2-Quasiconvexity versus 1-Quasiconvexity

Gianni Dal Maso — Sissa
Irene Fonseca — CMU
Giovanni Leoni — CMU
Massimiliano Morini — CMU

May 6, 2003

Abstract

In this paper it is shown that a smooth strictly 2-quasiconvex function with p-growth at infinity, $p > 1$, is the restriction to symmetric matrices of a 1-quasiconvex function with the same growth. As a consequence, lower semicontinuity results for second-order variational problems are deduced as corollaries of classical first order theorems.

1 Introduction

In recent years there has been a renewed interest in higher order variational problems motivated by various mathematical models in engineering and materials science: in connection with the so-called gradient theories of phase transitions within elasticity regimes (see [7], [17], [21]); in the study of equilibria of micromagnetic materials where mastery of second order energies (here accounting for the exchange energy) is required (see [6], [9], [21], [24]); in the theory of second order structured deformations (SOSD) (see [23]), in the Blake-Zisserman model for image segmentation in computer vision (see [5]); etc..

In the study of lower semicontinuity, relaxation and Γ-convergence problems for second order functional the natural notion of convexity, 2-quasiconvexity, was introduced by Meyers in [20] (see also [3], [14]). We recall that a real valued function f, defined on the space $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ of $n \times n$ symmetric matrices, is 2-*quasiconvex* if

$$\int_Q \left[f\left(A + \nabla^2 \phi\right) - f\left(A\right)\right]\, dx \geq 0$$

for every $A \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ and every $\phi \in C^2_c\left(Q\right)$, where $Q := (0,1)^n$ is the unit cube.

While lower semicontinuity properties of functionals depending *uniquely* on the second order derivatives can be proved easily, the case where lower order terms are present is significantly more difficult since the correct tools to handle the localization and truncation of gradients are still missing.

To bypass these difficulties one would be tempted to transform higher order into first order problems, where one uses the standard notion of quasiconvexity, called 1-quasiconvexity in this paper. We recall that a real valued function f, defined on the space $\mathbb{M}^{n\times n}$ of $n\times n$ matrices, is 1-*quasiconvex* if

$$\int_Q [f(A+\nabla\varphi)-f(A)]\,dx\geq 0$$

for every $A\in\mathbb{M}^{n\times n}$ and every $\varphi\in C_c^1(Q;\mathbb{R}^n)$.

Thus we are led to the following question.

(Q) *Is every 2-quasiconvex function the restriction of a 1-quasiconvex function to the space of symmetric matrices?*

A good indication of the plausibility of an affirmative answer is that it holds true for polyconvex functions as noticed by Dacorogna and Fonseca. Indeed if

$$f(A)=g(M(A))\qquad A\in\mathbb{M}^{n\times n}_{\text{sym}},$$

where g is a convex function and $M(A)$ stands for the vector whose components are all the minors of A, then the function

$$F(A):=g\left(\frac{M(A)+M(A)^t}{2}\right)\qquad A\in\mathbb{M}^{n\times n}$$

is a polyconvex extension of f to the whole space $\mathbb{M}^{n\times n}$ of $n\times n$ matrices.

It is known that 2-gradient Young measures, i.e. Young measures generated by second order gradients, may be characterized by duality via Jensen's inequality with respect to 2-quasiconvex functions (see [12]), just as gradient Young measures are characterized by duality with 1-quasiconvex functions (see [19]). Therefore, the understanding of the structure of 2-gradient Young measures helps the study of 2-quasiconvex functions, and, accordingly, the following result by Šverák in [25, Lemma 1] provides further evidence that 1-quasiconvexity and 2-quasiconvexity are somehow strictly linked: If a Young measure ν on $\mathbb{M}^{n\times n}$ is generated by a sequence $\{\nabla u_k\}$ of gradients, with $\{u_k\}$ bounded in $W^{1,p}(\Omega;\mathbb{R}^n)$ for some $p>1$, and $\operatorname{supp}\nu_x\subset\mathbb{M}^{n\times n}_{\text{sym}}$ for $\mathcal{L}^n$ a.e. $x\in\Omega$, then ν is generated also by a sequence $\{\nabla^2 w_k\}$, with $\{w_k\}$ bounded in $W^{2,p}(\Omega)$.

A partial answer to **(Q)** was given by Müller and Šverák (see [22]). Indeed, as an auxiliary result to construct a counter-example to regularity for elliptic systems, they proved that any smooth, strictly 2-quasiconvex function $f:\mathbb{M}^{2\times 2}_{\text{sym}}\to\mathbb{R}$, with bounded second derivatives, is the restriction of a 1-quasiconvex function. The main purpose of this paper is to extend their result to any space dimension and to a larger class of strictly 2-quasiconvex functions with p-growth at infinity, with $p>1$.

Theorem 1.1 *Let $f\in C^1\left(\mathbb{M}^{n\times n}_{\text{sym}}\right)$ satisfy the following conditions for suitable constants $p>1$, $\mu\geq 0$, $L\geq\nu>0$:*

(a) (strict 2-quasiconvexity)

$$\int_Q \left[f\left(A+\nabla^2\phi\right)-f\left(A\right)\right]\,dx \geq \nu\int_Q\left(\mu^2+\left|A\right|^2+\left|\nabla^2\phi\right|^2\right)^{\frac{p-2}{2}}\left|\nabla^2\phi\right|^2\,dx$$

for every $A\in\mathbb{M}^{n\times n}_{\mathrm{sym}}$ *and every* $\phi\in C^2_c\left(Q\right)$*;*

(b) (Lipschitz condition for gradients)

$$\left|\nabla f\left(A+B\right)-\nabla f\left(A\right)\right|\leq L\left(\mu^2+\left|A\right|^2+\left|B\right|^2\right)^{\frac{p-2}{2}}\left|B\right| \tag{1.1}$$

for every $A,\,B\in\mathbb{M}^{n\times n}_{\mathrm{sym}}$*.*

Then there exists a 1*-quasiconvex function* $F:\mathbb{M}^{n\times n}\to\mathbb{R}$ *such that*

$$F\left(A\right)=f\left(A\right)\qquad\forall A\in\mathbb{M}^{n\times n}_{\mathrm{sym}}, \tag{1.2}$$

$$\left|F\left(A\right)\right|\leq c_f\left(1+\left|A\right|^p\right)\qquad\forall A\in\mathbb{M}^{n\times n}, \tag{1.3}$$

for a suitable constant $c_f>0$ *depending on* f*.*

We remark that a 1-quasiconvex function F satisfying (1.2) and (1.3) is constructed explicitly if $p\geq 2$ (see (3.8)), while in the case $1<p<2$ it is defined as the quasiconvex envelope of a suitable extension of f to $\mathbb{M}^{n\times n}$.

The proof of the theorem relies on a Korn-type inequality for divergence-free vector fields and uses heavily the Lipschitz condition on the gradient of f. The use of Korn-type inequalities prevents us from obtaining a similar result for the case $p=1$, which, if valid, will require a different treatment.

We do not know if the result continues to hold without assuming (1.1). However, when condition (1.1) is dropped we can still prove the following weaker version of Theorem 1.1.

Theorem 1.2 *Let* $f:\mathbb{M}^{n\times n}_{\mathrm{sym}}\to\mathbb{R}$ *satisfy the following conditions for suitable constants* $p>1$*,* $\mu\geq 0$*,* $\nu>0$*,* $M>0$*:*

(a) (strict 2-quasiconvexity)

$$\int_Q \left[f\left(A+\nabla^2\phi\right)-f\left(A\right)\right]\,dx \geq \nu\int_Q\left(\mu^2+\left|A\right|^2+\left|\nabla^2\phi\right|^2\right)^{\frac{p-2}{2}}\left|\nabla^2\phi\right|^2\,dx$$

for every $A\in\mathbb{M}^{n\times n}_{\mathrm{sym}}$ *and every* $\phi\in C^2_c\left(Q\right)$*;*

(b) (growth condition)

$$\left|f\left(A\right)\right|\leq M\left(1+\left|A\right|^p\right) \tag{1.4}$$

for every $A\in\mathbb{M}^{n\times n}_{\mathrm{sym}}$*.*

Then there exists an increasing sequence $\{F_k\}$ *of* 1*-quasiconvex functions* $F_k : \mathbb{M}^{n\times n} \to \mathbb{R}$ *such that*

$$\lim_{k\to\infty} F_k(A) = f(A) \qquad \forall A \in \mathbb{M}^{n\times n}_{\text{sym}}, \tag{1.5}$$

$$|F_k(A)| \leq c_k(1+|A|^p) \qquad \forall A \in \mathbb{M}^{n\times n}, \tag{1.6}$$

for a suitable sequence of constants $\{c_k\}$ *depending only on* k *and on the structural constants* p, μ, ν, M, *but not on the specific function* f.

Theorem 1.2 allows us to reduce lower semicontinuity problems for 2-quasi convex normal integrands of the form $f = f(x, u, \nabla u, \nabla^2 u)$ to first order problems (see Section 4 for more details). Indeed, as a consequence of Theorem 1.2 we can prove the following result, which extends to the second order setting a lower semicontinuity property of 1-quasiconvex functions in $SBV(\Omega; \mathbb{R}^d)$ due to Ambrosio [2] and later generalized by Kristensen [18]. For the definition and properties of the space $SBH(\Omega)$ we refer to [4] and [5].

Theorem 1.3 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set and let*

$$f : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{M}^{n\times n}_{\text{sym}} \to [0, +\infty)$$

be an integrand which satisfies the following conditions:

(a) the function $f(x, \cdot, \cdot, \cdot)$ *is lower semicontinuous on* $\mathbb{R} \times \mathbb{R}^n \times \mathbb{M}^{n\times n}_{\text{sym}}$ *for* $\mathcal{L}^n$ *a.e.* $x \in \Omega$*;*

(b) the function $f(x, u, \xi, \cdot)$ *is* 2*-quasiconvex on* $\mathbb{M}^{n\times n}_{\text{sym}}$ *for* $\mathcal{L}^n$ *a.e.* $x \in \Omega$ *and every* $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$*;*

(c) there exist a locally bounded function $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ *and a constant* $p > 1$ *such that*

$$0 \leq f(x, u, \xi, A) \ \leq a(x, u, \xi)(1 + |A|^p)$$

for $\mathcal{L}^N$ *a.e.* $x \in \Omega$ *and every* $(u, \xi, A) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{M}^{n\times n}_{\text{sym}}$.

