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Characterization of the Multiscale Limit Associated with Bounded Sequences in BV

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Abstract

The notion of two-scale convergence for sequences of Radon measures with finite total variation is generalized to the case of multiple periodic length scales of oscillations. The main result concerns the characterization of $(n+1)$-scale limit pairs (u,U) of sequences $\{(u_\varepsilon\mathcal{L}^N{\lfloor_\Omega}, Du_\varepsilon{\lfloor_\Omega})\}_{\varepsilon>0}\subset\mathcal{M}(\Omega;\mathbb{R}^d)\times\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV(\Omega;\mathbb{R}^d)$. This characterization is useful in the study of the asymptotic behavior of periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation and described by $n\in\mathbb{N}$ microscales, undertaken in [10].

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1. Introduction and Main Results

The notion of two-scale convergence was first introduced by Nguetseng [13] and further developed by Allaire [1]. It was used to provide a mathematical rigorous justification of the formal asymptotic expansions that used to be commonly adopted in the study of homogenization problems (see, for example, [5], [12] and [14]).

In [2], Allaire and Briane extended that notion to the case of multiple separated scales of periodic oscillations. Precisely,

Definition 1.1.[†] *Let $n,N\in\mathbb{N}$, let $\Omega\subset\mathbb{R}^N$ be an open and bounded set, and let $Y:=[0,1]^N$. Let $\varrho_1,...,\varrho_n:(0,\infty)\to(0,\infty)$ satisfy for all $i\in\{1,\cdots,n\}$ and for all $j\in\{2,\cdots,n\}$,*

$$\lim_{\varepsilon\to0^+}\varrho_i(\varepsilon)=0,\quad \lim_{\varepsilon\to0^+}\frac{\varrho_j(\varepsilon)}{\varrho_{j-1}(\varepsilon)}=0. \tag{1.1}$$

A sequence $\{u_\varepsilon\}_{\varepsilon>0}\subset L^2(\Omega)$ is said to $(n+1)$-scale converge to a function $u_0\in L^2(\Omega\times Y_1\times\cdots\times Y_n)$, where each Y_i is a copy of Y, if for every $\varphi\in L^2(\Omega;C_\#(Y_1\times\cdots\times Y_n))$ we have

$$\lim_{\varepsilon\to0^+}\int_\Omega u_\varepsilon(x)\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\,\mathrm{d}x=\int_{\Omega\times Y_1\times\cdots\times Y_n}u_0(x,y_1,\cdots,y_n)\varphi(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,$$

† Here, and in the sequel, ε is a small parameter taking values on an arbitrary sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of positive numbers converging to zero. We write ε, $\{u_\varepsilon\}_{\varepsilon>0}$ and $\varepsilon\to0^+$ in place of ε_j, $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}}$ and $\varepsilon_j\to0^+$ as $j\to\infty$, respectively. Also, the subscript $\#$ stands for $Y_1\times\cdots\times Y_n$-periodic functions (or measures) with respect to the variables $(y_1,\cdots,y_n)$. We refer the reader to Section 2 for the notations used throughout this paper.

in which case we write $u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} u_0$.

Remark 1.2. *In the context of multiscale composites, the functions* $\varrho_1, ..., \varrho_n$ *stand for the length scales or scales of oscillation. The second condition in (1.1) is known as a separation of scales hypothesis.*

Also, Allaire and Briane [2] established a compactness result concerning this notion and provided the relationship between the $(n+1)$-scale limit and the usual weak limit in $L^2(\Omega)$ (see [2, Thms. 2.4 and 2.5]). Precisely,

Theorem 1.3. *Let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be a bounded sequence in* $L^2(\Omega)$. *Then, there exist a (not relabeled) subsequence of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a function* $u_0 \in L^2(\Omega \times Y_1 \times \cdots \times Y_n)$ *such that* $u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} u_0$. *Furthermore,* $u_\varepsilon \rightharpoonup \bar{u}_0$ *weakly in* $L^2(\Omega)$, *where* $\bar{u}_0(x) := \int_{Y_1 \times \cdots \times Y_0} u_0(x, y_1, \cdots, y_n)\, \mathrm{d}y_1 \cdots \mathrm{d}y_n$, *and* $\lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_{L^2(\Omega)} \geqslant \|u_0\|_{L^2(\Omega\times Y_1\times\cdots\times Y_n)} \geqslant \|\bar{u}_0\|_{L^2(\Omega)}$.

In general the $(n+1)$-scale limit differs from the weak limit in $L^2(\Omega)$, with the $(n+1)$-scale limit capturing more information on the oscillatory behavior of a bounded sequence in $L^2(\Omega)$ than its weak limit in $L^2(\Omega)$. The proof of Theorem 1.3 follows the arguments introduced in the case $n = 1$ treated in [1] (see also [13]).

Moreover, in order to study the asymptotic behavior of the solutions of certain partial differential equations with periodically oscillating coefficients in the space $H^1(\Omega)$, the $(n+1)$-scale limit of gradients was fully characterized in [2, Thm. 1.2]. Precisely,

Theorem 1.4. *Let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be a bounded sequence in* $H^1(\Omega)$. *Then there exist* $u \in H^1(\Omega)$ *and* n *functions* $u_i \in L^2(\Omega \times Y_1 \times \cdots \times Y_{i-1}; H^1_\#(Y_i))$, *for* $i \in \{1, \cdots, n\}$, *such that*

$$u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} u, \tag{1.2}$$

and, up to a not relabeled subsequence,

$$\nabla u_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \nabla u + \sum_{i=1}^{n} \nabla_{y_i} u_i. \tag{1.3}$$

Furthermore, given any $u \in H^1(\Omega)$ *and* $u_i \in L^2(\Omega \times Y_1 \times \cdots \times Y_{i-1}; H^1_\#(Y_i))$, $i \in \{1, \cdots, n\}$, *there exists a bounded sequence* $\{u_\varepsilon\}_{\varepsilon>0}$ *for which (1.2) and (1.3) hold.*

Remark 1.5. *In the theorem above, the function* u *is the weak limit in* $H^1(\Omega)$ *of the sequence* $\{u_\varepsilon\}_{\varepsilon>0}$. *The terms* $\nabla_{y_i} u_i$ *in (1.3) may be interpreted as the gradient limits at each scale.*

Remark 1.6. *Definition 1.1 and Theorem 1.4 admit simple generalizations to the cases* $L^p(\Omega)$ *and* $W^{1,p}(\Omega)$, *respectively, for any* $p \in (1, \infty)$.

Theorem 1.4 extends Prop. 1.14 (i) in [1] to the case in which $n \geqslant 2$, but its proof requires significant changes and is rather more difficult. By means of this result, Allaire and Briane [2] completely characterize the asymptotic behavior as $\varepsilon \to 0^+$ of solutions of the family of boundary value problems

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f, & \text{a.e. in } \Omega, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$, $A_\varepsilon(x) := A\big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\big)$, and A is a $N \times N$ matrix satisfying appropriate coercivity and boundedness hypotheses, and such that $A(x, \cdot)$ $Y_1 \times \cdots \times Y_n$-periodic (see [2, Thm. 1.3]).

A similar analysis was undertaken in [1] in the case $n = 1$. Also in [1] (see [1, Thms. 3.1 and 3.3]), Allaire provides a simple and elegant proof for the homogenized functional of a sequence $\{I_\varepsilon\}_{\varepsilon>0}$ of functionals of the form

$$u \in W_0^{1,p}(\Omega; \mathbb{R}^d) \mapsto I_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u(x)\Big)\, \mathrm{d}x.$$

Following this last approach, in [3] Amar extended the notion of two-scale convergence to the case of bounded sequences of Radon measures with finite total variation, and characterized the two-scale limit associated with a bounded sequence in $BV(\Omega)$ (see [3, Thm. 3.6]). Using this characterization, the asymptotic behavior as $\varepsilon \to 0^+$ of sequences of positively 1-homogeneous and periodically oscillating functionals with linear growth, defined in the space BV of functions of bounded variation, of the form

$$u \in BV(\Omega) \mapsto I_\varepsilon(u) := \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}Du}{\mathrm{d}\|Du\|}(x)\Big)\, \mathrm{d}\|Du\|(x)$$

is given in [3, Thm 4.1].

The purpose of the this paper is to extend the notion of two-scale convergence for sequences of Radon measures with finite total variation introduced in [3] to the case of multiple periodic length scales of oscillations, and to characterize the $(n+1)$-limit associated with a bounded sequence in $BV(\Omega;\mathbb{R}^d)$. Using some ideas of [2] and [3], we fully develop the underlying measure-theoretical background.

Definition 1.7. *Let $m, n, N \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^N$ be an open set and define $Y := (0,1)^N$. Let $\varrho_1, ..., \varrho_n$ be positive functions in $(0,\infty)$ satisfying (1.1). We say that a sequence $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^m)$ of Radon measures, with finite total variation in Ω, $(n+1)$-scale converges to a Radon measure $\mu_0 \in \big(C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))\big)' \simeq \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n;\mathbb{R}^m)$ with finite total variation in the product space $\Omega \times Y_1 \times \cdots \times Y_n$, where each Y_i is a copy of Y, if for all $\varphi \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_n;\mathbb{R}^m))$ we have*

$$\lim_{\varepsilon\to 0^+} \int_\Omega \varphi\Big(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\Big) \cdot \mathrm{d}\mu_\varepsilon(x) = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}\mu_0(x, y_1, \cdots, y_n),$$

in which case we write $\mu_\varepsilon \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \mu_0$.

This notion of convergence is justified due to a compactness result asserting that every bounded sequence $\{\mu_\varepsilon\}_{\varepsilon>0}$ in $\mathcal{M}(\Omega;\mathbb{R}^m)$ admits a $(n+1)$-scale convergent subsequence (see Theorem 3.2). One can also show that the weak-$\star$ limit in $\mathcal{M}(\Omega;\mathbb{R}^m)$ is the projection onto Ω of the $(n+1)$-scale limit, and so, in general the $(n+1)$-scale limit captures more information on the oscillations of $\{\mu_\varepsilon\}_{\varepsilon>0}$ than the weak-$\star$ limit in $\mathcal{M}(\Omega;\mathbb{R}^m)$ (see Proposition 3.3). The proofs of these two properties are simple generalizations of those in the case in which $n = 1$ (see [3]).

Definition 1.8. *For $d, i \in \mathbb{N}$, define the space $\mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$ of all $BV_\#\big(Y_i;\mathbb{R}^d\big)$-valued Radon measures $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$ with finite total variation, for which there exists a $\mathbb{R}^{d\times N}$-valued Radon measure $\lambda \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i;\mathbb{R}^{d\times N}\big)$, with finite total variation in the product space $\Omega \times Y_1 \times \cdots \times Y_i$, such that for all $B \in \mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_{i-1})$, $E \in \mathcal{B}(Y_i)$,*

$$\big(D_{y_i}(\boldsymbol{\mu}(B))\big)(E) = \lambda(B \times E). \tag{1.4}$$

We say that λ is the measure associated with $D_{y_i}\boldsymbol{\mu}$.

Note that since $\mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_{i-1}) \otimes \mathcal{B}(Y_i) = \mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_i)$, it follows that if $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, then there exists at most one measure $\lambda \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i;\mathbb{R}^{d\times N}\big)$ satisfying (1.4).

In Subsection 2.4 we will make more detailed considerations on the space $\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, $i \in \mathbb{N}$.

We now state our main result, which provides the characterization of $(n+1)$-scale limit pairs (u, U) of sequences $\big\{\big(u_\varepsilon \mathcal{L}^N{\lfloor_\Omega}, Du_\varepsilon{\lfloor_\Omega}\big)\big\}_{\varepsilon>0} \subset \mathcal{M}(\Omega;\mathbb{R}^d) \times \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV\big(\Omega;\mathbb{R}^d\big)$. We will assume a stronger separation of scales hypothesis than the one in (1.1), precisely (cf. [2]),

Definition 1.9. *The scales $\varrho_1, ..., \varrho_n$ are said to be well-separated if there exists $m \in \mathbb{N}$ such that for all $i \in \{2, \cdots, n\}$,*

$$\lim_{\varepsilon \to 0^+} \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)} \right)^m \frac{1}{\varrho_i(\varepsilon)} = 0. \tag{1.5}$$

The case in which $\varrho_i(\varepsilon) := \varepsilon^i$ is a simple example of well-separated scales. Indeed, it suffices to take $m = n+1$.

Theorem 1.10. *Let $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega; \mathbb{R}^d)$ be a sequence such that $u_\varepsilon \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(\Omega; \mathbb{R}^d)$ as $\varepsilon \to 0^+$, for some $u \in BV(\Omega; \mathbb{R}^d)$. Assume that the length scales $\varrho_1, ..., \varrho_n$ satisfy (1.1) and (1.5). Then*

a) $u_\varepsilon \mathcal{L}^N\lfloor_\Omega \xrightarrow[\varepsilon]{(n+1)\text{-}sc} \tau_u$, where $\tau_u \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^d\big)$ is the measure defined by

$$\tau_u := u\, \mathcal{L}^N\lfloor_\Omega \otimes \mathcal{L}^{nN}_{y_1, \cdots, y_n},$$

i.e., if $\varphi \in C_0\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_n; \mathbb{R}^d\big)\big)$ then

$$\langle \tau_u, \varphi \rangle = \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot u(x)\, \mathrm{d}x\mathrm{d}y_1 \cdots \mathrm{d}y_n.$$

b) there exist a subsequence $\{Du_{\varepsilon'}\}_{\varepsilon'>0}$ of $\{Du_\varepsilon\}_{\varepsilon>0}$ and n measures $\boldsymbol{\mu}_i \in \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i; \mathbb{R}^d)\big)$, $i \in \{1, \cdots, n\}$, such that

$$Du_{\varepsilon'} \xrightarrow[\varepsilon']{(n+1)\text{-}sc} \lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n},$$

where $\lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n} \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^{d \times N}\big)$ is the measure

$$\lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n} := Du\lfloor_\Omega \otimes \mathcal{L}^{nN}_{y_1, \cdots, y_n} + \sum_{i=1}^{n-1} \lambda_i \otimes \mathcal{L}^{(n-i)N}_{y_{i+1}, \cdots, y_n} + \lambda_n, \tag{1.6}$$

i.e., if $\varphi \in C_0\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_n; \mathbb{R}^{d \times N}\big)\big)$ then

$$\begin{aligned} \langle \lambda_{u, \boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n}, \varphi \rangle = &\int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}Du(x)\mathrm{d}y_1 \cdots \mathrm{d}y_n \\ &+ \sum_{i=1}^{n-1} \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_i(x, y_1, \cdots, y_i)\mathrm{d}y_{i+1} \cdots \mathrm{d}y_n \\ &+ \int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) : \mathrm{d}\lambda_n(x, y_1, \cdots, y_n), \end{aligned}$$

and each $\lambda_i \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^{d \times N}\big)$ is the measure associated with $D_{y_i} \boldsymbol{\mu}_i$, $i \in \{1, \cdots, n\}$.

The proof of Theorem 1.10 is not a simple generalization of the analogous result in the case $n = 1$ treated in [3]. When $n \geqslant 2$, and similarly to [2], some new arguments are needed. We also show that Theorem 1.10 fully characterizes the $(n+1)$-scale limit of bounded sequences in $BV(\Omega; \mathbb{R}^d)$, in that:

Proposition 1.11. *Let $u \in BV\big(\Omega; \mathbb{R}^d\big)$ and let $\boldsymbol{\mu}_i \in \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#\big(Y_i; \mathbb{R}^d\big)\big)$, $i \in \{1, \cdots, n\}$. Then there exists a bounded sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset BV\big(\Omega; \mathbb{R}^d\big)$ for which a) and b) of Theorem 1.10 hold (with ε' replaced by ε).*

Remark 1.12. *Proposition 1.11 together with Theorem 1.10 represent the BV version of Theorem 1.4.*

Using Theorem 1.10, in [10] we study the asymptotic behavior with respect to the $(n+1)$-scale convergence of first order derivatives and periodically oscillating functionals with linear growth, defined in the space BV

of functions of bounded variation and described by $n \in \mathbb{N}$ microscales. In the particular case in which $n = 1$, and as a corollary of our results in [10] we recover Thm. 4.1 in [3] under more general hypotheses.

This paper is organized as follows. In Section 2 we introduce the notation and we recall some basic properties of ($\mathbb{R}^m$-valued) Radon measures and of functions of bounded variation. We collect properties of integration with respect to certain Banach-valued measures, which seems to be hard to find in literature and that will play an important role in the subsequent section, Section 3. The latter is devoted to the proofs of Theorem 1.10 and of Proposition 1.11.

2. Notation and Preliminaries

2.1. Notation

In the sequel Z is a σ-compact separable metric space, Ω is an open subset of $\mathbb{R}^N$, $N \in \mathbb{N}$, and $Y := (0,1)^N$ is the reference cell. For each $i \in \mathbb{N}$, Y_i stands for a copy of Y. Given $x \in \mathbb{R}^N$, we write $[x]$ and $\langle x \rangle$ to denote the integer and the fractional part of x componentwise, respectively, so that $x = [x] + \langle x \rangle$ and $[x] \in \mathbb{Z}^N$, $\langle x \rangle \in Y$.

Let $n, m \in \mathbb{N}$. If $x, y \in \mathbb{R}^m$, then $x \cdot y$ stands for the Euclidean inner product of x and y, and $|x| := \sqrt{x \cdot x}$ for the Euclidean norm of x. The space of $(m \times n)$-dimensional matrices will be identified with $\mathbb{R}^{mn}$, and we write $\mathbb{R}^{m \times n}$. If $\xi = (\xi_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$, $\zeta = (\zeta_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n} \in \mathbb{R}^{m \times n}$, then

$$\xi : \zeta := \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} \zeta_{ij}$$

represents the inner product of ξ and ζ, while $|\xi| := \sqrt{\xi : \xi}$ denotes the norm of ξ. If $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, then $a \otimes b$ stands for the $(m \times n)$-dimensional rank-one matrix defined by $a \otimes b := (a_i b_j)_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$.

Let $g : \mathbb{R}^{nN} \to \mathbb{R}^m$ be a function. We denote the Lipschitz constant of g on a set $D \subset \mathbb{R}^{nN}$ by $\mathrm{Lip}(g; D)$; if D coincides with the domain of g we omit its dependence. We say that g is $Y_1 \times \cdots \times Y_n$-periodic if for all $i \in \{1, \cdots, n\}$, $\kappa \in \mathbb{Z}^N$, $y_1, ..., y_n \in \mathbb{R}^N$, one has $g(y_1, \cdots, y_i + \kappa, \cdots, y_n) = g(y_1, \cdots, y_i, \cdots, y_n)$.

We represent by $C(Z; \mathbb{R}^m)$ the space of all continuous functions $g : Z \to \mathbb{R}^m$, while $C_c(Z; \mathbb{R}^m)$ is the subspace of $C(Z; \mathbb{R}^m)$ of functions with compact support. The closure of $C_c(Z; \mathbb{R}^m)$ with respect to the supremum norm $\| \cdot \|_\infty$ is denoted by $C_0(Z; \mathbb{R}^m)$. It is well know that $C_0(Z; \mathbb{R}^m)$ is a separable Banach space, and that $g \in C_0(Z; \mathbb{R}^m)$ if, and only if, $g \in C(Z; \mathbb{R}^m)$ and for all $\eta > 0$ there exists a compact set $K_\eta \subset Z$ such that for all $z \in Z \backslash K_\eta$, $|g(z)| \leqslant \eta$. Moreover, if $Z \subset \mathbb{R}^N$ is an open and bounded set, then $C_0(Z; \mathbb{R}^m)$ coincides with the space of continuous functions on $\overline{Z}$ vanishing on ∂Z.

We write $C^k(Z; \mathbb{R}^m)$ (respectively, $C_c^k(Z; \mathbb{R}^m)$ and $C_0^k(Z; \mathbb{R}^m)$), $k \in \mathbb{N}$, to denote the space of all functions in $C(Z; \mathbb{R}^m)$ (respectively, $C_c(Z; \mathbb{R}^m)$ and $C_0(Z; \mathbb{R}^m)$) whose i^{th}-partial derivatives are continuous functions in Z for all $i \in \{1, \cdots, k\}$. We say that $g \in C^\infty(Z; \mathbb{R}^m)$ (respectively, $C_c^\infty(Z; \mathbb{R}^m)$ and $C_0^\infty(Z; \mathbb{R}^m)$) if for all $k \in \mathbb{N}$, $g \in C^k(Z; \mathbb{R}^m)$ (respectively, $C_c^k(Z; \mathbb{R}^m)$ and $C_0^k(Z; \mathbb{R}^m)$).

We will also consider the Banach spaces

$$C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m) := \big\{ g \in C(\mathbb{R}^{nN}; \mathbb{R}^m) : \ g \text{ is } Y_1 \times \cdots \times Y_n\text{-periodic} \big\}$$

endowed with the supremum norm $\| \cdot \|_\infty$, and $C_0(Z; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$, which is the closure with respect to the supremum norm $\| \cdot \|_\infty$ of $C_c(Z; C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$. The latter is the space of all functions $g : Z \times \mathbb{R}^{nN} \to \mathbb{R}^m$ such that for all $z \in Z$, $g(z, \cdot) \in C_\#(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$ and for all $y_1, ..., y_n \in \mathbb{R}^N$, $g(\cdot, y_1, \ldots, y_n) \in C_c(Z; \mathbb{R}^m)$. The spaces $C_\#^k(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$, $C_\#^\infty(Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$, $C_c^k(Z; C_\#^k(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$, $C_c^\infty(Z; C_\#^\infty(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$, $C_0^k(Z; C_\#^k(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$ and $C_0^\infty(Z; C_\#^\infty(Y_1 \times \cdots \times Y_n; \mathbb{R}^m))$ are now defined in an obvious way.

If $m = 1$ the co-domain will often be omitted (e.g., we write $C_0(Z)$ instead of $C_0(Z; \mathbb{R})$).

The letter $\mathcal{C}$ represents a generic positive constant, whose value may change from expression to expression.

Let $\rho \in C_c^\infty(\mathbb{R}^N)$ be the function defined by

$$\rho(x) := \begin{cases} c\, e^{\frac{1}{|x|^2-1}}, & |x| < 1, \\ 0, & |x| \geqslant 1, \end{cases}$$

where $c > 0$ is such that $\int_{\mathbb{R}^N} \rho(x)\,\mathrm{d}x = 1$. For each $0 < \varepsilon < 1$ let

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N}\rho\left(\frac{x}{\varepsilon}\right). \tag{2.1}$$

Then $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,\mathrm{d}x = 1, \quad \operatorname{supp}\rho_\varepsilon \subset \overline{B(0,\varepsilon)}, \quad \rho_\varepsilon \geqslant 0, \quad \rho_\varepsilon(-x) = \rho_\varepsilon(x), \tag{2.2}$$

for all $x \in \mathbb{R}^N$.

For $0 < \varepsilon < 1/2$, let η_ε denote the extension to $\mathbb{R}^N$ by $(-\frac{1}{2}, \frac{1}{2})^N$-periodicity of the function $\rho_{\varepsilon|(-\frac{1}{2},\frac{1}{2})^N}$. Then $\eta_\varepsilon \in C_\#^\infty(Y)$ is such that

$$\int_Q \eta_\varepsilon(y)\,\mathrm{d}y = 1, \quad \eta_\varepsilon \geqslant 0, \quad \eta_\varepsilon(-x) = \eta_\varepsilon(x), \tag{2.3}$$

for any unit cube $Q \subset \mathbb{R}^N$ and $x \in \mathbb{R}^N$.

2.2. Measure theory

For $m \in \mathbb{N}$, the m-dimensional Lebesgue measure is denoted by $\mathcal{L}^m$.

