Field Equations for Elastic Constituents Undergoing Disarrangements and Mixing

David R. Owen

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213, USA

March 20, 2007

Contents

1 Introduction

The goal of this article is the derivation of field equations that govern the isothermal evolution of elastic constituents undergoing disarrangements and mixing, described by means of the multiscale geometry afforded by structured deformations ([1], [2], [3]) in the broader context of "multiphase structured deformations." The refined geometrical and kinematical setting of multiphase structure deformations introduced here for the first time permits one to distinguish in a precise fashion between

- separate elastic constituents undergoing disarrangements, i.e., N elastic bodies that are not in contact and that each may undergo non-smooth submacroscopic geometrical changes (disarrangements) such as the formation of voids or the occurence of slips;
- intermingling elastic constituents, i.e., N elastic bodies that are split into numerous, small, affinely deforming and non-overlapping "constituent cells" in contact with cells of the same or of different constituents;
- mixing elastic constituents undergoing disarrangements, i.e., N elastic bodies that interpenetrate by occupying overlapping regions in space and that undergo disarrangements.

The framework of multiphase structured deformations provides field equations in the first and last cases above. For separate elastic constitutents undergoing disarrangements, each elastic constituent evolves via a time-parameterized family of structured deformations, and appropriate field relations [4] are summarized in Section 2 below. These relations generalize those of dynamical finite elasticity, allow each constituent to store energy via both deformations without disarrangements and deformations due to disarrangements, and also allow each constituent to experience dissipation during smooth dynamical processes. The capability of each constituent to undergo disarrangments by incorporating submacroscopic voids is significant, because the void fraction so created provides unoccupied space that potentially can be occupied by other constituents.

The desired field relations for mixing elastic constituents then amount to generalizations of the field relations [4] for a given, separate elastic constituent c, and I show here how the appearance of other constituents in the region occupied by constituent c causes the appearance of new terms or factors in those field relations. Multiphase structured deformations, introduced in Section 3, describe precisely how elastic constituents can interpenetrate and yet, through the presence of microvoids in each constituent, can provide enough room for the remaining constituents. The provision of room for other constituents is captured in the Accomodation Inequality (3.2), and the assertion that there is "enough" room is justified by the Approximation Lemma for Multiphase Structured Deformations stated in Section 4. This result generalizes the Approximation Lemma for Structured Deformations [1] and shows that injective, piecewise affine mappings can be used to approximate "macroscopically translational representatives" of a given multiphase structured deformation. The injectivity of the approximating mappings leads me to describe them as "intermingling approximations."

The use of intermingling approximations avoids our having to consider directly the interpenetration of different constituents, because no interpenetration occurs via intermingling approximations, i.e, at a given point in the range of an intermingling approximation, exactly one constituent is present. In Section 7 I use these intermingling approximations as well as the "intermingling fields" introduced in Sections 5 and 6 to identify contact forces exerted on a given deformed constituent cell by other deformed cells. Conditions are provided under which the contact forces converge to a "mixing force on constituent c" that describes the force per unit volume that the mixture exerts on constituent c and that enters into the equation of balance of linear momentum for constituent c in the presence of the remaining $N-1$ constituents. The "mixing force on constituent c" corresponds formally to the "momentum supply for constituent c" defined in ([5],Section 130; [6]) and to the "volume distributed force" considered in [7]. The formula (7.8) for the mixing force shows that it arises from a difference between the "constituent traction," computed when the constituent is separate from other constituents, and the "intermingling traction" exerted on the deforming cells of constituent c by other constituent cells.

Sections 8 and 9 provide a study of how the presence of other constituents requires a modification of the equation of balance of angular momentum and of the dissipation inequality that hold for constituent c when separated from the others. In particular, I adapt the analysis in Section 7 to obtain in Section 8 an expression for the "mixing moments" exerted on constituent c and to derive the equation of balance of angular momentum for constituent c within the mixture. A corresponding discussion in Section 9 identifies the "mixing power" expended on a given constituent and provides a dissipation inequality for that constituent within the mixture.

The formulas for the mixing forces, mixing moments, and mixing power permit us to determine their transformation properties under change of observer as well as restrictions on how they may depend upon the time-parameterized family of multiphase structured deformations that describes the macroscopic and submacroscopic evolution of all of the constituent. These observations set the stage for a study of constitutive relations for mixing forces, moments, and power to be undertaken in the future and to be built on earlier works on constitutive relations for mixtures, e.g., [5],[6],[7],[8],[9].

In Section 10, I introduce a special class of mixtures in which the mixing forces on a given constituent determine the mixing moments exerted on that constituent and, together with the spatial velocity field for that constituent, the mixing forces also determine the mixing power expended on that constituent. For these mixtures, the original field relations in Section 2 require only rather simple modifications to account for the influence of other constituents.

The point of view taken here emphasizes the treatment of each constituent as an individual continuous body whose particular evolution may be influenced by the intimate presence of other constituents. The present treatment ignores for the most part the possibility that the constituents, taken together, might be described as a new continuous body; thus, I emphasize "mixing" as opposed to "the mixture," in the spirit of Williams' study [7]. Nevertheless, in Section 11 a limit argument is used to obtain "mixing averages" for stress, velocity, and density. These are fields defined on the union of the regions occupied by the constituents and may be approximated as closely as desired by averages of stress, velocity, and density for intermingling approximations to the underlying multiphase structured deformations. I propose that mixing averages for stress and velocity be used to formulate boundary conditions imposed at points occupied by more than one constituent.

2 Field equations for separate elastic constituents

I assume that each of the N constituents of a mixture, when separated from the others, is an elastic body undergoing disarrangements and subject to the field equations proposed in [4]. For each $c = 1, \cdots, N$, the motion of the constituent c is described by means of a pair of smooth mappings $\chi_c : \mathcal{A}_c \times [0,T] \to \mathcal{E}$ and $G_c : \mathcal{A}_c \times [0,T] \to Lin\mathcal{V}$ such that the pair $(\chi_c(\cdot,t), G_c(\cdot,t))$ is a structured deformation for each $t \in [0,T]$. Here, $\mathcal{E}$ denotes three-dimensional physical space, $\mathcal{V}$ denotes its translation space, and $Lin\mathcal{V}$ denotes all linear mappings on $\mathcal{V}$. The field χ_c is called the *macro-*

scopic motion of constituent c, the tensor field G_c is called the *deformation without disarrangements* of constituent c, and the definition of "structured deformation" [1] requires that for each $t \in [0,T]$, $\chi_c(\cdot,t)$ is injective and that there is a positive number $\delta_c(t)$ such that

$$0 < \delta_c(t) \leq \det G_c(X,t) \leq \det \nabla\chi_c(X,t) \tag{2.1}$$

for all $X \in \mathcal{A}_c$. (The "permanent fracture site" described in the definition of structured deformation [1] is here taken to be the empty set and need not be displayed when denoting the motion of a constituent. Also, the definition of structured deformation requires that each reference region $\mathcal{A}_c$ be a piecewise fit region, defined in Section 6, and necessarily is bounded and open.)

As noted in [4] and explored in depth in [1], [10], the tensor field $M_c : \mathcal{A}_c \times [0,T] \to Lin\mathcal{V}$ defined by

$$M_c(X,t) := \nabla\chi_c(X,t) - G_c(X,t) \tag{2.2}$$

represents the contributions of submacroscopic disarrangements (non-smooth geometrical changes) to the macroscopic deformation $F_c := \nabla\chi_c$ of constituent c, and M_c is called the *deformation due to disarrangements* for constituent c. For the field theory proposed in [4], the volume density $\psi_c(X,t)$ of the Helmholtz free energy, measured per unit volume in the reference region $\mathcal{A}_c$, and the Piola-Kirchhoff stress $S_c(X,t)$ are related to the structured deformation $(\chi_c(\cdot,t), G_c(\cdot,t))$ of constituent c via the constitutive relations

$$\psi_c(X,t) = \Psi_c(G_c(X,t), M_c(X,t)) \tag{2.3}$$

$$S_c(X,t) = D_G\Psi_c(G_c(X,t), M_c(X,t)) + D_M\Psi_c(G_c(X,t), M_c(X,t)) \tag{2.4}$$

for all $(X,t) \in \mathcal{A}_c \times [0,T]$. In the last relation, D_G and D_M denoted differentiation with respect to G and M, respectively. The fact that G_c and M_c both transform under a change of observer in the same manner as F_c [4] yields the condition of material frame-indifference for the free energy response

$$\Psi_c(QG, QM) = \Psi_c(G, M) \tag{2.5}$$

for all $Q, G, M \in Lin\mathcal{V}$ with Q orthogonal and with G and M satisfying $0 < \det G \leq \det(G+M)$.

The field equations for constituent c, separated from the other constituents, now may be written in terms of the fields χ_c, F_c, G_c, and M_c ([4], equations (10.1) - (10.5)):

$$\rho_{c,ref}\,\ddot{\chi}_c = Div S_c + b_{c,ref} \tag{2.6}$$

$$D_G\Psi_c M_c^T + D_M\Psi_c F_c^T = 0 \tag{2.7}$$

$$sk(D_G\Psi_c M_c^T + D_M\Psi_c G_c^T) = 0 \tag{2.8}$$

$$D_G\Psi_c \cdot \dot{M}_c + D_M\Psi_c \cdot \dot{G}_c \geq 0 \tag{2.9}$$

$$0 < \delta_c < \det G_c \leq \det F_c. \tag{2.10}$$

In the equation of balance of linear momentum (2.6), the stress field S_c is given by (2.4), $\rho_{c,ref}$ denotes the mass density of constituent c in the reference region $\mathcal{A}_c$, $b_{c,ref}$ denotes the body force per unit volume in the reference region $\mathcal{A}_c$, and each superposed dot there, as well as in the dissipation inequality (2.9), denotes one differentiation with respect to time t. Div in (2.6) denotes the divergence with respect to position X in the reference region, and "sk" in (2.8) denotes the operation of taking the skew part $(A - A^T)/2$ of a tensor $A \in Lin\mathcal{V}$. The left-hand side of the dissipation inequality (2.9) represents the power expended by stresses without disarrangements at disarrangement sites and by stresses due to disarrangements away from disarrangement sites, and the relation (2.8) is equivalent to the assertion that the left-hand side of (2.9) is frame-indifferent [4]. Together, (2.8) and (2.5) imply that the Cauchy stress $(\det F_c)^{-1}S_c F_c^T$ for constituent c is symmetric and, consequently, that the balance of angular momentum for constituent c is satisfied [4].

