VARIATIONAL METHODS IN THE STUDY OF IMAGING, MICROMAGNETICS AND THIN FILMS

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ABSTRACT. The variational formulation of problems issuing from imaging, micromagnetics, membrane theories, foams, quantum dots and other physical applications often involve energies of different dimensionality, from bulk to interfacial terms, multiple scales, higher order derivatives, and discontinuous underlying fields. These present new challenges for the Calculus of Variations as existing theories usually do not apply.

Here we will give a brief tour of the variational formulation of problems issuing from imaging, micromagnetics and membrane theories. These, and other physical applications, often involve energies of different dimensionality, from bulk to interfacial terms, multiple scales, higher order derivatives, and discontinuous underlying fields. They present new challenges for the Calculus of Variations as existing theories usually do not apply.

1. IMAGING

A thorough study of the Mumford and Shah model [27] may be found in the book by Ambrosio, Fusco and Pallara [2]. The issue concerns the minimization of the functional

$$E(u,\Gamma) := \int_{\Omega\setminus\Gamma} \left(|\nabla u|^2 + \alpha|u-g|^2\right)\, dx + \beta\mathcal{H}^{N-1}(\Omega\cap\Gamma)$$

among all pairs (u,Γ), where Ω is an open bounded subset of $\mathbb{R}^N$ (typically a rectangle on the plane), $\Gamma \subset \overline{\Omega}$ is closed, $u \in C^1(\Omega\setminus\Gamma)$, $g : \Omega \to [0,1]$ represents the (data) grey level of an image, and α, β are positive parameters. In what follows, and without loss of generality, we set all parameters equal to 1.

In [15] De Giorgi and Ambrosio introduced the space SBV of special functions of bounded variation, i.e. those functions $u \in BV$, where BV is the space of functions of bounded variation, such that the Cantor part of their distributional derivative Du is null. Precisely, $u \in SBV(\Omega)$ if $u \in L^1(\Omega)$, Du is a finite Radon measure and

$$Du = \nabla u \mathcal{L}^N \lfloor \Omega + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \lfloor S(u),$$

where $\nabla u \in L^1(\Omega;\mathbb{R}^N)$ is the density of the Radon-Nikodym derivative of Du with respect to the N-dimensional Lebesgue measure restricted to Ω, $\mathcal{L}^N\lfloor\Omega$, $S(u)$ is the jump set of u with normal ν_u, u^+ and u^- are the traces of u on $S(u)$, and $\mathcal{H}^{N-1}$ is

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the $N-1$-dimensional Hausdorff measure. The space SBV provides a natural setup for free discontinuity problems, and in [16] the analysis (existence and regularity of solution) of the Mumford-Shah problem was carried out with the formulation

$$\inf\left\{\int_\Omega \left(|\nabla u|^2+|u-g|^2\right)\,dx+\mathcal{H}^{N-1}(\Omega\cap S(u)) : u\in SBV(\Omega)\right\}.$$

Later Rudin, Osher and Fatemi proposed in [29] a variant where

$$\int_\Omega |\nabla u|^2\,dx+\mathcal{H}^{N-1}(\Omega\cap S(u))$$

is replaced by the total variation $|Du|(\Omega)$. This preserves edges well but the images resulting have the *staircasing effect* where affine regions become piecewise constant. In order to resolve this phenomenon, in [10] Chan, Marquina and Mulet added a second order term and the energy becomes for $u\in W^{2,1}(\Omega)$

$$\inf\left\{\int_\Omega \left(|\nabla u|+|u-g|^2\right)\,dx+\psi(|\nabla u|)|D^2u|^2\right)\,dx$$

where $\psi:\mathbb{R}\to(0,+\infty)$ is a Borel function that preserves edges, in the sense that $\psi(t)\to 0$ when $t\to\infty$ (see also [11] for the treatment of the second-order Blake & Zisserman model in image segmentation). In [14] it is shown in one-dimension that it is energetically impossible to approach true edges (i.e. characteristic functions) using this model, although replacing $|D^2u|^2$ by $|D^2u|$ provides good analytical results and a representation of the relaxed energy

$$\mathcal{F}(u):=\inf\left\{\liminf_{n\to\infty} E(u_n) : u_n\to u \text{ in } L^1\right\}$$

is obtained in BV, where

$$E(u):=\inf\left\{\int_\Omega \left(|\nabla u|+|u-g|^2\right)\,dx+\psi(|\nabla u|)|D^2u|\right)\,dx.$$

This is, therefore, a good example of a functional on a space of (possibly) discontinuous fields, BV, involving higher (second) order derivatives and a competition between bulk and interfacial energies.

