# Multiscale Relaxation of Convex Functionals

Irene Fonseca
Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh, PA 15213, U.S.A.

Elvira Zappale
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata,
Università degli Studi di Salerno,
Fisciano (Sa) 84084, Italy.

January 25, 2002

**Abstract**

The $\Gamma$-limit of a family of functionals

$$u \mapsto \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D^s u\right)\, dx$$

is obtained for $s = 1, 2$ and when the integrand $f = f(x, y, v)$ is a continuous function, periodic in $x$ and $y$, and convex with respect to $v$. The 3-scale limits of second order derivatives are characterized.

*Keywords*: convexity, periodicity, multiscale limits, $\Gamma$-convergence, $\mathcal{A}$-quasiconvexity, measurability selection criterion
*2000 Mathematics Subject Classification*: 35G99, 49J40, 49J45, 74G65

## 1 Introduction

The method of two-scale convergence was introduced by Allaire [A] and Nguetseng [N1] to render rigorous formal asymptotics techniques commonly used by practitioners in the theory of homogenization. Although it is well understood that periodicity poses severe constraints on physical realistic models, it is also agreed that the understanding of the effective behavior of periodically structured composite materials may shed light into the more complex and mathematically challenging media. For this reason, the theory of two-scale convergence, as well as its sequel, the theory of (n+1)-scale convergence developed by Allaire and Briane [ABr], have played a very important role in the theory of PDEs and its applications in homogenization (see also [N1]).

We may say that it all started with the result below, first established by Nguetseng in [N1], and subsequently generalized by Allaire and recasted with a new proof (see [A], Theorem 0.1).

**Proposition 1.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$, and let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $1 \le p < +\infty$. There exist a subsequence (not relabelled) and a function $u_0(x, y) \in L^p(\Omega \times Q)$ such that*

$$\lim_{\varepsilon \to 0^+} \int_\Omega \psi\left(x, \frac{x}{\varepsilon}\right)\, u_\varepsilon(x)\, dx = \int_\Omega \int_Q \psi(x, y)\, u_0(x, y)\, dx\, dy \tag{1.1}$$

*for any smooth function $\psi(x, y)$ which is $Q-$periodic in $y$. The (sub)sequence $\{u_\varepsilon\}$ is said to* two-scale converge *to $u_0(x, y)$.*

Here, and in what follows, $Q := (0, 1)^N$, and $Y_i$ are copies of $Q$ for $i = 1, \ldots, n$, $n \in \mathbb{N}$. In [A], and drawing from (1.1), Allaire introduced the concept of *2-scale convergence*, which later he extended to *(n+1)-scale convergence* ($n \in \mathbb{N}$) in joint work with Briane [ABr].

**Definition 1.2.** *A bounded sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ is said to $(n+1)-$*scale converge *to a function $u_0 \in L^p(\Omega \times Y_1 \times \ldots \times Y_n)$, $1 \leq p < +\infty$, if*

$$\lim_{\varepsilon\to 0^+} \int_\Omega \varphi\left(x, \frac{x}{\varepsilon}, \ldots, \frac{x}{\varepsilon^n}\right) u_\varepsilon(x)\, dx = \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \varphi(x, y_1, \ldots, y_n) u_0(x, y_1, \ldots, y_n)\, dx\, dy_1 \ldots dy_n, \tag{1.2}$$

*for every $\varphi \in L^{p'}(\Omega; C_{\rm per}(Y_1 \times \ldots Y_n))$, $p' := \frac{p}{p-1}$, and we write*

$$u_\varepsilon \overset{\text{(n+1)-s}}{\rightharpoonup} u_0.$$

**Remark 1.3.** As shown in Corollary 5.4 in [A], admissible test functions $\varphi$ may also be taken to be measurable in the oscillating variable and continuous with respect to $x$, precisely, writing $\bar{\varphi}(y,x) := \varphi(x,y)$ then $\bar{\varphi} \in L^{p'}_{\rm per}(Y_1 \times \ldots Y_n; C(\overline{\Omega}))$.