The remaining field relation (2.7) was derived in [4] through the availability of both an additive and a multiplicative decomposition of the stress tensor $(\det K_c)S_c$, where $K_c = F_c^{-1}G_c$. This *consistency relation* is a tensorial equation that restricts how the macroscopic deformation $F_c = G_c + M_c$ may be apportioned between deformations G_c without disarrangements and deformations M_c due to disarrangements. Thus (2.7) obviates the need to impose additional constitutive relations such as an evolution equation for G_c or for M_c, as are imposed in theories of materials with internal variables (plasticity or viscoplasticity, for example).

We note that the field relations (2.6) - (2.10) reduce to those of dynamical, finite elasticity when $D_M\Psi_c(G, 0)$ is identically zero and when the body undergoes only classical deformations, i.e., when the field M_c vanishes [4]. The richer field theory provided by the field relations (2.6) - (2.10) allows an elastic body to incorporate voids at a submacroscopic level and, potentially, to accomodate other constituents of a mixture.

3 Multiphase structured deformations

In the description below of mixing for N elastic constituents I continue to assume for $c = 1, \ldots, N$ that geometrical changes experienced by constituent c are described by means of a time-parameterized family $t \mapsto (\chi_c(\cdot, t), G_c(\cdot, t))$ of structured deformations from a reference region $\mathcal{A}_c$. I follow the discussion of "mixing deformations" in [2] and define the *volume fraction* $\varphi_c(\cdot, t) : \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t) \to [0, 1]$ of constituent c by

$$\varphi_c(x, t) = \begin{cases} \dfrac{\det G_c(\chi_c(\cdot, t)^{-1}(x), t)}{\det F_c(\chi_c(\cdot, t)^{-1}(x), t)} & \text{if } x \in \chi_c(\mathcal{A}_c, t) \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\chi_c(\cdot, t)^{-1}$ denotes the inverse of the injective mapping $\chi_c(\cdot, t)$. The inequality (2.10) tells us, as expected, that the volume fraction takes values in the interval $[0, 1]$ and that $\varphi_c(x, t) > 0$ if and only if $x \in \chi_c(\mathcal{A}_c, t)$. I call $1 - \varphi_c$ the *void fraction of constituent* c and interpret $1 - \varphi_c(x, t)$ to be the fraction of volume at the point x at time t that is available for occupation by the other $N - 1$ constituents. Henceforth, I use the notation $N^{]} := \{1, \ldots, N\}$.

Definition: Let the reference regions $\mathcal{A}_c$ for $c \in N^{]}$ be pairwise disjoint and non-empty, and let $t \in [0, T]$ be given. The list $\big((\chi_c(\cdot, t), G_c(\cdot, t)) \mid c \in N^{]}\big)$ is called a *multiphase structured deformation* if the following two conditions hold:

(MphStd1) for every $c \in N^{]}$, $(\chi_c(\cdot, t), G_c(\cdot, t))$ is a structured deformation,

(MphStd2) (Accomodation Inequality) for every $x \in \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c, t)$,

$$\sum_{c=1}^{N} \varphi_c(x, t) \leq 1. \tag{3.2}$$

The pairwise disjointness of the reference regions required in the definition means that the constituents are separated in their reference regions, and (MphStd1) asserts that each constituent undergoes a structured deformation at the given time t. The Accomodation Inequality in (MphStd2) implies that the void fraction of each constituent is no less than the sum of the volume fractions of the $N-1$ remaining constituents, so that each constituent accomodates the others in a sense to be made precise in Theorem 4.1. (See [9] for an interpretation of (3.2) in the case of strict inequality.)

I assume from now on that at every time $t \in [0,T]$ the family $\big((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{]}\big)$ is a multiphase structured deformation and say that, at a given time $t \in [0,T]$, *mixing occurs* at $x \in \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c, t)$ if there exist distinct $c, c' \in N^{]}$ such that $x \in \chi_c(\mathcal{A}_c, t) \cap \chi_{c'}(\mathcal{A}_{c'}, t)$. Since the reference regions $\mathcal{A}_c$ for $c \in N^{]}$ are pairwise disjoint and since each macroscopic deformation $\chi_c(\cdot,t)$ is injective, mixing occurs at at least one $x \in \bigcup_{c=1}^{N} \mathcal{A}_c$ if and only if the mapping $\mu(\cdot,t) : \bigcup_{c=1}^{N} \mathcal{A}_c \to \mathcal{E}$ defined by

$$\mu(X,t) := \chi_c(X,t) \qquad \text{if } X \in \mathcal{A}_c \tag{3.3}$$

fails to be injective. Thus, while each constituent undergoes a motion χ_c for which $\chi_c(\cdot,t)$ is injective at every time t, the totality of constituents undergoes a motion μ for which $\mu(\cdot,t)$ may fail to be injective at certain times, signaling the interpenetration of at least two of the constituents and, according to this discussion, the occurence of mixing.

It is helpful to consider for each $t \in [0,T]$ two factorizations of the multiphase structured deformation $\big((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{]}\big)$. The first is obtained via the factorization ([1], [2]) of each constituent structured deformation

$$(\chi_c(\cdot,t), G_c(\cdot,t)) = (i_c(\cdot,t), H_c(\cdot,t)) \circ_{std} (\chi_c(\cdot,t), F_c(\cdot,t)) \tag{3.4}$$

in which

$$i_c(x,t) = x \text{ and } H_c(x,t) = G_c(\chi_c(\cdot,t)^{-1}(x),t) F_c(\chi_c(\cdot,t)^{-1}(x),t)^{-1} \tag{3.5}$$

for all $x \in \chi_c(\mathcal{A}_c, t)$. The factor $(\chi_c(\cdot,t), F_c(\cdot,t))$ in the right-hand side of (3.4) represents a classical deformation, since the gradient of $\chi_c(\cdot,t)$ is $F_c(\cdot,t)$, while the remaining factor $(i_c(\cdot,t), H_c(\cdot,t))$ represents a purely submacroscopic deformation that accounts for all of the disarrangements associated with the given structured deformation $(\chi_c(\cdot,t), G_c(\cdot,t))$. The operation "$\circ_{std}$" in (3.4) is the composition of structured deformations defined in [1]. Defining "constituentwise composition" $\diamond$ in the obvious way, we may use the factorization (3.4) of each constituent structured deformation to obtain a first factorization of the original multiphase structured deformation:

$$\begin{aligned}\left((\chi_c(\cdot,t),G_c(\cdot,t)) \mid c\in N^{]}\right) &= \left((i_c(\cdot,t),H_c(\cdot,t)) \mid c\in N^{]}\right)\\ &\quad \diamond \left((\chi_c(\cdot,t),F_c(\cdot,t)) \mid c\in N^{]}\right).\end{aligned} \tag{3.6}$$

We note that the factor $\left((\chi_c(\cdot,t),F_c(\cdot,t)) \mid c\in N^{]}\right)$ in (3.6) need not be a multiphase structured deformation, because the volume fraction of each constituent takes on only the values 0 and 1: wherever mixing occurs, the sum of the volume fractions is at least two, and the Accomodation Inequality is violated. Likewise, the factor $\left((i_c(\cdot,t),H_c(\cdot,t)) \mid c\in N^{]}\right)$ need not be a multiphase structured deformation, because the domains $\chi_c(\mathcal{A}_c,t)$ for $c\in N^{]}$ of the mappings $(i_c(\cdot,t),H_c(\cdot,t))$ need not be disjoint.

We now choose for each $t\in[0,T]$ and for $c\in N^{]}$, a translation $\tau_c(\cdot,t):\chi_c(\mathcal{A}_c,t)\to\mathcal{E}$ with the property that the closures of the ranges $\mathcal{T}_c(t):=\tau_c(\chi_c(\mathcal{A}_c,t),t)$ for $c\in N^{]}$ are pairwise disjoint. (Such choices are possible because, as noted earlier, each set $\chi_c(\mathcal{A}_c,t)$ is bounded.) I call $\left(\tau_c(\cdot,t) \mid c\in N^{]}\right)$ a family of *separating translations* for $\left((\chi_c(\cdot,t),G_c(\cdot,t)) \mid c\in N^{]}\right)$. If we write $I_c(x,t):=I$ for all $x\in\chi_c(\mathcal{A}_c,t)$, then the fact that the gradient of a translation at every point is the identity element I of $Lin\mathcal{V}$ implies that both $(\tau_c(\cdot,t),I_c(\cdot,t))$ and $(\tau_c(\cdot,t)^{-1},I_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1})$ are classical deformations from the regions $\chi_c(\mathcal{A}_c,t)$ and $\mathcal{T}_c(t)$, respectively, for $c\in N^{]}$. The factorization (3.4) then is equivalent to

$$\begin{aligned}(\chi_c(\cdot,t),G_c(\cdot,t)) &= (i_c(\cdot,t),H_c(\cdot,t))\circ_{std}(\tau_c(\cdot,t)^{-1},I_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1})\\ &\quad \circ_{std}(\tau_c(\cdot,t),I_c(\cdot,t))\ \circ_{std}(\chi_c(\cdot,t),F_c(\cdot,t))\\ &= (\tau_c(\cdot,t)^{-1},H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1})\\ &\quad \circ_{std}(\tau_c(\cdot,t)\circ\chi_c(\cdot,t),F_c(\cdot,t)).\end{aligned} \tag{3.7}$$