Then

$$\int_\Omega f(x, u, \nabla u, \nabla^2 u)\, dx \leq \liminf_{j\to\infty} \int_\Omega f(x, u_j, \nabla u_j, \nabla^2 u_j)\, dx$$

for every $u \in SBH(\Omega)$ *and any sequence* $\{u_j\} \subset SBH(\Omega)$ *converging to* u *in* $W^{1,1}(\Omega)$ *and such that*

$$\sup_j \left(\left\| \nabla^2 u_j \right\|_{L^p} + \int_{S(\nabla u_j)} \theta(|[\nabla u_j]|)\, d\mathcal{H}^{n-1} \right) < \infty, \tag{1.7}$$

where $\theta : [0, \infty) \to [0, \infty)$ *is a concave, nondecreasing function such that*

$$\lim_{t\to 0^+} \frac{\theta(t)}{t} = \infty,$$

and $[\nabla u_j]$ *denotes the jump of* ∇u_j *on the jump set* $S(\nabla u_j)$.

An analogous result has been proved in [11] in the space $BH(\Omega)$ in the case where (1.7) is replaced by

$$\sup_j \left\|\nabla^2 u_j\right\|_{L^p} < \infty \qquad \left|D_s^2 u_j\right|(\Omega) \to 0,$$

where $D_s^2 u_j$ is the singular part of the $\mathbb{M}^{n\times n}_{\mathrm{sym}}$-valued measure $D^2 u_j$. Note that condition (1.7) arises naturally in the context of free-discontinuity problems (see [2]).

As a corollary of Theorem 1.3 we have the following result.

Corollary 1.4 *Let Ω and f be as in Theorem 1.3. Then*

$$\int_\Omega f(x,u,\nabla u,\nabla^2 u)\,dx \le \liminf_{j\to\infty} \int_\Omega f(x,u_j,\nabla u_j,\nabla^2 u_j)\,dx$$

for every $u \in W^{2,p}(\Omega)$ and any sequence $\{u_j\} \subset W^{2,p}(\Omega)$ weakly converging to u in $W^{2,p}(\Omega)$.

In this generality Corollary 1.4 was proved in [11] and under stronger hypotheses in [14], [15], and [20].

Remark 1.5 All of the above are still valid in a vectorial setting, i.e., $u : \Omega \to \mathbb{R}^d$, with $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ replaced now by $\left(\mathbb{M}^{n\times n}_{\mathrm{sym}}\right)^d$. The proofs are entirely similar to those presented in this paper for the case $d = 1$, and we leave the obvious adaptations to the reader.

The paper is organized as follows. In Section 2 we present some auxiliary results including the Korn-type inequality mentioned above. In Section 3 we prove Theorems 1.1 and 1.2, while Theorem 1.3 is addressed in the last section.

2 Auxiliary results

We begin with some results on the Helmholtz Decomposition and on Korn's type inequalities.

A function $w : \mathbb{R}^n \to \mathbb{R}^d$ is said to be *Q–periodic* if $w(x + e_i) = w(x)$ for a.e. $x \in \mathbb{R}^n$ and every $i = 1,\dots,n$, where $(e_1,\dots,e_n)$ is the canonical basis of $\mathbb{R}^n$. The spaces of Q-periodic functions of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$, and $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ are denoted by $W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^n)$, $W^{2,p}_{\mathrm{per}}(Q)$, and $C^\infty_{\mathrm{per}}(Q;\mathbb{R}^n)$, respectively.

Lemma 2.1 (Helmholtz decomposition) *For every $p > 1$ and every $\varphi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^n)$ there exist two functions $\phi \in W^{2,p}_{\mathrm{per}}(Q)$ and $\psi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^n)$ such that*

$$\varphi = \nabla\phi + \psi, \quad \operatorname{div}\psi = 0.$$

The function ψ is uniquely determined, while ϕ is determined up to an additive constant.

Proof. Since, by periodicity, $\operatorname{div}\varphi$ has zero average on Q, there exists a Q-periodic solution ϕ of the equation $\Delta\phi = \operatorname{div}\varphi$, which is unique up to an additive constant. It is clear now that $\psi := \varphi - \nabla\phi$ is Q-periodic and $\operatorname{div}\psi = 0$. ■

Throughout the paper, for every $A \in \mathbb{M}^{n\times n}$, we denote the symmetric and antisymmetric parts of A by

$$A^s := \frac{A + A^t}{2}, \qquad A^a := \frac{A - A^t}{2},$$

where A^t is the transpose matrix of A.

If $\psi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^d$ is any function, then $\nabla\psi$ is a $d \times n$ matrix in $\mathbb{M}^{d\times n}$, with $(\nabla\psi)_{ij} := \frac{\partial\psi_i}{\partial x_j}$. Also, differential operators applied to matrix-valued fields are understood on a row-by-row basis, e.g., if $\psi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, then

$$\operatorname{div}\nabla\psi := (\operatorname{div}\nabla\psi_1, \ldots, \operatorname{div}\nabla\psi_n).$$

To simplify the notation, for any $\psi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, we set

$$\nabla\psi^t := (\nabla\psi)^t, \qquad \nabla\psi^s := (\nabla\psi)^s, \qquad \nabla\psi^a := (\nabla\psi)^a.$$

Lemma 2.2 *For every $p > 1$ there exists a constant $\gamma_{n,p} \geq 1$ such that*

$$\int_Q |\nabla\psi|^p \, dx \leq \gamma_{n,p} \int_Q |\nabla\psi^a|^p \, dx \tag{2.1}$$

for every Q-periodic function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞ with $\operatorname{div}\psi = 0$.

Proof. Since $\operatorname{div}\nabla\psi^t = \nabla(\operatorname{div}\psi) = 0$ we have

$$\Delta\psi = 2\operatorname{div}\left(\frac{\nabla\psi - \nabla\psi^t}{2}\right) = 2\operatorname{div}(\nabla\psi^a). \tag{2.2}$$

Hence (2.1) follows from standard L^p estimates for periodic solutions of the Poisson equation (see [16]). ■

Next we study the behavior of auxiliary functions of the type

$$g(x) := \left(\mu^2 + |x|^2\right)^{\frac{p}{2}} \tag{2.3}$$

defined on an arbitrary Hilbert space X.

Lemma 2.3 *For every $p > 1$ there exist two constants κ_p and K_p, with $0 < \kappa_p \leq 1 \leq K_p$, such that the following inequalities hold:*

$$\int_0^1 \left(\mu^2 + |x + ty|^2\right)^{\frac{p-2}{2}} (1 - t)\, dt \geq \kappa_p \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}}, \tag{2.4}$$

$$\int_0^1 \left(\mu^2 + |x + ty|^2\right)^{\frac{p-2}{2}} dt \leq K_p \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} \tag{2.5}$$

for every x, $y \in X$ and every constant $\mu \geq 0$.

Proof. Let us prove (2.4). If $1 < p \leq 2$, then (2.4) holds with $\kappa_p = 2^{\frac{p}{2}-2}$, since

$$\left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} \geq 2^{\frac{p-2}{2}} \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}}.$$

If $p > 2$, then we consider first the case where $|y|^2 \leq 4\left(\mu^2 + |x|^2\right)$. For $0 \leq t \leq 1/2$ we have

$$\begin{aligned}\left(\mu^2 + |x+ty|^2\right)^{\frac{1}{2}} &\geq \left(\mu^2 + |x|^2\right)^{\frac{1}{2}} - t\,|y| \geq (1-2t)\left(\mu^2 + |x|^2\right)^{\frac{1}{2}} \\ &\geq 5^{-\frac{1}{2}}\,(1-2t)\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{1}{2}},\end{aligned}$$

hence

$$\left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} \geq 5^{\frac{2-p}{2}}\,(1-2t)^{p-2}\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}}.$$

We deduce that (2.4) holds provided

$$\kappa_p \leq 5^{\frac{2-p}{2}} \int_0^{\frac{1}{2}} (1-2t)^{p-2}\,(1-t)\,dt = \frac{5^{\frac{2-p}{2}}}{4}\frac{2p-1}{p\,(p-1)}.$$

If $p > 2$ and $|y|^2 > 4\left(\mu^2 + |x|^2\right)$, then for $1/2 \leq t \leq 1$ we have

$$\begin{aligned}\left(\mu^2 + |x+ty|^2\right)^{\frac{1}{2}} &\geq t\,|y| - \left(\mu^2 + |x|^2\right)^{\frac{1}{2}} \geq \left(t - \frac{1}{2}\right)|y| \\ &\geq 2\cdot 5^{-\frac{1}{2}}\left(t - \frac{1}{2}\right)\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} \geq 5^{\frac{2-p}{2}}\,(2t-1)^{p-2}\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}}.$$

We conclude that (2.4) is verified with

$$\kappa_p \leq 5^{\frac{2-p}{2}} \int_{\frac{1}{2}}^{1} (2t-1)^{p-2}\,(1-t)\,dt = \frac{5^{\frac{2-p}{2}}}{4}\frac{1}{p\,(p-1)}.$$

This concludes the proof of (2.4).

Let us prove (2.5). If $p \geq 2$ then (2.5) holds with $K_p = 2^{\frac{p-2}{2}}$, since

$$\left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} \leq 2^{\frac{p-2}{2}} \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}}.$$

In the case $1 < p < 2$ we observe that

$$\left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} \leq \left|\left(\mu^2 + |x|^2\right)^{\frac{1}{2}} - t\,|y|\right|^{p-2}.$$

Let $a := \left(\mu^2 + |x|^2\right)^{\frac{1}{2}}$ and $b := |y|$. Then

$$\int_0^1 \left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} dt \leq \int_0^1 |a - tb|^{p-2}\, dt$$

and $\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} = \left(a^2 + b^2\right)^{\frac{p-2}{2}}$.

If $b \leq a$, then

$$\int_0^1 |a - tb|^{p-2}\, dt = \frac{a^{p-1} - (a-b)^{p-1}}{(p-1)\, b} \leq \frac{a^{p-2}}{p-1},$$

where the last inequality is obtained by comparing the difference quotients of the concave function $t \mapsto t^{p-1}$ in the intervals $[a-b, a]$ and $[0, a]$. Therefore we have that

$$\int_0^1 |a - tb|^{p-2}\, dt \leq \frac{a^{p-2}}{p-1} \leq \frac{2^{\frac{2-p}{2}}}{p-1} \left(a^2 + b^2\right)^{\frac{p-2}{2}},$$

and (2.5) is satisfied for $K_p \geq 2^{\frac{2-p}{2}}/(p-1)$.

