A-quasiconvexity with variable coefficients

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March 17, 2003

Abstract

It is shown that for integrals of the type

$$I(u,v) := \int_\Omega f(x,u(x),v(x))\,dx,$$

with $\Omega \subset \mathbb{R}^N$ open, bounded, and $f : \Omega\times\mathbb{R}^m\times\mathbb{R}^d \to [0,+\infty)$ Carathéodory satisfying a growth condition $0 \le f(x,u,v) \le C(1+|v|^p)$, for some $p \in (1,+\infty)$, a sufficient condition for lower semicontinuity along sequences $u_n \to u$ in measure, $v_n \rightharpoonup v$ in L^p, $\mathcal{A}v_n \to 0$ in $W^{-1,p}$ is the $\mathcal{A}_x$-quasiconvexity of $f(x,u,.)$. Here $\mathcal{A}$ is a variable coefficients operator of the form

$$\mathcal{A} := \sum_{i=1}^N A^{(i)}(x)\frac{\partial}{\partial x_i},$$

with $A^{(i)} \in C^\infty(\Omega;\mathcal{M}^{l\times d}) \cap W^{1,\infty}$, $i = 1,..N$, satisfying the condition

$$\operatorname{rank}\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right) = \text{const} \quad \text{for } x \in \Omega \text{ and } \omega \in \mathbb{R}^N \setminus \{0\},$$

and $\mathcal{A}_x$ denotes the constant coefficients operator one obtains by freezing x. Under additional regularity conditions on f it is proved that the condition above is also necessary. A characterization of the Young measures generated by bounded sequences $\{v_n\}$ in L^p satisfying the condition $\mathcal{A}v_n \to 0$ in $W^{-1,p}$ is obtained.

Key words $\mathcal{A}$-quasiconvexity, Young measures, lower semicontinuity.

AMS subject classification.
35D99, 35E99, 49J45

1 Introduction

Motivated in part by the study of equilibrium of certain advanced materials, recently there has been extensive research on minimization and relaxation of nonconvex multiple integrals of the type

$$u \to \int_\Omega f(x, u(x), \nabla u(x), .., \nabla^k u(x))\, dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, $u : \Omega \to \mathbb{R}^m$, $N, m \geq 1$, and $k \in \mathbb{N}$. One way of attacking this problem is to use the Direct Method of the Calculus of Variations, and a key step in that direction is to identify conditions on f that ensure lower semicontinuity for an appropriate topology. In the case where $k = 1$ it is known that sequential weak lower semicontinuity on $W^{1,p}$ is equivalent to quasiconvexity of $f(x, u, .)$ under appropriate growth and regularity conditions on f (see [2],[10],[12]).

Recently Fonseca and Müller [7], drawing from the theory of compensated compactness of Murat and Tartar ([11],[16]), extended this study to the more general setting

$$(u, v) \to \int_\Omega f(x, u(x), v(x))\, dx, \qquad \mathcal{A}v = 0,$$

where $u : \Omega \to \mathbb{R}^m$, $v : \Omega \to \mathbb{R}^d$ and $\mathcal{A}$ is a first order linear partial differential operator with constant coefficients and of constant rank, i.e.,

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}, \quad \text{and rank} \left(\sum_{i=1}^N A^{(i)} w_i \right) = \text{const} \quad \text{for every} \quad w \in S^{N-1},$$

where $A^{(i)}, i = 1, .., N$, are $\mathcal{M}^{l \times d}$ matrices. This setting includes the framework of (1.1), and also other situations like div $= 0$ or Maxwell Equations. In [7] it was shown that, under appropriate regularity and growth conditions on f, sequential lower semicontinuity of the functional on $L^p(\text{strong}) \times L^q(\text{weak})$ is equivalent to $\mathcal{A}$-quasiconvexity of $f(x, u, .)$. We recall that a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is $\mathcal{A}$ quasiconvex if and only if

$$f(v) \leq \int_Q f(v + w(x))\, dx$$

whenever $w \in C^\infty_{\text{per}}(\mathbb{R}^N, \mathbb{R}^d)$, $\int_Q w(x)\, dx$ and $\mathcal{A}w = 0$.

In this paper we generalize some of the results of [7] to the case of variable coefficients, precisely

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)}(x) \frac{\partial v}{\partial x_i},$$

where $A^{(i)} \in C^\infty(\Omega; \mathbb{M}^{l \times d}) \cap W^{1,\infty}$, and rank$\left(\sum_{i=1}^N A^{(i)}(x) w_i \right)$ =const for every $x \in \Omega$ and all $w \in R^N \setminus \{0\}$.

Given $x_0 \in \Omega$, denote by $\mathcal{A}_{x_0}$ the partial differential operator with constant coefficients that we obtain by freezing x_0, i.e.,

$$\mathcal{A}_{x_0} v := \sum_{i=1}^N A^{(i)}(x_0) \frac{\partial v}{\partial x_i}.$$

The following sufficient condition for lower semicontinuity holds.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $1 < q < +\infty$, and let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty[$ be a Carateódory function, with $0 \le f(x, u, v) \le a(x, u)(1 + |v|^q)$ for some locally bounded function $a : \Omega \times \mathbb{R}^m \to [0, +\infty)$ and for all $v \in \mathbb{R}^d$, a.e. $x \in \Omega$. Suppose that $f(x, u, .)$ is $\mathcal{A}_x$-quasiconvex for a.e. x in Ω and all $u \in \mathbb{R}^m$. Then*

$$\liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), v_n(x))\, dx \ge \int_\Omega f(x, u(x), v(x))\, dx$$

whenever $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^q(\Omega; \mathbb{R}^d)$, $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega; \mathbb{R}^l)$.

For the necessary condition we have the following.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $1 < q < +\infty$, and let $f : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be a continuous function satisfying the q-Lipschitz continuity condition*

$$|f(x, v_1) - f(x, v_2)| \le a(x) \left(1 + |v_1|^{q-1} + |v_2|^{q-1}\right) |v_1 - v_2|, \tag{1.2}$$

where $a \in L^\infty_{\mathrm{loc}}(\Omega)$. Suppose we have lower semicontinuity of the integral

$$\liminf \int_\Omega f(x, v_n(x))\, dx \ge \int_\Omega f(x, v(x))\, dx$$

for sequences $v_n \rightharpoonup v$ in $L^q(\Omega; \mathbb{R}^m)$, constrained by the system of PDEs in the following sense

$$\mathcal{A}v_n := \sum_{i=1}^N A^{(i)}(x) \frac{\partial v_n}{\partial x_i} \to 0 \quad in \quad W^{-1,q}(\Omega; \mathbb{R}^l). \tag{1.3}$$

Then $f(x, .)$ is $\mathcal{A}_x$-quasiconvex for all $x \in \Omega$.

We could not prove the necessary condition for exact solutions of the PDE, but only under the more restrictive condition (1.3). In the case of constant coefficients Fonseca and Müller [7] were able to prove the necessary condition for sequences in the kernel of $\mathcal{A}$. Using Fourier series representation they could construct a projection P onto the kernel of $\mathcal{A}$, using algebraic computations on the symbols, and to prove the estimate (continuity of the inverse)

$$||v - Pv||_{L^q} \le C_q ||\mathcal{A}v||_{W^{-1,q}}. \tag{1.4}$$

A major difficulty that arises when we deal with the variable coefficients setting is that to the composition of operators does not correspond the multiplication of symbols any more, only up to a regularizing operator. Thus in our case, using also Fourier analysis, we were just able to prove the estimate

$$||v - P_\eta v||_{L^q} \le C_q \left(||\mathcal{A}v||_{W^{-1,q}} + ||v||_{W^{-1,q}}\right), \tag{1.5}$$

where P_η is not a projection, $AP_\eta v \neq 0$ in general, but $AP_\eta v_n \to 0$ in $W^{-1,q}$ whenever $v_n \to 0$ in $W^{-1,q}$. We also emphasize that at least in the case $q = 2$ there exits a continuous projection onto the kernel of $\mathcal{A}$ but the continuity result (1.4) remains to be asserted, or at least the weaker estimate(1.5) with the P_η replaced by the projection P.

We also characterize the Young measures generated by bounded L^q sequences satisfying (1.3). In the case of constant coefficients similar characterization is provided for sequences in the kernel of the operator [7], in this way generalizing the result of the Kinderleher and Pedregal on gradients [8] [9] (in that case $\mathcal{A} = curl$). For the same reasons we detailed above we were unable to replace (1.3) by sequences in the kernel of the operator.

