Multiple Integrals under Differential Constraints: Two-Scale Convergence and Homogenization

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Abstract

Two-scale techniques are developed for sequences of maps $\{u_k\} \subset L^p(\Omega;\mathbb{R}^M)$ satisfying a linear differential constraint $\mathcal{A}u_k = 0$. These, together with Γ-convergence arguments and using the unfolding operator, provide a homogenization result for energies of the type

$$F_\varepsilon(u) := \int_\Omega f\big(x, \tfrac{x}{\varepsilon}, u(x)\big)\,dx \quad \text{with} \quad u \in L^p(\Omega;\mathbb{R}^M),\ \mathcal{A}u = 0,$$

that generalizes current results in the case where $\mathcal{A} = \operatorname{curl}$.

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1 Introduction

In this paper we study the limiting behavior of a family of energy functionals with periodic energy densities and underlying fields subject to differential constraints. We give an integral representation to

$$\mathcal{F}(u) := \inf\left\{ \liminf_{\varepsilon\to 0} \int_\Omega f\left(x, \frac{x}{\varepsilon}, u_\varepsilon\right)\,dx \;\middle|\; \begin{array}{l} u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^M), \\ \mathcal{A}u_\varepsilon = 0 \end{array} \right\}, \tag{1.1}$$

where $N, M \in \mathbb{N}$, $\Omega \subset \mathbb{R}^N$ is an open, bounded set, $1 < p < +\infty$, $u \in L^p(\Omega;\mathbb{R}^M)$, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^M \to [0,+\infty)$ satisfies

(H0) $f(x,\cdot,\xi)$ is measurable for every $(x,\xi) \in \Omega \times \mathbb{R}^M$ and $f(\cdot,y,\cdot)$ is continuous for a.e. $y \in \mathbb{R}^N$;

(H1) $f(x,\cdot,\xi)$ is Q-periodic for every $(x,\xi) \in \Omega \times \mathbb{R}^M$, with $Q := (0,1)^N$;

(H2) $0 \le f(x,y,\xi) \le C(1+|\xi|^p)$ for every $(x,\xi) \in \Omega \times \mathbb{R}^M$ and a.e. $y \in \mathbb{R}^N$,

and $\mathcal{A}$ is a first order partial differential operator of constant rank. Precisely, $\mathcal{A}$ maps $u = (u^1,\dots,u^M) : \Omega \to \mathbb{R}^M$ into $\mathcal{A}u = ((\mathcal{A}u)^1,\dots,(\mathcal{A}u)^L) : \Omega \to \mathbb{R}^L$, $L \in \mathbb{N}$, with

$$(\mathcal{A}u)^l := \sum_{i=1}^N \sum_{m=1}^M A_m^{li} \frac{\partial u^m}{\partial_{x_i}}, \quad l = 1,\dots,L,$$

and the coefficients $A^{li}_m \in \mathbb{R}$. The linear matrix valued function

$$\mathbb{A} : \mathbb{R}^N \to \mathbb{R}^{L\times M}, \ \ (\mathbb{A}(\xi))^l_m := \sum_{i=1}^N A^{li}_m \xi_i, \ \ l = 1, \dots, L, \ m = 1, \dots, M,$$

is related to $\mathcal{A}$ via the Fourier transform. We assume throughout that $\mathcal{A}$ satisfies Murat's condition of constant rank (see [23]), that is,

$$\text{the rank of } \mathbb{A}(\xi) \in \mathbb{R}^{L\times M} \text{ is constant as a function of } \xi \in \mathbb{R}^N \setminus \{0\}. \tag{1.2}$$

The study of lowersemicontinuity and relaxation of energy functionals of this type was initiated by Dacorogna [13], followed by Fonseca and Müller [18], and also Braides, Fonseca and Leoni [7], among others. In the latter, the homogenization of a family of functionals as considered in [1] was studied, with f independent of x, continuous in y (note that in (H0) we only ask measurability), and coercive (note that no coercivity is required here in (H2)). Therefore, this work generalizes previous results in the variational approach of homogenization for $\mathcal{A}$-free fields. We recall that important examples that are included in this general setting are the case of divergence free fields, in which $\mathcal{A}u = 0$ if and only if $\operatorname{div} u = 0$, and the case of gradients, in which $\mathcal{A}u = 0$ if and only if $\operatorname{curl} u = 0$.

The main theorem of this paper is

Theorem 1.1. *If (H0)-(H2) hold then for every $u \in L^p(\Omega; \mathbb{R}^M)$ with $\mathcal{A}u = 0$,*

$$\mathcal{F}(u) = \int_\Omega f_{\mathrm{hom}}(x, u(x))\, dx$$

where, for $x \in \Omega$ and $\xi \in \mathbb{R}^M$,

$$f_{\mathrm{hom}}(x, \xi) := \liminf_{n\to\infty} \inf_{v\in\mathcal{V}_\mathcal{A}} \int_Q f(x, ny, \xi + v(y))dy,$$

and $\mathcal{V}_\mathcal{A} := \left\{v \in L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M) \;\middle|\; \int_Q v = 0 \text{ and } \mathcal{A}v = 0\right\}$.

The proof may be found in Section 3, and the tools used here are Γ-convergence, as it was introduced by De Giorgi (see [15] and [16]), the notion of two-scale convergence for $\mathcal{A}$-free sequences, introduced in the case of gradients by Nguetseng (see [22], [24] and [25]), further developed by Allaire and Briane (see [2] and [1]) and many other authors (see also [19]), and extended here to the general $\mathcal{A}$-free setting in Section 2. Further, to prove the lower bound

$$\mathcal{F}(u) \geq \int_\Omega f_{\mathrm{hom}}(x, u(x))\, dx$$

we use the unfolding operator as proposed by Cioranescu, Damlamian and Griso (see [11] and [12]; see also Visintin [26], [27]).

In Section 2 we develop the concept of two-scale convergence for $\mathcal{A}$-free fields, and in Theorem 2.12 we give a complete characterization of weak two-scale limits of $\mathcal{A}$-free sequences. Precisely,

Theorem 1.2. *A function $w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M))$ is the weak two-scale limit of a $\mathcal{A}$-free sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^M)$ if and only if*

$$\mathcal{A}_{y_0}\bar{w}_0 = 0 \text{ and } \mathcal{A}_{y_1}\bar{w}_1 = 0,$$

where

$$\bar{w}_0(y_0) := \int_Q w(y_0, y_1)\, dy_1 \qquad \text{and } \bar{w}_1(y_0, y_1) := w(y_0, y_1) - \bar{w}_0(y_0)$$

for $y_0 \in \Omega$ and $y_1 \in Q$.

Recall that in the case of gradients, two-scale limits are of the form

$$(y_0, y_1) \in \Omega \times Q \mapsto \nabla v_0(y_0) + \nabla_{y_1} v_1(y_0, y_1),$$

with $v_1 \in L^p(\Omega; W^{1,p}_{\text{per}}(Q))$ (see [1], and see [28] for a generalization). In this context, using this together with Γ-limit techniques, Baia and Fonseca in [4] obtained the integral representation for the limit energy of a family of functionals

$$v \mapsto \int_\Omega f\left(x, \frac{x}{\varepsilon}, \nabla v_\varepsilon\right)\, dx \tag{1.3}$$

just as in Theorem 1.1, under conditions (H0), (H1), and with (H2) strengthened with a p-coercivity condition. We remark that the result obtained in this paper now extends that in [4] to the case in which f is not coercive. There is an extensive body of literature on homogenization of multiple integrals of the type (1.3), and in particular we refer to Braides and Defranceschi [6], Braides and Lukkassen [8], Lukkassen [21], Berlyand, Cioranescu and Golovaty [5], Babadjian and Baía [3], and the references therein.

2 Weak two-scale limits for $\mathcal{A}$-free sequences

Let $M, N \in \mathbb{N}$, let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $Q := (0,1)^N$ be the unit cube in $\mathbb{R}^N$. In the following, spaces of functions in $\mathbb{R}^N$ which are Q-periodic are denoted using a subscript "per", where $u : \mathbb{R}^N \to \mathbb{R}^M$ is said to be *Q-periodic* if $u(x + \zeta) = f(x)$ for all $\zeta \in \mathbb{Z}^N$ and all $x \in \mathbb{R}^N$. In particular, we use the space

$$L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M) := \left\{ u \in L^p_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^M) \,\middle|\, u \text{ is } Q\text{-periodic} \right\},$$

endowed with the norm of $L^p(Q, \mathbb{R}^M)$.

Definition 2.1 (weak two-scale convergence [24], [2]). Given a function $w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^{Nn};\mathbb{R}^M))$ and a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset L^p(\Omega;\mathbb{R}^M)$, we say that $\{u_\varepsilon\}$ *weakly two-scale converges to w*, or $u_\varepsilon \overset{2-s}{\rightharpoonup} w$ in $L^p(\Omega;\mathbb{R}^M)$ (with respect to the scales x and $\frac{x}{\varepsilon}$), if

$$\int_\Omega u_\varepsilon(x) \cdot \varphi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \underset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \int_Q w(y_0, y_1) \cdot \varphi(y_0, y_1)\, dy_1 dy_0, \tag{2.1}$$

for every $\varphi \in L^{p'}(\Omega; C_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M))$, where $p' := p/(p-1)$.

Here and in the following, if we talk about a "sequence" with index $\varepsilon > 0$, we understand that ε can be replaced with an arbitrary sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\varepsilon_k \to 0$ as $k \to \infty$. In particular, $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ (with respect to some notion of convergence) if and only if $u_{\varepsilon_k} \to u$ for every sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_k \to 0$ as $k \to \infty$.

Remark 2.2. Note that if $u_\varepsilon \overset{2-s}{\rightharpoonup} w$ in $L^p(\Omega;\mathbb{R}^M)$ then $u_\varepsilon \rightharpoonup \bar{w}_0$ in $L^p(\Omega;\mathbb{R}^M)$, where $\bar{w}_0(y_0) := \int_Q w(y_0,y_1)dy_1$ for $y_0 \in \Omega$.

Bounded sequences are compact with respect to two-scale weak convergence. Precisely (see [2], [22]):

Proposition 2.3. *Every bounded sequence in $L^p(\Omega;\mathbb{R}^M)$ has a subsequence which weakly two-scale converges to a limit in $L^p(\Omega;L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$.*

A simple example of a weakly two-scale convergent sequence is addressed next:

Proposition 2.4. *Given $u \in L^p(\Omega;C_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$ or $u \in L^p_{\mathrm{per}}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^M))$ (the second variable being the periodic one), the sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^M)$, with $u_\varepsilon(x) := u(x,\frac{x}{\varepsilon})$, is p-equiintegrable. It weakly two-scale converges to u, and it weakly converges in $L^p(\Omega;\mathbb{R}^M)$ to $x \in \Omega \mapsto \int_Q u(x,y)\,dy$.*

This result is an immediate consequence of the following lemma proved in [2] (see Lemma 5.2 and Corollary 5.4 in [2]).

Lemma 2.5. *Let $g \in L^1(\Omega;C_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$ or $g \in L^1_{\mathrm{per}}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^M))$. Then $\{g_\varepsilon\}$, with $g_\varepsilon(x) := g(x,\frac{x}{\varepsilon})$ (the second variable being the periodic one), is a bounded, equiintegrable sequence in $L^1(\Omega;\mathbb{R}^M)$ such that*

$$\int_\Omega g_\varepsilon(x)\,dx \to \int_\Omega\int_Q g(x,y)\,dxdy \quad \text{as } \varepsilon \to 0^+.$$

Remark 2.6. Equiintegrability of g_ε is not shown in [2], but it is a consequence of the following estimates: Let $E \subset \Omega$ be a measurable set. In the first case, i.e., if $g \in L^1(\Omega;C_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$, we have

$$\int_E |g_\varepsilon(x)|\,dx \le \int_E \sup_{y\in Q}|g(x,y)|\,dxdy.$$

On the other hand, if $g \in L^1_{\mathrm{per}}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^M))$, then

$$\int_E |g_\varepsilon(x)|\,dx \le \int_E \max_{x\in\overline{\Omega}}\left|g(x,\tfrac{y}{\varepsilon})\right|\,dy.$$

Note that since $\max_{x\in\overline{\Omega}}|g(x,\cdot)| \in L^1(Q;\mathbb{R}^M)$, $(\max_{x\in\overline{\Omega}}|g(x,\frac{\cdot}{\varepsilon})|)_\varepsilon$ is a weakly convergent sequence in L^1 by the Riemann-Lebesgue lemma (see [17], e.g.), and thus equiintegrable.

