

# CONVERGENCE OF NUMERICAL APPROXIMATIONS OF THE INCOMPRESSIBLE NAVIER STOKES EQUATIONS WITH VARIABLE DENSITY AND VISCOSITY\*

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**Abstract.** We consider numerical approximations of incompressible Newtonian fluids having variable, possibly discontinuous, density and viscosity. Since solutions of the equations with variable density and viscosity may not be unique, numerical schemes may not converge. If the solution is unique, then approximate solutions computed using the discontinuous Galerkin method to approximate the convection of the density and stable finite element approximations of the momentum equation converge to the solution. If the solution is not unique a sub-sequence of these approximate solutions will converge to a solution.

**Key words.** Navier-Stokes equations, transport equations, Taylor-Hood approximations.

**1. Introduction.** We consider numerical approximations of the incompressible Navier Stokes equations with variable density and viscosity,

$$(1.1) \quad \begin{aligned} \rho(v_t + (v \cdot \nabla)v) + \nabla p - \operatorname{div}(\mu(\rho)D(v)) &= \rho f, \\ \operatorname{div}(v) &= 0, \\ \rho_t + \operatorname{div}(\rho v) &= 0, \end{aligned}$$

on a bounded domain  $\Omega \subset \mathbb{R}^d$  with initial and boundary conditions

$$v|_{\partial\Omega} = 0, \quad v|_{t=0} = v_0, \quad \rho|_{t=0} = \rho_0.$$

These equations model the motion of mixtures of immiscible fluids having different densities and viscosities. The density and viscosity may be discontinuous so in general the solutions will not enjoy any regularity beyond that given by the basic estimates:

$$\frac{d}{dt} \int_{\Omega} (\rho/2)|v|^2 dx + \int_{\Omega} \mu(\rho)|D(v)|^2 dx = \int_{\Omega} \rho f \cdot v dx,$$

and  $\rho \in L^\infty[0, T; L^\infty(\Omega)]$ ; in particular,  $\rho$  does not have bounded variation. In this situation we can establish convergence of approximate numerical solutions; however, in the absence of additional regularity no rates of convergence can be guaranteed.

The existence of a weak solutions to (1.1) has been established by P. L. Lions [15]. Some additional regularity was proven by Antontsev, Kazhikhov and Monakhov [1], and Fujita and Kato [6] when the viscosity is constant and the initial density is bounded from below. In [4], Desjardins provides similar results under weaker assumptions, for instance, when the viscosity does not vary too much with the density.

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To establish the existence of solutions of the Navier Stokes equations with discontinuous density and viscosity sharp results for the convection equation governing the density are required. These were developed by DiPerna and Lions [5] who showed that the weak solutions of convection equations were unique even when the velocity was not Lipschitz, so that characteristics may not exist. They also showed that the solutions would converge strongly when the velocities converged weakly in  $L^2[0, T; H_0^1(\Omega)]$ . For technical reasons they only considered velocity fields  $v$  vanishing<sup>1</sup> on  $\partial\Omega$ , and for this reason we only consider Dirichlet boundary data for  $v$ . Currently uniqueness of solution to the coupled system can only be established if the velocity and density satisfy  $\nabla v \in L^2[0, T; L^\infty(\Omega)]$ ,  $v_t \in L^2[0, T; L^\infty(\Omega)]$  and  $\nabla \rho \in L^2[0, T; L^\infty(\Omega)]$ , see [15], so in general, do not expect uniqueness. In this situation we can only show that sub-sequences of approximate solutions converge to solutions of the Navier Stokes equations.

While there is a rich body of literature on numerical approximation of the classical (constant density and viscosity) Navier Stokes equations, very few results are available for the situation considered here. Algorithms proposed for the approximation of (1.1) include front tracking techniques [7, 8] and level set/phase field methods [2, 16, 17]. Recall that level set methods seek a smooth function  $\phi$  satisfying  $\phi_t + \text{div}(\phi v) = 0$  and compute  $\rho = H(\phi)$  where  $H(\cdot)$  is a suitable translation of the Heaviside graph. Numerical approximations typically approximate the Heaviside graph to give a smooth transition over several grid points. Since  $\phi$  is “smooth” accurate approximations can be computed; however, difficulties arise when attempting to estimate the accuracy of  $\rho = H(\phi)$ . Indeed, it is difficult to write down the approximate equation satisfied by  $\rho$  in this context. For this reason we chose to compute  $\rho$  directly using the discontinuous Galerkin method [9, 12]. Below the results of Walkington [18] are used where it was shown that approximations of the density computed using the discontinuous Galerkin method would converge strongly in  $L^2[0, T; L^2(\Omega)]$ . Traditionally the analysis of schemes for hyperbolic equations is based upon the (nonlinear) theory of Kruzkov [10], which requires the coefficients to be  $C^1$ . This guarantees that the solutions are regular, in the sense that they have bounded variation, and rates of convergence can be established [11]. This theory fails for the problem considered here since  $v$  is not  $C^1$  and  $\rho$  does not have bounded variation. This problem was circumvented in [18] where the analysis draws upon the (linear) DiPerna Lions theory [5]. We refer to [18] for further discussion and references on this topic.

It will assumed that the viscosity can be determined as a continuous function of the density,  $\mu = \mu(\rho)$ . Physically each material particle has an associated viscosity, so  $\mu$  should satisfy the convection equation  $\mu_t + v \cdot \nabla \mu = 0$ . If  $\mu = \mu(\rho)$  this equation is satisfied when  $\rho_t + \text{div}(\rho v) = 0$  and the fluid is incompressible,  $\text{div}(v) = 0$ . In order to model a mixture of fluids where different components have the same density but different viscosities the convection equation for  $\mu$  is computed independently using equation (2.2). This does not change the analysis below where the major difficulties are due to the coupling between the density and velocity in the convection terms.

**1.1. Weak Solutions and the Energy Estimate.** Since the solutions of equations (1.1) are not smooth we consider weak solutions. A pair  $(v, \rho)$  is a weak solution of (1.1) with initial data  $(v_0, \rho_0) \in L^2(\Omega) \times L^\infty(\Omega)$  if

$$v \in V = \{v \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)] \mid \text{div}(v) = 0\},$$

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<sup>1</sup>Lions and DiPerna also considered the periodic problem and the convection equation on all of  $\mathbb{R}^d$ .

$\rho \in \mathcal{R} = L^\infty[0, T; L^\infty(\Omega)]$ , and

$$\int_0^T \int_\Omega -\rho v \cdot w_t - (\rho v \otimes v) \cdot \nabla w + \mu(\rho) D(v) \cdot D(w) = \int_\Omega \rho_0 v_0 \cdot w(0) + \int_0^T \int_\Omega \rho f \cdot w,$$

$$(1.2) \quad \int_0^T \int_\Omega \rho(\psi_t + v \cdot \nabla \psi) = \int_\Omega \rho_0 \psi(0),$$

for all  $w \in \{w \in \mathcal{D}([0, T] \times \Omega) \mid \operatorname{div}(w) = 0\}$  and  $\psi \in \mathcal{D}([0, T] \times \Omega)$ . DiPerna and Lions [5, 15] established existence of solutions of this weak problem when  $\rho_0$  is non-negative. Their weak solutions satisfy the natural energy estimate

$$(1.3) \quad \frac{d}{dt} \int_\Omega (\rho/2) |v|^2 + \int_\Omega \mu(\rho) |D(v)|^2 \leq \int_\Omega \rho f \cdot v,$$

which may be derived by formally setting  $w = v$  in the weak statement of the momentum equation and  $\psi = |v|^2/2$  in the weak statement of the density equation.

**1.2. Outline.** In the next section we motivate, and then state, the numerical scheme used to approximate the Navier Stokes equations with variable density and viscosity (1.1). Requiring stability, consistency, and non-negativity of the density are shown to give rise to conflicting requirements. The scheme presented in Section 2.4 satisfies these requirements and is subsequently analyzed in Section 3.

**1.3. Notation.** Below  $\Omega \subset \mathbb{R}^d$  will be a bounded domain with unit outward normal  $n$ . We will consider a regular family of finite element meshes  $\{\mathcal{T}_h\}_{h>0}$  each of which is assumed to triangulate  $\Omega$  exactly. It is assumed that the finite elements have uniformly bounded aspect ratio, and the parameter  $h > 0$  represents the diameter of the largest element in  $\mathcal{T}_h$ . The space of polynomials of degree  $k$  on an element  $K \in \mathcal{T}_h$  is denoted  $\mathcal{P}_k(K)$ . For simplicity we assume that for each  $h > 0$  a uniform partition of  $[0, T]$  used with  $t^n = n\tau$  where  $\tau = T/N$ ,  $N \in \mathbb{N}$ , is assumed to converge to zero as  $h$  tends to zero. We will denote the approximate solutions by  $(v_h, \rho_h)$ ; in particular, the dependence upon  $\tau$  is implicit. If  $a \in \mathbb{R}$  the positive and negative parts are denoted by  $a^\pm$  with  $a^+ = \max(a, 0)$  and  $a^- = \min(a, 0)$ .

