Bending Moment in Membrane Theory

Guy Bouchitté
Département de Mathématiques, Université de Toulon et du Var-BP 132
83957 La Garde Cedex, France
bouchitte@univ-tln.fr

Irene Fonseca
Department of Mathematical Sciences, Carnegie Mellon University
Pittsburgh, PA 15213, USA
fonseca@andrew.cmu.edu

M.Luísa Mascarenhas
C.M.A.F., Universidade de Lisboa, Av. Prof. Gama Pinto 2
1649-003 Lisboa Codex, Portugal
mascar@ptmat.fc.ul.pt

Abstract. A model of nonlinear membrane is proposed where the external loading induces a density of bending moments. The mathematical justification of this model is obtained through a 3D-2D dimension reduction via a Γ-convergence analysis taking into account the Cosserat vector field.

AMS classification numbers : 35E99, 35M10, 49J45, 74B20, 74K15, 74K20, 74K35

Key words : Dimension reduction, Γ-convergence, thin films, relaxation, quasiconvexity.

1. Introduction and main results.

In this paper we deduce, via Γ-convergence, a 2-D membrane model from a 3-D nonlinear elasticity framework where we consider a class of surface forces generating the *bending moment*.

Let ω be an open bounded subset of $\mathbb{R}^2$ and let I be the interval $(-1/2, 1/2)$. Define $\Omega := \omega \times I$, $\Sigma^{\pm} := \omega \times \{\pm 1/2\}$, $\Gamma := \partial\omega \times I$ and, for each $\varepsilon > 0$, $\Omega_\varepsilon := \omega \times \varepsilon I$, $\Sigma_\varepsilon^{\pm} := \omega \times \{\pm\varepsilon\}$, $\Gamma_\varepsilon := \partial\omega \times \varepsilon I$.

In what follows $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure in $\mathbb{R}^N$, $N = 2, 3$, and $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. Greek indexes will be used to distinguish the first two components of a tensor, for instance (x_α) and (x_α, x_3), designates (x_1, x_2) and (x_1, x_2, x_3), respectively.

We write $M^{3\times 2}$ to denote the vector space of 3×2 real-valued matrices, and for $\overline{F} \in M^{3\times 2}$ and $b \in \mathbb{R}^3$, let $(\overline{F}|b)$ denote the 3×3 matrix whose first two columns are those of $\overline{F}$ and the last one is b.

As it is usual in the study of thin structures (e.g. [BB],[BF],[LDR1],[LDR2], [FF], [BJ], [BFF], [BrFF] ; see also [ABP],[FMJ]), the rescaled total energy of a deformation U of Ω_ε is given by

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\, d\tilde{x} - < F_\varepsilon\, U >$$

where $DU = (D_\alpha U | D_3 U)$ is the strain of a deformation $U \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$, W satisfies some suitable growth hypotheses (see (H1)) and F_ε stands for the external loading. Here we take F_ε as the sum of a bulk density and a surface density on $\Sigma_\varepsilon^{\pm}$, precisely,

$$F_\varepsilon := f_\varepsilon\ \mathcal{L}^3 \lfloor \Omega_\varepsilon + (g_\varepsilon^+\ \mathcal{H}^2 \lfloor \Sigma_\varepsilon^+ - g_\varepsilon^-\ \mathcal{H}^2 \lfloor \Sigma_\varepsilon^-), \tag{1.1}$$

where

$$f_\varepsilon := \frac{1}{\varepsilon} f\left(\tilde{x}_\alpha, \frac{\tilde{x}_3}{\varepsilon}\right), \qquad f \in L^{p'}(\Omega; \mathbb{R}^3)$$

and

$$g_\varepsilon^{\pm} := g_0^{\pm} + \frac{1}{\varepsilon} g\ , \qquad g_0^{\pm}, g \in L^{p'}(\omega; \mathbb{R}^3),$$

where $p' = p/(p-1)$, for a fixed $p \in (1, +\infty)$.

In the classical approach for recovering the membrane energy from a 3D model, the surface load term scaled with factor $\frac{1}{\varepsilon}$ is not present so that F_ε converges weakly-$\star$ to a surface load of density $\bar{f} + g_0^+ - g_0^-$ on $\omega \times \{0\}$. In our case, the limit of F_ε includes, in addition, a first order distribution which represents a *density of bending moments*.

Let $W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon;\mathbb{R}^3)$ (respectively $W^{1,p}_{\Gamma}(\Omega;\mathbb{R}^3)$) denote the space of functions in $W^{1,p}(\Omega_\varepsilon;\mathbb{R}^3)$ (respectively $W^{1,p}(\Omega;\mathbb{R}^3)$) that vanish on Γ_ε (respectively on Γ). Assuming that the deformations of the body satisfy a boundary condition of place on Γ_ε, the equilibrium problem under the load F_ε given in (1.1) can be formulated as a minimization problem

$$\inf_{U-\tilde{x}\in W^{1,p}_{\Gamma_\varepsilon}(\Omega_\varepsilon;\mathbb{R}^3)} \left\{ \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(DU)\ d\tilde{x} \ - < F_\varepsilon, U > \right\}. \tag{1.2}$$

In the sequel we will assume that the potential W is a Borel function satisfying the p-growth conditions

$$\text{(H1)} \qquad \frac{1}{C}|\xi|^p - C \le W(\xi) \le C(1+|\xi|^p),$$

for a certain positive constant C and for all $\xi \in M^{3\times 3}$.

The existence of a solution for problem (1.2) can be obtained via the direct method of the calculus of variations under the additional hypothesis that W is quasiconvex.

In order to pass to the limit in problem (1.2) as the thickness ε goes to zero, we perform the usual change of variables that transforms the problem from the thin, varying domain Ω_ε, into the fixed domain Ω : to each $\tilde{x} = (\tilde{x}_\alpha, \tilde{x}_3) \in \Omega_\varepsilon$ we associate $x = (x_\alpha, x_3) = \left(\tilde{x}_\alpha, \frac{1}{\varepsilon}\tilde{x}_3\right) \in \Omega$, and define u, $u^\pm$ and $u_{0,\varepsilon}$ by

$$u\Big(x_\alpha, x_3\Big) := U(\tilde{x}_\alpha, \tilde{x}_3)\ ,\quad u^\pm(x_\alpha) := u\left(x_\alpha, \pm\frac{1}{2}\right)\ ,\quad u_{0,\varepsilon}(x_\alpha, x_3) := (x_\alpha, \varepsilon x_3).$$

Taking into account (1.1), we may rewrite (1.2) as

$$(\mathcal{P}_\varepsilon) \qquad \inf_{u-u_{0,\varepsilon}\in W^{1,p}_{\Gamma}(\Omega;\mathbb{R}^3)} \left\{ \int_\Omega W\left(D_\alpha u \ \Big|\ \frac{1}{\varepsilon}D_3 u\right)\ dx \ - L_\varepsilon(u)\right\},$$

where now the work $L_\varepsilon(u)$ of the external loads is given by

$$L_\varepsilon(u) := \int_\Omega f\ u\ dx + \int_\omega (g_0^+\ u^+ - g_0^-\ u^-)dx_\alpha + \int_\omega g\ \left(\frac{u^+ - u^-}{\varepsilon}\right)\ dx_\alpha.$$

¿From (H1), together with the boundary conditions $u_\varepsilon = u_{0,\varepsilon}$ on Γ, it follows that any minimizing sequence $\{u_\varepsilon\}$ satisfies the bound

$$\sup_\varepsilon \left\{ \int_\Omega |D_\alpha u_\varepsilon|^p dx + \int_\Omega \frac{1}{\varepsilon^p}|D_3 u_\varepsilon|^p dx \right\} < +\infty.$$

This allow us to infer the weak compactness of $\{(u_\varepsilon, b_\varepsilon)\}$ in $W^{1,p}(\Omega;\mathbb{R}^3)\times L^p(\Omega;\mathbb{R}^3)$, where $b_\varepsilon := \frac{1}{\varepsilon}D_3 u_\varepsilon$, and therefore it is natural to expect a limit problem involving the corresponding weak limit (u,b). In fact, as it turns out, u will depend only on x_α (see Section 3), whereas the vector field b is, in general, no longer related with the bidimensional tangential strain tensor of u. In other words, u and b have to be considered as independent macroscopic entities.

We remark that, since our model includes a bending moment of density g, the limiting energy involves explicitly the average $\bar{b}$ of b, with respect to x_3. More precisely, recalling that $u = u(x_\alpha)$, we have $\lim_{\varepsilon\to 0} L_\varepsilon(u_\varepsilon) = L(u,\bar{b})$, where L is the linear function given by

$$L(u,\bar{b}) := \int_\omega \bar{f}\ u\ dx_\alpha + \int_\omega (g_0^+ - g_0^-)\ u\ dx_\alpha + \int_\omega g\ \bar{b}\ dx_\alpha \tag{1.3}$$

and where

$$\bar{b}(x_\alpha) := \int_I b(x_\alpha, x_3)\, dx_3 \quad \text{and} \quad \bar{f}(x_\alpha) := \int_I f(x_\alpha, x_3)\, dx_3. \tag{1.4}$$

We notice that the first equality in (1.4) is an immediate consequence of the fact that b is the weak limit of $b_\varepsilon = \frac{1}{\varepsilon} D_3 u_\varepsilon$ in $L^p(\Omega;\mathbb{R}^3)$ and $\bar{b}$ is the weak limit in $L^p(\omega;\mathbb{R}^3)$ of the sequence $\dfrac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}$.

In order to individualize this new variable $\bar{b}$ in the principal part of the total energy, we introduce the following functional $E_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3) \to \overline{\mathbb{R}}$, defined by

$$E_\varepsilon(u,\bar{b}) := \begin{cases} \int_\Omega W(D_\alpha u \,|\, \frac{1}{\varepsilon} D_3 u)\, dx & \text{if } \frac{1}{\varepsilon}\int_I D_3 u(x_\alpha, x_3)\, dx_3 = \bar{b}(x_\alpha), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.5}$$

Our aim is to prove that this functional Γ-converges, with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$, to some functional $E(u,\bar{b})$. As shown in Theorem 1.2 below, this limit energy admits an integral representation which we propose as an alternative model for non linear membranes with bending moments density. This new model is described by introducing the potential $\mathcal{Q}^*W(\overline{F}|b)$ defined, for every $\overline{F} \in M^{3\times 2}$ and $b \in \mathbb{R}^3$, as

$$\begin{aligned} \mathcal{Q}^*W(\overline{F}|b) := \inf_{(\varphi,\lambda)} \Big\{ \int_Q W(\overline{F} + D_\alpha\varphi | \lambda\, D_3\varphi)\, dx : \lambda \in \mathbb{R}\,, \\ \varphi \in W^{1,p}(Q;\mathbb{R}^3),\ \varphi(\cdot, x_3) \text{ is } Q'\text{-periodic } \mathcal{L}^1 \text{ a.e. } x_3 \in I,\ \lambda \int_Q D_3\varphi\, dx = b \Big\}, \end{aligned} \tag{1.6}$$

with $Q := I^3, Q' := I^2$.

The main properties of the potential $\mathcal{Q}^*W$ are summarized in the proposition below, proved in Section 2.