Theorem 1.3. *Let $1 < q < +\infty$ and let $\{\nu_x\}_{x\in\Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. Then there exists a q-equi-integrable sequence $\{v_n\}$ in $L^q(\Omega;\mathbb{R}^d)$ that generates the Young measure ν and satisfies $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega;\mathbb{R}^l)$ if and only if*

i) *there exists $v \in L^q(\Omega;\mathbb{R}^d)$ such that $\mathcal{A}v = 0$ and $v(x) = \langle \nu_x, Id\rangle$ a.e $x \in \Omega$;*

ii) $\int_\Omega \int_{\mathbb{R}^d} |z|^q \, d\nu_x(z)\, dx < +\infty$*;*

iii) *for a.e. $x \in \Omega$ and all continuous functions g that satisfy $|g(v)| \leq C(1+|v|^q)$ one has $\langle \nu_x, g\rangle \geq Q_{\mathcal{A}_x} g\left(\langle \nu_x, Id\rangle\right)$.*

2 Preliminaries

Here we present some notation that we will be using throughout the paper and also some results about Pseudodifferential Operators, Young measures and Linear Partial Differential Operators of constant coefficients and constant rank.

In the sequel $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, $Q := (0,1)^N$, $Q(x_0, r) := x_0 + r(-\frac{1}{2}, \frac{1}{2})^N$. The N-dimensional Lebesgue measure is $\mathcal{L}^N$. For a set A the function χ_A is

$$\chi_A := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Unless different thing is indicated, which will be clear from the context, the operator $\mathcal{A}$ refers to

$$\mathcal{A}v := \sum_{i=1}^N A^{(i)}(x) \frac{\partial v}{\partial x_i},$$

with $v : \Omega \to \mathbb{R}^d$, $A^{(i)} \in C^\infty(\Omega; \mathcal{M}^{l\times d}) \cap W^{1,\infty}$ for $i = 1, .., N$, and there exist r positive integer such that

$$\operatorname{rank}\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right) = r$$

for $x \in \Omega$ and $\omega \in \mathbb{R}^N \setminus \{0\}$. For $x_0 \in \Omega$, $\mathcal{A}_{x_0}$ is the constant coefficients operator with constant rank defined by

$$\mathcal{A}_{x_0} v := \sum_{i=1}^{N} A^{(i)}(x_0) \frac{\partial v}{\partial x_i}.$$

2.1 Pseudodifferential operators

We present some results on *Pseudodifferential Operators*, for more details and proofs we refer the reader to [14].

We start by introducing some notation. Given a function $u : \mathbb{R}^N \to \mathbb{C}$, we denote by ∂_j the partial derivative with respect to x_j, and by $D_j := -\underline{i}\partial_j$, where $\underline{i}$ is the imaginary unit. Given two functions u and v in $L^2(\mathbb{R}^N)$ we set

$$(u, v) := \int_{\mathbb{R}^N} u(x)\overline{v(x)}\, dx.$$

We denote by $\mathcal{S}$ the space of $C^\infty(\mathbb{R}^N)$ functions that are rapidly decreasing at infinity, i.e., a function φ belongs to $\mathcal{S}$ if $x^\alpha \partial^\beta \varphi$ are bounded in $\mathbb{R}^N$ for all pairs α, β of multiindices. The topology on $\mathcal{S}$ is defined by the norms ($k \in \mathbb{Z}_0^+$)

$$||\varphi||_k = \sup_{|\alpha+\beta| \leq k} ||x^\alpha \partial^\beta \varphi||_\infty.$$

We denote by $\mathcal{S}'$ the set of semilinear forms u (i.e. $(u, \alpha\varphi + \beta\psi) = \bar{\alpha}(u, \varphi) + \bar{\beta}(u, \psi)$) on $\mathcal{S}$ such that there exits $C \in \mathbb{R}$ and $M \in \mathbb{Z}_0^+$ verifying

$$|(u, \varphi)| \leq C ||\varphi||_M \qquad \text{for } \varphi \in \mathcal{S}.$$

For a function $u \in \mathcal{S}$, the Fourier transform $\hat{u}$ (or $\mathcal{F}u$) of u, is defined by the formula

$$\hat{u}(\lambda) := \int_{\mathbb{R}^N} u(x) e^{-\underline{i}x.\lambda}\, dx.$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1} u(\lambda) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} u(x) e^{\underline{i}x.\lambda}\, dx.$$

Given $s \in \mathbb{R}$ we denote by $L^{s,p}(\mathbb{R}^N)$ the image of $L^p(\mathbb{R}^N)$ under the linear mapping

$$J^s u = \mathcal{F}^{-1}\left(\left(1 + |\lambda|^2\right)^{-\frac{s}{2}} \mathcal{F}u \right).$$

If $u \in L^{s,p}(\mathbb{R}^N)$ then there exits a unique $\tilde{u} \in L^p(\mathbb{R}^N)$ with $u = J^s \tilde{u}$. The space $L^{s,p}(\mathbb{R}^N)$ is a Banach space with norm

$$||u||_{L^{s,p}} := ||\tilde{u}||_{L^p}.$$

The spaces $L^{s,p}$, with $p = 2$, coincide with $H^s(\mathbb{R}^N)$ for any $s \in \mathbb{R}$, and for $p \in (1, +\infty)$ and $s \in \mathbb{Z}$ they coincide with $W^{s,p}(\mathbb{R}^N)$. We have the duality relation

$$[L^{s,p}(\mathbb{R}^N)]^{'} = L^{-s,p'}(\mathbb{R}^N),$$

where $p' = \frac{p}{p-1}$.

For more details about the spaces $L^{s,p}$ we refer the reader to [1].

Let $q \in \mathbb{R}$ and let $b(x, \lambda)$ be a C^∞ complex-valued function on $\mathbb{R}^N \times \mathbb{R}^N$. We say that b is a *symbol of order-q*, and we write $b \in S^q$, if there exist constants $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_\lambda^\beta b(x, \lambda)| \leq C_{\alpha\beta} \Big(1 + |\lambda|^2\Big)^{\frac{q-|\beta|}{2}}, \tag{2.1}$$

for $(x, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N$, α, $\beta \in \mathbb{Z}_+^N$. We have $S^q \subset S^l$ for $q \leq l$, and define $S^\infty := \cup_q S^q$.

Given a symbol $b \in S^q$ we say that $b \sim \sum_j b_j$, with $j \in \mathbb{Z}^+$, if $b_j \in S^{q-j}$ and

$$b - \sum_{j<k} b_j \in S^{q-k}.$$

We define below two operations on symbols, the compound, $b\#c$, and the adjoint, $b^\star$.

Theorem 2.1. *Let $b \in S^q$ and $c \in S^l$. Then the oscillatory integrals*

$$b^\star(x, \lambda) := \frac{1}{(2\pi)^N} \int \bar{b}(x - y, \lambda - \eta) e^{-iy.\eta}\, dy\, d\eta,$$

$$b\#c(x, \lambda) := \frac{1}{(2\pi)^N} \int b(x, \lambda - \eta) c(x - y, \lambda) e^{-iy.\eta}\, dy\, d\eta$$

define symbols $b^\star \in S^q$ and $b\#c \in S^{q+l}$ with the following asymptotic expansions

$$b^\star \sim \sum_\alpha \frac{1}{\alpha!} \partial_\lambda^\alpha D_x^\alpha \bar{b}, \qquad b\#c \sim \sum_\alpha \frac{1}{\alpha!} \partial_\lambda^\alpha b D_x^\alpha c.$$

Remark 2.2. *For any $b \in S^\infty$ we have*

$$(b^\star)^\star = b.$$

For $t \in \mathbb{R}$, denote by τ^t the symbol $\tau^t(\lambda) = \Big(1 + |\lambda|^2\Big)^{\frac{t}{2}}$. We then have

i) $(\tau^t)^\star = \tau^t$*;*

ii) $\tau^{t_1} \# \tau^{t_2} = \tau^{t_1+t_2}$

We associate a pseudo-differential operator B (or $b(x,D)$) to the symbol $b(x,\lambda) \in S^q$ via the formula

$$B\varphi(x) := \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} b(x,\lambda)\hat{\varphi}(\lambda)e^{ix.\lambda}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^N).$$

The function $B\varphi \in \mathcal{S}(\mathbb{R}^N)$ and the application is continuous from $\mathcal{S}$ to $\mathcal{S}$ (see Theorem 3.1. in [14]).

The adjoint symbol is associated with the adjoint operator, it is the tool to extend the domain of a pseudodifferential operator to $S^{'}$, and the compound symbol is associated with composition, as the theorem below shows.