Here, we study those two-scale weak limits which are generated by sequences $\{u_\varepsilon\}$ satisfying a differential constraint $\mathcal{A}u_\varepsilon = 0$, where $\mathcal{A}$ denotes a homogeneous linear differential operator of first order mapping $u = (u^1, \dots, u^M) : \Omega \to \mathbb{R}^M$ into $\mathcal{A}u = ((\mathcal{A}u)^1, \dots, (\mathcal{A}u)^L) : \Omega \to \mathbb{R}^L$, with

$$(\mathcal{A}u)^l := \sum_{i=1}^{N} \sum_{m=1}^{M} A_m^{li} \frac{\partial u^m}{\partial_{x_i}}, \quad l = 1, \dots, L,$$

and the coefficients $A_m^{li} \in \mathbb{R}$. Its formal adjoint is denoted by $\mathcal{A}^*$, which maps $v = (v_1, \dots, v_L) : \Omega \to \mathbb{R}^L$ into $\mathcal{A}^* v : \Omega \to \mathbb{R}^M$, and is defined by

$$(\mathcal{A}^* v)_m := -\sum_{i=1}^{N} \sum_{l=1}^{L} A_m^{li} \frac{\partial v_l}{\partial_{x_i}}, \quad m = 1, \dots, M.$$

If $u \in C_c^1(\Omega; \mathbb{R}^M)$ and $v \in C_c^1(\Omega; \mathbb{R}^L)$, or $u \in C^1_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M)$ and $v \in C^1_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^L)$, then integration by parts yields

$$\int_\Omega \mathcal{A}u \cdot v = \int_\Omega u \cdot \mathcal{A}^* v \quad \text{or} \quad \int_Q \mathcal{A}u \cdot v = \int_Q u \cdot \mathcal{A}^* v,$$

respectively.

Below, it is understood that if we apply $\mathcal{A}$ to a vector field depending on multiple variables, then the variable on which $\mathcal{A}$ operates is indicated as a subscript, e.g., $\mathcal{A}_y u(x, y)$ means that for the purpose of the application of $\mathcal{A}$, $u(x, y)$ is considered as a function of y with x being a fixed parameter. The linear matrix valued function

$$\mathbb{A} : \mathbb{R}^N \to \mathbb{R}^{L \times M}, \quad (\mathbb{A}(\xi))^l_m := \sum_{i=1}^{N} A_m^{li} \xi_i, \quad l = 1, \dots, L, \ m = 1, \dots, M,$$

is related to $\mathcal{A}$ via the Fourier transform. As a consequence of constant rank condition (1.2), the orthogonal projection $\mathbb{P}(\xi) \in \mathbb{R}^{M \times M}$ onto the kernel of $\mathbb{A}(\xi)$ in $\mathbb{R}^M$ is 0-homogeneous and continuous as a function of $\xi \in \mathbb{R}^N \setminus \{0\}$. We set $\mathbb{P}(0)$ to be the identity matrix in $\mathbb{R}^{N \times N}$. By the Hörmander-Mikhlin multiplier theorem, $\mathbb{P}$ gives rise to a continuous projection operator $\mathcal{P} : L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M) \to L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M)$ onto the kernel of $\mathcal{A}$,

$$\mathcal{P}(u) := \mathcal{F}^{-1}(\mathbb{P}\mathcal{F}(u)),$$

where $\mathcal{F}$ is the Fourier transform. It turns out that

$$\|(I - \mathcal{P})u\|_{L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M)} \le C \|\mathcal{A}u\|_{W^{-1,p}(Q; \mathbb{R}^L)} \quad \text{for every } u \in L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M) \tag{2.2}$$

for some constant $C > 0$. For more details and a proof of (2.2), the reader is referred to [18].

Definition 2.7 (Notions of weak $\mathcal{A}$-differentiability and $\mathcal{A}$-free fields).

(i) If $u \in L^p(\Omega; \mathbb{R}^M)$, then we say that $\mathcal{A}u$ *exists in* L^p if there is a function $U \in L^p(\Omega; \mathbb{R}^L)$ such that

$$\int_\Omega u \cdot \mathcal{A}^* \varphi \, dy = \int_\Omega U \cdot \varphi \, dy \quad \text{for every } \varphi \in C_c^1(\Omega; \mathbb{R}^L).$$

In this case, we define $\mathcal{A}u := U$. We say that u is $\mathcal{A}$-free, or $\mathcal{A}u = 0$, if the preceding equation is satisfied with $U = 0$.

(ii) If $v \in L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M)$, then we say that $\mathcal{A}v$ *exists in* L^p_{per}, if there is a function $V \in L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^L)$ such that

$$\int_Q v \cdot \mathcal{A}^*\varphi \, dy = \int_Q V \cdot \varphi \, dy \quad \text{for every } \varphi \in C^1_{\text{per}}(\mathbb{R}^N;\mathbb{R}^L).$$

In this case, we define $\mathcal{A}v := V$. We say that v is $\mathcal{A}$-free, or $\mathcal{A}v = 0$, if the preceding equation is satisfied with $V = 0$.

(iii) If $w \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M))$, $w = w(y_0, y_1)$ with $(y_0, y_1) \in \Omega \times \mathbb{R}^N$ and $j \in \{0,1\}$, then we say that $\mathcal{A}_{y_j}u$ exists in $L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^L))$ if there exists a function $W_j \in L^p(\Omega; L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^L))$ such that

$$\int_\Omega \int_Q w \cdot \mathcal{A}^*_{y_j}\varphi \, dy_1 dy_0 = \int_\Omega \int_Q W_j \cdot \varphi \, dy_1 dy_0$$
$$\text{for every } \varphi \in C^1_c(\Omega; C^1_{\text{per}}(\mathbb{R}^N;\mathbb{R}^L)),$$

In this case, we define $\mathcal{A}_{y_j}w := W_j$.

The following extension result plays an important role in the variational theory of $\mathcal{A}$-free fields (see also [18]):

Lemma 2.8 ($\mathcal{A}$-free periodic extension)**.** *Let $D \subset Q := (0,1)^N \subset \mathbb{R}^N$ be open, let $1 < p < \infty$ and let $\mathcal{A}$ satisfy* (1.2)*. Then for every p-equiintegrable sequence $\{v_n\} \subset L^p(D;\mathbb{R}^M)$ with $v_n \rightharpoonup 0$ in $L^p(D;\mathbb{R}^M)$ and $\mathcal{A}v_n \to 0$ in $W^{-1,p}(D;\mathbb{R}^L)$, there exists an $\mathcal{A}$-free sequence $\{u_n\} \subset L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M)$, p-equiintegrable in Q, such that*

$$u_n - v_n \to 0 \text{ in } L^p(D;\mathbb{R}^M), \quad u_n \to 0 \text{ in } L^p(Q \setminus D;\mathbb{R}^M), \quad \int_Q u_n(x) = 0,$$

and $\|u_n\|_{L^p(Q;\mathbb{R}^M)} \le C\,\|v_n\|_{L^p(d;\mathbb{R}^M)}$ for all $n \in \mathbb{N}$ and some $C = C(\mathcal{A}) > 0$.

Proof. For every $k \in \mathbb{N}$ choose $\varphi_k \in C^\infty_c(D;[0,1])$ such that $\varphi_k(x) = 1$ whenever $\operatorname{dist}\big(x;\mathbb{R}^N \setminus D\big) \ge \frac{1}{k}$. Clearly,

$$\mathcal{A}(\varphi_k v_n) = \varphi_k \mathcal{A}v_n + \sum_{i=1}^N \sum_{m=1}^M v_n^m A^{li}_m \partial_i \varphi_k \underset{n\to\infty}{\longrightarrow} 0 \text{ in } W^{-1,p}(Q;\mathbb{R}^M),$$

for fixed k, since $v_n \rightharpoonup 0$ in L^p and L^p is compactly embedded in $W^{-1,p}$. Hence, we may choose a sequence $k(n) \to \infty$ such that

$$\mathcal{A}(\varphi_{k(n)}v_n) \underset{n\to\infty}{\longrightarrow} 0 \text{ in } W^{-1,p}(Q;\mathbb{R}^M).$$

For each n, $\varphi_{k(n)}v_n$ can be considered as an element of $L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M)$ by extending it to $\mathbb{R}^N$ Q-periodically. Let

$$\tilde{u}_n := \mathcal{P}(\varphi_{k(n)}v_n).$$

Then $\tilde{u}_n \in L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M)$, and by (2.2),

$$\|\tilde{u}_n\|_{L^p(Q;\mathbb{R}^M)} = \big\|\mathcal{P}(\varphi_{k(n)}v_n)\big\|_{L^p(Q;\mathbb{R}^M)} \le C\,\big\|\varphi_{k(n)}v_n\big\|_{L^p(Q;\mathbb{R}^M)} \le C\,\|v_n\|_{L^p(D;\mathbb{R}^M)}$$

by the continuity of $\mathcal{P}$ on L^p_{per}. Also, since $\{|v_n|^p\}$ is equiintegrable, we have

$$\begin{aligned}\|\tilde{u}_n - v_n\|_{L^p(D;\mathbb{R}^M)} &\le \left\|\mathcal{P}(\varphi_{k(n)}v_n) - \varphi_{k(n)}v_n\right\|_{L^p(Q;\mathbb{R}^M)} + \left\|(1-\varphi_{k(n)})v_n\right\|_{L^p(D;\mathbb{R}^M)} \\ &\le C\left\|\mathcal{A}(\varphi_{k(n)}v_n)\right\|_{W^{-1,p}(Q;\mathbb{R}^L)} + \left\|(1-\varphi_{k(n)})v_n\right\|_{L^p(D;\mathbb{R}^M)} \underset{n\to\infty}{\longrightarrow} 0,\end{aligned}$$

and

$$\begin{aligned}\|\tilde{u}_n\|_{L^p(Q\setminus D;\mathbb{R}^M)} &= \left\|\mathcal{P}(\varphi_{k(n)}v_n) - \varphi_{k(n)}v_n\right\|_{L^p(Q\setminus D;\mathbb{R}^M)} \\ &\le C\left\|\mathcal{A}(\varphi_{k(n)}v_n)\right\|_{W^{-1,p}(Q;\mathbb{R}^L)} \underset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

To verify the p-equiintegrability of $\{\tilde{u}_n\}$ in Q let $E \subset Q$, and observe that

$$\begin{aligned}\int_E |\tilde{u}_n|^p &\le C\int_E \left|\mathcal{P}(\varphi_{k(n)}v_n) - \varphi_{k(n)}v_n\right|^p + C\int_{E\cap D}|v_n|^p \\ &\le C\left\|\mathcal{A}(\varphi_{k(n)}v_n)\right\|^p_{W^{-1,p}(Q;\mathbb{R}^L)} + C\int_{E\cap D}|v_n|^p.\end{aligned}$$

It suffices to set $u_n(x) := \tilde{u}_n(x) - \int_Q \tilde{u}_n(y)\,dy$. $\square$

The following condition turns to be the characterization of weak two-scale limits of $\mathcal{A}$-free sequences (see Theorem 2.12 below).

Definition 2.9 (generalized $\mathcal{A}$-free fields with one microscale)**.** We say that $w \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$ is *generalized $\mathcal{A}$-free* if

$$\mathcal{A}_{y_0}\bar{w}_0 = 0 \quad\text{and}\quad \mathcal{A}_{y_1}\bar{w}_1 = 0, \tag{2.3}$$

where $\bar{w}_0 \in L^p(\Omega;\mathbb{R}^M)$ and $\bar{w}_1 \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$ are defined by

$$\bar{w}_0(y_0) := \int_Q w(y_0,y_1)dy_1 \quad\text{and}\quad \bar{w}_1(y_0,y_1) := w(y_0,y_1) - \bar{w}_0(y_0), \tag{2.4}$$

for $y_0 \in \Omega$ and $y_1 \in Q$.