The domain of the structured deformation $(\tau_c(\cdot,t)^{-1},H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1})$ on the right-hand side of (3.7) is $\mathcal{T}_c(t)$, and for $c\in N^{]}$ these regions are pairwise disjoint. Moreover, the range of $\tau_c(\cdot,t)\circ\chi_c(\cdot,t)$ on the right-hand side of (3.7) is $\mathcal{T}_c(t)$, so that the different constituents cannot undergo mixing by means of the structured deformations $(\tau_c(\cdot,t)\circ\chi_c(\cdot,t),F_c(\cdot,t))$ for $c\in N^{]}$. Consequently, the factorizations (3.7) for $c\in N^{]}$ yield the following alternative to (3.6):

$$\begin{aligned}&\left((\chi_c(\cdot,t),G_c(\cdot,t)) \mid c\in N^{]}\right)\\ =\ &\left((\tau_c(\cdot,t)^{-1},H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1}) \mid c\in N^{]}\right)\\ &\diamond\left((\tau_c(\cdot,t)\circ\chi_c(\cdot,t),F_c(\cdot,t)) \mid c\in N^{]}\right),\end{aligned} \tag{3.8}$$

in which both factors on the right hand side are multiphase structured deformations, and we have

Remark: Each multiphase structured deformation $\left((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{]}\right)$ is a composition of a multiphase structured deformation without mixing $\left((\tau_c(\cdot,t)\circ\chi_c(\cdot,t), F_c(\cdot,t)) \mid c \in N^{]}\right)$, in which each constituent undergoes a classical deformation, and a multiphase structured deformation $\left((\tau_c(\cdot,t)^{-1}, H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1}) \mid c \in N^{]}\right)$ whose volume fractions agree with those of $\left((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{]}\right)$ and in which each constituent macroscopically undergoes a translation.

These observations permit one to study the geometry of mixing in the simpler setting in which the c^{th} constituent at each time t macroscopically undergoes only the translation $\tau_c(\cdot,t)^{-1}$. I call the multiphase structured deformation $\left((\tau_c(\cdot,t)^{-1}, H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1}) \mid c \in N^{]}\right)$ in (3.8) a *translational representative* of $\left((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{]}\right)$, I call the region $\chi_c(\mathcal{A}_c,t)$ the c^{th} *constituent region* at time t, and I call $\mathcal{T}_c(t) := \tau_c(\chi_c(\mathcal{A}_c,t),t)$ the *translated* c^{th} *constituent region* at time t.

From the discussion given above the relation (3.3) and the factorization (3.8), we may conclude that mixing occurs at some $x \in \bigcup_{c=1}^{N} \tau_c(\cdot,t)^{-1}(\mathcal{T}_c(t)) = \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ if and only if the piecewise translational mapping $\mu_\mathcal{T}(\cdot,t) : \bigcup_{c=1}^{N} \mathcal{T}_c(t) \to \mathcal{E}$ defined by

$$\mu_\mathcal{T}(y,t) := \tau_c(\cdot,t)^{-1}(y) \quad \text{if } y \in \mathcal{T}_c(t) \tag{3.9}$$

fails to be injective. In Section 4, I shall approximate $\mu_\mathcal{T}(\cdot,t)$ by *injective,* piecewise affine mappings $h_\varepsilon(\cdot,t)$ that "intermingle" the N constituents without interpenetration and whose gradients $\nabla h_\varepsilon(\cdot,t)$ approximate $H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1}$ on the translated c^{th} constituent region $\mathcal{T}_c(t)$, for all $c \in N^{]}$.

4 Intermingling approximations

One of the principal results in the theory of structured deformations is the Approximation Lemma [1] that establishes, for each of the purely submacroscopic structured deformations $(i_c(\cdot,t), H_c(\cdot,t))$ in the factorization (3.6) and for each $\varepsilon > 0$, the existence of an *injective*, piecewise affine mapping $a_{c,\varepsilon}(\cdot,t)$ on the constituent region $\chi_c(\mathcal{A}_c,t)$ such that $\lim_{\varepsilon\to 0} a_{c,\varepsilon}(\cdot,t) = i_c(\cdot,t)$ and $\lim_{\varepsilon\to 0} \nabla a_{c,\varepsilon}(\cdot,t) = H_c(\cdot,t)$ (with convergence in the sense of

L^∞, i.e., essentially uniform convergence). Each approximation $a_{c,\varepsilon}(\cdot,t)$ provides a submacroscopic view of the structured deformation $(i_c(\cdot,t), H_c(\cdot,t))$, in that the constituent region $\chi_c(\mathcal{A}_c,t)$ is divided into a number of non-overlapping parallelepipeds, each of which undergoes via $a_{c,\varepsilon}(\cdot,t)$ an affine deformation whose gradient is determined by $H_c(\cdot,t)$ and whose translational part moves the parallepiped only slightly from its original position in $\chi_c(\mathcal{A}_c,t)$. As ε tends to zero, the number of parallelepipeds increases and the size decreases, supporting the attributive "submacroscopic."

The conclusion of the Approximation Lemma that $a_{c,\varepsilon}(\cdot,t)$ be injective rests crucially on the inequality (2.1) or, in the present context, on the condition

$$0 < \tilde{\delta}_c(t) < \det H_c(x,t) \leq 1 \tag{4.1}$$

that holds for all $x \in \chi_c(\mathcal{A}_c,t)$. This condition amounts to the assertion that the contribution $\det H_c(x,t)$ to macroscopic volume changes by smooth submacroscopic geometrical changes does not exceed the macroscopic volume change, $\det \nabla i_c(\cdot,t) = 1$. From the definition of the volume fractions $\varphi_c(\cdot,t)$ in (3.1) and from (3.5) we may conclude that the Accomodation Inequality (3.2) generalizes (4.1) from the case of a single constituent undergoing structured deformations to the case of N constituents undergoing multiphase structured deformations. In a future article I provide a proof of the following extension of the Approximation Lemma to multiphase structured deformations:

Theorem: (Approximation Lemma for Multiphase Structured Deformations) Let $t \in [0,T]$ and let a translational representative $\left((\tau_c(\cdot,t)^{-1}, H_c(\cdot,t) \circ \tau_c(\cdot,t)^{-1}) \mid c \in N^{\rceil}\right)$ of the multiphase structured deformation $\left((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{\rceil}\right)$ be given. For each $\varepsilon > 0$ there exists a cover of the disjoint union $\bigcup_{c=1}^{N} \mathcal{T}_c(t)$ by a collection $\mathbb{P}_\varepsilon(t)$ of mutually congruent, non-overlapping closed parallelepipeds P whose diameters tend to zero as ε tends to zero, and an injective mapping $h_\varepsilon(\cdot,t) : \bigcup_{P \in \mathbb{P}_\varepsilon(t)} \bigcup_{c=1}^{N} (\mathcal{T}_c(t) \cap IntP) \to \mathcal{E}$ such that

1. for each $P \in \mathbb{P}_\varepsilon(t)$ there is exacly one $c \in N^{\rceil}$ satisfying $\mathcal{T}_c(t) \cap IntP \neq \emptyset$; moreover,
$$h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t) \subset \chi_c(\mathcal{A}_c,t); \tag{4.2}$$
2. for each $P \in \mathbb{P}_\varepsilon(t)$ and for the unique $c \in N^{\rceil}$ satisfying $\mathcal{T}_c(t) \cap IntP \neq \emptyset$, the restriction $h_\varepsilon(\cdot,t)\mid_{\mathcal{T}_c(t)\cap IntP}$ is an affine deformation;

3. for each $P \in \mathbb{P}_\varepsilon(t)$ and for the unique $c \in N^{]}$ satisfying $\mathcal{T}_c(t) \cap IntP \neq \emptyset$,

$$\left\| (h_\varepsilon(\cdot,t) - \tau_c(\cdot,t)^{-1}) \mid_{\mathcal{T}_c(t)\cap IntP} \right\|_{L^\infty(\mathcal{T}_c(t)\cap IntP)} < \varepsilon, \tag{4.3}$$

$$\left\| (\nabla h_\varepsilon(\cdot,t) - H_c(\cdot,t) \circ \tau_c(\cdot,t)^{-1}) \mid_{\mathcal{T}_c(t)\cap IntP} \right\|_{L^\infty(\mathcal{T}_c(t)\cap IntP)} < \varepsilon. \tag{4.4}$$

We note that the first item in the Approximation Lemma amounts to the condition that the common diameter of the parallelpipeds $P \in \mathbb{P}_\varepsilon(t)$ can be made small enough that the interior $IntP$ meets exactly one of the pairwise disjoint, translated constituent regions $\mathcal{T}_1(t), \ldots, \mathcal{T}_N(t)$. Consequently, the entire collection $\mathbb{P}_\varepsilon(t)$ is partitioned into disjoint collections

$$\mathbb{P}_{\varepsilon,c}(t) := \{P \in \mathbb{P}_\varepsilon(t) \mid \mathcal{T}_c(t) \cap IntP \neq \emptyset\}, \tag{4.5}$$

and, for each $c \in N^{]}$ and $P \in \mathbb{P}_{\varepsilon,c}(t)$, I call the non-empty set $\mathcal{T}_c(t) \cap IntP$ a *(constituent) c-cell* for $h_\varepsilon(\cdot,t)$. The first two items then tell us that each c-cell $\mathcal{T}_c(t) \cap IntP$ undergoes an affine deformation, and its image $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ is included in the constituent region $\chi_c(\mathcal{A}_c,t)$. Moreover, because $h_\varepsilon(\cdot,t)$ is injective, for each $c, c' \in N^{]}$ and for each $P \in \mathbb{P}_{\varepsilon,c}(t)$ and $P' \in \mathbb{P}_{\varepsilon,c'}(t)$,

$$(c \neq c') \text{ or } (P \neq P') \;\Rightarrow\; h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t) \cap h_\varepsilon(\mathcal{T}_{c'}(t) \cap IntP', t) = \emptyset. \tag{4.6}$$

This condition tells us that the images of constituent cells of different constituents have no points in common, and the images of distinct constituent cells for the same constituent have no points in common. Hence, the mapping $h_\varepsilon(\cdot,t)$ causes constituent cells to intermingle without interpenetration, and the inequalities (4.3) and (4.4) lead us to call $h_\varepsilon(\cdot,t)$ an *intermingling approximation* for $\left((\tau_c(\cdot,t)^{-1}, H_c(\cdot,t) \circ \tau_c(\cdot,t)^{-1}) \mid c \in N^{]}\right)$. In different terms, (4.3) tells us that the *generally non-injective, piecewise translational mapping* $\mu_\mathcal{T}(\cdot,t)$ defined via $\left(\tau_c(\cdot,t)^{-1} \mid c \in N^{]}\right)$ in (3.9) is approximated by the *injective, piecewise affine mapping* $h_\varepsilon(\cdot,t)$ provided by the Approximation Lemma, while (4.4) tells us that the deformations without disarrangements $\left(H_c(\cdot,t) \circ \tau_c(\cdot,t)^{-1} \mid c \in N^{]}\right)$ are approximated by $\nabla h_\varepsilon(\cdot,t)$.