Divergences of vectors and matrices are denoted  $\operatorname{div}(v) = v_{i,i}$  and  $\operatorname{div}(T)_j = T_{ij,j}$ , and gradients of vector valued quantities are interpreted as matrices,  $(\nabla v)_{ij} = v_{i,j}$ . Here indices after the comma represent partial derivatives and the summation convention is used. The symmetric part of the velocity gradient (stretching tensor) is written as  $D(v)$ . Inner products of vectors  $v, w \in \mathbb{R}^d$  are written as  $v \cdot w$  and their tensor product  $v \otimes w$  is the matrix having components  $v_i w_j$ . The Frobenius inner product of two matrices  $A, B \in \mathbb{R}^{d \times d}$  is denoted by  $A \cdot B = \sum_{i,j} A_{ij} B_{ij}$ . We frequently use the elementary identities  $AB \cdot C = A \cdot CB^T = B \cdot A^T C$ .

Standard notation is adopted for the Lebesgue spaces,  $L^p(\Omega)$ , and the Sobolev spaces,  $W^{m,p}(\Omega)$  or  $H^m(\Omega)$ . The dual exponent to  $p$  will be denoted by  $p'$ ,  $1/p + 1/p' = 1$ . Solutions of the evolution equation will be functions from  $[0, T]$  into these spaces, and we adopt the usual notion,  $L^2[0, T; H^1(\Omega)]$ ,  $C[0, T; H^1(\Omega)]$ , etc. to indicate the temporal regularity of such functions. The space of  $C^\infty$  test functions having compact support in  $\Omega$  is denoted by  $\mathcal{D}(\Omega)$ . For vector valued quantities, such as the velocity  $v$ , we write  $v \in L^2(\Omega)$ , to indicate that each component lies in the specified space. The space  $H(\operatorname{div}; \Omega)$  is the set of vector valued functions in  $L^2[0, T; L^2(\Omega)]$  with divergence in  $L^2[0, T; L^2(\Omega)]$ . Strong convergence of a sequence will be indicated as  $\rho_h \rightarrow \rho$ , weak convergence by  $\rho_h \rightharpoonup \rho$ , and weak  $\star$  convergence by  $\rho_h \rightharpoonup^* \rho$ .

## 2. Construction of Numerical Schemes.

**2.1. Overview.** Convergence proofs of numerical schemes for linear partial differential equations are almost always a variant of old adage “stability and consistency imply convergence”. For nonlinear problems usually some form of compactness is also required. Our proof of convergence follows this line of argument; in particular, numerical schemes are constructed so that discrete versions of energy estimate (1.3) (and hence stability) hold.

The low regularity of the solution gives rise to many technical problems. If high order approximations of the density are used Gibbs phenomena arise, and stable approximations of the momentum equation require the density to be truncated or projected onto a set of strictly positive functions. Since the density has low regularity we can not establish consistency of such schemes. In this situation we are forced to resort piecewise constant approximations of the density which give rise to monotone schemes. Unfortunately, piecewise constant approximations of the density gives rise to a different consistency error; specifically, jump terms arise when the test functions are not continuous.

In the current context the key compactness result is that solutions of the equation for the density  $\rho$  will converge strongly in  $L^2[0, T; L^2(\Omega)]$  when the velocity converges weakly in  $L^2[0, T; H_0^1(\Omega)]$ , [5, 15]. The analogous statement for discontinuous Galerkin approximations of the density equation was established by Walkington in [18] and this result will be used below. Again the low regularity of the velocity, which appears as a non-constant coefficient in the density equation, gives rise to technical problems. Specifically, in order to establish strong convergence of the density the (approximate) velocity fields are required to have average divergence equal to zero on each element [18].

**2.2. Stability.** The natural energy estimate given in equation (1.3) was derived assuming that the balance of mass is satisfied *exactly*. Since the balance of mass is only approximately satisfied by numerical approximations, the energy estimate is not automatic. Also, numerical approximations of the density may not be non-negative, so even if an “energy estimate” holds it may not be useful. One way to circumvent these problems observe that if  $\rho_t + \text{div}(\rho v) = 0$  then

$$\rho (v_t + (v \cdot \nabla)v) = \frac{1}{2} \left( \rho v_t + (\rho v \cdot \nabla)v + (\rho v)_t + \text{div}(\rho v \otimes v) \right).$$

Taking the dot product of the right hand side with  $v$  vanishing on  $\partial\Omega$  and integrating gives  $(d/dt) \int (\rho |v|^2 / 2)$ . This identity holds independently of the equation for the balance of mass and also holds if different approximations of the velocity are used as coefficients of the convective terms. This motivates the following weak statement of the momentum equation.

$$(2.1) \quad \frac{1}{2} \int_{\Omega} \bar{\rho} v_t \cdot w + (\rho \bar{v} \cdot \nabla)v \cdot w + (\bar{\rho} v)_t \cdot w - (\rho \bar{v} \cdot \nabla)w \cdot v \\ + \int_{\Omega} -p \text{div}(w) + \mu(\rho) D(v) \cdot D(w) = \int_{\Omega} \bar{\rho} f \cdot w.$$

In the context of a numerical scheme,  $\rho$  is an approximation of the density which may not be positive, and  $\bar{\rho}$  is a non-negative projection or truncation of  $\rho$ . Similarly, in order to obtain stability of the convection equation, a projection,  $\bar{v}$ , of  $v$  onto a suitable subspace of  $H(\text{div}; \Omega)$  may be required for the the convection terms. [18]. Selecting  $w(t) = v(t)$  in the above equation

immediately gives

$$\frac{d}{dt} \int_{\Omega} (1/2) \bar{\rho} |v|^2 + \int_{\Omega} \mu(\rho) |D(v)|^2 = \int_{\Omega} \bar{\rho} f \cdot v.$$

**2.3. Consistency.** While numerical schemes based upon the weak statement (2.1) will “automatically” be stable, they are not “automatically” consistent. Specifically, in the absence of any estimates on  $v_t$  it is necessary to integrate the first term by parts. Then

$$\begin{aligned} \int_0^T \int_{\Omega} \bar{\rho} v_t \cdot w + (\rho \bar{v} \cdot \nabla) v \cdot w &= \int_0^T \int_{\Omega} -v \cdot (\bar{\rho} w)_t + (\rho \bar{v} \cdot \nabla) v \cdot w \\ &= \int_0^T \int_{\Omega} -(\bar{\rho} - \rho)_t v \cdot w - \bar{\rho} v \cdot w_t - (\rho_t (v \cdot w) - (\rho \bar{v} \cdot \nabla) v \cdot w) \end{aligned}$$

(1) If a *high order* approximation of the density equation is used it is possible to select  $v \cdot w$  as a test function in the Galerkin approximation of  $\rho_t + \operatorname{div}(\rho \bar{v}) = 0$ . Then

$$\int_0^T \int_{\Omega} \bar{\rho} v_t \cdot w + (\rho \bar{v} \cdot \nabla) v \cdot w = \int_0^T \int_{\Omega} -(\bar{\rho} - \rho)_t v \cdot w - \bar{\rho} v \cdot w_t - (\rho \bar{v} \otimes v) \cdot \nabla w,$$

and consistency requires the first term to vanish in the limit. For the continuous problem  $\rho$  is bounded in  $L^\infty[0, T; L^\infty(\Omega)]$  so that the momentum,  $\rho v$ , is bounded in  $L^2[0, T; L^2(\Omega)]$ . Since  $\rho_t + \operatorname{div}(\rho v) = 0$  it follows that  $\rho_t$  is bounded in  $L^2[0, T; H^{-1}(\Omega)]$ . Unfortunately  $L^\infty$  bounds could not be established for high order approximations of the density, so analogous estimates could not be established for the time derivative of the discrete density. For this reason we could not construct non-negative approximations,  $\bar{\rho}$ , for which  $(\bar{\rho} - \rho)_t$  converged to zero in  $L^2[0, T; H^{-1}(\Omega)]$ . In particular, we could not establish consistency of numerical schemes constructed using high order approximations of the density equation.