2. Micromagnetics

The continuum macroscopic behavior of a ferromagnetic body is modeled through micromagnetics. There is a vast literature in the Calculus of Variations on this subject, and we refer to [3], [12], [19], among many others.

According to Brown [8], equilibrium states of a body subject to an external magnetic field h_e correspond to (local) minimizers of the energy

$$E_\varepsilon^{\alpha,\beta,\gamma}(m):=\int_\Omega\left(\varepsilon^\alpha|\nabla m|^2+\frac{1}{\varepsilon^\beta}\varphi(m)\right)\,dx-\int_\Omega h_e\cdot m\,dx+\int_{\mathbb{R}^3}\frac{1}{\varepsilon^\gamma}|h|^2\,dx$$

where we set all physical constants equal to 1, $\alpha,\beta,\gamma\geq 0$, $\Omega\subset\mathbb{R}^3$ is the region occupied by the body, the magnetization $m:\mathbb{R}^3\to\mathbb{R}^3$ is set to be identically equal to zero outside Ω and it has constant magnitude m_s, the saturation magnetization that is a function of temperature and of the material properties, so that for *a.e.* $x\in\mathbb{R}^3$

$$|m(x)|=m_s\chi_\Omega(x).$$

Here we set $m_s = 1$. The induced magnetic field $h : \mathbb{R}^3 \to \mathbb{R}^3$ is related to m through Maxwell's equations for magnetostatics

$$\left\{ \begin{array}{ll} \operatorname{div}(m+h) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3), \\ \operatorname{curl} h = 0 & \text{in } \mathcal{D}'(\mathbb{R}^3), \end{array} \right. \tag{2.1}$$

where the equations hold in the sense of distributions. The various terms in the energy compete to lower it in different ways. The exchange energy $\int_\Omega \varepsilon^\alpha |\nabla m|^2 \, dx$ penalizes spatial changes of m and leads the body to prefer large regions of uniform magnetization (magnetic domains) separated by thin transition layers, the domain walls. The anisotropy term $\int_\Omega \frac{1}{\varepsilon^\beta} \varphi(m) \, dx$ induces m to align with the preferred crystallographic directions (easy axes), i.e. the zeros of the nonnegative density φ (a finite set of unit vectors on the sphere), the external field energy $\int_\Omega h_e \cdot m \, dx$ favors the alignment of m with h_e, and the field energy vanishes only when the magnetization is divergence free.

James and Kinderlehrer [23] studied the case of large ferromagnetic bodies where the interfacial, or exchange energy, is dominated by the other bulk terms, thus it is discarded, and $\beta = \gamma = 1$, therefore reducing the energy to

$$F(m) := \int_\Omega \frac{1}{\varepsilon^\beta} \varphi(m) \, dx - \int_\Omega h_e \cdot m \, dx + \int_{\mathbb{R}^3} \frac{1}{\varepsilon^\gamma} |h|^2 \, dx.$$

In [17] and [30] the asymptotic behavior as $\varepsilon \to 0$ of $\left\{ E_\varepsilon^{1,0,0} \right\}$ was investigated.

Currently, in collaboration with G. Bouchitté, G. Leoni and V. Millot we are identifying the Γ-limit of $\left\{ E_\varepsilon^{1,1,1} \right\}$ (see [13]) where, we recall,

$$E_\varepsilon(m) := \int_\Omega \left(\varepsilon |\nabla m|^2 + \frac{1}{\varepsilon} \varphi(m) \right) \, dx - \int_\Omega h_e \cdot m \, dx + \int_{\mathbb{R}^3} \frac{1}{\varepsilon} |h|^2 \, dx.$$

Note that in view of (2.1) m is a gradient up to a divergence-free field, and thus the energy above has the flavor of a multiscale energy involving second order derivatives.

For recent work on similar questions for several regimes of α, β and γ and for $\Omega \subset \mathbb{R}^2$, we refer to [1], [28].