5 Constituent stress, velocity and density fields

From a time-parameterized family $t \mapsto \left((\chi_c(\cdot,t), G_c(\cdot,t)) \mid c \in N^{]}\right)$ of multiphase structured deformations that describes the mixing of N constituents, we may recover for each $c \in N^{]}$ the time-parameterized family

$t \mapsto (\chi_c(\cdot,t), G_c(\cdot,t))$ of structured deformations that describes the geometrical changes, including disarrangements, of the c^{th} constituent. Next, without regard to the other constituents, we may recover by differentiation at each time t the velocity field $\dot{\chi}_c(\cdot,t)$ and, from (2.4), the Piola-Kirchhoff stress field $S_c(\cdot,t)$ for the c^{th} constituent. Both of the fields $\dot{\chi}_c(\cdot,t)$ and $S_c(\cdot,t)$ have domain the reference region $\mathcal{A}_c$, and we may compute from them in the standard way their spatial versions: for each $x \in \chi_c(\mathcal{A}_c,t)$

$$v_c(x,t) := \dot{\chi}_c(\chi_c(\cdot,t)^{-1}(x),t) \tag{5.1}$$

and

$$T_c(x,t) := (\det F_c(X,t))^{-1} S_c(X,t) F_c(X,t)^T \mid_{X=\chi_c(\cdot,t)^{-1}(x)}. \tag{5.2}$$

I call $v_c(\cdot,t)$ the c^{th} *constituent velocity field* at time t and $T_c(\cdot,t)$ the c^{th} *constituent stress field* at time t (or the c^{th} *Cauchy stress field* at time t). By means of the local form of mass conservation for the c^{th} constituent, we also may obtain the defining formula for the c^{th} *constituent density field* $\rho_c(\cdot,t)$: for each $x \in \chi_c(\mathcal{A}_c,t)$,

$$\rho_c(x,t) := (\det F_c(X,t))^{-1} \rho_{c,ref}(X) \mid_{X=\chi_c(\cdot,t)^{-1}(x)}. \tag{5.3}$$

The relations (5.1) - (5.3) and (2.4) imply that these three c^{th} constituent fields are determined by the time-parameterized family $t \mapsto (\chi_c(\cdot,t), G_c(\cdot,t))$ of structured deformations for constituent c and by the reference density $\rho_{c,ref}$, alone. Corresponding to the assumed smoothness of χ_c and G_c at the beginning of Section 2, I assume also that the constituent fields v_c, T_c and ρ_c are smooth functions on the trajectory of the c^{th} constituent.

We note for future reference that the balance of linear momentum (2.6) for constituent c, when separated from the other constituents, takes the standard spatial form in terms of the constituent fields:

$$\rho_c \dot{v}_c = div T_c + b, \tag{5.4}$$

where the given spatial body force b (measured per unit volume in the region $\bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$) and the referential body force $b_{c,ref}$ in (2.6) are related by the formula

$$b(x,t) = (\det F_c(X,t))^{-1} b_{c,ref}(X) \mid_{X=\chi_c(\cdot,t)^{-1}(x)}. \tag{5.5}$$

(Because b is specified as part of the environment, this field does not depend upon the particular constituent being considered, and (5.5) may be viewed as a definition of the referential body force field $b_{c,ref}$ in terms of b and the given macroscopic deformation F_c of constituent c.)

6 Intermingling stress, velocity and density fields

In this section I associate with each intermingling approximation a stress field, a velocity field, and a density field that will permit in the sequel (i) the identification of the forces and moments experienced by each constituent due to the presence of the others and (ii) the derivation of balance laws for each constituent in the presence of the others. Let $t \in [0,T]$, $\varepsilon > 0$, and an intermingling approximation $h_\varepsilon(\cdot,t) : \bigcup_{P \in \mathbb{P}_\varepsilon(t)} \bigcup_{c=1}^{N} (\mathcal{T}_c(t) \cap IntP) \to \mathcal{E}$ for the translational representative $\left((\tau_c(\cdot,t)^{-1}, H_c(\cdot,t) \circ \tau_c(\cdot,t)^{-1}) \mid c \in N^{]}\right)$ be given. From the discussion in Section 4, we may represent the range of $h_\varepsilon(\cdot,t)$ as the disjoint union of affinely deformed constituent cells:

$$Rng\, h_\varepsilon(\cdot,t) = \bigcup_{c=1}^{N} \bigcup_{P \in \mathbb{P}_{\varepsilon,c}(t)} h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t). \tag{6.1}$$

By (4.2) in the Approximation Theorem in Section 4, each deformed c-cell $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ is included in the constituent region $\chi_c(\mathcal{A}_c, t)$, so that the constituent fields T_c, v_c, and ρ_c are defined on $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$. To exploit this fact, we let $x \in Rng\, h_\varepsilon(\cdot,t)$ be given, and note that there is exactly one $c \in N^{]}$ and $P \in \mathbb{P}_{\varepsilon,c}(t)$ such that $x \in h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$. We may then define

$$T_\varepsilon(x,t) := T_c(x,t), \quad v_\varepsilon(x,t) := v_c(x,t), \quad \rho_\varepsilon(x,t) := \rho_c(x,t). \tag{6.2}$$

If $x \in (Ext\, Rng\, h_\varepsilon(\cdot,t)) \cap \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$, with $Ext\, Rng\, h_\varepsilon(\cdot,t)$ the exterior of $Rng\, h_\varepsilon(\cdot,t)$, then I define

$$T_\varepsilon(x,t) := 0, \quad v_\varepsilon(x,t) := 0, \quad \rho_\varepsilon(x,t) := 0. \tag{6.3}$$

Relations (6.2) and (6.3) define the *intermingling stress, velocity, and density fields* $T_\varepsilon(\cdot,t)$, $v_\varepsilon(\cdot,t)$, and $\rho_\varepsilon(\cdot,t)$ as tensor, vector, and scalar fields, respectively, with common domain $\bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t) \setminus Bdy\, Rng\; h_\varepsilon(\cdot,t)$.

The detailed definition of structured deformations and property (PF1) in [1] assure that $\bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ and $Rng\; h_\varepsilon(\cdot,t)$ are piecewise fit regions, i.e, that each is a finite union of bounded, regularly open sets of finite perimeter whose boundaries have zero volume. For a given $c \in N^{]}$ the smoothness of the constituent fields $T_c(\cdot,t)$, $v_c(\cdot,t)$, and $\rho_c(\cdot,t)$ assure ([10], Section 3) that at area-almost every point x of the essential boundary $Eby\,Rng\ h_\varepsilon(\cdot,t)$ each of the intermingling fields $T_\varepsilon(\cdot,t)$, $v_\varepsilon(\cdot,t)$, and $\rho_\varepsilon(\cdot,t)$ has two traces, corresponding to the two opposite unit normal vectors at x. Moreover, each of these points x of the essential boundary $Eby\,Rng\ h_\varepsilon(\cdot,t)$ corresponds to a point of contact of exactly two deformed constituent cells (from the same constituent or from different constituents) *or* corresponds to a point of contact of a deformed constituent cell and the set $(Ext\,Rng\,h_\varepsilon(\cdot,t))\cap\bigcup_{c=1}^{N}\chi_c(\mathcal{A}_c,t)$, which contains no deformed constituent cells. In Section 7, the traces of intermingling fields at such points enter into the identification of the forces exerted on a given constituent by the others and into the derivation of the equation of balance of linear momentum for that constituent in the prescence of the others.

7 Mixing forces and the balance of linear momentum

Let $c\in N^{]}$, $t\in[0,T]$, $\varepsilon>0$, $h_\varepsilon(\cdot,t):\bigcup_{c=1}^{N}\bigcup_{P\in\mathbb{P}_{\varepsilon,c}(t)}\mathcal{T}_c(t)\cap IntP\to\mathcal{E}$, and $P\in\mathbb{P}_{\varepsilon,c}(t)$ be given, and recall that $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ is an affinely deformed c-cell lying in the constituent region $\chi_c(\mathcal{A}_c,t)$. Moreover, $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ is disjoint from but may be in contact with the other deformed constituent cells $h_\varepsilon(\mathcal{T}_{c'}(t)\cap IntP',t)$ for $c'\neq c$ or $P'\neq P$. The defining relations (6.2) and (6.3) tell us that the intermingling fields $T_\varepsilon(\cdot,t)$, $v_\varepsilon(\cdot,t)$, and $\rho_\varepsilon(\cdot,t)$ agree with the constituent fields $T_c(\cdot,t)$, $v_c(\cdot,t)$, and $\rho_c(\cdot,t)$ on the deformed c-cell $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$, so that the linear momentum of the material points in $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ at time t is given by the integral $\int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}\rho_c(x,t)v_c(x,t)dV_x$, and the time derivative of the linear momentum of these material points, following the motion χ_c of constituent c, is $\int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}\rho_c(x,t)\dot{v}_c(x,t)dV_x$. Here, $\dot{v}_c=\frac{\partial}{\partial t}v_c+(grad v_c)v_c$ is the material time derivative of the constituent velocity field v_c. Moreover, the total body force at time t acting on the material points in $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ is the integral $\int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}b(x,t)dV_x$, where, as in (5.5), b is the spatial body force field measured per unit volume: $b(\cdot,t):\bigcup_{c=1}^{N}\chi_c(\mathcal{A}_c,t)\to\mathcal{V}$.