Proposition 1.1. *Let $\mathcal{Q}^*W$ be defined by (1.6). Under hypothesis* (H1) *:*
i) The following inequality holds

$$\mathcal{C}W \le \mathcal{Q}^*W \le \mathcal{Q}W,$$

where $\mathcal{C}W$ and $\mathcal{Q}W$ are the convex and quasiconvex envelopes of W, respectively.
ii) There exists a positive constant C such that for all $\overline{F} \in M^{3\times 2}$

$$\frac{1}{C}(|\overline{F}|^p + |b|^p) - C \le \mathcal{Q}^*W(\overline{F}\ |b) \le C\left(1 + |\overline{F}|^p + |b|^p\right). \tag{1.7}$$

iii) There holds

$$\mathcal{Q}^*(\mathcal{Q}W) = \mathcal{Q}^*W. \tag{1.8}$$

iv) Let $W_0 : M^{3\times 2} \to \mathbb{R}$ be given by

$$W_0(\overline{F}) := \inf_{b\in\mathbb{R}^3} \left\{ W(\overline{F}|b) :\ b \in \mathbb{R}^3 \right\}$$

and denote by $\mathcal{Q}_2 W_0$ its quasiconvex envelope. Then we have

$$\inf_{b\in\mathbb{R}^3} \mathcal{Q}^*W(\overline{F}\ |b) = \mathcal{Q}_2 W_0(\overline{F}). \tag{1.9}$$

Let us introduce the space $\mathcal{V}$, that we will identify with $W^{1,p}(\omega;\mathbb{R}^3)$,

$$\mathcal{V} := \{u \in W^{1,p}(\Omega;\mathbb{R}^3) \mid D_3 u(x) = 0 \text{ a.e. in } x \in \Omega\},$$

and consider the functional

$$E(u,\bar{b}) := \int_\omega \mathcal{Q}^*W(D_\alpha u \mid \bar{b}) dx_\alpha\ , \quad \text{if } (u,\bar{b}) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3).$$

We may now state the main results of this paper, which will be proved in Section 3 :

Theorem 1.2. *Under the hypothesis* (H1)*, the sequence* $\{E_\varepsilon\}$ Γ*-converges to* E*, as* $\varepsilon \to 0$*, precisely,*

i) if $u_\varepsilon \rightharpoonup u$ *in* $W^{1,p}(\Omega;\mathbb{R}^3)$ *and* $\dfrac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon} \rightharpoonup \bar{b}$ *in* $L^p(\omega;\mathbb{R}^3)$ *then*

$$\liminf_{\varepsilon\to 0} \int_\Omega W\left(D_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} D_3 u_\varepsilon\right.\right)\, dx \geq E(u,\bar{b})\ ;$$

ii) for every pair $(u,\bar{b})$ *in* $\mathcal{V}\times L^p(\omega;\mathbb{R}^3)$*, there exists a sequence* $\{u_\varepsilon\}$ *such that*

$$\left(u_\varepsilon, \frac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}\right) \rightharpoonup (u,\bar{b}) \quad , \quad \lim_{\varepsilon\to 0} \int_\Omega W\left(D_\alpha u_\varepsilon \left| \frac{1}{\varepsilon} D_3 u_\varepsilon\right.\right)\, dx = E(u,\bar{b})\ .$$

Corollary 1.3. *Let* W *satisfy hypothesis* (H1)*. Let* $f \in L^{p'}(\Omega,\mathbb{R}^3)$*,* $g_0^\pm, g \in L^{p'}(\omega,\mathbb{R}^3)$*, and let* u_ε *be a minimizer for* $(\mathcal{P}_\varepsilon)$*. Then the sequence* $\left\{\left(u_\varepsilon, \dfrac{u_\varepsilon^+ - u_\varepsilon^-}{\varepsilon}\right)\right\}$ *is weakly relatively compact in* $W^{1,p}(\Omega,\mathbb{R}^3)\times L^p(\omega,\mathbb{R}^3)$*. Furthermore, any cluster point* $(u,\bar{b})$ *of this sequence belongs to* $\mathcal{V}\times L^p(\omega,\mathbb{R}^3)$ *and is a solution of the problem*

$$(\mathcal{P}) \qquad \min_{\substack{u - x\in W_0^{1,p}(\omega;\mathbb{R}^3)\\ \bar{b}\in L^p(\omega,\mathbb{R}^3)}} \left\{\int_\omega \mathcal{Q}^* W(D_\alpha u \mid \bar{b}) dx_\alpha - L(u,\bar{b})\right\},$$

where $L(u,\bar{b})$ *is defined by (1.3).*

Remark 1.4. The boundary condition considered in $(\mathcal{P}_\varepsilon)$ were those treated in [LDR1] and [BJ]. It is easy to see that all the conclusions of this work still stand with obvious adaptations, if instead of taking $u_{0,\varepsilon}(x_\alpha, x_3) := (x_\alpha, \varepsilon x_3)$ for $(x_\alpha, x_3)\in\Omega$, we assume that $\{u_{0,\varepsilon}\}$ is weakly-$\star$ converging to some function u_0 in $W^{1,\infty}(\Omega;\mathbb{R}^3)$ and $\|D_3 u_{0,\varepsilon}\|_\infty \leq C\varepsilon$ for some $C>0$. Note that in the statement of problem $(\mathcal{P})$, in Corollary 1.3, the Dirichlet boundary condition will now read $u - u_0 \in W_0^{1,p}(\Omega;\mathbb{R}^3)$. Furthermore the above uniform bounds may be replaced by the condition of equi-integrability of the sequence $\{|D_\alpha u_{0,\varepsilon}|^p + |\frac{1}{\varepsilon} D_3 u_{0,\varepsilon}|^p\}$.

Remark 1.5. By Theorem 1.2 and by virtue of the coercivity condition in (1.7), it turns out that the functional

$$(u,\bar{b})\in\mathcal{V}\times L^p(\omega;\mathbb{R}^3)\mapsto \int_\omega \mathcal{Q}^* W(D_\alpha u \mid \bar{b})\ dx_\alpha$$

is weakly lower semicontinuous. Therefore, by [FM] (see also [FKP]), we conclude that $\mathcal{Q}^* W(\cdot \mid \bar{b})$ is quasiconvex and that $\mathcal{Q}^* W(\overline{F}\mid\cdot)$ is convex.

Remark 1.6. If no bending moments are present, that is if $g=0$, then $L(u,\bar{b})$ does not depend on $\bar{b}$ and we may perform explicitly the infimum in $(\mathcal{P})$ with respect to $\bar{b}$, by passing it under the integral sign. By (1.9), we are reduced to solving the following minimum problem in u

$$(\overline{\mathcal{P}}), \qquad \min_{u - x\in W_0^{1,p}(\omega;\mathbb{R}^3)} \left\{\int_\omega \mathcal{Q}_2 W_0(D_\alpha u) dx_\alpha - \left(\int_\omega \bar{f}\ u\ dx_\alpha + \int_\omega (g_0^+ - g_0^-)\ u\ dx_\alpha\right)\right\}.$$

and thus we have recovered, as a particular case, the membrane problem treated in [LDR1].

Remark 1.7. We stress the fact that strict inequalities between $\mathcal{C}W$, $\mathcal{Q}^*W$ and $\mathcal{Q}W$ may occur in part i) of Proposition 1.1. In particular, we may have $\mathcal{Q}^*W < \mathcal{Q}W$ even if W is quasiconvex. In other words, our model, which differs from the classical membrane model discussed in [LDR1] as it involves the Cosserat vector field b, cannot be derived by simply replacing the third column of the deformation tensor by b.

To show that we may have $\mathcal{Q}^*W < \mathcal{Q}W$, we consider as in [LDR1] a quasiconvex function W which is frame invariant and such that $W(\mathbb{I}) = 0$, where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^3$, whereas $W(F) > 0$ if $F \notin SO(3)$. As an example, we may take $W(F) := \sum_{i=1}^{i=3}(v_i(F)^2-1)_+^2 + |\det F - 1|$, where $v_i(F)$,$i = 1, 2, 3$, denote the the singular values of F. If $\mathcal{Q}^*W \equiv \mathcal{Q}W$, then by taking the infimum with respect to the third column and applying (1.9) we would obtain that $\mathcal{Q}_2W_0 \equiv W_0$, and this is incompatible with [Corollary 11, [LDR1]). Precisely, $\mathcal{Q}_2W_0(\overline{F}) = 0 < W_0(\overline{F})$ holds for all $\overline{F} \in M^{3\times 2}$ such that the eigenvalues $v_1(\overline{F}), v_2(\overline{F})$ of $\sqrt{\overline{F}^T\overline{F}}$ satisfy $0 \le v_1(\overline{F}) < v_2(\overline{F}) \le 1$.

The strict inequality between $\mathcal{C}W$ and $\mathcal{Q}^*W$ can be obtained by considering the potential introduced in [KoS]

$$W(F) := \begin{cases} 1 + |F - I|^2 & \text{if } F \neq \mathbb{I}, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\cdot|$ denotes the Euclidean norm. Recall that W is not quasiconvex. The identity $\mathcal{Q}^*W \equiv \mathcal{C}W$ would imply, once again by taking the infimum with respect to the third column and applying (1.9), that $\mathcal{Q}_2W_0 \equiv (\mathcal{C}W)_0$. A straightforward computation leads to

$$W_0(\overline{F}) = \begin{cases} 1 + |\overline{F} - \overline{I}|^2 & \text{if } \overline{F} \neq \overline{I}, \\ 0 & \text{otherwise,} \end{cases} \text{ and } (\mathcal{C}W)_0(\overline{F}) = \begin{cases} 1 + |\overline{F} - \overline{I}|^2 & \text{if } |\overline{F} - \overline{I}| \ge 1, \\ 2|\overline{F} - \overline{I}| & \text{if } |\overline{F} - \overline{I}| \le 1 \ , \end{cases}$$

where $\overline{\mathbb{I}}$ the 3×2 matrix associated with $\mathbb{I}$. For every $Z = (z_{ij}) \in M^{2\times 2}$ set

$$\overline{F}(Z) = \begin{pmatrix} 1 + z_{11} & z_{12} \\ z_{21} & 1 + z_{22} \\ 0 & 0 \end{pmatrix}.$$

Introduce also the 2D version of Kohn and Strang potential and its convexification

$$f(Z) := \begin{cases} 1 + |Z|^2 & \text{if } Z \neq 0, \\ 0 & \text{otherwise,} \end{cases} \text{ and } f^{**}(Z) = \begin{cases} 1 + |Z|^2 & \text{if } |Z| \le 1, \\ 2|Z| & \text{if } |Z| \ge 1. \end{cases}$$

Then one has $(\mathcal{C}W)_0(\overline{F}(Z)) = f^{**}(Z)$. On the other hand, it is easy to check that $\mathcal{Q}_2W_0(\overline{F}(Z)) = \mathcal{Q}f(Z)$ where, according to [KoS], the quasiconvex envelope $\mathcal{Q}f$ is given by

$$\mathcal{Q}f(Z) = \begin{cases} 1 + (\operatorname{tr} Z)^2 - 2\det Z & \text{if } |\operatorname{tr} Z| \ge 1, \\ 2\operatorname{tr} Z - 2\det Z & \text{otherwise,} \end{cases}$$

and, consequently, it differs from $f^{**}(Z)$.

2. Preliminary results.

2.1 Proof of Proposition 1.1.

(i) Recall that, for $\overline{F} \in M^{3\times 2}$ and $b \in \mathbb{R}^3$,

$$\mathcal{Q}W(\overline{F}|b) = \inf_{\psi} \left\{ \int_Q W(\overline{F} + D_\alpha\psi \ |b + \ D_3\psi) \ dx : \psi \in W^{1,p}_{\#}(Q;\mathbb{R}^3) \right\}, \tag{2.1}$$

where $W^{1,p}_{\#}(Q;\mathbb{R}^3)$ stands for the space of all $W^{1,p}(Q;\mathbb{R}^3)$ functions that are Q-periodic.