3. Thin Films

Following [7], we consider a thin 3D (elastic) domain represented by

$$\Omega(\varepsilon) := \left\{ (x_1, x_2, x_3) : (x_1, x_2) \in \omega \text{ and } |x_3| < \frac{\varepsilon}{2} f_\varepsilon(x_1, x_2) \right\},$$

where ω is a bounded domain of $\mathbb{R}^2$ and $f_\varepsilon(x_1, x_2)$ determines the ε-dependent profile $x_3 = \pm f_\varepsilon(x_1, x_2)$, $\varepsilon > 0$. Define $\Omega := \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\Gamma := \partial\omega \times \left(-\frac{1}{2}, \frac{1}{2}\right)$, and $\Gamma_\varepsilon := \partial\omega \times \left(-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\right)$.

The elastic energy density in $\Omega(\varepsilon)$ is $\mathcal{W}(\varepsilon)(x_1, x_2, x_3; \cdot)$, and the thin body may be subject to a body load with density $F(\varepsilon)(x_1, x_2, x_3)$ (and possibly surface loads as well on the $\Sigma_\varepsilon^\pm := \omega \times \{x_3 = \pm\varepsilon/2 f_\varepsilon(x_1, x_2)\}$). In order to reach equilibrium, the deformation $u(\varepsilon)$ seeks to minimize

$$w \mapsto \int_{\Omega(\varepsilon)} \mathcal{W}(\varepsilon)(x_1, x_2, x_3; Dw) \, dx - \int_{\Omega(\varepsilon)} F(\varepsilon) \cdot w \, dx$$

among all kinematically admissible fields w.

As it is usual, we reformulate this problem on a fixed domain through a change of variables,

$$\begin{aligned} \Omega_\varepsilon &:= \{(x_1,x_2,x_3) : (x_1,x_2,\varepsilon x_3) \in \Omega(\varepsilon)\}, \\ u_\varepsilon(x_1,x_2,x_3) &:= u(\varepsilon)(x_1,x_2,\varepsilon x_3), \\ W_\varepsilon(x_1,x_2,x_3;\cdot) &:= W(\varepsilon)(x_1,x_2,\varepsilon x_3;\cdot), \\ F_\varepsilon(x_1,x_2,x_3) &:= F(\varepsilon)(x_1,x_2,\varepsilon x_3), \end{aligned}$$

so, equivalently, u_ε seeks to minimize

$$v \mapsto \int_{\Omega_\varepsilon} W_\varepsilon\left(x_1,x_2,x_3; D_1 v \Big| D_2 v \Big| \frac{1}{\varepsilon} D_3 v\right) dx - \int_{\Omega_\varepsilon} F_\varepsilon \cdot v \, dx$$

among all kinematically admissible fields v on Ω_ε, where $(\xi_1|\xi_2|\xi_3)$, with $\xi_i \in \mathbb{R}^3$, $i = 1,2,3$, stands for the 3×3 matrix with columns ξ_1, ξ_2, ξ_3.

Under appropriate growth conditions, minimizers of E_ε – if they exist – will L^p-converge to minimizers of that $\Gamma(L^p)$–limit, and thus a characterization of the latter will provide the asymptotic effective energy for equilibria states of Ω_ε.

We adopt the following notation: Greek letters will run from 1 to 2 when taken as indices. Thus coordinates will be denoted by x_α, x_3, and $(F_\alpha|F_3)$ stands for the 3×3 matrix with column elements F_1, F_2, F_3 (3 vectors in $\mathbb{R}^3$). We will identify $W^{1,p}(\Omega) \cap \left\{u : \frac{\partial u}{\partial x_3} = 0\right\}$ with $W^{1,p}(\omega)$ $(\Omega := \omega \times (-1,1))$.

Here for simplicity we consider the case where no loads are present, $f_\varepsilon(x_\alpha) = 1$ for all $x_\alpha \in \omega$, $W(\varepsilon)(x_1,x_2,\varepsilon x_3;\cdot) = W(\cdot)$, and we are led to the study the limit of the family of scaled energies

$$u \mapsto E(u)_\varepsilon := \int_\Omega W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\right) dx,$$

with $W : \mathbb{R}^{3\times 3} \to [0,+\infty)$ a continuous function,

$$\frac{1}{C}|F|^p - C \le W(F) \le C(1+|F|^p) \tag{3.1}$$

for some $C > 0$, $1 < p < +\infty$, and for all $F \in \mathbb{R}^{3\times 3}$.