The Borel σ-algebra on Z is denoted by $\mathcal{B}(Z)$, and $\mathcal{M}(Z;\mathbb{R}^m)$ is the Banach space of all Radon measures $\lambda : \mathcal{B}(Z) \to \mathbb{R}^m$ endowed with the total variation norm $\|\cdot\|(Z)$, with

$$\|\lambda\|(Z) := \sup\left\{ \sum_{j=1}^\infty |\lambda(B_j)| \colon \{B_j\}_{j\in\mathbb{N}} \subset \mathcal{B}(Z) \text{ is a partition of } Z \right\}.$$

By Riesz Representation Theorem, the dual of $C_0(Z;\mathbb{R}^m)$ can be identified with $\mathcal{M}(Z;\mathbb{R}^m)$ through the duality pairing

$$\langle \lambda, \varphi \rangle_{\mathcal{M}(Z;\mathbb{R}^m), C_0(Z;\mathbb{R}^m)} = \int_Z \varphi(z)\cdot \mathrm{d}\lambda(z) := \sum_{i=1}^m \int_Z \varphi_i(z)\,\mathrm{d}\lambda_i(z),$$

where $\varphi = (\varphi_1, \cdots, \varphi_m)$ and $\lambda = (\lambda_1, \cdots, \lambda_m)$, so that the total variation of λ is alternatively given by

$$\|\lambda\|(Z) = \sup\left\{ \int_Z \varphi(z)\cdot \mathrm{d}\lambda(z) \colon\ \varphi \in C_0(Z;\mathbb{R}^m),\ \|\varphi\|_\infty \leqslant 1 \right\}.$$

We say that a sequence $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(Z;\mathbb{R}^m)$ weakly-$\star$ converges to some measure $\lambda \in \mathcal{M}(Z;\mathbb{R}^m)$, and we write $\lambda_j \overset{\star}{\rightharpoonup} \lambda$, if for all $\varphi \in C_0(Z;\mathbb{R}^m)$, $\int_Z \varphi(z)\cdot \mathrm{d}\lambda_j(z) \to \int_Z \varphi(z)\cdot \mathrm{d}\lambda(z)$ as $j \to \infty$. We recall that from every bounded sequence in $\mathcal{M}(Z;\mathbb{R}^m)$ we can extract a weakly-$\star$ convergent subsequence.

If $\varphi \in C_0(Z)$ and $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathcal{M}(Z;\mathbb{R}^m)$, then we set

$$\int_Z \varphi(z)\,\mathrm{d}\lambda(z) := \left(\int_Z \varphi(z)\,\mathrm{d}\lambda_1(z), \cdots, \int_Z \varphi(z)\,\mathrm{d}\lambda_m(z) \right).$$

If $\varphi = (\varphi_1, \cdots, \varphi_m) \in C_0(Z;\mathbb{R}^m)$ and $\lambda \in \mathcal{M}(Z;\mathbb{R})$, then we define

$$\int_Z \varphi(z)\,\mathrm{d}\lambda(z) := \left(\int_Z \varphi_1(z)\,\mathrm{d}\lambda(z), \cdots, \int_Z \varphi_m(z)\,\mathrm{d}\lambda(z) \right).$$

We write $\mathcal{M}_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ and $\mathcal{M}_{y\#}(Z\times Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ to denote the duals of $C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m)$ and $C_0(Z;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^m))$, respectively.

Let Z_1, Z_2 be two σ-compact separable metric spaces. We write $\mathcal{B}(Z_1)\otimes\mathcal{B}(Z_2)$ to represent the smallest σ-algebra that contains all sets of the form $B_1\times B_2$, where $B_1\in\mathcal{B}(Z_1)$, $B_2\in\mathcal{B}(Z_2)$. Since Z_1, Z_2 are separable metric spaces, we have that $\mathcal{B}(Z_1)\otimes\mathcal{B}(Z_2)=\mathcal{B}(Z_1\times Z_2)$. Let us also recall that by Carathéodory's Theorem (see, for example, [11]), given two positive measures $\lambda_1:\mathcal{B}(Z_1)\to[0,\infty]$, $\lambda_2:\mathcal{B}(Z_2)\to[0,\infty]$, we can construct an outer measure, *the product outer measure* $(\lambda_1\times\lambda_2)^*:2^{Z_1\times Z_2}\to[0,\infty]$, whose restriction to the σ-algebra $\mathcal{B}(Z_1)\times\mathcal{B}(Z_2)$ of the $(\lambda_1\times\lambda_2)^*$-measurable sets is a complete measure. The latter is known as the product measure of λ_1 and λ_2, and is denoted by $\lambda_1\times\lambda_2$. Moreover, it holds $\mathcal{B}(Z_1)\otimes\mathcal{B}(Z_2)\subset\mathcal{B}(Z_1)\times\mathcal{B}(Z_2)$, and for all $B_1\in\mathcal{B}(Z_1)$, $B_2\in\mathcal{B}(Z_2)$, one has

$$(\lambda_1\times\lambda_2)(B_1\times B_2)=\lambda_1(B_1)\lambda_2(B_2). \tag{2.4}$$

We denote by $\lambda_1\otimes\lambda_2$ the restriction of $\lambda_1\times\lambda_2$ to the σ-algebra $\mathcal{B}(Z_1)\otimes\mathcal{B}(Z_2)=\mathcal{B}(Z_1\times Z_2)$.

More generally, for $\lambda_1\in\mathcal{M}(Z_1;\mathbb{R})$, $\lambda_2\in\mathcal{M}(Z_2;\mathbb{R})$, we define

$$\lambda_1\otimes\lambda_2:=\lambda_1^+\otimes\lambda_2^++\lambda_1^-\otimes\lambda_2^--\lambda_1^+\otimes\lambda_2^--\lambda_1^-\otimes\lambda_2^+,$$

where $\lambda_1=\lambda_1^+-\lambda_1^-$ and $\lambda_2=\lambda_2^+-\lambda_2^-$ are the Hahn decompositions of λ_1 and λ_2, respectively. Note that $\lambda_1\otimes\lambda_2\in\mathcal{M}(Z_1\times Z_2;\mathbb{R})$ and (2.4) holds with $\lambda_1\times\lambda_2$ replaced by $\lambda_1\otimes\lambda_2$. Similarly, in the case in which $\lambda_1\in\mathcal{M}(Z_1;\mathbb{R})$ and $\lambda_2=(\lambda_2^1,\cdots,\lambda_2^m)\in\mathcal{M}(Z_2;\mathbb{R}^m)$, $\lambda_1\otimes\lambda_2$ is the measure in $\mathcal{M}(Z_1\times Z_2;\mathbb{R}^m)$ satisfying (2.4) (with $\lambda_1\times\lambda_2$ replaced by $\lambda_1\otimes\lambda_2$) defined by $\lambda_1\otimes\lambda_2:=(\lambda_1\otimes\lambda_2^1,\cdots,\lambda_1\otimes\lambda_2^m)$.

We recall the slicing decomposition of a Radon measure (see, for example, [9]). Let $\lambda\in\mathcal{M}(Z_1\times Z_2;\mathbb{R})$ be a finite, nonnegative Radon measure on $Z_1\times Z_2$. Represent by σ the canonical projection of λ onto Z_2, i.e., the measure defined by $\sigma(E):=\lambda(Z_1\times E)$, for all $E\in\mathcal{B}(Z_2)$. Then for σ-a.e. $z_2\in Z_2$ there exists a nonnegative Radon measure ν_{z_2} on Z_1 such that $\nu_{z_2}(Z_1)=1$, and such that for all bounded and continuous function g on $Z_1\times Z_2$, the mapping

$$z_2\mapsto\int_{Z_1}g(z_1,z_2)\,\mathrm{d}\nu_{z_2}(z_1)$$

is σ-measurable and

$$\int_{Z_1\times Z_2}g(z_1,z_2)\,\mathrm{d}\lambda(z_1,z_2)=\int_{Z_2}\left(\int_{Z_1}g(z_1,z_2)\,\mathrm{d}\nu_{z_2}(z_1)\right)\mathrm{d}\sigma(z_2). \tag{2.5}$$

2.3. *The space of functions of bounded variation*

A function $u:\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, is said to be a function of bounded variation if $u\in L^1(\Omega;\mathbb{R}^d)$ and its distributional derivative Du belongs to $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$, that is, if there exists a measure $Du\in\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$ such that for all $\phi\in C_c(\Omega)$, $j\in\{1,\cdots,d\}$ and $i\in\{1,\cdots,N\}$ one has

$$\int_\Omega u_j(x)\frac{\partial\phi}{\partial x_i}(x)\,\mathrm{d}x=-\int_\Omega\phi(x)\,\mathrm{d}D_iu_j(x),$$

where $u=(u_1,\cdots,u_d)$ and $Du_j=(D_1u_j,\cdots,D_Nu_j)$. The space of all such functions u is denoted by $BV(\Omega;\mathbb{R}^d)$, which is a Banach space when endowed with the norm $\|u\|_{BV(\Omega;\mathbb{R}^d)}:=\|u\|_{L^1(\Omega;\mathbb{R}^d)}+\|Du\|(\Omega)$.

We will also consider the space $BV_\#(Y;\mathbb{R}^d):=\{u\in BV_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^d):\ u \text{ is } Y\text{-periodic}\}$, endowed with the norm of $BV(Y;\mathbb{R}^d)$. Notice that if $u\in BV_\#(Y;\mathbb{R}^d)$, then $Du\in\mathcal{M}_\#(Y;\mathbb{R}^{d\times N})$.

We will consider the weak-$\star$ convergence in $BV(\Omega;\mathbb{R}^d)$. We recall that $\{u_j\}_{j\in\mathbb{N}}\subset BV(\Omega;\mathbb{R}^d)$ is said to weakly-$\star$ converge in $BV(\Omega;\mathbb{R}^d)$ to some $u\in BV(\Omega;\mathbb{R}^d)$ if $u_j\to u$ (strongly) in $L^1(\Omega;\mathbb{R}^d)$ and $Du_j\overset{\star}{\rightharpoonup}Du$ weakly-$\star$ in $\mathcal{M}(\Omega;\mathbb{R}^{d\times N})$. We recall also that from every bounded sequence in $BV(\Omega;\mathbb{R}^d)$ we can extract a weakly-$\star$ convergent subsequence.

2.4. Integration with respect to $BV_\#(Y;\mathbb{R}^d)$-valued Radon measures

In this subsection we will deal with integrals with respect to $BV_\#(Y;\mathbb{R}^d)$-valued Radon measures. We start by recalling the notion of Banach space-valued measures. For a more detailed exposition see, for example, [7].

Definition 2.1. *Let X be a Banach space. We say that $\boldsymbol{\mu}:\mathcal{B}(Z)\to X$ is a (X-valued) Radon measure if the following conditions are satisfied:*

i) $\boldsymbol{\mu}(\emptyset)=0$,

ii) Given any countable family $\{B_j\}_{j\in\mathbb{N}}$ of mutually disjoint Borel subsets of Z, the series $\sum_{j=1}^\infty \boldsymbol{\mu}(B_j)$ converges (in X) and

$$\boldsymbol{\mu}\Big(\bigcup_{j=1}^\infty B_j\Big)=\sum_{j=1}^\infty \boldsymbol{\mu}(B_j).$$

If, in addition, the condition

iii) The total variation of $\boldsymbol{\mu}$,

$$\|\boldsymbol{\mu}\|(Z):=\sup\left\{\sum_{j=1}^\infty \|\boldsymbol{\mu}(B_j)\|_X\colon \{B_j\}_{j\in\mathbb{N}}\subset\mathcal{B}(Z) \text{ is a partition of } Z\right\},$$

is finite,

is satisfied, then we say that $\boldsymbol{\mu}$ is a (X-valued) Radon measure with finite total variation, and we write $\boldsymbol{\mu}\in\mathcal{M}(Z;X)$.

Notice that if $\boldsymbol{\mu}\in\mathcal{M}(Z;X)$, then $\|\boldsymbol{\mu}\|:\mathcal{B}(Z)\to[0,\infty)$ defined by

$$\|\boldsymbol{\mu}\|(B):=\sup\left\{\sum_{j=1}^\infty \|\boldsymbol{\mu}(B_j)\|_X\colon \{B_j\}_{j\in\mathbb{N}}\subset\mathcal{B}(Z) \text{ is a partition of } B\right\},\quad B\in\mathcal{B}(Z),$$

is a finite positive Radon measure on Z.

We will be particularly interested in the case in which $Z=\Omega\times Y_1\times\cdots\times Y_{i-1}$ for some $i\in\mathbb{N}$, where

$$\Omega\times Y_1\times\cdots\times Y_{i-1}:=\Omega\quad\text{if } i=1,$$

and $X=BV_\#\big(Y_i;\mathbb{R}^d\big)$.

Let $\boldsymbol{\mu}\in\mathcal{M}\big(Z;BV_\#\big(Y;\mathbb{R}^d\big)\big)$ and $B\in\mathcal{B}(Z)$. Then $\boldsymbol{\mu}(B)\in BV_\#\big(Y;\mathbb{R}^d\big)$, and so $D_y(\boldsymbol{\mu}(B))\in\mathcal{M}_\#\big(Y;\mathbb{R}^{d\times N}\big)$. Moreover, it can be checked that the mapping $D_y\boldsymbol{\mu}: B\in\mathcal{B}(Z)\mapsto D_y\boldsymbol{\mu}(B):=D_y(\boldsymbol{\mu}(B))$ belongs to $\mathcal{M}\big(Z;\mathcal{M}_\#\big(Y;\mathbb{R}^{d\times N}\big)\big)$ in the sense of Definition 2.1.

According to the statement of Theorem 1.10 (see also Definition 1.8), the measures $\boldsymbol{\mu}\in\mathcal{M}\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$ for which there exists $\lambda\in\mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big)$ such that for all $B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$, $E\in\mathcal{B}(Y_i)$, we have

$$D_{y_i}(\boldsymbol{\mu}(B))(E)=\lambda(B\times E),\tag{2.6}$$

play an important role in the characterization of the multiscale limit of the sequence of distributional derivatives of a bounded sequence in $BV\big(\Omega;\mathbb{R}^d\big)$.

Example 2.2. *Fix $i\in\mathbb{N}$, let $\tau\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_{i-1};\mathbb{R})$, and let $v\in BV_\#(Y_i;\mathbb{R}^d)$. Then the mapping*

$$\boldsymbol{\mu}:B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})\mapsto\boldsymbol{\mu}(B):=\tau(B\times Y_1\times\cdots\times Y_{i-1})\,v$$

belongs to $\mathcal{M}\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_{\#}\big(Y_i;\mathbb{R}^d\big)\big)$*, with*

$$\|\boldsymbol{\mu}\|(\Omega \times Y_1 \times \cdots \times Y_{i-1}) = \|\tau\|(\Omega \times Y_1 \times \cdots \times Y_{i-1})\|v\|_{BV(Y_i;\mathbb{R}^d)}.$$

Observe also that for all $B \in \mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_{i-1})$*,* $(D_{y_i}\boldsymbol{\mu})(B) = D_{y_i}(\boldsymbol{\mu}(B)) = \tau(B)Dv$*. Moreover, defining* $\lambda := \tau \otimes Dv$*, we have that* $\lambda \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^{d\times N}\big)$ *and (2.6) holds. Thus,* $\boldsymbol{\mu} \in \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_{\#}\big(Y_i;\mathbb{R}^d\big)\big)$ *(see Definition 1.8).*

Our goal now is to give sense to the expression

$$\int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi(x, y_1, \cdots, y_i)\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})\mathrm{d}y_i, \tag{2.7}$$

whenever $\varphi \in C_0(\Omega; C_{\#}(Y_1 \times \cdots \times Y_i))$ and $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_{\#}\big(Y_i;\mathbb{R}^d\big)\big)$.

Step 1. We start by assuming that $i = 1$, and we write Y in place of Y_1. As it is usual when defining an integral, we will start by giving meaning to (2.7) for simple functions and then, using approximation arguments, we will extend such notion to more general functions. Let $s : \Omega \to \mathbb{R}$ be a Borel simple function, with

$$s := \sum_{i=1}^{m} c_i \chi_{B_i}, \tag{2.8}$$

where $m \in \mathbb{N}$, $c_1, ..., c_m \in \mathbb{R}$ are distinct and $B_1, ..., B_m \in \mathcal{B}(\Omega)$ are mutually disjoint. If $B \in \mathcal{B}(\Omega)$, then we define the integral of s over B with respect to $\boldsymbol{\mu}$, and we write $\int_B s(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, as the function in $BV_{\#}\big(Y;\mathbb{R}^d\big)$ given by

$$\int_B s(x)\,\mathrm{d}\boldsymbol{\mu}(x) := \sum_{i=1}^{m} c_i \boldsymbol{\mu}(B_i \cap B). \tag{2.9}$$

Let $\phi : \Omega \to \mathbb{R}$ be a bounded, Borel measurable function, and let $\{s_j\}_{j\in\mathbb{N}}$ be a sequence of Borel simple functions converging uniformly in Ω to ϕ, with $s_j := \sum_{i=1}^{m_j} c_i^{(j)} \chi_{B_i^{(j)}}$ as in (2.8). We have that

$$\int_Y \left|\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right|\mathrm{d}y = \int_Y \left|\sum_{i=1}^{m_j} c_i^{(j)} \boldsymbol{\mu}\Big(B_i^{(j)}\Big)\right|\mathrm{d}y \leqslant \sum_{i=1}^{m_j} \big|c_i^{(j)}\big| \Big\|\boldsymbol{\mu}\Big(B_i^{(j)}\Big)\Big\|_{L^1(Y;\mathbb{R}^d)}$$

and

$$\left\|D_y\left(\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)\right\|(Y) \leqslant \sum_{i=1}^{m_j} \big|c_i^{(j)}\big| \Big\|D_y\Big(\boldsymbol{\mu}\Big(B_i^{(j)}\Big)\Big)\Big\|(Y),$$

where we used (2.9). Consequently, using the definition of the total variation of $\boldsymbol{\mu}$,

$$\int_Y \left|\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right|\mathrm{d}y + \left\|D_y\left(\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)\right\|(Y) \leqslant \|s_j\|_\infty \|\boldsymbol{\mu}\|(\Omega) \tag{2.10}$$

and also

$$\int_Y \left|\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right|\mathrm{d}y \leqslant \sum_{i=1}^{m_j} \big|c_i^{(j)}\big| \|\boldsymbol{\mu}\|\Big(B_i^{(j)}\Big) = \int_\Omega |s_j(x)|\,\mathrm{d}\|\boldsymbol{\mu}\|(x). \tag{2.11}$$

Since $\sup_j \|s_j\|_\infty < \infty$ and $\boldsymbol{\mu}$ has finite total variation, we deduce from (2.10) that the sequence

$$\left\{\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right\}_{j\in\mathbb{N}}$$

is uniformly bounded in $BV_{\#}(Y;\mathbb{R}^d)$. Thus, up to a (not relabeled) subsequence, we may find $u \in BV_{\#}\big(Y;\mathbb{R}^d\big)$ such that

$$\int_\Omega s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \overset{\star}{\rightharpoonup}_j u \;\text{ weakly-}\star \text{ in } BV_{\#}\big(Y;\mathbb{R}^d\big).$$

Assume now that $\{t_j\}_{j\in\mathbb{N}}$ is another sequence of Borel simple functions converging uniformly in Ω to ϕ, and such that

$$\int_\Omega t_j(x)\,\mathrm{d}\boldsymbol{\mu}(x) \overset{\star}{\rightharpoonup}_j v \quad \text{weakly-}\star \text{ in } BV_\#\big(Y;\mathbb{R}^d\big),$$

for some $v \in BV_\#\big(Y;\mathbb{R}^d\big)$. Then $\{s_j - t_j\}_{j\in\mathbb{N}}$ is a sequence of Borel simple functions converging uniformly in Ω to 0, and so (2.10) ensures that $u = v$ for $\mathcal{L}^N$-a.e. $y \in \mathbb{R}^N$. This gives sense to the following definition.

Definition 2.3. *Let $\phi : \Omega \to \mathbb{R}$ be a bounded, Borel measurable function. If $B \in \mathcal{B}(\Omega)$ and $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$, then we define the integral of ϕ over B with respect to $\boldsymbol{\mu}$, and we write $\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, as the function in $BV_\#\big(Y;\mathbb{R}^d\big)$ given by*

$$\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) := \big(w\star\text{-}BV_\#\big(Y;\mathbb{R}^d\big)\big) - \lim_{j\to\infty} \int_B s_j(x)\,\mathrm{d}\boldsymbol{\mu}(x),$$

where $\{s_j\}_{j\in\mathbb{N}}$ is a sequence of Borel simple functions converging uniformly in Ω to ϕ.

The following lemma will be useful in the sequel. Its proof uses (2.10), (2.11), Definition 2.3, Lebesgue Dominated Convergence Theorem and the lower semicontinuity of the total variation.

Lemma 2.4. *Let $\phi : \Omega \to \mathbb{R}$ be a bounded, Borel measurable function, and let $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$. The following hold:*

i) $\displaystyle\int_Y \left| \int_\Omega \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right| \mathrm{d}y \leqslant \int_\Omega |\phi(x)|\,\mathrm{d}\|\boldsymbol{\mu}\|(x)$;

ii) If $\boldsymbol{\nu}$ is the set application given by $\boldsymbol{\nu}(B) := \displaystyle\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x)$, $B \in \mathcal{B}(\Omega)$, then $\boldsymbol{\nu} \in \mathcal{M}\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$, and $\|\boldsymbol{\nu}\|(B) \leqslant \|\phi\|_\infty \|\boldsymbol{\mu}\|(B)$ for all $B \in \mathcal{B}(\Omega)$.

Note that if $\phi : \Omega \to \mathbb{R}$ and $\psi : Y \to \mathbb{R}$ are a bounded, Borel measurable functions, then given $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$ and $B \in \mathcal{B}(\Omega)$, the integral

$$\int_{B\times Y} \phi(x)\psi(y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y := \int_Y \left(\int_B \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right)(y)\,\psi(y)\,\mathrm{d}y \tag{2.12}$$

is well defined in $\mathbb{R}^d$.

By considering first bounded, Borel simple functions, one can show that

$$\left| \sum_{i=1}^m \int_Y \left(\int_\Omega \phi_i(x)\,\mathrm{d}\boldsymbol{\mu}(x) \right)(y)\,\psi_i(y)\,\mathrm{d}y \right| \leqslant \left\| \sum_{i=1}^m \phi_i\psi_i \right\|_\infty \|\boldsymbol{\mu}\|(\Omega), \tag{2.13}$$

whenever $\phi_i : \Omega \to \mathbb{R}$, $\psi_i : Y \to \mathbb{R}$, $i \in \{1, \cdots, m\}$, are bounded, Borel functions.

In fact, for simplicity, assume that $m = 2$. Let s_1, s_2, t_1, t_2 be simple functions, and write

$$s_1 = \sum_{i=1}^{m_1} a_i\chi_{A_i}, \quad s_2 = \sum_{i=1}^{m_2} b_i\chi_{B_i}, \quad t_1 = \sum_{i=1}^{l_1} c_i\chi_{C_i}, \quad t_2 = \sum_{i=1}^{l_2} d_i\chi_{D_i},$$

with $m_1, m_2, l_1, l_2 \in \mathbb{N}$, $\{a_i\}_{i=1}^{m_1}$, $\{b_i\}_{i=1}^{m_2}$, $\{c_i\}_{i=1}^{l_1}$, $\{d_i\}_{i=1}^{l_2}$ finite collections of distinct real numbers, $\{A_i\}_{i=1}^{m_1}$, $\{B_i\}_{i=1}^{m_2} \subset \mathcal{B}(\Omega)$, and $\{C_i\}_{i=1}^{l_1}$, $\{D_i\}_{i=1}^{l_2} \subset \mathcal{B}(Y)$ finite collections of mutually disjoint sets.