Theorem 2.3. *For any b, $c \in S^\infty$ and φ , $\psi \in \mathcal{S}$ one has*

i) $(b^\star(x,D)\varphi,\psi) = (\varphi, b(x,D)\psi)$,

ii) $(b\#c(x,D)\varphi,\psi) = (b(x,D)c(x,D)\varphi,\psi)$.

Remark 2.4. *Given $b = b(x,\lambda) \in S^q$ and $c = c(\lambda) \in S^l$, the symbol correspondent to the composition $b(x,D)c(D)$ is the multiplication of the symbols, i.e.,*

$$b\#c(x,\lambda) = b(x,\lambda)c(\lambda).$$

However, the general case where the symbol c also depend on x, is more complicated. In that case, according to Theorems 2.1 and 2.3, all one can say is that

$$b\#c(x,\lambda) = b(x,\lambda)c(x,\lambda) + \textit{symbol of order } q+l-1$$

The domain of a pseudodifferential can be extended to $\mathcal{S}^{'}$, in the way we show below.

Definition 2.5. *Given a $b \in S^\infty$, we call* pseudodifferential operator of symbol *b, the operator $b(x,D) : \mathcal{S}^{'} \to \mathcal{S}^{'}$ defined by*

$$(b(x,D)u,\varphi) = (u, b^\star(x,D)\varphi), \qquad \textit{for} \quad u \in \mathcal{S}^{'}, \varphi \in \mathcal{S}$$

If $b \in S^q$ then $b(x,D)$ is said to have order q.

In particular we can define the action of a pseudodifferential operator on Sobolev spaces, and the continuity result below holds.

Theorem 2.6. *Let $b \in S^q$. Then for every $s \in \mathbb{R}$ there exists a constant C_s such that $b(x,D)u \in H^{s-q}$ for all $u \in H^s$, with*

$$||b(x,D)u||_{H^{s-q}} \le C_s ||u||_{H^s}.$$

For $p \neq 2$ a similar result holds if we replace the Sobolev spaces by the spaces $L^{s,p}$. In order to prove this we need the following result, due to Coifman and Meyer ([5]).

Theorem 2.7. *Let $b \in S^0$ and $p \in (1, +\infty)$. Then $b(x, D)u \in L^p(\mathbb{R}^N)$ for all $u \in L^p(\mathbb{R}^N)$, and*

$$||b(x, D)\varphi||_{L^p} \le C||\varphi||_{L^p}, \qquad \forall \varphi \in L^p(\mathbb{R}^N),$$

Theorem 2.8. *Let $b \in S^q$. Then for every $s \in \mathbb{R}$ there exists a constant C_s such that $b(x, D)u \in L^{s-q,p}$ for all $u \in L^{s,p}$, with*

$$||b(x, D)u||_{L^{s-q,p}} \le C_s||u||_{L^{s,p}}.$$

Proof. The proof is similar to the proof of Theorem 2.6 presented in [14].

Let $b \in S^q$. We first prove that

$$||b^\star(x, D)\varphi||_{L^{-s,p}} \le C||\varphi||_{L^{q-s,p}}, \qquad \text{for} \quad \varphi \in \mathcal{S}.$$

Note that $\mathcal{S} \subset L^{s,p}$ and

$$||\varphi||_{L^{s,p}} = ||\tau^s(D)\varphi||_{L^p}.$$

We have

$$\begin{aligned} ||b^\star(x, D)\varphi||_{L^{-s,p}} =& ||\tau^{-s}b^\star(x, D)\varphi||_{L^p} \\ &= ||\tau^{-s}(D)b^\star(x, D)\tau^{-q+s}(D)\tau^{q-s}(D)\varphi||_{L^p} \\ &\le C||\tau^{q-s}(D)\varphi||_{L^p} = C||\varphi||_{L^{q-s,p}}. \end{aligned}$$

Let $u \in L^{s,p}$. We now prove that

$$|(b(x, D)u, \varphi)| \le C||u||_{L^{s,p}}||\varphi||_{L^{m-s,p'}}, \qquad \forall \varphi \in \mathcal{S}.$$

Indeed,

$$\begin{aligned} |(b(x, D)u, \varphi)| &= |(u, b^\star(x, D)\varphi)| \\ &= |(u, \tau^s(D)\tau^{-s}(D)b^\star(x, D)\varphi)| \\ &= |(\tau^s(D)u, \tau^{-s}(D)b^\star(x, D)\varphi)| \\ &\le ||u||_{L^{s,p}}||b^\star(x, D)\varphi||_{L^{-s,p'}} \\ &\le C||u||_{L^{s,p}}||\varphi||_{L^{q-s,p'}}, \end{aligned}$$

thus $b(x, D)u \in L^{s-q,p}$ and

$$||b(x, D)u||_{L^{s-q,p}} \le C||u||_{L^{s,p}}.$$

□

In what follows we are interested in pseudodifferential operators associated with matrix-valued symbols. Given a matrix $B(x, \lambda) := [b_{jk}(x, \lambda)]_{j,k=1}^{s,t}$, we say that $B(x, \lambda) \in (S^q)^{s\times t}$ if $b_{jk}(x, \lambda) \in S^q$ for $j = 1, .., s$, $k = 1, .., t$. Given $u \in \mathcal{S}'(\mathbb{R}^N; \mathbb{R}^t)$ we define $Bu \in \mathcal{S}'(\mathbb{R}^N; \mathbb{R}^s)$ by

$$(Bu)_j := \sum_{k=1}^{t} b_{jk}(x, D)u_k, \qquad j = 1, .., s.$$

It is easy to check that all the results we presented above for scalar-valued symbols still hold for matrix-valued symbols.

We now derive some estimates that are useful to prove the necessary condition.

We denote by $A(x,\lambda)$ the symbol associated with the operator $\mathcal{A}$, i.e.

$$A(x,\lambda) := \sum_{i=1}^{N} A^{(i)}(x)\lambda_i,$$

and by $P(x,\lambda)$ the projection onto $\mathrm{Ker}(A(x,\lambda))$. Define $Q(x,\lambda)$ by the implicit equation

$$Q(x,\lambda)A(x,\lambda) := I_m - P(x;\lambda). \tag{2.2}$$

The function $Q(x,\lambda)$ is positively homogeneous of degree -1 in λ and using (1.4) we get that $Q(x,\lambda) \in C^\infty(\Omega \times \mathbb{R}^N \setminus \{0\}; \mathcal{M}^{m\times d})$. Define

$$Q_\eta(x,\lambda) := \eta(x)Q(x,\lambda)\chi(|\lambda|),$$

where $\chi : [0,+\infty) \to \mathbb{R}$ is a C^∞-function for which we can find numbers r, R, $0 < r < R < +\infty$, such that $\chi(|\lambda|) = 0$ for $|\lambda| < r$ and $\chi(|\lambda|) = 1$ for $|\lambda| > R$, and $\eta \in C_c^\infty(\Omega;[0,1])$, $\eta = 1$ on $\tilde{\Omega}$, for some open set $\tilde{\Omega} \subset\subset \Omega$. It is easy to check that

$$|\partial_x^\alpha \partial_\lambda^\beta Q_\eta(x,\lambda)| \le C_{\alpha,\beta}\Big(1+|\lambda|^2\Big)^{\frac{-1-|\beta|}{2}},$$

for $x \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}^N$. Thus $Q_\eta(x,\lambda)$ is a symbol of order -1 and we denote by Q_η the corresponding pseudo-differential operator.

We denote by $A_\eta(x,\lambda)$ the symbol

$$A_\eta(x,\lambda) := \sum_{i=1}^{N} \eta(x)A^{(i)}(x)\lambda_i,$$

and by $\mathcal{A}_\eta$ the corresponding operator.

By Remark 2.4., the compound operator $Q_\eta\mathcal{A}_\eta$ has order 0 and symbol

$$\eta(x)Q(x,\lambda)\chi(|\lambda|)A_\eta(x,\lambda) + \quad \text{symbol of order -1},$$

or, using (2.2),

$$\eta^2(x)I_m - \eta^2(x)P(x,\lambda)\chi(|\lambda|) + \quad \text{symbol of order -1}.$$

We denote by P_η the operator correspondent to the order 0 symbol $\eta^2(x)P(x,\lambda)\chi(|\lambda|)$, thus

$$u - P_\eta u = Q_\eta \mathcal{A}u + Ku,$$

for $u \in L^p(\Omega)$ with compact support in $\tilde{\Omega}$, where K is a pseudo-differential operator of order -1. Using Theorem 2.8, we get the estimates

$$||u - P_\eta u||_{L^p} \le C||\mathcal{A}u||_{W^{-1,p}} + C||u||_{W^{-1,p}}, \tag{2.3}$$

and

$$||\mathcal{A}P_\eta u||_{W^{-1,p}} \leq C||u||_{W^{-1,p}}, \tag{2.4}$$

where we have used the fact that $\mathcal{A}P_\eta$ is an operator of order 0, because of the relation

$$A(x,\lambda)P(x,\lambda) = 0.$$

2.2 Young measures

We present here some results about Young measures, for more details and proofs we refer the reader to [16], [4], [13].