With the Definition 1.2 in hand, Allaire and Briane in [ABr] extended Proposition 1.1 to read

**Proposition 1.4.** *Let $\Omega$ be an open subset of $\mathbb{R}^N$, and let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $1 \leq p < +\infty$. There exist a subsequence (not relabelled) and a function $u_0 \in L^p(\Omega \times Y_1 \times \ldots \times Y_n)$ such that $\{u_\varepsilon\}$ $(n+1)-$scale converges to $u_0$.*

**Remark 1.5.** The original result in [ABr] was stated for $L^2$ but it can be generalized easily to $L^p$, $1 \leq p < +\infty$. Note that if the sequence $\{u_\varepsilon\}$ is bounded in $L^\infty(\Omega)$, then $u_0 \in L^\infty(\Omega \times Y_1 \times \ldots \times Y_n)$. Indeed, extracting a further subsequence (not relabelled) such that, for all $k \in \mathbb{N}$, $\{u_\varepsilon\}$ $(n+1)-$scale converges to $u_k$ in $L^p(\Omega \times Y_1 \times \ldots \times Y_n)$, it follows that $u_k = u_0$ for all $k$ and (see [A], Proposition 1.6)

$$\|u_0\|_{L^\infty(\Omega\times Y_1\times\ldots\times Y_n)} = \lim_{k\to\infty} \|u_0\|_{L^p(\Omega\times Y_1\times\ldots\times Y)} \leq \liminf_{k\to\infty} \limsup_{\varepsilon\to 0^+} \|u_\varepsilon\|_{L^p(\Omega)} \leq \sup_{\varepsilon>0} \|u_\varepsilon\|_{L^\infty(\Omega)} < +\infty.$$

For this reason, a bounded sequence $\{u_\varepsilon\} \subset L^\infty(\Omega)$ is said to $(n+1)$*-scale converge to* $u_0 \in L^\infty(\Omega \times Y_1 \times \ldots \times Y_n)$ if (1.2) holds for every $\varphi \in L^1(\Omega; C_{\rm per}(Y_1 \times \ldots Y_n))$.

A class of functions for which it is easy to identify the $(n+1)$-scale limit is that of constant functions. We start by recalling the well known Riemann-Lebesgue Lemma.

**Lemma 1.6.** *Let $w \in L^p_{\rm per}(Q; \mathbb{R}^d), 1 \leq p < +\infty$, and let $w_\varepsilon(x) := w\left(\frac{x}{\varepsilon}\right)$, $\varepsilon > 0$. If $E \subset \mathbb{R}^N$ is a measurable set then*

$$w_\varepsilon \rightharpoonup \int_Q w(y)\, dy \text{ in } L^p_{\rm loc}(E, \mathbb{R}^d) \ (\overset{\star}{\rightharpoonup} \text{ if } p = \infty). \tag{1.3}$$

Donato [Do] (see also [AHZ]) has extended (1.3) to the realm of multiscale convergence by showing that

$$\int_\Omega \varphi\left(x, \frac{x}{\varepsilon}, \ldots \frac{x}{\varepsilon^n}\right) dx \to \int_\Omega \int_{Y_1} \int_{Y_n} \varphi(x, y_1, \ldots, y_n)\, dx\, dy_1 \ldots dy_n \tag{1.4}$$

whenever $\varphi \in L^{p'}(\Omega; C_{\rm per}(Y_1 \times \cdots \times Y_n))$. In particular, a sequence identically constant $(n+1)-$scale converges to the constant itself.

Also, Allaire and Briane have characterized fully the $(n+1)$-scale limits of gradients in the theorem below (see [ABr], Thm 2.6).

**Theorem 1.7.** *For any bounded sequence $\{u_\varepsilon\} \subset W^{1,p}(\Omega)$ there exist $u \in W^{1,p}(\Omega)$, $u_1 \in L^p(\Omega; W^{1,p}_{\rm per}(Y_1))$ and $n$ functions $u_k \in L^p(\Omega\times Y_1 \times \ldots \times Y_{n-1}; W^{1,p}_{\rm per}(Y_n))$, $k = 1, \ldots, n$, such that, up to a subsequence,*

$$u_\varepsilon \overset{\text{(n+1)-s}}{\rightharpoonup} u, \tag{1.5}$$

$$Du_\varepsilon \overset{\text{(n+1)-s}}{\rightharpoonup} Du(x) + \sum_{k=1}^n D_{y_k} u_k(x, y_1, \ldots, y_n). \tag{1.6}$$

*Moreover, any $(n+1)-$tuple limit $(u, u_1, \ldots, u_n) \in W^{1,p}(\Omega) \times L^p(\Omega; W^{1,p}_{\rm per}(Y_1)) \times \cdots \times \times L^p(\Omega\times Y_1\times \ldots \times Y_{n-1}; W^{1,p}_{\rm per}(Y_n))$ may be $(n+1)-$scale attained, i.e. there exists a bounded sequence $\{u_\varepsilon\}$ in $W^{1,p}(\Omega; \mathbb{R}^d)$ for which (1.5) and (1.6) hold.*