For each $\psi \in W^{1,p}_{\#}(Q;\mathbb{R}^3)$, define $\varphi(x) := \psi(x) + bx_3$. Since $D_\alpha\varphi = D_\alpha\psi$, $D_3\varphi = D_3\psi + b$ and $\int_I D_3\psi dx_3 = 0$, we obtain, in view of (1.6) with $\lambda = 1$,

$$\begin{aligned} \mathcal{Q}^*W(\overline{F}|b) &\le \int_Q W(\overline{F} + D_\alpha\varphi \mid D_3\varphi) \ dx \\ &= \int_Q W(\overline{F} + D_\alpha\psi \mid b + D_3\psi) \ dx. \end{aligned}$$

Taking the infimum in all ψ we get from (2.1)

$$\mathcal{Q}^*W(\overline{F}|b) \leq \mathcal{Q}W(\overline{F}|b).$$

Consider now (φ, λ) such that $\varphi \in W^{1,p}(Q;\mathbb{R}^3)$, $\varphi(\cdot, x_3)$ is Q'-periodic for $\mathcal{L}^1$ a.e. $x_3 \in I$ and $\lambda \int_Q D_3\varphi dx_3 = b$. Define $\psi(x) := \varphi(x) - bx_3/\lambda$. We have $D_\alpha\psi = D_\alpha\varphi$, $D_3\psi = D_3\varphi - b/\lambda$ and $\int_Q D_\alpha\psi dx = \int_Q \lambda D_3\psi dx = 0$. Then, by Jensen inequality,

$$\begin{aligned}\int_Q W(\overline{F} + D_\alpha\varphi \mid \lambda D_3\varphi)\, dx &= \int_Q W(\overline{F} + D_\alpha\psi \mid b + \lambda D_3\psi)\, dx \\ &\geq \int_Q \mathcal{C}W(\overline{F} + D_\alpha\psi \mid b + \lambda D_3\psi)\, dx \\ &\geq \mathcal{C}W(\overline{F}, b).\end{aligned}$$

Taking the infimum in all (φ, λ), we deduce that $\mathcal{C}W(\overline{F}, b) \leq \mathcal{Q}^*W(\overline{F}|b)$.

(ii) The bounds in (1.7) are a straightforward consequence of hypothesis (H1), of the convexity of $|\xi|^p$, and of the inequalities

$$\mathcal{C}W \leq \mathcal{Q}^*W \leq \mathcal{Q}W \leq W.$$

(iii) To prove (1.8), first note that trivially $\mathcal{Q}^*W(\overline{F}|b) \geq \mathcal{Q}^*(\mathcal{Q}W)(\overline{F}|b)$.

To show the converse inequality we prove the auxiliary result

$$(\mathcal{Q}W_\varepsilon)(\xi) = (\mathcal{Q}W)_\varepsilon(\xi), \text{ with } W_\varepsilon(\xi) := W\left(\xi_\alpha \middle| \frac{1}{\varepsilon}\, \xi_3\right). \tag{2.2}$$

Recall that, under the growth hypothesis (H1), the quasiconvex envelop of W is given by

$$\mathcal{Q}W(\xi) = \inf\left\{\frac{1}{\mathcal{L}^3(U)}\int_U W(\xi + D\varphi)\, dx \mid \varphi \in W_0^{1,p}(U;\mathbb{R}^3)\right\}, \tag{2.3}$$

provided $\mathcal{L}^3(\partial U) = 0$, where U is any open bounded subset of $\mathbb{R}^3$.

Let $\delta > 0$ and choose $\varphi \in W_0^{1,p}(Q;\mathbb{R}^3)$ such that

$$\begin{aligned}(\mathcal{Q}W_\varepsilon)(\xi) &\geq \int_Q W_\varepsilon\ (\xi + D\varphi)\, dx - \delta \\ &= \int_Q W\left(\xi_\alpha + D_\alpha\varphi \;\middle|\; \frac{1}{\varepsilon}\xi_3 + \frac{1}{\varepsilon}D_3\varphi\right)\, dx - \delta.\end{aligned}$$

Changing variables ($y_\alpha = x_\alpha, y_3 = \varepsilon x_3$) and letting $\psi(y) := \varphi(y_\alpha, \varepsilon^{-1}\, y_3)$, $U_\varepsilon := Q' \times \varepsilon I$, we obtain

$$\begin{aligned}(\mathcal{Q}W_\varepsilon)(\xi) &\geq \frac{1}{\varepsilon}\int_{Q'\times\varepsilon I} W\left(\xi_\alpha + D_\alpha\psi \;\middle|\; \frac{1}{\varepsilon}\xi_3 + D_3\psi\right)\, dy - \delta \\ &= \frac{1}{\mathcal{L}^3(U_\varepsilon)}\int_{U_\varepsilon} W\left(\xi_\alpha + D_\alpha\psi \;\middle|\; \frac{1}{\varepsilon}\xi_3 + D_3\psi\right)\, dy - \delta. \\ &\geq (\mathcal{Q}W)_\varepsilon(\xi) - \delta.\end{aligned}$$

Letting $\delta \to 0$ we conclude that $(\mathcal{Q}W_\varepsilon)(\xi) \geq (\mathcal{Q}W)_\varepsilon(\xi)$. The opposite inequality is proven in a similar way.

Fix $\delta > 0$ and let $\varphi \in W^{1,p}(Q;\mathbb{R}^3)$, $\varphi(\cdot, x_3)$ $Q'-$ periodic, and $\lambda \in \mathbb{R}$, satisfying $\lambda \int_Q D_3\varphi\, dx = b$, and

$$\mathcal{Q}^*(\mathcal{Q}W)(\overline{F}|b) \geq \int_Q (\mathcal{Q}W)(\overline{F} + D_\alpha\ \varphi \mid \lambda\ D_3\varphi)\, dx - \delta. \tag{2.4}$$

It is easy to show that for any $\varphi \in W^{1,p}(Q;\mathbb{R}^3)$, $\varphi(\cdot, x_3)$ $Q'-$ periodic, there exists a sequence of piecewise affine functions $\{\varphi_n\}$ such that $\varphi_n(\cdot, x_3)$ is $Q'-$ periodic and

$$\lim_n \|\varphi_n - \varphi\|_{W^{1,p}(Q;\mathbb{R}^3)} = 0.$$

Let $\{\varphi_n\}$ be such a sequence, with respect to φ, and consider

$$\psi_n(x) := \varphi_n(x) + \left(\frac{b}{\lambda} - \int_Q D_3\varphi_n \; dx\right) x_3.$$

Then $\psi_n(\cdot, x_3)$ is still $Q'-$ periodic, $\lambda \int_Q D_3\psi_n \; dx = b$ and $\lim_n \|\psi_n - \varphi\|_{W^{1,p}(Q;\mathbb{R}^3)} = 0$. Up to a subsequence, and almost everywhere in Q, we have

$$C(1 + |\nabla\psi_n|^p) - \mathcal{Q}W(\overline{F} + D_\alpha\psi_n \; |\lambda D_3\psi_n) \to C(1 + |\nabla\varphi|^p) - \mathcal{Q}W(\overline{F} + D_\alpha\varphi \; |\lambda D_3\varphi),$$

where we have used the fact that, by, (H1), QW is p-Lipschitz continuous (see [M]), and using Fatou's Lemma we obtain

$$\limsup_n \int_Q \mathcal{Q}W(\overline{F} + D_\alpha\psi_n \; |\lambda D_3\psi_n) \leq \int_Q \mathcal{Q}W(\overline{F} + D_\alpha\varphi \; |\lambda D_3\varphi). \tag{2.5}$$

From (2.4) and (2.5) we get, for some n big enough,

$$\mathcal{Q}^*(\mathcal{Q}W)(\overline{F}|b) \geq \int_Q (\mathcal{Q}W)(\overline{F} + D_\alpha \; \psi_n \mid \lambda \; D_3\psi_n) \; dx - 2\delta. \tag{2.6}$$

Let $D\psi_n = \sum_{i=1}^{k} \chi_{U_i} \; (A_i \mid \eta_i)$, where χ_{U_i} denotes the characteristic function of U_i, $\mathcal{L}^N(\partial U_i) = 0$, $A_i \in M^{3\times 2}$ and $\eta_i \in \mathbb{R}^3$, for all $i = 1, \cdots, k$. Recalling (2.2) and (2.3), inequality (2.6) becomes, for suitable $\theta_i \in W_0^{1,p}(U_i;\mathbb{R}^3)$,

$$\begin{aligned}
\mathcal{Q}^*(\mathcal{Q}W)(\overline{F}|b) &\geq \sum_{i=1}^{k} \mathcal{L}^N(U_i) \; \mathcal{Q}W(\overline{F} + A_i \; |\lambda \; \eta_i) - 2\delta \\
&= \sum_{i=1}^{k} \mathcal{L}^N(U_i) \; (\mathcal{Q}W)_{\lambda^{-1}}(\overline{F} + A_i \mid \eta_i) - 2\delta \\
&= \sum_{i=1}^{k} \mathcal{L}^N(U_i) \; (\mathcal{Q}W_{\lambda^{-1}})(\overline{F} + A_i \mid \eta_i) - 2\delta \\
&\geq \sum_{i=1}^{k} \int_{U_i} W_{\lambda^{-1}}(\overline{F} + A_i + D_\alpha \; \theta_i \mid \eta_i + \; D_3\theta_i) \; dx - 3\delta \\
&= \int_Q W(\overline{F} + D_\alpha \; \Lambda \mid \lambda \; D_3 \; \Lambda) \; dx - 3\delta \geq \mathcal{Q}^*W(\overline{F}|b) - 3\delta,
\end{aligned}$$

where $\Lambda := \psi_n + \sum_{i=1}^{k} \chi_{U_i}\theta_i$. Since $\theta_i|_{\partial U_i} = 0$, $\Lambda(\cdot, x_3)$ is $Q'-$ periodic and $\lambda \int_Q D_3\Lambda dx = b$. Letting $\delta \to 0$, we conclude that $\mathcal{Q}^*(\mathcal{Q}W)(\overline{F}|b) \geq \mathcal{Q}^*W(\overline{F}|b)$.

(iv) Using the definition of quasiconvex envelope of W_0, we have

$$\begin{aligned}
\mathcal{Q}_2W_0(\overline{F}) &= \inf_{\substack{\varphi \in W^{1,p}(Q';\mathbb{R}^3) \\ \varphi \text{ is } Q'-\text{periodic}}} \int_{Q'} \inf_{c \in \mathbb{R}^3} W(\overline{F} + D_\alpha\varphi(x_\alpha)|c) \; dx_\alpha \\
&= \inf_{\substack{\varphi \in W^{1,p}(Q';\mathbb{R}^3) \\ \varphi \text{ is } Q'-\text{periodic} \\ b \in L^p(Q';\mathbb{R}^3)}} \int_{Q'} W(\overline{F} + D_\alpha\varphi(x_\alpha)|b(x_\alpha)) \; dx_\alpha,
\end{aligned} \tag{2.7}$$

where the last equality is justified using the Measurable Selection Theorem (see [CV]) and the coercivity hypothesis (H1). The second equality in (2.7) still holds if we consider test function depending in x_3, and thus

$$\begin{aligned}
\mathcal{Q}_2 W_0(\overline{F}) &= \inf_{\substack{\varphi \in W^{1,p}(Q;\mathbb{R}^3) \\ \varphi \text{ is } Q'-\text{periodic} \\ b \in L^p(Q;\mathbb{R}^3)}} \int_{Q' \times I} W(\overline{F} + D_\alpha \varphi(x_\alpha, x_3) | b(x_\alpha, x_3)) \, dx_\alpha \, dx_3 \\
&\leq \inf_{\substack{(\lambda, c) \in \mathbb{R} \times \mathbb{R}^3 \\ \varphi \in W^{1,p}(Q;\mathbb{R}^3) \\ \varphi \text{ is } Q'-\text{periodic} \\ \lambda \int_Q D_3 \varphi dx = c}} \int_Q W(\overline{F} + D_\alpha \varphi(x) | \lambda \, D_3 \varphi(x)) \, dx \qquad (2.8) \\
&= \inf_{c \in \mathbb{R}^3} \mathcal{Q}^* W(\overline{F} \, | c).
\end{aligned}$$

Then $\mathcal{Q}_2 W_0(\overline{F}) \leq \inf_{c \in \mathbb{R}^3} \mathcal{Q}^* W(\overline{F} \, | c)$. To prove the opposite inequality, in view of (1.8) we may admit, without loss of generality, that W is quasiconvex and this, together with hypothesis (H1), implies that W is locally Lipschitz (see [M]). In fact, if the inequality holds for the quasiconvex envelope of W, we will have

$$\inf_{c \in \mathbb{R}^3} \mathcal{Q}^* W(\overline{F} \, | c) = \inf_{c \in \mathbb{R}^3} \mathcal{Q}^* (\mathcal{Q} W)(\overline{F} \, | c) \leq \mathcal{Q}_2 (\mathcal{Q} W)_0 \; (\overline{F} \, | c) \leq \mathcal{Q}_2 W_0(\overline{F} \, | c).$$