Proposition 2.10. *Let $\{u_\varepsilon\}$ be a bounded, $\mathcal{A}$-free sequence in $L^p(\Omega;\mathbb{R}^M)$ which weakly two-scale converges to a function $w \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$. Then w is generalized $\mathcal{A}$-free.*

Proof. Fix $\psi \in C^1_c(\Omega;\mathbb{R}^M)$. We have

$$\begin{aligned}0 \;=\; \lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(x)\cdot\mathcal{A}^*\psi(x)\,dx \;&=\; \int_\Omega\int_Q w(y_0,y_1)\cdot\mathcal{A}^*\psi(y_0)\,dy_1dy_0 \\ &=\; \int_\Omega \bar{w}_0\cdot\mathcal{A}^*\psi(y_0)\,dy_0,\end{aligned}$$

where we used the facts that u_ε is $\mathcal{A}$-free, (2.1) and $(2.4)_1$, in this order. This establishes $(2.3)_1$. Next, define $\varphi(y_0,y_1) := \psi(y_0)\phi(y_1)$ for arbitrary functions

$\psi \in C_c^1(\Omega;\mathbb{R}^N)$ and $\phi \in C^1_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^L)$. Since u_ε is $\mathcal{A}$-free, we have that

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(x)\cdot \varepsilon\mathcal{A}^*\Big[\varphi\Big(x,\frac{x}{\varepsilon}\Big)\Big]\,dx \\
&= \lim_{\varepsilon\to 0^+}\int_\Omega u_\varepsilon(x)\cdot \Big[\varepsilon(\mathcal{A}^*\psi)(x)\phi\Big(\frac{x}{\varepsilon}\Big)+\psi(x)(\mathcal{A}^*\phi)\Big(\frac{x}{\varepsilon}\Big)\Big]\,dx \\
&= \int_\Omega\int_Q w(y_0,y_1)\cdot\psi(y_0)\mathcal{A}^*_{y_1}\phi(y_1)\,dy_1dy_0 \\
&= \int_\Omega\Big(\int_Q \bar{w}_1\cdot\mathcal{A}^*_{y_1}\phi(y_1)\,dy_1\Big)\psi(y_0)\,dy_0,
\end{aligned}$$

where we used the fact that $\int_Q \mathcal{A}^*_{y_1}\phi(y_1)\,dy_1 = 0$. This yields $(2.3)_2$. □

In the case of gradients (i.e., curl-free fields) and divergence-free fields, (2.3) is known to characterize the weak two-scale limits of $\mathcal{A}$-free sequences in L^p (see [1] and [2]). Precisely, weak two-scale limits of gradients of $W^{1,p}(\Omega)$-bounded functions are all functions of the form $(y_0,y_1)\mapsto \nabla_{y_0}u_0(y_0)+\nabla_{y_1}u_1(y_0,y_1)$ with $u_0\in W^{1,p}(\Omega)$, $u_1\in L^p(\Omega;W^{1,p}_{\mathrm{per}}(\mathbb{R}^N))$, and weak two-scale limits of divergence free fields in $L^p(\Omega;\mathbb{R}^N)$ are all functions $w\in L^p(\Omega;L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^N))$ such that $\operatorname{div}_{y_0}\bar{w}_0=0$ and $\operatorname{div}_{y_1}\bar{w}_1(y_0,y_1)=0$. To prove the corresponding result for a general $\mathcal{A}$ satisfying (1.2), we reconstruct a suitable $\mathcal{A}$-free sequence from a given generalized $\mathcal{A}$-free function.

Proposition 2.11. *Let $Y\subset\mathbb{R}^N$ be an open cube compactly containing Ω, and suppose that $\mathcal{A}$ satisfies (1.2). Then for every $w\in L^p(\Omega;L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$, $w=w(y_0,y_1)$, such that $\int_Q w(\cdot,y_1)\,dy_1=0$ (i.e., $w=\bar{w}_1$ and $\bar{w}_0=0$) and $\mathcal{A}_{y_1}w=0$, there exists an $\mathcal{A}$-free, p-equiintegrable sequence $\{u_\varepsilon\}\subset L^p(Y;\mathbb{R}^M)$ such that the restriction of u_ε to Ω weakly two-scale converges to w and $u_\varepsilon\rightharpoonup 0$ weakly in $L^p(Y;\mathbb{R}^M)$.*

Proof. **Step 1:** Suppose first that $w\in C^1(\mathbb{R}^N;C^1_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$. Define

$$v_\varepsilon(x) := \int_Q w\big(x+\varepsilon y,\tfrac{x}{\varepsilon}\big)\,dy,\ \ x\in\overline{\Omega}.$$

Then $v_\varepsilon\in C^1(\overline{\Omega};\mathbb{R}^M)$. By Hölder's inequality, we have

$$|v_\varepsilon(x)|^p \le \int_Q \big|w\big(x+\varepsilon y,\tfrac{x}{\varepsilon}\big)\big|^p\,dy,$$

and by the volume-preserving change of variables

$$S_\varepsilon:\mathbb{R}^N\times Q\to\mathbb{R}^N\times\overline{Q},\ \ S_\varepsilon(x,y):=\big(x+\varepsilon y,\tfrac{x}{\varepsilon}-\big\lfloor\tfrac{x}{\varepsilon}\big\rfloor\big),$$

we get the bound

$$\|v_\varepsilon\|_{L^p(E;\mathbb{R}^M)}\le\|w\|_{L^p(S_\varepsilon(E\times Q);\mathbb{R}^M)}\le\|w\|_{L^p(E_\varepsilon\times Q;\mathbb{R}^M)} \tag{2.5}$$

for every measurable $E\subset\Omega$, with $E_\varepsilon:=\{x\in\mathbb{R}^N\mid \operatorname{dist}(x;E)<\varepsilon\}$. In particular, $\{v_\varepsilon\}$ is p-equiintegrable in Ω. Next, we claim that

$$\begin{aligned}
&v_\varepsilon \overset{2-s}{\rightharpoonup} w \text{ in } L^p(\Omega;\mathbb{R}^M),\\
&v_\varepsilon\rightharpoonup 0 \text{ in } L^p(\Omega;\mathbb{R}^M) \text{ and } \mathcal{A}v_\varepsilon\to 0 \text{ strongly in } W^{-1,p}(\Omega;\mathbb{R}^L).
\end{aligned}\tag{2.6}$$

To prove (2.6), we set $\tilde{v}_\varepsilon(x) := w(x, \frac{x}{\varepsilon})$. Clearly, $\{\tilde{v}_\varepsilon\} \subset C^1(\mathbb{R}^N; \mathbb{R}^M)$ is locally bounded in L^∞. Since $\mathcal{A}_{y_1} w = 0$, we have

$$\mathcal{A}\tilde{v}_\varepsilon(x) - \mathcal{A}v_\varepsilon(x) = \int_Q \Big(\mathcal{A}_{y_0} w\big(x, \tfrac{x}{\varepsilon}\big) - \mathcal{A}_{y_0} w\big(x + \varepsilon y, \tfrac{x}{\varepsilon}\big)\Big) dy \tag{2.7}$$

and thus $\mathcal{A}v_\varepsilon - \mathcal{A}\tilde{v}_\varepsilon \to 0$ in $L^p(\Omega; \mathbb{R}^M)$, where we used the fact that $\mathcal{A}_{y_0} w$ is uniformly continuous in $\Omega_1 \times \mathbb{R}^N$, with $\Omega_1 := \{x \in \mathbb{R}^N \mid \operatorname{dist}(x; \Omega) < 1\}$. In addition, by Proposition 2.4 we obtain

$$\mathcal{A}\tilde{v}_\varepsilon = \mathcal{A}_{y_0} w\big(\cdot, \tfrac{\cdot}{\varepsilon}\big) \underset{\varepsilon \to 0^+}{\rightharpoonup} \int_Q \mathcal{A}_{y_0} w(\cdot, y_1) dy_1 = \mathcal{A} \int_Q w(\cdot, y_1) dy_1 = 0 \tag{2.8}$$

weakly in $L^p(\Omega; \mathbb{R}^M)$. By (2.7) and (2.8), we deduce that $\lim_\varepsilon \mathcal{A}v_\varepsilon = 0$ weakly in $L^p(\Omega; \mathbb{R}^L)$, and thus strongly in $W^{-1,p}(\Omega; \mathbb{R}^L)$ by Sobolev's compact embedding theorem. Again using the uniformly continuity of w in $\Omega_1 \times \mathbb{R}^N$, we have that $v_\varepsilon - \tilde{v}_\varepsilon \to 0$ in $L^p(\Omega; \mathbb{R}^M)$ and, in particular, $v_\varepsilon \overset{2-s}{\rightharpoonup} w$ and $v_\varepsilon \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^M)$ (note that $\bar{w}_0 = 0$). This completes the proof of (2.6), which now allows us to apply Lemma 2.8 to $\{v_\varepsilon\}$ (with $D := \Omega$ and Y in place of Q, if necessary translate and rescale). We thus get an $\mathcal{A}$-free, p-equiintegrable sequence $\{u_\varepsilon\} \subset L^p(Y; \mathbb{R}^M)$ such that $v_\varepsilon - u_\varepsilon \to 0$ in $L^p(\Omega; \mathbb{R}^M)$ and $u_\varepsilon \to 0$ in $L^p(Y \setminus \Omega; \mathbb{R}^M)$. In particular, $\{u_\varepsilon\}$ weakly two-scale converges to w in Ω and weakly converges to zero in Y. In addition, in view of (2.5) we obtain

$$\begin{aligned} \|u_\varepsilon\|_{L^p(E;\mathbb{R}^M)} &\le \|u_\varepsilon\|_{L^p(Y\setminus\Omega;\mathbb{R}^M)} + \|u_\varepsilon - v_\varepsilon\|_{L^p(E\cap\Omega;\mathbb{R}^M)} + \|v_\varepsilon\|_{L^p(E\cap\Omega;\mathbb{R}^M)} \\ &\le \sigma_\varepsilon + \|w\|_{L^p(S_\varepsilon((E\cap\Omega)\times Q);\mathbb{R}^M)}, \end{aligned} \tag{2.9}$$

for every $0 < \varepsilon \le 1$ and every measurable $E \subset Y$, where

$$\sigma_\varepsilon := \|u_\varepsilon\|_{L^p(Y\setminus\Omega;\mathbb{R}^M)} + \|u_\varepsilon - v_\varepsilon\|_{L^p(\Omega;\mathbb{R}^M)} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Consequently, $\{u_\varepsilon\}$ is p-equiintegrable in Y.

Step 2: We use a mollification and a diagonalization argument to reduce the general case to the previous step. For every $y_1 \in \mathbb{R}^N$ extend $w(\cdot, y_1)$ by zero outside Ω, whence $\mathcal{A}_{y_1} w = 0$ in $\mathbb{R}^N \times \mathbb{R}^N$ and $\int_Q w(\cdot, y_1) dy_1 = 0$ in $\mathbb{R}^N$. Standard mollification of $w = w(y_0, y_1)$ by convolution (twice, first in y_1 and then in y_0) yields a sequence $\{w_j\} \subset C_c^\infty(\mathbb{R}^N; C_{\text{per}}^\infty(\mathbb{R}^N; \mathbb{R}^M))$ such that $\|w_j\|_{L^p(\mathbb{R}^N; L^p(Q;\mathbb{R}^M))} \le \|w\|_{L^p(\Omega; L^p(Q;\mathbb{R}^M))}$, $w_j \to w$ in L^p, and each w_j satisfies satisfies

$$\mathcal{A}_{y_1} w_j = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^N \quad \text{and} \quad \int_Q w_j(\cdot, y_1) dy_1 = 0 \text{ in } \mathbb{R}^N,$$

since both properties are invariant under convolution in y_1. As to the latter, note that for arbitrary $z \in \mathbb{R}^N$, $\int_Q w(\cdot, z + y_1) dy_1 = \int_Q w(\cdot, y_1) dy_1 = 0$, since w is Q-periodic in y_1.

By Step 1, for each j there exists an $\mathcal{A}$-free sequence $\{u_{j,\varepsilon}\} \subset L^p(Y; \mathbb{R}^M)$ such that

$$u_{j,\varepsilon} \overset{2-s}{\rightharpoonup} w_j \text{ in } L^p(\Omega; \mathbb{R}^M) \quad \text{and} \quad u_{j,\varepsilon} \rightharpoonup 0 \text{ in } L^p(Y; \mathbb{R}^M)$$

as $\varepsilon \to 0^+$. Moreover, by (2.9),

$$\|u_{j,\varepsilon}\|_{L^p(E;\mathbb{R}^M)} \le \sigma_{j,\varepsilon} + \|w_j\|_{L^p(S_\varepsilon((E\cap\Omega)\times Q);\mathbb{R}^M)} \tag{2.10}$$

for every measurable $E \subset Y$, with an error term satisfying $\sigma_{j,\varepsilon} \to 0$ as $\varepsilon \to 0^+$ for fixed j. Further,

$$\|w_j\|_{L^p(S_\varepsilon((E\cap\Omega)\times Q);\mathbb{R}^M)} \le \|w_j - w\|_{L^p(\mathbb{R}^N\times Q;\mathbb{R}^M)} + \|w\|_{L^p(S_\varepsilon((E\cap\Omega)\times Q);\mathbb{R}^M)}. \tag{2.11}$$

Since the spaces of test functions for weak two-scale convergence and weak convergence in L^p, i.e., $L^{p'}(\Omega; C_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$ and $L^{p'}(\Omega;\mathbb{R}^M)$, repectively, are both separable, a diagonalizing argument yields an integer valued function $\varepsilon \mapsto j(\varepsilon)$ such that $j(\varepsilon) \to \infty$, the sequence $\{u_\varepsilon\}$, with $u_\varepsilon := u_{j(\varepsilon),\varepsilon}$, weakly two-scale converges to w in Ω and weakly converges to zero in Y, and $\sigma_{j(\varepsilon),\varepsilon} \to 0$ as $\varepsilon \to 0^+$. In particular, (2.10) and (2.11) imply that u_ε is p-equiintegrable on Y. □

We conclude that $\mathcal{A}$-free weak two scale limits are characterized as follows.