It can be shown that

$$s_1t_1 + s_2t_2 = \sum_{i=1}^{\bar{m}} \kappa_i\chi_{E_i}\chi_{F_i},$$

where for all $i \in \{1, \cdots, \bar{m}\}$, $\kappa_i \in \mathbb{R}$ and $|\kappa_i| \leqslant \|s_1 t_1 + s_2 t_2\|_\infty$, $\{E_i\}_{i=1}^{\bar{m}}$ is a family of mutually disjoint Borel subsets of Ω, and for all $i \in \{1, \cdots, \bar{m}\}$, $F_i \in \mathcal{B}(Y)$.

Thus,

$$\begin{aligned}
&\left|\int_Y \left(\int_\Omega s_1(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\,t_1(y)\,\mathrm{d}y + \int_Y \left(\int_\Omega s_2(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\,t_2(y)\,\mathrm{d}y\right| \\
&\quad = \left|\int_Y \left(\sum_{i=1}^{m_1} a_i\big(\boldsymbol{\mu}(A_i)\big)(y)\right)\left(\sum_{i=1}^{l_1} c_i\chi_{C_i}(y)\right) + \left(\sum_{i=1}^{m_2} b_i\big(\boldsymbol{\mu}(B_i)\big)(y)\right)\left(\sum_{i=1}^{l_2} d_i\chi_{D_i}(y)\right)\mathrm{d}y\right| \\
&\quad = \left|\int_Y \sum_{i=1}^{\bar{m}} \kappa_i\big(\boldsymbol{\mu}(E_i)\big)(y)\,\chi_{F_i}(y)\,\mathrm{d}y\right| \leqslant \|s_1t_1 + s_2t_2\|_\infty \sum_{i=1}^{\bar{m}} \int_{F_i} \left|\big(\boldsymbol{\mu}(E_i)\big)(y)\right|\mathrm{d}y \\
&\quad \leqslant \|s_1t_1 + s_2t_2\|_\infty \sum_{i=1}^{\bar{m}} \int_Y \left|\big(\boldsymbol{\mu}(E_i)\big)(y)\right|\mathrm{d}y \leqslant \|s_1t_1 + s_2t_2\|_\infty \|\boldsymbol{\mu}\|(\Omega),
\end{aligned}$$

from which we deduce (2.13) for simple functions. To prove the general case, if $\phi_i : \Omega \to \mathbb{R}$, $\psi_i : Y \to \mathbb{R}$, $i \in \{1, \cdots, m\}$, are bounded, Borel functions, then for each $j \in \mathbb{N}$ we can find $s_j^{(i)} : \Omega \to \mathbb{R}$ and $t_j^{(i)} : Y \to \mathbb{R}$, Borel simple functions, such that $s_j^{(i)} \to \phi_i$ uniformly in Ω as $j \to \infty$, and $t_j^{(i)} \to \psi_i$ uniformly in Y as $j \to \infty$. By definition,

$$\int_\Omega \phi_i(x)\,\mathrm{d}\boldsymbol{\mu}(x) = \big(w\star\text{-}BV_\#\big(Y;\mathbb{R}^d\big)\big) - \lim_{j\to\infty} \int_\Omega s_j^{(i)}(x)\,\mathrm{d}\boldsymbol{\mu}(x),$$

so that the uniform convergence $t_j^{(i)} \to \psi_i$ in Y entails

$$\lim_{j\to\infty} \int_Y \left(\int_\Omega s_j^{(i)}(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\,t_j^{(i)}(y)\,\mathrm{d}y = \int_Y \left(\int_\Omega \phi_i(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\,\psi_i(y)\,\mathrm{d}y,$$

for all $i \in \{1, \cdots, m\}$. To conclude, it suffices to pass to the limit as $j \to \infty$ the inequality

$$\left|\sum_{i=1}^m \int_Y \left(\int_\Omega s_j^{(i)}(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\,t_j^{(i)}(y)\,\mathrm{d}y\right| \leqslant \left\|\sum_{i=1}^m s_j^{(i)} t_j^{(i)}\right\|_\infty \|\boldsymbol{\mu}\|(\Omega)$$

established above for simple functions.

We are finally in position to give sense to (2.7) (for $i = 1$).

Definition 2.5. *Let $\varphi \in C_0\big(\Omega; C_\#(Y)\big)$ and $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$ be given. We define*

$$\int_{\Omega\times Y} \varphi(x,y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y := \lim_{j\to\infty} \left\{\sum_{i=1}^{m_j} \int_Y \left(\int_\Omega \phi_i^{(j)}(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\,\psi_i^{(j)}(y)\,\mathrm{d}y\right\}, \tag{2.14}$$

where for each $j \in \mathbb{N}$, $m_j \in \mathbb{N}$, and for all $i \in \{1, ..., m_j\}$, $\phi_i^{(j)} \in C_0(\Omega)$, $\psi_i^{(j)} \in C_\#(Y)$, and $\{\varphi_j\}_{j\in\mathbb{N}}$, with $\varphi_j := \sum_{i=1}^{m_j} \phi_i^{(j)}\psi_i^{(j)}$, converges to φ in $C_0\big(\Omega; C_\#(Y)\big)$.

Remark 2.6. (i) *Given $\varphi \in C_0\big(\Omega; C_\#(Y)\big)$, the existence of a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ as in Definition 2.5 is a consequence of the Stone-Weierstrass Theorem.*

(ii) *Note that (2.14) reduces to (2.12) when $\varphi(x,y) = \phi(x)\psi(y)$ with $\phi \in C_0(\Omega)$, $\psi \in C_\#(Y)$.*

(iii) *Estimate (2.13) ensures that the limit in the Definition 2.5 exists and does not depend on the approximating sequence. Moreover,*

$$\left|\int_{\Omega\times Y} \varphi(x,y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y\right| \leqslant \|\varphi\|_\infty \|\boldsymbol{\mu}\|(\Omega), \tag{2.15}$$

for all $\varphi \in C_0(\Omega; C_\#(Y))$ *, and*

$$\varphi \in C_0(\Omega; C_\#(Y)) \mapsto \int_{\Omega\times Y} \varphi(x,y)\,\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y$$

defines a linear continuous functional.

(iv) *We could have considered the more general setting in which* $\varphi \in C(\Omega; C_\#(Y)) \cap L^\infty(\Omega\times Y)$*. In this case, (iii) above still holds with "*$\varphi \in C_0(\Omega; C_\#(Y))$*" replaced by "*$\varphi \in C(\Omega; C_\#(Y)) \cap L^\infty(\Omega\times Y)$*".*

Next we prove an integration by parts formula for measures in $\mathcal{M}_\star\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$.

Lemma 2.7. *Let* $\boldsymbol{\mu} \in \mathcal{M}_\star\big(\Omega; BV_\#\big(Y;\mathbb{R}^d\big)\big)$*,* $\phi \in C_0(\Omega)$ *and* $\psi \in C^1_\#(Y)$ *be given. Then*

$$\int_Y \left(\int_\Omega \phi(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y) \otimes \nabla\psi(y)\,\mathrm{d}y = -\int_{\Omega\times Y} \phi(x)\,\psi(y)\,\mathrm{d}\lambda(x,y), \tag{2.16}$$

where $\lambda \in \mathcal{M}_{y\#}\big(\Omega\times Y;\mathbb{R}^{d\times N}\big)$ *is the measure associated with* $D_y\boldsymbol{\mu}$*.*

Proof. Fix $B \in \mathcal{B}(\Omega)$, and let $\lambda_B \in \mathcal{M}_\#(Y;\mathbb{R})$ be the (projection) measure defined by $\lambda_B(\cdot) := \lambda(B\times\cdot)$. We have that

$$\begin{aligned}\int_Y \left(\int_\Omega \chi_B(x)\,\mathrm{d}\boldsymbol{\mu}(x)\right)(y)\otimes\nabla\psi(y)\,\mathrm{d}y &= \int_Y (\boldsymbol{\mu}(B))(y)\otimes\nabla\psi(y)\,\mathrm{d}y = -\int_Y \psi(y)\,\mathrm{d}D_y(\boldsymbol{\mu}(B))(y)\\ &= -\int_Y \psi(y)\,\mathrm{d}\lambda_B(y) = -\int_{B\times Y}\psi(y)\,\mathrm{d}\lambda(x,y) = -\int_{\Omega\times Y}\chi_B(x)\,\psi(y)\,\mathrm{d}\lambda(x,y),\end{aligned} \tag{2.17}$$

where we have used the fact that $\boldsymbol{\mu}(B) \in BV_\#(Y;\mathbb{R}^d)$ and the slicing decomposition of a Radon measure (see (2.5)) applied to $\lambda_{\lfloor B\times Y}$.

Since any function in $C_0(\Omega)$ can be approximated with respect to the uniform convergence in Ω by Borel simple functions, (2.16) follows from (2.17) and Definition 2.3. ◻

Step 2. We define (2.7) recursively for an arbitrary $i\in\mathbb{N}$. Fix $i \geqslant 2$, and let $\vartheta \in C_0(\Omega; C_\#(Y_1\times\cdots\times Y_{i-1}))$ and $\boldsymbol{\mu} \in \mathcal{M}\big(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#\big(Y_i,\mathbb{R}^d\big)\big)$.

Proceeding as before (see (2.9) and Definition 2.3), we define the integral of ϑ over $B \in \mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$ with respect to $\boldsymbol{\mu}$, and we write $\int_B \vartheta(x,y_1,\cdots,y_{i-1})\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})$, as the function in $BV_\#\big(Y_i;\mathbb{R}^d\big)$ given by

$$\begin{aligned}&\int_B \phi(x,y_1,\cdots,y_{i-1})\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\\ &\qquad := \big(w\star\text{-}BV_\#\big(Y_i;\mathbb{R}^d\big)\big) - \lim_{j\to\infty}\int_B s_j(x,y_1,\cdots,y_{i-1})\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1}),\end{aligned}$$

where $\{s_j\}_{j\in\mathbb{N}}$ is a sequence of Borel simple functions $s_j : \Omega\times\mathbb{R}^{(i-1)N} \to \mathbb{R}$, $Y_1\times\cdots\times Y_{i-1}$-periodic in the variables $(y_1,\cdots,y_{i-1})$, converging uniformly in $\Omega\times Y_1\times\cdots\times Y_{i-1}$ to ϑ.

Let $\varphi \in C_0(\Omega; C_\#(Y_1\times\cdots\times Y_i))$, and take a sequence $\{\varphi_j\}_{j\in\mathbb{N}}$ converging to φ in $C_0(\Omega; C_\#(Y_1\times\cdots\times Y_i))$, where each φ_j is of the form $\varphi_j(x,y_1,\cdots,y_{i-1},y_i) = \sum_{k=1}^{m_j}\vartheta_k^{(j)}(x,y_1,\cdots,y_{i-1})\psi_k^{(j)}(y_i)$ with $m_j\in\mathbb{N}$, and for all $k\in\{1,\cdots,m_j\}$, $\vartheta_k^{(j)} \in C_0(\Omega; C_\#(Y_1\times\cdots\times Y_{i-1}))$, $\psi_k^{(j)} \in C_\#(Y_i)$. Once again proceeding as before (see (2.12) and Definition 2.5) we can give sense to the expression

$$\sum_{k=1}^{m_j}\int_{Y_i}\left(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}}\vartheta_k^{(j)}(x,y_1,\cdots,y_{i-1})\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\right)(y_i)\,\psi_k^{(j)}(y_i)\,\mathrm{d}y_i \tag{2.18}$$

in $\mathbb{R}^d$, and prove that the limit of (2.18) as $j \to \infty$ exists and is independent of the approximating sequence. We then define

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_i} &\varphi(x, y_1, \cdots, y_i)\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})\mathrm{d}y_i \\ &:= \lim_{j\to\infty} \sum_{k=1}^{m_j} \int_{Y_i} \Big(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \Big)(y_i)\, \psi_k^{(j)}(y_i)\, \mathrm{d}y_i.\end{aligned} \tag{2.19}$$

Similarly, if $\varphi \in C_0\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_i; \mathbb{R}^d\big)\big)$, then we set

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_i} &\varphi(x, y_1, \cdots, y_i) \cdot \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})\mathrm{d}y_i \\ &:= \lim_{j\to\infty} \sum_{k=1}^{m_j} \int_{Y_i} \Big(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \Big)(y_i) \cdot \psi_k^{(j)}(y_i)\, \mathrm{d}y_i,\end{aligned} \tag{2.20}$$

where $\varphi_j(x, y_1, \cdots, y_{i-1}, y_i) := \sum_{k=1}^{m_j} \vartheta_k^{(j)}(x, y_1, \cdots, y_{i-1})\psi_k^{(j)}(y_i)$ with $m_j \in \mathbb{N}$, and for all $k \in \{1, \cdots, m_j\}$, $\vartheta_k^{(j)} \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_{i-1}))$, $\psi_k^{(j)} \in C_\#\big(Y_i, \mathbb{R}^d\big)$, converges to φ in $C_0\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_i; \mathbb{R}^d\big)\big)$ as $j \to \infty$.

If, in particular, $\boldsymbol{\mu} \in \mathcal{M}_\star\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#\big(Y_i, \mathbb{R}^d\big)\big)$ then similar arguments to those of Lemma 2.7 ensure that for all $\vartheta \in C_0(\Omega; C_\#(Y_1 \times \cdots \times Y_{i-1}))$, $\psi \in C^1_\#(Y_i)$ and $\theta \in C^1_\#\big(Y_i; \mathbb{R}^N\big)$ one has

$$\begin{aligned}\int_{Y_i} \Big(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} &\vartheta(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \Big)(y_i) \otimes \nabla\psi(y_i)\, \mathrm{d}y_i \\ &= -\int_{\Omega\times Y_1\times\cdots\times Y_i} \vartheta(x, y_1, \cdots, y_{i-1})\, \psi(y_i)\, \mathrm{d}\lambda(x, y_1, \cdots, y_i),\end{aligned} \tag{2.21}$$

where $\lambda \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^{d\times N}\big)$ is the measure associated with $D_{y_i}\boldsymbol{\mu}$, and for all $k \in \{1, \cdots, d\}$,

$$\begin{aligned}\int_{Y_i} \Big(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} &\vartheta(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}_k(x, y_1, \cdots, y_{i-1}) \Big)(y_i)\, \mathrm{div}\, \theta(y_i)\, \mathrm{d}y_i \\ &= -\int_{\Omega\times Y_1\times\cdots\times Y_i} \vartheta(x, y_1, \cdots, y_{i-1})\, \theta(y_i) \cdot \mathrm{d}\lambda_{(k)}(x, y_1, \cdots, y_i),\end{aligned} \tag{2.22}$$

where $\lambda_{(k)}$ denotes the k^{th} row of λ and $\boldsymbol{\mu}_k$ denotes the k^{th} component of $\boldsymbol{\mu}$.

Remark 2.8. *As observed in Remark 2.6 (iv), in (2.20) we may consider the more general setting in which* $\varphi \in C\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_i; \mathbb{R}^d\big)\big) \cap L^\infty\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^d\big)$. *In this case, the functions* $\vartheta_k^{(j)}$ *are to be taken in* $C(\Omega; C_\#(Y_1 \times \cdots \times Y_i)) \cap L^\infty(\Omega \times Y_1 \times \cdots \times Y_i)$, *and, as before, the correspondent limit in (2.20) is independent of the approximating sequence (with respect to the supremum norm* $\|\cdot\|_\infty$ *in* $\Omega \times Y_1 \times \cdots \times Y_i$*).*

Moreover,

$$F(\varphi) := \int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi(x, y_1, \cdots, y_i) \cdot \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1})\mathrm{d}y_i$$

for $\varphi \in C\big(\Omega; C_\#\big(Y_1 \times \cdots \times Y_i; \mathbb{R}^d\big)\big) \cap L^\infty\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^d\big)$, *defines a linear continuous functional, and we have*

$$\big|F(\varphi)\big| \leqslant \|\varphi\|_\infty \|\boldsymbol{\mu}\|(\Omega \times Y_1 \times \cdots \times Y_{i-1}).$$

Furthermore, proceeding as in Lemma 2.4 and (2.19), in the particular case in which φ *is scalar and does not depend on* y_i, *then*

$$\begin{aligned}\int_{Y_i} \Big| \int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} &\varphi(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}) \Big| \mathrm{d}y_i \\ \leqslant \int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} &|\varphi(x, y_1, \cdots, y_{i-1})|\, \mathrm{d}\|\boldsymbol{\mu}\|(x, y_1, \cdots, y_{i-1}),\end{aligned}$$

and if we define for all $B \in \mathcal{B}(\Omega \times Y_1 \times \cdots \times Y_{i-1})$,

$$\boldsymbol{\nu}(B) := \int_B \varphi(x, y_1, \cdots, y_{i-1})\, \mathrm{d}\boldsymbol{\mu}(x, y_1, \cdots, y_{i-1}),$$

then we have that $\boldsymbol{\nu} \in \mathcal{M}\big(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#\big(Y_i; \mathbb{R}^d\big)\big)$, and $\|\boldsymbol{\nu}\|(B) \leqslant \|\varphi\|_\infty \|\boldsymbol{\mu}\|(B)$.

3. Multiscale Convergence in BV

The main goal of this section is to characterize $(n+1)$-scale limit pairs (u, U) associated with sequences $\{(u_\varepsilon \mathcal{L}^N{\lfloor_\Omega}, Du_\varepsilon{\lfloor_\Omega})\}_{\varepsilon>0} \subset \mathcal{M}(\Omega; \mathbb{R}^d) \times \mathcal{M}(\Omega; \mathbb{R}^{d\times N})$ whenever $\{u_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $BV(\Omega; \mathbb{R}^d)$.

We start by establishing some properties concerning the notion of multiscale convergence for sequences of measures, introduced in Definition 1.7.

Let $n \in \mathbb{N}$ be fixed. In the sequel, $\varrho_1, ..., \varrho_n : (0, \infty) \to (0, \infty)$ satisfy (1.1).

Remark 3.1. *The $(n+1)$-scale limit μ_0 may depend on the sequence $\{\varepsilon\}$. Indeed, let $n = 1$, $\varrho_1(\varepsilon) = \varepsilon$ for all $\varepsilon > 0$, let $\Omega \subset \mathbb{R}^N$ be open and bounded, and let $\vartheta \in C_\#(Y)$. Define $\mu_\varepsilon := \vartheta\big(\frac{\cdot}{\varepsilon}\big)\mathcal{L}^N{\lfloor_\Omega}$. If $\varphi \in C_0(\Omega; C_\#(Y))$, then by the Riemann-Lebesgue Lemma (see [8])*

$$\lim_{\varepsilon\to 0^+} \int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}\mu_\varepsilon(x) = \lim_{\varepsilon\to 0^+} \int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\vartheta\Big(\frac{x}{\varepsilon}\Big)\, \mathrm{d}x = \int_{\Omega\times Y} \varphi(x, y)\vartheta(y)\, \mathrm{d}x\mathrm{d}y =: \langle \mathcal{L}^N{\lfloor_\Omega} \otimes \vartheta\mathcal{L}^N_y, \varphi\rangle$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, \mathrm{d}\mu_{\varepsilon^2}(x) &= \lim_{\varepsilon\to 0^+} \int_\Omega \varphi\Big(x, \frac{x}{\varepsilon}\Big)\vartheta\Big(\frac{x}{\varepsilon^2}\Big)\, \mathrm{d}x = \int_{\Omega\times Y_1\times Y_2} \varphi(x, y_1)\vartheta(y_2)\, \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{\Omega\times Y} \varphi(x, y)\bigg(\int_{Y_2} \vartheta(y_2)\, \mathrm{d}y_2\bigg)\, \mathrm{d}x\mathrm{d}y =: \langle \bar{\vartheta}\mathcal{L}^N{\lfloor_\Omega} \otimes \mathcal{L}^N_y, \varphi\rangle,\end{aligned}$$

where $\bar{\vartheta} := \int_Y \vartheta(y)\, \mathrm{d}y$. Hence $\mu_\varepsilon \xrightharpoonup[\varepsilon]{2\text{-}sc} \mathcal{L}^N{\lfloor_\Omega} \otimes \vartheta\mathcal{L}^N_y$, while $\mu_{\varepsilon^2} \xrightharpoonup[\varepsilon]{2\text{-}sc} \bar{\vartheta}\mathcal{L}^N{\lfloor_\Omega} \otimes \mathcal{L}^N_y$. This example shows that it may be the case that $\mu_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \mu_0$ and $\mu_{\varepsilon'} \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \lambda_0$, with $\varepsilon' \prec \varepsilon$, but $\mu_0 \neq \lambda_0$. What we can guarantee is that $\mu_{\varepsilon'} \xrightharpoonup[\varepsilon']{(n+1)\text{-}sc} \mu_0$. This is due to the dependence of the test functions on the length scales.

The notion of $(n+1)$-scale convergence is justified in view of the following compactness result. The proof is a straightforward generalization of that of [3, Thm 3.5] (see also [1]).

Theorem 3.2. *Let $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ be a bounded sequence. Then there exist a subsequence $\{\mu_{\varepsilon'}\}_{\varepsilon'>0}$ of $\{\mu_\varepsilon\}_{\varepsilon>0}$ and a measure $\mu_0 \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$ such that $\mu_{\varepsilon'} \xrightharpoonup[\varepsilon']{(n+1)\text{-}sc} \mu_0$.*

As in the cases studied in [1], [2], and [3], the $(n+1)$-scale limit contains more information on the oscillations of a bounded sequence in $\mathcal{M}(\Omega; \mathbb{R}^m)$ than its weak-$\star$ limit, in that the latter is the canonical projection of the $(n+1)$-scale limit onto Ω.

Proposition 3.3. *Let $\{\mu_\varepsilon\}_{\varepsilon>0} \subset \mathcal{M}(\Omega; \mathbb{R}^m)$ and $\mu_0 \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^m)$ be such that $\mu_\varepsilon \xrightharpoonup[\varepsilon]{(n+1)\text{-}sc} \mu_0$. Then $\mu_\varepsilon \overset{\star}{\rightharpoonup} \bar{\mu}_0$ weakly-$\star$ in $\mathcal{M}(\Omega; \mathbb{R}^m)$ as $\varepsilon \to 0^+$, where $\bar{\mu}_0 \in \mathcal{M}(\Omega; \mathbb{R}^m)$ is the measure defined for all $B \in \mathcal{B}(\Omega)$ by*

$$\bar{\mu}_0(B) := \mu_0(B \times Y_1 \times \cdots \times Y_n).$$

Moreover, $\|\bar{\mu}_0\|(\Omega) \leqslant \|\mu_0\|(\Omega \times Y_1 \times \cdots \times Y_n) \leqslant \liminf_{\varepsilon\to 0^+} \|\mu_\varepsilon\|(\Omega)$.

The proof of Proposition 3.3 is a simple generalization of [3, Lemmas 3.3 and 3.4].

Remark 3.4. *In view of Proposition 3.3, since every weakly-$\star$ convergent sequence in $\mathcal{M}(\Omega; \mathbb{R}^m)$ is bounded, the same holds for any $(n+1)$-scale convergent sequence in $\mathcal{M}(\Omega; \mathbb{R}^m)$.*

Assume that $\{u_\varepsilon\}_{\varepsilon>0} \subset BV(\Omega; \mathbb{R}^d)$ is a bounded sequence. By Theorem 3.2, there exist subsequences of $\big\{u_\varepsilon \mathcal{L}^N{\lfloor_\Omega}\big\}_{\varepsilon>0}$ and $\{Du_\varepsilon\}_{\varepsilon>0}$ that $(n+1)$-scale converge. Theorem 1.10 provides a characterization of these

$(n+1)$-scale limits as well as the relationship between them. To prove it we need an auxiliary lemma, which is an extension of [3, Thm. 2.5] (see also [2, Lemma 3.7]).