If $a < b$, then

$$\int_0^1 |a - tb|^{p-2}\, dt = \frac{a^{p-1} + (b-a)^{p-1}}{(p-1)\, b} \leq \frac{2^{2-p} b^{p-2}}{p-1}.$$

On the other hand, in this case we have $b^{p-2} \leq 2^{\frac{2-p}{2}} \left(a^2 + b^2\right)^{\frac{p-2}{2}}$, so that (2.5) holds for $K_p \geq 2^{\frac{3}{2}(2-p)}/(p-1)$. ■

Lemma 2.4 *For every $p > 1$, there exist two constants $\theta_p > 0$ and $\Theta_p > 0$ such that for every $\mu \geq 0$ the function g defined in* (2.3) *satisfies the following inequalities*

$$\begin{aligned} \theta_p \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 &\leq g(x+y) - g(x) - \nabla g(x) \cdot y \\ &\leq \Theta_p \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 \end{aligned}$$

for every x, $y \in X$.

Proof. By continuity it is enough to prove the statement when 0 does not belong to the segment joining x and $x + y$. In this case the function

$$h(t) := \left(\mu^2 + |x+ty|^2\right)^{\frac{p}{2}}$$

belongs to $C^\infty([0,1])$, and Taylor's formula with integral remainder yields

$$h(1) - h(0) - h'(0) = \int_0^1 h''(t)(1-t)\, dt.$$

By direct computation we see that

$$
\begin{aligned}
p\left((p-1)\wedge 1\right)\left(\mu^{2}+|x+ty|^{2}\right)^{\frac{p-2}{2}}|y|^{2} &\leq h''(t)\\
&\leq p\left((p-1)\vee 1\right)\left(\mu^{2}+|x+ty|^{2}\right)^{\frac{p-2}{2}}|y|^{2}.
\end{aligned}
$$

The conclusion follows from Lemma 2.3. ■

In the proof of Theorem 1.1 we will need the following extension of Lemma 2.4 to the family of functions

$$
g_{\beta}(x,y):=\left(\mu^{2}+|x|^{2}+\beta^{2}|y|^{2}\right)^{\frac{p}{2}},\qquad \beta\geq 0,
$$

defined on the product of two Hilbert spaces X and Y.

Lemma 2.5 *Let $p>1$, $\beta\geq 0$ and $\mu\geq 0$. Then*

$$
\begin{aligned}
&g_{\beta}(x+\xi,y+\eta)-g_{\beta}(x,y)-\nabla_{x}g_{\beta}(x,y)\cdot\xi-\nabla_{y}g_{\beta}(x,y)\cdot\eta\\
\geq&\theta_{p}\left(\mu^{2}+|x|^{2}+|\xi|^{2}+\beta^{2}|y|^{2}+\beta^{2}|\eta|^{2}\right)^{\frac{p-2}{2}}\left(|\xi|^{2}+\beta^{2}|\eta|^{2}\right)
\end{aligned}
$$

for every $x,\ \xi\in X,\ y,\ \eta\in Y$, where θ_p is the first constant in Lemma 2.4. Therefore, if $p\geq 2$, we have

$$
\begin{aligned}
&g_{\beta}(x+\xi,y+\eta)-g_{\beta}(x,y)-\nabla_{x}g_{\beta}(x,y)\cdot\xi-\nabla_{y}g_{\beta}(x,y)\cdot\eta\\
&\geq\theta_{p}\left(\mu^{2}+|x|^{2}+|\xi|^{2}\right)^{\frac{p-2}{2}}|\xi|^{2}+\frac{\theta_{p}\beta^{2}}{2}\left(\mu^{2}+|x|^{2}\right)^{\frac{p-2}{2}}|\eta|^{2}+\frac{\theta_{p}\beta^{p}}{2}|\eta|^{p}
\end{aligned}
$$

for every $x,\ \xi\in X,\ y,\ \eta\in Y$.

Proof. Observing that $g_{\beta}(x,y)=g_{1}(x,\beta y)$, the inequality can be obtained by applying Lemma 2.4 to the Hilbert space $X\times Y$. ■

We continue with some technical lemmas which are used in the proofs of Theorems 1.1 and 1.2.

Lemma 2.6 *Let X be a Hilbert space and let $1<p\leq 2$. Then for every $\mu\geq 0$ and every $0<\varepsilon<1$ we have*

$$
\begin{aligned}
\left(\mu^{2}+|x+y|^{2}\right)^{\frac{p-2}{2}}|x+y|^{2}&\leq 2\left(\mu^{2}+|x|^{2}\right)^{\frac{p-2}{2}}|x|^{2}+2\left(\mu^{2}+|y|^{2}\right)^{\frac{p-2}{2}}|y|^{2},\\
\varepsilon^{\frac{2-p}{2}}\left(\mu^{2}+|y|^{2}\right)^{\frac{p-2}{2}}|y|^{2}&\leq\left(\mu^{2}+|x|^{2}+|y|^{2}\right)^{\frac{p-2}{2}}|y|^{2}+\varepsilon\left(\mu^{2}+|x|^{2}\right)^{\frac{p-2}{2}}|x|^{2}
\end{aligned}
$$

for every $x,\ y\in X$.

Proof. Since the mapping $t \mapsto \left(\mu^2+t\right)^{\frac{p-2}{2}} t$ is nondecreasing, while the mapping $t \mapsto \left(\mu^2+t\right)^{\frac{p-2}{2}}$ is nonincreasing, we have

$$\begin{aligned}\left(\mu^2+|x+y|^2\right)^{\frac{p-2}{2}} |x+y|^2 &\leq \left(\mu^2+2\,|x|^2+2\,|y|^2\right)^{\frac{p-2}{2}} \left(2\,|x|^2+2\,|y|^2\right) \\ &\leq 2\left(\mu^2+|x|^2\right)^{\frac{p-2}{2}} |x|^2+2\left(\mu^2+|y|^2\right)^{\frac{p-2}{2}} |y|^2,\end{aligned}$$

which proves the first inequality. For the same reason, we have

$$\begin{aligned}\varepsilon^{\frac{2-p}{2}}\left(\mu^2+|y|^2\right)^{\frac{p-2}{2}} |y|^2 &\leq \varepsilon^{\frac{2-p}{2}}\left(\mu^2+\varepsilon\,|x|^2+|y|^2\right)^{\frac{p-2}{2}} \left(\varepsilon\,|x|^2+|y|^2\right) \\ &\leq \left(\mu^2+|x|^2+|y|^2\right)^{\frac{p-2}{2}} \left(\varepsilon\,|x|^2+|y|^2\right) \\ &\leq \varepsilon\left(\mu^2+|x|^2\right)^{\frac{p-2}{2}} |x|^2+\left(\mu^2+|x|^2+|y|^2\right)^{\frac{p-2}{2}} |y|^2,\end{aligned}$$

and this concludes the proof. ■

Lemma 2.7 *Let $1<p\leq 2$. Then*

$$b^p \leq 8\varepsilon^{\frac{p-2}{p}}\left(\mu^2+a^2+b^2\right)^{\frac{p-2}{2}} b^2+\varepsilon a^p+\varepsilon\mu^p \tag{2.6}$$

for every $a\geq 0$, $b\geq 0$, $\mu\geq 0$, and $0<\varepsilon<1$.

Proof. By Young's inequality with exponents $2/p$ and $2/(2-p)$ we have that

$$\begin{aligned}b^p &= \left(2^{\frac{2-p}{2}}\varepsilon^{\frac{p-2}{2}}\left(\mu^2+b^2\right)^{\frac{p(p-2)}{4}} b^p\right)\left(2^{\frac{p-2}{2}}\varepsilon^{\frac{2-p}{2}}\left(\mu^2+b^2\right)^{\frac{p(2-p)}{4}}\right) \\ &\leq 2\varepsilon^{\frac{p-2}{p}}\left(\mu^2+b^2\right)^{\frac{p-2}{2}} b^2+\frac{\varepsilon}{2}\left(\mu^2+b^2\right)^{\frac{p}{2}}.\end{aligned} \tag{2.7}$$

If $a^2\leq\mu^2+b^2$, we have

$$\left(\mu^2+b^2\right)^{\frac{p-2}{2}} \leq 2\left(\mu^2+a^2+b^2\right)^{\frac{p-2}{2}},$$

and so from (2.7) we obtain

$$b^p \leq 4\varepsilon^{\frac{p-2}{p}}\left(\mu^2+a^2+b^2\right)^{\frac{p-2}{2}} b^2+\frac{\varepsilon}{2}\left(\mu^p+b^p\right).$$

Subtracting $(\varepsilon/2)\, b^p$ to both sides we obtain (2.6).

On the other hand, if $a^2>\mu^2+b^2$, then from (2.7) with $\left(\mu^2+b^2\right)$ replaced by $\left(\mu^2+a^2+b^2\right)$ we obtain

$$\begin{aligned}b^p &\leq 2\varepsilon^{\frac{p-2}{p}}\left(\mu^2+a^2+b^2\right)^{\frac{p-2}{2}} b^2+\frac{\varepsilon}{2}\left(\mu^2+a^2+b^2\right)^{\frac{p}{2}} \\ &\leq 2\varepsilon^{\frac{p-2}{p}}\left(\mu^2+a^2+b^2\right)^{\frac{p-2}{2}} b^2+\varepsilon a^p,\end{aligned}$$

which proves (2.6). ■

The next lemma shows that condition (1.1) in Theorem 1.1 can be obtained from a suitable bound on the second derivatives of f. This is trivial in the case $p \geq 2$, but requires some work in the case $1 < p < 2$.

Lemma 2.8 *Let X be a Hilbert space, and let $f \in C^1(X) \cap C^2(X \setminus \{0\})$. Assume that there exist two constants $p > 1$, $C > 0$, and $\mu \geq 0$ such that*

$$\left|\nabla^2 f(x)\right| \leq C\left(\mu^2 + |x|^2\right)^{\frac{p-2}{2}} \tag{2.8}$$

for every $x \in X \setminus \{0\}$. Then

$$|\nabla f(x+y) - \nabla f(x)| \leq K_p C\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y| \tag{2.9}$$

for every x, $y \in X$, where K_p is the second constant in Lemma 2.3.