(2) If *piecewise constant* approximations of the density are used then numerical approximations of  $\rho$  are non-negative so it is possible to select  $\bar{\rho} = \rho$ . The first term in (2.1) then becomes

$$\int_0^T \int_{\Omega} \rho v_t \cdot w + (\rho \bar{v} \cdot \nabla) v \cdot w = \int_0^T \int_{\Omega} -\rho v \cdot w_t - \rho_t v \cdot w + (\rho \bar{v} \cdot \nabla) v \cdot w.$$

To establish consistency we would like to multiply the Galerkin approximation of  $\rho_t + \operatorname{div}(\rho \bar{v}) = 0$  by  $v \cdot w$ . When the density is approximated using piecewise constant functions we must first approximate  $v \cdot w$  by a (discontinuous) piecewise constant function. This leads to an expression of the form

$$\int_0^T \int_{\Omega} \rho v_t \cdot w + (\rho \bar{v} \cdot \nabla) v \cdot w = \int_0^T \int_{\Omega} -\rho v \cdot w_t - (\rho \bar{v} \otimes v) \cdot \nabla w + \text{“jump terms”},$$

and the scheme is consistent provided the “jump terms” vanish in the limit. In Section 3 we show that the jump terms do vanish in the limit which establishes consistency.

**2.4. Scheme.** In light of the above discussion we will consider approximations of the incompressible Navier Stokes equations having variable density and viscosity where the density is approximated using piecewise constant approximations in space and time, and the momentum equation is approximated using the implicit Euler scheme with velocity-pressure spaces satisfying

the Babuska-Brezzi condition. In order to minimize the technicalities it will be assumed that the pressure space contains the (discontinuous) piecewise constant functions on each triangulation. Relaxing this condition is considered in Section 4. Since the accuracy of the piecewise constant approximation of the density is formally first order, at each discrete time we can first advance the density and then the velocity and pressure without further loss of accuracy. In this situation the linear systems for the density and velocity/pressure can be decoupled.

Given a triangulation  $\mathcal{T}_h$  of  $\Omega$  and time step  $\tau = T/N$  let

$$\mathcal{R}_h = \{\rho \in L^2(\Omega) \mid \rho|_K \in \mathbb{R}, \forall K \in \mathcal{T}_h\}.$$

If  $\rho^0$  is the projection of  $\rho(0)$  onto  $\mathcal{R}_h$ , the (piecewise constant) discontinuous Galerkin approximation of  $\rho(t^n)$  satisfies  $\rho^n \in \mathcal{R}_h$ , and

$$(2.2) \quad \int_K \rho^n \psi^n + \tau \int_{\partial K} (\rho_-^n (v^{n-1} \cdot n)^+ + \rho_+^n (v^{n-1} \cdot n)^-) \psi^n = \int_K \rho^{n-1} \psi^n,$$

for  $K \in \mathcal{T}_h$  and  $\psi^n \in \mathbb{R}$ . Here  $v \cdot n = (v \cdot n)^+ + (v \cdot n)^-$  are the positive and negative parts of  $v \cdot n$  and  $\rho_\pm^n(x) = \lim_{s \searrow 0} \rho^n(x \pm sn)$  so that the middle term gives the ‘‘upwind’’ value of  $\rho^n$ .

To march the velocity forward, let

$$V_h \subset \{v \in H_0^1(\Omega) \mid v|_K \in \mathcal{P}_k(K), K \in \mathcal{T}_h\},$$

and

$$P_h \subset \{p \in L^2(\Omega)/\mathbb{R} \mid p|_K \in \mathcal{P}_\ell(K), K \in \mathcal{T}_h\},$$

be a pair spaces satisfying the Babuska-Brezzi condition and let  $v^0$  be the  $L^2(\Omega)$  projection of  $v(0)$  onto  $V_h$ . Then the approximations,  $(v^n, p^n) \in V_h \times P_h$ , of  $(v(t^n), p(t^n))$  are the solution of

$$(2.3) \quad \begin{aligned} & \frac{1}{2} \int_\Omega \rho^{n-1} \left( \frac{v^n - v^{n-1}}{\tau} \right) \cdot w + (\rho^n v^{n-1} \cdot \nabla) v^n \cdot w + \left( \frac{(\rho v)^n - (\rho v)^{n-1}}{\tau} \right) \cdot w - (\rho^n v^{n-1} \cdot \nabla) w \cdot v^n \\ & + \int_\Omega -p^n \operatorname{div}(w) + \mu^n D(v^n) \cdot D(w) = \int_\Omega \rho^n f^n \cdot w, \end{aligned}$$

$$\int_\Omega \operatorname{div}(v^n) q = 0,$$

for all  $(w, q) \in V_h \times P_h$ . In the above  $f^n$  is an approximation of the average of  $f$  on  $(t^{n-1}, t^n)$  and  $\mu^n = \mu(\rho^n)$ .

### 3. Analysis of the Numerical Scheme.

**3.1. Estimates.** To establish stability of the scheme (2.2)-(2.3) we first state the natural energy estimate the scheme was designed to satisfy.

**Notation:** If  $\{v_n\}_{n=0}^N \subset V_h$  and  $\{\rho^n\}_{n=0}^N \subset \mathcal{R}_h$  we let  $v_h \in L^2[-\tau, T; V_h]$  and  $\rho_h \in L^2[-\tau, T; \mathcal{R}_h]$  denote the piecewise constant functions taking values  $v^n$  and  $\rho^n$  on  $(t^{n-1}, t^n]$  respectively.

LEMMA 3.1. *Let  $(\rho_h, v_h, p_h)$  be the approximate solution of equations (1.1) computed using the scheme (2.2)-(2.3). Then*

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \rho^n |v^n|^2 + \frac{1}{2} \sum_{i=1}^n \int_{\Omega} \rho^{n-1} |v^n - v^{n-1}|^2 + \sum_{i=1}^n \tau \int_{\Omega} \mu^n |D(v^n)|^2 \\ = \frac{1}{2} \int_{\Omega} \rho^0 |v^0|^2 + \sum_{i=1}^n \tau \int_{\Omega} \rho^n f^n \cdot v^n. \end{aligned}$$

*Let the pressure space contains the piecewise constant functions. If  $0 < c \leq \rho(0) \leq C$  and  $0 < c \leq \mu(\rho) \leq C$  for constants  $c, C \in \mathbb{R}$ ,  $v_0 \in L^2(\Omega)$ , and  $f \in L^2[0, T; L^2(\Omega)]$ , then  $\{v_h\}_{h>0}$  is bounded in  $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$  and*

$$\int_{\tau}^T \|v_h(t) - v_h(t - \tau)\|_{L^2(\Omega)}^2 \leq C(v_0, f)\tau.$$

The first estimate follows directly upon substituting  $w = v^n$  and  $q = p^n$  into equations (2.3). The assumption on the pressure space guarantees that the scheme for the density is monotone [18, Theorem 6.1], so the bounds on the initial data are preserved,

$$(3.1) \quad \min_{\Omega} \rho^0 \leq \rho^n(x) \leq \max_{\Omega} \rho^0, \quad x \in \Omega.$$

The bounds on  $\{v_h\}_{h>0}$  then follow from the energy estimate.

**3.2. Consistency of the Density Equation.** To establish compactness of the sequence  $\{v_h\}_{h>0}$  in  $L^2[0, T; L^2(\Omega)]$  it is necessary use test functions  $\psi$  in the discrete density equation (2.2) which are not piecewise constant. This gives rise to consistency errors which are estimated in this section. The following lemma provides explicit expressions for these errors.

LEMMA 3.2. *Let  $\rho_h \in \mathcal{R}_h$  satisfy equation (2.2). If  $\psi \in H_0^1(\Omega)$  and  $\bar{\psi} \in \mathcal{R}_h$  is the function taking the average value of  $\psi$  on each element  $K \in \mathcal{T}_h$  then*

$$\int_{\Omega} (\rho^n - \rho^{n-1})\psi - \tau \int_{\Omega} \rho^n v^{n-1} \cdot \nabla \psi = \tau \int_{\Omega} \rho^n (\psi - \bar{\psi}) \operatorname{div}(v^{n-1}) + \tau \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho^n] (v^{n-1} \cdot n)^- (\psi - \bar{\psi}),$$

where the value of  $\bar{\psi}$  on  $\partial K$  is taken as  $\bar{\psi}|_K$ , (that is the trace from inside  $K$ ) and  $[\rho^n] = \rho_+^n - \rho_-^n$ .