Using the notion of 2-scale convergence Allaire provided a simple and elegant proof for the "homogenized" Γ-limit of a functional (see [A] Theorems 3.1 and 3.3; see also [Ma1])

$$I_\varepsilon(u) := \int_\Omega W\left(\frac{x}{\varepsilon}, Du\right)\, dx.$$

Precisely,

**Theorem 1.8.** *Let $W : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be an integrand satisfying*

(i) *$W(\cdot,\xi)$ is measurable and $Q$-periodic for all $\xi \in \mathbb{R}^{d\times N}$;*

(ii) *$W(x,\cdot)$ is convex and $C^1$ for a.e. $x \in \mathbb{R}^N$;*

(iii) *$\frac{1}{C}|\xi|^p \le W(x,\xi) \le C(1+|\xi|^p)$ for some $p>1, C>0$, for all $\xi \in \mathbb{R}^N$ and for a.e. $x \in \mathbb{R}^N$.*

*Then*

$$\Gamma(L^p(\Omega)) - \lim_{\varepsilon\to+0} I_\varepsilon(u) = \int_\Omega W_{\text{hom}}(Du)dx$$

*for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, where*

$$W_{\text{hom}}(\xi) := \inf\left\{\int_Q W(y,\xi + D\varphi(y))\,dy : \varphi \in W^{1,p}_{\text{per}}(Q;\mathbb{R}^d)\right\}.$$

Note that in Theorems 3.1 and 3.3 in [A] Allaire assumed further a growth condition on $\frac{\partial W}{\partial \xi}(x,\xi)$ (see (4.8)), but this latter hypothesis is superfluous in view (iii) and the convexity of $W(x,\cdot)$.

It is our aim in this paper to extend Theorem 1.8 to multiscale, higher order, convex, and periodic variational problems. As before, let $\Omega$ be an open, bounded subset of $\mathbb{R}^N$, let $s \in \{1,2\}$ and consider a function

$$f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^s_* \to [0,+\infty)$$

where $\mathbb{R}^1_* := \mathbb{R}^{d\times N}$ and $\mathbb{R}^2_* := (\text{Sym}(\mathbb{R}^N,\mathbb{R}^N))^d$, with $\text{Sym}(\mathbb{R}^N,\mathbb{R}^N)$ being the space of all linear symmetric transformations from $\mathbb{R}^N$ onto $\mathbb{R}^N$.

We seek to characterize the asymptotic behavior as $\varepsilon \to 0^+$ of functionals

$$u \in W^{s,p}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D^s u(x)\right) dx$$

by providing an integral representation for the effective energy.

Our main result is the following:

**Theorem 1.9.** *Let $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^s_* \to [0,+\infty)$ be an integrand satisfying the assumptions:*

(H1) *$f$ is continuous in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^s_*$;*

(H2) *$f$ is separately $Q-$periodic in $x$ and $y$;*

(H3) *$f(x,y,\cdot)$ is convex for all $x,y \in \mathbb{R}^N$;*

(H4) *$\frac{1}{C}|\xi|^p \le f(x,y,\xi) \le C(1+|\xi|^p)$ for some $p>1, C>0$ and for all $(x,y,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^s_*$.*

*Then*

$$\Gamma(L^p(\Omega)) - \lim_{\varepsilon\to0^+} J_\varepsilon(u,\Omega) = \int_\Omega \overline{f^s_{\text{hom}}}(D^s u)dx \tag{1.7}$$

*for every $u \in W^{s,p}(\Omega;\mathbb{R}^d)$, where*

$$\overline{f^s_{\text{hom}}}(\xi) := \inf\left\{\int_Q f^s_{\text{hom}}(x,\xi + D^s\varphi(x))\,dx : \varphi \in W^{s,p}_{\text{per}}(Q;\mathbb{R}^d)\right\} \tag{1.8}$$

*and*

$$f^s_{\mathrm{hom}}(x,\xi)=\inf\left\{\int_Q f(x,y,\xi+D^s\psi(y))\,dy : \psi\in W^{s,p}_{\mathrm{per}}(Q;\mathbb{R}^d)\right\}. \tag{1.9}$$