Theorem 2.9. *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let $\{z_n\}$ be a sequence of measurable functions, $z_n : E \to \mathbb{R}^d$. Then there exists a subsequence $\{z_{n_k}\}$ and a weak* measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ such that the following hold:*

i) *$v_x \geq 0$, $||\nu_x||_{\mathcal{M}} \leq 1$ for a.e. $x \in E$;*

ii) *One has i') $||\nu_x||_{\mathcal{M}} = 1$ for a.e. $x \in E$ if and only if*

$$\lim_{M\to+\infty} \sup_k \mathcal{L}^N\left(\{|z_{n_k}| \geq M\}\right) = 0; \tag{2.5}$$

iii) *if $K \subset \mathbb{R}^d$ is a compact subset and $\operatorname{dist}(z_{n_k}, K) \to 0$ in measure then*

$$\operatorname{supp}\nu_x \subset K \quad \textit{for a.e. } x \in E;$$

iv) *if i') holds then in iii) one may replace 'if' by 'if and only if';*

v) *if $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory integrand, bounded from below, then*

$$\liminf_{k\to+\infty} \int_\Omega f(x, z_{n_k}(x))\,dx \geq \int_\Omega \bar{f}(x)\,dx$$

where

$$\bar{f}(x) := \langle \nu_x, f(x,.)\rangle = \int_{\mathbb{R}^d} f(x,y)d\nu_x(y);$$

vi) *if i') holds and if f is as in v), then*

$$\liminf_{k\to+\infty} \int_\Omega f(x, z_{n_k}(x))\,dx = \int_\Omega \bar{f}(x)\,dx < +\infty$$

if and only if $\{f(., z_{n_k}(.))\}$ is equi-integrable. In this case

$$f(., z_{n_k}(.)) \rightharpoonup \bar{f} \quad \textit{in } L^1(\Omega).$$

The map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ is called the Young measure generated by the sequence $\{z_{n_k}\}$. The Young measure ν is said to be homogeneous if there is $\nu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_x = \nu_0$ for a.e. $x \in E$.

Remark 2.10. *Condition (2.5) holds if for some* $p > 0$

$$\sup_{n\in\mathbb{N}} \int_E |z_n|^p \, dx < +\infty$$

Proposition 2.11. *If* $\{v_n\}$ *generates a Young measure* ν *and if* $\omega_n \to \omega$ *in measure then* $\{v_n + \omega_n\}$ *generates the 'translated' Young measure*

$$\tilde{\nu}_x := \Gamma_{\omega(x)} \nu_x$$

where

$$\langle \Gamma_a \mu, \varphi \rangle := \langle \mu, \varphi(. + a) \rangle$$

for $a \in \mathbb{R}^d$, $\varphi \in C_0(\mathbb{R}^d)$. *In particular, if* $\omega_n \to 0$ *in measure then* $\{v_n + \omega_n\}$ *generates the Young measure* ν.

Proposition 2.12. *If* $\{v_n\}$ *generates a Young measure* ν *and* $u_n \to u$ *a.e. in* Ω *then the pair* $\{(u_n, v_n)\}$ *generates the Young measure* μ *defined by*

$$\mu_x := \delta_{u(x)} \otimes \nu_x, \quad \textit{a.e. } x \in \Omega$$

2.3 Operators with constant coefficients

In this subsection we present some results about operators of the form

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i},$$

with

$$\operatorname{rank}\left(\sum_{i=1}^{N} A^{(i)} \omega_i \right) = \text{const},$$

for all $\omega \in \mathbb{R}^N \setminus \{0\}$. For more details and proofs we refer the reader to [7].

We recall that for this kind of operators there exists a continuous projection $\mathbb{T} : L^q(T_N; \mathbb{R}^d) \to L^q(T_N; \mathbb{R}^d)$, where $1 < q < +\infty$ and $L^q(T_N; \mathbb{R}^d)$ denotes the space of functions $v : \mathbb{R}^N \to \mathbb{R}^d$, $Q-$periodic, $v \in L^q(Q)$. The operator $\mathbb{T}$ has the following properties

Lemma 2.13. **i)** $\mathcal{A}(\mathbb{T}v) = 0$;

ii) $||v - \mathbb{T}v||_{L^q} \leq C_q ||\mathcal{A}v||_{W^{-1,q}}$ *for every* $v \in L^q(T_N; \mathbb{R}^d)$ *such that* $\int_Q v \, dx = 0$;

iii) *if* $\{v_n\}$ *is q-equi-integrable then* $\{\mathbb{T}v_n\}$ *is also q-equi-integrable.*

The result below is due to Fonseca and Müller ([7]).

Theorem 2.14. *Let $1 \leq q < +\infty$, and let $\nu = \{\nu\}_{x\in\Omega}$ be a weakly measurable family of probability measures on $\mathbb{R}^d$. There exists a q-equi-integrable sequence $\{v_n\}$ in $L^q(\Omega;\mathbb{R}^d)$ that generates the Young measure ν and satisfies $\mathcal{A}v_n = 0$ in Ω if and only if the following three conditions hold:*

i) *there exists $v \in L^q(\Omega;\mathbb{R}^d)$ such that $\mathcal{A}v = 0$ and*

$$v(x) = \langle \nu_x, Id\rangle \quad a.e.\ x \in \Omega$$

ii)

$$\int_\Omega \langle \nu_x, |z|^q\rangle\, dx$$

iii) *for a.e. $x \in \Omega$ and all continuous functions g that satisfy $|g(v)| \leq C\,(1 + |v|^q)$ for some $C > 0$ and all $v \in \mathbb{R}^d$ one has*

$$\langle \nu_x, g\rangle \geq Q_{\mathcal{A}}g(\langle \nu_x, Id\rangle),$$

where for $v \in \mathbb{R}^d$

$$Q_{\mathcal{A}}g(v) := \inf\left\{\int_Q f(v + \omega(x))\, dx : \omega \in L^q(T_N;\mathbb{R}^d) \cap \mathrm{Ker}\mathcal{A}\right\}.$$

3 The sufficient condition

We now prove Theorem 1.1.

Proof. By extracting a subsequence we can assume

$$L := \liminf \int_\Omega f(x, u_n(x), v_n(x))\, dx = \lim \int_\Omega f(x, u_n(x), v_n(x))\, dx.$$

By extracting another subsequence we can assume that the pair $\{(u_n, v_n)\}$ generates a Young measure $\{\mu_x = \delta_{u(x)} \otimes \nu_x\}_{x\in\Omega}$, where $\{\nu_x\}_{x\in\Omega}$ is the Young measure associated to v_n. We have

$$L \geq \int_\Omega \int_{\mathbb{R}^m\times\mathbb{R}^d} f(x, \eta, \xi)\, d\mu_x(\eta, \xi) = \int_\Omega \int_{\mathbb{R}^d} f(x, u(x), \xi)\, d\nu_x(\xi).$$

Now we truncante the sequence v_n to get $q-$equi-integrability. As v_n is a bounded sequence in L^q we have

$$\int_\Omega \langle \nu_x, |z|^q\rangle\, dx < +\infty.$$

Consider the following family of truncation functions

$$\tau_k(z) := \begin{cases} z & \text{if} \quad |z| \leq k \\ k\frac{z}{|z|} & \text{if} \quad |z| > k, \end{cases}$$

and we have

$$\lim_k \lim_n \int_\Omega |\tau_k(v_n)|^q \, dx = \lim_k \int_\Omega \langle \nu_x, |\tau_k(.)|^q \rangle \, dx = \int_\Omega \langle \nu_x, |z|^q \rangle \, dx.$$