Let $\varphi \in W^{1,p}(Q';\mathbb{R}^3)$ and $b \in L^p(Q';\mathbb{R}^3)$ be admissible functions for the infimum in the right hand side of (2.7). Let (b_n) be a sequence in $W_0^{1,p}(Q';\mathbb{R}^3)$, such that $b_n \to b$ in $L^p(Q';\mathbb{R}^3)$ such that there exists $\lim_n \|D_\alpha b_n\|_{L^p(Q';\mathbb{R}^3)}$ and define

$$\tilde{\varphi}_n(x) := \varphi(x_\alpha) + \frac{x_3}{\lambda_n} b_n(x_\alpha),$$

where

$$\lambda_n := \begin{cases} n & \text{if } \lim_n \|D_\alpha b_n\|_{L^p(Q';\mathbb{R}^3)} < +\infty, \\ n \, \|D_\alpha b_n\|_{L^p(Q';\mathbb{R}^3)} & \text{otherwise.} \end{cases}$$

We easily see that $\tilde{\varphi}_n \in W^{1,p}(Q;\mathbb{R}^3)$ is Q'-periodic, $\lambda_n \int_Q D_3 \tilde{\varphi}_n dx = \int_{Q'} b_n dx_\alpha =: c_n \in \mathbb{R}^3$ and

$$W(\overline{F} + D_\alpha \tilde{\varphi}_n(x) | \lambda_n D_3 \tilde{\varphi}_n(x)) = W\left(\overline{F} + D_\alpha \varphi(x_\alpha) + \frac{x_3}{\lambda_n} D_\alpha b_n(x_\alpha) \middle| b_n(x_\alpha)\right).$$

Since $\frac{x_3}{\lambda_n} D_\alpha b_n(x_\alpha) \to 0$ and $b_n \to b$ in $L^p(Q;\mathbb{R}^3)$, the p-Lipschitz continuity of W yields

$$\begin{aligned}
\int_{Q'} W(\overline{F} + D_\alpha \varphi(x_\alpha) \; | b(x_\alpha)) \, dx_\alpha &= \lim_n \int_Q W(\overline{F} + D_\alpha \tilde{\varphi}_n(x) | \lambda_n D_3 \tilde{\varphi}_n(x)) \, dx \\
&\geq \limsup_n \mathcal{Q}^* W(\overline{F} \, | c_n) \qquad (2.9) \\
&\geq \inf_{c \in \mathbb{R}^3} \mathcal{Q}^* W(\overline{F} \, | c).
\end{aligned}$$

Taking the infimum in the left hand side of (2.9), over all the admissible φ in $W^{1,p}(Q';\mathbb{R}^3)$ and b in $L^p(Q';\mathbb{R}^3)$, and using (2.7), by (2.9) yields

$$\mathcal{Q}_2 W_0(\overline{F}) \geq \inf_{c \in \mathbb{R}^3} \mathcal{Q}^* W(\overline{F} \, | c),$$

which, together with (2.8), completes the proof of (1.9). □

2.2. Properties of the Γ-limit.

We start by localizing the functionals E_ε introduced in (1.5). Representing by $\mathcal{A}(\omega)$ the family of all open subsets of ω, define

$$E_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to \overline{\mathbb{R}},$$

$$E_\varepsilon(u,\bar b,A) := \begin{cases} \int_{A\times I} W(D_\alpha u \,|\frac{1}{\varepsilon} D_3 u)\; dx & \text{if } \frac{1}{\varepsilon}\int_I D_3 u(x_\alpha,x_3)\; dx_3 = \bar b(x_\alpha),\\ +\infty & \text{otherwise.}\end{cases}$$

In order to prove that the sequence $\{E_\varepsilon(\cdot,\cdot,A)\}$ Γ-converges to some functional $E_0(\cdot,\cdot,A)$ for all $A\in\mathcal{A}(\omega)$, it is enough to show that any given sequence $\{\varepsilon_k\}$ converging to 0^+ admits a subsequence $\{\varepsilon_{k_n}\}$ such that the Γ-lower limit of $E_{\varepsilon_{k_n}}$ given by

$$E^-(u,\bar b,A) := \inf\Big\{\liminf_n \int_{A\times I} W\left(D_\alpha u_n \,|\lambda_n\; D_3 u_n\right) dx \mid u_n \rightharpoonup u \,\text{in w - } W^{1,p}(A\times I;\mathbb{R}^3), \\ \lambda_n \int_I D_3 u_n\; dx_3 \rightharpoonup \bar b, \text{ in w - } L^p(A;\mathbb{R}^3)\Big\}, \tag{2.10}$$

where for simplicity of notations we write $\lambda_n := (\varepsilon_{k_n})^{-1}$, coincides with $E_0(\cdot,\cdot,A)$ for all $(u,\bar b)$ in $\mathcal{V}\times L^p(\omega;\mathbb{R}^3)$.

We will denote $E^-(u,\bar b) := E^-(u,\bar b,\omega)$.

Lemma 2.1. *Let W be a Borel function, satisfying hypothesis* (H1). *Any sequence $\{\varepsilon_k\}$ converging to 0^+ admits a subsequence $\{\varepsilon_{k_n}\}$ such that for each $(u,\bar b)\in\mathcal{V}\times L^p(\omega;\mathbb{R}^3)$, the set function $E^-(u,\bar b,\cdot)$, defined in (2.10) (with $\lambda_n := (\varepsilon_{k_n})^{-1}$), is the trace of a Radon measure, absolutely continuous with respect to the 2-dimensional Lebesgue measure.*

Proof. We first prove that

$$E^-(u,\bar b,A) \le C\left(\mathcal{L}^2(A) + \int_A |D_\alpha u|^p\; dx_\alpha + \int_A |\bar b|^p\; dx_\alpha\right) \tag{2.11}$$

for some positive constant C and for all $(u,\bar b,A)\in\mathcal{V}\times L^p(\omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$.

Suppose that $\bar b\in W^{1,p}(\omega;\mathbb{R}^3)$ and $\lambda_n\to+\infty$, where as in (2.10) we set $\lambda_n := (\varepsilon_{k_n})^{-1}$. Define $u_n(x) := u(x_\alpha) + \dfrac{x_3}{\lambda_n}\;\bar b(x_\alpha)$. The sequence $\{u_n\}$ satisfies $u_n \to u \,\text{in}\, W^{1,p}(\Omega;\mathbb{R}^3)$ and $\lambda_n\; D_3 u_n = \bar b$. Consequently

$$\begin{aligned} &\int_{A\times I} |D_\alpha u_n|^p\; dx \to \int_A |D_\alpha u|^p\; dx_\alpha\,,\\ &{\lambda_n}^p \int_{A\times I} |D_3 u_n|^p\; dx = \int_A |\bar b|^p\; dx_\alpha\,. \end{aligned} \tag{2.12}$$

By hypothesis (H1) we get

$$\limsup_n \int_{A\times I} W(D_\alpha u_n \,|\lambda_n\; D_3 u_n) dx \\ \le \lim_n C\left(\mathcal{L}^2(A) + \int_{A\times I} |D_\alpha u_n|^p\; dx_\alpha + {\lambda_n}^p \int_{A\times I} |D_3 u_n|^p\; dx_\alpha\right)$$

which, together with (2.12), yields (2.11) for $\bar b\in W^{1,p}(\omega;\mathbb{R}^3)$. In view of the coercivity condition in (H1) together with the Dirichlet condition we deduce that the Γ- lower limit $E^-(u,\cdot,A)$ is lower semicontinuous (see [DM]) and by density we obtain (2.11) for all $\bar b\in L^p(\omega;\mathbb{R}^3)$.

Fix a sequence $\{\varepsilon_k\}$ converging to 0^+, and let $\mathcal{R}(\omega)$ be the countable subfamily of $\mathcal{A}(\omega)$ obtained by taking all finite unions of open squares in ω with faces parallel to the axes, centered at $x_\alpha\in\omega\cap\mathbb{Q}^2$ and with rational edge length. Since $L^p(\Omega;\mathbb{R}^3)$ is a separable metric space, using Kuratowski's Compactness Theorem (see [DM]) and a diagonal argument, we may find a subsequence $\{\varepsilon_{k_n}\}$ of $\{\varepsilon_k\}$, such that $\Gamma-\lim_n E_{\varepsilon_{k_n}}(u,\bar b,A)$ exists for all $u\in\mathcal{V}$ and for all $A\in\mathcal{R}(\omega)$.

Let $E^-(u,\bar b,\cdot)$ be defined by (2.10) (with $\lambda_n := (\varepsilon_{k_n})^{-1}$) . To prove that $E^-(u,\bar b,\cdot)$ is the trace of a Borel measure, we begin by showing that

$$E^-(u,\bar b,A) \le E^-(u,\bar b,B) + E^-(u,\bar b,A\setminus\overline{U}) \tag{2.13}$$

for all A, B, $U \in \mathcal{A}(\omega)$, with $U \subset\subset B \subset\subset A$, and for all $u \in \mathcal{V}$. Fix $\eta > 0$ and find $\{v_n\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that $v_n \to u$ in $L^p((A\setminus\overline{U})\times I;\mathbb{R}^3)$, $\lambda_n \int_I D_3 v_n \rightharpoonup \bar b$, and

$$\liminf_n \int_{(A\setminus\overline{U})\times I} W\left(D_\alpha v_n|\lambda_n D_3 v_n\right)\, dx \le E^-(u,\bar b,A\setminus\overline{U}) + \eta. \tag{2.14}$$

Extract a subsequence $\{n_j\}$ for which

$$\lim_j \int_{(A\setminus\overline{U})\times I} W\left(D_\alpha v_{n_j}|\lambda_{n_j} D_3 v_{n_j}\right)\, dx = \liminf_n \int_{(A\setminus\overline{U})\times I} W\left(D_\alpha v_n|\lambda_n D_3 v_n\right)\, dx. \tag{2.15}$$

Let $B_0 \in \mathcal{R}(\omega)$ be such that $U \subset\subset B_0 \subset\subset B$. Then $E^-(u,\bar b,B_0) = \Gamma - \lim_j E_{\lambda_{n_j}^{-1}}(u,\bar b,B_0)$, and thus there exists a sequence $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^3)$ such that $u_j \to u$ in $L^p(B_0\times I;\mathbb{R}^3)$, $\lambda_{n_j}\int_I D_3 u_j \rightharpoonup \bar b$, and

$$\lim_j \int_{B_0\times I} W\left(D_\alpha u_j|\lambda_{n_j} D_3 u_j\right)\, dx = E^-(u,\bar b,B_0). \tag{2.16}$$

Consider $G_n(v,A) := \int_{A\times I} \left(1+|D_\alpha v|^p + \lambda_n^p\, |D_3 v|^p\right)\, dx$. Due to the coercivity hypothesis (H1), we may extract a bounded subsequence, that we will not relabel, from the sequences of measures $\nu_j := G_{n_j}(v_{n_j},\cdot) + G_{n_j}(u_j,\cdot)$ restricted to $B_0\setminus\overline{U}$, converging in the sense of distributions to some Radon measure ν defined on $B_0\setminus\overline{U}$.