**Remark 1.10.** (i) It is not clear what is the natural regularity that $f=f(x,y,\xi)$ should satisfy with respect to $x$ and $y$. A thorough discussion of the difficulties encountered when only measurability is required on the pair $(x,y)$ may be found in Allaire [A]. Indeed, in this case not even the measurability of, say, $f(x/\varepsilon, x/\varepsilon^2, 0)$ is guaranteed, letting alone the validity of results such as (1.4) (see also [Do2]). As Allaire points out, this measurability is ensured when there is continuity on least in one of the arguments, and here we choose it to be $x$ (see (A1), (A2) below).
(ii) Theorem 1.9 was proven in the case where $s=1$ by Braides and Lukkassen (see [BL], Theorem 1.1; see also [BrD]), where (H1) was on one hand relaxed to read
(H1)$'$ $f(\cdot,y,\cdot)$ is continuous for a.e. $y\in\mathbb{R}^N$ and $f(x,\cdot,\xi)$ is measurable for all $(x,\xi)\in\Omega\times\mathbb{R}^{d\times N}$,

and on the other hand it was strengthened with the additional uniform continuity assumption
(H5)$|f(x,y,\xi)-f(x',y,\xi)|\le\omega(|x-x'|)(a(y)+f(x,y,\xi))$ for all $x,x'\in\mathbb{R}^N$, a.e. $y\in\mathbb{R}^N$, for all $\xi\in\mathbb{R}^{d\times N}$, where $\omega$ is a continuous, positive function with $\omega(0)=0$.

Note that this latter condition essentially entails some sort of decoupling between different scales.
(iii) As it can be seen directly from the proof of Theorem 1.9, by analogy with the statement of Theorem 1.8, it is possible to replace condition H1) with the assumptions
(A1) $f(\cdot,y,\cdot)$ is continuous for a.e. $y\in\mathbb{R}^N$;
(A2) $f(x,\cdot,\xi)$ is measurable for all $(x,\xi)\in\Omega\times\mathbb{R}^s_*$;
(A3) $\dfrac{\partial f}{\partial\xi}(x,y,\xi)$ exists for all $x\in\Omega$, a.e. $y\in\mathbb{R}^N$, and for all $\xi\in\mathbb{R}^{d\times N}$, and it satisfies (A1), (A2).
(iv) Theorem 1.9 has also been extended to the non convex case and for $s=1$ by Braides and Defranceschi [BrD] under assumptions (H1)$'$, (H2), (H4) and (H5), and by Fonseca and Leoni [FL] under the hypotheses (H1), (H2) and (H4). The latter approach relies entirely on the blow-up method.

The main idea of the proof of Theorem 1.9 is to combine a three-scale convergence generalization of Allaire's 2-scale argument with iterated integrals to obtain the lower bound in (1.7) (see [A] Theorem 3.1 and 3.3; see also [MT]), with the blow-up method introduced in [FM1] to assert the upper bound. To this end, we extend Theorem 1.7 to second order derivatives to read

**Theorem 1.11.** *If $\{u_\varepsilon\}$ is a bounded sequence in $W^{2,p}(\Omega;\mathbb{R}^d)$, $\varepsilon>0$, then there exists a subsequence (not relabelled) converging weakly in $W^{2,p}(\Omega;\mathbb{R}^d)$ to a function $u$, and there exist $U\in L^p(\Omega;W^{2,p}(Q;\mathbb{R}^d))$ and $W\in L^p(\Omega\times Q;W^{2,p}(Q;\mathbb{R}^d))$ such that*

(i) *$U-A(x)y\in L^p(\Omega;W^{2,p}_{\mathrm{per}}(Q;\mathbb{R}^d))$ for some $A\in L^p(\Omega;\mathbb{R}^{d\times N})$;*

(ii) *$W-C(x,y)z\in L^p(\Omega\times Q;W^{2,p}_{\mathrm{per}}(Q;\mathbb{R}^d))$ for some $C\in L^p(\Omega\times Q;\mathbb{R}^{d\times N})$;*

(iii) *$u_\varepsilon\overset{3\text{-s}}{\rightharpoonup}u$, $Du_\varepsilon\overset{3\text{-s}}{\rightharpoonup}Du$, and*

$$\frac{\partial^2 u_\varepsilon}{\partial x_i\partial x_j}\overset{3\text{-s}}{\rightharpoonup}\frac{\partial^2 u}{\partial x_i\partial x_j}+\frac{\partial^2 U}{\partial y_i\partial y_j}(x,y)+\frac{\partial^2 W}{\partial z_i\partial z_j}(x,y,z). \tag{1.10}$$

*Conversely, given $u\in W^{2,p}(\Omega;\mathbb{R}^d)$, $U\in L^p(\Omega;W^{2,p}(Q;\mathbb{R}^d))$, and $W\in L^p(\Omega\times Q;W^{2,p}(Q;\mathbb{R}^d))$ satisfying (i), (ii), there exists a bounded sequence $\{u_\varepsilon\}\subset W^{2,p}(\Omega;\mathbb{R}^d)$ for which (iii) holds.*