We can then find a sequence $\hat{v}_k := \tau_k(v_{n_k})$ such that

$$||\hat{v}_k - v_{n_k}||_{L^s} \to 0, \qquad \lim_{k\to+\infty} \int_\Omega |\hat{v}_k|^q \, dx = \int_\Omega \langle \nu_x, |z|^q \rangle \, dx,$$

for $1 < s < p$. The sequence $\hat{v}_k$ also generates the Young measure ν, it is $q-$equi-integrable and

$$\mathcal{A}\hat{v}_k \to 0 \quad \text{in} \quad W^{-1,s}.$$

Now choose a point $x_0 \in \Omega$ such that $f(x_0, u(x_0), .)$ is $\mathcal{A}_{x_0}$-quasiconvex,

$$\begin{aligned} &\lim_{r\to 0} \frac{1}{r^N} \int_{Q(x_0,r)} |\langle \nu_x, |z|^q \rangle - \langle \nu_{x_0}, |z|^q \rangle| \, dx = 0, \\ &\lim_{r\to 0} \frac{1}{r^N} \int_{Q(x_0,r)} |v(x) - v(x_0)|^q \, dx = 0, \end{aligned} \tag{3.1}$$

and

$$\lim_{r\to 0} \int_Q |\langle \nu_{x_0+rz}, \varphi \rangle - \langle \nu_{x_0}, \varphi \rangle| \, dz = 0 \tag{3.2}$$

for a countable number of φ in $C_0(\mathbb{R}^d)$. Define $w_{k,r} \in L^q(Q; \mathbb{R}^d)$ by $w_{k,r}(z) := \hat{v}_k(x_0 + rz)$. Using (3.1) and (3.2), we have

$$\lim_{r\to 0} \lim_{k\to+\infty} \int_Q |\hat{v}_k(x_0 + rz)|^q \, dz = \langle \nu_{x_0}, |z|^q \rangle,$$

$$\sum_{i=1}^N A^{(i)}(x_0 + rz) \frac{\partial(\hat{v}_k(x_0 + rz))}{\partial z_i} \to 0 \quad \text{in} \quad W^{-1,s} \quad \text{as } k \to +\infty$$

$$\lim_{r\to 0} \lim_{k\to+\infty} \int_Q (\hat{v}_k(x_0 + rz) - v(x_0)) \, \Psi(z) \, dz = 0,$$

for every $\Psi \in L^{q'}$,

$$\lim_{r\to 0} \lim_{k\to+\infty} \int_Q \zeta(z)\varphi\left(\hat{v}_k(x_0 + rz)\right) \, dz = \langle \nu_{x_0}, \varphi \rangle \int_Q \zeta(z) \, dz,$$

for ζ in $C_c(Q)$ and φ in the countable subset of $C_0(\mathbb{R}^d)$ for which (3.2) holds.

Then using an appropriate diagonalization can find a sequence $\omega_k \in L^q(Q; \mathbb{R}^d)$ such that

$$\omega_k \rightharpoonup v(x_0) \quad \text{in} \quad L^q, \qquad \sum_{i=1}^N A^{(i)}(x_0 + r_k z) \frac{\partial \omega_k(z)}{\partial z_i} \to 0 \quad \text{in} \quad W^{-1,s} \tag{3.3}$$

and
$$\lim_{k\to+\infty}\int_Q \eta(z)\varphi(\omega_k)\,dz = \langle \nu_{x_0},\varphi\rangle \int_Q \eta(z)\,dz,$$
for η and φ in a countable dense subset of $L^1(Q)$ and $C_0(\mathbb{R}^d)$, respectively, and
$$\lim_{k\to+\infty}\int_Q |\omega_k(z)|^q\,dz = \langle \nu_{x_0}, |z|^q\rangle,$$
thus ω_k generates the Young measure ν_{x_0} and it is q-equi-integrable.

Now we prove that
$$\mathcal{A}_{x_0}\omega_k = \sum_{i=1}^N A^{(i)}(x_0)\frac{\partial \omega_k}{\partial z_i} \to 0 \quad \text{in} \quad W^{-1,s}. \tag{3.4}$$
In fact we have
$$\begin{aligned}\mathcal{A}_{x_0}\omega_k =& \sum_{i=1}^N \frac{\partial}{\partial z_i}\left[\left(A^{(i)}(x_0) - A^{(i)}(x_0 + r_k z)\right)\omega_k(z)\right]\\ &+ r_k \sum_{i=1}^N \frac{\partial A^{(i)}}{\partial x_i}(x_0 + r_k z)\omega_k(z) + \sum_{i=1}^N A^{(i)}(x_0+r_k z)\frac{\partial \omega_k}{\partial z_i}\end{aligned}$$
and all the terms go to 0 in $W^{-1,s}$. The first because of the s-equi-integrability of ω_k and the continuity of the coefficients which imply
$$\left(A^{(i)}(x_0) - A^{(i)}(x_0+r_k z)\right)\omega_k(z) \to 0 \quad \text{in} \quad L^s,$$
the second because $r_k \to 0$ and the boundedness of ω_k in L^s, and the third because of (3.3).

Next we modify ω_k in order to get Q-periodicity. We consider an increasing sequence of smooth cut-off functions $\varphi^j \in C_c^\infty(Q)$, $\varphi^j \nearrow 1$ and we do a appropriate diagonalization of $\varphi^j\omega_k$, in order to get a new sequence $\tilde{\omega}_k \in L^q(Q)$, q-equi-integrable, that still generates the homogeneous Young measure ν_{x_0}, and verifies
$$\mathcal{A}_{x_0}\tilde{\omega}_k \to 0 \quad \text{in} \quad W^{-1,s}.$$
Now we just have to project $\{\tilde{\omega}_k\}$ into the kernel of $\mathcal{A}_{x_0}$, i.e., we apply Lemma 2.13. to get
$$\hat{\omega}_k := \mathbb{T}\left[\tilde{\omega}_k - v(x_0) - \int_Q (\tilde{\omega}_k(x) - v(x_0))\,dx\right] + v(x_0),$$
Q-periodic, q-equi-integrable, $\hat{\omega}_k \rightharpoonup v(x_0)$, $\int_Q \hat{\omega}_k(y)\,dy = v(x_0)$, $\hat{\omega}_k$ still generates ν_{x_0} and $\mathcal{A}_{x_0}\hat{\omega}_k = 0$. Thus
$$\int_{\mathbb{R}^d} f(x_0,u(x_0),\xi)d\nu_{x_0}(\xi) = \lim_k \int_Q f(x_0,u(x_0),\hat{\omega}_k(y))\,dy \geq f(x_0,u(x_0),v(x_0)),$$

from which we get

$$L \geq \int_\Omega f(x, u(x), v(x))\, dx.$$

$\square$

Remark 3.1. *Using a similar prove one can obtain the same result of Theorem 1.1. for systems in the divergence form with L^∞ coefficients*

$$\mathcal{A}v := \sum_{i=1}^N \frac{\partial\left(A^{(i)}(x)v(x)\right)}{\partial x_i},$$

and rank$\left(\sum_{i=1}^N A^{(i)}(x)\omega_i\right)$ =const, *for a.e. $x \in \Omega$ and all $\omega \in \mathbb{R}^N \setminus \{0\}$, i.e., if $f(x, u, .)$ is $\mathcal{A}_x$-quasiconvex for a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$ then we have lower semicontinuity for sequences $u_n \to u$ in measure, $v_n \rightharpoonup v$ in L^q, $\mathcal{A}v_n \to 0$ in $W^{-1,q}$. In the prove one uses the approximate continuity of the coefficients at a.e. $x \in \Omega$.*

However, in this case, we were unable to prove that the sufficient condition is also necessary.