Lemma 3.5. *Let $\lambda \in \mathcal{M}_{y\#}(\Omega \times Y_1 \times \cdots \times Y_n; \mathbb{R}^N)$ be given. The following conditions are equivalent:*

i) for all $i \in \{1, \cdots, n\}$ there exists a measure $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i))$ such that

$$\lambda = \begin{cases} \lambda_1 & \text{if } n = 1, \\ \displaystyle\sum_{i=1}^{n-1} \lambda_i \otimes \mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n} + \lambda_n & \text{if } n \geqslant 2, \end{cases}$$

where each $\lambda_i \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^N\big)$ is the measure associated with $D_{y_i}\boldsymbol{\mu}_i$;

ii) for all $\varphi \in C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1 \times \cdots \times Y_n; \mathbb{R}^N\big)\big)$ such that $\operatorname{div}_{y_n} \varphi = 0$ and, if $n \geqslant 2$, for all $k \in \{1, \cdots, n-1\}$, $x \in \Omega$, $y_i \in Y_i$, $i \in \{1, \cdots, n\}$,

$$\int_{Y_{k+1}\times\cdots\times Y_n} \operatorname{div}_{y_k} \varphi(x, y_1, \cdots, y_n)\, \mathrm{d}y_{k+1} \cdots \mathrm{d}y_n = 0,$$

we have

$$\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}\lambda(x, y_1, \cdots, y_n) = 0.$$

Proof. We will give the proof only for $n = 2$, the argument being easily adapted for any $n \in \mathbb{N}$.

Step 1. Assume first that *i)* holds, and let $\varphi \in C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$ be such that $\operatorname{div}_{y_2} \varphi = 0$ and

$$\int_{Y_2} \operatorname{div}_{y_1} \varphi(x, y_1, y_2)\, \mathrm{d}y_2 = 0.$$

Using the decomposition of λ as in *i)*, we have

$$\int_{\Omega\times Y_1\times Y_2} \varphi \cdot \mathrm{d}\lambda(x, y_1, y_2) = \int_{\Omega\times Y_1\times Y_2} \varphi \cdot \mathrm{d}\lambda_1(x, y_1)\mathrm{d}y_2 + \int_{\Omega\times Y_1\times Y_2} \varphi \cdot \mathrm{d}\lambda_2(x, y_1, y_2). \tag{3.1}$$

We will show that both integrals on the right-hand side of (3.1) are equal to zero. Let $\{\varphi_j\}_{j\in\mathbb{N}}$ be a sequence of the form $\varphi_j(x, y_1, y_2) = \sum_{k=1}^{m_j} \phi_k^{(j)}(x)\psi_k^{(j)}(y_1)\theta_k^{(j)}(y_2)$, where $m_j \in \mathbb{N}$ and for all $k \in \{1, \cdots, m_j\}$, $\phi_k^{(j)} \in C_c^\infty(\Omega)$, $\psi_k^{(j)} \in C_\#^\infty(Y_1)$, $\theta_k^{(j)} \in C_\#^\infty(Y_2; \mathbb{R}^N)$, converging to φ in $C_0^\infty\big(\Omega; C_\#^\infty\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$. Then,

$$\int_{Y_2} \operatorname{div}_{y_1} \varphi_j\, \mathrm{d}y_2 = \sum_{k=1}^{m_j} \left(\phi_k^{(j)} \nabla\psi_k^{(j)} \cdot \int_{Y_2} \theta_k^{(j)}\, \mathrm{d}y_2\right) \to \int_{Y_2} \operatorname{div}_{y_1} \varphi\, \mathrm{d}y_2 = 0 \text{ in } C_0(\Omega; C_\#(Y_1)), \tag{3.2}$$

$$\operatorname{div}_{y_2} \varphi_j = \sum_{k=1}^{m_j} \phi_k^{(j)} \psi_k^{(j)} \operatorname{div} \theta_k^{(j)} \to \operatorname{div}_{y_2} \varphi = 0 \text{ in } C_0(\Omega; C_\#(Y_1 \times Y_2)). \tag{3.3}$$

The convergence $\varphi_j \to \varphi$ in $C_0\big(\Omega; C_\#\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$ and Lemma 2.7 (see also Remark 2.8) yield

$$\begin{aligned} \int_{\Omega\times Y_1\times Y_2} \varphi(x, y_1, y_2) \cdot \mathrm{d}\lambda_1(x, y_1)\mathrm{d}y_2 &= \lim_{j\to\infty} \int_{\Omega\times Y_1\times Y_2} \varphi_j(x, y_1, y_2) \cdot \mathrm{d}\lambda_1(x, y_1)\mathrm{d}y_2 \\ &= \lim_{j\to\infty} \left\{ \sum_{k=1}^{m_j} \int_{\Omega\times Y_1} \phi_k^{(j)}(x)\psi_k^{(j)}(y_1)\, \mathrm{d}\lambda_1(x, y_1) \cdot \int_{Y_2} \theta_k^{(j)}(y_2)\, \mathrm{d}y_2 \right\} \\ &= \lim_{j\to\infty} \left\{ -\sum_{k=1}^{m_j} \int_{Y_1} \left(\int_\Omega \phi_k^{(j)}(x)\, \mathrm{d}\boldsymbol{\mu}_1(x)\right)(y_1)\, \nabla\psi_k^{(j)}(y_1)\, \mathrm{d}y_1 \cdot \int_{Y_2} \theta_k^{(j)}(y_2)\, \mathrm{d}y_2 \right\} \\ &= \lim_{j\to\infty} \left\{ -\sum_{k=1}^{m_j} \int_{Y_1} \left(\int_\Omega \phi_k^{(j)}(x)\, \mathrm{d}\boldsymbol{\mu}_1(x)\right)(y_1)\, \tilde{\psi}_k^{(j)}(y_1)\, \mathrm{d}y_1 \right\}, \end{aligned} \tag{3.4}$$

where $\tilde{\psi}_k^{(j)} := \nabla\psi_k^{(j)} \cdot \int_{Y_2} \theta_k^{(j)}\,\mathrm{d}y_2$. By (3.2), $\sum_{k=1}^{m_j} \phi_k^{(j)}\tilde{\psi}_k^{(j)} \to 0$ in $C_0(\Omega; C_\#(Y_1))$, and so, using (3.4) and Definition 2.5, we obtain

$$\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_1(x,y_1)\mathrm{d}y_2 = \int_{\Omega\times Y_1} 0\,\mathrm{d}\boldsymbol{\mu}_1(x)\mathrm{d}y_1 = 0. \tag{3.5}$$

Similarly, in view of (2.19), (2.22) and (3.3), we get

$$\begin{aligned}
&\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2) = \lim_{j\to\infty}\int_{\Omega\times Y_1\times Y_2} \varphi_j(x,y_1,y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2)\\
&\quad= \lim_{j\to\infty}\left\{\sum_{k=1}^{m_j}\int_{\Omega\times Y_1\times Y_2} \phi_k^{(j)}(x)\psi_k^{(j)}(y_1)\theta_k^{(j)}(y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2)\right\}\\
&\quad= \lim_{j\to\infty}\left\{-\sum_{k=1}^{m_j}\int_{Y_2}\left(\int_{\Omega\times Y_1} \phi_k^{(j)}(x)\psi_k^{(j)}(y_1)\,\mathrm{d}\boldsymbol{\mu}_2(x,y_1)\right)(y_2)\,\mathrm{div}\,\theta_k^{(j)}(y_2)\,\mathrm{d}y_2\right\}\\
&\quad= \int_{\Omega\times Y_1\times Y_2} 0\,\mathrm{d}\boldsymbol{\mu}_2(x,y_1)\mathrm{d}y_2 = 0.
\end{aligned} \tag{3.6}$$

From (3.1), (3.5) and (3.6), we conclude that

$$\int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda(x,y_1,y_2) = 0,$$

which proves *ii)*.

Step 2. Conversely, assume by contradiction that *ii)* holds but $\lambda \notin \mathcal{E}$, where $\mathcal{E}$ is the space of all measures $\tau \in \mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$ for which there exist two measures $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ and $\boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega\times Y_1; BV_\#(Y_2))$ such that

$$\tau = \lambda_1 \otimes \mathcal{L}^N_{y_2} + \lambda_2,$$

where $\lambda_1 \in \mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ and $\lambda_2 \in \mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$ are the measures associated with $D_{y_1}\boldsymbol{\mu}_1$ and $D_{y_2}\boldsymbol{\mu}_2$, respectively.

Note that $\mathcal{E}$ is a vectorial subspace of $\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N)$. We claim that it is weakly-$\star$ closed.

Substep 2a. Assume that the claim holds. Recalling that in a Banach space, a convex set is weakly closed if, and only if, it is closed, then by a corollary to the Hahn–Banach Theorem (see, for example, [6, Cor. I.8]), there exists a function $\varphi \in C_0(\Omega; C_\#(Y_1\times Y_2;\mathbb{R}^N))$ such that for all $\tau\in\mathcal{E}$,

$$\begin{aligned}
\langle \tau,\varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))} &= \int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1,y_2)\cdot\mathrm{d}\tau(x,y_1,y_2) = 0,\\
\langle \lambda,\varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))} &= \int_{\Omega\times Y_1\times Y_2}\varphi(x,y_1,y_2)\cdot\mathrm{d}\lambda(x,y_1,y_2) \neq 0.
\end{aligned} \tag{3.7}$$

Let $f\in C_c^\infty(\Omega)$, $g\in C_\#^\infty(Y_1)$ and $h \in C_\#^\infty(Y_2)$ be arbitrary. Define $\boldsymbol{\mu}_1 : \mathcal{B}(\Omega)\to BV_\#(Y_1)$, $\boldsymbol{\mu}_2 : \mathcal{B}(\Omega\times Y_1)\to BV_\#(Y_2)$ by

$$\boldsymbol{\mu}_1(B) := \left(\int_B f(x)\,\mathrm{d}x\right)g,\ \ B\in\mathcal{B}(\Omega),\quad \boldsymbol{\mu}_2(E) := \left(\int_E f(x)g(y_1)\,\mathrm{d}x\mathrm{d}y_1\right)h,\ \ E\in\mathcal{B}(\Omega\times Y_1).$$

Clearly, $\boldsymbol{\mu}_1 \in \mathcal{M}(\Omega; BV_\#(Y_1))$ and $\boldsymbol{\mu}_2 \in \mathcal{M}(\Omega\times Y_1; BV_\#(Y_2))$. Moreover, for all $B\in\mathcal{B}(\Omega)$, $E\in\mathcal{B}(\Omega\times Y_1)$,

$$D_{y_1}(\boldsymbol{\mu}_1(B)) = \left(\int_B f(x)\,\mathrm{d}x\right)\nabla g\mathcal{L}^N{}_{\lfloor Y_1},\quad D_{y_2}(\boldsymbol{\mu}_2(E)) = \left(\int_E f(x)g(y_1)\,\mathrm{d}x\mathrm{d}y_1\right)\nabla h\mathcal{L}^N{}_{\lfloor Y_2}.$$

Hence $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ and $\boldsymbol{\mu}_2 \in \mathcal{M}_\star(\Omega \times Y_1; BV_\#(Y_2))$, with

$$\lambda_1 = f\mathcal{L}^N_{\lfloor\Omega} \otimes \nabla g\mathcal{L}^N{}_{\lfloor Y_1} \quad \text{and} \quad \lambda_2 = \left(fg\mathcal{L}^N_{\lfloor\Omega} \otimes \mathcal{L}^N{}_{\lfloor Y_1}\right) \otimes \nabla h\mathcal{L}^N{}_{\lfloor Y_2},$$

respectively. Thus $\lambda_1 \otimes \mathcal{L}^N_{y_2}$, $\lambda_2 \in \mathcal{E}$, and so by the first condition in (3.7), and denoting by $\langle \cdot\,,\,\cdot \rangle$ the duality pairing in the sense of distributions, we conclude that

$$\begin{aligned} 0 &= \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_1(x,y_1)\mathrm{d}y_2 = \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot (f(x)\nabla g(y_1))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= \int_{\Omega\times Y_1} \left(\int_{Y_2} \varphi(x,y_1,y_2)\,\mathrm{d}y_2\right)\cdot (f(x)\nabla g(y_1))\,\mathrm{d}x\mathrm{d}y_1 = -\left\langle \int_{Y_2} \mathrm{div}_{y_1}\,\varphi\,\mathrm{d}y_2, fg\right\rangle, \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot \mathrm{d}\lambda_2(x,y_1,y_2) = \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\cdot (f(x)g(y_1)\nabla h(y_2))\,\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &= -\left\langle \mathrm{div}_{y_2}\,\varphi, fgh\right\rangle. \end{aligned}$$

The arbitrariness of $f \in C_c^\infty(\Omega)$, $g \in C_\#^\infty(Y_1)$ and $h \in C_\#^\infty(Y_2)$ yields

$$\int_{Y_2} \mathrm{div}_{y_1}\,\varphi\,\mathrm{d}y_2 = 0 \quad \text{and} \quad \mathrm{div}_{y_2}\,\varphi = 0, \tag{3.8}$$

in the sense of distributions.

Substep 2b. We show that (3.8) and *ii)* contradict the second condition in (3.7). We will derive such contradiction by proving that there exists a sequence $\{\varphi_j\}_{j\in\mathbb{N}} \subset C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$ such that $\mathrm{div}_{y_2}\,\varphi_j = 0$, $\int_{Y_2} \mathrm{div}_{y_1}\,\varphi_j\,\mathrm{d}y_2 = 0$ and $\varphi_j \to \varphi$ in $C_0\big(\Omega; C_\#\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$ as $j \to \infty$.

Let $0 < \varepsilon < 1/2$, and let $\rho_\varepsilon \in C_c(\mathbb{R}^N)$ and $\eta_\varepsilon \in C_\#(Y)$ be the functions introduced in Subsection 2.1 (see (2.1), (2.2) and (2.3)). For $x \in \Omega$, $y_1, y_2 \in \mathbb{R}^N$, define

$$\varphi_\varepsilon(x,y_1,y_2) := \int_{Y_1\times Y_2} \varphi(x,y_1',y_2')\eta_\varepsilon(y_1-y_1')\eta_\varepsilon(y_2-y_2')\,\mathrm{d}y_1'\mathrm{d}y_2'.$$

Then $\varphi_\varepsilon \in C_0\big(\Omega; C_\#^\infty\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$ and $\varphi_\varepsilon \to \varphi$ in $C_0\big(\Omega; C_\#\big(Y_1 \times Y_2; \mathbb{R}^N\big)\big)$ as $\varepsilon \to 0^+$. Moreover, by (3.8) $\mathrm{div}_{y_2}\,\varphi_\varepsilon = 0$ in $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$ and $\int_{Y_2} \mathrm{div}_{y_1}\,\varphi_\varepsilon\,\mathrm{d}y_2 = 0$ in $\Omega \times \mathbb{R}^N$.

Extend φ_ε to $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ by zero outside $\Omega \times \mathbb{R}^N \times \mathbb{R}^N$, and for each $j \in \mathbb{N}$ let

$$\begin{aligned} &K_j := \left\{x \in \Omega\colon\ |x| \leqslant j,\ \mathrm{dist}(x, \mathbb{R}^N\backslash\Omega) \geqslant \frac{2}{j}\right\}, \quad \varphi_j^{(\varepsilon)}(x,y_1,y_2) := \varphi_\varepsilon(x,y_1,y_2)\chi_{K_j}(x), \\ &\tilde{\varphi}_j^{(\varepsilon)}(x,y_1,y_2) := \int_{\mathbb{R}^N} \varphi_j^{(\varepsilon)}(x',y_1,y_2)\rho_{\frac{1}{j}}(x-x')\,\mathrm{d}x', \end{aligned}$$

for all $(x,y_1,y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, where $\rho_{\frac{1}{j}}$ is the function given by (2.1) with ε replaced by $1/j$. Notice that $K_j \subset K_{j+1}$, and $\cup_{j\in\mathbb{N}} K_j = \Omega$. Moreover, since $\mathrm{supp}\,\rho_{\frac{1}{j}} \subset \overline{B(0,1/j)}$ we have

$$\begin{aligned} \mathrm{supp}\,\tilde{\varphi}_j^{(\varepsilon)} &\subset \left\{(x,y_1,y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N\colon\ \mathrm{dist}(x,K_j) \leqslant \frac{1}{j}\right\} \\ &\subset \left\{x \in \Omega\colon\ \mathrm{dist}(x,\partial\Omega) \geqslant \frac{1}{j}\right\} \times \mathbb{R}^N \times \mathbb{R}^N. \end{aligned}$$

Hence,

$$\tilde{\varphi}_j^{(\varepsilon)} \in C_c^\infty\big(\Omega; C_\#^\infty(Y_1 \times Y_2; \mathbb{R}^N)\big), \quad \mathrm{div}_{y_2}\,\tilde{\varphi}_j^{(\varepsilon)} = 0, \quad \int_{Y_2} \mathrm{div}_{y_1}\,\tilde{\varphi}_j^{(\varepsilon)}\,\mathrm{d}y_2 = 0.$$

Furthermore, arguing as in [11, Thm 2.78], we have that $\tilde\varphi_j^{(\varepsilon)} \to \varphi_\varepsilon$ in $C_0\big(\Omega; C_\#\big(Y_1\times Y_2;\mathbb{R}^N\big)\big)$ as $j\to\infty$. Finally, using a diagonalization argument we can find a subsequence $j_\varepsilon \prec j$ such that $\tilde\varphi_\varepsilon := \tilde\varphi_{j_\varepsilon}^{(\varepsilon)} \in C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1\times Y_2;\mathbb{R}^N\big)\big)$, $\operatorname{div}_{y_2}\tilde\varphi_\varepsilon = 0$, $\int_{Y_2}\operatorname{div}_{y_1}\tilde\varphi_\varepsilon\,\mathrm{d}y_2 = 0$ and $\tilde\varphi_\varepsilon \to \varphi$ in $C_0\big(\Omega; C_\#\big(Y_1\times Y_2;\mathbb{R}^N\big)\big)$ as $\varepsilon\to 0^+$. Using *ii)*,

$$0 = \int_{\Omega\times Y_1\times Y_2} \tilde\varphi_\varepsilon(x,y_1,y_2)\,\mathrm{d}\lambda(x,y_1,y_2) \underset{\varepsilon\to 0^+}{\longrightarrow} \int_{\Omega\times Y_1\times Y_2} \varphi(x,y_1,y_2)\,\mathrm{d}\lambda(x,y_1,y_2),$$

which contradicts the second condition in (3.7).

It remains to prove the claim, i.e. $\mathcal{E}$ is weakly-$\star$ closed.

Substep 2c. We start by proving that the set $\mathcal{E}_1$ of all measures $\tau\in\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$ for which there exists a measure $\boldsymbol{\mu}_1 \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ such that τ is the measure associated with $D_{y_1}\boldsymbol{\mu}_1$ (i.e., for all $B\in\mathcal{B}(\Omega)$, $E\in\mathcal{B}(Y_1)$, $\tau(B\times E) = D_{y_1}(\boldsymbol{\mu}_1(B))(E)$) is weakly-$\star$ closed.

Since the weak-$\star$ topology is metrizable on every closed ball of $\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$, by the Krein–Smulian Theorem to prove that $\mathcal{E}_1$ is weakly-$\star$ closed it suffices to show that $\mathcal{E}_1$ is sequentially weakly-$\star$ closed. Let $\{\tau_j\}_{j\in\mathbb{N}}\subset\mathcal{E}_1$ and $\tau\in\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$ be such that $\tau_j \overset{\star}{\rightharpoonup} \tau$ weakly-$\star$ in $\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$ as $j\to\infty$, that is, for all $\varphi\in C_0\big(\Omega; C_\#(Y_1;\mathbb{R}^N)\big)$ we have

$$\lim_{j\to\infty}\int_{\Omega\times Y_1}\varphi(x,y_1)\,\mathrm{d}\tau_j(x,y_1) = \int_{\Omega\times Y_1}\varphi(x,y_1)\,\mathrm{d}\tau(x,y_1).$$

We want to prove that $\tau\in\mathcal{E}_1$. Let $\{\boldsymbol{\mu}_j^{(1)}\}_{j\in\mathbb{N}}\subset\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ be such that τ_j is the measure associated with $D_{y_1}\boldsymbol{\mu}_j^{(1)}$ for each $j\in\mathbb{N}$.

Fix $j\in\mathbb{N}$, and let $\tilde{\boldsymbol{\mu}}_j^{(1)}:\mathcal{B}(\Omega)\to BV_\#(Y_1)$ be defined by

$$\tilde{\boldsymbol{\mu}}_j^{(1)}(B) := \boldsymbol{\mu}_j^{(1)}(B) - \int_{Y_1}\boldsymbol{\mu}_j^{(1)}(B)\,\mathrm{d}y_1,\ \ B\in\mathcal{B}(\Omega).$$

It can be seen that each $\tilde{\boldsymbol{\mu}}_j^{(1)}$ satisfies conditions *i)* and *ii)* of Definition 2.1. Moreover, for all $B\in\mathcal{B}(\Omega)$, $D_{y_1}\big(\tilde{\boldsymbol{\mu}}_j^{(1)}(B)\big) = D_{y_1}\big(\boldsymbol{\mu}_j^{(1)}(B)\big)$ and

$$\begin{aligned}\|\tilde{\boldsymbol{\mu}}_j^{(1)}\|(\Omega) &= \sup\left\{\sum_{i=1}^\infty \|\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i)\|_{BV_\#(Y_1)}\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\}\\ &\leqslant 2\sup\left\{\sum_{i=1}^\infty \|\boldsymbol{\mu}_j^{(1)}(B_i)\|_{BV_\#(Y_1)}\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\} = 2\|\boldsymbol{\mu}_j^{(1)}\|(\Omega)<\infty.\end{aligned}$$

Thus $\tilde{\boldsymbol{\mu}}_j^{(1)}\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$, being τ_j the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}_j^{(1)}$. Furthermore,

$$\begin{aligned}\|\tilde{\boldsymbol{\mu}}_j^{(1)}\|_{\mathcal{M}(\Omega;L^{1\star}_\#(Y_1))} &= \sup\left\{\sum_{i=1}^\infty \|\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i)\|_{L^{1\star}_\#(Y_1)}\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\}\\ &\leqslant \mathcal{C}\sup\left\{\sum_{i=1}^\infty \|D_{y_1}(\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i))\|(Y_1)\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega) \text{ is a partition of } \Omega\right\}\\ &= \mathcal{C}\sup_{\substack{\{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega)\\ \text{partition of }\Omega}}\sum_{i=1}^\infty \sup_{\substack{\{E_k\}_{k\in\mathbb{N}}\subset\mathcal{B}(Y_1)\\ \text{partition of }Y_1}}\sum_{k=1}^\infty \big|D_{y_1}(\tilde{\boldsymbol{\mu}}_j^{(1)}(B_i))(E_k)\big|\\ &= \mathcal{C}\sup_{\substack{\{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega)\\ \text{partition of }\Omega}}\sum_{i=1}^\infty \sup_{\substack{\{E_k\}_{k\in\mathbb{N}}\subset\mathcal{B}(Y_1)\\ \text{partition of }Y_1}}\sum_{k=1}^\infty |\tau_j(B_i\times E_k)|\\ &\leqslant \mathcal{C}\sup_{\substack{\{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega)\\ \text{partition of }\Omega}}\sum_{i=1}^\infty \sup_{\substack{\{E_k\}_{k\in\mathbb{N}}\subset\mathcal{B}(Y_1)\\ \text{partition of }Y_1}}\sum_{k=1}^\infty \|\tau_j\|(B_i\times E_k) \leqslant \mathcal{C}\|\tau_j\|(\Omega\times Y_1),\end{aligned} \tag{3.9}$$

where $1^\star$ is the Sobolev conjugate of N, and where we have used a Poincaré inequality in BV (see [4, Rmk 3.50]) taking into account that for each $B \in \mathcal{B}(\Omega)$, $\tilde{\boldsymbol{\mu}}_j^{(1)}$ is a function in $BV_\#(Y_1)$ with zero mean value.