In order to identify the contact force that all of the other intermingling constituent cells exert at time t on the given deformed c-cell $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$, we recall that at (area) almost every point x in the essential boundary $Eby\ h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ of the given deformed c-cell, the intermingling stress field $T_\varepsilon(\cdot,t)$ has an exterior trace $T_\varepsilon^+(\cdot,t)$ and an interior trace $T_\varepsilon^-(\cdot,t)$. Because $T_\varepsilon(\tilde{x},t) = T_c(\tilde{x},t)$ for all $\tilde{x} \in h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ and $T_c(\cdot,t)$ is a smooth function on $\chi_c(\mathcal{A}_c,t)$, it follows that $T_\varepsilon^-(x,t) = T_c(x,t)$ for (area) almost every $x \in Eby\ h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$. However, the exterior trace satisfies $T_\varepsilon^+(x,t) = 0$ if x is a point of contact of $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ and $(Ext\,Rng\,h_\varepsilon(\cdot,t)) \cap \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ or, $T_\varepsilon^+(x,t) = T_{c'}(x,t)$, if x is a point of contact of $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ and $h_\varepsilon(\mathcal{T}_{c'}(t) \cap IntP', t)$ for some $c' \in N^{\rceil}$. In a similar way, $v_\varepsilon(\cdot,t)$ and $\rho_\varepsilon(\cdot,t)$ may be assigned at x inner traces $v_c(x,t)$ and $\rho_c(x,t)$ as well as outer traces $v_{c'}(x,t)$ and $\rho_{c'}(x,t)$ or 0.

I denote by $s_{\varepsilon,c}(x,t)$ the *intermingling traction at x*, i.e., the traction exerted at $x \in Ebyh_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ by the deformed constituent cell in contact with $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ at x. In general, the intermingling traction arises from bilateral contacts between constituent cells, and I assume that $s_{\varepsilon,c}(x,t)$ is determined in a frame-indifferent manner by the inner and outer traces of the intermingling fields $v_\varepsilon(\cdot,t)$ and $T_\varepsilon(\cdot,t)$ at x, as well as by the outer normal $n(x)$ to $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ at x. Assuming that $s_{\varepsilon,c}(\cdot,t)$ is integrable on $Ebyh_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$, we may use the Gauss-Green formula to write the *contact force exerted on* $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ in the form

$$\begin{aligned}
& \int_{Ebyh_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)} s_{\varepsilon,c}(x,t)dA_x \\
= \; & \int_{Ebyh_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)} (s_{\varepsilon,c}(x,t) - T_c(x,t)n(x))dA_x \\
& + \int_{Ebyh_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)} T_c(x,t)n(x)dA_x \qquad (7.1) \\
= \; & \int_{Ebyh_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)} (s_{\varepsilon,c}(x,t) - T_c(x,t)n(x))dA_x \\
& + \int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)} divT_c(x,t)dV_x.
\end{aligned}$$

I assume now for the given constituent c that all of the constituent fields $T_{c'}$, $v_{c'}$, $\rho_{c'}$, for $c' \in N^{\lceil}$, and the body force field b satisfy: for each $t \in [0,T]$ there is a sequence $m \mapsto \varepsilon_m > 0$ tending to zero such that, for all $m \in \mathbb{N}$ and for all $P \in \mathbb{P}_{\varepsilon_m,c}(t)$, the balance of linear momentum holds for the material points in $h_{\varepsilon_m}(\mathcal{T}_c(t) \cap IntP, t)$ at time t:

$$\begin{aligned}
& \int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} \rho_c(x,t)\dot{v}_c(x,t)dV_x \\
= \; & \int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} b(x,t)dV_x + \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} s_{\varepsilon_m,c}(x,t)dA_x.
\end{aligned} \qquad (7.2)$$

Using (7.1) we obtain for every $m \in \mathbb{N}$ and $P \in \mathbb{P}_{\varepsilon_m,c}(t)$:

$$\begin{aligned} &\textstyle\int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} \left(\rho_c \dot{v}_c - b - divT_c\right)(x,t)dV_x \\ = \ &\textstyle\int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (s_{\varepsilon_m,c}(x,t) - T_c(x,t)n(x))dA_x. \end{aligned} \tag{7.3}$$

Now let $r > 0$ and $y_0 \in \mathcal{T}_c(t)$ be given such that the closure of the ball $B(y_0, r)$ centered at y_0 of radius r is included in $\mathcal{T}_c(t)$, and let $m \in \mathbb{N}$ be given. We apply (7.3) to each deformed c-cell $h_{\varepsilon_m}(\mathcal{T}_c(t) \cap IntP, t)$ for which $P \in \mathbb{P}_{\varepsilon_m,c}(t)$ satisfies $B(y_0, r) \cap IntP \neq \emptyset$, and we note that, because the collection $\mathbb{P}_{\varepsilon_m,c}(t)$ covers $\mathcal{T}_c(t)$, those $P \in \mathbb{P}_{\varepsilon_m,c}(t)$ satisfying $B(y_0, r) \cap IntP \neq \emptyset$ cover $B(y_0, r)$. Summing both sides of (7.3) over such P and using the injectivity of $h_{\varepsilon_m}(\cdot, t)$, we obtain

$$\begin{aligned} &\textstyle\int_{h_{\varepsilon_m}(\cup_P(\mathcal{T}_c(t)\cap IntP),t)} \left(\rho_c \dot{v}_c - b - divT_c\right)(x,t)dV_x \\ = \ &\textstyle\sum_P \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (s_{\varepsilon_m,c}(x,t) - T_c(x,t)n(x))dA_x. \end{aligned} \tag{7.4}$$

The fact that h_{ε_m} is piecewise affine and injective permits us to use the Change of Variables Formula in the volume integral on the left-hand side of (7.4) to obtain the formula

$$\begin{aligned} &\textstyle\int_{h_{\varepsilon_m}(\cup_P(\mathcal{T}_c(t)\cap IntP),t)} \left(\rho_c \dot{v}_c - b - divT_c\right)(x,t)dV_x \\ = \ &\textstyle\int_{\cup_P(\mathcal{T}_c(t)\cap IntP)} \left(\rho_c \dot{v}_c - b - divT_c\right)(h_{\varepsilon_m}(y,t),t) \det \nabla h_{\varepsilon_m}(y,t)dV_y. \end{aligned} \tag{7.5}$$

Condition 3 in Theorem 4.1 and the fact that those $P \in \mathbb{P}_{\varepsilon_m,c}(t)$ satisfying $B(y_0, r) \cap IntP \neq \emptyset$ form a cover of $B(y_0, r)$ by parallelepipeds with common diameter tending to zero imply that the right-hand side of (7.5) approaches the volume integral

$$\int_{B(y_0,r)} \left(\det H_c(\rho_c \dot{v}_c - b - divT_c)\right)(\tau_c(\cdot,t)^{-1}(y),t)dV_y$$

as m tends to ∞. Recalling that $\tau_c(\cdot, t)^{-1}$ is a translation of $\mathcal{T}_c(t)$ onto $\chi_c(\mathcal{A}_c, t)$, we may apply the Change of Variables Formula to the last integral and use both (7.4) and (7.5) to conclude that

$$\begin{aligned} &\textstyle\lim_{m\to\infty} \sum_P \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (s_{\varepsilon_m,c}(x,t) - T_c(x,t)n(x))dA_x \\ = \ &\textstyle\int_{B(\tau_c(\cdot,t)^{-1}(y_0),r)} \left(\det H_c(\rho_c \dot{v}_c - b - divT_c)\right)(x,t)dV_x. \end{aligned} \tag{7.6}$$

The fact that $y_0 \in \mathcal{T}_c(t)$ and $r > 0$ are arbitrary (subject to the constraint $B(y_0, r) \subset \mathcal{T}_c(t)$) means that $\tau_c(\cdot, t)^{-1}(y_0) \in \chi_c(\mathcal{A}_c, t)$ and $r > 0$ are arbitrary, and the assumed smoothness of the fields appearing in the integrand permit us to obtain the following local version of (7.6) that holds for all x in $\chi_c(\mathcal{A}_c, t)$:

$$(\det H_c(\rho_c \dot{v}_c - b - divT_c))(x, t) = f_c(x, t), \tag{7.7}$$

where

$$f_c(x, t) := \lim_{r\to 0} \frac{1}{\text{vol}B(x,r)} \lim_{m\to\infty} \sum_P \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} s^{\Delta}_{\varepsilon_m,c}(x', t) dA_{x'}. \tag{7.8}$$

The sum on the right-hand side of (7.8) is taken over all $P \in \mathbb{P}_{\varepsilon_m,c}(t)$ satisfying $B(\tau_c(x, t), r) \cap IntP \neq \emptyset$. I call the field $f_c(\cdot, t)$ defined in (7.8) the *(volume density of) mixing force exerted on constituent c at time t*. The integrand

$$s^{\Delta}_{\varepsilon_m,c}(x', t) := s_{\varepsilon_m,c}(x', t) - T_c(x', t)\, n(x') \tag{7.9}$$

in (7.8) is the *excess traction* on constituent c, i.e., the difference between the intermingling traction $s_{\varepsilon,c}(x', t)$ and the constituent traction $T_c(x', t)n(x')$, so that a non-zero mixing force arises when, in the limit and on average, the excess traction is non-zero.