It turns out that (see [24], [25] and [26]) the $\Gamma(L^1)$-limit of

$$E_\varepsilon(u) := \begin{cases} \int_\Omega W\left(\nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\right) dx & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ +\infty & \text{otherwise}, \end{cases}$$

is the functional

$$J_0(u) := \begin{cases} \int_\omega Q_2\overline{W}(\nabla_\alpha u)\, dx & \text{if } u \in W^{1,p}(\omega;\mathbb{R}^3), \\ +\infty & \text{otherwise}, \end{cases}$$

where $\overline{W}(\overline{F}) := \min\{W(\overline{F}|F_3) : F_3 \in \mathbb{R}^3\}$, $\overline{F}$ is a 3×2 matrix, and

$$Q_2\overline{W}(\overline{F}) := \inf\left\{\int_{Q'} \overline{W}(\overline{F} + D_\alpha\varphi(x))\, dx_\alpha : \ \varphi \in W^{1,\infty}_{\#}(Q';\mathbb{R}^3)\right\},$$

Next we take into account the possibility of fracture in a thin film and, accordingly, we add to the formulation above a crack initiation energy. Precisely, we consider

$$\int_{\Omega_\varepsilon \setminus K} W(\nabla v)\, dx + \int_{K\cap\Omega_\varepsilon} \vartheta(v^+ - v^-, \nu(K))\, d\mathcal{H}^2,$$

where $W : \mathbb{M}^{3\times3} \to [0,+\infty)$ and $\vartheta : (\mathbb{R}^3 \setminus \{0\}) \times S^2 \to [0,+\infty)$ denote bulk and surface energy density, respectively, K denotes an unknown crack surface, $\mathcal{H}^2(K)$ is the 2-dimensional Hausdorff measure of K, $\nu(K)$ is the normal to the crack surface in the reference configuration, $v \in W^{1,p}(\Omega_\varepsilon \setminus K; \mathbb{R}^3)$ is the deformation, defined in the complement of the crack, and $v^{\pm}$ are the traces of u on both sides of K. The surface energy density ϑ is a fracture initiation term defined on $\mathbb{R}^3 \times S^2$. In Griffith's theory [22] ϑ is constant; we have added a possible dependence on $\nu(K)$ to account for anisotropy, and a dependence on the jump $u^+ - u^-$ to include Barenblatt's approach [4]. Again, by performing a change of variables, and extending ϑ to $\mathbb{R}^3 \times \mathbb{R}^3$ as a homogeneous function of degree 1 in the variable ν, we arrive at the re-scaled energy

$$\int_{\Omega\setminus K_\varepsilon} W\Big(\nabla_\alpha u\Big|\frac{1}{\varepsilon}\nabla_3 u\Big)\,dx + \int_{K_\varepsilon\cap\Omega} \vartheta\Big(u^+ - u^-, \nu_\alpha(K_\varepsilon), \frac{1}{\varepsilon}\nu_3(K_\varepsilon)\Big)\,d\mathcal{H}^2.$$

Just as in the case of imaging problems (see Section 1), it is well known that free discontinuity problems formulated in terms of variable domains $\Omega\setminus K_\varepsilon$ and unknown admissible field u are very difficult to handle, and it is more convenient to set up the problem within the $SBV(\Omega;\mathbb{R}^3)$ framework, by replacing K_ε by the set $S(u)$ of essential discontinuity points for u and interpreting ∇u as an approximate gradient, thus reducing the simultaneous variation in the crack site and in the deformation to a variation on the discontinuous function u only. Precisely, $GSBV(\Omega)$ is defined as the space of scalar functions in $L^1(\Omega)$ such that for all $T > 0$ the truncations $u_T := (-T) \wedge (u \vee T)$ belong to $SBV(\Omega)$. We say that $u \in GSBV(\Omega;\mathbb{R}^m)$ if every component of u is in $GSBV(\Omega)$ (see [2]). Let $p > 1$. $SBV_p(\Omega;\mathbb{R}^m)$ and $GSBV_p(\Omega;\mathbb{R}^m)$ are defined as the subspaces of functions u of $SBV(\Omega;\mathbb{R}^m)$ and $GSBV(\Omega;\mathbb{R}^m)$, respectively, such that

$$\mathcal{H}^{N-1}(S(u)\cap\Omega) < +\infty \qquad \text{and} \qquad \nabla u \in L^p(\Omega;\mathbb{M}^{m\times N}).$$