Theorem 2.12. *Let $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be open and bounded, and suppose that $\mathcal{A}$ satisfies* (1.2). *Then $w \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M))$ is the weak two-scale limit of some bounded, $\mathcal{A}$-free sequence $\{u_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^M)$ if and only if w is generalized $\mathcal{A}$-free.*

Proof. By Proposition 2.10, if w is the weak two-scale limit of an $\mathcal{A}$-free sequence then w is generalized $\mathcal{A}$-free. Conversely, given a generalized $\mathcal{A}$-free w, we apply Proposition 2.11 to $\bar{w}_1$ (using the notation of Definition 2.9). This yields an $\mathcal{A}$-free, bounded sequence $\{\tilde{u}_\varepsilon\}$ weakly two-scale converging to $\bar{w}_1$. Hence, $u_\varepsilon := \bar{w}_0 + \tilde{u}_\varepsilon$ weakly two-scale converges to $w = \bar{w}_0 + \bar{w}_1$. □

3 $\mathcal{A}$-free homogenization with one microscale

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, let $1 < p < \infty$ and $\varepsilon \in (0,1]$, and consider the functional

$$F_\varepsilon(u) := \int_\Omega f\Big(x, \frac{x}{\varepsilon}, u(x)\Big)\, dx \quad \text{for } u \in \mathcal{U}_\mathcal{A} := \big\{ u \in L^p(\Omega;\mathbb{R}^M) \,\big|\, \mathcal{A}u = 0 \big\},$$

where $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ satisfies the conditions (H0)–(H2) listed in the introduction.

Our main result is the following:

Theorem 3.1. *If $\mathcal{A}$ satisfies* (1.2) *and (H0)–(H2) hold, then*

$$F_{\mathrm{hom}}(u) := \Gamma - \lim\nolimits_{\varepsilon\to 0^+} F_\varepsilon(u),$$

the Γ-limit in the sense of De Giorgi with respect to weak convergence in L^p, exists for every $u \in \mathcal{U}_\mathcal{A}$. Moreover,

$$F_{\mathrm{hom}}(u) = \liminf_{n\to\infty} \inf_{w\in\mathcal{W}_\mathcal{A}} \int_\Omega \int_Q f(x, ny, u(x) + w(x,y))\, dy dx,$$

where $\mathcal{W}_\mathcal{A} := \big\{ w \in L^p(\Omega; L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M)) \,\big|\, \int_Q w(\cdot,y)dy = 0,\ \mathcal{A}_y w = 0 \big\}$.

We recall that the $\Gamma(L^p\text{-weak})$-limit of F_ε exists at u if

$$\begin{aligned}
\Gamma - \liminf F_\varepsilon(u) &:= \inf\left\{\liminf F_\varepsilon(u_\varepsilon) \;\middle|\; \{u_\varepsilon\} \subset \mathcal{U}_{\mathcal{A}},\ u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^M)\right\},\\
\Gamma - \limsup F_\varepsilon(u) &:= \inf\left\{\limsup F_\varepsilon(u_\varepsilon) \;\middle|\; \{u_\varepsilon\} \subset \mathcal{U}_{\mathcal{A}},\ u_\varepsilon \rightharpoonup u \text{ in } L^p(\Omega;\mathbb{R}^M)\right\}
\end{aligned}$$

coincide (see [14]). The corollary below provides an integral representation for F_{hom}.

Corollary 3.2. *Under the asssumptions of Theorem 3.1, if $u \in \mathcal{U}_{\mathcal{A}}$ then*

$$F_{\text{hom}}(u) = \int_\Omega f_{\text{hom}}(x, u(x))\, dx,$$

where for $x \in \Omega$ and $\xi \in \mathbb{R}^M$,

$$f_{\text{hom}}(x,\xi) := \liminf_{n\to\infty} \inf_{v\in\mathcal{V}_{\mathcal{A}}} \int_Q f(x, ny, \xi + v(y))dy$$

and $\mathcal{V}_{\mathcal{A}} := \{v \in L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M) \mid \int_Q v = 0$ and $\mathcal{A}v = 0\}$.

Remark 3.3. For $n \in \mathbb{N}$, $u \in \mathcal{U}_{\mathcal{A}}$, $w \in \mathcal{W}_{\mathcal{A}}$, $v \in \mathcal{V}_{\mathcal{A}}$ and $\xi \in \mathbb{R}^N$ define

$$\begin{aligned}
\hat{F}(n,u,w) &:= \int_\Omega \int_Q f(x, ny, u(x) + w(x,y))\, dydx,\\
\hat{f}(n,x,\xi,v) &:= \int_Q f(x, ny, \xi + v(y))\, dy,
\end{aligned}$$

Then for every $k, n \in \mathbb{N}$ we have that

$$\inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(kn,u,w) \le \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n,u,w) \quad \text{for every } u \in \mathcal{U}_{\mathcal{A}}, \text{ and} \tag{3.1}$$

$$\inf_{v\in\mathcal{V}_{\mathcal{A}}} \hat{f}(kn,x,\xi,v) \le \inf_{v\in\mathcal{V}_{\mathcal{A}}} \hat{f}(n,x,\xi,v) \quad \text{for every } (x,\xi) \in \Omega \times \mathbb{R}^M. \tag{3.2}$$

Indeed, if $w \in \mathcal{W}_{\mathcal{A}}$ then $\bar{w}(x,y) := w(x,ky) \in \mathcal{W}_{\mathcal{A}}$, and (H1) together with the periodicity of $w(x,\cdot)$ and a change of variables yield $\hat{F}(kn,u,\bar{w}) = \hat{F}(n,u,w)$. Similarly, if $v \in \mathcal{V}_{\mathcal{A}}$ then $\bar{v}(y) := v(ky) \in \mathcal{V}_{\mathcal{A}}$ and $\hat{f}(kn,x,\xi,\bar{v}) = \hat{f}(n,x,\xi,v)$. In particular, $n \mapsto \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n!,u,w)$ is decreasing, $\hat{F}(n!,u,w) \le \hat{F}(n,u,w)$ and so $\lim_n \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n!,u,w)$ exists. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n!,u,w) \ge \liminf_{n\in\mathbb{N}} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n,u,w) = F_{\text{hom}}(u)\\
&\quad \ge \inf_{n\in\mathbb{N}} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n,u,w) \ge \inf_{n\in\mathbb{N}} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n!,u,w) = \lim_{n\to\infty} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n!,u,w).
\end{aligned}$$

We conclude that

$$F_{\text{hom}}(u) = \inf_{n\in\mathbb{N}} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n,u,w) = \lim_{n\to\infty} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \hat{F}(n!,u,w)$$

and, similarly,

$$f_{\text{hom}}(x,\xi) = \inf_{n\in\mathbb{N}} \inf_{v\in\mathcal{V}_{\mathcal{A}}} \hat{f}(n,x,\xi,v) = \lim_{n\to\infty} \inf_{v\in\mathcal{V}_{\mathcal{A}}} \hat{f}(n!,x,\xi,v).$$

The following lemma is an important ingredient in the proof of Theorem 3.1.

Lemma 3.4 ($\mathcal{A}$-free decomposition lemma [18]). *Suppose that $\mathcal{A}$ satisfies (1.2), let $D \subset \mathbb{R}^N$ be open and bounded, let $1 < p < \infty$ and let $\{u_n\} \subset L^p(D;\mathbb{R}^M)$ be a bounded sequence of $\mathcal{A}$-free functions. Then there exists a subsequence $\{u_{k(n)}\}$ of $\{u_n\}$ and a bounded, $\mathcal{A}$-free, p-equiintegrable sequence $\{v_n\} \subset L^p(D;\mathbb{R}^M)$, such that $u_{k(n)} - v_n \to 0$ in L^q for every $q \in [1, p)$.*

The proof of Theorem 3.1 is divided into establishing the upper and the lower bounds for $\Gamma - \lim F_\varepsilon$. We will use the following two technical results. For their proofs, the reader is referred to Appendix B.

Proposition 3.5. *Let f satisfy (H0)–(H2), let $\Omega' \subset\subset \Omega$ be open, let $\varepsilon_n \to 0^+$ as $n \to \infty$, let $\{g_n\} \subset L^\infty(\Omega, \mathbb{R}^N)$ be such that $\|g_n - id\|_{L^\infty} \to 0$, and let $\{v_n\}, \{w_n\} \subset L^p(\Omega;\mathbb{R}^M)$ be bounded sequences.*

(i) If $\{v_n\}, \{w_n\}$ are p-equiintegrable and $\|v_n - w_n\|_{L^p(\Omega;\mathbb{R}^M)} \to 0$ then

$$\lim_{n\to\infty} \int_\Omega \Big[f\Big(x, \frac{x}{\varepsilon_n}, v_n(x)\Big) - f\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big) \Big]\, dx = 0$$

and

$$\lim_{n\to\infty} \int_{\Omega'} \Big[f\Big(g_n(x), \frac{x}{\varepsilon_n}, v_n(x)\Big) - f\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big) \Big]\, dx = 0.$$

(ii) If $\{w_n\}$ is p-equiintegrable and $v_n - w_n \to 0$ in measure then

$$\liminf_{n\to\infty} \int_\Omega f\Big(x, \frac{x}{\varepsilon_n}, v_n(x)\Big)\, dx \geq \liminf_{n\to\infty} \int_\Omega f\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big)\, dx.$$

Proposition 3.6. *Let f satisfy (H0)–(H2), let $\Omega' \subset\subset \Omega$ be open, let $\{g_\nu\} \subset L^\infty(\Omega' \times Q, \mathbb{R}^N)$, and let $\{v_{\nu,n} \mid \nu, n \in \mathbb{N}\}$, $\{w_{\nu,n} \mid \nu, n \in \mathbb{N}\} \subset L^p(\Omega' \times Q;\mathbb{R}^M)$ be p-equiintegrable sets.*

(i) If $\sup_{n\in\mathbb{N}} \|v_{\nu,n} - w_{\nu,n}\|_{L^p(\Omega';\mathbb{R}^M)} \to 0$ and $g_\nu(x,y) - x \to 0$ uniformly in $(x,y) \in \Omega' \times Q$ as $\nu \to \infty$, then

$$\int_{\Omega'} \int_Q \Big[f\Big(g_\nu(x), ny, v_{\nu,n}(x,y)\Big) - f\Big(x, ny, w_{\nu,n}(x,y)\Big) \Big]\, dydx \underset{\nu\to\infty}{\longrightarrow} 0,$$

uniformly in $n \in \mathbb{N}$.

(ii) If $\|v_{\nu,n} - w_{\nu,n}\|_{L^p(\Omega';\mathbb{R}^M)} \to 0$ as $n \to \infty$ for every ν then

$$\int_{\Omega'} \int_Q \Big[f\Big(x, ny, v_{\nu,n}(x,y)\Big) - f\Big(x, ny, w_{\nu,n}(x,y)\Big) \Big]\, dydx \underset{n\to\infty}{\longrightarrow} 0,$$

for every $\nu \in \mathbb{N}$.