Proof. By continuity it is enough to prove the statement when 0 does not belong to the segment joining x and $x + y$. In this case by (2.8) we have

$$|\nabla f(x+y) - \nabla f(x)| \leq C|y| \int_0^1 \left(\mu^2 + |x+ty|^2\right)^{\frac{p-2}{2}} dt,$$

and the conclusion follows from Lemma 2.3. ■

The estimate given by the following lemma will be crucial in the proof of Theorem 1.1.

Lemma 2.9 *Let X be a Hilbert space and let $f \in C^1(X)$. Assume that there exist $p > 1$ and $\mu \geq 0$ such that*

$$|\nabla f(x+y) - \nabla f(x)| \leq \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y| \tag{2.10}$$

for every x, $y \in X$. If $1 < p \leq 2$, then for every $\varepsilon > 0$ there exists $c_{\varepsilon,p} > 0$, depending only on ε and p, such that

$$\begin{aligned} &|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\ &\quad \leq \varepsilon\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + c_{\varepsilon,p}\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |z|^2 \end{aligned} \tag{2.11}$$

for every x, y, $z \in X$.

If $p \geq 2$, then for every $\varepsilon > 0$ there exists $c_{\varepsilon,p} > 0$, depending only on ε and p, such that

$$\begin{aligned} &|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\ &\quad \leq \varepsilon\left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + c_{\varepsilon,p}\left(\mu^2 + |x|^2\right)^{\frac{p-2}{2}} |z|^2 + c_{\varepsilon,p}|z|^p \end{aligned} \tag{2.12}$$

for every x, y, $z \in X$.

Proof. Let us consider first the case $1 < p \leq 2$. Clearly (2.10) implies that

$$|\nabla f(x+y) - \nabla f(x)| \leq \left(\mu^2 + |y|^2\right)^{\frac{p-2}{2}} |y|$$

for every $x,\ y \in X$. By the Mean Value Theorem we have

$$\begin{aligned} |f(x+y) - f(x) - \nabla f(x) \cdot y| &\leq |\nabla f(x+ty) - \nabla f(x)|\,|y| \\ &\leq \left(\mu^2 + t^2 |y|^2\right)^{\frac{p-2}{2}} t\,|y|^2 \end{aligned}$$

for some $t \in [0,1]$. Since the function $t \mapsto \left(\mu^2 + t^2 |y|^2\right)^{\frac{p-2}{2}} t$ is nondecreasing, we obtain

$$|f(x+y) - f(x) - \nabla f(x) \cdot y| \leq \left(\mu^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 \tag{2.13}$$

for every $x,\ y \in X$.

By (2.10) and (2.13) we have

$$\begin{aligned} &|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\ &\quad \leq |f(x+y+z) - f(x+y) - \nabla f(x+y) \cdot z| \\ &\quad\quad + |\nabla f(x+y) \cdot z - \nabla f(x) \cdot z| \\ &\quad \leq \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |z|^2 + \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|\,|z|\,. \end{aligned} \tag{2.14}$$

We now estimate the last term. If $|z| \leq \mu$ then for every $\varepsilon > 0$ we have

$$\begin{aligned} \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|\,|z| &\leq \varepsilon \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 \\ &\quad + \frac{1}{4\varepsilon} \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |z|^2 \\ &\leq \varepsilon \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + \frac{1}{4\varepsilon} \mu^{p-2} |z|^2 \\ &\leq \varepsilon \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + \frac{1}{\varepsilon} \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |z|^2\,. \end{aligned} \tag{2.15}$$

If $|z| > \mu$, let $q := p/(p-1)$ be the conjugate exponent of p. Since $\frac{p-2}{2} + \frac{1}{2} - \frac{1}{q} = \frac{p-2}{2q}$ we have

$$\begin{aligned} \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|\,|z| &= \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^{1-\frac{2}{q}} |y|^{\frac{2}{q}} |z| \\ &\leq \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2q}} |y|^{\frac{2}{q}} |z| \\ &\leq \varepsilon \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + \frac{1}{p\,(q\varepsilon)^{p-1}} |z|^p \\ &\leq \varepsilon \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + k_{\varepsilon,p} \left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |z|^2 \end{aligned} \tag{2.16}$$

for some constant $k_{\varepsilon,p}$ depending only on ε and p.

Let us consider now the case $p \geq 2$. By the Mean Value Theorem and by the Cauchy Inequality we have

$$
\begin{aligned}
&|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\
&\quad \leq |f(x+y+z) - f(x+y) - \nabla f(x+y) \cdot z| \\
&\qquad + |\nabla f(x+y) \cdot z - \nabla f(x) \cdot z| \\
&\quad \leq \left(\mu^2 + 2|x|^2 + 2|y|^2 + |z|^2\right)^{\frac{p-2}{2}} |z|^2 + \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|\,|z| \\
&\quad \leq 6^{\frac{p-2}{2}} \left(\mu^2 + |x|^2\right)^{\frac{p-2}{2}} |z|^2 + 6^{\frac{p-2}{2}} |y|^{p-2} |z|^2 + 3^{\frac{p-2}{2}} |z|^p \\
&\qquad + \frac{\varepsilon}{2} \left(\mu^2 + |x|^2 + |y|^2\right)^{\frac{p-2}{2}} |y|^2 + k_{\varepsilon,p} \left(\mu^2 + |x|^2\right)^{\frac{p-2}{2}} |z|^2 + k_{\varepsilon,p} |y|^{p-2} |z|^2
\end{aligned}
$$

for some constant $k_{\varepsilon,p}$ depending only on ε and p. The conclusion follows from Young' inequality with exponents $p/(p-2)$ and $p/2$. ∎

The following lemma states an elementary property of strictly 2-quasiconvex functions.

Lemma 2.10 *Assume that $f : \mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfies the strict 2-quasiconvexity condition (a) of Theorem 1.1 for some constants $p > 1$, $\mu \geq 0$, $\nu > 0$, and let $g : \mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ be the function defined by*

$$
g(A) := \left(\mu^2 + |A|^2\right)^{\frac{p}{2}}. \tag{2.17}
$$

Then the function $f_\lambda := f - \lambda g$ is 2-quasiconvex for $\lambda \leq \nu/\Theta_p$, where Θ_p is the second constant in Lemma 2.4.

Proof. Let $A \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ and $\phi \in C^2_c(Q)$. Since, by periodicity,

$$
\int_Q \nabla g(A) \cdot \nabla^2 \phi \, dx = 0,
$$

we have

$$
\begin{aligned}
\int_Q \left[f_\lambda\left(A + \nabla^2\phi\right) - f_\lambda(A)\right] dx &= \int_Q \left[f\left(A + \nabla^2\phi\right) - f(A)\right] dx \\
&- \lambda \int_Q \left[g\left(A + \nabla^2\phi\right) - g(A) + \nabla g(A) \cdot \nabla^2\phi\right] dx \\
&\geq (\nu - \lambda\Theta_p) \int_Q \left(\mu^2 + |A|^2 + \left|\nabla^2\phi\right|^2\right)^{\frac{p-2}{2}} \left|\nabla^2\phi\right|^2 dx \geq 0,
\end{aligned}
$$

which concludes the proof. ∎

In the proof of Theorem 1.1 we need the following generalization of Lemma 2.2.

Lemma 2.11 *For every $p > 1$ there exists a constant $\tau_{n,p} \geq 1$ such that*

$$\int_Q \left(\mu^2 + |\nabla\psi^s|^2\right)^{\frac{p-2}{2}} |\nabla\psi^s|^2 \, dx \leq \tau_{n,p} \int_Q \left(\mu^2 + |\nabla\psi^a|^2\right)^{\frac{p-2}{2}} |\nabla\psi^a|^2 \, dx \tag{2.18}$$

for every constant $\mu \geq 0$ and every Q-periodic function $\psi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞ with $\operatorname{div} \psi = 0$.

Proof. Let μ and ψ be as in the statement of the lemma. In the case $p \geq 2$ by Lemma 2.2 we have

$$\int_Q |\nabla\psi|^p \, dx \leq \gamma_{n,p} \int_Q |\nabla\psi^a|^p \, dx,$$

and so

$$\begin{aligned}
&\int_Q \left(\mu^2 + |\nabla\psi|^2\right)^{\frac{p-2}{2}} |\nabla\psi|^2 \, dx \\
&\quad \leq 2^{\frac{p-2}{2}} \left\{\mu^{p-2} \int_Q |\nabla\psi|^2 \, dx + \int_Q |\nabla\psi|^p \, dx\right\} \\
&\quad \leq 2^{\frac{p-2}{2}} \left\{\mu^{p-2}\gamma_{n,2} \int_Q |\nabla\psi^a|^2 \, dx + \gamma_{n,p} \int_Q |\nabla\psi^a|^p \, dx\right\} \\
&\quad \leq \tau_{n,p} \int_Q \left(\mu^2 + |\nabla\psi^a|^2\right)^{\frac{p-2}{2}} |\nabla\psi^a|^2 \, dx
\end{aligned}$$

with $\tau_{n,p} := 2^{\frac{p-2}{2}} (\gamma_{n,2} + \gamma_{n,p},)$.