4 The necessary condition

In this section we prove Theorem 1.2

Proof. Fix x_0 in Ω, $c \in \mathbb{R}^d$, and let $r > 0$ be such that $Q(x_0, 2r) \subset\subset \Omega$. Let $\omega \in C^\infty(\mathbb{R}^N; \mathbb{R}^m)$, Q-periodic, satisfying

$$\int_Q \omega(y)\, dy = 0 \qquad \mathcal{A}_{x_0}\omega := \sum_{i=1}^N A^{(i)}(x_0)\frac{\partial \omega}{\partial y_i} = 0. \tag{4.1}$$

Using the uniform continuity of f on compact sets we can choose n large enough such that

$$|f(x, v) - f(x', v)| < \varepsilon \quad \text{for } x, x' \in \overline{Q(x_0, r)}, \quad v \in \overline{Q(0, c + ||\omega||_\infty)}, |x - x'| < \frac{1}{n}.$$

Decompose

$$Q(x_0, r) = \cup_{j=1}^{n^N} Q(x_j, \frac{r}{n}),$$

where the equality above is up to a $\mathcal{L}^N$-negligible set. Consider $\varphi \in C_c^\infty(Q(x_0, r), [0, 1])$ such that $\mathcal{L}^N(Q(x_0, r) \cap \{\varphi \neq 1\}) < \varepsilon r^N$. Define

$$u_m(x) := \begin{cases} \varphi(x)\omega^\star\left(\frac{mn(x - x_j)}{r}\right) & \text{if } x \in Q(x_j, \frac{r}{n}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega^\star(y) := \omega(y + (\frac{1}{2}, .., \frac{1}{2}))$. We have

$$
\begin{aligned}
\mathcal{A}u_m &= \mathcal{A}u_m - \mathcal{A}_{x_0}u_m + \mathcal{A}_{x_0}u_m \\
&= \sum_{i=1}^{N}\sum_{j=1}^{n^N} \frac{\partial}{\partial x_i}\left(\left(A^{(i)}(x) - A^{(i)}(x_0)\right)\varphi(x)\omega^\star\left(mn\frac{x-x_j}{r}\right)\right)\chi_{Q(x_j,\frac{r}{n})} \\
&\quad - \sum_{i=1}^{N}\sum_{j=1}^{n^N} \varphi(x)\frac{\partial\left(A^{(i)}(x) - A^{(i)}(x_0)\right)}{\partial x_i}\omega^\star\left(mn\frac{x-x_j}{r}\right)\chi_{Q(x_j,\frac{r}{n})} \\
&\quad + \sum_{i=1}^{N}\sum_{j=1}^{n^N} A^{(i)}(x_0)\frac{\partial\varphi}{\partial x_i}\omega^\star\left(mn\frac{x-x_j}{r}\right)\chi_{Q(x_j,\frac{r}{n})} \\
&\quad + \varphi(x)\sum_{i=1}^{N}\sum_{j=1}^{n^N} A^{(i)}(x_0)\frac{\partial\left(\omega^\star\left(mn\frac{x-x_j}{r}\right)\right)}{\partial x_i}\chi_{Q(x_j,\frac{r}{n})} \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{4.2}
$$

As

$$
\omega^\star\left(mn\frac{x-x_j}{r}\right) \rightharpoonup 0 \quad \text{in} \quad L^q\left(Q(x_j, \frac{r}{n})\right) \qquad \text{as } m \to +\infty,
$$

we have $I_2, I_3 \to 0$ in $W^{-1,q}$ as $m \to +\infty$, and by (4.1) $I_4 = 0$. Moreover

$$
\begin{aligned}
||I_1||_{W^{-1,q}} &\leq \sum_{i=1}^{N}\sum_{j=1}^{n^N}\left|\left|\left(A^{(i)}(x) - A^{(i)}(x_0)\right)\omega^\star\left(mn\frac{x-x_j}{r}\right)\varphi(x)\right|\right|_{L^q\left(Q(x_j,\frac{r}{n})\right)} \\
&\leq C\sum_{i=1}^{N}\left(\int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q\,dx\right)^{\frac{1}{q}},
\end{aligned}
\tag{4.3}
$$

where C is independent of m.

Now consider $\eta \in C_c^\infty(\Omega; [0,1])$, $\eta = 1$ on $Q(x_0, r)$ and define

$$
v_m := P_\eta u_m.
$$

As P_η is an operator of order 0, by Theorem 2.8, we have

$$
||v_m||_{L^q} \leq C||u_m||_{L^q}, \tag{4.4}
$$

$$
||v_m||_{W^{-1,q}} \leq C||u_m||_{W^{-1,q}},
$$

thus, up to a subsequence,

$$
v_m \rightharpoonup 0 \quad \text{in} \quad L^q.
$$

Moreover, by (2.4),

$$
\mathcal{A}v_m \to 0 \quad \text{in} \quad W^{-1,q}.
$$

As the pseudodifferential operators are non-local, we need to localize the sequence $\{v_m\}$. For that consider $\eta_r \in C_c^\infty(Q(x_0,2r);[0,1])$, $\eta_r = 1$ in $Q(x_0,r)$, and define

$$\tilde{v}_m := \eta_r v_m.$$

We have

$$\tilde{v}_m \rightharpoonup 0 \quad \text{in} \quad L^q, \qquad \mathcal{A}\tilde{v}_m \to 0 \quad \text{in} \quad W^{-1,q},$$

thus, by the lower semicontinuity we have

$$\liminf \int_\Omega f(x, c+\tilde{v}_m(x))\,dx \geq \int_\Omega f(x,c)\,dx. \tag{4.5}$$

On the other hand, using (1.2), (2.3), (4.3), (4.4), and Hölder's inequality, we get

$$\begin{aligned}
&\left|\int_\Omega f(x,c+\tilde{v}_m(x))\,dx - \int_\Omega f(x,c+u_m(x))\right| \\
&\leq C\int_\Omega |\tilde{v}_m(x)-u_m(x)|\left(1+|c+\tilde{v}_m|^{q-1}+|c+u_m|^{q-1}\right)\,dx \\
&\leq C\int_{Q(x_0,2r)} |\tilde{v}_m(x)-u_m(x)|\left(1+|\tilde{v}_m|^{q-1}+|u_m|^{q-1}\right)\,dx \\
&\leq C\left(\int_{Q(x_0,2r)} |\tilde{v}_m(x)-u_m(x)|^q\,dx\right)^{\frac{1}{q}}\left(r^{\frac{N}{q'}}+\left(\int_{Q(x_0,2r)}|\tilde{v}_m|^q\,dx\right)^{\frac{1}{q'}}\right. \\
&\qquad\qquad\left.+\left(\int_{Q(x_0,2r)}|u_m|^q\,dx\right)^{\frac{1}{q'}}\right) \\
&\leq C\left(\int_\Omega |v_m(x)-u_m(x)|^q\,dx\right)^{\frac{1}{q}}\left(r^{\frac{N}{q'}}+\left(\int_{Q(x_0,r)}|u_m|^q\,dx\right)^{\frac{1}{q'}}\right) \\
&\leq C\left(||\mathcal{A}u_m||_{W^{-1,q}}+||u_m||_{W^{-1,q}}\right)\left(r^{\frac{N}{q'}}+r^{\frac{N}{q'}}\left(\int_Q|\omega(mz)|^q\right)^{\frac{1}{q'}}dz\right) \\
&\leq Cr^{\frac{N}{q'}}\left(\sum_{i=1}^N\int_{Q(x_0,r)}\left|A^{(i)}(x)-A^{(i)}(x_0)\right|^q\,dx\right)^{\frac{1}{q}}+Cr^{\frac{N}{q'}}||u_m||_{W^{-1,q}},
\end{aligned} \tag{4.6}$$

where C is independent of m. Thus using (4.5) and (4.6) we have

$$\begin{aligned}
&\limsup_m \int_\Omega f(x,c+u_m(x))\,dx + Cr^{\frac{N}{q'}}\left(\sum_{i=1}^N\int_{Q(x_0,r)}\left|A^{(i)}(x)-A^{(i)}(x_0)\right|^q\,dx\right)^{\frac{1}{q}} \\
&\qquad\geq \int_\Omega f(x,c)\,dx.
\end{aligned}$$

or, equivalently,

$$\limsup_m \int_{Q(x_0,r)} f(x, c + u_m(x))\, dx + Cr^{\frac{N}{q'}} \left(\sum_{i=1}^N \int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q \, dx \right)^{\frac{1}{q}}$$
$$\geq \int_{Q(x_0,r)} f(x, c)\, dx.$$

We now estimate the first term above, using the continuity of f and Riemann-Lebesgue lemma,

$$\begin{aligned}
&\limsup_m \int_{Q(x_0,r)} f(x, c + u_m(x))\, dx \\
&\quad \leq \limsup_m \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} f\left(x, c + \omega^\star(mn\frac{x - x_j}{r})\right) dx + 2M\varepsilon r^n \\
&\quad \leq \limsup_m \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} f\left(x_j, c + \omega^\star(mn\frac{x - x_j}{r})\right) dx + (2M+1)\varepsilon r^N \\
&\quad \leq \limsup_m \sum_{j=1}^{n^N} \frac{r^N}{n^N} \int_Q f(x_j, c + \omega(my))\, dy + (2M+1)\varepsilon r^N \\
&\quad \leq \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} \left(\int_Q f(x_j, c + \omega(y))\, dy \right) dx + (2M+1)\varepsilon r^N \\
&\quad \leq \sum_{j=1}^{n^N} \int_{Q(x_j,\frac{r}{n})} \left(\int_Q f(x, c + \omega(y))\, dy \right) dx + (2M+2)\varepsilon r^N \\
&\quad \leq \int_{Q(x_0,r)} \left(\int_Q f(x, c + \omega(y))\, dy \right) dx + O(\varepsilon) r^N,
\end{aligned} \tag{4.7}$$

where $M := \sup\{f(x, v) : x \in \overline{Q(x_0, r)}, |v| \leq c + ||\omega||_\infty\}$. Thus dividing by r^N and using (4.7) we get

$$\frac{1}{r^N} \int_{Q(x_0,r)} \left(\int_Q f(x, c + \omega(y))\, dy \right) dx + O(\varepsilon)$$
$$+ C\left(\sum_{i=1}^N \frac{1}{r^N} \int_{Q(x_0,r)} |A^{(i)}(x) - A^{(i)}(x_0)|^q \, dx \right)^{\frac{1}{q}} \geq \frac{1}{r^N} \int_{Q(x_0,r)} f(x, c)\, dx.$$