Proposition 3.7 (upper bound). *Assume that (H0)–(H2) hold. Then for every $n \in \mathbb{N}$, every $\delta > 0$, every $u \in \mathcal{U}_\mathcal{A}$, and every $w \in \mathcal{W}_\mathcal{A}$, there exists a sequence $\{u_\varepsilon\} \subset \mathcal{U}_\mathcal{A}$ such that $u_\varepsilon \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^M)$ as $\varepsilon \to 0^+$, and*

$$\lim_{\varepsilon\to 0^+} \int_\Omega f(x, \tfrac{x}{\varepsilon}, u_\varepsilon)dx \leq \int_\Omega \int_Q f(x, ny, u(x) + w(x,y))\, dydx + \delta. \tag{3.3}$$

Proof. **Step 1:** Assume first that $u \in C(\overline{\Omega};\mathbb{R}^M)$ and $w \in L^p_{\rm per}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^M)) \cap \mathcal{W}_\mathcal{A}$ (we write $w(x,y)$ with y being the periodic variable). For fixed $n \in \mathbb{N}$,

$$g(x,y) := f(x, ny, u(x) + w(x,y)),\ x \in \overline{\Omega},\ y \in \mathbb{R}^N \text{ (the periodic variable)},$$

is a function in $L^1_{\rm per}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^M))$. Moreover, the sequence $\{g_\varepsilon\} \subset L^1(\Omega)$, with $g_\varepsilon(x) := g\big(x, \frac{x}{n\varepsilon}\big)$, is bounded in L^1, and by Lemma 2.5, with $v_\varepsilon(x) := w\big(x, \frac{x}{\varepsilon}\big)$, we have

$$\int_\Omega f\big(x, \tfrac{x}{\varepsilon}, u(x) + v_\varepsilon(x)\big)dx = \int_\Omega g_\varepsilon(x)\,dx \underset{\varepsilon\to 0^+}{\longrightarrow} \int_\Omega \int_Q g(x,y)\,dydx. \tag{3.4}$$

In particular, (3.3) holds with equality for $\delta = 0$ and $u_\varepsilon(x) := u(x) + v_\varepsilon(x)$. Note that $\{v_\varepsilon\}$ is a p-equiintegrable sequence and, by Proposition 2.4, $v_\varepsilon \rightharpoonup \int_Q w(\cdot,y)dy = 0$ weakly in L^p. In addition, still by Proposition 2.4,

$$\mathcal{A}v_\varepsilon = \mathcal{A}_{y_0} w\Big(\cdot, \frac{\cdot}{n\varepsilon}\Big) \underset{\varepsilon\to 0^+}{\rightharpoonup} \int_Q \mathcal{A}_{y_0} w(\cdot, y_1)dy_1 = \mathcal{A}\int_Q w(\cdot,y_1)dy_1 = 0$$

in $L^p(\Omega;\mathbb{R}^L)$, and hence $\mathcal{A}v_\varepsilon \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^L)$. Due to (a rescaled version of) Lemma 2.8, there exists an $\mathcal{A}$-free, p-equiintegrable sequence $\{\tilde{v}_\varepsilon\} \subset L^p(\Omega;\mathbb{R}^M)$ such that

$$\tilde{v}_\varepsilon - v_\varepsilon = \tilde{v}_\varepsilon - w\Big(\cdot, \frac{\cdot}{n\varepsilon}\Big) \underset{\varepsilon\to 0^+}{\longrightarrow} 0 \text{ in } L^p(\Omega;\mathbb{R}^M).$$

In particular, $\tilde{v}_\varepsilon \rightharpoonup 0$ in $L^p(\Omega;\mathbb{R}^M)$. By Proposition 3.5 (i) we have

$$\lim_{\varepsilon\to 0^+} \int_\Omega f(x, \tfrac{x}{\varepsilon}, u(x) + \tilde{v}_\varepsilon(x))dx = \lim_{\varepsilon\to 0^+} \int_\Omega f(x, \tfrac{x}{\varepsilon}, u(x) + v_\varepsilon(x))dx$$

and this, together with (3.4), concludes the proof.

Step 2: The case of a general $u \in \mathcal{U}_\mathcal{A}$ and $w \in \mathcal{W}_\mathcal{A}$ follows by density and diagonalization arguments. More precisely, we extend w to $\mathbb{R}^N$ by setting $w(y_0,y_1) := 0$ for $y_0 \in \mathbb{R}^N \setminus \Omega$. Mollifying in y_0 results in a function $\tilde{w}$ which, in particular, belongs to $L^p_{\rm per}(\mathbb{R}^N;C(\overline{\Omega};\mathbb{R}^M)) \cap \mathcal{W}_\mathcal{A}$ and is close to w in the topology of $L^p(\Omega;L^p(\mathbb{R}^N;\mathbb{R}^M))$. Note that by (H0) and (H2), the right hand side of (3.3) is continuous in w. Similarly, $u \in \mathcal{U}_\mathcal{A}$ can be replaced by a $\tilde{u} \in C(\overline{\Omega};\mathbb{R}^M)$ close to u in L^p (which in general does not satisfy $\mathcal{A}u = 0$ anymore, but this is irrelevant here). □

Proposition 3.8 (lower bound 1)**.** *Assume that (H0)–(H2) hold. Then for every sequence $\varepsilon_n \to 0^+$, every $u \in \mathcal{U}_\mathcal{A}$ and every sequence $\{u_n\} \subset \mathcal{U}_\mathcal{A}$ with $u_n \rightharpoonup 0$ in $L^p(\Omega;\mathbb{R}^M)$, there exist a family of functions $V = \{v_{\nu,n} \mid \nu, n \in \mathbb{N}\} \subset \mathcal{U}_\mathcal{A}$ such that*

$$\begin{gathered} V \textit{ is p-equiintegrable, } v_{\nu,n} \underset{n\to\infty}{\rightharpoonup} 0 \textit{ weakly in } L^p \textit{ for every } \nu \in \mathbb{N}, \textit{ and} \\ \liminf_{n\to\infty} \int_\Omega f(x, \tfrac{x}{\varepsilon_n}, u + u_n)\,dx \geq \sup_{\nu\in\mathbb{N}} \liminf_{n\to\infty} \int_\Omega f\big(x, \nu n x, u + v_{\nu,n}\big)\,dx. \end{gathered} \tag{3.5}$$

Proof. Fix $\nu \in \mathbb{N}$. In the following, we repeatedly extract subsequences of n (with all sequences depending on ν) without further mentioning and without relabeling n. In particular, we may always assume that a limes inferior in n is a limit. For our definition of $v_{\nu,n}$ below, we set $v_{\nu,n} := 0$ if n does not match a value attained by the final (ν-dependent) subsequence of n.

By Lemma 3.4, there exists a p-equiintegrable sequence $\{\tilde{u}_n\} \subset \mathcal{U}_{\mathcal{A}}$ such that $\tilde{u}_n - u_n \to 0$ in $L^q(\Omega;\mathbb{R}^M)$ for $q < p$ and, in particular, $\tilde{u}_n \rightharpoonup 0$ in L^p. By Proposition 3.5 (ii), we have that

$$\lim_{n\to\infty} \int_\Omega f(x, \tfrac{x}{\varepsilon_n}, u + u_n)dx \geq \lim_{n\to\infty} \int_\Omega f(x, \tfrac{x}{\varepsilon_n}, u + \tilde{u}_n)dx. \tag{3.6}$$

If $\tilde{k}_{\nu,n} := \frac{1}{\nu\varepsilon_n}$ is a sequence of integers (strictly increasing after selecting a suitable subsequence of n), then we define $v_{\nu,k} := u_n$ if $k = \tilde{k}_{\nu,n}$, $v_{\nu,k} := 0$ otherwise. If this is not the case, then for $n \in \mathbb{N}$, let $\lfloor\frac{1}{\nu\varepsilon_n}\rfloor$ denote the largest integer smaller than $\frac{1}{\nu\varepsilon_n}$ and let

$$\theta_{\nu,n} := \nu\varepsilon_n \lfloor\tfrac{1}{\nu\varepsilon_n}\rfloor \in [0,1], \quad \text{whence } k_{\nu,n} := \theta_{\nu,n}\tfrac{1}{\nu\varepsilon_n} \in \mathbb{N}_0 \text{ and } \theta_{\nu,n} \to 1 \text{ as } n \to \infty. \tag{3.7}$$

Choose an open cube $Y \subset \mathbb{R}^N$ which compactly contains Ω. By (a rescaled version of) Lemma 2.8 applied to the sequence $\{\tilde{u}_n\}$, we can find an $\mathcal{A}$-free, bounded, p-equiintegrable sequence $\{\bar{u}_n\} \subset L^p(Y;\mathbb{R}^M)$ such that $\bar{u}_n - \tilde{u}_n \to 0$ in $L^p(\Omega;\mathbb{R}^M)$ and $\bar{u}_n \to 0$ in $L^p(Y \setminus \Omega;\mathbb{R}^M)$. Using Proposition 3.5 (i) and (3.7) we obtain

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega f(x, \tfrac{x}{\varepsilon_n}, u + \tilde{u}_n)dx &= \lim_{n\to\infty} \int_\Omega f(x, \tfrac{x}{\varepsilon_n}, u + \bar{u}_n)dx \\ &= \lim_{n\to\infty} \int_\Omega f(x, \nu\tfrac{k_{\nu,n}}{\theta_{\nu,n}}x, u + \bar{u}_n)dx.\end{aligned} \tag{3.8}$$

For any fixed $\Omega' \subset\subset \Omega$, $\theta_{\nu,n}\Omega' \subset \Omega$ if n is large enough (depending on ν). Since $f \geq 0$, a change of variables yields that

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega f\big(x, \nu k_{\nu,n}\tfrac{1}{\theta_{\nu,n}}x, u(x) + \bar{u}_n(x)\big)dx \\ &\geq \liminf_{n\to\infty} \int_{\theta_{\nu,n}\Omega'} f\big(x, \nu k_{\nu,n}\tfrac{1}{\theta_{\nu,n}}x, u(x) + \bar{u}_n(x)\big)dx \\ &= \liminf_{n\to\infty} (\theta_{\nu,n})^N \int_{\Omega'} f\big(\theta_{\nu,n}x, \nu k_{\nu,n}x, u(\theta_{\nu,n}x) + \bar{u}_n(\theta_{\nu,n}x)\big)dx \\ &= \liminf_{n\to\infty} \int_{\Omega'} f\big(x, \nu k_{\nu,n}x, u(x) + \bar{u}_n(\theta_{\nu,n}x)\big)dx,\end{aligned} \tag{3.9}$$

where we used Proposition 3.5 (i) and, by (3.6), the facts that $\theta_{\nu,n} \to 1$ and $u(\theta_{\nu,n}\cdot) \to u$ in $L^p(\Omega';\mathbb{R}^M)$. In view of (3.6)–(3.9), we conclude that

$$\lim_{n\to\infty} \int_\Omega f(x, \tfrac{x}{\varepsilon_n}, u_n)dx \geq \liminf_{n\to\infty} \int_{\Omega'} f\big(x, \nu k_{\nu,n}x, u(x) + \bar{u}_n(\theta_{\nu,n}x)\big)dx.$$

Letting Ω' approach Ω such that $|\Omega\setminus\Omega'|\to 0$, using (H2) and the equiintegrability of $\{|\bar u_n|^p\}$, we infer that

$$\lim_{n\to\infty}\int_\Omega f(x,\tfrac{x}{\varepsilon_n},u_n)dx \ge \liminf_{n\to\infty}\int_\Omega f\big(x,\nu k_{\nu,n}x,u(x)+v_{\nu,k_{\nu,n}}(x)\big)dx,$$

with $v_{\nu,k_{\nu,n}}(x):=\bar u_n(\theta_{\nu,n}x)$ for $x\in\Omega$, $\nu\in\mathbb{N}$ and $n\ge n_0(\nu)$, where $n_0(\nu)$ is chosen large enough such that $\theta_{\nu,n}\ge\frac12$ and $\theta_{\nu,n}\Omega\subset Y$ for every $n\ge n_0(\nu)$. In particular, $\{v_{\nu,k_{\nu,n}}\mid \nu\in\mathbb{N},\ n\ge n_0(\nu)\}$ is p-equiintegrable on Ω, just like $\{\bar u_n\}$ is p-equiintegrable on Y. Moreover, $v_{\nu,k_{\nu,n}}\in\mathcal{U}_\mathcal{A}$ and $v_{\nu,k_{\nu,n}}\rightharpoonup 0$ weakly in L^p as $n\to\infty$. $\square$

Proposition 3.9 (lower bound 2). *Assume that (H0)–(H2) hold. Then for every $u\in\mathcal{U}_\mathcal{A}$, and every family $\{v_{\nu,n}\mid \nu,n\in\mathbb{N}\}\subset\mathcal{U}_\mathcal{A}$ satisfying* (3.5),

$$\begin{aligned}\liminf_{\nu\to\infty}\liminf_{n\to\infty}&\int_\Omega f\big(x,\nu nx,u+v_{\nu,n}\big)\,dx\\ &\ge \liminf_{n\to\infty}\inf_{w\in\mathcal{W}_\mathcal{A}}\int_\Omega\int_Q f(x,ny,u(x)+w(x,y))dydx.\end{aligned}$$

The proof of this Proposition uses a strategy similar to that in Lemma 2.9 in [10], although we work under slightly weaker assumptions on f, strongly relying on the p-equiintegrability of $\{v_{\nu,n}\}$. In particular, we use the so-called *unfolding operator* T_δ ([11], [10]; see also [12], [26]): For $\delta>0$, $T_\delta: L^1(\Omega;\mathbb{R}^M)\to L^1(\Omega;L^1_{\text{per}}(\mathbb{R}^N;\mathbb{R}^M))$ is defined by

$$T_\delta(v)(x,y):=v(\delta\lfloor\tfrac{x}{\delta}\rfloor+\delta(y-\lfloor y\rfloor))\ \text{ for } x\in\Omega \text{ and } y\in\mathbb{R}^N,$$

where v is extended by zero outside of Ω and, as before, for $t\in\mathbb{R}$, $\lfloor t\rfloor$ denotes the largest integer less then or equal to t. To arguments in $\mathbb{R}^N$ with $N>1$, $\lfloor\cdot\rfloor$ is applied component-wise. Note that

$$T_\delta v(x,y)=T_\delta v\big(\delta\lfloor\tfrac{x}{\delta}\rfloor,y\big)\ \text{ for every } \delta>0,\ (x,y)\in\Omega\times\mathbb{R}^N. \tag{3.10}$$

Further properties of T_δ are collected in Appendix A.