*Proof.* Select  $\psi^n = \bar{\psi}|_K$  in equation (2.2) and sum over all of the simplices  $K \in \mathcal{T}_h$  to get

$$(3.2) \quad \int_{\Omega} \rho^n \psi + \tau \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_-^n (v^{n-1} \cdot n)^+ + \rho_+^n (v^{n-1} \cdot n)^-) \bar{\psi} = \int_{\Omega} \rho^{n-1} \psi.$$

In the middle term  $\rho_{\pm}^n(x) = \lim_{s \searrow 0} \rho^n(x \pm sn)$  and  $\bar{\psi}|_{\partial K} = \bar{\psi}|_K$ . If  $\mathcal{E}_0$  denotes all of the interior edges (faces in 3d) of the elements, then the middle term may be written as

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_-^n (v^{n-1} \cdot n)^+ + \rho_+^n (v^{n-1} \cdot n)^-) \bar{\psi} = \sum_{e \in \mathcal{E}_0} \int_e -(\rho_-^n (v^{n-1} \cdot N)^+ + \rho_+^n (v^{n-1} \cdot N)^-) [\bar{\psi}].$$

Here  $N$  is one of the normals to  $e$ ,  $\rho_{\pm}^n(x) = \lim_{s \searrow 0} \rho^n(x \pm sN)$  and  $[\bar{\psi}] = \bar{\psi}_+ - \bar{\psi}_-$ . Integrals over the edges  $e \subset \partial\Omega$  vanish since  $\int_e v \cdot n = 0$  on boundary edges. If  $\psi : \bar{\Omega} \rightarrow \mathbb{R}$  is continuous and vanishes on  $\partial\Omega$  then  $[\psi] = 0$  on each edge  $e \in \mathcal{E}_0$ , so  $[\bar{\psi}] = [\bar{\psi} - \psi]$ . Reversing the above calculation shows

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_-^n(v^{n-1} \cdot n)^+ + \rho_+^n(v^{n-1} \cdot n)^-) \bar{\psi} \\
&= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_-^n(v^{n-1} \cdot n)^+ + \rho_+^n(v^{n-1} \cdot n)^-) (\bar{\psi} - \psi) \\
&= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_-^n(v^{n-1} \cdot n) + (\rho_+^n - \rho_-^n)(v^{n-1} \cdot n)^-) (\bar{\psi} - \psi) \\
&= \sum_{K \in \mathcal{T}_h} \int_K \operatorname{div}(\rho^n v^{n-1}(\bar{\psi} - \psi)) + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_+^n - \rho_-^n)(v^{n-1} \cdot n)^- (\bar{\psi} - \psi) \\
&= - \int_{\Omega} (\rho^n v^{n-1} \cdot \nabla \psi + \rho^n (\psi - \bar{\psi}) \operatorname{div}(v^{n-1})) + \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho^n](v^{n-1} \cdot n)^- (\psi - \bar{\psi})
\end{aligned}$$

The last step used the property that  $\rho^n$  and  $\psi^n$  are constant on each element  $K \in \mathcal{T}_h$ . The lemma follows upon substituting this expression into (3.2).  $\square$

The following corollary expresses the weak statement satisfied by the discrete density  $\rho_h$  in a more convenient form. Given a sequence of functions  $\{\psi^n\}_{n=0}^N \subset \mathcal{R}_h$ , recall the convention that  $\psi_h : (-\tau, T] \rightarrow \mathcal{R}_h$  is the function taking values  $\psi_h(t) = \psi^n$  for  $t \in ((n-1)\tau, n\tau]$ .

**COROLLARY 3.3.** *Let  $\rho_h \in \mathcal{R}_h$  satisfy equation (2.2),  $\{\psi^n\}_{n=0}^N \subset H_0^1(\Omega)$  and let  $\{\bar{\psi}^n\}_{n=0}^N \subset \mathcal{R}_h$  be the piecewise constant approximations of  $\{\psi^n\}_{n=0}^N$ . Then*

$$\begin{aligned}
(3.3) \quad & \sum_{j=m+1}^n \int_{\Omega} (\rho^j - \rho^{j-1}) \psi^j - \int_{t^m}^{t^n} \int_{\Omega} \rho_h v_h(\cdot - \tau) \cdot \nabla \psi_h \\
&= \int_{t^m}^{t^n} \int_{\Omega} \rho_h (\psi_h - \bar{\psi}_h) \operatorname{div}(v_h(\cdot - \tau)) + \int_{t^m}^{t^n} \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho_h](v_h(\cdot - \tau) \cdot n)^- (\psi_h - \bar{\psi}_h),
\end{aligned}$$

where the value of  $\bar{\psi}_h$  on  $\partial K$  is taken as  $\bar{\psi}_h|_K$ .

The two terms on the right of represent the consistency error of the piecewise constant DG scheme. The first of these is easy to bound, and the following lemma will be used to bound the last term.

**LEMMA 3.4.** *Let  $K \subset \mathbb{R}^d$  be a simplex,  $v \in \mathcal{P}_{\ell}(K)^d$ ,  $\psi \in \mathcal{P}_{\ell}(K)$  and  $p, q \geq 1$ . Then there exists a constant  $C$  depending only upon  $d, p, q, \ell$  and the aspect ratio of  $K$  such that*

$$\int_{\partial K} |v \cdot n| |\psi - \bar{\psi}|^q \leq C \|v\|_{L^{p'}(K)} h_K^{q-1} |\psi|_{W^{1,pq}(K)}^q,$$

where  $\bar{\psi} = (1/|K|) \int_K \psi$  is the average of  $\psi$  on  $K$  and  $h_K$  is the diameter of  $K$ .

*Proof.* Let  $\hat{K}$  be the usual reference simplex and  $\chi(\xi) = x_0 + B\xi$  be an affine mapping of  $\hat{K}$  to  $K$ . We use a hat to denote the natural correspondence between functions defined on  $K$  and  $\hat{K}$ ,



$\hat{\psi} = \psi \circ \chi$ . Writing the integral over the boundary as the sum over the faces  $e \subset \partial K$ , gives

$$\begin{aligned} \int_{\partial K} |v \cdot n| |\psi - \bar{\psi}|^q &= \sum_{e \subset \partial K} \int_e |v \cdot n| |\psi - \bar{\psi}|^q \\ &= \sum_{\hat{e} \subset \partial \hat{K}} \frac{|e|}{|\hat{e}|} \int_{\hat{e}} |\hat{v} \cdot n| |\hat{\psi} - \bar{\psi}|^q \\ &\leq C \sum_{\hat{e} \subset \partial \hat{K}} |e| \|\hat{v}\|_{L^{p'}(\hat{K})} \|\hat{\psi} - \bar{\psi}\|_{L^{qp}(\hat{K})}^q \\ &\leq C \sum_{\hat{e} \subset \partial \hat{K}} |e| \|\hat{v}\|_{L^{p'}(\hat{K})} |\hat{\psi}|_{W^{1,pq}(\hat{K})}^q. \end{aligned}$$

To obtain the third line the trace theorem was used and the finite dimensionality of  $\mathcal{P}_\ell(\hat{K})$  allows the use of the indicated norms. The last line follows from the Poincare inequality and the observation that the average of  $\psi$  is the average of  $\hat{\psi}$ .

Since

$$\|\hat{v}\|_{L^{p'}(\hat{K})} = (|\hat{K}|/|K|)^{1/p'} \|v\|_{L^{p'}(K)}, \quad |\hat{\psi}|_{W^{1,pq}(\hat{K})} \leq C(|\hat{K}|/|K|)^{1/pq} h_K |\psi|_{W^{1,pq}(K)},$$

and  $|e| \leq C|K|/h_K$ , where  $C$  depends upon the aspect ratio of  $K$ , the lemma follows.  $\square$

**3.3. Compactness.** The energy estimate shows that  $\{v_h\}_{h>0}$  is bounded in  $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$ . A result of J. L. Lions [13, 14] states that compactness of the sequence in  $L^2[0, T; L^2(\Omega)]$  will follow if

$$\int_\delta^T \|v_h(t) - v_h(t - \delta)\|_{L^2(\Omega)} \leq C\delta^\alpha,$$

for  $0 \leq \delta \leq T$  and some  $\alpha > 0$ .

We recall J. L. Lions' argument [13] which shows that weak solutions of the the Navier Stokes equations with variable density and viscosity satisfy this inequality. This proof carries over to Galerkin approximations with a few modifications which will be considered subsequently.