Since $\sup_{j\in\mathbb{N}} \|\tau_j\|(\Omega \times Y_1) < \infty$, and as $\mathcal{M}(\Omega; L^{1^\star}_\#(Y_1)) \simeq \big(C_0(\Omega; L^N_\#(Y_1))\big)'$ (see, for example, [7, p.182]), from (3.9) we deduce the existence of a (not relabeled) subsequence of $\{\tilde{\boldsymbol{\mu}}_j^{(1)}\}_{j\in\mathbb{N}}$ and of a measure $\tilde{\boldsymbol{\mu}} \in \mathcal{M}(\Omega; L^{1^\star}_\#(Y_1))$ such that

$$\tilde{\boldsymbol{\mu}}_j^{(1)} \overset{\star}{\rightharpoonup} \tilde{\boldsymbol{\mu}} \ \text{ weakly-}\star \text{ in } \mathcal{M}(\Omega; L^{1^\star}_\#(Y_1)).$$

In particular, for all $\varphi \in C_0(\Omega; C_\#(Y_1))$ we have

$$\lim_{j\to\infty} \int_{\Omega\times Y_1} \varphi(x, y_1)\, \mathrm{d}\tilde{\boldsymbol{\mu}}_j^{(1)}(x)\mathrm{d}y_1 = \int_{\Omega\times Y_1} \varphi(x, y_1)\, \mathrm{d}\tilde{\boldsymbol{\mu}}(x)\mathrm{d}y_1, \tag{3.10}$$

where the integrals are to be understood in the sense of Subsection 2.4.

We want to prove that $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$ and that τ is the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}$, thus proving that $\tau \in \mathcal{E}$. We start by showing that $\tilde{\boldsymbol{\mu}} \in \mathcal{M}_\star(\Omega; BV_\#(Y_1))$. Let $\phi \in C_0(\Omega)$ and $\psi \in C^1_\#\big(Y_1; \mathbb{R}^N\big)$ be given. Taking into account that τ_j is the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}_j^{(1)}$, Lemma 2.7 and the weak-$\star$ convergence $\tau_j \overset{\star}{\rightharpoonup} \tau$ in $\mathcal{M}_{y\#}\big(\Omega \times Y_1; \mathbb{R}^N\big)$, we have

$$\begin{aligned}\lim_{j\to\infty} \int_{\Omega\times Y_1} \phi(x) \operatorname{div}\psi(y_1)\, \mathrm{d}\tilde{\boldsymbol{\mu}}_j^{(1)}(x)\mathrm{d}y_1 &= \lim_{j\to\infty} \int_{Y_1} \bigg(\int_\Omega \phi(x)\, \mathrm{d}\tilde{\boldsymbol{\mu}}_j^{(1)}(x)\bigg)(y_1) \operatorname{div}\psi(y_1)\, \mathrm{d}y_1 \\ = -\lim_{j\to\infty} \int_{\Omega\times Y_1} \phi(x)\, \psi(y_1)\cdot \mathrm{d}\tau_j(x, y_1) &= -\int_{\Omega\times Y_1} \phi(x)\, \psi(y_1)\cdot \mathrm{d}\tau(x, y_1).\end{aligned} \tag{3.11}$$

From (3.10) and (3.11), we get

$$\int_{Y_1} \bigg(\int_\Omega \phi(x)\, \mathrm{d}\tilde{\boldsymbol{\mu}}(x)\bigg)(y_1) \operatorname{div}\psi(y_1)\, \mathrm{d}y_1 = -\int_{\Omega\times Y_1} \phi(x)\, \psi(y_1)\cdot \mathrm{d}\tau(x, y_1), \tag{3.12}$$

for all $\phi \in C_0(\Omega)$ and $\psi \in C^1_\#\big(Y_1; \mathbb{R}^N\big)$.

We claim that for all $B \in \mathcal{B}(\Omega)$ and $\psi \in C^1_\#\big(Y_1; \mathbb{R}^N\big)$, we have

$$\int_{Y_1} \tilde{\boldsymbol{\mu}}(B)(y_1) \operatorname{div}\psi(y_1)\, \mathrm{d}y_1 = -\int_{Y_1} \psi(y_1)\cdot \mathrm{d}\tau_B(y_1), \tag{3.13}$$

where $\tau_B(\cdot) := \tau(B \times \cdot)$, thus showing that $\tilde{\boldsymbol{\mu}}(B) \in BV_\#(Y_1)$ with $D_{y_1}(\tilde{\boldsymbol{\mu}}(B)) = \tau_B$.

Indeed, proceeding as in Lemma 2.4, it can be proved that for all bounded, Borel measurable functions $\phi : \Omega \to \mathbb{R}$, we have

$$\int_Y \bigg| \int_\Omega \phi(x)\, \mathrm{d}\tilde{\boldsymbol{\mu}}(x) \bigg|\, \mathrm{d}y \leqslant \int_\Omega |\phi(x)|\, \mathrm{d}\|\tilde{\boldsymbol{\mu}}\|(x). \tag{3.14}$$

Fix $\delta > 0$. Since $\|\tilde{\boldsymbol{\mu}}\| \in \mathcal{M}(\Omega; \mathbb{R})$ and $\|\tau\| \in \mathcal{M}_{\#y}(\Omega \times Y_1; \mathbb{R})$ are positive, finite Radon measures, we may find an open set $A_\delta \supset B$ and a closed set $C_\delta \subset B$ such that

$$\|\tilde{\boldsymbol{\mu}}\|(A_\delta \backslash C_\delta) < \delta, \quad \|\tau\|(A_\delta \backslash C_\delta) < \delta. \tag{3.15}$$

By Urysohn's Lemma, we may also find a function $\phi_\delta \in C_0(\Omega; [0, 1])$ such that $\phi_\delta = 0$ in $\Omega\backslash A_\delta$ and $\phi_\delta = 1$ in C_δ. Then, in view of (3.14),

$$\begin{aligned}&\bigg| \int_{Y_1} \bigg(\int_\Omega \phi_\delta(x)\, \mathrm{d}\tilde{\boldsymbol{\mu}}(x)\bigg)(y_1) \operatorname{div}\psi(y_1)\, \mathrm{d}y_1 - \int_{Y_1} \tilde{\boldsymbol{\mu}}(B)(y_1) \operatorname{div}\psi(y_1)\, \mathrm{d}y_1 \bigg| \\ &\qquad \leqslant \mathcal{C}\|\nabla\psi\|_\infty \int_{Y_1} \bigg| \int_\Omega \big(\phi_\delta(x) - \chi_B(x)\big)\mathrm{d}\tilde{\boldsymbol{\mu}}(x) \bigg|\, \mathrm{d}y_1 \leqslant 2\mathcal{C}\|\nabla\psi\|_\infty \|\tilde{\boldsymbol{\mu}}\|(A_\delta\backslash C_\delta).\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we get

$$\lim_{\delta\to 0^+}\int_{Y_1}\Big(\int_\Omega \phi_\delta(x)\,\mathrm{d}\tilde{\boldsymbol{\mu}}(x)\Big)(y_1)\,\mathrm{div}\,\psi(y_1)\,\mathrm{d}y_1=\int_{Y_1}\tilde{\boldsymbol{\mu}}(B)(y_1)\,\mathrm{div}\,\psi(y_1)\,\mathrm{d}y_1. \tag{3.17}$$

Similarly,

$$\lim_{\delta\to 0^+}\int_{\Omega\times Y_1}\phi_\delta(x)\,\psi(y_1)\cdot\mathrm{d}\tau(x,y_1)=\int_{Y_1}\psi(y_1)\cdot\mathrm{d}\tau_B(y_1). \tag{3.18}$$

Considering (3.12) with ϕ replaced by ϕ_δ, passing to the limit as $\delta\to 0^+$ taking into account (3.17) and (3.18), we deduce (3.13). In particular, for all $B\in\mathcal{B}(\Omega)$, $E\in\mathcal{B}(Y_1)$,

$$D_{y_1}(\tilde{\boldsymbol{\mu}}(B))(E)=\tau_B(E)=\tau(B\times E). \tag{3.19}$$

To conclude that $\tilde{\boldsymbol{\mu}}\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ it remains to prove that $\tilde{\boldsymbol{\mu}}$ has finite total variation. As in (3.9), by (3.19) we get

$$\sup\left\{\sum_{i=1}^\infty\|D_{y_1}(\tilde{\boldsymbol{\mu}}(B_i))\|(Y_1)\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega)\text{ is a partition of }\Omega\right\}\leqslant\|\tau\|(\Omega\times Y_1).$$

Consequently,

$$\begin{aligned}\|\tilde{\boldsymbol{\mu}}\|(\Omega)&=\sup\left\{\sum_{i=1}^\infty\|\tilde{\boldsymbol{\mu}}(B_i)\|_{BV_\#(Y_1)}\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega)\text{ is a partition of }\Omega\right\}\\&\leqslant\mathcal{C}\sup\left\{\sum_{i=1}^\infty\big(\|\tilde{\boldsymbol{\mu}}(B_i)\|_{L^{1\star}_\#(Y_1)}+\|D_{y_1}(\tilde{\boldsymbol{\mu}}(B_i))\|(Y_1)\big)\colon\ \{B_i\}_{i\in\mathbb{N}}\subset\mathcal{B}(\Omega)\text{ is a partition of }\Omega\right\}\\&\leqslant\mathcal{C}\left(\sup_{j\in\mathbb{N}}\|\tau_j\|(\Omega\times Y_1)+\|\tau\|(\Omega\times Y_1)\right)<\infty,\end{aligned}$$

where we have also used (3.9). Thus, $\tilde{\boldsymbol{\mu}}\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ and τ is the measure associated with $D_{y_1}\tilde{\boldsymbol{\mu}}$, which shows that $\tau\in\mathcal{E}_1$, and this concludes the proof that $\mathcal{E}_1$ is a weakly-$\star$ closed subspace of $\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$.

Substep 2d. Similarly to Substep 2c, one can show that the space $\mathcal{E}_2$ of all measures $\tau\in\mathcal{M}_{y\#}\big(\Omega\times Y_1\times Y_2;\mathbb{R}^N\big)$ for which there exists a measure $\boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$ such that τ is the measure associated with $D_{y_2}\boldsymbol{\mu}_2$ (i.e., for all $B\in\mathcal{B}(\Omega\times Y_1)$, $E\in\mathcal{B}(Y_2)$, $\tau(B\times E)=D_{y_2}(\boldsymbol{\mu}_2(B))(E)$) is weakly-$\star$ closed.

Substep 2e. We are now in position to prove that $\mathcal{E}$ is a weakly-$\star$ closed vectorial subspace of $\mathcal{M}_{y\#}\big(\Omega\times Y_1\times Y_2;\mathbb{R}^N\big)$. As before, it suffices to show that $\mathcal{E}$ is sequentially weakly-$\star$ closed. Let $\{\tau_j\}_{j\in\mathbb{N}}\subset\mathcal{E}$ be a sequence such that $\tau_j\overset{\star}{\rightharpoonup}\tau$ weakly-$\star$ in $\mathcal{M}_{y\#}\big(\Omega\times Y_1\times Y_2;\mathbb{R}^N\big)$ as $j\to\infty$. We want to prove that $\tau\in\mathcal{E}$.

For each $j\in\mathbb{N}$ write $\tau_j=\tau_j^{(1)}\otimes\mathcal{L}^N_{y_2}+\tau_j^{(2)}$, where $\tau_j^{(1)}\in\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$ and $\tau_j^{(2)}\in\mathcal{M}_{y\#}\big(\Omega\times Y_1\times Y_2;\mathbb{R}^N\big)$ are the measures associated with $D_{y_1}\boldsymbol{\mu}_j^{(1)}$ and $D_{y_2}\boldsymbol{\mu}_j^{(2)}$ for some $\boldsymbol{\mu}_j^{(1)}\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ and $\boldsymbol{\mu}_j^{(2)}\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$, respectively.

Let $\vartheta\in C_0\big(\Omega;C_\#\big(Y_1;\mathbb{R}^N\big)\big)$ be such that $\|\vartheta\|_\infty\leqslant 1$. Then ϑ can be seen as an element of $C_0\big(\Omega;C_\#\big(Y_1\times Y_2;\mathbb{R}^N\big)\big)$, still with norm less than or equal to 1. Moreover,

$$\begin{aligned}\langle\tau_j,\vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))}&=\int_{\Omega\times Y_1\times Y_2}\vartheta(x,y_1)\,\mathrm{d}\tau_j(x,y_1,y_2)\\&=\int_{\Omega\times Y_1\times Y_2}\vartheta(x,y_1)\,\mathrm{d}\tau_j^{(1)}(x,y_1)\mathrm{d}y_2+\int_{\Omega\times Y_1\times Y_2}\vartheta(x,y_1)\,\mathrm{d}\tau_j^{(2)}(x,y_1,y_2)\\&=\int_{\Omega\times Y_1}\vartheta(x,y_1)\,\mathrm{d}\tau_j^{(1)}(x,y_1)=\langle\tau_j^{(1)},\vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1;\mathbb{R}^N))},\end{aligned}$$

since $\int_{\Omega\times Y_1\times Y_2}\vartheta(x,y_1)\,\mathrm{d}\tau_j^{(2)}(x,y_1,y_2)=0$ by (2.21) (with $i=2$ and $\psi\equiv 1$). This implies that

$$\begin{aligned}
&\|\tau_j\|(\Omega\times Y_1\times Y_2)\\
&\quad=\sup\Big\{\langle\tau_j,\varphi\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))}\colon\ \varphi\in C_0\big(\Omega;C_\#\big(Y_1\times Y_2;\mathbb{R}^N\big)\big),\ \ \|\varphi\|_\infty\leqslant 1\Big\}\\
&\quad\geqslant\ \sup\Big\{\langle\tau_j,\vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1\times Y_2;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1\times Y_2;\mathbb{R}^N))}\colon\ \vartheta\in C_0\big(\Omega;C_\#\big(Y_1;\mathbb{R}^N\big)\big),\ \ \|\vartheta\|_\infty\leqslant 1\Big\}\\
&\quad=\sup\Big\{\langle\tau_j^{(1)},\vartheta\rangle_{\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N),C_0(\Omega;C_\#(Y_1;\mathbb{R}^N))}\colon\ \vartheta\in C_0\big(\Omega;C_\#\big(Y_1;\mathbb{R}^N\big)\big),\ \ \|\vartheta\|_\infty\leqslant 1\Big\}\\
&\quad=\|\tau_j^{(1)}\|(\Omega\times Y_1).
\end{aligned}$$

Hence $\{\tau_j^{(1)}\}_{j\in\mathbb{N}}$ is a bounded sequence in $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$, and so there exist a subsequence $\{\tau_{j_k}^{(1)}\}_{k\in\mathbb{N}}$ of $\{\tau_j^{(1)}\}_{j\in\mathbb{N}}$ and a measure $\tau_1\in\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$ such that $\tau_{j_k}^{(1)}\overset{\star}{\rightharpoonup}\tau_1$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1;\mathbb{R}^N)$. Since $\tau_{j_k}^{(1)}\in\mathcal{E}_1$ for all $k\in\mathbb{N}$, and $\mathcal{E}_1$ is a weakly-$\star$ closed subspace of $\mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^N\big)$ (see Substep 2c), we conclude that $\tau_1\in\mathcal{E}_1$. Let $\boldsymbol{\mu}_1\in\mathcal{M}_\star(\Omega;BV_\#(Y_1))$ be such that τ_1 is the measure associated with $D_{y_1}\boldsymbol{\mu}_1$.

Next, write $\tau_{j_k}^{(2)}=\tau_{j_k}-\tau_{j_k}^{(1)}\otimes\mathcal{L}_{y_2}^N$, so that $\tau_{j_k}^{(2)}\overset{\star}{\rightharpoonup}\tau-\tau_1\otimes\mathcal{L}_{y_2}^N=:\tau_2$ weakly-$\star$ in $\mathcal{M}_{y\#}\big(\Omega\times Y_1\times Y_2;\mathbb{R}^N\big)$. Since $\tau_{j_k}^{(2)}\in\mathcal{E}_2$ for all $k\in\mathbb{N}$, by Substep 2c we conclude that $\tau_2\in\mathcal{E}_2$. Thus we can find $\boldsymbol{\mu}_2\in\mathcal{M}_\star(\Omega\times Y_1;BV_\#(Y_2))$ such that τ_2 is the measure associated with $D_{y_2}\boldsymbol{\mu}_2$. Finally,

$$\tau=\tau_1\otimes\mathcal{L}_{y_2}^N+\tau_2\in\mathcal{E},$$

and this concludes the proof of the claim. □

Proof of Theorem 1.10. *a)* We claim that for all $\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^d\big)\big)$ we have

$$\lim_{\varepsilon\to0^+}\int_\Omega\varphi\left(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\right)\cdot u_\varepsilon(x)\,\mathrm{d}x=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\cdot u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{3.20}$$

If $\varphi\in C(\Omega;C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^d))$, then by Riemann-Lebesgue's Lemma

$$\varphi\left(\,\cdot\,,\frac{\cdot}{\varrho_1(\varepsilon)},\cdots,\frac{\cdot}{\varrho_n(\varepsilon)}\right)\overset{\star}{\rightharpoonup}\int_{Y_1\times\cdots\times Y_n}\varphi(\,\cdot\,,y_1,\cdots,y_n)\,\mathrm{d}y_1\cdots\mathrm{d}y_n \tag{3.21}$$

weakly-$\star$ in $L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, from which (3.20) follows since by hypothesis $u_\varepsilon\to u$ (strongly) in $L^1\big(\Omega;\mathbb{R}^d\big)$, and since if $\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^d\big)\big)$ then (3.21) holds weakly-$\star$ in $L^\infty(\Omega;\mathbb{R}^d)$.

b) By reasoning component by component, we may assume without loss of generality that $d=1$. Since $\{Du_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $\mathcal{M}(\Omega;\mathbb{R}^N)$, by Theorem 3.2, and up to a subsequence (not relabeled),

$$Du_\varepsilon\xrightarrow[\varepsilon]{(n+1)\text{-}sc}\mu_0, \tag{3.22}$$

for some $\mu_0\in\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^N)$.

We claim that if $\varphi\in C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1\times\cdots\times Y_n;\mathbb{R}^N\big)\big)$ is such that $\operatorname{div}_{y_n}\varphi=0$ and, if $n\geqslant 2$, for all $k\in\{1,\cdots,n-1\}$, $x\in\Omega$, $y_i\in Y_i$, $i\in\{1,\cdots,n\}$,

$$\int_{Y_{k+1}\times\cdots\times Y_n}\operatorname{div}_{y_k}\varphi(x,y_1,\cdots,y_n)\,\mathrm{d}y_{k+1}\cdots\mathrm{d}y_n=0, \tag{3.23}$$

then we have

$$\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\cdot\mathrm{d}\mu_0(x,y_1,\cdots,y_n)=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\cdot\mathrm{d}Du(x)\mathrm{d}y_1\cdots\mathrm{d}y_n. \tag{3.24}$$

If the claim holds, then by Lemma 3.5 there exist n measures $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i))$, $i \in \{1, \cdots, n\}$, such that

$$\mu_0 - Du_{\lfloor \Omega} \otimes \mathcal{L}^{nN}_{y_1,\cdots,y_n} = \sum_{i=1}^{n-1} \lambda_i \otimes \mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n} + \lambda_n,$$

where each $\lambda_i \in \mathcal{M}_{y\#}\big(\Omega \times Y_1 \times \cdots \times Y_i; \mathbb{R}^N\big)$ is the measure associated with $D_{y_i}\boldsymbol{\mu}_i$. This will establish statement *b)*.

Let us prove (3.24). Let $\varphi \in C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1 \times \cdots \times Y_n; \mathbb{R}^N\big)\big)$ be such that $\operatorname{div}_{y_n} \varphi = 0$. Using the fact that $u_\varepsilon \in BV(\Omega)$ we obtain

$$\begin{aligned}
&\int_\Omega \varphi\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) \cdot \mathrm{d}Du_\varepsilon(x) \\
&\quad = -\int_\Omega (\operatorname{div}_x \varphi)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x - \sum_{k=1}^{n-1} \frac{1}{\varrho_k(\varepsilon)} \int_\Omega (\operatorname{div}_{y_k} \varphi)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x.
\end{aligned} \tag{3.25}$$

By *a)* and Fubini's Theorem, we deduce that

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \int_\Omega (\operatorname{div}_x \varphi)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x &= \int_{\Omega \times Y_1 \times \cdots \times Y_n} (\operatorname{div}_x \varphi)(x, y_1, \cdots, y_n) u(x)\,\mathrm{d}x\mathrm{d}y_1 \cdots \mathrm{d}y_n \\
&= -\int_{\Omega \times Y_1 \times \cdots \times Y_n} \varphi(x, y_1, \cdots, y_n) \cdot \mathrm{d}Du(x)\mathrm{d}y_1 \cdots \mathrm{d}y_n.
\end{aligned} \tag{3.26}$$

We claim that, if in addition φ is such that for $n \geqslant 2$ and for all $k \in \{1, \cdots, n-1\}$,

$$\int_{Y_{k+1} \times \cdots \times Y_n} \operatorname{div}_{y_k} \varphi(x, y_1, \cdots, y_n)\,\mathrm{d}y_{k+1} \cdots \mathrm{d}y_n = 0,$$

then for all $k \in \{1, \cdots, n-1\}$,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varrho_k(\varepsilon)} \int_\Omega (\operatorname{div}_{y_k} \varphi)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\,\mathrm{d}x = 0. \tag{3.27}$$

Assume that (3.27) holds. Then passing (3.25) to the limit as $\varepsilon \to 0^+$, from (3.22), (3.26) and (3.27) we get (3.24), which concludes the proof of Theorem 1.10.

It remains to establish (3.27). The main ideas to prove (3.27) are those of [2, Thm. 3.3, Cor. 3.4], which we will include here for the sake of completeness. Let $n \geqslant 2$, fix $k \in \{1, \cdots, n-1\}$ and define $\vartheta_k := \operatorname{div}_{y_k} \varphi$. By (3.23), we can write

$$\vartheta_k(x, y_1, \cdots, y_n) = \sum_{i=k+1}^{n} \vartheta_i^{(k)}(x, y_1, \cdots, y_i),$$

where the functions $\vartheta_i^{(k)}$ are given by the inductive formulae

$$\begin{cases}
\vartheta_n^{(k)} := \vartheta_k - \displaystyle\int_{Y_n} \vartheta_k\,\mathrm{d}y_n, \\
\vartheta_i^{(k)} := \displaystyle\int_{Y_{i+1} \times \cdots \times Y_n} \vartheta_k\,\mathrm{d}y_{i+1} \cdots \mathrm{d}y_n - \int_{Y_i \times \cdots \times Y_n} \vartheta_k\,\mathrm{d}y_i \cdots \mathrm{d}y_n \quad \text{if } i \in \{k+1, \cdots, n-1\}.
\end{cases}$$

By construction, for each $i \in \{k+1, \cdots, n\}$ one has

$$\vartheta_i^{(k)} \in \mathcal{O}_i := \left\{\vartheta \in C_c^\infty(\Omega; C_\#^\infty(Y_1 \times \cdots \times Y_i)) : \int_{Y_i} \vartheta(x, y_1, \cdots, y_i)\,\mathrm{d}y_i = 0\right\}.$$

Moreover, for $n \geqslant 2$ and $k \in \{1, \cdots, n-1\}$,

$$\frac{1}{\varrho_k(\varepsilon)} \int_\Omega (\operatorname{div}_{y_k} \varphi)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\, \mathrm{d}x = \frac{1}{\varrho_k(\varepsilon)} \int_\Omega \vartheta_k \left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_n(\varepsilon)}\right) u_\varepsilon(x)\, \mathrm{d}x$$
$$= \sum_{i=k+1}^{n} \frac{\varrho_i(\varepsilon)}{\varrho_k(\varepsilon)} \frac{1}{\varrho_i(\varepsilon)} \int_\Omega \vartheta_i^{(k)} \left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) u_\varepsilon(x)\, \mathrm{d}x.$$

Hence, using the boundedness of $\{u_\varepsilon\}_{\varepsilon>0}$ in $BV(\Omega)$ and (1.1), to prove (3.27) it suffices to show that for each $i \in \{k+1, \cdots, n\}$ there exists a constant $\mathcal{C}_i = \mathcal{C}\big(\vartheta_i^{(k)}\big)$, independent of ε, such that

$$\left| \frac{1}{\varrho_i(\varepsilon)} \int_\Omega \vartheta_i^{(k)} \left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) u_\varepsilon(x)\, \mathrm{d}x \right| \leqslant \mathcal{C}_i \|u_\varepsilon\|_{BV(\Omega)}. \tag{3.28}$$

Fix $i \in \{k+1, \cdots, n\}$. To simplify the notation, in the remaining part of the proof we will drop the dependence on i and k of the function $\vartheta_i^{(k)}$, so that $\vartheta_i^{(k)} = \vartheta \in \mathcal{O}_i$.