Proof of Proposition 3.9. Fix $u\in\mathcal{U}_\mathcal{A}$ and let $\{v_{\nu,n}\mid \nu,n\in\mathbb{N}\}$ satisfy (3.5). Moreover, fix $\Omega'\subset\subset\Omega$, and for $z\in\mathbb{Z}^N$ and $\nu\in\mathbb{N}$ define

$$Q_{\nu,z}:=\tfrac{1}{\nu}z+\tfrac{1}{\nu}Q\subset\mathbb{R}^N \text{ for } z\in\mathbb{Z}^N,\ Z_\nu:=\{z\in\mathbb{Z}^N\mid Q_{\nu,z}\cap\Omega'\neq\emptyset\},$$

and set

$$I_{\nu,n}:=\int_\Omega f\big(x,\nu nx,u+v_{\nu,n}\big)\,dx.$$

Note that if ν is large enough such that the distance of Ω' to $\partial\Omega$ is at least $\frac{\sqrt{N}}{\nu}$, then $Q_{\nu,z}\subset\Omega$ for every $z\in Z_\nu$. Since $f\ge 0$, a change of variables and the definition of

$T_{\frac{1}{\nu}}$ yield

$$I_{\nu,n} \geq \sum_{z\in Z_\nu} \int_{Q_{\nu,z}} f(x, \nu n x, u + v_{\nu,n}) dx$$
$$= \sum_{z\in Z_\nu} \frac{1}{\nu^N} \int_Q f\Big(\tfrac{z+y}{\nu}, n(z+y), T_{\frac{1}{\nu}}(u)(\tfrac{z}{\nu}, y) + T_{\frac{1}{\nu}}(v_{\nu,n})(\tfrac{z}{\nu}, y)\Big)\, dy$$
$$= \sum_{z\in Z_\nu} \int_{Q_{\nu,z}} \int_Q f\Big(\tfrac{\lfloor \nu x\rfloor + y}{\nu}, ny, T_{\frac{1}{\nu}}(u)(\tfrac{\lfloor \nu x\rfloor}{\nu}, y) + T_{\frac{1}{\nu}}(v_{\nu,n})(\tfrac{\lfloor \nu x\rfloor}{\nu}, y)\Big)\, dydx$$

where we used (H1) and the facts that $|Q_{\nu,z}| = \frac{1}{\nu^N}$ and $z = \lfloor \nu x\rfloor$ for $x \in Q_{\nu,z}$. Thus, using (3.10) we have that

$$\begin{aligned} I_{\nu,n} &\geq \sum_{z\in Z_\nu} \int_{Q_{\nu,z}} \int_Q f\Big(\tfrac{\lfloor \nu x\rfloor + y}{\nu}, ny, T_{\frac{1}{\nu}}(u)(x, y) + T_{\frac{1}{\nu}}(v_{\nu,n})(x, y)\Big)\, dydx \\ &\geq \int_{\Omega'} \int_Q f\Big(\tfrac{\lfloor \nu x\rfloor + y}{\nu}, ny, T_{\frac{1}{\nu}}(u)(x, y) + T_{\frac{1}{\nu}}(v_{\nu,n})(x, y)\Big)\, dydx \end{aligned} \tag{3.11}$$

if ν is sufficiently large such that $\big|\Omega' \setminus \bigcup_{z\in Z_\nu} Q_{\nu,z}\big| = 0$. By Proposition A.2,

$$U := \Big\{ T_{\frac{1}{\nu}}(u) + T_{\frac{1}{\nu}}(v_{\nu,n}) \;\Big|\; \nu \in \mathbb{N},\ n \in \mathbb{N}\Big\} \cup \Big\{ T_{\frac{1}{\nu}}(v_{\nu,n}) \;\Big|\; \nu \in \mathbb{N},\ n \in \mathbb{N}\Big\},$$

is a p-equiintegrable subset of $L^p(\Omega' \times Q; \mathbb{R}^M)$. Moreover, as $\nu \to \infty$, $\frac{\lfloor \nu x\rfloor + y}{\nu} \to x$ uniformly in $(x, y) \in \Omega \times Q$, and $T_{\frac{1}{\nu}}(u) \to u$ in $L^p(\Omega' \times Q; \mathbb{R}^M)$, the latter by Proposition A.1. Hence, (3.11) and Proposition 3.6 (i) imply that

$$\begin{aligned} I_{\nu,n} &\geq \sigma_\nu + \int_{\Omega'} \int_Q f\Big(x, ny, u(x) + T_{\frac{1}{\nu}}(v_{\nu,n})(x, y)\Big)\, dydx \\ &= \sigma_\nu + \sum_{z\in Z_\nu} \int_{\Omega'\cap Q_{\nu,z}} \int_Q f\Big(x, ny, u(x) + T_{\frac{1}{\nu}}(v_{\nu,n})(\tfrac{z}{\nu}, y)\Big)\, dydx, \end{aligned} \tag{3.12}$$

with an error term $\sigma_\nu = \sigma_\nu(\Omega')$ which is independent of n and satisfies $\sigma_\nu \to 0$ as $\nu \to \infty$ for fixed Ω'. Moreover, for fixed ν and $z \in Z_\nu$, $\hat{v}_{\nu,z,n}(y) := T_{\frac{1}{\nu}}(v_{\nu,n})(\frac{z}{\nu}, y) = v_{\nu,n}\big(\frac{1}{\nu}(z + y)\big)$, $y \in Q$, is a p-equiintegrable, $\mathcal{A}$-free sequence in $L^p(Q; \mathbb{R}^M)$ with $\hat{v}_{\nu,z,n} \rightharpoonup 0$ weakly in L^p as $n \to \infty$. By Lemma 2.8, there exists a p-equiintegrable, $\mathcal{A}$-free sequence $\{w_{\nu,z,n}\} \subset L^p_{\mathrm{per}}(\mathbb{R}^N; \mathbb{R}^M)$ such that $\hat{v}_{\nu,z,n} - w_{\nu,z,n} \to 0$ in $L^p(Q; \mathbb{R}^M)$ as $n \to \infty$ and $\int_Q w_{\nu,z,n}(y)\, dy = 0$. By Proposition 3.6 (ii), we infer that

$$\begin{aligned} &\int_{\Omega'\cap Q_{\nu,z}} \int_Q f\Big(x, ny, u(x) + T_{\frac{1}{\nu}}(v_{\nu,n})(\tfrac{z}{\nu}, y)\Big)\, dydx \\ &\qquad = \tau_{z,n,\nu} + \int_{\Omega'\cap Q_{\nu,z}} \int_Q f\Big(x, ny, u(x) + w_{\nu,z,n}(y)\Big)\, dydx \end{aligned}$$

for every $z \in Z_\nu$, where $\lim_n \tau_{z,n,\nu} = 0$. In view of (3.12), we obtain that

$$\begin{aligned} \liminf_{n\to\infty} I_{\nu,n} &\geq \sigma_\nu + \liminf_{n\to\infty} \sum_{z\in Z_\nu} \int_{\Omega'\cap Q_{\nu,z}} \int_Q f\big(x, ny, u(x) + w_{\nu,z,n}(y)\big)\, dydx \\ &= \sigma_\nu + \liminf_{n\to\infty} \int_{\Omega'} \int_Q f\big(x, ny, u(x) + w_{\nu,n}(x, y)\big)\, dydx, \end{aligned}$$

where

$$w_{\nu,n}(x,y) := \sum_{z\in Z_\nu} \chi_{\Omega'\cap Q_{\nu,z}}(x) w_{\nu,z,n}(y),$$

$\chi_{\Omega'\cap Q_{\nu,z}}(x) := 1$ if $x \in \Omega' \cap Q_{\nu,z}$, and $\chi_{\Omega'\cap Q_{\nu,z}}(x) := 0$ elsewhere. Clearly, $w_{\nu,n} \in \mathcal{W}_{\mathcal{A}}$, and thus, with

$$\kappa_{\Omega\setminus\Omega'} := -\sup_{n\in\mathbb{N}} \int_{\Omega\setminus\Omega'} \int_Q f(x, ny, u(x))\, dxdy,$$

we have that

$$\begin{aligned}\liminf_{n\to\infty} I_{\nu,n} &\geq \kappa_{\Omega\setminus\Omega'} + \sigma_\nu + \liminf_{n\to\infty} \int_\Omega \int_Q f\big(x, ny, u(x) + w_{\nu,n}(x,y)\big)\, dydx \\ &\geq \kappa_{\Omega\setminus\Omega'} + \sigma_\nu + \liminf_{n\to\infty} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \int_\Omega \int_Q f\big(x, ny, u(x) + w(x,y)\big)\, dydx.\end{aligned}$$

To conclude let $\nu \to \infty$ and then let Ω' approach Ω such that $|\Omega \setminus \Omega'| \to 0$, using (H2) to ensure that $\kappa_{\Omega\setminus\Omega'} \to 0$ as $|\Omega \setminus \Omega'| \to 0$. $\square$

Proof of Theorem 3.1. Given $u \in \mathcal{U}_{\mathcal{A}}$, $\varepsilon_k \to 0^+$, $\{u_k\} \subset \mathcal{U}_{\mathcal{A}}$ with $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^M)$ and setting

$$J(u) := \liminf_{n\to\infty} \inf_{w\in\mathcal{W}_{\mathcal{A}}} \int_\Omega \int_Q f(x, ny, u(x) + w(x,y))\, dydx,$$

we may assume w.l.o.g. that $\liminf \int_\Omega f\big(x, \frac{x}{\varepsilon_k}, u_k\big)\, dx = \lim \int_\Omega f\big(x, \frac{x}{\varepsilon_k}, u_k\big)\, dx$, and by Proposition 3.8 and Proposition 3.9, it follows that

$$\lim_{k\to\infty} \int_\Omega f\Big(x, \frac{x}{\varepsilon_k}, u_k(x)\Big)\, dx \geq J(u).$$

Therefore $\Gamma - \liminf F_\varepsilon(u) \geq J(u)$.

Conversely, if $\delta > 0$ and $\varepsilon_k \to 0^+$, and if $n \in \mathbb{N}$, $w \in \mathcal{W}_{\mathcal{A}}$ are such that

$$\int_\Omega \int_Q f(x, ny, u(x) + w(x,y))\, dydx - \delta \leq J(u),$$

then using Proposition 3.7 we find $\{u_k\} \subset \mathcal{U}_{\mathcal{A}}$, $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^M)$ such that

$$\lim_{k\to\infty} \int_\Omega f\Big(x, \frac{x}{\varepsilon_k}, u_k(x)\Big)\, dx \leq \int_\Omega \int_Q f(x, ny, u(x) + w(x,y))\, dydx - \delta \leq J(u) + 2\delta.$$

Letting $\delta \to 0$, we conclude that $\Gamma - \limsup F_\varepsilon(u) \leq J(u)$. $\square$

The proof of Corollary 3.2 relies on a measurable selection criterion which is a simplified variant of Theorem III.6 in [9].