We note that constituent c is not present in the region $\bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t) \setminus \chi_c(\mathcal{A}_c, t)$, and it is then reasonable to extend the mixing force field $f_c(\cdot, t)$ from the constituent region $\chi_c(\mathcal{A}_c, t)$ to $\bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t)$ by assigning to it the value 0 outside of the constituent region $\chi_c(\mathcal{A}_c, t)$. With this extension and the definition (3.1) of the volume fraction $\varphi_c(\cdot, t)$, the relation (7.7) becomes

$$(\varphi_c(\rho_c \dot{v}_c - b - divT_c))(x, t) = f_c(x, t), \tag{7.10}$$

valid for all $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t)$.

We recall that relation (5.4) is the spatial version of the balance of linear momentum (2.6) for constituent c, when separated from the other constituents, and relation (7.10) now provides the desired extension of (5.4) to the case when the other $N - 1$ constituents are present. Accordingly, I call (7.10) the equation of *balance of linear momentum for constituent c within the mixture.* Comparing (5.4) and (7.10) permits us to conclude that the list formed by the constituent fields $T_c(\cdot, t)$, $v_c(\cdot, t)$, $\rho_c(\cdot, t)$, and the body force field $b(\cdot, t)$ must now be augmented by the scalar field φ_c, the volume fraction of constituent c, and the vector field f_c, the mixing force on constituent

c. The definitions (3.1) and (7.8), with due regard to the extension by zero in the latter definition and the assumed frame-indifference of the intermingling traction field $s_{\varepsilon,c}$, show that the volume fraction and the mixing force are objective fields, i.e., the volume fraction is invariant under a change of observer and the mixing force undergoes the very rotation that specifies a given change of observer. In particular, the mixing force differs from the body force in this regard, because the body force transforms in the same manner as the inertial forces under a change of observer and, hence, is not an objective field.

The assumptions on the intermingling traction $s_{\varepsilon,c}$ in the paragraph containing (7.1) along with the relations (7.8), (6.2), (6.3), (5.2), and (2.4) show that the mixing force field $f_c(\cdot,t)$ is determined by the list of fields $F_{c'}(\cdot,t)$, $G_{c'}(\cdot,t)$, and $\dot{\chi}_{c'}(\cdot,t)$ for $c' \in N^]$ and, apparently, by the sequence $m \mapsto h_{\varepsilon_m}(\cdot,t)$ of intermingling approximations that satisfy the conditions in Theorem 4.1 and that provide the balance equations. However, the equation of balance of linear momentum (7.10), itself, shows that $f_c(\cdot,t)$ does not depend upon the particular sequence of intermingling approximations that provide the balance relations (7.2) and that satisfy the conditions in Theorem 4.1. Finally, (7.8) (and its extension by 0) tells us that, for a given $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'},t)$, the mixing force density $f_c(x,t)$ depends upon the fields $F_{c'}(\cdot,t)$, $G_{c'}(\cdot,t)$, and $\dot{\chi}_{c'}(\cdot,t)$ for those $c' \in N^]$ for which $x \in \chi_{c'}(\mathcal{A}_{c'},t)$ and, for each such c', $f_c(x,t)$ depends only upon their values in an arbitrarily small neighborhood of $\chi_{c'}(\cdot,t)^{-1}(x) \in \mathcal{A}_{c'}$.

These observations provide a basis for formulating in a systematic manner constitutive equations for the mixing force on each constitutuent. Although such a formulation is not undertaken in the present study, it is worth observing that the resulting constitutive equations for the mixing force densities $f_c(x,t)$, $c \in N^]$, together with the balance equations (7.10) for the constituents $c \in N^]$, yield N balance equations that replace the original N balance equations (2.6) governing each constituent when separated from the others. A particular instance of the new balance equations is described in Section 10.

8 Mixing moments and the balance of angular momentum

The relations (2.5) and (2.8) are among the relations that restrict the motion of constituent c, when separated from the others, and, as remarked in Section

2, these two conditions imply that the law of balance of angular momentum for constituent c is satisfied, again when separated from the others. Becase each of (2.5) and (2.8) is a statement of frame-indifference, rather than a balance law, I shall continue to require that these relations hold, even when constituent c is intermingling with or mixing with the other constituents. This permits us to maintain in this broader context the condition that the Cauchy stress T_c be symmetric.

In the notation of Section 7, the angular momentum about a given point x_o of the material points in the region $h_\varepsilon(\mathcal{T}_c(t) \cap IntP, t)$ is $\int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}(x - x_o) \times \rho_c(x,t)v_c(x,t)dV_x$, and the time derivative of the angular momentum of these material points, following the motion χ_c of constituent c, is $\int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times\rho_c(x,t)\dot{v}_c(x,t)dV_x$. Moreover, the moment about x_o of the body force at time t acting on the material points in $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ is the integral $\int_{h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times b(x,t)dV_x$, where, as in (5.5), b is the spatial body force field measured per unit volume: $b(\cdot,t) : \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t) \to \mathcal{V}$. Finally, the assumptions made in Section 7 tell us that *the moment about x_o of the contact force exerted on $h_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)$ by other constituent cells* is given by the surface integral $\int_{Ebyh_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times s_{\varepsilon,c}(x,t)dA_x$, and a standard argument employing the symmetry of T_c and the Gauss-Green formula permits us to rewrite the moment about x_o of that contact force as

$$\begin{aligned}&\int_{Ebyh_\varepsilon(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times(s_{\varepsilon,c}(x,t)-T_c(x,t)\,n(x))dA_x\\&+\int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times divT_c(x,t)dV_x\end{aligned} \tag{8.1}$$

I assume now that the constituent fields $T_{c'}$, $v_{c'}$, $\rho_{c'}$ for $c' \in N^{[}$, the body force field b, and the point x_o are such that for each $t\in[0,T]$ there is a sequence $m\mapsto\varepsilon_m>0$ tending to zero such that, for all $m\in\mathbb{N}$ and for all $P\in\mathbb{P}_{\varepsilon_m,c}(t)$, not only that the balance of linear momentum holds for the material points in $h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)$ at time t but also that the balance of angular momentum about x_o holds for those points:

$$\begin{aligned}&\int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times(\rho_c\dot{v}_c-b-divT_c)\,(x,t)dV_x\\=\;&\int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)}(x-x_o)\times(s_{\varepsilon_m,c}(x,t)-T_c\,(x,t)\,n(x))dA_x,\end{aligned} \tag{8.2}$$

where the alternative expression (8.1) for the moment of the contact forces has been used to obtain the terms involving T_c on both sides of this relation.

If we use (7.10) to replace $\rho_c \dot{v}_c - b - divT_c$ by $f_c / \det H_c$ on the left-hand side of (8.2), employ the definition (7.9) of the excess traction, and follow the reasoning in Section 7 we obtain the following analogue of (7.6):

$$\begin{aligned} &\lim_{m\to\infty} \sum_P \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (x - x_o) \times s^{\Delta}_{\varepsilon_m,c}(x,t) dA_x \\ = \quad &\int_{B(\tau_c(\cdot,t)^{-1}(y_0),r)} (x - x_o) \times f_c(x,t) dV_x \end{aligned} \tag{8.3}$$

which holds for every $y_0 \in \mathcal{T}_c(t)$ and for all $r > 0$ sufficiently small. In the same manner as relation (7.7) followed from (7.6), we may conclude that for every $x \in \chi_c(\mathcal{A}_c, t)$ there holds

$$m_c(x, t; x_o) = (x - x_o) \times f_c(x, t) \tag{8.4}$$

where the vector on the left-hand side is the *(volume density of) mixing moments about x_o at time t exerted on constituent c at x:*

$$\begin{aligned} &m_c(x, t; x_o) \\ := \quad &\lim_{r\to 0} \lim_{m\to\infty} \frac{\sum_P \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (x' - x_0) \times s^{\Delta}_{\varepsilon_m,c}(x',t) dA_{x'}}{\text{vol}B(x,r)}. \end{aligned} \tag{8.5}$$

The sum on the right-hand side of (8.5) is taken over all $P \in \mathbb{P}_{\varepsilon_m,c}(t)$ satisfying $B(\tau_c(x,t), r) \cap IntP \neq \emptyset$.

If we extend $m_c(\cdot, t; x_o)$ from the constituent region $\chi_c(\mathcal{A}_c, t)$ to $\bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t)$ as the constant field with value 0, then the corresponding extension of the mixing force $f_c(\cdot, t)$ made in Section 7 implies that

$$m_c(x, t; x_o) = (x - x_o) \times f_c(x, t) \qquad \text{for all } x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t). \tag{8.6}$$

I call this relation the equation of *balance of angular momentum for constituent c within the mixture.* The definitions (7.8) and (8.5) of the mixing force and mixing moment densities imply that $m_c(x, t; x_o) = m_c(x, t; x) + (x - x_o) \times f_c(x, t)$ for all $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t)$, and we may conclude that *the balance of angular momentum* (8.6) *for constituent c within the mixture is equivalent to*

$$m_c(x, t; x) = 0 \qquad \text{for all } x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t), \tag{8.7}$$

i.e., *at each point* $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'},t)$ *the mixing moment about the point* x, *itself, exerted on constituent* c *must vanish.*

The definition (8.5) of $m_c(x,t;x_o)$ and the objectivity of the mixing force f_c under change of observer noted in Section 7 imply that the mixing moment m_c also transforms objectively. Moreover, the discussion at the end of Section 7 on the dependence of $f_c(x,t)$ on the fields $F_{c'}(\cdot,t)$, $G_{c'}(\cdot,t)$, and $\dot{\chi}_{c'}(\cdot,t)$ with $c' \in N^{]}$ leads also to the conclusion that $m_c(x,t;x_o)$ depends upon these fields for those $c' \in N^{]}$ for which $x \in \chi_{c'}(\mathcal{A}_{c'},t)$ and, for each such c', $m_c(x,t;x_o)$ depends only upon the values of these three fields in an arbitrarily small neighborhood of $\chi_{c'}(\cdot,t)^{-1}(x) \in \mathcal{A}_{c'}$.