As shown in [2, Lemma 3.6], there exists a linear operator $S : \vartheta \in \mathcal{O}_i \mapsto S\vartheta \in \mathcal{O}_i^N$ such that $\operatorname{div}_{y_i}(S\vartheta) = \vartheta$ and $\|S\vartheta\|_\infty \leqslant \mathcal{C}\|\vartheta\|_\infty$, for some constant $\mathcal{C}$. Then we can write

$$\begin{aligned} &\frac{1}{\varrho_i(\varepsilon)} \vartheta\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) \\ &\quad = \operatorname{div}\left((S\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right)\right) - \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right) \frac{1}{\varrho_i(\varepsilon)} (T_\varepsilon \vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right), \end{aligned} \tag{3.29}$$

where T_ε is the linear operator given by

$$T_\varepsilon \vartheta := \varrho_{i-1}(\varepsilon) \operatorname{div}_x(S\vartheta) + \sum_{j=1}^{i-1} \frac{\varrho_{i-1}(\varepsilon)}{\varrho_j(\varepsilon)} \operatorname{div}_{y_j}(S\vartheta).$$

Note that $T_\varepsilon \vartheta \in \mathcal{O}_i$. Indeed, $T_\varepsilon \vartheta \in \mathcal{O}_i$ inherits the same regularity of $S\vartheta$, and

$$\int_{Y_i} \operatorname{div}_x(S\vartheta)\, \mathrm{d}y_i = \operatorname{div}_x \int_{Y_i} S\vartheta\, \mathrm{d}y_i = 0, \quad \int_{Y_i} \operatorname{div}_{y_j}(S\vartheta)\, \mathrm{d}y_i = \operatorname{div}_{y_j} \int_{Y_i} S\vartheta\, \mathrm{d}y_i = 0,$$

for all $j \in \{1, \cdots, i-1\}$, and so $\int_{Y_i} T_\varepsilon \vartheta\, \mathrm{d}y_i = 0$.

Let us now analyze the right-hand side of (3.29). On the one hand we have that

$$\begin{aligned} \left| \int_\Omega \operatorname{div}\left((S\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right)\right) u_\varepsilon(x)\, \mathrm{d}x \right| &= \left| - \int_\Omega (S\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) \cdot \mathrm{d}Du_\varepsilon(x) \right| \\ &\leqslant \|S\vartheta\|_\infty \|Du_\varepsilon\|(\Omega) \leqslant \mathcal{C}\|\vartheta\|_\infty \|Du_\varepsilon\|(\Omega). \end{aligned}$$

On the other hand, the function $\frac{1}{\varrho_i(\varepsilon)}(T_\varepsilon\vartheta)\big(\cdot, \frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_i(\varepsilon)}\big)$ is of the same type as the function $\frac{1}{\varrho_i(\varepsilon)}\vartheta\big(\cdot, \frac{\cdot}{\varrho_1(\varepsilon)}, \cdots, \frac{\cdot}{\varrho_i(\varepsilon)}\big)$.

Applying (3.29) to $T_\varepsilon\vartheta$ instead of ϑ, and reiterating this process m times, with m as in (1.5), we get

$$\begin{aligned} &\frac{1}{\varrho_i(\varepsilon)} \vartheta\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right) \\ &\quad = \sum_{j=0}^{m-1} (-1)^j \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^j \operatorname{div}\left((S(T_\varepsilon)^j\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right)\right) \\ &\qquad + (-1)^m \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)} ((T_\varepsilon)^m\vartheta)\left(x, \frac{x}{\varrho_1(\varepsilon)}, \cdots, \frac{x}{\varrho_i(\varepsilon)}\right). \end{aligned} \tag{3.30}$$

Reasoning as above,

$$\begin{aligned} &\left|\int_\Omega (-1)^j \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^j \operatorname{div}\Big((S(T_\varepsilon)^j\vartheta)\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)\Big)\,u_\varepsilon(x)\,\mathrm{d}x\right| \\ &\qquad\leqslant \mathcal{C}\left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^j \|(T_\varepsilon)^j\vartheta\|_\infty\|Du_\varepsilon\|(\Omega) \leqslant \mathcal{C}\|(T_\varepsilon)^j\vartheta\|_\infty\|Du_\varepsilon\|(\Omega) \end{aligned} \tag{3.31}$$

for all $j\in\{0,\cdots,m-1\}$, while

$$\begin{aligned} &\left|\int_\Omega (-1)^m \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)}((T_\varepsilon)^m\vartheta)\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)\,u_\varepsilon(x)\,\mathrm{d}x\right| \\ &\qquad\leqslant \left(\frac{\varrho_i(\varepsilon)}{\varrho_{i-1}(\varepsilon)}\right)^m \frac{1}{\varrho_i(\varepsilon)}\|(T_\varepsilon)^m\vartheta\|_\infty\|u_\varepsilon\|_{L^1(\Omega)} \leqslant \mathcal{C}\|(T_\varepsilon)^m\vartheta\|_\infty\|u_\varepsilon\|_{L^1(\Omega)}, \end{aligned} \tag{3.32}$$

where we used (1.1) and (1.5).

Finally using the definition of the operator T_ε, we deduce that for all $j\in\{0,\cdots,m\}$,

$$\sup_{\varepsilon>0}\|(T_\varepsilon)^j\vartheta\|_\infty \leqslant \mathcal{C}\big(\|\mathcal{S}\vartheta\|_{C^j(\Omega;C^j_\#(Y_1\times\cdots\times Y_i;\mathbb{R}^N))} + \|\vartheta\|_{C^j(\Omega;C^j_\#(Y_1\times\cdots\times Y_i))}\big), \tag{3.33}$$

so that (3.28) follows from (3.30)–(3.33). □

The proof of the converse of Theorem 1.10, that is, of Proposition 1.11, is hinged on a version for $BV_\#(Y;\mathbb{R}^d)$-valued measures of the classical Meyers–Serrin's (density) Theorem. We will need some auxiliary results.

For $0<\varepsilon<1/2$, let $\rho_\varepsilon\in C_c^\infty(\mathbb{R}^N)$ and $\eta_\varepsilon\in C^\infty_\#(Y)$ be functions satisfying (2.2) and (2.3), respectively. Fix $i\in\{1,\cdots,n\}$, let $\boldsymbol{\mu}\in\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i,\mathbb{R}^d\big)\big)$ and denote by λ the measure associated with $D_{y_i}\boldsymbol{\mu}$. We define

$$\begin{aligned} &\psi^\varepsilon_{\boldsymbol{\mu}}(x,y_1,\cdots,y_i) \\ &\quad:= \int_{Y_i}\bigg(\int_{\Omega\times Y_1\times\cdots\times Y_{i-1}} \rho_\varepsilon(x-x')\prod_{\kappa=1}^{i-1}\eta_\varepsilon(y_\kappa-y'_\kappa)\,\mathrm{d}\boldsymbol{\mu}(x',y'_1,\cdots,y'_{i-1})\bigg)(y'_i)\,\eta_\varepsilon(y_i-y'_i)\,\mathrm{d}y'_i, \end{aligned} \tag{3.34}$$

for $x\in\Omega_\varepsilon:=\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)>\varepsilon\}$ and $y_1,...,y_i\in\mathbb{R}^N$.

Lemma 3.6. *The function $\psi^\varepsilon_{\boldsymbol{\mu}}$ defined in (3.34) belongs to $C^\infty\big(\Omega_\varepsilon;C^\infty_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$.*

Proof. The proof is similar to the usual mollification case (see, for example, [4]). It is done by induction on the order of the derivative, and the key ingredients are the difference quotients and the Lebesgue Dominated Convergence Theorem, taking into account the regularity of ρ_ε and η_ε. □

Lemma 3.7. *Let $\Omega'\subset\subset\Omega$ be an open, bounded set, and let $\psi^\varepsilon_{\boldsymbol{\mu}}$ be the function defined in (3.34). Then $\psi^\varepsilon_{\boldsymbol{\mu}}\mathcal{L}^{(i+1)N}\lfloor_{\Omega'\times Y_1\times\cdots\times Y_i}\overset{\star}{\rightharpoonup}_\varepsilon\ \boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_i}$ weakly-$\star$ in $\mathcal{M}_{\#y}\big(\Omega'\times Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)$, that is, for all $\varphi\in C_0\big(\Omega';C_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$ we have*

$$\begin{aligned} &\lim_{\varepsilon\to0^+}\int_{\Omega'\times Y_1\times\cdots\times Y_i}\varphi(x,y_1,\cdots,y_i)\cdot\psi^\varepsilon_{\boldsymbol{\mu}}(x,y_1,\cdots,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i \\ &\qquad=\int_{\Omega'\times Y_1\times\cdots\times Y_i}\varphi(x,y_1,\cdots,y_i)\cdot\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i, \end{aligned}$$

where the last integral is to be understood in the sense of Subsection 2.4.

Lemma 3.8. *Let $\Omega' \subset\subset \Omega$ be an open, bounded set, and let $\psi^\varepsilon_{\boldsymbol{\mu}}$ be the function defined in (3.34). Then $\nabla_{y_i}\psi^\varepsilon_{\boldsymbol{\mu}}\mathcal{L}^{(i+1)N}\lfloor_{\Omega'\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup} \lambda$ weakly-$\star$ in $\mathcal{M}_{\#y}\big(\Omega'\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big)$ as $\varepsilon\to 0^+$, and*

$$\lim_{\varepsilon\to 0^+}\int_{\Omega'\times Y_1\times\cdots\times Y_i} |\nabla_{y_i}\psi^\varepsilon_{\boldsymbol{\mu}}(x,y_1,\cdots,y_i)|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i = \|\lambda\|(\Omega'\times Y_1\times\cdots\times Y_i).$$

Proof. Fix $x\in\Omega_\varepsilon$ and $y_1,\cdots,y_i\in\mathbb{R}^N$. Set $\widetilde{Y} := Y_1\times\cdots\times Y_{i-1}$, $Y := Y_i$, $\tilde{y} := (y_1,\cdots,y_{i-1})$, $y := y_i$, and $\bar\eta_\varepsilon(\tilde y) := \prod_{\kappa=1}^{i-1}\eta_\varepsilon(y_\kappa)$. Notice that due to (2.3), for all $\tilde y'\in\mathbb{R}^{(i-1)N}$, $y'\in\mathbb{R}^N$, we have

$$\int_{\widetilde Y}\bar\eta_\varepsilon(\tilde y-\tilde y')\,\mathrm{d}\tilde y = 1, \quad \int_Y \eta_\varepsilon(y-y')\,\mathrm{d}y = 1. \tag{3.35}$$

Using (2.21) and (3.35), we get

$$\begin{aligned}
\nabla_y\psi^\varepsilon_{\boldsymbol{\mu}}(x,\tilde y,y) &= \int_Y\bigg(\int_{\Omega\times\widetilde Y}\rho_\varepsilon(x-x')\bar\eta_\varepsilon(\tilde y-\tilde y')\,\mathrm{d}\boldsymbol{\mu}(x',\tilde y')\bigg)(y')\otimes\nabla_y\eta_\varepsilon(y-y')\,\mathrm{d}y'\\
&= -\int_Y\bigg(\int_{\Omega\times\widetilde Y}\rho_\varepsilon(x-x')\bar\eta_\varepsilon(\tilde y-\tilde y')\,\mathrm{d}\boldsymbol{\mu}(x',\tilde y')\bigg)(y')\otimes\nabla_{y'}\eta_\varepsilon(y-y')\,\mathrm{d}y'\\
&= \int_{\Omega\times\widetilde Y\times Y}\rho_\varepsilon(x-x')\bar\eta_\varepsilon(\tilde y-\tilde y')\eta_\varepsilon(y-y')\,\mathrm{d}\lambda(x',\tilde y',y').
\end{aligned}$$

Hence $\nabla_{y_i}\psi^\varepsilon_{\boldsymbol{\mu}} = \varphi_\varepsilon * \lambda$ in $\Omega_\varepsilon\times\mathbb{R}^{iN}$, where $\varphi_\varepsilon(x,y_1,\cdots,y_i) := \rho_\varepsilon(x)\prod_{\kappa=1}^{i}\eta_\varepsilon(y_i)$, and well known results on mollification of measures yield the desired convergences (see, for example, [4, Thm. 2.2]). □

Remark 3.9. *Let $\phi\in C_c(\Omega)$ and $\boldsymbol{\mu}\in\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$ be given, and define $\boldsymbol{\nu}(B) := \int_B\phi(x)\,\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})$ for all $B\in\mathcal{B}(\Omega\times Y_1\times\cdots\times Y_{i-1})$. By Remark 2.8, $\boldsymbol{\nu}\in\mathcal{M}\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$. Note that $\operatorname{supp}\boldsymbol{\nu}\subset\operatorname{supp}\phi\times\mathbb{R}^{(i-1)N}$.*

Considering first functions $\tilde\varphi$, φ of the form $\tilde\varphi(x,y_1,\cdots,y_i) = \tilde\vartheta(x,y_1,\cdots,y_{i-1})\tilde\psi(y_i)$ and $\varphi(x,y_1,\cdots,y_i) = \vartheta(x,y_1,\cdots,y_{i-1})\psi(y_i)$ with $\tilde\vartheta,\vartheta\in C_0(\Omega;C_\#(Y_1\times\cdots\times Y_{i-1}))$, $\tilde\psi\in C_\#(Y_i)$ and $\psi\in C^1_\#(Y_i)$, using (2.21), arguing component by component, and finally considering a density argument, we conclude that $\boldsymbol{\nu}\in\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, with $\tau := \phi\,\mathrm{d}\lambda$ being the measure associated with $D_{y_i}\boldsymbol{\nu}$, so that

$$\begin{aligned}
&\int_{\Omega\times Y_1\times\cdots\times Y_i}\tilde\varphi(x,y_1,\cdots,y_i)\cdot\mathrm{d}\boldsymbol{\nu}(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i\\
&\qquad= \int_{\Omega\times Y_1\times\cdots\times Y_i}\big(\tilde\varphi(x,y_1,\cdots,y_i)\phi(x)\big)\cdot\mathrm{d}\boldsymbol{\mu}(x,y_1,\cdots,y_{i-1})\mathrm{d}y_i, &(3.36)\\
&\int_{\Omega\times Y_1\times\cdots\times Y_i}\varphi(x,y_1,\cdots,y_i):\mathrm{d}\tau(x,y_1,\cdots,y_i)\\
&\qquad= \int_{\Omega\times Y_1\times\cdots\times Y_i}\big(\varphi(x,y_1,\cdots,y_i)\phi(x)\big):\mathrm{d}\lambda(x,y_1,\cdots,y_i), &(3.37)
\end{aligned}$$

for all $\tilde\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$ and $\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big)\big)$.

Notice that the domain of the function $\psi^\varepsilon_{\boldsymbol{\nu}}$ given by (3.34), is $\Omega_\varepsilon\times\mathbb{R}^{iN}$. In order to have it defined on the whole $\Omega\times\mathbb{R}^{iN}$, we extend $\boldsymbol{\nu}$ by zero. Precisely, for $B\in\mathcal{B}(\mathbb{R}^N\times Y_1\times\cdots\times Y_{i-1})$, let $\bar{\boldsymbol{\nu}}(B) := \boldsymbol{\nu}(B\cap\Omega\times Y_1\times\cdots\times Y_{i-1})$. Then $\bar{\boldsymbol{\nu}}\in\mathcal{M}_\star\big(\mathbb{R}^N\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$, and $\operatorname{supp}\bar{\boldsymbol{\nu}} = \operatorname{supp}\boldsymbol{\nu}$.

In this setting, the function $\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}$ defined in (3.34) (with $\boldsymbol{\mu}$ and Ω replaced by $\bar{\boldsymbol{\nu}}$ and $\mathbb{R}^N$, respectively) belongs to $C^\infty_c\big(\mathbb{R}^N;C^\infty_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$. Furthermore,

$$\operatorname{supp}\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}\subset\Omega\times\mathbb{R}^{iN}\quad\textit{for all } \varepsilon>0 \textit{ small enough}, \tag{3.38}$$

since for all $y_1,...,y_i \in \mathbb{R}^N$, $\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}(\cdot, y_1, \cdots, y_i) = 0$ *in* $\{x \in \mathbb{R}^N\colon\, \mathrm{dist}(x, \operatorname{supp}\phi) > \varepsilon\}$. *Arguing as in Lemmas 3.7 and 3.8, we conclude that*

$$\begin{aligned}
&\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}\mathcal{L}^{(i+1)N}{}_{\lfloor \Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup}_\varepsilon \boldsymbol{\nu}\mathcal{L}^N{}_{\lfloor Y_i} \text{ weakly-}\star \text{ in } \mathcal{M}_{\#y}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^d\big),\\
&\nabla_{y_i}\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}\mathcal{L}^{(i+1)N}{}_{\lfloor \Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup}_\varepsilon \tau \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big),\\
&\lim_{\varepsilon\to 0^+}\int_{\Omega\times Y_1\times\cdots\times Y_i} |\nabla_{y_i}\psi^\varepsilon_{\bar{\boldsymbol{\nu}}}(x,y_1,\cdots,y_i)|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i = \|\tau\|(\Omega\times Y_1\times\cdots\times Y_i).
\end{aligned}\tag{3.39}$$

Proposition 3.10. *Fix* $i \in \{1,\cdots,n\}$, *and let* $\boldsymbol{\mu} \in \mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$. *Denote by* λ *the measure associated with* $D_{y_i}\boldsymbol{\mu}$. *Then there exists a sequence* $\{\psi_j\}_{j\in\mathbb{N}} \subset C^\infty\big(\Omega; C^\infty_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big) \cap L^1\big(\Omega\times Y_1\times\cdots\times Y_{i-1}; W^{1,1}\big(Y_i;\mathbb{R}^d\big)\big)$ *satisfying*

$$\begin{aligned}
&\psi_j\mathcal{L}^{(i+1)N}{}_{\lfloor \Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup}_j \boldsymbol{\mu}\mathcal{L}^N{}_{\lfloor Y_i} \text{ weakly-}\star \text{ in } \mathcal{M}_{\#y}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^d\big),\\
&\nabla_{y_i}\psi_j\mathcal{L}^{(i+1)N}{}_{\lfloor \Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup}_j \lambda \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big),\\
&\lim_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_i} |\nabla_{y_i}\psi_j(x,y_1,\cdots,y_i)|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i = \|\lambda\|(\Omega\times Y_1\times\cdots\times Y_i).
\end{aligned}\tag{3.40}$$

Proof. For simplicity we will assume that $i = 1$. The case $i \geqslant 2$ may be treated similarly.

Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_k \subset\subset \Omega_{k+1}$ and

$$\Omega = \bigcup_{k=1}^\infty \Omega_k,$$

and consider a smooth partition of unity subordinated to the open cover $\big\{\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big\}_{k\in\mathbb{N}}$ of Ω, where $\Omega_0 := \emptyset$, that is, a sequence $\{\phi_k\}_{k\in\mathbb{N}}$ such that

$$\phi_k \in C^\infty_c\big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}};[0,1]\big), \qquad \sum_{k=1}^\infty \phi_k(x) = 1 \text{ for all } x\in\Omega. \tag{3.41}$$

For each $k\in\mathbb{N}$, define $\boldsymbol{\nu}_k := \phi_k\,\mathrm{d}\boldsymbol{\mu}$ in the sense of Remark 3.9. In particular, $\operatorname{supp}\boldsymbol{\nu}_k \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big)$. Let $\{\tilde{\varphi}_j\}_{j\in\mathbb{N}}$ and $\{\varphi_j\}_{j\in\mathbb{N}}$ be dense in $C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^d\big)\big)$ and $C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$, respectively.

By induction and by (3.38) and (3.39) (with $\boldsymbol{\nu}$ replaced by $\boldsymbol{\nu}_k$), given $j\in\mathbb{N}$ we can find a sequence $\{\varepsilon_k^{(j)}\}_{k\in\mathbb{N}}$ of positive numbers converging to zero, with $\varepsilon_k^{(j)} < \varepsilon_k^{(j-1)}$ (and $\varepsilon_k^{(0)} := 1/2$), such that for all $k\in\mathbb{N}$ and $l\in\{1,\cdots,j\}$ we have

$$\operatorname{supp}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k} \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\,\big)\times\mathbb{R}^N, \tag{3.42}$$

$$\left|\int_{\Omega\times Y_1}\tilde{\varphi}_l(x,y_1)\cdot\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1}\tilde{\varphi}_l(x,y_1)\cdot\mathrm{d}\boldsymbol{\nu}_k(x)\mathrm{d}y_1\right| \leqslant \frac{1}{j\,2^k}, \tag{3.43}$$

$$\left|\int_{\Omega\times Y_1}\varphi_l(x,y_1):\nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 - \int_{\Omega\times Y_1}\varphi_l(x,y_1):\mathrm{d}\tau_k(x,y_1)\right| \leqslant \frac{1}{j\,2^k},$$

$$\left|\int_{\Omega\times Y_1}\left|\nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\right|\mathrm{d}x\mathrm{d}y_1 - \|\tau_k\|(\Omega\times Y_1)\right| \leqslant \frac{1}{2^k}, \tag{3.44}$$

where τ_k is the measure associated with $D_{y_1}\boldsymbol{\nu}_k$. For every open, bounded $\Omega' \subset\subset \Omega$ only finitely many $\Omega_{k+1}\backslash\overline{\Omega_{k-1}}$ cover Ω', and so, in view of (3.42), for each $j\in\mathbb{N}$ the function ψ_j defined by

$$\psi_j(x,y_1) := \sum_{k=1}^\infty \psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1) \tag{3.45}$$

belongs to $C^\infty\big(\Omega; C^\infty_\#\big(Y_1;\mathbb{R}^d\big)\big)$, with $\nabla_{y_1}\psi_j = \sum_{k=1}^\infty \nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}$. Moreover, $\psi_j \in L^1\big(\Omega; W^{1,1}\big(Y_1;\mathbb{R}^d\big)\big)$ and

$$\sup_{j\in\mathbb{N}} \|\psi_j\|_{L^1(\Omega\times Y_1;\mathbb{R}^d)} =: M < \infty, \quad \sup_{j\in\mathbb{N}} \|\nabla_{y_1}\psi_j\|_{L^1(\Omega\times Y_1;\mathbb{R}^{d\times N})} =: \tilde{M} < \infty. \tag{3.46}$$

Indeed, thanks to (3.42), and defining $\psi^{\varepsilon_0^{(j)}}_{\bar{\boldsymbol{\nu}}_0} := 0$, we obtain

$$\begin{aligned}\int_{\Omega\times Y_1} |\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 &\leqslant \sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} |\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \\ &\leqslant \sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} \Big|\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1) + \psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1) + \psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1,\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}\int_{\Omega\times Y_1} |\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 &\leqslant \sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} |\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \\ &\leqslant \sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} \Big|\nabla_{y_1}\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1) + \nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1) + \nabla_{y_1}\psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1.\end{aligned} \tag{3.48}$$

We have that

$$\begin{aligned}&\int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} \Big|\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1 \\ &\quad= \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} \bigg|\int_{Y_1}\bigg(\int_{\mathbb{R}^N} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\bar{\nu}_k(x')\bigg)(y_1')\,\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}y_1'\bigg|\,\mathrm{d}x\mathrm{d}y_1 \\ &\quad\leqslant \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})}\int_{Y_1}\bigg[\int_{Y_1}\bigg|\bigg(\int_{\Omega} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\bar{\nu}_k(x')\bigg)(y_1')\bigg|\,\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}y_1\bigg]\mathrm{d}y_1'\mathrm{d}x \\ &\quad= \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})}\bigg[\int_{Y_1}\bigg|\bigg(\int_{\Omega} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\bar{\nu}_k(x')\bigg)(y_1')\bigg|\,\mathrm{d}y_1'\bigg]\mathrm{d}x \\ &\quad\leqslant \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})}\int_{\Omega} \rho_{\varepsilon_k^{(j)}}(x-x')\,\mathrm{d}\|\bar{\nu}_k\|(x')\,\mathrm{d}x \leqslant \|\bar{\nu}_k\|(\Omega_{k+1}\setminus\overline{\Omega}_{k-1}) \leqslant \|\boldsymbol{\mu}\|(\Omega_{k+1}\setminus\overline{\Omega}_{k-1}),\end{aligned}$$

where we used Fubini's Theorem, (2.3), Lemma 2.4 (see also Remark 2.8), (3.42) and (2.2) in this order. Thus,

$$\sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} \Big|\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1 \leqslant 2\|\boldsymbol{\mu}\|(\Omega). \tag{3.49}$$

Similarly,

$$\begin{aligned}&\sum_{k=1}^\infty \int_{(\Omega_k\setminus\overline{\Omega}_{k-1})\times Y_1} \Big|\psi^{\varepsilon_{k-1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k-1}}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1 \leqslant 2\|\boldsymbol{\mu}\|(\Omega), \\ &\sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_k)\times Y_1} \Big|\psi^{\varepsilon_{k+1}^{(j)}}_{\bar{\boldsymbol{\nu}}_{k+1}}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1 \leqslant 2\|\boldsymbol{\mu}\|(\Omega).\end{aligned} \tag{3.50}$$

From (3.47), (3.49) and (3.50), we deduce the first condition in (3.46). To prove the second condition in (3.46), we observe that from (3.42), (3.44), (3.41) and equality $\tau_k = \phi_k\,\mathrm{d}\lambda$ (see Remark 3.9), we have that

$$\begin{aligned}\sum_{k=1}^\infty \int_{(\Omega_{k+1}\setminus\overline{\Omega}_{k-1})\times Y_1} \Big|\nabla_{y_1}\psi^{\varepsilon_k^{(j)}}_{\bar{\boldsymbol{\nu}}_k}(x,y_1)\Big|\,\mathrm{d}x\mathrm{d}y_1 &\leqslant \sum_{k=1}^\infty \Big(\|\tau_k\|(\Omega_{k+1}\setminus\overline{\Omega}_{k-1}) + \frac{1}{2^k}\Big) \\ &\leqslant \sum_{k=1}^\infty \|\lambda\|(\Omega_{k+1}\setminus\overline{\Omega}_{k-1}) + 1 \leqslant 2\|\lambda\|(\Omega) + 1.\end{aligned}$$

Arguing as above, and taking into account (3.48),

$$\int_{\Omega\times Y_1} |\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant 6\|\lambda\|(\Omega\times Y_1)+3,$$

which concludes the proof of (3.46).

