The following trace theorem is a particular case of [27, Theorem 15.17].

Theorem 6.9. *Let $\Omega \subset \mathbb{R}_+^2$ be a bounded Lipschitz open set and $S = \partial\Omega \cap \{y = 0\}$. If $1 < p < 2$, then for all $u \in W^{1,p}(\Omega)$, denoting by Tr the trace operator, we have*

$$\|Tr(u)\|_{L^{\frac{p}{2-p}}(S)} \leq C \|u\|_{W^{1,p}(\Omega)}$$

for a suitable constant $C > 0$.

We conclude the appendix with the following elliptic estimate.

Lemma 6.10. *Let $M > 0$, $c_0 > 0$. Let $h_1, h_2 \in AP \cap C_{\#}^{1,\alpha}([0, b])$ for some $\alpha \in (0, 1)$, with $\|h_i\|_{C^{1,\alpha}([0, b])} \leq M$ and $h_i \geq c_0$, and let u_1 and u_2 the corresponding elastic equilibria in Ω_{h_1} and Ω_{h_2} , respectively. Then,*

$$\|E(u_1(\cdot, h_1(\cdot))) - E(u_2(\cdot, h_2(\cdot)))\|_{C^{1,\alpha}([0, b])} \leq C \|h_1 - h_2\|_{C^{1,\alpha}([0, b])} \quad (6.11)$$

for some constant $C > 0$ depending only on M, c_0 , and α .

Proof. Recall that by minimality

$$\int_{\Omega_{h_1}} \mathbb{C}E(u_1) : E(w) dz = 0, \quad \int_{\Omega_{h_2}} \mathbb{C}E(u_2) : E(w) dz = 0$$

for all $w \in LD_{\#}(\Omega_{h_i}; \mathbb{R}^2)$ with $w(x, 0) = 0$. Let $\Phi : \Omega_{h_1} \rightarrow \Omega_{h_2}$ be a diffeomorphism such that $\|\Phi - Id\|_{C^{1,\alpha}(\Omega_{h_1})} \leq C\|h_2 - h_1\|_{C^{1,\alpha}([0,b])}$. Then, it is easily checked that

$$\int_{\Omega_{h_1}} \mathbb{C}E(u_2 \circ \Phi^{-1}) : E(w) dz = \int_{\Omega_{h_1}} d : \nabla w dz$$

with $\|d\|_{C^{0,\alpha}(\Omega_{h_1})} \leq C\|h_2 - h_1\|_{C^{1,\alpha}([0,b])}$. Setting $v := u_1 - u_2 \circ \Phi^{-1}$ it follows that

$$\int_{\Omega_{h_1}} \mathbb{C}E(v) : E(w) dz = - \int_{\Omega_{h_1}} d : \nabla w dz. \quad (6.12)$$

From standard elliptic regularity (see [21, Proposition 8.9]), it follows that

$$\|v\|_{C^{1,\alpha}(\Omega_{h_1})} \leq C \left(\|v\|_{H^1(\Omega_{h_1})} + \|d\|_{C^{0,\alpha}(\overline{\Omega}_{h_1})} \right),$$

for some constant $C > 0$ depending only M , c_0 , and α . Since by (6.12), using also Korn's and Poincaré inequality, we have $\|v\|_{H^1(\Omega_{h_1})} \leq C\|d\|_{C^{0,\alpha}(\overline{\Omega}_{h_1})}$, the conclusion follows. \square

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