**J. L. Lions' Compactness Argument.** Beginning with the weak statement of the momentum equation (c.f. equation (2.3))

$$\int_\Omega (1/2)(\rho v_t + (\rho v)_t) \cdot w + (1/2)(\rho v \cdot \nabla) v \cdot w - (1/2)(\rho v \cdot \nabla) w \cdot v - p \operatorname{div}(w) + \mu D(v) \cdot D(w) = \int_\Omega \rho f \cdot w,$$

the identity  $\rho v_t = (\rho v)_t - \rho_t v$  is used to obtain

$$\int_\Omega (\rho v)_t \cdot w = \int_\Omega \rho f \cdot w + (1/2)\rho_t(v \cdot w) - (1/2)(\rho v \cdot \nabla) v \cdot w + (1/2)(\rho v \cdot \nabla) w \cdot v + p \operatorname{div}(w) - \mu D(v) \cdot D(w).$$

The second term on the right hand side can be eliminated upon writing the weak statement of the balance of mass as

$$(3.4) \quad \int_\Omega \rho_t \psi = \int_\Omega \rho v \cdot \nabla \psi,$$

and selecting  $\psi = v.w$ , to give

$$\int_{\Omega} (\rho v)_t . w = \int_{\Omega} \rho f . w + (\rho v . \nabla) w . v + p \operatorname{div}(w) - \mu D(v) \cdot D(w).$$

Integrating this expression with respect to  $s \in (t - \delta, t)$  and letting  $w = w(t)$  be independent of  $s$  gives

$$\int_{\Omega} \rho v|_{t-\delta}^t . w(t) = \int_{t-\delta}^t \int_{\Omega} \rho f . w(t) + (\rho v . \nabla) w(t) . v + p \operatorname{div}(w(t)) - \mu D(v) \cdot D(w(t)) \, ds.$$

Integrating the weak statement of the balance of mass (3.4) with respect to  $s \in (t - \delta, t)$  and setting  $\psi = v(t).w(t)$  shows

$$\int_{\Omega} \rho|_{t-\delta}^t v(t) . w(t) = \int_{t-\delta}^t \int_{\Omega} \rho v . \nabla(v(t) . w(t)).$$

Subtracting this equation from the previous one and observing that

$$\rho v|_{t-\delta}^t . w(t) - \rho|_{t-\delta}^t v(t) . w(t) = \rho(t - \delta)(v(t) - v(t - \delta)) . w(t)$$

gives

$$(3.5) \quad \int_{\Omega} \rho(t - \delta)(v(t) - v(t - \delta)) . w(t) \\ = \int_{t-\delta}^t \int_{\Omega} \rho f . w(t) + (\rho v . \nabla) w(t) . v + p \operatorname{div}(w(t)) - \mu D(v) \cdot D(w(t)) - \rho v . \nabla(v(t) . w(t)) \, ds.$$

Upon electing  $w(t) = v(t) - v(t - \delta)$  the left hand side dominates  $\|v(t) - v(t - \delta)\|_{L^2(\Omega)}^2$  when  $\rho$  is bounded below by  $c > 0$ . The right hand side is estimated using the following lemma.

**LEMMA 3.5.** *Let  $\Omega \subset \mathbb{R}^d$  with  $d = 2$  or  $3$  and  $v, w \in L^2[0, T; H_0^1(\Omega)] \cap L^\infty[0, T; L^2(\Omega)]$ ,  $\rho, \mu \in L^\infty[0, T; L^\infty(\Omega)]$ , and  $f \in L^2[0, T; L^2(\Omega)]$ . Then there exists a constant  $C > 0$  and  $\alpha \in (0, 1)$  such that*

$$\left| \int_{\delta}^T \int_{t-\delta}^t \int_{\Omega} \rho f . w(t) + (\rho v . \nabla) w(t) . v - \mu D(v) \cdot D(w(t)) - \rho v . \nabla(v(t) . w(t)) \, ds \, dt \right| \leq C \delta^\alpha,$$

for  $0 < \delta < T$ . Here  $C$  depends only upon  $d, T$ , and  $f, \rho, \mu, v$ , and  $w$  through the norms stated in the hypotheses.

This lemma follows from elementary applications of Holders inequality and the Sobolev embedding theorem,  $\|v\|_{L^4(\Omega)} \leq \|v\|_{L^2(\Omega)}^\beta \|\nabla v\|_{L^2(\Omega)}^{1-\beta}$  where  $\beta = 1/2$  and  $\beta = 1/4$  for  $d = 2$  and  $3$  respectively.

**Compactness for the Discrete Problem.** The calculations above can be replicated for numerical solutions computed using equations (2.2)-(2.3) provided the discrete weak statement of the balance of mass (3.3) is used in place of equation (3.4). This gives rise to four extra terms on the right hand side of equation (3.5).

**LEMMA 3.6.** *Let  $\{(\rho_h, v_h)\}_{h>0}$  be numerical approximations of the Navier Stokes equations with variable density and viscosity computed using equations (2.2)-(2.3) over a quasi regular family of triangulations  $\{\mathcal{T}_h\}_{h>0}$  of  $\Omega \subset \mathbb{R}^d$  with  $d = 2$  or  $3$ . Assume that*

- $v^0 \in L^2(\Omega)$ ,  $\rho^0 \in L^\infty(\Omega)$  satisfies  $0 < c \leq \rho^0(x) \leq C$ , and  $f \in L^2[0, T; L^2(\Omega)]$ .
- $\mu : \mathbb{R} \rightarrow \mathbb{R}^+$  is continuous.
- The spaces for the velocity and pressure satisfy the Babuska Brezzi condition and the pressure space contains the piecewise constant functions.

Then there exists a constant  $C > 0$  independent of  $h$  and  $\alpha \in (0, 1)$  such that

$$\int_{\delta}^T \|v_h(t) - v_h(t - \delta)\|_{L^2(\Omega)}^2 \leq C\delta^\alpha,$$

for  $0 < \delta < T$ .

*Proof.* Since  $\{v_h\}_{h>0}$  are piecewise constant in time it suffices to consider  $\delta$  a multiple of the time step  $\tau$ . Writing  $(t - \delta, t) = (t^m, t^n)$  and  $w(t) = v^n - v^m = w^{mn}$ , the discrete analogue of equation (3.5) is

$$\begin{aligned} & \int_{\Omega} \rho^m (v^n - v^m) \cdot w^{mn} \\ &= \int_{t^m}^{t^n} \int_{\Omega} \rho_h f \cdot w^{mn} + (\rho_h v_h(\cdot - \tau) \cdot \nabla) w^{mn} \cdot v_h - \mu_h D(v_h) \cdot D(w^{mn}) - \rho_h v_h \cdot \nabla (v^n \cdot w^{mn}) \, ds \\ & \quad + \int_{t^m}^{t^n} \int_{\Omega} \rho_h (v_h \cdot w^{mn} - \overline{v_h \cdot w^{mn}}) \operatorname{div}(v_h(\cdot - \tau)) - \rho_h (v^n \cdot w^{mn} - \overline{v^n \cdot w^{mn}}) \operatorname{div}(v_h(\cdot - \tau)) \, ds \\ & \quad + \tau \sum_{j=m+1}^n \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho^j] (v^{j-1} \cdot n)^- (v^j \cdot w^{mn} - \overline{v^j \cdot w^{mn}}) - [\rho^j] (v^{j-1} \cdot n)^- (v^n \cdot w^{mn} - \overline{v^n \cdot w^{mn}}) \end{aligned}$$

The last two lines represent the consistency errors associated with the density equation and the term involving the pressure vanishes since  $w^{mn} = v^n - v^m$  is discretely divergence free. (Recall that  $\bar{\psi}$  is the piecewise constant function having average value of  $\psi$  on each element  $K \in \mathcal{T}_h$ .)