## 2 Preliminaries

In what follows $\Omega$ is an open, bounded domain in $\mathbb{R}^N$, $Q:=(0,1)^N$, $Y_1,\ldots,Y_n$, are $n$ identical copies of $Q$, with $n\in\mathbb{N}$, $Q(x,\varepsilon):=x+\varepsilon Q$ for $x\in\mathbb{R}^N$ and $\varepsilon>0$, $\mathbb{R}^1_*:=\mathbb{R}^d$ and $\mathbb{R}^2_*:=(\mathrm{Sym}(\mathbb{R}^N,\mathbb{R}^N))^d$, where $\mathrm{Sym}(\mathbb{R}^N,\mathbb{R}^N)$ is the space of all linear symmetric transformations from $\mathbb{R}^N$ onto $\mathbb{R}^N$. $C^k_c(\mathbb{R}^N;\mathbb{R}^d)$ is the space of $k$-differentiable $\mathbb{R}^d$-valued functions in $\Omega$ with compact support, and $C^k_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)$ stands for the space of $Q$-periodic functions in $C^k(\mathbb{R}^N;\mathbb{R}^d)$, $1 \le k \le +\infty$. Recall that $f$ is said to be *$Q$-periodic* if $f(x + k\mathbf{e}_i) = f(x)$ for all $x$, all $k \in \mathbb{Z}^k$, and for all $i = 1, \dots, N$, where $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$. $C_c(\mathbb{R}^N)$ stands for the space of continuous functions with compact support in $\mathbb{R}^N$, $C_{\rm per}(\mathbb{R}^N)$ is the space of periodic continuous functions on $\mathbb{R}^N$, $L^p_{\rm per}(\mathbb{R}^N;\mathbb{R}^d)$ is the closure in $L^p_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d)$ of $C^\infty_{\rm per}(\mathbb{R}^N;\mathbb{R}^d)$, and $W^{k,p}_{\rm per}(\Omega;\mathbb{R}^d)$ is the closure in $W^{k,p}_{\rm loc}(\mathbb{R}^N;\mathbb{R}^d)$ of $C^\infty_{\rm per}(\mathbb{R}^N;\mathbb{R}^d)$. $\mathcal{L}^N$ is the $N-$dimensional Lebesgue measure, and $\mathcal{A}(\Omega)$ is the family of open subsets of $\Omega$. $C$ will stand for a generic positive constant which may vary from expression to expression within the same formula.

The following well known lowersemicontinuity result will be used in the proof of the continuity of $f_{\rm hom}$ (for a proof we refer the reader to [I1], [I2]).

**Theorem 2.1.** *Let $\Omega$ be an open, bounded subset of $\mathbb{R}^N$, and let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty)$, $f = f(x, u, v)$, be a function Lebesgue measurable with respect to $x$, and Borel measurable with respect to $(u, v)$. Suppose further that $f(x, \cdot, \cdot)$ is lower semicontinuous for a.e. $x \in \Omega$, $f(x, u, \cdot)$ is convex for a.e. $x \in \Omega, u \in \mathbb{R}^d$, and there exists $(u_0, v_0) \in L^q(\Omega;\mathbb{R}^d) \times L^p(\Omega;\mathbb{R}^m)$ with $1 \le p, q < +\infty$ such that*

$$\int_\Omega f(x, u_0(x), v_0(x))\, dx < +\infty.$$

*If $u_n \to u$ in $L^q(\Omega;\mathbb{R}^d)$ and $v_n \rightharpoonup v$ in $L^p(\Omega;\mathbb{R}^m)$ then*

$$\int_\Omega f(x, u, v)\, dx \le \int_\Omega \liminf_{n\to\infty} f(x, u_n, v_n)\, dx.$$

Next we recall De Giorgi's notion of $\Gamma$*−convergence.* Let $(X, d)$ be a metric space and let $\{F_\varepsilon\}$ be a family of functionals defined on $(X, d)$. We say that a functional $\mathcal{F} : X \times \mathcal{A}(\Omega) \to [0, +\infty]$ is the $\Gamma(X)$–lim inf (resp. the $\Gamma(X)$–lim sup) of the family $\{F_\varepsilon\}$ if for every sequence $\{\varepsilon_n\}$ converging to $0^+$

$$\mathcal{F}(u, \Omega) = \inf\left\{\liminf_{n\to+\infty} (\text{ resp. } \limsup_{n\to+\infty}) F_{\varepsilon_n}(u_n, \Omega) : u_n \to u \text{ in } (X, d)\right\},$$

and we write

$$\mathcal{F} = \Gamma(X) - \liminf_{\varepsilon\to0^+} F_\varepsilon \ (\text{resp. } \mathcal{F} = \Gamma(X) - \limsup_{\varepsilon\to0^+} F_\varepsilon).$$

$\mathcal{F}$ is the $\Gamma(X)$*-limit of the family* $\{F_\varepsilon\}$, and we write

$$\mathcal{F} = \Gamma(X) - \lim_\varepsilon F_\varepsilon,$$

if $\Gamma(X) - \liminf$ and $\Gamma(X) - \limsup$ coincide.