Now we prove the first convergence in (3.40). Let $\tilde\varphi \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^d\big)\big)$ be given, and fix $\eta>0$. There exists $m\in\mathbb{N}$ such that

$$\|\tilde\varphi-\tilde\varphi_m\|_{C_0(\Omega;C_\#(Y_1;\mathbb{R}^d))}\leqslant\eta.$$

Using (3.46), (3.45), (3.41), (3.42), (2.15) (see also Remark 2.8), (3.36) and (3.43), we obtain for any $j>m$

$$\begin{aligned}
&\left|\int_{\Omega\times Y_1}\tilde\varphi(x,y_1)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1-\int_{\Omega\times Y_1}\tilde\varphi(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\quad\leqslant\left|\int_{\Omega\times Y_1}\big(\tilde\varphi(x,y_1)-\tilde\varphi_m(x,y_1)\big)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1\right|\\
&\qquad+\left|\int_{\Omega\times Y_1}\tilde\varphi_m(x,y_1)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1-\int_{\Omega\times Y_1}\tilde\varphi_m(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\qquad+\left|\int_{\Omega\times Y_1}\big(\tilde\varphi_m(x,y_1)-\tilde\varphi(x,y_1)\big)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|\\
&\quad\leqslant\eta M+\sum_{k=1}^{\infty}\left|\int_{\Omega\times Y_1}\tilde\varphi_m(x,y_1)\cdot\psi_{\boldsymbol{\nu}_k}^{\varepsilon_k^{(j)}}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1-\int_{\Omega\times Y_1}\big(\tilde\varphi_m(x,y_1)\phi_k(x)\big)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1\right|+\eta\|\boldsymbol{\mu}\|(\Omega)\\
&\quad\leqslant\mathcal{C}\eta+\frac{1}{j}\cdot
\end{aligned}$$

Letting first $j\to\infty$ and then $\eta\to0^+$, we conclude that

$$\lim_{j\to\infty}\int_{\Omega\times Y_1}\tilde\varphi(x,y_1)\cdot\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1=\int_{\Omega\times Y_1}\tilde\varphi(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1.$$

Since $\tilde\varphi\in C_0\big(\Omega;C_\#\big(Y_1;\mathbb{R}^d\big)\big)$ was taken arbitrarily, this proves that

$$\psi_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1}\overset{\star}{\rightharpoonup}_j\boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_1}\ \text{ weakly-}\star\text{ in }\ \mathcal{M}\big(\Omega\times Y_1;\mathbb{R}^d\big).$$

The proof of the convergence

$$\nabla_{y_1}\psi_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1}\overset{\star}{\rightharpoonup}_j\lambda\ \text{ weakly-}\star\text{ in }\ \mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^{d\times N}\big). \tag{3.51}$$

is similar.

Using the lower semicontinuity of the total variation, convergence (3.51) yields

$$\liminf_{j\to\infty}\int_{\Omega\times Y_1}|\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1\geqslant\|\lambda\|(\Omega\times Y_1). \tag{3.52}$$

To prove the converse inequality, let $\varphi\in C_c\big(\Omega;C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$ be such that $\|\varphi\|_\infty\leqslant1$. Using similar arguments to those in the proof of Lemma 3.8, Fubini's Theorem, the symmetry of $\rho_{\varepsilon_k^{(j)}}$ and $\eta_{\varepsilon_k^{(j)}}$ with respect to the origin, (3.37) and the inclusion $\operatorname{supp}\varphi \subset \Omega_l \times \mathbb{R}^N$ for some $l \in \mathbb{N}$, we deduce that

$$\begin{aligned}
\int_{\Omega\times Y_1} \varphi(x,y_1) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 &= \sum_{k=1}^{l}\int_{\Omega\times Y_1} \varphi(x,y_1) : \nabla_{y_1}\psi_{\boldsymbol{\nu}_k}^{\varepsilon_k^{(j)}}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1\\
&= \sum_{k=1}^{l}\int_{\Omega\times Y_1} \varphi(x,y_1) : \left[\int_{\mathbb{R}^N\times Y_1} \rho_{\varepsilon_k^{(j)}}(x-x')\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}\bar\tau_k(x',y_1')\right]\mathrm{d}x\mathrm{d}y_1\\
&= \sum_{k=1}^{l}\int_{\mathbb{R}^N\times Y_1}\left[\int_{\Omega\times Y_1} \varphi(x,y_1)\rho_{\varepsilon_k^{(j)}}(x-x')\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}x\mathrm{d}y_1\right] : \mathrm{d}\bar\tau_k(x',y_1')\\
&= \sum_{i=1}^{l}\int_{\mathbb{R}^N\times Y_1}\big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}})*\varphi\big)(x',y_1') : \mathrm{d}\bar\tau_k(x',y_1') = \sum_{k=1}^{l}\int_{\Omega\times Y_1}\big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}})*\varphi\big)(x,y_1) : \mathrm{d}\tau_k(x,y_1)\\
&= \int_{\Omega\times Y_1}\sum_{k=1}^{l}\Big[\big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}})*\varphi\big)(x,y_1)\,\phi_k(x)\Big] : \mathrm{d}\lambda(x,y_1) = \int_{\Omega\times Y_1}\bar\varphi_j(x,y_1) : \mathrm{d}\lambda(x,y_1),
\end{aligned}\tag{3.53}$$

where $\bar\varphi_j(x,y_1) := \sum_{k=1}^{l}\Big[\big((\rho_{\varepsilon_k^{(j)}}\eta_{\varepsilon_k^{(j)}})*\varphi\big)(x,y_1)\,\phi_k(x)\Big]$. Notice that $\|\bar\varphi_j\|_\infty \leqslant 1$. Indeed, for all $x \in \Omega$, $y_1 \in Y_1$, we have

$$\begin{aligned}
|\bar\varphi_j(x,y_1)| &= \left|\sum_{k=1}^{l}\left(\int_{\Omega\times Y_1}\rho_{\varepsilon_k^{(j)}}(x-x')\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\varphi(x',y_1')\,\mathrm{d}x'\mathrm{d}y_1'\,\phi_k(x)\right)\right|\\
&\leqslant \|\varphi\|_\infty\sum_{k=1}^{l}\left(\int_{\Omega\times Y_1}\rho_{\varepsilon_k^{(j)}}(x-x')\eta_{\varepsilon_k^{(j)}}(y_1-y_1')\,\mathrm{d}x'\mathrm{d}y_1'\,\phi_k(x)\right) \leqslant \|\varphi\|_\infty\sum_{k=1}^{l}\phi_k(x) \leqslant 1,
\end{aligned}$$

where we used (2.2), (2.3), (3.41) and the condition $\|\varphi\|_\infty \leqslant 1$. Taking the supremum over $x \in \Omega$ and $y_1 \in Y_1$, we get $\|\bar\varphi_j\|_\infty \leqslant 1$. Moreover, $\bar\varphi_j \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$ and so, from (3.53), we deduce that

$$\int_{\Omega\times Y_1}\varphi(x,y_1) : \nabla_{y_1}\psi_j(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega\times Y_1). \tag{3.54}$$

By density, taking into account (3.46) and using Lebesgue Dominated Convergence Theorem, we conclude that (3.54) holds for all $\varphi \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$ with $\|\varphi\|_\infty \leqslant 1$. Hence

$$\int_{\Omega\times Y_1}|\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega\times Y_1),$$

which together with (3.52) yield

$$\lim_{j\to\infty}\int_{\Omega\times Y_1}|\nabla_{y_1}\psi_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 = \|\lambda\|(\Omega\times Y_1). \qquad \square$$

Corollary 3.11. *Fix $i \in \{1,\cdots,n\}$, and let $\boldsymbol{\mu} \in \mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$. Denote by λ the measure associated with $D_{y_i}\boldsymbol{\mu}$. Then there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$ satisfying (3.40).*

Proof. As in the previous proof, we may assume without loss of generality that $i = 1$. Let $\{\psi_k\}_{k\in\mathbb{N}} \subset C^\infty\big(\Omega; C_\#^\infty\big(Y_1;\mathbb{R}^d\big)\big)$ be the sequence given by Proposition 3.10. Let $\{\Omega_j\}_{j\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_j \subset\subset \Omega_{j+1}$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$, and let $\{\phi_j\}_{j\in\mathbb{N}}$ be a sequence of cut-off functions $\phi_j \in C_c^\infty(\Omega;[0,1])$ satisfying $\phi_j = 1$ in Ω_j and $\phi_j = 0$ in $\Omega\backslash\Omega_{j+1}$, for all $j \in \mathbb{N}$. Define

$$\tilde\psi_{j,k}(x,y_1) := \phi_j(x)\psi_k(x,y_1).$$

We have that $\tilde{\psi}_{j,k} \in C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1;\mathbb{R}^d\big)\big)$. Let $\tilde{\varphi} \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^d\big)\big)$ and $\varphi \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$ be given. Then for all $j\in\mathbb{N}$, $\tilde{\varphi}\phi_j \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^d\big)\big)$ and $\varphi\phi_j \in C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$. Using the first two convergences in (3.40), Remark 2.6 (iii) (see also Remark 2.8), the convergence $\lim_{j\to\infty}\|\boldsymbol{\mu}\|(\Omega\backslash\Omega_j) = 0$, the pointwise convergence $\phi_j \to 1$ in Ω, and Lebesgue Dominated Convergence Theorem, we get

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_{\Omega\times Y_1}\tilde{\varphi}(x,y_1)\cdot\tilde{\psi}_{j,k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 = \lim_{j\to\infty}\lim_{k\to\infty}\int_{\Omega\times Y_1}\big(\tilde{\varphi}(x,y_1)\phi_j(x)\big)\cdot\psi_k(x,y_1)\,\mathrm{d}x\mathrm{d}y_1$$
$$= lim_{j\to\infty}\int_{\Omega\times Y_1}\big(\tilde{\varphi}(x,y_1)\phi_j(x)\big)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1 = \int_{\Omega\times Y_1}\tilde{\varphi}(x,y_1)\cdot\mathrm{d}\boldsymbol{\mu}(x)\mathrm{d}y_1,$$

and

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_{\Omega\times Y_1}\varphi(x,y_1):\nabla_{y_1}\tilde{\psi}_{j,k}(x,y_1)\,\mathrm{d}x\mathrm{d}y_1 = \lim_{j\to\infty}\lim_{k\to\infty}\int_{\Omega\times Y_1}\varphi(x,y_1):(\phi_j(x)\nabla_{y_1}\psi_k(x,y_1))\,\mathrm{d}x\mathrm{d}y_1$$
$$= \lim_{j\to\infty}\int_{\Omega\times Y_1}(\varphi(x,y_1)\phi_j(x)):\mathrm{d}\lambda(x,y_1) = \int_{\Omega\times Y_1}\varphi(x,y_1):\mathrm{d}\lambda(x,y_1).$$

On the other hand,

$$\int_{\Omega\times Y_1}|\nabla_{y_1}\tilde{\psi}_{j,k}(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 = \int_{\Omega\times Y_1}|\phi_j(x)\nabla_{y_1}\psi_k(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \int_{\Omega\times Y_1}|\nabla_{y_1}\psi_k(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1,$$

and so

$$\limsup_{j\to\infty}\limsup_{k\to\infty}\int_{\Omega\times Y_1}|\nabla_{y_1}\tilde{\psi}_{j,k}(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega\times Y),$$

where we have used the third convergence in (3.40). Using a diagonal argument together with the separability of the spaces $C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^d\big)\big)$ and $C_0\big(\Omega; C_\#\big(Y_1;\mathbb{R}^{d\times N}\big)\big)$, we can find a subsequence $k_j \prec k$ such that $\tilde{\psi}_j := \tilde{\psi}_{j,k_j} \in C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1;\mathbb{R}^d\big)\big)$ and

$$\begin{aligned}
&\psi_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup}_j \boldsymbol{\mu}\mathcal{L}^N\lfloor_{Y_1} \text{ weakly-}\star\text{ in } \mathcal{M}\big(\Omega\times Y_1;\mathbb{R}^d\big),\\
&\nabla_{y_1}\tilde{\psi}_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup}_j \lambda \text{ weakly-}\star\text{ in } \mathcal{M}_{y\#}\big(\Omega\times Y_1;\mathbb{R}^{d\times N}\big),\\
&\limsup_{j\to\infty}\int_{\Omega\times Y_1}|\nabla_{y_1}\tilde{\psi}_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \leqslant \|\lambda\|(\Omega\times Y_1).
\end{aligned}$$

Finally, the convergence $\nabla_{y_1}\tilde{\psi}_j\mathcal{L}^{2N}\lfloor_{\Omega\times Y_1} \overset{\star}{\rightharpoonup}_j \lambda$ implies

$$\liminf_{j\to\infty}\int_{\Omega\times Y_1}|\nabla_{y_1}\tilde{\psi}_j(x,y_1)|\,\mathrm{d}x\mathrm{d}y_1 \geqslant \|\lambda\|(\Omega\times Y_1),$$

which concludes the proof. □

Corollary 3.12. *Assume that $\partial\Omega$ is Lipschitz. Let $u \in BV(\Omega;\mathbb{R}^d)$ and for each $i\in\{1,\cdots,n\}$, let $\boldsymbol{\mu}_i \in \mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1}; BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$. Then there exist sequences $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}} \subset C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$ satisfying*

$$\begin{aligned}
&u_j \overset{\star}{\rightharpoonup}_j u \text{ weakly-}\star\text{ in } BV(\Omega;\mathbb{R}^d), \quad \lim_{j\to\infty}\int_\Omega|\nabla u_j(x)|\,\mathrm{d}x = \|Du\|(\Omega),\\
&\Big(\nabla u_j + \sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}\Big)\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup}_j \lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n} \text{ weakly-}\star\text{ in } \mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big),\\
&\lim_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_j(x) + \sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n = \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n),
\end{aligned}$$
(3.55)

where $\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}$ is the measure defined in (1.6).

Proof. We will proceed in two steps.

Step 1. We first prove that there are sequences $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\Omega;\mathbb{R}^d) \cap W^{1,1}(\Omega;\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}} \subset C^\infty\big(\Omega; C^\infty_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$ satisfying (3.55).

Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_k \subset\subset \Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^\infty \Omega_k$, and consider a smooth partition of unity $\{\phi_k\}_{k\in\mathbb{N}}$ subordinated to the open cover $\big\{\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\big\}_{k\in\mathbb{N}}$ of Ω, where $\Omega_0 := \emptyset$, as in (3.41).

For each $k\in\mathbb{N}$ and $i\in\{1,\cdots,n\}$, define $\boldsymbol{\nu}_i^k := \phi_k\,\mathrm{d}\boldsymbol{\mu}_i$ in the sense of Remark 3.9, and let $\{\varphi_j^{(i)}\}_{j\in\mathbb{N}}$ be dense in $C_0\big(\Omega; C_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big)\big)$. Arguing as in the proof of Proposition 3.10 and as in [4, Thm 3.9], for each $j\in\mathbb{N}$ we can find a sequence $\{\varepsilon_k^{(j)}\}_{k\in\mathbb{N}}$ of positive numbers converging to zero, with $\varepsilon_k^{(j)} < \varepsilon_k^{(j-1)}$ (and $\varepsilon_k^{(0)} := 1/2$), such that for all $k\in\mathbb{N}$, $l\in\{1,\cdots,j\}$ and $i\in\{1,\cdots,n\}$ one has

$$
\begin{aligned}
&\operatorname{supp}\big(\rho_{\varepsilon_k^{(j)}} * (u\phi_k)\big) \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\,\big),\\
&\int_\Omega \big[|\rho_{\varepsilon_k^{(j)}} * (u\phi_k) - u\phi_k| + |\rho_{\varepsilon_k^{(j)}} * (u\otimes\nabla\phi_k) - u\otimes\nabla\phi_k|\big]\,\mathrm{d}x \leqslant \frac{1}{j\,2^k},\\
&\operatorname{supp}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}} \subset \big(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\,\big)\times\mathbb{R}^{iN},\\
&\bigg|\int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi_l^{(i)}(x,y_1,...,y_i) : \nabla_{y_i}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}(x,y_1,...,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i\\
&\qquad\qquad - \int_{\Omega\times Y_1\times\cdots\times Y_i} \varphi_l^{(i)}(x,y_1,...,y_i) : \mathrm{d}\tau_i^k(x,y_1,...,y_i)\bigg| \leqslant \frac{1}{j\,2^k},\\
&\bigg|\int_{\Omega\times Y_1\times\cdots\times Y_i} \Big|\nabla_{y_i}\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}(x,y_1,...,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_i - \|\tau_i^k\|(\Omega\times Y_1\times\cdots\times Y_i)\bigg| \leqslant \frac{1}{2^k},
\end{aligned}
\tag{3.56}
$$

where $\psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}$ were introduced in (3.34) and τ_i^k is the measure associated with $D_{y_i}\boldsymbol{\nu}_i^k$.

Similarly to the proof of Proposition 3.10 and as in [4, Thm 3.9], for each $j\in\mathbb{N}$ and $i\in\{1,\cdots,n\}$ the functions u_j and $\psi_j^{(i)}$ defined by

$$
u_j(x) := \sum_{k=1}^\infty ((\rho_{\varepsilon_k^{(j)}} * (u\phi_k))(x), \quad \psi_j^{(i)}(x,y_1,\cdots,y_i) := \sum_{k=1}^\infty \psi_{\bar{\boldsymbol{\nu}}_i^k}^{\varepsilon_k^{(j)}}(x,y_1,\cdots,y_i), \tag{3.57}
$$

belong to $C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$ and $C^\infty\big(\Omega; C^\infty_\#\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$, respectively, and are such that

$$
\begin{aligned}
&u_j \to_j u \ \text{ in } L^1(\Omega;\mathbb{R}^d), \quad \lim_{j\to\infty}\int_\Omega |\nabla u_j(x)|\,\mathrm{d}x = \|Du\|(\Omega),\\
&\sup_{j\in\mathbb{N}} \|\nabla_{y_i}\psi_j^{(i)}\|_{L^1(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N})} < \infty, \qquad (3.58)\\
&\nabla_{y_i}\psi_j^{(i)}\mathcal{L}^{(i+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_i} \overset{\star}{\rightharpoonup}_j \lambda_i \ \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_i;\mathbb{R}^{d\times N}\big). \qquad (3.59)
\end{aligned}
$$

In particular, $u_j \overset{\star}{\rightharpoonup} u$ weakly-$\star$ in $BV(\Omega;\mathbb{R}^d)$. In turn, this implies that $\nabla u_j\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup} Du\lfloor_\Omega \otimes \mathcal{L}^{nN}_{y_1,\cdots,y_n}$ weakly-$\star$ in $\mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)$ as $j\to\infty$. Also, convergences (3.59) imply that $\nabla_{y_i}\psi_j^{(i)}\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup} \lambda_i\otimes\mathcal{L}^{(n-i)N}_{y_{i+1},\cdots,y_n}$ weakly-$\star$ in $\mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)$ as $j\to\infty$. Hence,

$$
\Big(\nabla u_j + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}\Big)\mathcal{L}^{(n+1)N}\lfloor_{\Omega\times Y_1\times\cdots\times Y_n} \overset{\star}{\rightharpoonup}_j \lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n} \ \text{ weakly-}\star \text{ in } \mathcal{M}_{y\#}\big(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big).
$$

Using the lower semicontinuity of the total variation,

$$\liminf_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\Big|\nabla u_j(x)+\sum_{i=1}^{n}\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\geqslant\|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n). \tag{3.60}$$

Finally, let $\varphi\in C_c\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)\big)$ with $\|\varphi\|_\infty\leqslant 1$ be given. Let $m\in\mathbb{N}$ be such that $\operatorname{supp}\varphi\subset\Omega_m\times\mathbb{R}^{iN}$. Taking into account (2.3), similar arguments to those of Proposition 3.10 (see (3.53)) show that

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_n}&\varphi(x,y_1,\cdots,y_n):\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\&=\int_{\Omega\times Y_1\times\cdots\times Y_n}\bar\varphi_j(x,y_1,\cdots,y_n):\mathrm{d}\lambda_i(x,y_1,\cdots,y_i)\mathrm{d}y_{i+1}\cdots\mathrm{d}y_n,\end{aligned} \tag{3.61}$$

where $\bar\varphi_j(x,y_1,\cdots,y_n):=\sum_{k=1}^m\Big[\big((\rho_{\varepsilon_k^{(j)}}\prod_{i=1}^n\eta_{\varepsilon_k^{(j)}})*\varphi\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big]$ is such that

$$\bar\varphi_j\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)\big),\quad\|\bar\varphi_j\|_\infty\leqslant 1. \tag{3.62}$$

On the other hand, using the identity

$$\nabla u_j=\sum_{k=1}^{\infty}\rho_{\varepsilon_k^{(j)}}*(\phi_k\,\mathrm{d}Du)+\sum_{k=1}^{\infty}\big[\rho_{\varepsilon_k^{(j)}}*(u\otimes\nabla\phi_k)-u\otimes\nabla\phi_k\big],$$

the estimate (3.56) and the condition $\|\varphi\|_\infty\leqslant 1$, we deduce that

$$\begin{aligned}\int_{\Omega\times Y_1\times\cdots\times Y_n}&\varphi(x,y_1,\cdots,y_n):\nabla u_j(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\&\leqslant\sum_{k=1}^{m}\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\big(\rho_{\varepsilon_k^{(j)}}*(\phi_k\,\mathrm{d}Du)\big)(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n+\frac{1}{j}.\end{aligned} \tag{3.63}$$

In turn, using (2.2), (2.3) and Fubini's Theorem,

$$\begin{aligned}\sum_{k=1}^{m}&\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\big(\rho_{\varepsilon_k^{(j)}}*(\phi_k\,\mathrm{d}Du)\big)(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\&=\sum_{k=1}^{m}\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\Big(\int_{\mathbb{R}^N}\rho_{\varepsilon_k^{(j)}}(x-x')\phi_k(x')\,\mathrm{d}Du(x')\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\&=\sum_{k=1}^{m}\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\Big(\int_{\mathbb{R}^N\times Y_1\times\cdots\times Y_n}\phi_k(x')\rho_{\varepsilon_k^{(j)}}(x-x')\\&\qquad\qquad\prod_{i=1}^{n}\eta_{\varepsilon_k^{(j)}}(y_i-y_i')\,\mathrm{d}Du(x')dy_1'\cdots\mathrm{d}y_n'\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\&=\sum_{k=1}^{m}\int_{\mathbb{R}^N\times Y_1\times\cdots\times Y_n}\Big[\Big(\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n)\rho_{\varepsilon_k^{(j)}}(x'-x)\\&\qquad\qquad\prod_{i=1}^{n}\eta_{\varepsilon_k^{(j)}}(y_i'-y_i)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\Big)\phi_k(x')\Big]:\mathrm{d}Du(x')\mathrm{d}y_1'\cdots\mathrm{d}y_n'\\&=\int_{\Omega\times Y_1\times\cdots\times Y_n}\bar\varphi_j(x',y_1',\cdots,y_n'):\mathrm{d}Du(x')\mathrm{d}y_1'\cdots\mathrm{d}y_n'.\end{aligned} \tag{3.64}$$

Thus, from (3.61), (3.63) and (3.64) we conclude that

$$\begin{aligned} &\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \Big(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &\qquad \leqslant \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar\varphi_j(x,y_1,\cdots,y_n) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n) + \frac{1}{j} \\ &\qquad \leqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n) + \frac{1}{j}, \end{aligned} \tag{3.65}$$

where in the last inequality we have used (3.62). Lebesgue Dominated Convergence Theorem, (3.58) and an approximation argument ensure that for all $\varphi\in C_0\big(\Omega;C_\#\big(Y;\mathbb{R}^{d\times N}\big)\big)$ with $\|\varphi\|_\infty\leqslant 1$ one has

$$\begin{aligned} &\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \Big(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &\qquad \leqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n) + \frac{1}{j}. \end{aligned}$$

Hence,

$$\limsup_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n} \Big|\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big|\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \leqslant \|\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n),$$

which, together with (3.60), concludes Step 1.