We bound the first term in each of the last two lines since the second is bounded similarly. Since  $\|\bar{\psi}\|_{L^p(\Omega)} \leq \|\psi\|_{L^p(\Omega)}$  for any  $\psi \in L^p(\Omega)$  the first term on the second to last line may be bounded as

$$\begin{aligned} & \int_{t^m}^{t^n} \int_{\Omega} \rho_h (v_h \cdot w^{mn} - \overline{v_h \cdot w^{mn}}) \operatorname{div}(v_h(\cdot - \tau)) \, ds \\ & \leq 2 \|\rho_h\|_{L^\infty[0, T; L^\infty(\Omega)]} \int_{t^m}^{t^n} \|v_h\|_{L^4(\Omega)} \|w^{mn}\|_{L^4(\Omega)} \|\operatorname{div}(v_h(\cdot - \tau))\|_{L^2(\Omega)} \, ds \\ & \leq C \int_{t^m}^{t^n} \|\nabla v_h\|_{L^2(\Omega)}^{1-\beta} \|\operatorname{div}(v_h(\cdot - \tau))\|_{L^2(\Omega)} \, ds \|\nabla w^{mn}\|_{L^2(\Omega)}^{1-\beta} \\ & \leq C \int_{t^m}^{t^n} \|\nabla v_h\|_{L^2(\Omega)}^{1-\beta} \|\nabla v_h(\cdot - \tau)\|_{L^2(\Omega)} \, ds \|\nabla w^{mn}\|_{L^2(\Omega)}^{1-\beta} \\ & \leq C \|\nabla v_h\|_{L^2[0, T; L^2(\Omega)]}^{2-\beta} (t^n - t^m)^{\beta/2} \|\nabla w^{mn}\|_{L^2(\Omega)}^{1-\beta} \end{aligned}$$

Since  $v_h, w^{mn} \in L^\infty[0, T; L^2(\Omega)]$ , quantities involving  $\|v_h\|_{L^2(\Omega)}$  and  $\|w^{mn}\|_{L^2(\Omega)}$  have been absorbed into the constant  $C$ . Integrating with respect to  $t^n \in (\delta, T)$  and recalling that  $t^m = t^n - \delta$  and  $w^{mn} = v_h(t^n) - v_h(t^n - \delta)$  shows that this term by a constant of the form  $C\delta^{\beta/2}$  with  $C$  independent of  $h$ .

To estimate the first jump term use Lemma 3.4 with  $q = 1$  to obtain

$$\begin{aligned}
& \tau \sum_{j=m+1}^n \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho^j] (v^{j-1} \cdot n)^- (v^j \cdot w^{mn} - \overline{v^j \cdot w^{mn}}) \\
& \leq C \|\rho_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \tau \sum_{j=m+1}^n \sum_{K \in \mathcal{T}_h} \int_K \|v^{j-1}\|_{L^{p'}(K)} |v^j \cdot w^{mn}|_{W^{1,p}(K)} \\
& \leq C \|\rho_h\|_{L^\infty[0,T;L^\infty(\Omega)]} \int_{t^m}^{t^n} \|v_h(\cdot - \tau)\|_{L^{p'}(\Omega)} |v_h \cdot w^{mn}|_{W^{1,p}(\Omega)}.
\end{aligned}$$

When  $p < 2$  the terms of the form  $\nabla(v \cdot w)$  can be estimated as

$$\begin{aligned}
|\nabla(v \cdot w)|_{W^{1,p}(\Omega)} & \leq \| |v| |\nabla w| + |\nabla v| |w| \|_{L^p(\Omega)} \\
& \leq \| |v| |\nabla w| \|_{L^p(\Omega)} + \| |\nabla v| |w| \|_{L^p(\Omega)} \\
& \leq \|v\|_{L^{2p/(2-p)}(\Omega)} \|\nabla w\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \|w\|_{L^{2p/(2-p)}(\Omega)}.
\end{aligned}$$

Letting  $p = 4/3$ , so that  $2p/(2-p) = 4$ , the first jump term becomes

$$\begin{aligned}
& \tau \sum_{j=m+1}^n \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho^j] (v^{j-1} \cdot n)^- (v^j \cdot w^{mn} - \overline{v^j \cdot w^{mn}}) \\
& \leq C \int_{t^m}^{t^n} \|v_h(\cdot - \tau)\|_{L^4(\Omega)} \left( \|v_h\|_{L^4(\Omega)} \|\nabla w^{mn}\|_{L^2(\Omega)} + \|\nabla v_h\|_{L^2(\Omega)} \|w^{mn}\|_{L^4(\Omega)} \right) ds \\
& \leq C \int_{t^m}^{t^n} \|\nabla v_h(\cdot - \tau)\|_{L^2(\Omega)}^{1-\beta} \left( \|\nabla v_h\|_{L^2(\Omega)}^{1-\beta} \|\nabla w^{mn}\|_{L^2(\Omega)} + \|\nabla v_h\|_{L^2(\Omega)} \|\nabla w^{mn}\|_{L^2(\Omega)}^{1-\beta} \right) ds \\
& \leq C \left( \|\nabla v_h\|_{L^2[0,T;L^2(\Omega)]}^{2(1-\beta)} (t^n - t^m)^\beta \|\nabla w^{mn}\|_{L^2(\Omega)} \right. \\
& \quad \left. + \|\nabla v_h\|_{L^2[0,T;L^2(\Omega)]}^{2-\beta} (t^n - t^m)^{\beta/2} \|\nabla w^{mn}\|_{L^2(\Omega)}^{1-\beta} \right).
\end{aligned}$$

Integrating with respect to  $t^n \in (\delta, T)$  bounds this term by a constant of the form  $C\delta^{\beta/2}$  with  $C$  independent of  $h$ .  $\square$

**3.4. Convergence.** The bound on the sequence  $\{v_h\}_{h>0}$  and the compactness result of J. L. Lions [13, 14] allows passage to a subsequence for which

$$v_h \rightharpoonup^* v \text{ in } L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)], \quad \text{and} \quad v_h \rightarrow v \text{ in } L^2[0, T; L^2(\Omega)].$$

In this situation Theorem 5.1 of [18] states that corresponding densities  $\{\rho_h\}_{h>0}$  converge to a limit  $\rho$  in  $L^2[0, T; L^2(\Omega)]$ . We will show that the pair  $(v, \rho)$  is a solution of (1.1).

Note that since  $\{\rho_h\}_{h>0} \subset L^\infty[0, T; L^\infty(\Omega)]$  is bounded and converges in  $L^2[0, T; L^2(\Omega)]$  it also converges in  $L^p[0, T; L^p(\Omega)]$  for  $p < \infty$ . Similarly, since  $\{v_h\}_{h>0}$  is bounded in  $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$  and converges in  $L^2[0, T; L^2(\Omega)]$ , the Sobolev embedding theorem and elementary interpolation show that  $v_h$  converges in  $L^p[0, T; L^q(\Omega)]$  for any pair  $p, q \geq 1$  satisfying  $1/2 < 1/q + 2/dp$ .

**THEOREM 3.7.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ , be a bounded Lipschitz and  $\{\mathcal{T}_h\}_{h>0}$  be a regular family of quasi-uniform triangulations of  $\Omega$ . Let  $f \in L^2[0, T; L^2(\Omega)]$ ,  $v_0 \in L^2(\Omega)$ , and  $\rho_0 \in L^\infty(\Omega)$  satisfy*

$0 < c \leq \rho_0(x) \leq C$  for positive constants  $c$  and  $C$ . Assume that the viscosity,  $\mu : \mathbb{R} \rightarrow (0, \infty)$ , is a continuous non-negative function of the density.

Let  $\{(v_h, \rho_h)\}_{h>0}$  be the approximate solution of equations (1.1) computed using the scheme presented in Section 2.4 with time steps  $\tau$  converging to zero as  $h \rightarrow 0$ . In particular, assume that the density is computed using the piecewise constant discontinuous Galerkin method and that the velocity-pressure spaces satisfy the Babuska-Brezzi condition and that the pressure space contains the piecewise constant functions.

Then, after passing to a subsequence, the densities  $\{\rho_h\}$  converge strongly in  $L^2[0, T; L^2(\Omega)]$ , and the velocities  $\{v_h\}$  converge strongly in  $L^2[0, T; L^2(\Omega)]$  and weakly star in  $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$  to a weak solution of equations (1.1) with initial data  $(v_0, \rho_0)$  and homogeneous Dirichlet boundary data on the velocity. If the solution of equation (1.1) is unique then the whole sequence  $\{(v_h, \rho_h)\}_{h>0}$  converges.

*Proof.* Notice that the hypotheses of Lemma 3.1 are satisfied since monotonicity of the scheme (2.2) for computing the density guarantees that  $0 < c \leq \rho_h(x, t) \leq C$ . Also,  $\mu : \mathbb{R} \rightarrow (0, \infty)$  is continuous so  $\mu_h = \mu(\rho_h)$  satisfies a similar inequality.