The following result is well known result in the theory of $\Gamma$-convergence, and for the convenience of the reader its proof may be found in the Appendix (see also [BhFF], [BrFF]).

**Proposition 2.2.** *Let $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty)$, $\varepsilon > 0$, be a sequence of functionals satisfying the hypotheses*

(i) *$F_\varepsilon(u, \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;*

(ii) *$F_\varepsilon(u, A) = F_\varepsilon(v, A)$ whenever $u = v$ a.e. in $A \in \mathcal{A}(\Omega)$;*

(iii) *there exists a positive constant $C$ such that for all $A \in \mathcal{A}(\Omega)$*

$$\frac{1}{C}\left(\int_A |Du|^p\, dx\right) \le F_\varepsilon(u, A) \le C\int_A (1 + |Du|^p)\, dx, \tag{2.1}$$

*for every $\varepsilon > 0$.*

*For every sequence $\{\varepsilon_n\}$ converging to $0^+$ there exists a subsequence $\{\varepsilon_j\}$ such that the functional $\mathcal{F}_{\{\varepsilon_j\}}(u, A)$ is the $\Gamma(L^p(A))$ limit of $\{F_{\varepsilon_j}(\cdot, A)\}$ for every $A \in \mathcal{A}(\Omega)$ and $u \in W^{1,p}(A;\mathbb{R}^d)$, where for any sequence $\{\delta_n\}$ converging to $0^+$*

$$\mathcal{F}_{\{\delta_n\}}(u, A) := \inf\left\{\liminf_{n\to\infty} F_{\delta_n}(u, A) : u_n \to u \text{ in } L^p(A)\right\}.$$

For a comprehensive study of $\Gamma$–convergence we refer to [Da] and [BrD].

The proof of Theorem 1.11 uses a version of Aumann's Measurability Selection Principle and a simple corollary of the theory of $\mathcal{A}$-quasiconvexity (see [D], [FM2], [BrFL]). We recall

**Theorem 2.3.** *Let $(X, \mathcal{M})$ be a measurable space with $\mu$ a positive, finite and complete measure, and let $Z$ be a complete, separable metric space. Let $F : X \to \{C \subset Z : C \neq \emptyset, C \text{ is closed}\}$ be a multifunction such that $\{(x, y) \in X \times Z : y \in F(x)\} \in \mathcal{M} \times \beta(Z)$, where $\beta(Z)$ is the Borel $\sigma$-algebra of $Z$. Then there exists a sequence of measurable functions $f_n : X \to Z$ such that*

$$F(x) = \overline{\{f_n(x) : n \in \mathbb{N}\}} \quad \text{for } \mu \text{ a.e. } x \in X.$$

As in [FM2], let

$$\mathcal{A} : L^q(\Omega; \mathbb{R}^l) \to W^{-1,q}(\Omega; \mathbb{R}^m), \quad \mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i},$$

be a constant–rank, first order linear partial differential operator, with $A^{(i)} : \mathbb{R}^l \to \mathbb{R}^m$ linear transformations, $i = 1, \dots, N$. We recall that $\mathcal{A}$ satisfies the *constant-rank* property if there exists $r \in \mathbb{N}$ such that (see [Mu])

$$\operatorname{rank} \mathbb{A}w = r \quad \text{for all } w \in S^{N-1},$$

where

$$\mathbb{A}w := \sum_{i=1}^{N} w_i \, A^{(i)}, \qquad w \in \mathbb{R}^N.$$

For each $w \in \mathbb{R}^N$ the operator $\mathbb{P}(w) : \mathbb{R}^l \to \mathbb{R}^l$ is the orthogonal projection of $\mathbb{R}^l$ onto $\ker \mathbb{A}(w)$, and $\mathbb{S}(w) : \mathbb{R}^m \to \mathbb{R}^l$ is defined by $\mathbb{S}(w)\, \mathbb{A}(w)\, z := z - \mathbb{P}(w)\, z$ for $z \in \mathbb{R}^l$ and $\mathbb{S} \equiv 0$ on $(\text{range}(\mathbb{A}(w))^{\perp}$. It may be shown that $\mathbb{P} : \mathbb{R}^N \setminus \{0\} \to \text{Lin}(\mathbb{R}^l; \mathbb{R}^l)$ is smooth and homogeneous of degree zero and $\mathbb{S} : \mathbb{R}^N \setminus \{0\} \to \text{Lin}(\mathbb{R}^m; \mathbb{R}^l)$ is smooth and homogeneous of degree $-1$.