For every $t>0$ let $B_t := \{x_\alpha \in B_0|\ \mathrm{dist}(x_\alpha,\partial B_0) > t\}$. Define, for $0<\delta<\eta$, the subsets $L_\delta := B_{\eta-2\delta}\setminus\overline{B_{\eta+\delta}}$. Consider a smooth cut-off function $\varphi_\delta \in \mathcal{C}_0^\infty(B_{\eta-\delta};[0,1])$ such that $\varphi_\delta(x_\alpha)=1$ on B_η. As the thickness of the strip L_δ is of order δ, we have an upper bound of the form $\|D\varphi_\delta\|_{L^\infty(B_{\eta-\delta})} \le C/\delta$. Define

$$w_j(x) := u_j(x)\varphi_\delta(x_\alpha) + v_{n_j}(x)(1-\varphi_\delta(x_\alpha)).$$

Clearly this sequence converges to u in $L^p(A\times I;\mathbb{R}^3)$ as $j\to\infty$, satisfies

$$D_\alpha w_j = \varphi_\delta D_\alpha u_j + (1-\varphi_\delta) D_\alpha v_{n_j} + D_\alpha\varphi_\delta \otimes (u_j - v_{n_j}),\ \ D_3 w_j = \varphi_\delta D_3 u_j + (1-\varphi_\delta) D_3 v_{n_j}$$

and $\lambda_{n_j}\int_I D_3 w_j \rightharpoonup \bar b$. By the growth condition (H1), we have the estimate

$$\begin{aligned}
&\int_{A\times I} W\left(D_\alpha w_j|\lambda_{n_j} D_3 w_j\right) dx \\
&\quad\le \int_{B_\eta\times I} W\left(D_\alpha u_j|\lambda_{n_j} D_3 u_j\right) dx + \int_{(A\setminus\overline{B_{\eta-\delta}})\times I} W\left(D_\alpha v_{n_j}|\lambda_{n_j} D_3 v_{n_j}\right) dx \\
&\qquad + C\left(G_{n_j}(v_{n_j},L_\delta) + G_{n_j}(u_j,L_\delta)\right) + \frac{1}{\delta^p}\int_{L_\delta\times I} |v_{n_j}-u_j|^p\, dx \\
&\quad\le \int_{B_0\times I} W\left(D_\alpha u_j|\lambda_{n_j} D_3 u_j\right) dx + \int_{(A\setminus\overline{U})\times I} W\left(D_\alpha v_{n_j}|\lambda_{n_j} D_3 v_{n_j}\right) dx \\
&\qquad + C\left(G_{n_j}(v_{n_j},L_\delta) + G_{n_j}(u_j,L_\delta)\right) + \frac{1}{\delta^p}\int_{L_\delta\times I} |v_{n_j}-u_j|^p\, dx
\end{aligned}$$

Passing to the limit and using (2.14), (2.15) and (2.16), we have

$$\begin{aligned}E^-(u,\bar b,A)&\le E^-(u,\bar b,B_0)+E^-(u,\bar b,A\setminus\overline{U})+\eta+C\nu(\overline{L_\delta})\\&\le E^-(u,\bar b,B)+E^-(u,\bar b,A\setminus\overline{U})+\eta+C\nu(\overline{L_\delta}),\end{aligned}$$

where we have used the fact that

$$E^-(u,\bar b,A)\le\liminf_n E_{\lambda_n^{-1}}(\overline{w}_n,\bar b_n,A)\le\liminf_j\int_{A\times I}W\left(D_\alpha w_j|\lambda_{n_j}D_3w_j\right)dx,$$

where $(\overline{w}_n,\bar b_n):=(w_j,\lambda_{n_j}\int_I D_3w_j\ dx_3)$ if $n=n_j$, and $(\overline{w}_n,\bar b_n):=(u,\ \bar b)$ if $n\ne n_j$. We let δ go to zero to obtain

$$E^-(u,\bar b,A)\le E^-(u,\bar b,B)+E^-(u,\bar b,A\setminus\overline{U})+\eta+C\nu(\partial B_\eta).$$

It suffices to choose a subsequence $\{\eta_n\}$ such that $\eta_n\to0^+$ and $\nu(\partial B_{\eta_n})=0$, to conclude the proof of (2.13).

Finally we prove that $E^-(u,\bar b,\cdot)$ is the trace of a Borel measure. Fix $(u,\bar b)\in\mathcal{V}\times L^p(\omega;\mathbb{R}^3)$ and let $\{u_n\}$ satisfy, up to a subsequence of $\{\lambda_n\}$ that we will not relabel,

$$E^-(u,\bar b)=\lim_n\int_\Omega W(D_\alpha u_n\ |\lambda_n\ D_3u_n)dx. \tag{2.17}$$

Let $\mu_n:=\left(\int_I W(D_\alpha u_n\ |\lambda_nD_3u_n)\ dx_3\right)\mathcal{L}^2\lfloor\Omega$, and let μ be defined, up to a subsequence, as the limit of $\{\mu_n\}$ in the sense of distributions . We prove that $E^-(u,\bar b,A)=\mu(A)$ for all $A\in\mathcal{A}(\omega)$.

We have, from the definition of E^-,

$$E^-(u,\bar b,A)\le\liminf_n\int_{A\times I}W(D_\alpha u_n\ |\lambda_nD_3u_n)dx\le\mu(\overline{A}). \tag{2.18}$$

Fix $\varepsilon>0$ and let U_ε and B_ε be open sets such that $U_\varepsilon\subset\subset B_\varepsilon\subset\subset A$ and, invoking (2.11), $E^-(u,\bar b,A\backslash\overline{U}_\varepsilon)<\varepsilon$. Inequalities (2.13) and (2.18) yield

$$\begin{aligned}E^-(u,\bar b,A)&\le\varepsilon+E^-(u,\bar b,B_\varepsilon)\\&\le\varepsilon+\mu(\overline{B}_\varepsilon)\le\varepsilon+\mu(A).\end{aligned}$$

Letting $\varepsilon\to0$ we obtain

$$E^-(u,\bar b,A)\le\mu(A). \tag{2.19}$$

Conversely, for $\varepsilon>0$ let $A_\varepsilon\subset\subset A$ be such that $\mu(A\setminus\overline{A}_\varepsilon)<\varepsilon$. Since by (2.17) $\mu(\omega)=E^-(u,\bar b)$, by (2.13) and (2.19) we obtain

$$\begin{aligned}\mu(A)&\le\mu(\overline{A}_\varepsilon)+\varepsilon\\&=\mu(\omega)-\mu(\omega\setminus\overline{A}_\varepsilon)+\varepsilon\\&\le E^-(u,\bar b)-E^-(u,\bar b,\omega\setminus\overline{A}_\varepsilon)+\varepsilon\\&\le E^-(u,\bar b,A)+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to0$ we conclude that $\mu(A)\le E^-(u,\bar b,A)$. □

The following lemma is based on a standard argument due to E. De Giorgi and allow us to restrict ourselves, in (2.10), to sequences $\{u_n\}$ having the same trace as its limit u.

Lemma 2.2. *Let W be a Borel function, satisfying hypothesis* (H1), *let A be an open subset of ω, and let $L \in \mathbb{R}$. Consider $u \in \mathcal{V}$, a sequence $\{u_n\}$ in $W^{1,p}(A \times I;\mathbb{R}^3)$ and a sequence $\{\lambda_n\}$ in $\mathbb{R}$ such that $u_n \to u$ in $L^p(A \times I;\mathbb{R}^3)$, $\lambda_n \to +\infty$, $\lambda_n \int_I D_3\varphi_n \, dx_3 \rightharpoonup \bar{b}$, weakly in $L^p(A;\mathbb{R}^3)$ and*

$$\lim_n \int_{A\times I} W(D_\alpha u_n(x) \mid \lambda_n \, D_3 u_n(x)) \, dx = L. \tag{2.20}$$

Then, there exist a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ and a sequence $\{\tilde{u}_k\}$ in $W^{1,p}(A \times I;\mathbb{R}^3)$ satisfying $\tilde{u}_k = u$ on a neighborhood of $\partial A \times I$, $\tilde{u}_k \to u$ in $L^p(A \times I;\mathbb{R}^3)$, $\lambda_{n_k} \displaystyle\int_I D_3\tilde{u}_k \, dx_3 \rightharpoonup \bar{b}$ weakly in $L^p(A;\mathbb{R}^3)$, and

$$\limsup_k \int_{A\times I} W(D_\alpha \tilde{u}_k(x) \mid \lambda_{n_k} \, D_3 \tilde{u}_k(x)) \, dx \le L.$$

Proof. For $v \in W^{1,p}(\Omega;\mathbb{R}^3)$ and $A \in \mathcal{A}(\omega)$ define, as in Lemma 2.1,

$$G_n(v, A) := \int_{A\times I} (1 + |D_\alpha v|^p + \lambda_n^p \, |D_3 v|^p) \, dx.$$

Let $\{u_n\}$ satisfy (2.20).

Due to the coercivity hypothesis (H1), from the sequences of measures $\nu_n := G_n(u_n,\cdot) + G_n(u,\cdot)$ we may extract a bounded subsequence $\{\nu_{n_j}\}$ converging in the sense of distributions to some Radon measure ν.

For $A \in \mathcal{A}(\omega)$ and $t > 0$, let $A_t := \{x_\alpha \in A | \operatorname{dist}(x_\alpha, \partial A) > t\}$. Fix $\eta > 0$ and for every $0 < \delta < \eta$ define the subsets $L_\delta := A_{\eta-2\delta} \setminus \overline{A_{\eta+\delta}}$. Consider a smooth cut-off function $\varphi_\delta \in \mathcal{C}_0^\infty(A_{\eta-\delta};[0,1])$ such that $\varphi_\delta(x_\alpha) = 1$ on A_η. As the thickness of the strip L_δ is of order δ, we have an upper bound of the form $\|D\varphi_\delta\|_{L^\infty(A_{\eta-\delta})} \le C/\delta$. Define

$$w_j^{\eta,\delta}(x) := u_{n_j}(x)\varphi_\delta(x_\alpha) + u(x_\alpha)(1 - \varphi_\delta(x_\alpha)).$$

We have

$$D_\alpha w_j^{\eta,\delta} = \varphi_\delta D_\alpha u_{n_j} + (1-\varphi_\delta) D_\alpha u + D_\alpha\varphi_\delta \otimes (u_{n_j} - u), \;\; D_3 w_j^{\eta,\delta} = \varphi_\delta D_3 u_{n_j}$$

and, passing to the limit in j,

$$\lim_j w_j^{\eta,\delta} = u \;\text{ in }\; L^p(A\times I;\mathbb{R}^3)\,,\;\; \lambda_{n_j} \int_I D_3 w_j^{\eta,\delta} \, dx_3 \rightharpoonup \varphi_\delta \bar{b} \;\text{ weakly in }\; L^p(A;\mathbb{R}^3). \tag{2.21}$$

By the growth condition (H1), we have the estimate

$$\begin{aligned}
&\int_{A\times I} W\left(D_\alpha w_j^{\eta,\delta} | \lambda_{n_j} D_3 w_j^{\eta,\delta}\right) dx \\
&\quad\le \int_{A\times I} W\left(D_\alpha u_{n_j} | \lambda_{n_j} D_3 u_{n_j}\right) dx + \int_{(A\setminus\overline{A_{\eta-\delta}})\times I} W\left(D_\alpha u \; |0\right) dx \\
&\qquad + C\left(G_{n_j}(u_{n_j}, L_\delta) + G_{n_j}(u, L_\delta)\right) + \frac{1}{\delta^p}\int_{L_\delta\times I} |u_{n_j} - u|^p \, dx.
\end{aligned}$$

Passing to the limit in j, we get

$$\begin{aligned}
\limsup_j \int_{A\times I} W\left(D_\alpha w_j^{\eta,\delta} | \lambda_{n_j} D_3 w_j^{\eta,\delta}\right) dx \le L + \int_{(A\setminus\overline{A_{\eta-\delta}})\times I} W\left(D_\alpha u|0\right) dx \\
+ C\nu(\overline{L}_\delta),
\end{aligned}$$

and then, passing to the limit as δ goes to zero,

$$\limsup_{\delta}\limsup_{j}\int_{A\times I} W\left(D_\alpha w_j^{\eta,\delta}|\lambda_{n_j} D_3 w_j^{\eta,\delta}\right)dx \le L + \int_{(A\setminus\overline{A_\eta})\times I} W\left(D_\alpha u|0\right)dx + C\nu(\partial A_\eta). \tag{2.22}$$