By letting $r \to 0$ and using the arbitrariness of ε we get

$$f(x_0, c) \leq \int_Q f(x_0, c + \omega(y))\, dy,$$

i.e., $f(x_0, .)$ is $\mathcal{A}_{x_0}$-quasiconvex. □

5 Characterization of Young measures

We now prove Theorem 1.3. The idea is to split the domain in small cubes, approach the variable coefficients operator by one with constant coefficients in each cube, apply in each cube the theorem about characterization of Young measures generated by bounded sequences in L^q that are in the kernell of an operator with constant coefficients (Theorem 2.14), and then use an appropriate diagonalization.

Proof. We assume without loss of generality that

$$\langle \nu_x, Id \rangle = 0.$$

Otherwise we work with translated measure.

Consider $\{\xi_h\}_{h=1}^{+\infty}$ a dense countable subset of $L^1(\Omega)$, $\xi_0(x) = 1$, $\{\varphi_l\}_{l=1}^{+\infty}$ a dense countable subset of $C_0(\mathbb{R}^d)$ and $\varphi_0(z) = |z|^q$. Given $\alpha \in \mathbb{N}$ we can find $\gamma > 0$ such that

$$\int_B |\xi_h(x)|\, dx ||\varphi_l||_\infty < \frac{1}{\alpha} \quad \text{for } h, l = 1, ..., \alpha, \tag{5.1}$$

and

$$\int_B \langle \nu_x, |z|^q \rangle\, dx < \frac{1}{\alpha} \tag{5.2}$$

when $\mathcal{L}^N(B) < \gamma$.

For each $\alpha \in \mathbb{N}$ we consider a compact set K_α such that $\mathcal{L}^N(\Omega \setminus K_\alpha) < \min\{\frac{1}{\alpha^2}, \gamma/3\}$ and the functions

$$x \to \langle \nu_x, |z|^q \rangle, \qquad x \to \langle \nu_x, \varphi_l \rangle \quad l = 1, ..., \alpha,$$

are continuous in K_α. We consider disjoint cubes $Q_i \subset\subset \Omega$ of side $\frac{1}{m_\alpha}$, for an appropriate integer m_α, such that $\mathcal{L}^N(\Omega \setminus \cup Q_i) < \min\{\frac{1}{\alpha^2}, \gamma/3\}$ and

$$\sup_{x,x' \in Q_i \cap K_\alpha} |A^{(j)}(x) - A^{(j)}(x')|^q < \frac{1/\alpha}{NC_1} \quad j = 1, .., N, \tag{5.3}$$

$$\sup_{x,x' \in Q_i \cap K_\alpha} |\langle \nu_x, \varphi_l \rangle - \langle \nu_{x'}, \varphi_l \rangle| < \frac{1/\alpha}{||\xi_h||} \quad h, l = 1, ..., \alpha, \tag{5.4}$$

and

$$\sup_{x,x' \in Q_i \cap K_\alpha} |\langle \nu_x, |z|^q \rangle - \langle \nu_{x'}, |z|^q \rangle| < \frac{1/\alpha}{|\Omega|}, \tag{5.5}$$

where $C_1 := 2 \int_\Omega \langle \nu_y, |z|^p \rangle\, dy + 1$. By considering less cubes and a smaller compact set $\hat{K}_\alpha$, if necessary, we may assume that for each cube Q_i we have

$$\mathcal{L}^N(Q_i \cap \hat{K}_\alpha) \geq \frac{\mathcal{L}^N(Q_i)}{2}, \tag{5.6}$$

and $\hat{K}_\alpha \subset \overline{\cup Q_i}$. It is easy to check that

$$\mathcal{L}^N(\Omega \setminus \hat{K}_\alpha) < \min\{3/\alpha^2, \gamma\}, \qquad \mathcal{L}^N(\Omega \setminus \cup Q_i) < \min\{3/\alpha^2, \gamma\}.$$

In each cube Q_i we pick up a point $x_i \in \hat{K}_\alpha \cap Q_i$ that fulfills the conditions below

$$\langle \nu_{x_i}, |z|^q \rangle \leq \frac{1}{\mathcal{L}^N(\hat{K}_\alpha \cap Q_i)} \int_{\hat{K}_\alpha \cap Q_i} \langle \nu_y, |z|^q \rangle \, dy, \qquad \langle \nu_{x_i}, \mathrm{Id} \rangle = 0,$$

$$\langle \nu_{x_i}, g \rangle \geq Q_{\mathcal{A}_{x_i}} g(0), \tag{5.7}$$

for every continuous g satisfying $|g(v)| \leq C(1 + |v|^q)$. Now we apply Theorem 2.14 and get a $q-$equi-integrable sequence $\hat{v}^i_{\alpha,n} \in L^p(Q_i; \mathbb{R}^d)$ that generates the homogeneous Young measure ν_{x_i} and satisfies $\mathcal{A}_{x_i} v^i_{\alpha,n} = 0$. Using an appropriate sequence of cut-off functions, $\eta^s \in C^\infty_c(Q_i)$, $\eta^s \nearrow 1$, and diagonalizing $\eta^s \hat{v}^i_{\alpha,n}$, one can construct a new sequence, $v^i_{\alpha,n}$, such that $v^i_{\alpha,n} = 0$ on a neighborhood of ∂Q_i, q-equi-integrable, also generating ν_{x_i} and

$$\mathcal{A}_{x_i} v^i_{\alpha,n} \to 0 \quad \text{in} \quad W^{-1,q}(Q_i; \mathbb{R}^l) \quad \text{as} \quad n \to +\infty.$$

Define

$$v_{\alpha,n} := \begin{cases} v^i_{\alpha,n} & \text{if } x \in Q_i, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\mathcal{A}_\alpha v_{\alpha,n} := \sum_i \left(\sum_{j=1}^N \frac{\partial \left(A^{(j)}(x_i) v^i_{\alpha,n} \right)}{\partial x_j} \right) \to 0 \quad \text{in} \quad W^{-1,q}$$

We claim that

$$\int_\Omega |v_{\alpha,n}|^q \, dx \leq C_1 \tag{5.8}$$

for all $\alpha \in \mathbb{N}$ and n large enough. As $\left(v^i_{\alpha,n}\right)_n$ generates ν_{x_i} and it is $q-$equi-integrable, we know that

$$\int_{Q_i} |v^i_{\alpha,n}|^q \, dx \to \langle \nu_{x_i}, |z|^q \rangle \mathcal{L}^N(Q_i).$$

By (5.6) and (5.7),

$$\begin{aligned} \langle \nu_{x_i}, |z|^q \rangle \mathcal{L}^N(Q_i) &\leq \frac{\mathcal{L}^N(Q_i)}{\mathcal{L}^N(Q_i \cap \hat{K}_\alpha)} \int_{Q_i \cap \hat{K}_\alpha} \langle \nu_y, |z|^q \rangle \, dy \\ &\leq 2 \int_{Q_i} \langle \nu_y, |z|^q \rangle \, dy, \end{aligned}$$

and for n large enough

$$\int_\Omega |v_{\alpha,n}|^q \, dx \leq 2 \int_\Omega \langle \nu_y, |z|^q \rangle \, dy + 1 = C_1.$$

We claim that

$$\sum_i \int_{Q_i \setminus \hat{K}_\alpha} |v^i_{\alpha,n}|^q \, dx \leq F(\alpha), \tag{5.9}$$

for some F satisfying the condition $F(\alpha) \to 0$ as $\alpha \to 0$ and n large enough. Using the $q-$equi-integrability of $\left(v^i_{\alpha,n}\right)_n$ and (5.7) we have

$$\begin{aligned}\int_{Q_i \setminus \hat{K}_\alpha} |v^i_{\alpha,n}|^q \, dx \to \quad & \langle \nu_{x_i}, |z|^q \rangle \mathcal{L}^N(Q_i \setminus \hat{K}_\alpha) \\ & \leq \frac{\mathcal{L}^N(Q_i \setminus \hat{K}_\alpha)}{\mathcal{L}^N(Q_i \cap \hat{K}_\alpha)} \int_{Q_i \cap \hat{K}_\alpha} \langle \nu_y, |z|^q \rangle \, dy\end{aligned}$$