Let  $w \in \mathcal{D}([0, T] \times \Omega)$  be divergence free and let  $w^n$  be the Stokes projection of  $w(t^n)$  onto the space

$$\tilde{V}_h = \{v_h \in V_h \mid \int_{\Omega} \operatorname{div}(v_h) q_h = 0, \forall q_h \in P_h\},$$

and let  $w_h \in L^2[0, T; V_h]$  be the piecewise constant function taking values  $w^n$  on  $(t^{n-1}, t^n]$  and  $\hat{w}_h \in C[0, T; V_h]$  be the corresponding piecewise linear interpolant. Since the pair  $(V_h, P_h)$  satisfies the Babuska-Brezzi condition,  $w_h$  and  $\hat{w}_h$  converge to  $w$  in  $L^\infty[0, T; H_0^1(\Omega)]$  and  $W^{1, \infty}[0, T; H_0^1(\Omega)]$  respectively. Selecting  $w^n$  as the test function in (2.3) and summing over  $n$  gives

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^N \int_{\Omega} (\rho^{n-1} - \rho^n) v^n \cdot w^n + \sum_{n=1}^N \int_{\Omega} (\rho v)^{n-1} \cdot (w^{n-1} - w^n) \\ & + \frac{\tau}{2} \sum_{n=1}^N \int_{\Omega} (\rho^n v^{n-1} \cdot \nabla) v^n \cdot w^n - (\rho^n v^{n-1} \cdot \nabla) w^n \cdot v^n \\ & + \tau \sum_{n=1}^N \int_{\Omega} \mu^n D(v^n) \cdot D(w^n) = \int_{\Omega} \rho^0 v^0 \cdot w^0 + \tau \sum_{n=1}^N \int_{\Omega} \rho^n f^n \cdot w. \end{aligned}$$

To obtain the first line we used the identity

$$\frac{1}{2} \left( \rho^{n-1} (v^n - v^{n-1}) \cdot w^n + ((\rho v)^n - (\rho v)^{n-1}) \cdot w^n \right) = \frac{1}{2} (\rho^{n-1} - \rho^n) v^n \cdot w^n + ((\rho v)^n - (\rho v)^{n-1}) \cdot w^n,$$

and summed the second term by parts. The upper limit of the summation vanishes since  $w \in \mathcal{D}([0, T] \times \Omega)$  implies  $w^N = 0$ . Recalling the notation that  $v_h(t) \in V_h$  is the function taking on value  $v^n$  on  $(t^{n-1}, t^n]$ , we find

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^N \int_{\Omega} (\rho^{n-1} - \rho^n) v^n \cdot w^n - \int_0^T \int_{\Omega} \rho_h(\cdot - \tau) v_h(\cdot - \tau) \hat{w}_t \\ & + \frac{1}{2} \int_0^T \int_{\Omega} (\rho_h v_h(\cdot - \tau) \cdot \nabla) v_h \cdot w_h - (\rho_h v_h(\cdot - \tau) \cdot \nabla) w_h \cdot v_h \end{aligned}$$

$$+ \int_0^T \int_{\Omega} \mu_h D(v_h) \cdot D(w_h) = \int_{\Omega} \rho^0 v^0 \cdot w^0 + \int_0^T \int_{\Omega} \rho_h f_h \cdot w_h.$$

Selecting  $\psi = v_h \cdot w_h$  in Corollary 3.3 shows that the the first term can be rewritten as

$$\sum_{n=1}^N \int_{\Omega} (\rho^{n-1} - \rho^n) v^n \cdot w^n = - \int_0^T \int_{\Omega} \rho_h v_h(\cdot - \tau) \cdot \nabla(v_h \cdot w_h) - e_h,$$

where

$$e_h = \int_0^T \int_{\Omega} \rho_h (v_h \cdot w_h - \overline{v_h \cdot w_h}) \operatorname{div}(v_h(\cdot - \tau)) + \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho_h] (v_h(\cdot - \tau) \cdot n)^- (v_h \cdot w_h - \overline{v_h \cdot w_h})$$

is the consistency error. Then

$$\begin{aligned} & - \int_0^T \int_{\Omega} \rho_h(\cdot - \tau) v_h(\cdot - \tau) \hat{w}_t + (\rho_h v_h(\cdot - \tau) \otimes v_h) \cdot \nabla w_h \\ & + \int_0^T \int_{\Omega} \mu_h D(v_h) \cdot D(w_h) = \int_{\Omega} \rho^0 v^0 \cdot w^0 + \int_0^T \int_{\Omega} \rho_h f_h \cdot w_h + e_h. \end{aligned}$$

Now pass to a sub-sequence, along which  $v_h$  and  $\rho_h$  converge in  $L^2[0, T; L^2(\Omega)]$  and  $v_h \rightharpoonup v$  in  $L^2[0, T; H_0^1(\Omega)]$ . Since  $\rho_h, \mu_h \in L^\infty[0, T; L^\infty(\Omega)]$  they converge in  $L^p[0, T; L^p(\Omega)]$  for  $1 \leq p < \infty$ , and  $v_h$  converges in  $L^p[0, T; L^4(\Omega)]$  for  $p < 8/3$ . This is sufficient to pass to the limit term-by-term in the above equation and the theorem will follow provided  $e_h \rightarrow 0$ .

It suffices to show that the consistency error  $e_h$  vanishes as  $h \rightarrow 0$ . The first term in  $e_h$  is bounded using classical estimates for piecewise constant approximations [3],

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho_h (v_h \cdot w_h - \overline{v_h \cdot w_h}) \operatorname{div}(v_h(\cdot - \tau)) \\ & \leq C \|\rho_h\|_{L^\infty[0, T; L^\infty(\Omega)]} \|\operatorname{div}(v_h)\|_{L^2[0, T; L^2(\Omega)]} \|v_h \cdot w_h - \overline{v_h \cdot w_h}\|_{L^2[0, T; L^2(\Omega)]} \\ & \leq C \|\rho_h\|_{L^\infty[0, T; L^\infty(\Omega)]} \|\operatorname{div}(v_h)\|_{L^2[0, T; L^2(\Omega)]} \|v_h \cdot w_h\|_{L^2[0, T; W^{1, p}(\Omega)]} h^{1+d(1/2-1/p)}. \end{aligned}$$

As in the proof of Lemma 3.6

$$\|v_h \cdot w_h\|_{W^{1, 4/3}(\Omega)} \leq \|v_h\|_{L^4(\Omega)} \|\nabla w_h\|_{L^2(\Omega)} + \|\nabla v_h\|_{L^2(\Omega)} \|w_h\|_{L^4(\Omega)} \leq C \|v_h\|_{H^1(\Omega)} \|w_h\|_{H^1(\Omega)}.$$

It follows that  $\|v_h \cdot w_h\|_{L^2[0, T; W^{1, 4/3}]} \leq \|v_h\|_{L^2[0, T; H^1(\Omega)]} \|w_h\|_{L^\infty[0, T; H^1(\Omega)]}$  is bounded so

$$\begin{aligned} & \int_0^T \int_{\Omega} \rho_h (v_h \cdot w_h - \overline{v_h \cdot w_h}) \operatorname{div}(v_h(\cdot - \tau)) \\ & \leq C \|\rho_h\|_{L^\infty[0, T; L^\infty(\Omega)]} \|\operatorname{div}(v_h)\|_{L^2[0, T; L^2(\Omega)]} \|v_h\|_{L^2[0, T; H^1(\Omega)]} \|w_h\|_{L^\infty[0, T; H^1(\Omega)]} h^{1-d/4} \\ & \rightarrow 0. \end{aligned}$$

The second term of  $e_h$  is bounded using Lemma 3.4 with  $q \leq 2$ ,

$$\int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} [\rho_h] (v_h(\cdot - \tau) \cdot n)^- (v_h \cdot w_h - \overline{v_h \cdot w_h})$$

$$\begin{aligned}
&\leq \left( \|\rho_h\|_{L^\infty[0,T;L^\infty(\Omega)]}^{q'-2} \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} |v_h(\cdot - \tau) \cdot n| [\rho_h]^2 \right)^{1/q'} \left( \int_0^T \sum_{K \in \mathcal{T}_h} \int_{\partial K} |v_h(\cdot - \tau)| (v_h \cdot w_h - \overline{v_h \cdot w_h})^q \right)^{1/q} \\
&\leq C \|\rho_h\|_{L^\infty[0,T;L^\infty(\Omega)]}^{1-2/q'} (J_h^N)^{1/q'} \left( \int_0^T \int_\Omega \|v_h(\cdot - \tau)\|_{L^{p'}(\Omega)} |v_h \cdot w_h|_{W^{1,pq}(\Omega)}^q h^{q-1} \right)^{1/q} \\
&\leq C \|\rho_h\|_{L^\infty[0,T;L^\infty(\Omega)]}^{1-2/q'} (J_h^N)^{1/q'} \|v_h\|_{L^{r'}[0,T;L^{p'}(\Omega)]}^{1/q} |v_h \cdot w_h|_{L^{rq}[0,T;W^{1,pq}(\Omega)]} h^{1-1/q}.
\end{aligned}$$

where  $q' \geq 2$  and

$$J_h^N = \sum_{e \in \mathcal{E}_0} \int_0^T \int_e |v_h(\cdot - \tau) \cdot n| [\rho_h]^2.$$