Choosing a subsequence $\eta_k \to 0^+$ such that $\nu(\partial A_{\eta_k}) = 0$, and $\delta_k < \eta_k$, we obtain, from (2.22),

$$\limsup_{k}\ \limsup_{j}\int_{A\times I} W\left(D_\alpha w_j^{\eta_k\delta_k}|\lambda_{n_j} D_3 w_j^{\eta_k\delta_k}\right)dx \le L. \tag{2.23}$$

Since the weak topology is metrizable in bounded sets and, by (H1) and (2.23), $\{\lambda_n \int_I D_3 w_j^{\eta_k\delta_k}\ dx\}$ is bounded in $L^p(A;\mathbb{R}^3)$, (2.21) and (2.23) yield the existence of subsequence $\{j_k\}$ such that $\tilde{u}_k := w_{j_k}^{\eta_k\delta_k}$ satisfies $\tilde{u}_k = u$ on a neighborhood of $\partial A\times I$, $\tilde{u}_k \to u$ in $L^p(A\times I;\mathbb{R}^3)$, $\lambda_{n_{j_k}} \displaystyle\int_I D_3\tilde{u}_k\ dx_3 \rightharpoonup \bar{b}$ weakly in $L^p(A;\mathbb{R}^3)$, and

$$\limsup_{k}\int_{A\times I} W(D_\alpha\tilde{u}_k(x)\mid \lambda_{n_{j_k}}\ D_3\tilde{u}_k(x))\ dx \le L.$$

□

Lemma 2.3. *The infimum in (2.10) remains unchanged if we replace W by its quasiconvex envelope $\mathcal{Q}W$.*

Proof. Fix $(u,\bar{b},A)\in\mathcal{V}\times L^p(\omega;\mathbb{R}^3)\times\mathcal{A}(\omega)$ and define

$$E'(u,\bar{b},A) := \inf\Big\{\liminf_{n}\int_{A\times I}\mathcal{Q}W\left(D_\alpha u_n\ \middle|\lambda_n\ D_3 u_n\right)\ dx\Big| \\ u_n \rightharpoonup u \text{ in } W^{1,p}(A\times I;\mathbb{R}^3),\ \lambda_n\int_I D_3 u_n(\cdot,x_3)\ dx_3 \rightharpoonup \bar{b},\text{ in } L^p(A;\mathbb{R}^3)\Big\}$$

Since, by the definition of quasiconvex envelope, $W \ge \mathcal{Q}W$, it is obvious that $E^-(u,\bar{b},A)\ge E'(u,\bar{b},A)$.

We prove the opposite inequality. For fixed $\delta > 0$, let $u_n \rightharpoonup u$ weakly in $W^{1,p}(A\times I;\mathbb{R}^3)$ and $\lambda_n \displaystyle\int_I D_3 u_n(x_\alpha,x_3)\ dx_3 \rightharpoonup \bar{b}$ weakly in $L^p(A;\mathbb{R}^3)$, such that

$$E'(u,\bar{b},A) \ge \lim_{n}\int_{A\times I}\mathcal{Q}W\left(D_\alpha u_n\ \middle|\lambda_n D_3 u_n\right)\ dx - \delta. \tag{2.24}$$

In view of (2.2) and by the relaxation Theorem of Acerbi and Fusco (see [AF]), for each n, there exists a sequence $\{u_{n,k}\}$ converging to u_n weakly in $W^{1,p}(A\times I;\mathbb{R}^3)$ such that

$$\int_{A\times I}\mathcal{Q}W\left(D_\alpha u_n\ \middle|\lambda_n D_3 u_n\right)\ dx = \lim_{k}\int_{A\times I} W\left(D_\alpha u_{n,k}\ \middle|\lambda_n D_3 u_{n,k}\right)\ dx. \tag{2.25}$$

¿From (2.24) and (2.25) we have

$$E'(u,\bar{b},A) \ge \lim_{n}\lim_{k}\int_{A\times I} W\left(D_\alpha u_{n,k}\ \middle|\lambda_n D_3 u_{n,k}\right)\ dx - \delta \tag{2.26}$$

with

$$\lim_{n}\lim_{k}\|u_{n,k}-u\|_{L^p(A\times I;\mathbb{R}^3)} \tag{2.27}$$

and, for the weak topology of $L^p(A;\mathbb{R}^3)$,

$$\lim_{n}\lim_{k}\int_I \lambda_n\ D_3 u_{n,k}\ dx_3 = \bar{b}. \tag{2.28}$$

In view of hypothesis (H1) and the convexity of $|\cdot|^p$ we have

$$\sup_n \sup_k \left\| \int_I \lambda_n \; D_3 u_{n,k} \; dx_3 \right\|_{L^p(A;\mathbb{R}^3)} < +\infty. \tag{2.29}$$

Since the weak topology is metrizable in bounded sets of $L^p(A;\mathbb{R}^3)$, (2.26), (2.27), (2.28) and (2.29) yield the existence of a diagonal sequence $\{u_{n,k_n}\}$ satisfying $u_{n,k_n} \to u$ in $L^p(A\times I;\mathbb{R}^3)$ (or weakly in $W^{1,p}(A\times I;\mathbb{R}^3)$, in view of the coercivity hypothesis (H1)), $\displaystyle\int_I \lambda_n \; D_3 u_{n,k_n} \; dx_3 \rightharpoonup \bar{b}$ weakly in $L^p(A;\mathbb{R}^3)$, and realizing the double limit in the right hand side of (2.26). Consequently we have

$$\begin{aligned} E'(u,\bar{b},A) &\geq \lim_n \int_{A\times I} W\left(D_\alpha u_{n,k_n} \,\Big|\, \lambda_n \; D_3 u_{n,k_n}\right) \; dx - \delta \\ &\geq E^-(u,\bar{b},A) - \delta. \end{aligned}$$

Letting δ go to zero, the conclusion follows . □

3. Proofs of the main results.

Proof of Theorem 1.2. First we notice that, by Lemma 2.3 and (1.8) we may assume with no loss of generality that W is quasiconvex. Then by (H1) (see [M]) we may also assume that W satisfies

(H2) $$|W(\xi) - W(\xi')| \leq C \; (1 + |\xi|^{p-1} + |\xi'|^{p-1})|\xi - \xi'|,$$

for certain $C > 0$ and for all $\xi, \xi' \in M^{3\times 3}$.

Now let $\{\varepsilon_k\}$ be a sequence converging to 0^+ and choose a subsequence $\{\varepsilon_{k_n}\}$ in agreement with Lemma 2.1. Recalling (2.10) we have

$$\begin{aligned} E^-(u,\bar{b}) =& E^-(u,\bar{b},\omega) \\ =& \inf\left\{\liminf_n \int_\Omega W\left(D_\alpha u_n \,|\lambda_n \; D_3 u_n\right) dx \mid u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^3),\right. \\ & \left.\lambda_n \int_I D_3 u_n \; dx_3 \rightharpoonup \bar{b}, \text{ in } L^p(\omega;\mathbb{R}^3)\right\}, \end{aligned}$$

where $\lambda_n = (\varepsilon_{k_n})^{-1}$. As we have noticed in the beginning of Subsection 2.2, in order to establish the Γ-convergence of E_ε to E it is enough to show that

$$E^- \equiv E.$$

This identity is a consequence of the Lemmas 3.1 and 3.2 below, applied to $A = \omega$. □

Lemma 3.1. *Let W be a Borel function, satisfying hypotheses* (H1) *and* (H2)*. Consider the functional E^-, defined in (2.10) for the sequence $\{\lambda_n\}$ given by Lemma 2.1, and $\mathcal{Q}^*W$ defined in (1.6). Then,*

$$E^-(u,\bar{b},A) \geq \int_A \mathcal{Q}^*W(D_\alpha u \,|\bar{b}) \; dx_\alpha, \tag{3.1}$$

for all $(u,\bar{b},A) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$.

Proof.
Step 1. We begin by proving the Lemma for $A = Q'$ and for $u(x_\alpha) := \overline{F}x_\alpha + u_0$, with $\overline{F} \in M^{3\times 2}$ and $u_0, \bar{b} \in \mathbb{R}^3$.

Assume that $E^-(u,\bar{b},Q') < +\infty$. By Lemma 2.2 we may restrict ourselves, in (2.10), to sequences having the same trace as its limit. Then consider $\{\varphi_n\}$ in $W^{1,p}(Q;\mathbb{R}^3)$ such that

$$\varphi_n = 0 \,\text{on}\, \partial Q' \times I,\ \ \varphi_n \rightharpoonup 0 \ \text{ in } W^{1,p}(Q;\mathbb{R}^3),\ \ \lambda_n \int_I D_3\varphi_n \ dx_3 \rightharpoonup \bar{b} \ \text{ in } L^p(Q';\mathbb{R}^3).$$

Define

$$\tilde{\varphi}_n(x) := \varphi_n(x) + x_3\left(\frac{\bar{b}}{\lambda_n} - \int_Q D_3\varphi_n \ dx\right).$$

By (H1) $\{\varphi_n\}$ and, consequently, $\{\tilde{\varphi}_n\}$ are bounded in $W^{1,p}(Q;\mathbb{R}^3)$, $\displaystyle\int_Q \lambda_n \ D_3\tilde{\varphi}_n \ dx = \bar{b}$, $D_\alpha\varphi_n = D_\alpha\tilde{\varphi}_n$ and $\tilde{\varphi}_n(\cdot,x_3)$ is Q'-periodic. Then,

$$\int_Q W(\overline{F} + D_\alpha\tilde{\varphi}_n|\lambda_n \ D_3\tilde{\varphi}_n) \ dx \geq \mathcal{Q}^*W(\overline{F}|b). \tag{3.2}$$

On the other hand,

$$\lim_n \|(\overline{F} + D_\alpha\varphi_n|\lambda_n \ D_3\varphi_n) - (\overline{F} + D_\alpha\tilde{\varphi}_n|\lambda_n \ D_3\tilde{\varphi}_n)\|_{L^\infty(Q;\mathbb{R}^3)} = \lim_n \left|\int_Q \lambda_n \ D_3\varphi_n \ dx - \bar{b}\right| = 0,$$

which yields, in view of (H2) and since $\{\varphi_n\}$ and $\{\tilde{\varphi}_n\}$ are bounded in $W^{1,p}(Q;\mathbb{R}^3)$,

$$\lim_n \int_Q W(\overline{F} + D_\alpha\tilde{\varphi}_n|\lambda_n \ D_3\tilde{\varphi}_n) \ dx = \lim_n \int_Q W(\overline{F} + D_\alpha\varphi_n|\lambda_n \ D_3\varphi_n) \ dx. \tag{3.3}$$

¿From (3.2) and (3.3) we conclude that

$$\liminf_n \int_Q W(\overline{F} + D_\alpha\varphi_n|\lambda_n \ D_3\varphi_n) \ dx \geq \mathcal{Q}^*W(\overline{F}|b). \tag{3.4}$$

We complete the proof of step 1 by taking the infimum over all the admissible sequences $\{\varphi_n\}$ in (3.4).

Step 2. Now we establish the general case.