These spaces are natural domains for the treatment of energies with bulk and surface contributions in the case where the bulk energy density grows superlinearly at infinity. This being said, the $3D$-energy of a deformation $v \in GSBV_p(\Omega_\varepsilon;\mathbb{R}^3)$ of the thin film occupying Ω_ε as a reference configuration is given by

$$\int_{\Omega_\varepsilon} W(\nabla v)\,dx + \int_{S(v)\cap\Omega_\varepsilon} \vartheta(v^+ - v^-, \nu(v))\,d\mathcal{H}^2$$

for functions $v \in GSBV_p(\Omega_\varepsilon;\mathbb{R}^3)$. We extend ϑ by positive 1-homogeneity as follows:

$$\vartheta(z,\eta) = \begin{cases} |\eta|\vartheta\Big(z, \frac{\eta}{|\eta|}\Big) & \text{if } \eta \neq 0, \\ 0 & \text{if } \eta = 0. \end{cases}$$

Changing variables and setting

$$u(x_\alpha, x_3) := v(x_\alpha, \varepsilon x_3), \quad (\nu_\alpha, \nu_3) := \nu,$$

clearly $u \in GSBV_p(\Omega;\mathbb{R}^3)$ and the integral above becomes

$$\begin{aligned}
&\int_{\Omega_\varepsilon} W\Big(\nabla_\alpha u\left(x_\alpha, \frac{x_3}{\varepsilon}\right) \Big| \frac{1}{\varepsilon}\nabla_3 u\left(x_\alpha, \frac{x_3}{\varepsilon}\right)\Big)\, dx \\
&\quad + \int_{S(v)\cap\Omega_\varepsilon} \vartheta\Big(u^+\left(x_\alpha, \frac{x_3}{\varepsilon}\right) - u^-\left(x_\alpha, \frac{x_3}{\varepsilon}\right), \\
&\qquad \nu_\alpha(u)\left(x_\alpha, \frac{x_3}{\varepsilon}\right), \frac{1}{\varepsilon}\nu_3(u)\left(x_\alpha, \frac{x_3}{\varepsilon}\right)\Big) \frac{\varepsilon}{\sqrt{\varepsilon^2|\nu_\alpha|^2(v) + \nu_3^2(v)}}\, d\mathcal{H}^2 \\
&= \varepsilon\Big[\int_\Omega W\Big(\nabla_\alpha u(x) \Big| \frac{1}{\varepsilon}\nabla_3 u(x)\Big)\, dx \\
&\quad + \int_{S(u)\cap\Omega} \vartheta\Big(u^+(x) - u^-(x), \nu_\alpha(u)(x), \frac{1}{\varepsilon}\nu_3(u)(x)\Big)\, d\mathcal{H}^2\Big].
\end{aligned}$$

Consider the ε-scaled $3D$-energies $J_\varepsilon : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by
(3.2)

$$J_\varepsilon(u) := \begin{cases} \int_\Omega W\Big(\nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\Big)\, dx + \int_{S(u)} \vartheta\Big(u^+ - u^-, \nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u)\Big)\, d\mathcal{H}^2 & \text{if } u \in GSBV_p(\Omega;\mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

We introduce the space

$$\mathcal{V} := \{u \in SBV_p(\Omega;\mathbb{R}^3) :\ \nabla_3 u = 0 \text{ a.e. and } \nu_3(u) = 0\ \mathcal{H}^2\text{-a.e.}\}.$$

Note that if $u \in \mathcal{V}$ then $D_3 u = 0$ in the sense of distributions, so that $\mathcal{V}$ can be identified with $SBV_p(\omega;\mathbb{R}^3)$.