9 Mixing power and the dissipation inequality

The relations (2.4), (2.6), and (2.9) hold when constituent c is separated from the other constituents and imply that the power expended on each part of the constituent region $\chi_c(\mathcal{A}_c,t)$ is no less than the rate of change of Helmholtz free energy plus the rate of change of kinetic energy of that part. We now impose this inequality on intermingling constituents and use the methods explained in Sections 7 and 8 to identify the "mixing power" and an appropriate extension of (2.9). We first note that the power expended by intermingling constituents on a deformed c-cell is

$$\int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} v_c(x,t)\cdot b(x,t)dV_x + \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} v_c(x,t)\cdot s_{\varepsilon_m,c}(x,t)dA_x,$$

which may be rewritten in the form

$$\begin{aligned}&\int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} \left(v_c\cdot(b+divT_c)+T_c\cdot gradv_c\right)(x,t)dV_x\\&+\int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} v_c(x,t)\cdot s^{\Delta}_{\varepsilon_m,c}(x,t)dA_x,\end{aligned} \tag{9.1}$$

while the rate of change of kinetic energy plus Helmholtz free energy of the material points in $h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)$, following the motion χ_c, is given by

$$\int_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} \left(\rho_c(v_c\cdot\dot{v}_c+\dot{\psi}_{sp,c})\right)(x,t)dV_x. \tag{9.2}$$

The spatial field $\psi_{sp,c}$ is the Helmholtz free energy per unit mass, and $\dot{\psi}_{sp,c}$ is the material time derivative of $\psi_{sp,c}$. I assume that the power expended

(9.1) on $h_{\varepsilon_m}(\mathcal{T}_c(t) \cap IntP, t)$ is no less than the volume integral in (9.2), for all m and P as in Sections 7 and 8. The equation of balance of linear moment (7.10) permits us to simplify the resulting inequality to read

$$\int\limits_{h_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (-v_c \cdot \tfrac{f_c}{\det H_c} + T_c \cdot gradv_c - \rho_c \dot{\psi}_{sp,c})dV + \int\limits_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} v_c \cdot s^{\Delta}_{\varepsilon_m,c} dA \geq 0 \tag{9.3}$$

where I have omitted the variables of integration. The arguments used in Section 7 to deduce (7.7) from (7.3) yield the inequality

$$\Big(p_c - v_c \cdot f_c + \det H_c(T_c \cdot gradv_c - \rho_c \dot{\psi}_{sp,c})\Big) \,|_{(x,t)} \geq 0 \tag{9.4}$$

for all $x \in \chi_c(\mathcal{A}_c, t)$, where the number

$$p_c(x,t) := \lim_{r\to 0}\liminf_{m\to\infty} \frac{\sum_p \int_{Ebhy_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} v_c(x',t) \cdot s^{\Delta}_{\varepsilon_m,c}(x',t) dA_{x'}}{\text{vol}B(x,r)} \tag{9.5}$$

is called the *mixing power expended at time t on constituent c at the point x*. The "lim inf" occurs in (9.5), rather than a double limit as in (7.8) and (8.5), because the relation (9.3) is an inequality, rather than an equality. Consequently, we may not infer in the present case that the double limit exists. If we extend the mixing power to the region $\bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t) \backslash \chi_c(\mathcal{A}_c, t)$ as the zero field, then (9.4) may be written for all $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'}, t)$ in the final form

$$\Big(\varphi_c(T_c \cdot gradv_c - \rho_c \dot{\psi}_{sp,c}) + p_c - v_c \cdot f_c\Big) \,|_{(x,t)} \geq 0. \tag{9.6}$$

I call this inequality the *dissipation inequality for constituent c within the mixture* and note that when the term $p_c - v_c \cdot f_c$ vanishes, the inequality (9.6) is equivalent to the dissipation inequality (2.9) for constituent c, when separate from the other constituents. Therefore, the field $p_c - v_c \cdot f_c$ *is the contribution of mixing to the total internal dissipation density of constituent c within the mixture*: $\varphi_c(T_c \cdot gradv_c - \rho_c \dot{\psi}_{sp,c}) + p_c - v_c \cdot f_c$. The relations

(9.5) and (7.8) permit one to write $(p_c - v_c \cdot f_c)(x,t)$ in an analogous form

$$\begin{aligned} & p_c(x,t) - v_c(x,t) \cdot f_c(x,t) \\ = & \lim_{r\to 0}\lim_{m\to\infty}\inf \frac{\sum_P \int_{Ebyh_{\varepsilon_m}(\mathcal{T}_c(t)\cap IntP,t)} (v_c(x',t)-v_c(x,t))\cdot s_{\varepsilon_m,c}(x',t)dA_{x'}}{\mathrm{vol}B(x,r)} \end{aligned} \tag{9.7}$$

I call $p_c - v_c \cdot f_c$ the *reduced mixing power expended on constituent* c, and the formula (9.7) along with (8.5) - (8.7) and a routine calculation yield the following remark.

Remark: The equation of balance of angular momentum (8.7) implies that the reduced mixing power $p_c - v_c \cdot f_c$ transforms objectively under a change of observer. Because (2.8) and (2.5) imply that $\varphi_c(T_c \cdot gradv_c - \rho_c\dot{\psi}_{sp,c})$ transforms objectively, we conclude that the total volume density $\varphi_c(T_c \cdot gradv_c - \rho_c\dot{\psi}_{sp,c}) + p_c - v_c \cdot f_c$ of internal dissipation associated with constituent c in the mixture transforms objectively.

This remark enables one to impose constitutive relations on the reduced mixing power $p_c - v_c \cdot f_c$ along the lines indicated at the end of Sections 7 and of Section 8, and a simple illustration is presented in the next section.

10 Field equations for f_c-determined mixing

A specific but reasonably broad description of mixing within the present theory arises when the (yet to be specified) constitutive relation for the mixing force f_c defined in (7.8) determines the constitutive relations both for the mixing moment m_c defined in (8.5) and for the reduced mixing power $p_c - v_c \cdot f_c$ given by (9.7) in such a way that

1. the equation of balance of angular moment (8.4) is satisfied identically for a preassigned reference point x_o:

$$m_c(x,t;x_o) = (x - x_o) \times f_c(x,t) \tag{10.1}$$

for all $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'},t)$ *and*

2. the reduced mixing power vanishes:

$$p_c(x,t) - v_c(x,t) \cdot f_c(x,t) = 0 \tag{10.2}$$

for all $x \in \bigcup_{c'=1}^{N} \chi_{c'}(\mathcal{A}_{c'},t)$.

When this is the case, I say that *constituent c undergoes f_c-determined mixing*, and we may now record for constituent c the resulting field relations for all $t \in [0,T]$ and for $X \in \mathcal{A}_c$:

$$(\rho_{c,ref}\,\ddot{\chi}_c - DivS_c - b_{c,ref})\mid_{(X,t)}= \tfrac{f_c}{\varphi_c}\mid_{(\chi_c(X,t),t)} \qquad (10.3)$$

$$\left(D_G\Psi_c M_c^T + D_M\Psi_c F_c^T\right)\mid_{(X,t)}= 0 \qquad (10.4)$$

$$\left(sk(D_G\Psi_c M_c^T + D_M\Psi_c G_c^T)\right)\mid_{(X,t)}= 0 \qquad (10.5)$$

$$\left(D_G\Psi_c \cdot \dot{M}_c + D_M\Psi_c \cdot \dot{G}_c\right)\mid_{(X,t)}\geq 0 \qquad (10.6)$$

$$\sum_{c'=1}^{N} \varphi_{c'}\mid_{(\chi_c(X,t),t)}\leq 1 \qquad (10.7)$$

The last relation (10.7) is the Accomodation Inequality (3.2), which implies the inequality "$\leq$" in (2.10), and the first relation (10.3) is the equation of balance of linear momentum (7.7) for constituent c within the mixture. The remaining three relations (10.4) - (10.6) are precisely the relations (2.7) - (2.9) in Section 2 for constituent c, when separated from the mixture. Thus, we have

Remark: When constituent c undergoes f_c-determined mixing, the field relations that govern its evolution are the relations (2.6) - (2.10) that would govern its evolution when separated from the other constituents, *except* that the balance of linear momentum (2.6) is replaced by (10.3) containing the additional term f_c/φ_c, and the inequality (2.10) is replaced by the Accomodation Inequality (10.7). The definition (7.8) of f_c as well as the Accomodation Inequality provide coupling between the evolution of constituent c and the evolution of the other constituents of the mixture.