We consider now the case $1 < p < 2$. Let $E_\mu := \{|\nabla\psi^a| \leq \mu\}$ and $E^\mu := \{|\nabla\psi^a| > \mu\}$, and let $\Psi_\mu := 1_{E_\mu} \nabla\psi^a$ and $\Psi^\mu := 1_{E^\mu} \nabla\psi^a$, where 1_E is the characteristic function of the set E. Note that Ψ_μ and Ψ^μ are periodic vector-fields. Let ψ_μ and ψ^μ be periodic solutions of the equations

$$\Delta\psi_\mu = 2 \operatorname{div} \Psi_\mu \qquad \text{and} \qquad \Delta\psi^\mu = 2 \operatorname{div} \Psi^\mu.$$

From the first equation we get

$$\int_Q |\nabla\psi_\mu|^2 \, dx \leq 4 \int_Q |\Psi_\mu|^2 \, dx. \tag{2.19}$$

Standard L^p estimates for periodic solutions of the Poisson equation (see [16]) yield a constant $\tilde{\gamma}_{n,p} \geq 4$ such that

$$\int_Q |\nabla\psi^\mu|^p \, dx \leq \tilde{\gamma}_{n,p} \int_Q |\Psi^\mu|^p \, dx. \tag{2.20}$$

From (2.19) we obtain

$$\begin{aligned}\int_Q \left(\mu^2+|\nabla\psi_\mu|^2\right)^{\frac{p-2}{2}} |\nabla\psi_\mu|^2\, dx &\le \mu^{p-2}\int_Q |\nabla\psi_\mu|^2\, dx \\ &\le 4\mu^{p-2}\int_Q |\Psi_\mu|^2\, dx \qquad (2.21)\\ &\le 8\int_{Q\cap E_\mu} \left(\mu^2+|\nabla\psi^a|^2\right)^{\frac{p-2}{2}} |\nabla\psi^a|^2\, dx,\end{aligned}$$

where the last inequality follows from the fact that $|\Psi_\mu| = |\nabla\psi^a|\, 1_{E_\mu} \le \mu$. From (2.20) we obtain

$$\begin{aligned}\int_Q \left(\mu^2+|\nabla\psi^\mu|^2\right)^{\frac{p-2}{2}} |\nabla\psi^\mu|^2\, dx &\le \int_Q |\nabla\psi^\mu|^p\, dx \\ &\le \tilde{\gamma}_{n,p}\int_Q |\Psi^\mu|^p\, dx \qquad (2.22)\\ &\le 2\tilde{\gamma}_{n,p}\int_{Q\cap E^\mu} \left(\mu^2+|\nabla\psi^a|^2\right)^{\frac{p-2}{2}} |\nabla\psi^a|^2\, dx,\end{aligned}$$

where the last inequality follows from the fact that $|\Psi^\mu| = |\nabla\psi^a|\, 1_{E^\mu} \ge \mu 1_{E^\mu}$. By (2.2) we have

$$\Delta\left(\psi_\mu+\psi^\mu\right) = 2\,\mathrm{div}\,\nabla\psi^a = \Delta\psi,$$

and since $\psi_\mu+\psi^\mu-\psi$ is a periodic function we deduce that $\nabla\psi = \nabla\psi_\mu+\nabla\psi^\mu$. Finally, from Lemma 2.6 and using (2.21) and (2.22) we obtain

$$\int_Q \left(\mu^2+|\nabla\psi|^2\right)^{\frac{p-2}{2}} |\nabla\psi|^2\, dx \le \tau_{n,p}\int_Q \left(\mu^2+|\nabla\psi^a|^2\right)^{\frac{p-2}{2}} |\nabla\psi^a|^2\, dx,$$

with $\tau_{n,p} = 2\tilde{\gamma}_{n,p}$. Since $|\nabla\psi^s| \le |\nabla\psi|$ and the mapping $t \mapsto \left(\mu^2+t\right)^{\frac{p-2}{2}} t$ is nondecreasing, inequality (2.18) follows. ■

3 Proofs

Proof of Theorem 1.1. We begin by observing that (1.1) gives

$$|f(A)| \le k_f\left(1+|A|^p\right) \qquad \forall A\in\mathbb{M}^{n\times n}_{\mathrm{sym}} \tag{3.1}$$

for a suitable constant k_f depending on f.
Step 1: We first consider the case $1<p<2$. Let $g:\mathbb{M}^{n\times n}\to\mathbb{R}$ be the function defined by

$$g(A) := \left(\mu^2+|A^a|^2\right)^{\frac{p}{2}} - \mu^p.$$

Given a constant $\beta > 0$, to be chosen at the end of the proof, let $G : \mathbb{M}^{n\times n} \to \mathbb{R}$ be the function defined by

$$G(A) := f(A^s) + \beta g(A^a), \tag{3.2}$$

and let F be its 1-quasiconvexification, i.e., (see, e.g., [8])

$$F(A) = \inf \left\{ \int_Q G(A + \nabla\varphi(x))\, dx : \varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^n) \right\}, \tag{3.3}$$

for all $A \in \mathbb{M}^{n\times n}$.

We want to prove that for every $\varepsilon > 0$ there exists $\beta > 0$ such that

$$\int_Q \left[G\left(A + \nabla\varphi\right) - G\left(A\right) \right] dx \geq -\varepsilon \left(\mu^2 + |A^a|^2 \right)^{\frac{p-2}{2}} |A^a|^2 \tag{3.4}$$

for every $A \in \mathbb{M}^{n\times n}$ and for every $\varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^n)$. In view of (3.3) this will imply that for every $A \in \mathbb{M}^{n\times n}$ we have

$$G(A) - \varepsilon \left(\mu^2 + |A^a|^2 \right)^{\frac{p-2}{2}} |A^a|^2 \leq F(A) \leq G(A) \tag{3.5}$$

which yields (1.2). Inequality (1.3) follows from (3.1), (3.2) and (3.5).

Let us prove (3.4). Fix $\varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^n)$ and consider the periodic Helmholtz decomposition

$$\varphi = \nabla\phi + \psi$$

given by Lemma 2.1. Following the argument used by Müller and Šverák in the proof of Lemma 4.2 in [22], we have

$$\begin{aligned}
&\int_Q \left[G\left(A + \nabla\varphi\right) - G\left(A\right) \right] dx \\
&\quad = \int_Q \left[f\left(A^s + \nabla^2\phi + \nabla\psi^s\right) - f\left(A^s + \nabla^2\phi\right) \right] dx \\
&\qquad + \int_Q \left[f\left(A^s + \nabla^2\phi\right) - f\left(A^s\right) \right] dx \\
&\qquad + \beta \int_Q \left[\left(\mu^2 + |A^a + \nabla\psi^a|^2 \right)^{\frac{p}{2}} - \left(\mu^2 + |A^a|^2 \right)^{\frac{p}{2}} \right] dx \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned} \tag{3.6}$$

Since $\nabla f\left(A^s\right)$ is a symmetric matrix we have $\nabla f\left(A^s\right) \cdot \nabla\psi^s = \nabla f\left(A^s\right) \cdot \nabla\psi$, and therefore, by periodicity,

$$\int_Q \nabla f\left(A^s\right) \cdot \nabla\psi^s\, dx = 0.$$

Hence

$$I_1 = \int_Q \left[f\left(A^s + \nabla^2\phi + \nabla\psi^s\right) - f\left(A^s + \nabla^2\phi\right) - \nabla f\left(A^s\right) \cdot \nabla\psi^s \right] dx.$$

By Lemma 2.9 we have

$$I_1 \geq -\nu \int_Q \left(\mu^2 + |A^s|^2 + \left|\nabla^2 \phi\right|^2\right)^{\frac{p-2}{2}} \left|\nabla^2 \phi\right|^2 \, dx$$
$$- c_{\nu,p,L} \int_Q \left(\mu^2 + |\nabla \psi^s|^2\right)^{\frac{p-2}{2}} |\nabla \psi^s|^2 \, dx,$$

while the strict 2-quasiconvexity of f (condition (a)) yields

$$I_2 \geq \nu \int_Q \left(\mu^2 + |A^s|^2 + \left|\nabla^2 \phi\right|^2\right)^{\frac{p-2}{2}} \left|\nabla^2 \phi\right|^2 \, dx,$$

and so, using Lemma 2.11, we obtain

$$\begin{aligned} I_1 + I_2 &\geq -c_{\nu,p,L} \int_Q \left(\mu^2 + |\nabla \psi^s|^2\right)^{\frac{p-2}{2}} |\nabla \psi^s|^2 \, dx \\ &\geq -c_{\nu,p,L} \, \tau_{n,p} \int_Q \left(\mu^2 + |\nabla \psi^a|^2\right)^{\frac{p-2}{2}} |\nabla \psi^a|^2 \, dx. \end{aligned} \tag{3.7}$$

Since $\nabla g(A^a)$ is an antisymmetric matrix we have $\nabla g(A^a) \cdot \nabla \psi^a = \nabla g(A^a) \cdot \nabla \psi$, and therefore, by periodicity,

$$\int_Q \nabla g(A^a) \cdot \nabla \psi^a \, dx = 0.$$

Hence, by Lemma 2.4 and Lemma 2.6, for every $0 < \delta < 1$ we obtain

$$\begin{aligned} I_3 &= \beta \int_Q \left[g\left(A^a + \nabla \psi^a\right) - g\left(A^a\right) - \nabla g(A^a) \cdot \nabla \psi^a\right] dx \\ &\geq \beta \theta_p \int_Q \left(\mu^2 + |A^a|^2 + |\nabla \psi^a|^2\right)^{\frac{p-2}{2}} |\nabla \psi^a|^2 \, dx \\ &\geq \beta \theta_p \delta^{\frac{2-p}{2}} \int_Q \left(\mu^2 + |\nabla \psi^a|^2\right)^{\frac{p-2}{2}} |\nabla \psi^a|^2 \, dx - \beta \theta_p \delta \left(\mu^2 + |A^a|^2\right)^{\frac{p-2}{2}} |A^a|^2. \end{aligned}$$

Choosing $\beta > 0$ and $0 < \delta < 1$ so that

$$\beta \theta_p \delta^{\frac{2-p}{2}} \geq c_{\nu,p,L} \, \tau_{n,p}, \qquad \beta \theta_p \delta \leq \varepsilon$$

we obtain

$$I_1 + I_2 + I_3 \geq -\varepsilon \left(\mu^2 + |A^a|^2\right)^{\frac{p-2}{2}} |A^a|^2,$$

which, together with (3.6), yields (3.4).

Step 2: Let us consider now the case $p \geq 2$. Let $\lambda := \nu / \Theta_p$, where Θ_p is the second constant in Lemma 2.4. Given a constant $\beta > 0$, to be chosen at the end of the proof, let $F : \mathbb{M}^{n \times n} \to \mathbb{R}$ be the function defined by

$$F(A) := f(A^s) - \lambda \left(\mu^2 + |A^s|^2\right)^{\frac{p}{2}} + \lambda \left(\mu^2 + |A^s|^2 + \beta^2 |A^a|^2\right)^{\frac{p}{2}}. \tag{3.8}$$

It is clear that (1.2) holds, while (1.3) follows from (3.1).

It remains to prove that, for some $\beta > 0$, the function F is 1-quasiconvex, i.e.,

$$\int_Q [F(A + \nabla\varphi) - F(A)]\, dx \geq 0 \tag{3.9}$$

for every $A \in \mathbb{M}^{n\times n}$ and for every $\varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^n)$.