$J_h^N$  measures the jumps in the density across the inter-element boundaries  $e \in \mathcal{E}_0$ , and it was shown in [18, Theorem 5.1] that, under the hypotheses assumed above,  $J_h^N \rightarrow 0$  as  $h$  (and  $\tau$ ) tend to zero. The parameters  $p$ ,  $q$  and  $r$  are selected so that the norms of  $v_h$  and  $v_h \cdot w_h$  are bounded. If

$$p = 26/21, \quad p' = 26/5, \quad q = 14/13, \quad q' = 14, \quad r = 13/7, \quad r' = 13/6,$$

then  $1/2 = 1/p' + 2/dr'$  when  $d = 3$ , so  $\|v_h\|_{L^{r'}[0,T;L^{p'}(\Omega)]}$  and

$$|v_h \cdot w_h|_{L^{rq}[0,T;W^{1,pq}(\Omega)]} = |v_h \cdot w_h|_{L^2[0,T;W^{1,4/3}(\Omega)]}$$

are bounded, and the second term in  $e_h$  vanishes as  $h \rightarrow 0$ .  $\square$

**4. Projections of the Velocity Field.** In order to guarantee that the piecewise constant DG scheme is monotone and convergent, the average divergence of the velocity field in equation (2.2) must vanish on each simplex  $K \in \mathcal{T}_h$ . Above we assumed space  $P_h$  contain the piecewise constant functions so that solution  $v_h$  of the approximate momentum equation (2.3) automatically satisfy this condition. In this section projections of the velocity field  $v_h \in V_h$  onto a space  $\bar{V}_h \subset H(\Omega; \text{div})$  having average divergence on each element equal to zero are considered when  $P_h$  does not contain the piecewise constant functions. In this case the density and velocity/pressure are approximated by  $\rho^n \in \mathcal{R}_h$  satisfying

$$(4.1) \quad \int_K \rho^n \psi^n + \tau \int_{\partial K} (\rho_-^n (\bar{v}^{n-1} \cdot n)^+ + \rho_+^n (\bar{v}^{n-1} \cdot n)^-) \psi^n = \int_K \rho^{n-1} \psi^n,$$

for  $K \in \mathcal{T}_h$  and  $\psi^n \in \mathbb{R}$ , and  $(v^n, p^n) \in V_h \times P_h$  satisfying

$$\begin{aligned}
(4.2) \quad &\frac{1}{2} \int_\Omega \rho^{n-1} \left( \frac{v^n - v^{n-1}}{\tau} \right) \cdot w + (\rho^n \bar{v}^{n-1} \cdot \nabla) v^n \cdot w + \left( \frac{(\rho v)^n - (\rho v)^{n-1}}{\tau} \right) \cdot w - (\rho^n \bar{v}^{n-1} \cdot \nabla) w \cdot v^n \\
&+ \int_\Omega -p^n \text{div}(w) + \mu^n D(v^n) \cdot D(w) = \int_\Omega \rho^n f^n \cdot w,
\end{aligned}$$

$$\int_\Omega \text{div}(v^n) q = 0,$$

for all  $(w, q) \in V_h \times P_h$ .

Writing  $\bar{v}^{n-1} = P_{\bar{V}_h} v^{n-1}$  where

$$\bar{V}_h \subset \{v_h \in H(\Omega; \text{div}) \mid \int_K \text{div}(v_h) = 0, K \in \mathcal{T}_h\},$$

examination of the proofs shows that the modified scheme will also converge if the projection  $P_{\bar{V}_h} : V_h \rightarrow \bar{V}_h$  satisfies the following hypotheses.

ASSUMPTION 4.1.

1. There exists  $\ell \in \mathbb{N}$  independent of  $h$  such that  $\bar{v}_h|_K \in \mathcal{P}_\ell(K)$  for each  $K \in \mathcal{T}_h$ .
2. For each  $\bar{v}_h \in \bar{V}_h$

$$\int_K \text{div}(\bar{v}_h) = 0, \text{ and } \int_{\partial K \cap \partial \Omega} \bar{v}_h \cdot n = 0, \quad K \in \mathcal{T}_h.$$

3. If  $v_h \in V_h$  and  $\bar{v}_h = P_{\bar{V}_h} v_h$  then there exists  $C > 0$  independent of  $h$  such that  $\|\text{div}(\bar{v}_h)\|_{L^2(\Omega)} \leq C \|v_h\|_{H^1(\Omega)}$ .
4. If  $v_h \in V_h$  and  $\bar{v}_h = P_{\bar{V}_h} v_h$  then there exists  $C > 0$  independent of  $h$  such that  $\|\bar{v}_h\|_{L^2(\Omega)} \leq C \|v_h\|_{L^2(\Omega)}$  and  $\|\bar{v}_h\|_{L^6(\Omega)} \leq C \|\nabla v_h\|_{H^1(\Omega)}$ .
5. Let  $\{v_h\}_{h>0}$ ,  $v_h \in V_h$  be bounded in  $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$  and  $\bar{v}_h = P_{\bar{V}_h} v_h$ . If  $v_h \rightarrow v$  in  $L^2[0, T; L^2(\Omega)]$  then  $\bar{v}_h \rightarrow v$ .

**Stokes Projections.** If  $(\bar{V}_h, \bar{P}_h) \subset H_0^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$  is a family of finite element spaces constructed on  $\mathcal{T}_h$  which satisfy the Babuska Brezzi condition and if  $\bar{P}_h$  contains the piecewise constant functions then the Stokes projection  $P_{\bar{V}_h} : V_h \rightarrow \bar{V}_h$  satisfies Assumptions 4.1.

The Stokes projection of  $v_h \in V_h$ , is computed from the unique solution  $(\bar{v}_h, \bar{p}_h) \in (\bar{V}_h, \bar{P}_h)$  of

$$(4.3) \quad a(\bar{v}_h, \bar{w}_h) + b(\bar{p}_h, \bar{w}_h) + b(\bar{q}_h, \bar{v}_h) = a(v_h, \bar{w}_h)$$

for all  $(\bar{w}_h, \bar{q}_h) \in (\bar{V}_h, \bar{P}_h)$ . The bilinear forms  $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$  and  $b : L^2(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$  are defined by

$$a(v, w) = (v, w)_{H^1(\Omega)}, \quad b(p, v) = (p, \text{div}(v))_{L^2(\Omega)}.$$

By construction the average of  $\text{div}(\bar{v}_h)$  vanishes on each simplex  $K \in \mathcal{T}_h$ . The next lemma establishes estimates  $\bar{v}_h$  needed to show that the continuity properties of Assumption 4.1 are also satisfied by this construction.

LEMMA 4.2. Let  $\Omega \subset \mathbb{R}^d$  be sufficiently regular to guarantee  $H^2(\Omega)^d \times H^1(\Omega)$  regularity of the Stokes operator, and let  $\{\mathcal{T}_h\}_{h>0}$  be a regular quasi uniform family of triangulations of  $\Omega$ .

Let  $(V_h, P_h)$  and  $(\bar{V}_h, \bar{P}_h) \subset H_0^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$  be families of finite element spaces constructed on  $\mathcal{T}_h$  which satisfy the Babuska Brezzi condition, and let  $(\bar{v}_h, \bar{p}_h) \in (\bar{V}_h, \bar{P}_h)$  be the Stokes projection of  $v_h \in H^1(\Omega)$  satisfying  $b(q_h, v_h) = 0$  for all  $q_h \in P_h$ . Then

- $\|\bar{v}_h\|_{H^1(\Omega)} \leq \|v_h\|_{H^1(\Omega)}$ , and
- $\|\bar{v}_h - v_h\|_{L^2(\Omega)} \leq C \|v_h\|_{H^1(\Omega)} h \leq C \|v_h\|_{L^2(\Omega)}$ .

The first statement of the lemma follows upon setting  $\bar{w}_h = \bar{v}_h$  in equation (4.3), and the Aubin Nitsche trick and inverse inequalities are used to establish the second statement. The Sobolev embedding theorem guarantees

$$\|\bar{v}_h\|_{L^6(\Omega)} \leq C \|\bar{v}_h\|_{H^1(\Omega)} \leq C \|v_h\|_{H^1(\Omega)}.$$

It follows that the Stokes projection  $v_h \rightarrow \bar{v}_h$  satisfies Assumptions 4.1.



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