11 Mixing averages of intermingling fields; boundary conditions

Let $t \in [0,T]$ and $\varepsilon > 0$ be given, and let $\lambda_\varepsilon(\cdot,t)$ denote any one of the intermingling fields $\rho_\varepsilon(\cdot,t)$, $v_\varepsilon(\cdot,t)$, or $T_\varepsilon(\cdot,t)$ defined in Section 6 by means of a given intermingling approximation $h_\varepsilon(\cdot,t)$ for $\left((\tau_c(\cdot,t)^{-1}, H_c(\cdot,t)\circ\tau_c(\cdot,t)^{-1})\right.$ $\left.\mid c \in N^{]}\right)$. Further, let $x_0 \in \bigcup_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ be given and put

$$C(x_0,t) := \left\{c \in N^{]} \mid x_0 \in \chi_c(\mathcal{A}_c,t)\right\}, \qquad (11.1)$$

the set of constituents present at x_0 at time t. Then $C(x_0,t)$ is a non-empty subset of $N^\rceil$, and $\bigcap_{c\in C(x_0,t)} \chi_c(\mathcal{A}_c,t)$ is a non-empty open subset of $\mathcal{E}$. Hence, we may choose $r>0$ such that the ball $B(x_0,r)$ centered at x_0 of radius r is included in $\bigcap_{c\in C(x_0,t)} \chi_c(\mathcal{A}_c,t)$, and we consider the formula

$$\begin{aligned}
&\int_{h_\varepsilon(\bigcup_{c\in C(x_0,t)} \tau_c(B(x_0,r),t)\ ,\ t)} \lambda_\varepsilon(x,t)\,dV_x \\
&= \sum_{c\in C(x_0,t)} \int_{h_\varepsilon(\tau_c(B(x_0,r),t)\,,t)} \lambda_\varepsilon(x,t)\,dV_x \qquad (11.2)\\
&= \sum_{c\in C(x_0,t)} \int_{\tau_c(B(x_0,r),t)} \det\nabla h_\varepsilon(y,t)\ \lambda_\varepsilon(h_\varepsilon(y,t),t)\,dV_y.
\end{aligned}$$

The injectivity and piecewise affine properties of $h_\varepsilon(\cdot,t)$ provided in Theorem 4.1 and the Change of Variables Formula justify the two steps in the last computation, and item 3 of Theorem 4.1 permits us to pass to the limit in (11.2) and to write

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{h_\varepsilon(\bigcup_{c\in C(x_0,t)} \tau_c(B(x_0,r),t)\ ,\ t)} \lambda_\varepsilon(x,t)\,dV_x \\
&= \sum_{c\in C(x_0,t)} \int_{\tau_c(B(x_0,r),t)} ((\det H_c)\ \lambda_c)\,(\tau_c(\cdot,t)^{-1}(y),t)\,dV_y.
\end{aligned} \qquad (11.3)$$

The fact that $\tau_c(\cdot,t)^{-1}$ is a translation permits us to employ the Change of Variables Formula again to obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{h_\varepsilon(\bigcup_{c\in C(x_0,t)} \tau_c(B(x_0,r),t)\ ,\ t)} \lambda_\varepsilon(x,t)\,dV_x \\
&= \sum_{c\in C(x_0,t)} \int_{B(x_0,r)} ((\det H_c)\ \lambda_c)\,(x,t)\,dV_x \\
&= \sum_{c\in C(x_0,t)} \int_{B(x_0,r)} (\varphi_c\ \lambda_c)\,(x,t)\,dV_x \\
&= \int_{B(x_0,r)} (\sum_{c\in C(x_0,t)} \varphi_c\ \lambda_c)(x,t)\,dV_x.
\end{aligned} \qquad (11.4)$$

The same reasoning applies when $\lambda_\varepsilon(\cdot,t)$ is replaced by the constant field 1, so that

$$\begin{aligned}\lim_{\varepsilon\to 0} vol\, h_\varepsilon(\underset{c\in C(x_0,t)}{\cup}\tau_c(B(x_0,r),t)\ ,\ \ t) &= \lim_{\varepsilon\to 0}\int_{h_\varepsilon(\bigcup_{c\in C(x_0,t)}\tau_c(B(x_0,r),t)\ ,\ t)} 1\, dV_x \\ &= \int_{B(x_0,r)}\sum_{c\in C(x_0,t)}\varphi_c(x,t)\ dV_x. \quad (11.5)\end{aligned}$$

The last two relations tell us that

$$\lim_{\varepsilon\to 0}\frac{\int_{h_\varepsilon(\underset{c\in C(x_0,t)}{\cup}\tau_c(B(x_0,r),t)\ ,\ t)}\lambda_\varepsilon(x,t)\, dV_x}{vol\, h_\varepsilon(\bigcup_{c\in C(x_0,t)}\tau_c(B(x_0,r),t)\ ,\ \ t)}$$

$$= \frac{\int_{B(x_0,r)}(\sum_{c\in C(x_0,t)}\varphi_c\ \lambda_c)(x,t)\, dV_x}{\int_{B(x_0,r)}\sum_{c\in C(x_0,t)}\varphi_c(x,t)\ dV_x}$$

and, hence, that

$$\begin{aligned}&\lim_{r\to 0}\lim_{\varepsilon\to 0}\frac{\int_{h_\varepsilon(\underset{c\in C(x_0,t)}{\cup}\tau_c(B(x_0,r),t)\ ,\ t)}\lambda_\varepsilon(x,t)\, dV_x}{vol\, h_\varepsilon(\underset{c\in C(x_0,t)}{\cup}\tau_c(B(x_0,r),t)\ ,\ \ t)} \\ &= \frac{\sum_{c\in C(x_0,t)}\varphi_c\,(x_0,t)\,\lambda_c(x_0,t)}{\sum_{c\in C(x_0,t)}\varphi_c(x_0,t)}.\end{aligned} \quad (11.6)$$

We note that the definition (3.1) of the volume fractions and (2.10) imply that the denominator of the second fraction in (11.6) is positive and equals $\sum_{c=1}^{N}\varphi_c(x_0,t)$. For $c\in N^{]}\setminus C(x_0,t)$ the constituent field $\lambda_c(\cdot,t)$ is not defined at x_0, but the volume fraction $\varphi_c(x_0,t)$ is then zero, and the numerator $\sum_{c\in C(x_0,t)}\varphi_c\,(x_0,t)\,\lambda_c(x_0,t)$ of the second fraction may be replaced without ambiguity by $\sum_{c=1}^{N}\varphi_c\,(x_0,t)\,\lambda_c(x_0,t)$. Consequently we may define

$$\bar{\lambda}(x_0,t) := \frac{\sum\limits_{c=1}^{N} \varphi_c\,(x_0,t)\,\lambda_c(x_0,t)}{\sum\limits_{c=1}^{N} \varphi_c(x_0,t)}$$

$$= \lim_{r\to 0}\lim_{\varepsilon\to 0} \frac{\int\limits_{h_\varepsilon(\underset{c\in C(x_0,t)}{\cup} \tau_c(B(x_0,r),t)\,,t)} \lambda_\varepsilon(x,t)\,dV_x}{vol\, h_\varepsilon\left(\underset{c\in C(x_0,t)}{\cup} \tau_c(B(x_0,r)\,,t)\,,\,t\right)}, \tag{11.7}$$

and I call $\bar{\lambda}(x_0,t)$ the *mixing average of the intermingling field* $\lambda_\varepsilon(\cdot,t)$ *at* x_0. Since $x_0 \in \bigcup\limits_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ is arbitrary, the mixing average field $\bar{\lambda}(\cdot,t)$ is defined on the union $\bigcup\limits_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ of the constituent regions. In this manner, we have obtained via (11.7) mixing average fields $\bar{\rho}(\cdot,t)$, $\bar{v}(\cdot,t)$, and $\bar{T}(\cdot,t)$ for the density, velocity, and Cauchy stress as limits of volume averages of the corresponding intermingling fields $\rho_\varepsilon(\cdot,t)$, $v_\varepsilon(\cdot,t)$, or $T_\varepsilon(\cdot,t)$. Of course, the first formula in (11.7) that defines the mixing average shows that $\bar{\lambda}(\cdot,t)$ depends only upon the given multiphase structured deformation $\left((\chi_c(\cdot,t),G_c(\cdot,t)) \mid c\in N^{]}\right)$ through the constituent fields $\left(\lambda_c(\cdot,t) \mid c\in N^{]}\right)$ and the volume fractions $\left(\varphi_c(\cdot,t) \mid c\in N^{]}\right)$, while $\bar{\lambda}(\cdot,t)$ does not depend upon the particular family of intermingling approximations $(h_\varepsilon(\cdot,t) \mid \varepsilon>0\,)$ that satisfies the conditions in Theorem 4.1 and appears in the second formula in (11.7).

I propose now that mixing averages be employed in formulating boundary conditions on velocity or on stress for the field relations obtained in the previous sections. For example, requiring that the traction $\bar{T}(x,t)n(x)$ associated with the mixing average stress $\bar{T}(\cdot,t)$ have a particular value at a point $x \in Eby \bigcup\limits_{c=1}^{N} \chi_c(\mathcal{A}_c,t)$ imposes (via (11.7) and through observations made in Section 5) corresponding restrictions on the fields $\left((\chi_c(\cdot,\cdot),G_c(\cdot,\cdot)) \mid c\in N^{]}\,\right)$ at points in the disjoint union $\bigcup\limits_{c=1}^{N} Eby\mathcal{A}_c$ that map into x at time t.

Acknowledgement

The author thanks W. O. Williams for valuable discussions and the Center for Nonlinear Analysis (NSF Grant No. DMS-0405343) for its support during the preparation of this paper.

References

[1] Del Piero, G., and D. R. Owen, Structured deformations of continua, *Arch. Rational Mech. Anal.* **124**, (1993) 99-155.

[2] Del Piero, G., and D. R. Owen, *Structured Deformations*, Quaderni dell' Istituto Nazionale di Alta Matematica, Gruppo Nazionale di fisica Matematica No. 58 (2000).

[3] *Multiscale Modeling in Continuum Mechanics and Structured Deformations*, ed. G. Del Piero and D. R. Owen, Springer- Wien New York (2004).

[4] Deseri, L., and D. R. Owen, Toward a field theory for elastic bodies undergoing disarrangements, *J. Elasticity* **70**, (2003) 197-236.

[5] Truesdell, C., and W. Noll, *The Non-Linear Field Theories of Mechanics*, Second edition, Springer-Verlag, Berlin Heidelberg (1992).

[6] Samohýl, I., and M. Šilhavý, Mixture invariance and its applications, *Arch. Rational Mech. Anal.* **109**, (1990) 299-321.

[7] Williams, W. O., Constitutive equations for flow of an incompressible fluid through a porous medium, *Q. Appl. Math.* **36**, (1978) 255-267.

[8] Sampaio, R., and W. O. Williams, On the viscosities of liquid mixtures, *ZAMP* **28**, (1977) 607-614.

[9] Sampaio, R., and W. O. Williams, Thermodynamics of diffusing mixtures, *J. de Mécanique* **18**, (1979) 19-45.

[10] Del Piero, G., and D. R. Owen, Integral-gradient formulae for structured deformations, *Arch. Rational Mech. Anal.* **131**, (1995) 121-138.