Step 2. We prove that the sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}$ may be taken in $C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$ and $C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$, respectively.

The argument is similar to that of Corollary 3.11. Let $\{u_j\}_{j\in\mathbb{N}}$ and $\{\psi_j\}_{j\in\mathbb{N}}$ be the sequences constructed in Step 1. Let $\{\Omega_k\}_{k\in\mathbb{N}}$ be a sequence of open sets such that $\Omega_k\subset\subset\Omega_{k+1}$ and $\Omega=\bigcup_{k=1}^\infty\Omega_k$, and let $\{\theta_k\}_{k\in\mathbb{N}}$ be a sequence of cut-off functions $\theta_k\in C_c^\infty(\Omega;[0,1])$ satisfying for all $k\in\mathbb{N}$, $\theta_k=1$ in Ω_k. Define

$$\psi_{j,k}^{(i)}(x,y_1,\cdots,y_i) := \theta_k(x)\psi_j^{(i)}(x,y_1,\cdots,y_i).$$

We have that $\psi_{j,k}^{(i)}\in C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$, with $\nabla_{y_i}\psi_{j,k}^{(i)}=\theta_k\nabla_{y_i}\psi_j^{(i)}$. For each $j\in\mathbb{N}$, let $\{u_k^{(j)}\}_{k\in\mathbb{N}}\subset C^\infty\big(\overline{\Omega};\mathbb{R}^d\big)$ be a sequence such that

$$u_k^{(j)}\to_k u_j \ \text{ in } W^{1,1}\big(\Omega;\mathbb{R}^d\big). \tag{3.66}$$

We observe that here, and only here, we use the hypothesis that $\partial\Omega$ is Lipschitz. We have that

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_\Omega\Big|u_k^{(j)}(x)-u(x)\Big|\,\mathrm{d}x=0,\quad \lim_{j\to\infty}\lim_{k\to\infty}\int_\Omega\Big|\nabla u_k^{(j)}(x)\Big|\,\mathrm{d}x=\|Du\|(\Omega). \tag{3.67}$$

Let $\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)\big)$ be given. Using on the one hand convergence (3.66), and on the other hand the pointwise convergence $\theta_k\to_k 1$ in Ω together with Lebesgue Dominated Convergence Theorem and taking into account estimate (3.58), we obtain

$$\begin{aligned} &\lim_{j\to\infty}\lim_{k\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \Big(\nabla u_k^{(j)}(x) + \sum_{i=1}^n \nabla_{y_i}\psi_{j,k}^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &\quad = \lim_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \Big(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n \\ &\quad = \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n), \end{aligned} \tag{3.68}$$

where in the last equality we have used Step 1. By similar arguments,

$$\begin{aligned}
&\limsup_{j\to\infty}\limsup_{k\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\left|\nabla u_k^{(j)}(x)+\sum_{i=1}^n\nabla_{y_i}\psi_{j,k}^{(i)}(x,y_1,\cdots,y_i)\right|\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\quad\leqslant \lim_{j\to\infty}\lim_{k\to\infty}\left\{\int_\Omega\left|\nabla u_k^{(j)}(x)-\nabla u_j(x)\right|\mathrm{d}x\right.\\
&\qquad\left.+\int_{\Omega\times Y_1\times\cdots\times Y_n}\left|\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\right|\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n+\int_\Omega(1-\theta_k(x))|\nabla u_j(x)|\,\mathrm{d}x\right\}\\
&\quad=\lim_{j\to\infty}\int_{\Omega\times Y_1\times\cdots\times Y_n}\left|\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\right|\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\
&\quad=\|\lambda_{u,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).
\end{aligned}\tag{3.69}$$

From (3.67), (3.68) and (3.69), using the separability of $C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)\big)$ and a diagonal argument, and finally the lower semicontinuity of the total variation, we can find sequences as in the statement of Corollary 3.12. □

Remark 3.13. *As it was observed within the previous proof, if $\partial\Omega$ fails to be Lipschitz, then Corollary 3.12 holds replacing the condition "$\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega};\mathbb{R}^d)$" by "$\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$".*

We are now in place to prove Proposition 1.11.

Proof of Proposition 1.11. Let $u\in BV\big(\Omega;\mathbb{R}^d\big)$ and for $i\in\{1,\cdots,n\}$, let $\boldsymbol{\mu}_i\in\mathcal{M}_\star\big(\Omega\times Y_1\times\cdots\times Y_{i-1};BV_\#\big(Y_i;\mathbb{R}^d\big)\big)$. Let $\{u_j\}_{j\in\mathbb{N}}\subset C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}\subset C_c^\infty\big(\Omega;C_\#^\infty\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$ be sequences satisfying (3.55).

For each $\varepsilon>0$ and $j\in\mathbb{N}$, define

$$u_{\varepsilon,j}(x):=u_j(x)+\sum_{i=1}^n\varrho_i(\varepsilon)\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big),\ \ x\in\Omega.$$

Then $u_{\varepsilon,j}\in W^{1,1}\big(\Omega;\mathbb{R}^d\big)$, and

$$\begin{aligned}
\nabla u_{\varepsilon,j}(x)&=\nabla u_j(x)+\sum_{i=1}^n\varrho_i(\varepsilon)\,\nabla_x\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)\\
&\quad+\sum_{i=2}^n\sum_{k=1}^{i-1}\frac{\varrho_i(\varepsilon)}{\varrho_k(\varepsilon)}\nabla_{y_k}\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big).
\end{aligned}$$

Let $\tilde\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^d\big)\big)$ and $\varphi\in C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)\big)$ be given. Since for fixed $j\in\mathbb{N}$ and $i\in\{1,\cdots,n\}$, and for all $(y_1,\cdots,y_i)\in\mathbb{R}^{iN}$, $x\mapsto\psi_j^{(i)}(x,y_1,\cdots,y_i)$ has compact support in $\mathbb{R}^N$, from (1.1) and (3.21) we deduce that

$$\begin{aligned}
\lim_{\varepsilon\to0^+}\int_\Omega\tilde\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_{\varepsilon,j}(x)\,\mathrm{d}x&=\lim_{\varepsilon\to0^+}\int_\Omega\tilde\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_j(x)\,\mathrm{d}x\\
&=\int_{\Omega\times Y_1\times\cdots\times Y_n}\tilde\varphi(x,y_1,\cdots,y_n)\cdot u_j(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_\Omega\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big):\nabla u_{\varepsilon,j}(x)\,\mathrm{d}x\\
&\quad=\lim_{\varepsilon\to0^+}\int_\Omega\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big):\Big(\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big)\Big)\,\mathrm{d}x\\
&\quad=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\Big(\nabla u_j(x)+\sum_{i=1}^n\nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\Big)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n.
\end{aligned}$$

Thus, in view of (3.55),

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \tilde\varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_{\varepsilon,j}(x)\,\mathrm{d}x=\int_{\Omega\times Y_1\times\cdots\times Y_n}\tilde\varphi(x,y_1,\cdots,y_n)\cdot u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n, \quad (3.70)$$

and

$$\begin{aligned}&\lim_{j\to\infty}\lim_{\varepsilon\to 0^+}\int_\Omega \varphi\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_i(\varepsilon)}\Big):\nabla u_{\varepsilon,j}(x)\,\mathrm{d}x\\&\quad=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi(x,y_1,\cdots,y_n):\mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n).\end{aligned} \quad (3.71)$$

We claim that we may find a sequence $\{j_\varepsilon\}_{\varepsilon>0}$ such that $j_\varepsilon\to\infty$ as $\varepsilon\to 0^+$, and if we define $v_\varepsilon:=u_{\varepsilon,j_\varepsilon}$, then $\{v_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $W^{1,1}\big(\Omega;\mathbb{R}^d\big)$ satisfying *a)* and *b)* of Theorem 1.10.

In fact, let $\{\tilde\varphi_m\}_{m\in\mathbb{N}}$ and $\{\varphi_m\}_{m\in\mathbb{N}}$ be dense in $C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^d\big)\big)$ and $C_0\big(\Omega;C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\big)\big)$, respectively. For each $\varepsilon>0$, $j,m\in\mathbb{N}$, define

$$\begin{aligned}\tilde\Psi_{\varepsilon,j,m}&:=\int_\Omega\tilde\varphi_m\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big)\cdot u_{\varepsilon,j}(x)\,\mathrm{d}x,\\ \tilde L_m&:=\int_{\Omega\times Y_1\times\cdots\times Y_n}\tilde\varphi_m(x,y_1,\cdots,y_n)\cdot u(x)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,\\ \Psi_{\varepsilon,j,m}&:=\int_\Omega\varphi_m\Big(x,\frac{x}{\varrho_1(\varepsilon)},\cdots,\frac{x}{\varrho_n(\varepsilon)}\Big):\nabla u_{\varepsilon,j}(x)\,\mathrm{d}x,\\ L_m&:=\int_{\Omega\times Y_1\times\cdots\times Y_n}\varphi_m(x,y_1,\cdots,y_n):\mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n).\end{aligned}$$

By (3.70) and (3.71), for all $m\in\mathbb{N}$, we have

$$\lim_{j\to\infty}\lim_{\varepsilon\to 0^+}\tilde\Psi_{\varepsilon,j,m}=\tilde L_m,\qquad \lim_{j\to\infty}\lim_{\varepsilon\to 0^+}\Psi_{\varepsilon,j,m}=L_m. \quad (3.72)$$

For each $\varepsilon>0$, $j\in\mathbb{N}$, set

$$\Theta_{\varepsilon,j}:=\sum_{m=1}^\infty\left[\frac{1}{2^m}\left(\frac{|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}{1+|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}+\frac{|\Psi_{\varepsilon,j,m}-L_m|}{1+|\Psi_{\varepsilon,j,m}-L_m|}\right)\right].$$

Fix $\delta>0$, and let $m_\delta\in\mathbb{N}$ be such that $\displaystyle\sum_{m=m_\delta+1}^\infty\frac{1}{2^m}\leqslant\delta/2$. Then,

$$0\leqslant\Theta_{\varepsilon,j}\leqslant\sum_{m=1}^{m_\delta}\left[\frac{1}{2^m}\left(\frac{|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}{1+|\tilde\Psi_{\varepsilon,j,m}-\tilde L_m|}+\frac{|\Psi_{\varepsilon,j,m}-L_m|}{1+|\Psi_{\varepsilon,j,m}-L_m|}\right)\right]+\delta$$

and so, using (3.72),

$$0\leqslant\limsup_{j\to\infty}\limsup_{\varepsilon\to 0^+}\Theta_{\varepsilon,j}\leqslant\delta,\quad 0\leqslant\limsup_{j\to\infty}\liminf_{\varepsilon\to 0^+}\Theta_{\varepsilon,j}\leqslant\delta.$$

Letting $\delta\to 0^+$, we obtain

$$\lim_{j\to\infty}\limsup_{\varepsilon\to 0^+}\Theta_{\varepsilon,j}=\lim_{j\to\infty}\liminf_{\varepsilon\to 0^+}\Theta_{\varepsilon,j}=0.$$

By a diagonalization argument, we may find a sequence $\{j_\varepsilon\}_{\varepsilon>0}$ such that $j_\varepsilon\to\infty$ as $\varepsilon\to 0^+$, and

$$\lim_{\varepsilon\to 0^+}\Theta_{\varepsilon,j_\varepsilon}=0. \quad (3.73)$$

This way, given $m \in \mathbb{N}$, by definition of $\Theta_{\varepsilon,j_\varepsilon}$ and by (3.73), we have

$$0 \leqslant \frac{1}{2^m}\left(\frac{|\tilde{\Psi}_{\varepsilon,j_\varepsilon,m} - \tilde{L}_m|}{1+|\tilde{\Psi}_{\varepsilon,j_\varepsilon,m} - \tilde{L}_m|} + \frac{|\Psi_{\varepsilon,j_\varepsilon,m} - L_m|}{1+|\Psi_{\varepsilon,j_\varepsilon,m} - L_m|}\right) \leqslant \Theta_{\varepsilon,j_\varepsilon} \underset{\varepsilon\to 0^+}{\longrightarrow} 0,$$

which implies

$$\lim_{\varepsilon\to 0^+} \tilde{\Psi}_{\varepsilon,j_\varepsilon,m} = \tilde{L}_m, \quad \lim_{\varepsilon\to 0^+} \Psi_{\varepsilon,j_\varepsilon,m} = L_m. \tag{3.74}$$

Finally, the existence of a sequence $\{v_\varepsilon\}_{\varepsilon>0}$ as claimed above follows from (3.74), taking into account the boundedness of $\{u_{\varepsilon,j_\varepsilon}\}_{\varepsilon>0}$ in $W^{1,1}(\Omega;\mathbb{R}^d)$. □

We finish this section by proving an extension of Corollary 3.12 to the case in which Ω is bounded, and that will play an important role in our application to homogenization [10].

Proposition 3.14. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set such that $\partial\Omega$ is Lipschitz. Let $u \in BV(\Omega;\mathbb{R}^d)$ and for each $i \in \{1,\cdots,n\}$, let $\boldsymbol{\mu}_i \in \mathcal{M}_\star(\Omega \times Y_1 \times \cdots \times Y_{i-1}; BV_\#(Y_i;\mathbb{R}^d))$. Then there exist sequences $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\overline{\Omega};\mathbb{R}^d)$ and $\{\psi_j^{(i)}\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ satisfying (3.55), and such that*

$$\begin{aligned} &\tilde{\lambda}_j \overset{\star}{\rightharpoonup} \tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n} \ \text{weakly-}\star \text{ in } \mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}),\\ &\lim_{j\to\infty} \|\tilde{\lambda}_j\|(\Omega\times Y_1\times\cdots\times Y_n) = \|\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n),\end{aligned} \tag{3.75}$$

where, for any $B \in \mathcal{B}(\Omega\times Y_1\times\cdots\times Y_n)$,

$$\begin{aligned} &\tilde{\lambda}_j(B) := \left(\int_B \left(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\right)\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n, \mathcal{L}^{(n+1)N}(B)\right),\\ &\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}(B) := \left(\lambda_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}(B), \mathcal{L}^{(n+1)N}(B)\right).\end{aligned}$$

Proof. The proof is very similar to that of Corollary 3.12. We will just point out the main differences.

In Step 1 of the proof of Corollary 3.12, for each $j \in \mathbb{N}$ we require the sequence $\{\varepsilon_k^{(j)}\}_{k\in\mathbb{N}}$ to satisfy the additional conditions

$$\operatorname{supp}\left(\rho_{\varepsilon_k^{(j)}} * \phi_k\right) \subset \left(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\right), \quad \sup_{x\in\Omega}\left|\phi_k(x) - \rho_{\varepsilon_k^{(j)}} * \phi_k(x)\right| \leqslant \frac{1}{j\,2^k}. \tag{3.76}$$

This is possible since if $\phi \in C(\Omega)$, then $\rho_\varepsilon * \phi$ converges uniformly to ϕ as $\varepsilon \to 0^+$ on every compact subset of Ω, and $\operatorname{supp}\phi_k \subset \left(\Omega_{k+1}\backslash\overline{\Omega_{k-1}}\right)$.

Defining $u_j \in C^\infty(\Omega;\mathbb{R}^d)\cap W^{1,1}(\Omega;\mathbb{R}^d))$ and $\psi_j^{(i)} \in C^\infty(\Omega; C_\#^\infty(Y_1\times\cdots\times Y_i;\mathbb{R}^d))$ as in (3.57), then (3.55) holds. Moreover, we clearly have $\tilde{\lambda}_j \overset{\star}{\rightharpoonup} \tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}$ weakly-$\star$ in $\mathcal{M}_{y\#}(\Omega\times Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R})$ as $j\to\infty$, which in turn implies that

$$\liminf_{j\to\infty} \|\tilde{\lambda}_j\|(\Omega\times Y_1\times\cdots\times Y_n) \geqslant \|\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).$$

Furthermore, given $\psi = (\varphi,\theta) \in C_c(\Omega; C_\#(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}))\times C_c(\Omega; C_\#(Y_1\times\cdots\times Y_n))$ with $\|\psi\|_\infty \leqslant 1$, then by (3.65)

$$\begin{aligned} &\int_{\Omega\times Y_1\times\cdots\times Y_n} \psi(x,y_1,\cdots,y_n)\cdot \mathrm{d}\tilde{\lambda}_j(x,y_1,\cdots,y_n)\\ &\quad= \int_{\Omega\times Y_1\times\cdots\times Y_n} \varphi(x,y_1,\cdots,y_n) : \left(\nabla u_j(x) + \sum_{i=1}^n \nabla_{y_i}\psi_j^{(i)}(x,y_1,\cdots,y_i)\right)\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &\qquad+ \int_{\Omega\times Y_1\times\cdots\times Y_n} \theta(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\\ &\quad\leqslant \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar{\varphi}_j(x,y_1,\cdots,y_n) : \mathrm{d}\lambda_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n) + \frac{1}{j}\\ &\qquad+ \int_{\Omega\times Y_1\times\cdots\times Y_n} \theta(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n,\end{aligned}$$

where $\bar{\varphi}_j(x,y_1,\cdots,y_n) := \sum_{k=1}^m \Big[\big((\rho_{\varepsilon_k^{(j)}} \prod_{i=1}^n \eta_{\varepsilon_k^{(j)}}) * \varphi\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big]$. Similarly, setting

$$\bar{\theta}_j(x,y_1,\cdots,y_n) := \sum_{k=1}^m \Big[\big((\rho_{\varepsilon_k^{(j)}} \prod_{i=1}^n \eta_{\varepsilon_k^{(j)}}) * \theta\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big],$$

then, using (3.76) and Fubini's Theorem, we deduce that

$$\left|\int_{\Omega\times Y_1\times\cdots\times Y_n} \theta(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n - \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar{\theta}_j(x,y_1,\cdots,y_n)\,\mathrm{d}x\mathrm{d}y_1\cdots\mathrm{d}y_n\right| \leqslant \frac{\mathcal{L}^N(\Omega)}{j}.$$

Hence, defining $\bar{\psi}_j(x,y_1,\cdots,y_n) := \sum_{k=1}^m \Big[\big((\rho_{\varepsilon_k^{(j)}} \prod_{i=1}^n \eta_{\varepsilon_k^{(j)}}) * \psi\big)(x,y_1,\cdots,y_n)\,\phi_k(x)\Big]$, we conclude that

$$\begin{aligned}
&\int_{\Omega\times Y_1\times\cdots\times Y_n} \psi(x,y_1,\cdots,y_n)\cdot \mathrm{d}\tilde{\lambda}_j(x,y_1,\cdots,y_n) \\
&\qquad \leqslant \int_{\Omega\times Y_1\times\cdots\times Y_n} \bar{\psi}_j(x,y_1,\cdots,y_n)\cdot \mathrm{d}\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\ldots,\boldsymbol{\mu}_n}(x,y_1,\cdots,y_n) + \frac{1+\mathcal{L}^N(\Omega)}{j} \\
&\qquad \leqslant \|\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n) + \frac{1+\mathcal{L}^N(\Omega)}{j},
\end{aligned} \tag{3.77}$$

where in the last inequality we have used the fact that $\bar{\psi}_j \in C_0\big(\Omega; C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}\big)\big)$ and $\|\bar{\psi}_j\|_\infty \leqslant 1$. Using a density argument, together with Lebesgue Dominated Convergence Theorem, we deduce that (3.77) holds for every $\psi \in C_0\big(\Omega; C_\#\big(Y_1\times\cdots\times Y_n;\mathbb{R}^{d\times N}\times\mathbb{R}\big)\big)$ with $\|\psi\|_\infty \leqslant 1$. Consequently,

$$\limsup_{j\to\infty} \|\tilde{\lambda}_j\|(\Omega\times Y_1\times\cdots\times Y_n) \leqslant \|\tilde{\lambda}_{u,\boldsymbol{\mu}_1,\cdots,\boldsymbol{\mu}_n}\|(\Omega\times Y_1\times\cdots\times Y_n).$$

Thus (3.75) holds. We proceed as in Step 2 of Corollary 3.12 to prove that the sequence $\{u_j\}_{j\in\mathbb{N}}$ may be taken in $C^\infty(\overline{\Omega};\mathbb{R}^d)$ and that the sequences $\{\psi_j^{(i)}\}_{j\in\mathbb{N}}$ may be taken in $C_c^\infty\big(\Omega; C_\#^\infty\big(Y_1\times\cdots\times Y_i;\mathbb{R}^d\big)\big)$. □

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