Lemma 3.10. *Let Z be a separable metric space, let T be a measurable space and let $\Gamma : T \to 2^Z$ be a multifunction such that $\Gamma(t) \subset Z$ is nonempty and open for every $t \in T$, and $\{t \in T \mid z \in \Gamma(t)\}$ is measurable for every $z \in Z$. Then Γ admits a measurable selection, i.e., there exists a measurable function $\gamma : T \to Z$ such that $\gamma(t) \in \Gamma(t)$ for every $t \in T$.*

Proof. Let $Z_0 := \{z_k \mid k \in \mathbb{N}\}$ be a countable dense subset of Z. Since $\Gamma(t) \neq \emptyset$ and it is open, $\Gamma(t) \cap Z_0 \neq \emptyset$ for all $t \in T$. We define $\gamma(t) := z_{k(t)}$ if $k(t)$ is the smallest integer such that $z_{k(t)} \in \Gamma(t)$, so that the function γ attains values only in the countable set Z_0. Moreover,

$$\gamma^{-1}(z_k) = \{t \mid z_k \in \Gamma(t)\} \setminus \bigcup_{j=1}^{k-1} \{t \mid z_j \in \Gamma(t)\}$$

is measurable for every $k \in \mathbb{N}$, whence γ is measurable. $\square$

Proof of Corollary 3.2. Fix $u \in \mathcal{U}_{\mathcal{A}}$. Let $\mathcal{V}_0 \subset \mathcal{V}_{\mathcal{A}}$ be a countable subset which is dense in $\mathcal{V}_{\mathcal{A}}$ with respect to the topology of L^p_{per} (for instance, choose a countable dense subset in $L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M)$, and project it onto $\mathcal{V}_{\mathcal{A}}$ using $v \mapsto P(v) := \mathcal{P}v - \int_Q (\mathcal{P}v) dy$). For every $n \in \mathbb{N}$, by Lebesgue's Dominated Convergence Theorem and by (H2), we have for a.e. $x \in \Omega$

$$J_n(x, u(x)) := \inf_{v \in \mathcal{V}_{\mathcal{A}}} \int_Q f(x, ny, u(x) + v(y))\, dy = \inf_{v \in \mathcal{V}_0} \int_Q f(x, ny, u(x) + v(y))\, dy.$$

In particular, the functions $x \in \Omega \mapsto J_n(x, u(x))$ and $x \in \Omega \mapsto f_{\mathrm{hom}}(x, u(x)) = \liminf_{n\to\infty} J_n(x, u(x))$ are measurable. By Theorem 3.1 and since $w(x, \cdot) \in \mathcal{V}_{\mathcal{A}}$ if $w \in \mathcal{W}_{\mathcal{A}}$, we have

$$\begin{aligned}
F_{\mathrm{hom}}(u) &= \liminf_{n\to\infty} \inf_{w \in \mathcal{W}_{\mathcal{A}}} \int_\Omega \int_Q f(x, ny, u(x) + w(x, y))\, dydx \\
&\geq \liminf_{n\to\infty} \int_\Omega \Big(\inf_{v \in \mathcal{V}_{\mathcal{A}}} \int_Q f(x, ny, u(x) + v(y))\, dy \Big) dx \\
&\geq \int_\Omega \Big(\liminf_{n\to\infty} \inf_{v \in \mathcal{V}_{\mathcal{A}}} \int_Q f(x, ny, u(x) + v(y))\, dy \Big) dx \\
&= \int_\Omega f_{\mathrm{hom}}(x, u(x))\, dx,
\end{aligned}$$

where we used Fatou's lemma.

To prove the converse inequality, fix $\delta > 0$, and for $m \in \mathbb{N}$ set

$$\Omega_{m,\delta} := \bigcup_{n \in \{1,\dots,m\}} \big\{ x \in \Omega \,\big|\, J_n(x, u(x)) < f_{\mathrm{hom}}(x, u(x)) + \delta \big\}.$$

Since $|\Omega \setminus \Omega_{m,\delta}| \to 0$ as $m \to \infty$, there exists $n_\delta \in \mathbb{N}$ such that $|\Omega \setminus \Omega_{m,\delta}| \leq \delta$ for $m \geq n_\delta$. Consider the sets

$$\Gamma_\delta(x) := \Big\{ v \in \mathcal{V}_{\mathcal{A}} \,\Big|\, \int_Q f(x, n_\delta!\, y, u(x) + v(y))\, dy < f_{\mathrm{hom}}(x, u(x)) + \delta \Big\}$$

which are open in $L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M)$ (by (H2) and Lebesgue's Dominated Convergence Theorem) and nonempty for $x \in \Omega_{n_\delta,\delta}$ (by Remark 3.3, using the fact that $n \mapsto \inf_{v \in \mathcal{V}_{\mathcal{A}}} \int_Q f(x, n!\, y, u(x) + v(y))\, dy$ is decreasing). In addition, $\{x \in \Omega_{n_\delta,\delta} \mid v \in \Gamma_\delta(x)\}$ is measurable for every $v \in L^p_{\mathrm{per}}(\mathbb{R}^N, \mathbb{R}^M)$. We now apply Lemma 3.10 with $T := \Omega_{n_\delta,\delta}$ and $Z := L^p_{\mathrm{per}}(\mathbb{R}^N, \mathbb{R}^M)$ to find a measurable selection $\bar{w} : \Omega_{n_\delta,\delta} \to L^p_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^M)$ of

Γ_δ. Moreover, $\bar{w} \in L^p(\Omega'_\delta; L^p_{\rm per}(\mathbb{R}^N;\mathbb{R}^M))$ for a suitable measurable set $\Omega'_\delta \subset \Omega_{n_0,\delta}$ such that

$$\left|\Omega \setminus \Omega'_\delta\right| \le 2\delta. \tag{3.13}$$

For a.e. $x \in \Omega'_\delta$, $\bar{w}(x) \in \Gamma_\delta(x)$, and thus, since $f \ge 0$, we have

$$\int_{\Omega'_\delta}\int_Q f(x, n_\delta!\, y, u(x) + \bar{w}(x,y))\, dydx \le \int_\Omega f_{\rm hom}(x,u(x))\, dx + \delta\, |\Omega|\, .$$

Extending $\bar{w}(x,\cdot) := 0$ for $x \in \Omega \setminus \Omega'_\delta$, we have that $\bar{w} \in \mathcal{W}_\mathcal{A}$ and so,

$$\begin{aligned}
&\inf_{w\in\mathcal{W}_\mathcal{A}} \int_\Omega\int_Q f(x, n_\delta!\, y, u(x) + w(x,y))\, dydx \\
&\le \int_{\Omega'_\delta}\int_Q f(x, n_\delta!\, y, u(x) + \bar{w}(x,y))\, dydx + \int_{\Omega\setminus\Omega'_\delta}\int_Q f(x, n_\delta!\, y, u(x))\, dydx \\
&\le \int_\Omega f_{\rm hom}(x,u(x))\, dx + \delta\, |\Omega| + \int_{\Omega\setminus\Omega'_\delta}\int_Q f(x, n_\delta!\, y, u(x))\, dydx.
\end{aligned}$$

Letting $\delta \to 0^+$, by (H2) and Remark 3.3 we conclude that

$$F_{\rm hom}(u) \le \int_\Omega f_{\rm hom}(x,u(x))\, dx. \qquad \square$$

Appendix A Properties of the unfolding operator

We recall the definition of the unfolding operator (see [11] and [10]; see also [12] and [26]): Let $\Omega \subset \mathbb{R}^N$ be open. For $\gamma > 0$ and $v \in L^p(\Omega)$ (v extended by zero outside Ω), set

$$T_\gamma(v)(x,y) := v\big(\gamma\big\lfloor \tfrac{x}{\gamma}\big\rfloor + \gamma(y - \lfloor y\rfloor)\big) \quad \text{for } x\in\Omega \text{ and } y \in \mathbb{R}^N,$$

where as before, $\lfloor\cdot\rfloor$ is defined as the component-wise integer part of its argument. Note that if v has support in $K \subset\subset \mathbb{R}^N$ then

$$\operatorname{supp} T_\gamma(v) \subset \left\{x \in \mathbb{R}^N \,\middle|\, \operatorname{dist}(x;K) \le \gamma\sqrt{N}\right\} \times \mathbb{R}^N. \tag{A.1}$$

Indeed, if $\gamma\big\lfloor \frac{x}{\gamma}\big\rfloor + \gamma(y - \lfloor y\rfloor) \in K$ for some $y \in \mathbb{R}^N$, then

$$\operatorname{dist}(x;K) \le \left|x - \gamma\big\lfloor \tfrac{x}{\gamma}\big\rfloor - \gamma(y - \lfloor y\rfloor)\right| = \gamma\left|\big(\tfrac{x}{\gamma} - \big\lfloor\tfrac{x}{\gamma}\big\rfloor\big) - \big(y - \lfloor y\rfloor\big)\right| \le \gamma \operatorname{diam} Q = \gamma\sqrt{N}.$$

Proposition A.1. *Let $1 \le p < \infty$. Then for every $\gamma > 0$, $T_\gamma : L^p(\Omega) \to L^p(\Omega\times Q)$ is linear. Moreover, for every $v \in L^p(\Omega)$ (extended by zero outside Ω),*

$$\|T_\gamma(v)\|_{L^p(\Omega\times Q)} \le \|T_\gamma(v)\|_{L^p(\mathbb{R}^N\times Q)} = \|v\|_{L^p(\mathbb{R}^N)} = \|v\|_{L^p(\Omega)} \tag{A.2}$$

and

$$\int_{\mathbb{R}^N}\int_Q |v(x) - T_\gamma(v)(x,y)|^p\, dydx \to 0 \quad \text{as } \gamma \to 0^+. \tag{A.3}$$

Proof. The first equality in (A.2) is a consequence of Fubini's theorem and a change of variables, and the remaining assertions in (A.2) are trivial. For the proof of (A.3), fix $\varepsilon > 0$ and choose a sequence $\{v_n\} \subset C_c^\infty(\mathbb{R}^N)$ with $v_n \to v$ in $L^p(\mathbb{R}^N)$. Since T_γ is linear and by (A.2), there exists $m \in \mathbb{N}$ such that for every $n \geq m$ and every $\gamma > 0$,

$$\|v_n - v\|_{L^p(\mathbb{R}^N)} = \|T_\gamma(v_n) - T_\gamma(v)\|_{L^p(\mathbb{R}^N \times Q)} < \tfrac{1}{3}\varepsilon.$$

Hence, with $C_m := \left|\{x \in \mathbb{R}^N \mid \operatorname{dist}(x; \operatorname{supp} v_m) \leq \sqrt{N}\}\right|^{\frac{1}{p}}$ and using (A.1), for $\gamma \leq 1$ we obtain

$$\begin{aligned} \|v - T_\gamma(v)\|_{L^p(\mathbb{R}^N \times Q)} &\leq \|v_m - T_\gamma(v_m)\|_{L^p(\mathbb{R}^N \times Q)} + \tfrac{2}{3}\varepsilon && \text{(A.4)} \\ &\leq C_m \|v_m - T_\gamma(v_m)\|_{L^\infty(\mathbb{R}^N \times Q)} + \tfrac{2}{3}\varepsilon && \text{(A.5)} \\ &= C_m \sup_{x \in \mathbb{R}^N,\ y \in Q} \left|v_m(x) - v_m(\gamma\lfloor\tfrac{x}{\gamma}\rfloor + \gamma y)\right| + \tfrac{2}{3}\varepsilon. && \text{(A.6)} \end{aligned}$$

Since v_m is uniformly continuous in $\mathbb{R}^N$ and

$$\left|x - (\gamma\lfloor\tfrac{x}{\gamma}\rfloor + \gamma y)\right| = \gamma\left|\left(\tfrac{x}{\gamma} - \lfloor\tfrac{x}{\gamma}\rfloor\right) - y\right| < \gamma \operatorname{diam}(Q) = \gamma\sqrt{N},$$

by (A.4) we conclude that $\|v - T_\gamma(v)\|_{L^p(\mathbb{R}^N \times Q)} < \varepsilon$ for $0 < \gamma < \gamma_0(m)$ with some $\gamma_0(m) > 0$ sufficiently small. $\square$

Proposition A.2. *Let $1 \leq p < \infty$, let $B \subset \mathbb{R}^N$ be a bounded set, and let $V \subset L^p(\mathbb{R}^N)$ be a p-equiintegrable set of functions with support in B. Then $\{T_\gamma v \mid \gamma \in (0,1],\ v \in V\} \subset L^p(\mathbb{R}^N \times Q)$ is also p-equiintegrable.*

Proof. Let $\delta > 0$ and choose $\eta > 0$ such that

$$\sup_{v \in V} \int_F |v|^p < \frac{\delta}{2}. \tag{A.7}$$

for every measurable $F \subset \mathbb{R}^N$ with $|F| < \eta$. Let $t_0 >> 1$ be such that

$$\sup_{v \in V} |\{|v(x)| > t_0\}| < \eta, \tag{A.8}$$

and let $\tau > 0$ be such that

$$\tau t_0^p < \frac{\delta}{2} \tag{A.9}$$

Consider a measurable set $E \subset \mathbb{R}^N \times Q$ such that $|E| < \tau$. For $F \subset \mathbb{R}^N$ measurable and $\gamma > 0$ define

$$T_\gamma F := \left\{(x, y) \in \mathbb{R}^N \times Q \,\middle|\, \gamma\left\lfloor\tfrac{x}{\gamma}\right\rfloor + \gamma y \in F\right\}.$$

Note that

$$T_\gamma(\chi_F) = \chi_{T_\gamma(F)}, \tag{A.10}$$

and for every $v \in V$ and $t \geq 0$,

$$T_\gamma\{|v| \geq t\} = \{|T_\gamma v| \geq t\} \quad \text{and} \quad T_\gamma\{|v| > t\} = \{|T_\gamma v| > t\}. \tag{A.11}$$

Hence, for $v \in V$ we have that

$$\begin{aligned}\int_E |T_\gamma(v)|^p &\leq \int_{E\cap\{|T_\gamma(v)|\leq t_0\}} t_0^p + \int_{\{|T_\gamma(v)|>t_0\}} |T_\gamma v|^p \\ &\leq \tau t_0^p + \int_{\mathbb{R}^N\times Q} \chi_{T_\gamma\{|v|>t_0\}} |T_\gamma(v)|^p \\ &\leq \frac{\delta}{2} + \int_{\mathbb{R}^N\times Q} \left|T_\gamma(\chi_{\{|v|>t_0\}}v)\right|^p,\end{aligned}$$

where we used (A.9), (A.10) and (A.11), in this order, and the fact that $T_\gamma(fg) = T_\gamma(f)T_\gamma(g)$ for functions $f \in L^\infty(\Omega)$, $g \in L^p(\Omega;\mathbb{R}^M)$. By (A.2), (A.7) and (A.8), this implies that

$$\int_E |T_\gamma(v)|^p \leq \frac{\delta}{2} + \int_{\mathbb{R}^N} \left|\chi_{\{|v|\geq t_0\}}v\right|^p = \frac{\delta}{2} + \int_{\{|v|\geq t_0\}} |v|^p < \delta.$$

□

Appendix B Some results on uniform continuity

Below, we collect several auxiliary results on the continuity of Nemytskii operators associated to the function f introduced in Section 3.