We introduce the operators

$$\mathcal{S} : L^p_{\text{per}}(Q; \mathbb{R}^m) \to W^{1,p}_{\text{per}}(Q; \mathbb{R}^l), \quad \mathcal{T} : L^p_{\text{per}}(Q; \mathbb{R}^l) \to L^p_{\text{per}}(Q; \mathbb{R}^l)$$

given by

$$\mathcal{S}v(x) := \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \mathbb{S}(\lambda)\, \hat{v}(\lambda)\, e^{2\pi i x \cdot \lambda}$$

and

$$\mathcal{T}v(x) := \sum_{\lambda \in \mathbb{Z}^N} \hat{v}(\lambda)\, e^{2\pi i x \cdot \lambda} - \sum_{\lambda \in \mathbb{Z}^N \setminus \{0\}} \mathbb{S}(\lambda)\, \mathbb{A}(\lambda)\, \hat{v}(\lambda)\, e^{2\pi i x \cdot \lambda} \tag{2.2}$$

whenever

$$v(x) = \sum_{\lambda \in \mathbb{Z}^N} \hat{v}(\lambda)\, e^{2\pi i x \cdot \lambda},$$

and where the Fourier coefficients of a function $f \in L^1_{\text{per}}(Q)$ are defined by

$$\hat{f}(\lambda) := \int_Q f(x)\, e^{-2\pi i x \cdot \lambda} dx, \quad \lambda \in \mathbb{Z}^N.$$

The following proposition may be found in [FM2].

**Proposition 2.4.** *$\mathcal{T} : L^p_{\text{per}}(Q; \mathbb{R}^l) \to L^p_{\text{per}}(Q; \mathbb{R}^l)$ is a bounded linear operator that vanishes on constant mappings and $\mathcal{S} : L^p_{\text{per}}(Q; \mathbb{R}^m) \to W^{1,p}_{\text{per}}(Q; \mathbb{R}^l)$ is a pseudo differential bounded operator of order $-1$ such that*

(i) *if* $v \in L^p_{\text{per}}\left(Q;\mathbb{R}^l\right)$ *then* $\mathcal{T}\circ\mathcal{T}v = \mathcal{T}v$ *and* $\mathcal{A}\left(\mathcal{T}v\right) = 0$;

(ii) $||v - \mathcal{T}v||_{L^p_{\text{per}}(Q;\mathbb{R}^l)} \le C_p||\mathcal{A}\left(v\right)||_{W^{-1,p}_{\text{per}}(Q;\mathbb{R}^m)}$ *for all* $v \in L^p_{\text{per}}\left(Q;\mathbb{R}^l\right)$ *such that* $\int_Q v\,dx = 0$, *for some* $C_p > 0$;

(iii) $v - \mathcal{T}v = \mathcal{S}\mathcal{A}v$ *for all* $v \in L^p_{\text{per}}\left(Q;\mathbb{R}^l\right)$.

Just as in the case of first order gradients (see [K], [FMP]), $L^p$ bounded, $\mathcal{A}$-constrained sequences may be modified on small sets so as to render them equi-integrable. Precisely (see [FM2]),

**Proposition 2.5.** *Let* $1 < p < +\infty$, *let* $\{V_n\}$ *be a bounded sequence in* $L^p(\Omega;\mathbb{R}^l)$ *such that* $\mathcal{A}V_n \to 0$ *in* $W^{-1,p}(\Omega;\mathbb{R}^l)$, $V_n \rightharpoonup V$ *in* $L^p(\Omega;\mathbb{R}^l)$. *Then there exists a sequence* $\{v_n\} \subset L^p(\Omega;\mathbb{R}^l) \cap \ker(\mathcal{A})$ *such that* $\{|v_n|^p\}$ *is equi-integrable,*

$$\lim_{n\to\infty} \mathcal{L}^N\left(\{x \in \Omega : V_n(x) \neq v_n(x)\}\right) = 0, \tag{2.3}$$

*and*

$$\int_\Omega v_n\,dx = \int_\Omega V\,dx, \quad ||v_n - V_n||_{L^s(\Omega)} \to 0 \quad \textit{for all } 1 \le s < p.$$

*Moreover, if* $\Omega = Q$ *then* $v_n - V \in L^p_{\text{per}}(\mathbb{R}^N;\mathbb{R}^l) \cap \ker(\mathcal{A})$.