Fix $(u,\bar{b},A) \in \mathcal{V} \times L^p(\omega,\mathbb{R}^3) \times \mathcal{A}(\omega)$. Let $\{u_n\}$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(A \times I;\mathbb{R}^3)$, $\lambda_n \int_I D_3u_n \ dx_3 \rightharpoonup \bar{b}$ weakly in $L^p(A;\mathbb{R}^3)$, and

$$E(u,\bar{b},A) = \lim_n \int_{A\times I} W(D_\alpha u_n \ |\lambda_n \ D_3u_n) \ dx.$$

Define the sequence of measures $\mu_n := \left(\displaystyle\int_I W(D_\alpha u_n \ |\lambda_n \ D_3u_n) \ dx_3\right)\mathcal{L}^2\lfloor A$. Since $\{\mu_n\}$ is bounded, it converges weakly-$\star$, up to a subsequence, to some measure μ. Represent by ρ the absolutely continuous part of μ with respect to the 2-dimensional Lebesgue measure. To prove (3.1) it suffices to show that, for a.e. $x_0 \in A$,

$$\rho(x_0) \geq \mathcal{Q}^*W(D_\alpha u(x_0),\bar{b}(x_0)) \tag{3.5}$$

It is known that, for a.e. $x_0 \in A$,

$$\rho(x_0) = \lim_{\varepsilon\to 0}\frac{\mu(x_0+\varepsilon Q')}{\varepsilon^2}, \tag{3.6}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{p+2}}\int_{x_0+\varepsilon Q'} |u(x) - u(x_0) - \nabla u(x_0)(x-x_0)|^p \ dx_\alpha = 0, \tag{3.7}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{x_0+\varepsilon Q'} |\bar{b}(x) - \bar{b}(x_0)|^p \ dx_\alpha = 0. \tag{3.8}$$

Let x_0 satisfy (3.6), (3.7) and (3.8). Let $\{\varepsilon\}$ represent a sequence converging to zero such that, for all ε,

$$\mu(\partial(x_0 + \varepsilon Q')) = 0. \tag{3.9}$$

Using (3.6), the definition of μ and (3.9), we have

$$\begin{aligned} \rho(x_0) &= \lim_{\varepsilon\to 0}\lim_n \frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I} W(D_\alpha u_n \,|\lambda_n\, D_3 u_n)\, dx \\ &= \lim_{\varepsilon\to 0}\lim_n \int_Q W(D_\alpha u_n(x_0+\varepsilon y_\alpha, y_3)\,|\lambda_n\, D_3 u_n(x_0+\varepsilon y_\alpha, y_3))\, dy \\ &= \lim_{\varepsilon\to 0}\lim_n \int_Q W(D_\alpha u_{n,\varepsilon}\,|\lambda_n\, D_3 u_{n,\varepsilon})\, dy, \end{aligned} \tag{3.10}$$

where $u_{n,\varepsilon}(y) := \dfrac{u_n(x_0+\varepsilon y_\alpha, y_3) - u(x_0)}{\varepsilon}$.

Since $u_n \to u$ in $L^p(A\times I;\mathbb{R}^3)$, (3.7) yields

$$\lim_{\varepsilon\to 0}\lim_n \|u_{n,\varepsilon}(\cdot) - \nabla u(x_0)\cdot\|^p_{L^p(Q;\mathbb{R}^3)} = 0. \tag{3.11}$$

We also have for $\varphi \in L^{p'}(Q';\mathbb{R}^3)$, and as $\lambda_n \int_I D_3 u_n\, dx_3 \rightharpoonup \bar b$ weakly in $L^p(A;\mathbb{R}^3)$,

$$\begin{aligned} \lim_{\varepsilon\to 0}\lim_n \int_Q \lambda_n\, D_3 u_{n,\varepsilon}(y)\,\varphi(y)\, dy &= \lim_{\varepsilon\to 0}\lim_n \frac{1}{\varepsilon^2}\int_{(x_0+\varepsilon Q')\times I} \lambda_n D_3 u_n(x)\varphi\left(\frac{x_\alpha - x_0}{\varepsilon}\right) dx \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{x_0+\varepsilon Q'} \bar b(x_\alpha)\,\varphi\left(\frac{x_\alpha - x_0}{\varepsilon}\right)\, dx_\alpha \\ &= b(x_0)\int_{Q'}\varphi(y_\alpha)\, dy_\alpha, \end{aligned} \tag{3.12}$$

where we have used (3.8).

By means of a standard diagonalization process, from (3.10), (3.11) and (3.12), we construct $\tilde u_k := u_{\varepsilon_k, n_k}$ and λ_{n_k} such that

$$\tilde u_k(y) \to D_\alpha u(x_0) y \text{ in } L^p(Q;\mathbb{R}^3), \qquad \lambda_{n_k}\int_I D_3\tilde u_k\, dy_3 \rightharpoonup \bar b(x_0) \text{ weakly in } L^p(Q';\mathbb{R}^3)$$

and

$$\rho(x_0) = \lim_k \int_Q W(D_\alpha \tilde u_k\,|\lambda_{n_k}\, D_3\tilde u_k)\, dy. \tag{3.13}$$

Since

$$\lim_k \int_Q W(D_\alpha \tilde u_k\,|\lambda_{n_k}\, D_3 \tilde u_k)\, dy \geq E^-(D_\alpha u(x_0) y, \bar b(x_0), Q'),$$

(3.5) follows from (3.13) and Step 1. □

Lemma 3.2. *Under the hypotheses of Lemma 3.1, we have*

$$E^-(u,\bar b, A) \leq \int_A \mathcal{Q}^* W(D_\alpha u\,|\bar b)\, dx_\alpha \tag{3.14}$$

for all $(u,\bar b,A)\in\mathcal V\times L^p(\omega;\mathbb R^3)\times\mathcal A(\omega)$.

Proof. In view of Proposition 1.1 iii) and of Lemma 2.3, without loss of generality, we may assume in (3.14) that the density W is a quasiconvex function. Therefore, by (H1) W is p-Lipschitz which easily entails the continuity of the function

$$W^{(\lambda)}(\overline F\ |b):=\inf_{\substack{\varphi\in W^{1,p}(Q;\mathbb R^3)\\ \varphi\text{ is }Q'-\text{periodic}}}\left\{\int_Q W(\overline F+D_\alpha\varphi|\lambda\ D_3\varphi)\ dx:\lambda\int_Q D_3\varphi\ dx=b\right\},$$

where $\lambda\in\mathbb R$ and $(\overline F,b)\in M^{3\times2}\times\mathbb R$. Note that

$$\mathcal Q^*W(\overline F\mid b)=\inf_{\lambda\in\mathbb R}W^{(\lambda)}(\overline F\ |b).\tag{3.15}$$

We claim that for each fixed λ and for all $(u,\bar b,A)\in W^{1,p}(\omega;\mathbb R^3)\times L^p(\omega;\mathbb R^3)\times\mathcal A(\omega)$,

$$E^-(u,\bar b,A)\le\int_A W^{(\lambda)}(D_\alpha u\mid\bar b)\ dx_\alpha.\tag{3.16}$$

Suppose that (3.16) holds. Since, by Lemma 2.1, $E^-(u,\bar b,\cdot)$ is a measure absolutely continuous with respect to the 2-dimensional Lebesgue measure, we can write $E^-(u,\bar b,\cdot)=\rho\mathcal L^2\lfloor\omega$ for some function $\rho\in L^1(\omega)$. Let $x_0\in\omega$ be a Lebesgue point for ρ, $D_\alpha u$ and $\bar b$. Using (3.15), we may find a sequence $\{\lambda_j\}$ such that

$$Q^*W(D_\alpha u(x_0)|\bar b(x_0))=\inf_j W^{(\lambda_j)}(D_\alpha u(x_0)|\bar b(x_0)).$$

Now (3.16) yields

$$\rho(x_\alpha)\le W^{(\lambda_j)}(D_\alpha u(x_\alpha)\mid\bar b(x_\alpha))$$

for $\mathcal L^2$ a.e. $x_\alpha\in\omega$ and for all $j\in\mathbb N$, and in turn we conclude that

$$\rho(x_0)\le\mathcal Q^*W(\overline F|\bar b)(x_0).$$

Given the arbitrariness of x_0, we deduce that

$$E^-(u,\bar b,A)=\int_A\rho(x_\alpha)\ dx_\alpha\le\int_A\mathcal Q^*W(D_\alpha u(x_\alpha)\mid\bar b(x_\alpha))\ dx_\alpha,$$

which establishes (3.14).

We now prove claim (3.16) in three steps.

Step 1. First we consider the case where u is affine, $u=\overline F x_\alpha$, and $\bar b$ constant.

Let φ be admissible for $W^{(\lambda)}(\overline F|\bar b)$. Using the Q'-periodicity of φ, we define $\varphi_n:\mathbb R^2\times I\to\mathbb R^3$ by $\varphi_n(x):=\dfrac{\lambda}{\lambda_n}\varphi\left(\dfrac{\lambda_n}{\lambda}x_\alpha,x_3\right)$, where $\{\lambda_n\}$, $\lambda_n\to+\infty$, is the sequence given by Lemma 2.1.

We have $\varphi_n\in W^{1,p}(A\times I;\mathbb R^3)$, $\varphi_n\to0$ in $L^p(A\times I;\mathbb R^3)$ and, by the Riemann-Lebesgue Lemma,

$$\lambda_n\int_I D_3\varphi_n\ dx_3=\lambda\int_I D_3\varphi\left(\frac{\lambda_n}{\lambda}x_\alpha,x_3\right)dx_3\rightharpoonup\lambda\int_I\int_{Q'}D_3\varphi(y_\alpha,x_3)\ dy_\alpha\ dx_3=\bar b,$$

$$\begin{aligned}E^-(u,\bar b,A)&\le\lim_n\int_{A\times I}W(\overline F+D_\alpha\varphi_n\mid\lambda_nD_3\varphi_n)\ dx\\&=\lim_n\int_A\int_I W\left(\overline F+D_\alpha\varphi\left(\frac{\lambda_n}{\lambda}x_\alpha,x_3\right)\ \Big|\ \lambda D_3\varphi\left(\frac{\lambda_n}{\lambda}x_\alpha,x_3\right)\right)dx_3\ dx_\alpha\\&=\int_A\int_{Q'}\int_I W(\overline F+D_\alpha\varphi(y_\alpha,x_3)\mid\lambda D_3\varphi(y_\alpha,x_3))\ dx_3\ dy_\alpha\ dx_\alpha\\&=\mathcal L^2(A)\int_Q W(\overline F+D_\alpha\varphi\mid\lambda D_3\varphi)\ dx.\end{aligned}$$

Taking the infimum over all admissible φ , we obtain

$$E^-(u,\bar{b},A) \le \mathcal{L}^2(A) W^{(\lambda)}(\overline{F} \mid \bar{b}) = \int_A W^{(\lambda)}(\overline{F} \mid \bar{b})\; dx_\alpha.$$

Step 2. We prove the claim for u piecewise affine and $\bar{b}$ piecewise constant.

Let $\{A_j\}_{j=1,\cdots,k}$ be a finite and measurable partition of A, such that u and $\bar{b}$ are affine and constant, respectively, on each A_j, $j = 1, \cdots, k$. By *Step 1* we have, for all $j = 1 \cdots, k$,

$$E^-(u,\bar{b},A_j) \le \int_{A_j} W^{(\lambda)}(D_\alpha u \mid \bar{b})\; dx_\alpha.$$

By Lemma 2.1 $E^-(u,\bar{b},\cdot)$ is a measure, thus

$$E^-(u,\bar{b},A) = \sum_{j=1}^{k} E^-(u,\bar{b},A_j) \le \sum_{j=1}^{k} \int_{A_j} W^{(\lambda)}(D_\alpha u \mid \bar{b})\; dx_\alpha = \int_A W^{(\lambda)}(D_\alpha u \mid \bar{b})\; dx_\alpha.$$

Step 3. We prove the claim for an arbitrary $(u,\bar{b},A) \in W^{1,p}(\omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3) \times \mathcal{A}(\omega)$.