Let f_λ be the 2-quasiconvex function defined in Lemma 2.10, and let

$$g_\beta(A) = \hat{g}_\beta(A^s, A^a) := \left(\mu^2 + |A^s|^2 + \beta^2 |A^a|^2\right)^{\frac{p}{2}},$$

so that

$$F(A) = f_\lambda(A^s) + \lambda g_\beta(A).$$

Let us prove (3.9). Fix a Q-periodic function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞ and consider the periodic Helmholtz decomposition

$$\varphi = \nabla\phi + \psi$$

given by Lemma 2.1. Then we have

$$\begin{aligned}
&\int_Q [F(A + \nabla\varphi) - F(A)]\, dx \\
&= \int_Q [f_\lambda(A^s + \nabla\varphi^s) - f_\lambda(A^s + \nabla\varphi^s - \nabla\psi^s)]\, dx \\
&\quad + \int_Q \left[f_\lambda\left(A^s + \nabla^2\phi\right) - f_\lambda(A^s)\right]\, dx \\
&\quad + \lambda \int_Q [g_\beta(A + \nabla\varphi) - g_\beta(A)]\, dx \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.10}$$

Since $\nabla f_\lambda(A^s)$ is a symmetric matrix, we have $\nabla f_\lambda(A^s) \cdot \nabla\psi^s = \nabla f_\lambda(A^s) \cdot \nabla\psi$, and therefore, by periodicity,

$$\int_Q \nabla f_\lambda(A^s) \cdot \nabla\psi^s\, dx = 0.$$

Hence

$$I_1 = -\int_Q [f_\lambda(A^s + \nabla\varphi^s - \nabla\psi^s) - f_\lambda(A^s + \nabla\varphi^s) + \nabla f_\lambda(A^s) \cdot \nabla\psi^s]\, dx.$$

Since the function g defined in (2.17) clearly satisfies condition (2.8) , by Lemma 2.8 and (1.1) it follows that (1.1) still holds for the function f_λ for a suitable constant $M > 0$ in place of L. We are now in position to apply Lemma 2.9 to obtain a constant $\sigma = \sigma_{p,M}$ such that

$$\begin{aligned} I_1 \geq & -\lambda\theta_p \int_Q \left(\mu^2 + |A^s|^2 + |\nabla\varphi^s|^2\right)^{\frac{p-2}{2}} |\nabla\varphi^s|^2 \, dx \\ & - \sigma \left(\mu^2 + |A^s|^2\right)^{\frac{p-2}{2}} \int_Q |\nabla\psi^s|^2 \, dx - \sigma \int_Q |\nabla\psi^s|^p \, dx, \end{aligned}$$

and so, using Lemma 2.2, we obtain

$$\begin{aligned} I_1 \geq & -\lambda\theta_p \int_Q \left(\mu^2 + |A^s|^2 + |\nabla\varphi^s|^2\right)^{\frac{p-2}{2}} |\nabla\varphi^s|^2 \, dx \\ & - \sigma\gamma_{n,2} \left(\mu^2 + |A^s|^2\right)^{\frac{p-2}{2}} \int_Q |\nabla\psi^a|^2 \, dx - \sigma\gamma_{n,p} \int_Q |\nabla\psi^a|^p \, dx. \end{aligned} \tag{3.11}$$

On the other hand, the 2-quasiconvexity of f_λ yields

$$I_2 \geq 0. \tag{3.12}$$

Since, by periodicity,

$$\int_Q \nabla g_\beta(A) \cdot \nabla\varphi \, dx = 0,$$

by Lemma 2.5 we have

$$\begin{aligned} I_3 &= \lambda \int_Q \left[g_\beta\left(A + \nabla\varphi\right) - g_\beta\left(A\right) - \nabla g_\beta(A) \cdot \nabla\varphi\right] dx \\ &\geq \lambda\theta_p \int_Q \left(\mu^2 + |A^s|^2 + |\nabla\varphi^s|^2\right)^{\frac{p-2}{2}} |\nabla\varphi^s|^2 \, dx \\ &+ \frac{\lambda\theta_p\beta^2}{2} \left(\mu^2 + |A^s|^2\right)^{\frac{p-2}{2}} \int_Q |\nabla\psi^a|^2 \, dx + \frac{\lambda\theta_p\beta^p}{2} \int_Q |\nabla\psi^a|^p \, dx. \end{aligned} \tag{3.13}$$

Choosing $\beta > 0$ so that

$$\frac{\lambda\theta_p\beta^2}{2} \geq \sigma\gamma_{n,2}, \qquad \frac{\lambda\theta_p\beta^p}{2} \geq \sigma\gamma_{n,p},$$

by (3.11), (3.12), and (3.13), we obtain

$$I_1 + I_2 + I_3 \geq 0,$$

which together with (3.10) yields (3.9). ■

Proof of Theorem 1.2. Since the function $t \mapsto f\left(A + ta \otimes b + tb \otimes a\right)$ is convex on $\mathbb{R}$ for every $A \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ and every $a,\, b \in \mathbb{R}^n$ (see, e.g., [[12]]), from the growth condition (b) it follows that there exists a constant $L > 0$ depending only on M and p such that

$$|f\left(A + B\right) - f\left(A\right)| \leq L \left(1 + |A|^{p-1} + |B|^{p-1}\right) |B| \tag{3.14}$$

for every A, $B \in \mathbb{M}^{n\times n}_{\rm sym}$. Given a constant $\beta > 0$, to be chosen at the end of the proof, let $G : \mathbb{M}^{n\times n} \to \mathbb{R}$ be the function defined by

$$G(A) := f(A^s) + \beta\,|A^a|^p\,, \tag{3.15}$$

and let F be its 1-quasiconvexification.

We want to prove that there exist two increasing sequences of positive numbers $\{\beta_k\}$ and $\{\lambda_k\}$, depending only on k, p, μ, ν, M, but not on the specific function f, such that the corresponding functions G_k satisfy

$$\int_Q \left[G_k\left(A + \nabla\varphi\right) - G_k\left(A\right)\right]\,dx \geq -\frac{1}{k}\,|A^s|^p - \lambda_k\,|A^a|^p - \frac{1}{k} \tag{3.16}$$

for every $A \in \mathbb{M}^{n\times n}$ and for every Q-periodic function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞. This will imply (see, e.g., [8]) that for every $A \in \mathbb{M}^{n\times n}$ we have

$$G_k(A) - \frac{1}{k}\,|A^s|^p - \lambda_k\,|A^a|^p - \frac{1}{k} \leq F_k(A) \leq G_k(A) \tag{3.17}$$

which yields (1.5) and (1.6) since $G_k(A) = f\,(A)$ whenever $A \in \mathbb{M}^{n\times n}_{\rm sym}$.

Let us prove (3.16). Fix a Q-periodic function $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞ and consider the periodic Helmholtz decomposition

$$\varphi = \nabla\phi + \psi$$

given by Lemma 2.1. Then we have

$$\begin{aligned}
&\int_Q \left[G\left(A + \nabla\varphi\right) - G\left(A\right)\right]dx \\
&\quad= \int_Q \left[f\left(A^s + \nabla^2\phi + \nabla\psi^s\right) - f\left(A^s + \nabla^2\phi\right)\right]\,dx \\
&\qquad+ \int_Q \left[f\left(A^s + \nabla^2\phi\right) - f\left(A^s\right)\right]\,dx \\
&\qquad+ \beta\int_Q \left[|A^a + \nabla\psi^a|^p - |A^a|^p\right]\,dx \\
&\quad=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.18}$$

By (3.14) and by Cauchy's inequality, for every $\delta > 0$ there exists a constant $c_{\delta,p,L} > 0$ such that

$$\begin{aligned}
I_1 &\geq -L\int_Q \left(1 + \left|A^s + \nabla^2\phi\right|^{p-1} + |\nabla\psi^s|^{p-1}\right)|\nabla\psi^s|\,\,dx \\
&\geq -\delta - \delta\,|A^s|^p - \delta\int_Q \left|\nabla^2\phi\right|^p\,dx - c_{\delta,p,L}\int_Q |\nabla\psi^s|^p\,dx.
\end{aligned}$$

Hence, using Lemma 2.2 we obtain

$$I_1 \geq -\delta - \delta\,|A^s|^p - \delta\int_Q \left|\nabla^2\phi\right|^p\,dx - c_{\delta,p,L}\gamma_{n,p}\int_Q |\nabla\psi^a|^p\,dx.$$

If $p \geq 2$ then we have

$$I_1 \geq -\delta - \delta \left| A^s \right|^p - \delta \int_Q \left(\mu^2 + \left| A^s \right|^2 + \left| \nabla^2 \phi \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla^2 \phi \right|^2 \, dx$$
$$- c_{\delta,p,L} \gamma_{n,p} \int_Q \left(\left| A^a \right|^2 + \left| \nabla \psi^a \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla \psi^a \right|^2 \, dx.$$

If $1 < p \leq 2$ then by Lemma 2.7, and by Lemma 2.6 with $\mu = 0$, we obtain for every $0 < \varepsilon < 1$

$$I_1 \geq -\delta \left(1 + \varepsilon \mu^p \right) - \delta \left(1 + \varepsilon \right) \left| A^s \right|^p$$
$$- 8 \delta \varepsilon^{\frac{p-2}{p}} \int_Q \left(\mu^2 + \left| A^s \right|^2 + \left| \nabla^2 \phi \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla^2 \phi \right|^2 \, dx$$
$$- c_{\delta,p,L} \gamma_{n,p} \varepsilon^{\frac{p-2}{2}} \int_Q \left(\left| A^a \right|^2 + \left| \nabla \psi^a \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla \psi^a \right|^2 \, dx - c_{\delta,p,L} \gamma_{n,p} \varepsilon^{\frac{p}{2}} \left| A^a \right|^p.$$

In both cases there exists a sequence of positive numbers $\{\lambda_k\}$, depending only on p, μ, ν, M, such that for every k

$$I_1 \geq -\frac{1}{k} - \frac{1}{k} \left| A^s \right|^p - \nu \int_Q \left(\mu^2 + \left| A^s \right|^2 + \left| \nabla^2 \phi \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla^2 \phi \right|^2 \, dx \tag{3.19}$$
$$- \lambda_k \int_Q \left(\left| A^a \right|^2 + \left| \nabla \psi^a \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla \psi^a \right|^2 \, dx - \lambda_k \left| A^a \right|^p.$$

The strict 2-quasiconvexity of f (condition (a)) yields

$$I_2 \geq \nu \int_Q \left(\mu^2 + \left| A^s \right|^2 + \left| \nabla^2 \phi \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla^2 \phi \right|^2 \, dx. \tag{3.20}$$

Since, by periodicity,

$$\int_Q A^a \cdot \nabla \psi^a \, dx = 0,$$

by Lemma 2.4 with $\mu = 0$, we have

$$I_3 = \beta \int_Q \left[\left| A^a + \nabla \psi^a \right|^p - \left| A^a \right|^p - p \left| A^a \right|^{p-2} A^a \cdot \nabla \psi^a \right] dx \tag{3.21}$$
$$\geq \beta \theta_p \int_Q \left(\left| A^a \right|^2 + \left| \nabla \psi^a \right|^2 \right)^{\frac{p-2}{2}} \left| \nabla \psi^a \right|^2 \, dx.$$

Choosing $\beta_k > 0$ so that $\beta_k \theta_p \geq \lambda_k$, from (3.19), (3.20), and (3.21), we obtain

$$I_1 + I_2 + I_3 \geq -\frac{1}{k} - \frac{1}{k} \left| A^s \right|^p - \lambda_k \left| A^a \right|^p,$$

which together with (3.18) gives (3.16). ∎

Remark 3.1 It is clear from the proof of Theorem 1.2 that if f is nonnegative the we may take F_k to be also nonnegative.