Let $J_\alpha := \{ i : \alpha \mathcal{L}^N(Q_i \setminus \hat{K}_\alpha) > \mathcal{L}^N(Q_i \cap \hat{K}_\alpha)\}$, we then have

$$\sum_{i \in J_\alpha} \mathcal{L}^N(Q_i \cap \hat{K}_\alpha) \leq \sum_{i \in J} \alpha \mathcal{L}^N(Q_i \setminus \hat{K}_\alpha) \leq \alpha \mathcal{L}^N(\Omega \setminus \hat{K}_\alpha) < \frac{1}{\alpha}.$$

Thus

$$\begin{aligned}\sum_i \int_{Q_i \setminus \hat{K}_\alpha} |v^i_{\alpha,n}|^q \, dx &\leq \sum_i \frac{\mathcal{L}^N(Q_i \setminus \hat{K}_\alpha)}{\mathcal{L}^N(Q_i \cap \hat{K}_\alpha)} \int_{Q_i \cap \hat{K}_\alpha} \langle \nu_y, |z|^q \rangle \, dy + \frac{1}{\alpha} \\ &\leq \sum_{i \in J_\alpha} \int_{Q_i \cap \hat{K}_\alpha} \langle \nu_y, |z|^q \rangle \, dy + \frac{1}{\alpha} \int_\Omega \langle \nu_y, |z|^q \rangle \, dy + \frac{1}{\alpha},\end{aligned}$$

for n large enough, from which we get (5.8).

As

$$\begin{aligned}\mathcal{A}_\alpha v_{\alpha,n} - \mathcal{A} v_{\alpha,n} &= \sum_i \left(\sum_{j=1}^N \left(A^{(j)}(x_i) - A^{(j)}(x) \right) \frac{\partial v^i_{\alpha,n}}{\partial x_j} \right) \\ &= \sum_i \left(\sum_{j=1}^N \frac{\partial \left(\left(A^{(j)}(x_i) - A^{(j)}(x) \right) v^i_{\alpha,n} \right)}{\partial x_j} + \sum_{j=1}^N \frac{\partial A^{(j)}(x)}{\partial x_j} v^i_{\alpha,n} \right)\end{aligned}$$

and

$$\begin{aligned}\sum_i \sum_j \int_{Q_i \cap \hat{K}_\alpha} \left| A^j(x) - A^j(x_i) \right|^q |v^i_{\alpha,n}|^q \, dx &\leq \frac{\frac{1}{\alpha}}{C_1} \int_\Omega |v_{\alpha,n}|^q \, dx < \frac{1}{\alpha} \\ \sum_i \sum_j \int_{Q_i \setminus \hat{K}_\alpha} \left| A^j(x) - A^j(x_i) \right|^q |v^i_{\alpha,n}|^q \, dx &\leq 2^q ||A||^q_\infty N F(\alpha),\end{aligned}$$

we conclude that for n large enough

$$||\mathcal{A}_\alpha v_{\alpha,n} - \mathcal{A} v_{\alpha,n}||_{-1,q} \leq 2 ||A||^q_\infty N F(\alpha) + \frac{2}{\alpha}. \tag{5.10}$$

We now prove that for n large enough

$$\left|\int_\Omega \xi_h(x)\varphi_l(v_{\alpha,n})\,dx - \int_\Omega \xi_h(x)\langle\nu_x,\varphi_l\rangle\,dx\right| \le \frac{6}{\alpha} \quad \text{for } h,l=1,..,\alpha \tag{5.11}$$

Indeed, as $n\to+\infty$,

$$\int_\Omega \xi_h(x)\varphi_l(v_{\alpha,n})\,dx \to \sum_i \langle\nu_{x_i},\varphi_l\rangle \int_{Q_i}\xi_h(x)\,dx + \varphi_l(0)\int_{\Omega\setminus\cup Q_i}\xi_h(x)\,dx,$$

and

$$\begin{aligned}
&\left|\int_\Omega \xi_h(x)\langle\nu_x,\varphi_l\rangle\,dx - \sum_i\langle\nu_{x_i},\varphi_l\rangle\int_{Q_i}\xi_h(x)\,dx - \varphi_l(0)\int_{\Omega\setminus\cup Q_i}\xi_h(x)\,dx\right| \\
&\quad\le \int_{\Omega\setminus\cup Q_i}|\xi_h(x)\langle\nu_x,\varphi_l\rangle|\,dx + \sum_i\int_{Q_i\cap\hat K_\alpha}|\xi_h(x)\left(\langle\nu_x,\varphi_l\rangle - \langle\nu_{x_i},\varphi_l\rangle\right)|\,dx \\
&\quad+ \sum_i\int_{Q_i\setminus\hat K_\alpha}|\xi_h(x)\langle\nu_{x_i},\varphi_l\rangle|\,dx + \sum_i\int_{Q_i\setminus\hat K_\alpha}|\xi_h(x)\langle\nu_x,\varphi_l\rangle|\,dx \\
&\quad+ |\varphi_l(0)|\int_{\Omega\setminus\cup Q_i}|\xi_h(x)|\,dx,
\end{aligned}$$

using (5.1) and (5.4) we get (5.11). A similar argument can be done in order to obtain

$$\left|\int_\Omega |v_{\alpha,n}(x)|^q\,dx - \int_\Omega\langle\nu_x,|z|^q\rangle\,dx\right| \le F(\alpha)+\frac{4}{\alpha},$$

for n large enough. Then, using appropriate diagonalization, we can find a sequence $w_\alpha := v_{\alpha,n_\alpha}\in L^q$, $w_\alpha \rightharpoonup 0$ in L^q, $\mathcal{A}w_\alpha\to 0$ in $W^{-1,q}$, verifying

$$\lim_\alpha \int_\Omega \xi_h(x)\varphi_l(w_\alpha(x))\,dx = \int_\Omega \xi_h(x)\langle\nu_x,\varphi_l\rangle\,dx,$$

for all $h,l\in\mathbb{N}$, and

$$\lim_\alpha\int_\Omega |w_\alpha(x)|^q\,dx = \int_\Omega\langle\nu_x,|z|^q\rangle\,dx,$$

thus w_k generates the Young measure ν and it is q-equi-integrable.

For the necessary condition we can use an argument similar to the proof of Theorem 1.1, in order to get that at a.e. $x_0\in\Omega$, the homogeneuos Young measure ν_{x_0} is generated by a sequence $\omega_k\in L^q(Q)$, Q-periodic, q-equi-integrable, $\omega_k\rightharpoonup v(x_0)$ in L^q, $\int_Q\omega_k(x)\,dx = v(x_0)$, $\mathcal{A}_{x_0}\omega_k=0$. Then

$$\lim_k\int_Q g(\omega_k(x))\,dx = \langle\nu_{x_0},g\rangle \ge Q_{\mathcal{A}_{x_0}}(v(x_0)),$$

which proves iii); i) and ii) are trivial. □

Remark 5.1. *Using a similar prove one can obtain the same result of Theorem 1.3. for systems written in the divergence form with L^∞ coefficients*

$$\mathcal{A}v := \sum_{i=1}^{N} \frac{\partial\left(A^{(i)}(x)v(x)\right)}{\partial x_i}$$

and $\operatorname{rank}\left(\sum_{i=1}^{N} A^{(i)}(x)\omega_i\right) = \text{const}$, *for a.e.* $x \in \Omega$ *and all* $\omega \in \mathbb{R}^N$. *In the proof we can use Lusin's Theorem to get continuity of the coefficients out of a set of small measure.*

Acknowledgments

I would like to thank I. Fonseca for proposing this problem and for some helpful conversations. I also would like to thank S. Müller, whose initial involvement in this just in collaboration with I. Fonseca, led the way to the present work. My research was partially funded by Fundação para a Ciência e para a Tecnologia (Grant PRAXIS XXI BD/15642/98). I also would like to thank the Center for Nonlinear Analysis (NSF Grant No. DMS-9803791), Carnegie Mellon University, Pittsburgh and the Departamento de Matemática, Instituto Superior Técnico, Lisbon, for their support during the preparation of this paper.

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