Let $\{(u_n,\bar{b}_n)\}$ be a sequence such that u_n are piecewise affine functions, $\bar{b}_n$ are constants and $\{(u_n,\bar{b}_n)\}$ converges to $(u,\bar{b})$ in $W^{1,p}(A;\mathbb{R}^3) \times L^p(A;\mathbb{R}^3)$. Since $E^-(\cdot,\cdot,A)$ is lower semicontinuous, we have

$$E^-(u,\bar{b},A) \le \liminf_n E^-(u_n,\bar{b}_n,A) \le \liminf_n \int_A W^{(\lambda)}(D_\alpha u_n \mid \bar{b}_n)\; dx_\alpha. \tag{3.17}$$

Since $(\overline{F} \mid \bar{b}) \mapsto W^{(\lambda)}(\overline{F} \mid \bar{b})$ is continuous and satisfies p-growth conditions, we have

$$\lim_n \int_A W^{(\lambda)}(D_\alpha u_n \mid \bar{b}_n)\; dx_\alpha = \int_A W^{(\lambda)}(D_\alpha u \mid \bar{b})\; dx_\alpha,$$

which, together with (3.17), concludes the proof of (3.16). □

Proof of Corollary 1.3. Consider the problem $(\mathcal{P}_\varepsilon)$. Taking $u_{0,\varepsilon}$ as an admissible test function, we see that the infimum of $(\mathcal{P}_\varepsilon)$ is uniformly bounded in ε, so any sequence of minimizers $\{u_\varepsilon\}$ will satisfy

$$\sup_\varepsilon \int_\Omega W\left(D_\alpha u_\varepsilon \,\Big|\, \frac{1}{\varepsilon} D_3 u_\varepsilon\right)\; dx - L_\varepsilon(u_\varepsilon) < +\infty \;\text{ and }\; u_\varepsilon = u_{0,\varepsilon} \;\text{ on } \Gamma.$$

By the coercivity assumption in (H_1) we have

$$\sup_\varepsilon \left\{ \int_\Omega |D_\alpha u_\varepsilon|^p dx + \int_\Omega \frac{1}{\varepsilon^p} |D_3 u_3|^p dx \right\} < +\infty.$$

Therefore $\{u_\varepsilon\}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $D_3 u_\varepsilon \to 0$ in $L^p(\Omega;\mathbb{R}^3)$, as $\varepsilon \to 0$. On the other hand, $\{b_\varepsilon\}$, $b_\varepsilon := \dfrac{1}{\varepsilon} D_3 u_\varepsilon$, is bounded in $L^p(\Omega;\mathbb{R}^3)$, therefore we conclude that the sequence $\{(u_\varepsilon, b_\varepsilon)\}$ is weakly compact in $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\Omega;\mathbb{R}^3)$, and all cluster points (u,b) belong to $\mathcal{V} \times L^p(\Omega;\mathbb{R}^3)$. Since $u_{0,\varepsilon}|_\Gamma$ converges uniformly to $u_0|_\Gamma$, $u_0(x) := (x_\alpha, 0)$, the limit u satisfies $u = u_0$ on Γ. Also, with $\bar{b} := \int_I b(x_\alpha, x_3)\; dx_3$, it follows that $\bar{b}_\varepsilon := \displaystyle\int_I \frac{1}{\varepsilon} D_3 u_\varepsilon dx_3 \rightharpoonup \bar{b}$ weakly in $L^p(\omega;\mathbb{R}^3)$ and $L_\varepsilon(u_\varepsilon) \to L(u,\bar{b})$, as $\varepsilon \to 0$.

Introduce the functionals Φ_ε and Φ given by

$$\Phi_\varepsilon(u,\bar{b}) := \begin{cases} E_\varepsilon(u,\bar{b}) - L_\varepsilon(u) & \text{if } u = u_{0,\varepsilon} \text{ on } \Gamma, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\Phi(u,\bar b) := \begin{cases} \int_\omega \mathcal{Q}^* W(D_\alpha u\,|\bar b)\,dx_\alpha - L(u,\bar b) & \text{if } u = u_0 \text{ on } \Gamma, \\ +\infty & \text{otherwise.} \end{cases}$$

We will show that Φ_ε Γ-converges to Φ, therefore any cluster point of a minimizing sequence of $(\mathcal{P}_\varepsilon)$ will be a solution of $(\mathcal{P})$ (see [DM]), and this will complete the proof of Corollary 1.3. By the Kuratowski's Compactness Theorem (see [DM]), it is enough to prove that the Γ-lower limit of any subsequence of Φ_ε coincides with Φ.

Let $(u,\bar b) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3)$ and $(u_\varepsilon, \bar b_\varepsilon) \rightharpoonup (u,\bar b)$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$. If $\liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon,\bar b_\varepsilon) < +\infty$, then $u_\varepsilon = u_{0,\varepsilon}$ on Γ for infinitely many ε. Since $u_\varepsilon|_\Gamma \to u|_\Gamma$ weakly in $L^p(\Gamma;\mathbb{R}^3)$ and $u_{0,\varepsilon}|_\Gamma \to u_0|_\Gamma$ uniformly, we have $u|_\Gamma = u_0|_\Gamma$ and, in view of Theorem 1.2, we deduce that

$$\liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon,\bar b_\varepsilon) \geq \Phi(u,\bar b). \tag{3.18}$$

To complete the proof we need to construct, for each $(u,\bar b) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3)$ such that $u = u_0$ on Γ, a sequence $(\bar u_\varepsilon, \bar b_\varepsilon) \rightharpoonup (u,\bar b)$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3) \times L^p(\omega;\mathbb{R}^3)$, satisfying

$$\liminf_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon,\bar b_\varepsilon) = \Phi(u,\bar b),$$

or, equivalently, since $\lim_{\varepsilon\to 0} L_\varepsilon(u_\varepsilon) = L(u,\bar b)$ will trivially hold, we must find $(u_\varepsilon,\bar b_\varepsilon)$ such that

$$\liminf_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon,\bar b_\varepsilon) = \int_A \mathcal{Q}^* W(D_\alpha u\,|\bar b)\,dx_\alpha\ , \quad u_\varepsilon = u_{0,\varepsilon} \text{ on } \Gamma. \tag{3.19}$$

Let $(u,\bar b) \in \mathcal{V} \times L^p(\omega;\mathbb{R}^3)$ such that $u = u_0$ on Γ. In view of Theorem 1.2, there exists a sequence $\{u_\varepsilon\}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\displaystyle\int_I \frac{1}{\varepsilon} D_3 u_\varepsilon dx_3 \rightharpoonup \bar b$ weakly in $L^p(\omega;\mathbb{R}^3)$ and

$$\int_A \mathcal{Q}^* W(D_\alpha u\,|\bar b)\,dx_\alpha = \lim_n \int_\Omega W\left(D_\alpha u_\varepsilon\,\left|\frac{1}{\varepsilon} D_3 u_\varepsilon\right.\right)\,dx. \tag{3.20}$$

In order to obtain the boundary conditions in (3.20), we associate to each deformation u_ε the displacement $v_\varepsilon := u_\varepsilon - u_{0,\varepsilon}$. We then rewrite (3.20) as

$$\int_A \mathcal{Q}^* W(D_\alpha u\,|\bar b)\,dx_\alpha = \lim_n \int_\Omega W\left(e_\alpha + D_\alpha v_\varepsilon\,\left|e_3 + \frac{1}{\varepsilon} D_3 v_\varepsilon\right.\right)\,dx, \tag{3.21}$$

where $v_\varepsilon \rightharpoonup u - u_0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$ and $\displaystyle\int_I \frac{1}{\varepsilon} D_3 v_\varepsilon \rightharpoonup \bar b - e_3$ weakly in $L^p(\omega;\mathbb{R}^3)$. By Lemma 2.2, applied to $\widetilde{W}(\xi) := W(\mathbb{I} + \xi)$ and to $\{v_\varepsilon\}$, there exist two subsequences, $\varepsilon_k \to 0$ and $\{\overline{v}_k\}$ in $W^{1,p}(\Omega;\mathbb{R}^3)$, satisfying $\overline{v}_k = 0$ on Γ, $\overline{v}_k \rightharpoonup u - u_0$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$, $\displaystyle\int_I \frac{1}{\varepsilon_k} D_3 \overline{v}_k\,dx_3 \rightharpoonup \bar b - e_3$ weakly in $L^p(\omega;\mathbb{R}^3)$ and

$$\liminf_k \int_\Omega W\left(e_\alpha + D_\alpha \overline{v}_k\,\left|e_3 + \frac{1}{\varepsilon_k} D_3 \overline{v}_k\right.\right)\,dx \leq \lim_n \int_\Omega W\left(e_\alpha + D_\alpha v_\varepsilon\,\left|e_3 + \frac{1}{\varepsilon} D_3 v_\varepsilon\right.\right)\,dx.$$

Then, defining $\overline{u}_k := \overline{v}_k + u_{0\varepsilon_k}$, from (3.18) and (3.21), we conclude (3.19). $\square$

Acknowledgments. The research of G. Bouchitté was partially supported by ICCTI-MENESR 166-B2, the research of I. Fonseca by the National Science Foundation under Grants No. DMS–9731957, DMS–0103798, and the research of M.L. Mascarenhas by ICCTI-PRAXIS XXI, FEDER-PRAXIS/2/2.1/MAT/125/94, PRAXIS-FEDER/3/3.1/CTM/10/94 and ICCTI-MENESR 166-B2. The authors acknowledge the hospitality of the Universities of Toulon (ANLA), Carnegie Mellon and its Center for Nonlinear Analysis (CNA, Grant No. DMS–9803791), and Lisbon (CMAF), where this work was undertaken.

References.

[AF] Acerbi, E. & N. Fusco, Semicontinuity problems in the Calculus of Variations. *Arch. Rat. Mech. Anal.* **86** (1984), 125–145.

[ABP] Anzellotti, G. & S. Baldo & D. Percivale, Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity. *Asymptotic Anal.* **9** (1994), 61–100.

[BB] Bhattacharya, K. & A. Braides, Thin films with many small cracks. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.***458** (2002), 823–840.

[BFF] Bhattacharya, K. & I. Fonseca & G. Francfort, An asymptotic study of the debonding of thin films. *Arch. Rat. Mech. Anal.* **161** (2002), 205–229.

[BJ] Bhattacharya, K. & R. D. James, A theory of thin films of martensitic materials with applications to microactuators. *J. Mech. Phys. Solids* **47** (1999), 531–576.

[BF] Braides, A.& I. Fonseca,*Appl. Math. Optim. 44* **3** (2001), 299–323.

[BrFF] Braides, A. & I. Fonseca & G. Francfort, 3D-2D asymptotic analysis for inhomogeneous thin films. *Indiana Univ. Math. J.* **49** (2000), 1367–1404.

[CV] Castaing, M. & M. Valadier, *Convex Analysis and Measurable Multifunctions.* Sringer-Verlag, Lecture Notes in Math. **580**, Berlin, 1977.

[DM] Dal Maso, G., *An Introduction to Γ-Convergence.* Birkhäuser, Boston, 1993.

[FF] Fonseca, I. & G. Francfort, On the inadequacy of the scaling of linear elasticity for 3D-2D asymptotics in a nonlinear setting. *J. Math. Pures Appl.* **(9) 80** (2001), 547–562.

[FKP] Fonseca, I. & D. Kindherlerer & P. Pedregal, Energy functionals depending on elastic strain and chemical composition. *Calc. Var. Partial Differential Equations* **2** (1994), 283–13.

[FM] Fonseca, I. & S. Müller, Relaxation of quasiconvex functionals in $\mathrm{BV}(\Omega, \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Rational Mech. Anal.* **123** (1993), 1–49.

[FMJ] Friesecke, G. & S. Müller & R.D. James Rigorous derivation of nonlinear plate theory and geometric rigidity. *C. R. Math. Acad. Sci. Paris* **334** (2002), 173–178.

[KoS] Kohn, R and Strang, Optimal design and relaxation of variational problems I, II, III. *Comm. Pure Appl. Math.* **39** (1986), 113–137, 139–182, 353–377.

[LDR1] Le Dret, H. & A. Raoult, The nonlinear membrane model as a variational limit of nonlinear three-dimensional elasticity. *J. Math. Pures Appl.* **74** (1995), 549–578.

[LDR2] Le Dret, H. & A. Raoult, Variational convergence for nonlinear shell models with directors and related semicontinuity and relaxation results.*Arch. Ration. Mech. Anal.* **154** (2000), 101–134.

[M] Marcellini, Paolo. Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. Manuscripta Math. 51 (1985), no. 1-3, 1–28.