4 Lower semicontinuity

The proof of Theorem 1.3 relies on the so-called Decomposition Lemma (see [13]).

Lemma 4.1 (Decomposition Lemma) *Let Ω be a bounded open set in $\mathbb{R}^n$, let $p > 1$, and let $\{u_k\}$ be a sequence weakly converging to a function u in $W^{1,p}(\Omega;\mathbb{R}^n)$. Then there exists a subsequence (not relabeled) and a sequence $\{v_k\}$ weakly converging to u in $W^{1,p}(\Omega;\mathbb{R}^n)$ such that $v_k = u$ in a neighborhood of $\partial\Omega$, $\{|\nabla v_k|^p\}$ is equi-integrable, and $\mathcal{L}^n(\{u_k \neq v_k\}) \to 0$.*

The following simple lemma may be found in [11], however we include its proof for the convenience of the reader.

Lemma 4.2 *Let $D \subset \mathbb{R}^m$ be an open set and let*

$$f : D \times \mathbb{M}^{d\times n} \to \mathbb{R}$$

be a lower semicontinuous function such that for every $v \in D$ the function $f(v,\cdot)$ is continuous. Then for every $\bar{v} \in D$, $\varepsilon > 0$, and $L > 0$ there exists $\delta = \delta(\bar{v},\varepsilon,L) \in (0,1)$ such that

$$f(\bar{v},A) \leq f(v,A) + \varepsilon$$

for every $(v,A) \in D \times \mathbb{M}^{d\times n}$, with $|v-\bar{v}| \leq \delta$ and $|A| \leq L$.

Proof. Assume, for contradiction, that there exist $\bar{v} \in D$, $L > 0$, $\bar{\varepsilon} > 0$, and a sequence

$$\{(v_k, A_k)\} \subset D \times \overline{B_{d\times n}(0,L)},$$

such that

$$\bar{\varepsilon} + f(v_k, A_k) < f(\bar{v}, A_k) \tag{4.1}$$

and $(v_k, A_k) \to (\bar{v}, \bar{A})$ as $k \to \infty$, for some $\bar{A} \in \overline{B_{d\times n}(0,L)}$. Since the function $f(\bar{v},\cdot)$ is continuous and f is lower semicontinuous, for any $\varepsilon < \frac{1}{2}\bar{\varepsilon}$ there exists $\delta > 0$ such that

$$|f(\bar{v},A) - f(\bar{v},\bar{A})| \leq \varepsilon, \qquad f(\bar{v},\bar{A}) \leq f(v,A) + \varepsilon$$

for all $(v,A) \in D \times \overline{B_{d\times n}(0,L)}$ with

$$|v-\bar{v}| + |A-\bar{A}| \leq \delta.$$

Thus for all n sufficiently large, also by (4.1), we have

$$\bar{\varepsilon} + f(v_k,A_k) < f(\bar{v},A_k) \leq f(\bar{v},\bar{A}) + \varepsilon \leq f(v_k,A_k) + 2\varepsilon,$$

which is a contradiction. ∎

Although the following lemma is well known to experts, its proof is not easy to find in the literature and so we present it below for the reader's convenience.

Lemma 4.3 *Let $D \subset \mathbb{R}^m$ be an open set and let*

$$f : D \times \mathbb{M}^{d\times n} \to \mathbb{R}$$

be a lower semicontinuous function which satisfies the following conditions:

(a) for every $v \in D$ the function $f(v,\cdot)$ is continuous in $\mathbb{M}^{d\times n}$;

(b) there exist a locally bounded function $a : D \to [0,+\infty)$, a lower semicontinuous function $b : D \to (0,+\infty)$, a locally bounded function $c : D \to [0,+\infty)$, and a constant $p > 1$ such that

$$b\,(v)\,|A|^p - c(v) \le f(v,A) \le a(v)(1+|A|^p)$$

for every $(v,A) \in D \times \mathbb{M}^{d\times n}$.

For every $v \in D$, let $\mathcal{Q}f(v,\cdot)$ be the 1-quasiconvexification of the function $f(v,\cdot)$. Then $\mathcal{Q}f$ is lower semicontinuous on $D \times \mathbb{M}^{d\times n}$.

Proof. By conditions (a) and (b) for every $(v,A) \in D\times\mathbb{M}^{d\times n}$, we have (see [8])

$$\mathcal{Q}f(v,A) = \inf\left\{ \int_Q f(v, A + \nabla\varphi(x))\,dx : \varphi \in C^1_c(Q;\mathbb{R}^d)\right\}. \tag{4.2}$$

By replacing $f(v,A)$ with $f(v,A) + \tilde{c}(v)$, where $\tilde{c}$ is any continuous function with $\tilde{c} \ge c$, we may assume without loss of generality that $f \ge 0$.

We begin by showing that for every fixed $A \in \mathbb{M}^{d\times n}$ the function $\mathcal{Q}f(\cdot,A)$ is lower semicontinuous. Without loss of generality we may assume that $A = 0$. Let $\{v_k\} \subset D$ be a sequence converging to some $\bar{v} \in D$. If

$$\liminf_{k\to\infty} \mathcal{Q}f(v_k,0) = \infty,$$

then there is nothing to prove. Thus, without loss of generality, we may assume that

$$\liminf_{k\to\infty} \mathcal{Q}f(v_k,0) = \lim_{k\to\infty} \mathcal{Q}f(v_k,0) < \infty,$$

and

$$C := \sup_k \mathcal{Q}f(v_k,0) < \infty. \tag{4.3}$$

By (4.2) for every fixed $0 < \varepsilon < 1$ and for every $k \in \mathbb{N}$ there exists $\varphi_k \in C^1_c(Q;\mathbb{R}^d)$ such that

$$\mathcal{Q}f(v_k,0) + \varepsilon \ge \int_Q f(v_k, \nabla\varphi_k(x))\,dx. \tag{4.4}$$

Hence, by condition (b) and (4.3), we have

$$\begin{aligned} C + 1 &\ge b\,(v_k) \int_Q |\nabla\varphi_k(x)|^p\,dx - c(v_k) \\ &\ge b_0 \int_Q |\nabla\varphi_k(x)|^p\,dx - c_0, \end{aligned}$$

where $b_0 := \inf_k b(v_k) > 0$ and $c_0 := \sup_k c(v_k) < \infty$, since b is lower semicontinuous and c is locally bounded. By the Decomposition Lemma (see Lemma 4.1), there exists a subsequence of $\{\varphi_k\}$ (not relabeled) and a sequence $\{w_k\}$ weakly converging to some function w in $W_0^{1,p}\left(Q;\mathbb{R}^d\right)$ such that $\{|\nabla w_k|^p\}$ is equi-integrable, and $\mathcal{L}^n\left(\{\varphi_k \neq w_k\}\right) \to 0$. Hence we may find $L \geq 1$ and $\bar{k} \in \mathbb{N}$ such that

$$\mathcal{L}^n\left(\{\varphi_k \neq w_k\}\right) + \int_{\{|\nabla w_k| \geq L\}} |\nabla w_k|^p \, dx \leq \varepsilon, \tag{4.5}$$

for every $k \geq \bar{k}$. By Lemma 4.2 there exists $\delta = \delta(\bar{v}, L, \varepsilon) \in (0,1)$ such that

$$f(\bar{v}, B) \leq f(v, B) + \varepsilon$$

for all $(v, B) \in D \times \mathbb{M}^{d\times n}$, with $|v - \bar{v}| \leq \delta$ and all $|B| \leq L$. Therefore

$$\begin{aligned} \mathcal{Q}f(v_k, 0) + \varepsilon &\geq \int_{\{\varphi_k = w_k\}\cap\{|\nabla w_k| < L\}} f(v_k, \nabla w_k(x))\, dx \\ &\geq \int_{\{\varphi_k = w_k\}\cap\{|\nabla w_k| < L\}} f(\bar{v}, \nabla w_k(x))\, dx - \varepsilon \\ &\geq \int_{\{\varphi_k = w_k\}} f(\bar{v}, \nabla w_k(x))\, dx - \varepsilon(2 + a(\bar{v})), \end{aligned} \tag{4.6}$$

where in the last inequality we have used (4.5) and condition (b). Since $\{|\nabla w_k|^p\}$ is equi-integrable and $\mathcal{L}^n\left(\{\varphi_k \neq w_k\}\right) \to 0$, by condition (b) we have

$$\liminf_{k\to\infty} \int_{\{\varphi_k = w_k\}} f(\bar{v}, \nabla w_k(x))\, dx = \liminf_{k\to\infty} \int_Q f(\bar{v}, \nabla w_k(x))\, dx,$$

hence letting $k \to \infty$ in (4.6) yields

$$\begin{aligned} \liminf_{k\to\infty} \mathcal{Q}f(v_k, 0) + \varepsilon &\geq \liminf_{k\to\infty} \int_Q f(\bar{v}, \nabla w_k(x))\, dx - \varepsilon(2 + a(\bar{v})) \\ &\geq \mathcal{Q}f(\bar{v}, 0) - \varepsilon(2 + a(\bar{v})). \end{aligned}$$

It is now sufficient to let $\varepsilon \to 0^+$ to conclude that $v \mapsto \mathcal{Q}f(v, A)$ is lower semicontinuous.