Let $W : \mathbb{M}^{3\times 3} \to [0,+\infty)$ and $\vartheta : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3 \to [0,+\infty)$ be continuous functions, W satisfies (3.1), ϑ is symmetric and positively homogeneous of degree 1 in the second variable, i.e.,

$$\vartheta(z,\nu) = \vartheta(-z,-\nu), \qquad \vartheta(z,t\nu) = t\vartheta(z,\nu) \text{ for all } t > 0,\ z,\nu \in \mathbb{R}^3,$$

and

$$\frac{1}{C}(1+|z|) \le \vartheta(z,\nu) \le C(1+|z|)$$

for all $F \in \mathbb{M}^{3\times 3}$, $z \in \mathbb{R}^3$, $\nu \in S^2$, and for some $C > 0$ and $1 < p < +\infty$. Suppose, in addition, that ϑ satisfies the Lipschitz condition

$$|\vartheta(z,\nu) - \vartheta(z',\nu)| \le L\,|z - z'|$$

for all $F, F' \in \mathbb{M}^{3\times 3}$, $z, z' \in \mathbb{R}^3$, $\nu \in S^2$, and for some $C > 0$. It was shown in [7] that the functionals J_ε defined in (3.2) Γ-converge with respect to the $L^1(\Omega;\mathbb{R}^3)$ convergence as $\varepsilon \to 0^+$ to the functional $J_0 : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ given by

$$J_0(u) := \begin{cases} \displaystyle\int_\omega Q_2\overline{W}(\nabla_\alpha u)\, dx_\alpha + \int_{S(u)\cap\omega} R_2\overline{\vartheta}(u^+ - u^-, \nu_\alpha(u))\, d\mathcal{H}^1 & \text{if } u \in \mathcal{V}, \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\overline{\vartheta}(z,\eta) := \inf\{\vartheta(z;\eta,\xi) :\ \xi \in \mathbb{R}\}$$

for $z \in \mathbb{R}^3$ and $\eta \in \mathbb{R}^2 \setminus \{0\}$,

$$R_2\overline{\vartheta}(z,\eta) := \inf\left\{\int_{S(u)\cap Q'_\eta} \widehat{\vartheta}(u^+ - u^-, \nu(u))\, d\mathcal{H}^1 :\ u \in GSBV_{\#}(Q'_\eta;\mathbb{R}^3, z)\right\}$$

for all $z \in \mathbb{R}^3$ and $\eta \in S^1$, where Q'_η is a unit square on the plane, centered at zero and with two faces orthogonal to η, and $GSBV_{\#}(Q_\nu;\mathbb{R}^m, z)$ denotes the space of functions $u \in GSBV_\infty(Q_\nu;\mathbb{R}^m)$ which are 1-periodic in the direction orthogonal to ν, with $\nabla u = 0$ a.e., and such that $u(x) = \pm z/2$ if $\langle x, \nu\rangle = \pm 1/2$.

We remark that if the sequence $\{E_\varepsilon(u_\varepsilon)\}$ is bounded then not only we deduce that $\{u_\varepsilon\}$ will converge weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ to some $u \in W^{1,p}(\omega;\mathbb{R}^3)$ (up to the extraction of a subsequence and invoking some form of Poincaré-Friedrichs inequality), but also $\left\{\frac{1}{\varepsilon}\nabla_3 u_\varepsilon\right\}$ will converge weakly in $L^p(\Omega;\mathbb{R}^3)$ to some Cosserat vector b. In [6] the effective energy is obtained keeping track of both the membrane deformation u and the bending moment $\int_{-1/2}^{1/2} b(x_\alpha, x_3)\, dx_3$. In [18] we introduced interfacial energy as in [5], [31], while tracking down the cross-sectional behavior as in [6] without without averaging through the cross-section. We obtained a membrane whose constitutive behavior depends intrinsically upon the strength of the vanishing interfacial energy

Although all the analytical statements above were under condition (3.1), we note that this is incompatible with natural hypotheses in the context of nonlinear elasticity, where ruling out interpenetration of matter leads to strain energy densities W which blow-up as the determinant of the strain F goes to zero. Clearly this precludes a polynomial-type control from above. A thorough analysis of 3D-2D dimension reduction under assumptions (geometric rigidity) acceptable in 3D nonlinear elasticity was pioneered by Friesecke, James and Müller (see [20], [21]).

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