Proposition B.1 (Scorza-Dragoni, e.g. see [20]). *Let $\Omega \subset \mathbb{R}^N$ be open, let $S_1 \subset\subset \Omega$, let $S_2 \subset\subset \mathbb{R}^M$ and suppose that f satisfies (H0). Then for every $\delta > 0$, there exists a compact set $K_\delta \subset Q := (0,1)^N$ such that $|Q \setminus K_\delta| < \delta$ and f is uniformly continuous on $S_1 \times K_\delta \times S_2$.*

Proposition B.2. *Let $1 \leq p < \infty$, assume that (H0)–(H2) hold, let $\lambda \in (0,1]$, let $\Omega' \subset \Omega$ be measurable, and let $V \subset L^p(\Omega';\mathbb{R}^M)$ be p-equiintegrable. Then the functions*

$$\boldsymbol{f}_\lambda : V \to L^1(\Omega'), \quad \boldsymbol{f}_\lambda(v)(x) := f\Big(x, \frac{x}{\lambda}, v(x)\Big) \text{ for } x \in \Omega',$$

are uniformly continuous, uniformly in λ. Moreover, for every compact $\Omega'' \subset \Omega$ and $A := L^1(\Omega';\Omega'') \subset L^1(\Omega';\mathbb{R}^N)$, the functions

$$\boldsymbol{g}_\lambda : A \times V \to L^1(\Omega'), \quad \boldsymbol{g}_\lambda(a,v)(x) := f\Big(a(x), \frac{x}{\lambda}, v(x)\Big) \text{ for } x \in \Omega',$$

are uniformly continuous, uniformly in λ.

Proof. Let $\varepsilon > 0$. By the p-equiintegrability of V there exists a set $\Omega''' = \Omega'''(\varepsilon) \subset \Omega'$ such that $\Omega''' \subset\subset \Omega$ and

$$\sup_{v\in V} \int_{\Omega'\setminus\Omega'''} f\big(x, \tfrac{x}{\lambda}, v(x)\big)\, dx < \frac{\varepsilon}{4}.$$

Hence, it suffices to show that

$$\|\boldsymbol{f}_\lambda(v_1) - \boldsymbol{f}_\lambda(v_2)\|_{L^1(\Omega''')} < \frac{\varepsilon}{2} \text{ if } \|v_1 - v_2\|_{L^p(\Omega''';\mathbb{R}^M)} < \delta,$$

for a suitable $\delta = \delta(\varepsilon) > 0$ independent of $v_1, v_2 \in V$ and $\lambda \in (0,1]$. This is a special case of the second part of the assertion.

For a proof of the second part of the assertion, let $a_1, a_2 \in A$, $v_1, v_2 \in V$, and $\lambda \in (0,1]$, and fix $\varepsilon > 0$. We want to show that

$$\|\boldsymbol{g}_\lambda(a_1, v_1) - \boldsymbol{g}_\lambda(a_2, v_2)\|_{L^1(\Omega')} < \varepsilon \quad \text{if } \|a_1 - a_2\|_{L^1(\Omega';\mathbb{R}^N)} + \|v_1 - v_2\|_{L^p(\Omega';\mathbb{R}^M)} < \delta,$$

for a suitable $\delta = \delta(\varepsilon) > 0$ independent of a_1, a_2, v_1, v_2 and λ. Due to the p-equintegrability of V and (H2), there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for every every $E \subset \Omega'$ measurable, and for all $a \in A$, $v \in V$,

$$\int_E \Big| f\Big(a(x), \frac{x}{\lambda}, v(x)\Big)\Big|\, dx < \frac{\varepsilon}{9}, \quad \text{provided that } |E| < \delta_1. \tag{B.1}$$

In view of the p-equiintegrability of V, there exists $R = R(\varepsilon) > 0$ large enough such that

$$\sup_{v \in V} |\{|v| \geq R\}| < \delta_1. \tag{B.2}$$

Let $L > 0$ be sufficiently large so that $\Omega' \subset [-L, L]^N$. By the periodicity of f with respect to its second variable and by Proposition B.1 applied with $S_1 := \Omega''$ and $S_2 := \overline{B}_R(0) \subset \mathbb{R}^M$, there exists a compact set $K = K(\varepsilon, L) \subset Q$ and $\delta_2 = \delta_2(\varepsilon, L) > 0$, such that

$$|Q \setminus K| < (2L+2)^{-N}\delta_1, \tag{B.3}$$

and for every $x_1, x_2 \in \Omega''$, every $y \in \mathbb{Z}^N + K$ and every $\xi_1, \xi_2 \in \overline{B}_R(0)$,

$$|f(x_1, y, \xi_1) - f(x_2, y, \xi_2)| < \frac{\varepsilon}{9\,|\Omega'|} \quad \text{if } |x_1 - x_2| + |\xi_1 - \xi_2| < \delta_2. \tag{B.4}$$

By (B.3), with $m_\lambda := \lfloor \frac{L}{\lambda} \rfloor + 1$, we have that

$$\begin{aligned}
\big|\Omega' \setminus \lambda(\mathbb{Z}^N + K)\big| &\leq \big|[-L, L]^N \setminus \lambda(\mathbb{Z}^N + K)\big| \\
&\leq \lambda^N \big|\big[-m_\lambda, m_\lambda\big]^N \setminus (\mathbb{Z}^N + K)\big| \\
&= \lambda^N \big(2m_\lambda\big)^N |Q \setminus K| \\
&\leq (2L+2)^N |Q \setminus K| < \delta_1.
\end{aligned} \tag{B.5}$$

Finally, there exists $\delta = \delta(\varepsilon)$ such that

$$|\{|a_1 - a_2| + |v_1 - v_2| \geq \delta_2\}| < \delta_1 \quad \text{if } \|a_1 - a_2\|_{L^1} + \|v_1 - v_2\|_{L^p} < \delta. \tag{B.6}$$

Define

$$\begin{aligned}
\tilde{S} := [\Omega' \cap \lambda(\mathbb{Z}^N + K)] &\cap \{|v_1| < R\} \cap \{|v_2| < R\} \\
&\cap \{|a_1 - a_2| + |v_1 - v_2| < \delta_2\},
\end{aligned}$$

whence $\Omega' \setminus \tilde{S}$ is a union of four sets, each of which has measure less than δ_1, due to (B.2), (B.5), and (B.6), respectively. By (B.1) and (B.4), we infer that

$$\begin{aligned}
&\int_{\Omega'} \Big| f\Big(a_1(x), \frac{x}{\lambda}, v_1(x)\Big) - f\Big(a_2(x), \frac{x}{\lambda}, v_2(x)\Big)\Big|\, dx \\
&\quad \leq \frac{8}{9}\varepsilon + \int_{\tilde{S}} \Big| f\Big(a_1(x), \frac{x}{\lambda}, v_1(x)\Big) - f\Big(a_2(x), \frac{x}{\lambda}, v_2(x)\Big)\Big|\, dx < \varepsilon
\end{aligned}$$

whenever $\|a_1 - a_2\|_{L^1(\Omega')} + \|v_1 - v_2\|_{L^p(\Omega')} < \delta$. $\square$

Proposition B.3. *Let $1 \le p < \infty$, assume that (H0)–(H2) hold, let $Q = (0,1)^N$, let $\lambda \in (0,1]$, let $\Omega' \subset \Omega$ be measurable, and let $\tilde{V} \subset L^p(\Omega' \times Q; \mathbb{R}^M)$ be p-equiintegrable. Then for every $\Omega'' \subset\subset \Omega$ and $\tilde{A} := L^1(\Omega' \times Q; \Omega'') \subset L^1(\Omega' \times Q; \mathbb{R}^N)$, the functions*

$$\boldsymbol{h}_\lambda : \tilde{A} \times \tilde{V} \to L^1(\Omega' \times Q), \quad \boldsymbol{h}_\lambda(a,v)(x,y) := f\Big(a(x,y), \frac{y}{\lambda}, v(x,y)\Big)$$

are uniformly continuous, uniformly in λ.

Proof. This is analogous to the proof of Proposition B.2. We omit the details. □

Proof of Proposition 3.5. (i) The first part of (i) follows from the uniform equi-continuity of $\{\boldsymbol{g}_\lambda\}$ obtained in Proposition B.2, with a compact set Ω'' satisfying $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. For the second part use the uniform equi-continuity of $\{\boldsymbol{f}_\lambda\}$ obtained in Proposition B.2 with $\Omega' := \Omega$.

(ii) Let $\varepsilon > 0$. For $v \in L^p(\Omega)$ and $h > 0$ consider the truncated function $v^{[h]} := \max\{\min\{v,h\}, -h\}$, whereas for $v \in L^p(\Omega; \mathbb{R}^M)$, $v^{[h]}$ is defined component-wise. Since $\{w_n\}$ is p-equiintegrable, so is

$$W := \{w_n : n \in \mathbb{N}\} \cup \{w_n^{[h]} : n \in \mathbb{N},\ h > 0\},$$

and $w_n^{[h]} - w_n \to 0$ in L^p as $h \to \infty$, uniformly in $n \in \mathbb{N}$. Hence, by the first part of (i), there exists $H = H(\varepsilon) > 0$ such that

$$\int_\Omega \Big| f\Big(x, \frac{x}{\varepsilon_n}, w_n^{[H]}(x)\Big) - f\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big) \Big|\, dx < \frac{\varepsilon}{3}, \tag{B.7}$$

for every $n \in \mathbb{N}$. Since $\{v_n^{[H]}\}$ and $\{w_n^{[H]}\}$ are p-equiintegrable and $w_n^{[H]} - v_n^{[H]} \to 0$ in L^p, again the first part of (i) yields

$$\int_\Omega \Big| f\Big(x, \frac{x}{\varepsilon_n}, v_n^{[H]}\Big) - f\Big(x, \frac{x}{\varepsilon_n}, w_n^{[H]}(x)\Big) \Big|\, dx \underset{n\to\infty}{\longrightarrow} 0. \tag{B.8}$$

Finally, since $f \ge 0$ and $\big|\{v_n \ne v_n^{[H]}\}\big| \to 0$ as $n \to \infty$, we have that

$$\begin{aligned} \int_\Omega f\Big(x, \frac{x}{\varepsilon_n}, v_n\Big)\, dx - \int_\Omega f\Big(x, \frac{x}{\varepsilon_n}, v_n^{[H]}(x)\Big)\, dx \\ \ge -\int_{\{v_n \ne v_n^{[H]}\}} f\Big(x, \frac{x}{\varepsilon_n}, v_n^{[H]}\Big)\, dx \underset{n\to\infty}{\longrightarrow} 0, \end{aligned} \tag{B.9}$$

where we used (H2) and the p-equiintegrability of $v_n^{[H]}$. Combining (B.7)–(B.9), we infer that

$$\int_\Omega f\Big(x, \frac{x}{\varepsilon_n}, v_n(x)\Big)\, dx \ge \int_\Omega f\Big(x, \frac{x}{\varepsilon_n}, w_n(x)\Big)\, dx - \varepsilon$$

for every n large enough. □

Proof of Proposition 3.6. Both assertions are immediate consequences of the uniform equi-continuity of $\{\boldsymbol{h}_\lambda\}$ obtained in Proposition B.3, with a compact set Ω'' satisfying $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. □

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