Finally, we observe that the continuity of $A \mapsto \mathcal{Q}f(v, A)$ is an immediate consequence of the quasiconvexity of $\mathcal{Q}f(v, \cdot)$. Since the coefficient $a(v)$ in (b) is locally bounded, the functions $A \mapsto \mathcal{Q}f(v, A)$ have the same local modulus of continuity when v varies in a compact subset of D. Since we have seen that $v \mapsto \mathcal{Q}f(v, A)$ is lower semicontinuous on D, this implies that $\mathcal{Q}f$ is lower semicontinuous on $D \times \mathbb{M}^{d\times n}$. ■

Proof of Theorem 1.3. Fix $u \in W^{1,1}(\Omega)$ and let $\{u_j\} \subset SBH(\Omega)$ be any sequence converging to u in $W^{1,1}(\Omega)$ and satisfying (1.7). Without loss of generality we may assume that

$$\liminf_{j\to\infty} \int_\Omega f(x, u_j, \nabla u_j, \nabla^2 u_j)\, dx = \lim_{j\to\infty} \int_\Omega f(x, u_j, \nabla u_j, \nabla^2 u_j)\, dx.$$

Fix $\varepsilon \in (0,1)$ and let

$$f_\varepsilon(x,u,\xi,A) := f(x,u,\xi,A) + \varepsilon\,|A|^p\,.$$

The function f_ε satisfies all the conditions of Theorem 1.2, hence there exists an increasing sequence $\{F_{k,\varepsilon}\}$ of functions $F_{k,\varepsilon} : \Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{M}^{n\times n} \to [0,+\infty)$ such that for $\mathcal{L}^n$ a.e. $x\in\Omega$ and for every $(u,\xi)\in\mathbb{R}\times\mathbb{R}^n$ the function $F_{k,\varepsilon}(x,u,\xi,\cdot)$ is 1-quasiconvex,

$$\lim_{k\to\infty} F_{k,\varepsilon}(x,u,\xi,A) = f_\varepsilon(x,u,\xi,A) \qquad \forall A\in\mathbb{M}^{n\times n}_{\mathrm{sym}}, \tag{4.7}$$

$$0 \le F_{k,\varepsilon}(x,u,\xi,A) \le c_k(x,u,\xi)\,(1+|A|^p) \qquad \forall A\in\mathbb{M}^{n\times n}. \tag{4.8}$$

Let $C := \sup_j \left\|\nabla^2 u_j\right\|^p_{L^p}$. For every fixed $\varepsilon\in(0,1)$ and $k\in\mathbb{N}$, we have

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega f(x,u_j,\nabla u_j,\nabla^2 u_j)\,dx &\ge \liminf_{j\to\infty}\int_\Omega f_\varepsilon(x,u_j,\nabla u_j,\nabla^2 u_j)\,dx - \varepsilon C \qquad (4.9)\\ &\ge \liminf_{j\to\infty}\int_\Omega F_{k,\varepsilon}(x,u_j,\nabla u_j,\nabla^2 u_j)\,dx - \varepsilon C,\end{aligned}$$

where we used the fact that $f_\varepsilon \ge F_{k,\varepsilon}$. We note that, in view of the construction in the proof of Theorem 1.2, the function $F_{k,\varepsilon}$ is defined as 1-quasiconvexification of $f_\varepsilon(x,u,\xi,A^s)+\beta\,|A^a|^p$, and so, by the previous lemma, we have that $F_{k,\varepsilon}$ is a normal integrand. Define

$$G_{k,\varepsilon} : \Omega\times(\mathbb{R}\times\mathbb{R}^n)\times\left(\mathbb{R}^n\times\mathbb{M}^{n\times n}\right)\to[0,\infty)$$

as

$$G_{k,\varepsilon}\left(x,(w_1,w_2),(\xi_1,\xi_2)\right) := F_{k,\varepsilon}\left(x,w_1,w_2,\xi_2\right).$$

Indeed, with $w_j := (u_j,\nabla u_j)$ then

$$\int_\Omega F_{k,\varepsilon}(x,u_j,\nabla u_j,\nabla^2 u_j)\,dx$$

reduces to

$$\int_\Omega G_{k,\varepsilon}(x,w_j,\nabla w_j)\,dx.$$

It is clear that $\{w_j\}\subset SBV(\Omega,\mathbb{R}\times\mathbb{R}^n)$ and that

$$\sup_j\left(\|\nabla w_j\|_{L^p} + \int_{S(w_j)}\theta(|[w_j]|)\,d\mathcal{H}^{n-1}\right) < \infty.$$

Hence we may apply Theorem 1.2 in [18] to obtain

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega f(x,u_j,\nabla u_j,\nabla^2 u_j)\,dx &\ge \liminf_{j\to\infty}\int_\Omega G_{k,\varepsilon}(x,w_j,\nabla w_j)\,dx - \varepsilon C\\ \ge \int_\Omega G_{k,\varepsilon}(x,w,\nabla w)\,dx - \varepsilon C &= \int_\Omega F_{k,\varepsilon}(x,u,\nabla u,\nabla^2 u)\,dx - \varepsilon C,\end{aligned}$$

where we have used (4.9), and $w := (u, \nabla u)$.

By Lebesgue's Monotone Convergence Theorem, letting $k \to \infty$ in the previous inequality and using (4.7) gives

$$\lim_{j\to\infty} \int_\Omega f(x, u_j, \nabla u_j, \nabla^2 u_j)\, dx \geq \int_\Omega f_\varepsilon(x, u, \nabla u, \nabla^2 u)\, dx - \varepsilon C.$$

It now suffices to let $\varepsilon \to 0^+$. ■

Acknowledgments

The research of I. Fonseca was partially supported by the National Science Foundation under Grant No. DMS–0103799. The work of Gianni Dal Maso is part of the Project "Calculus of Variations" 2002, supported by the Italian Ministry of Education, University, and Research.

This work was undertaken when G. Dal Maso visited the Center for Nonlinear Analysis (NSF Grant No. DMS–9803791), Carnegie Mellon University, Pittsburgh, PA, USA. The authors thank the Center for Nonlinear Analysis for its support during the preparation of this paper.

I. Fonseca was engaged in several stimulating discussions on the subject of this paper with B. Dacorogna during her visit to EPFL (Lausanne, Switzerland) in Fall 2001. G. Leoni wishes to thank J. Kristensen for pointing out reference [22].

The authors wish to thank D. Kinderlehrer for many interesting conversations on the subject of this paper.

References

[1] E. Acerbi, N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rational Mech. Anal. 86 (1984), no. 2, 125–145.

[2] E. Ambrosio, N. Fusco, D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[3] J. M. Ball, J. C. Currie, P. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal. 41 (1981), no. 2, 135–174.

[4] M. Carriero, A. Leaci, F. Tomarelli, *Special bounded Hessian and elastic-plastic plate*, Rend. Accad. Naz. Sci. XL Mem. Mat. (4) 25 (1992), 233–258.

[5] M. Carriero, A. Leaci, F. Tomarelli, *Strong minimizers of Blake & Zisserman functional*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 15 (1997), no. 1-2, 257–285.

[6] R. Choksi, R. V. Kohn, F. Otto, *Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy,* Comm. Math. Phys. 201 (1999), no. 1, 61–79.

[7] S. Conti, I. Fonseca, G. Leoni, *A Γ-convergence result for the two-gradient theory of phase transitions*, Comm. Pure Appl. Math. 55 (2002), no. 7, 857–936.

[8] B. Dacorogna, *Direct methods in the calculus of variations*, Applied Mathematical Sciences, 78. Springer-Verlag, Berlin, 1989.

[9] A. DeSimone, *Energy minimizers for large ferromagnetic bodies*, Arch. Rational Mech. Anal. 125 (1993), 99–143.

[10] I. Fonseca, G. Leoni, J. Malý, R. Paroni, *A note on Meyers' theorem in $W^{k,1}$*, Trans. Amer. Math. Soc. 354 (2002), no. 9, 3723–3741.

[11] I. Fonseca, G. Leoni, R. Paroni, *On lower semicontinuity in BH^p and 2-quasiconvexification*, Research Report no. 01-CNA-013, Carnegie Mellon University, Pittsburgh. To appear on Calc. Var. and Partial Differential Equations.

[12] I. Fonseca, S. Müller, *A-quasiconvexity, lower semicontinuity, and Young measures*, SIAM J. Math. Anal. 30 (1999), no. 6, 1355–1390.

[13] I. Fonseca, S. Müller, P. Pedregal, *Analysis of concentration and oscillation effects generated by gradients*, SIAM J. Math. Anal. 29 (1998), no. 3, 736–756.

[14] N. Fusco, *Quasiconvexity and semicontinuity for higher-order multiple integrals (Italian)*, Ricerche Mat. 29 (1980), no. 2, 307–323.

[15] M. Guidorzi, L. Poggiolini, *Lower semicontinuity for quasiconvex integrals of higher order.* NoDEA Nonlinear Differential Equations Appl. 6 (1999), no. 2, 227–246.

[16] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[17] R. V. Kohn, S. Müller, *Surface energy and microstructure in coherent phase transitions*, Comm. Pure Appl. Math. 47 (1994), no. 4, 405–435.

[18] J. Kristensen, *Lower semicontinuity in spaces of weakly differentiable functions*, Math. Ann. 313 (1999), no. 4, 653–710.

[19] D. Kinderlehrer, P. Pedregal, *Characterizations of Young measures generated by gradients*, Arch. Rational Mech. Anal. 115 (1991), no. 4, 329–365.

[20] N. Meyers, *Quasi-convexity and lower semi-continuity of multiple variational integrals of any order,* Trans. Amer. Math. Soc. 119 (1965), 125–149.

[21] S. Müller, *Variational models for microstructures and phase transitions*, Lecture Notes, MPI Leipzig, 1998.

[22] S. Müller, V. Šverák, *Convex integration for Lipschitz mappings and counterexamples to regularity*, Preprint Nr. 26/1999, Max-Planck Institute, Leipzig.

[23] D. R. Owen, R. Paroni, *Second-order structured deformations*, Arch. Ration. Mech. Anal. 155 (2000), no. 3, 215–235.

[24] T. Rivière, S. Serfaty, *Limiting domain wall energy for a problem related to micromagnetics*, Comm. Pure Appl. Math. 54 (2001), no. 3, 294–338.

[25] V. Šverák, *New examples of quasiconvex functions*, Arch. Ration. Mech. Anal. 119 (1992), 293–300.

S.I.S.S.A,
Trieste, Italy
dalmaso@sissa.it

Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh, PA 15213, USA
fonseca@andrew.cmu.edu
giovanni@andrew.cmu.edu
morini@andrew.cmu.edu