The relevance of this general framework, as emphasized by Tartar (see [T1], [T2], [T3], [T4], [T5]), lies on the fact that in continuum mechanics and electromagnetism PDEs other than $\operatorname{curl} v = 0$ arise naturally, and this calls for a relaxation theory which encompasses PDE constraints of the type $\mathcal{A}v = 0$. Included in this general setting are gradients of arbitrary order. Indeed, for gradients of order zero (i.e., unconstrained fields) it suffices to set $\mathcal{A}v \equiv 0$. Here, due to Jensen's inequality $\mathcal{A}$-quasiconvexity reduces to convexity. For gradients of order one, $\mathcal{A}v = 0$ if and only if $\operatorname{curl} v = 0$. Note that $w \in C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{R}^l)$ is such that curl $w = 0$ and $\int_Q w(y)\,dy = 0$ if and only if there exists $\varphi \in C^\infty_{\text{per}}(\mathbb{R}^N;\mathbb{R}^d)$ such that $\nabla\varphi = v$, where $l = d \times N$. Thus, in this case we recover the well known notion of *quasiconvexity* introduced by [Mo] (see also [D]). For higher order gradients we replace the target space $\mathbb{R}^l$ by an appropriate finite dimensional vector space $E^d_s$ of $d$-tuples of symmetric $s$-linear maps on $\mathbb{R}^N$, and it is then possible to find a first order linear partial differential operator $\mathcal{A}$ such that $v \in L^p(\Omega;E^d_s)$ and $\mathcal{A}v = 0$ if and only if there exists $\varphi \in W^{s,p}(\Omega;\mathbb{R}^d)$ such that $v = \nabla^s\varphi$.

# 3 Proof of Theorem 1.11

We are now in position to characterize the multiscale limit for the Hessian tensor. In order to simplify the exposition, we consider the case where $n = 2$, although the proof may be carried out for arbitrary $n \in \mathbb{N}$ with the obvious adaptations.

*Proof of Theorem 1.11.* By working with each coordinate separately, without loss of generality we may assume that $u_\varepsilon$ is scalar valued. Applying Theorem 1.7 to the sequence $\{(u_\varepsilon, Du_\varepsilon)\} \subset W^{1,p}(\Omega;\mathbb{R}\times\mathbb{R}^N)$, we get

$$u_\varepsilon(x) \overset{3\text{-s}}{\rightharpoonup} u(x), \quad Du_\varepsilon(x) \overset{3\text{-s}}{\rightharpoonup} Du(x)$$

and

$$\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial u_\varepsilon}{\partial x_j}\right) \overset{3\text{-s}}{\rightharpoonup} \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \frac{\partial U^{(j)}}{\partial y_i}(x,y) + \frac{\partial W^{(j)}}{\partial z_i}(x,y,z), \tag{3.1}$$

for some $U^{(j)} \in L^p(\Omega;W^{1,p}_{\text{per}}(Y_1))$ and $W^{(j)} \in L^p(\Omega\times Y_1;W^{1,p}_{\text{per}}(Y_2))$, j=1, ..., N.

For any function $\varphi \in C^\infty_c(\Omega;C^\infty_{\text{per}}(Y_1\times Y_2))$ we have

$$\int_\Omega\int_{Y_1}\int_{Y_2}\left(\frac{\partial U^{(j)}}{\partial y_i}(x,y) - \frac{\partial U^{(i)}}{\partial y_j}(x,y) + \frac{\partial W^{(j)}}{\partial z_i}(x,y,z) - \frac{\partial W^{(i)}}{\partial z_j}(x,y,z)\right)\varphi(x,y,z)\,dx\,dy\,dz = 0,$$

where we have used the fact that $\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} = \frac{\partial^2 u_\varepsilon}{\partial x_j \partial x_i}$. Choosing $\varphi(x,y,z) := \theta(x)\psi(y,z)$ with $\theta \in C_c^\infty(\Omega)$, $\psi \in C_{\mathrm{per}}^\infty(Y_1 \times Y_2)$, we get for a.e. $x \in \Omega$

$$\int_{Y_1}\int_{Y_2}\left(\frac{\partial U^{(j)}}{\partial y_i}(x,y) - \frac{\partial U^{(i)}}{\partial y_j}(x,y) + \frac{\partial W^{(j)}}{\partial z_i}(x,y,z) - \frac{\partial W^{(i)})}{\partial z_j}(x,y,z)\right)\psi(y,z)dydz = 0. \tag{3.2}$$

If, in addition, $\psi(y,z) = \psi(y)$ then (3.2) becomes

$$\begin{aligned} 0 &= \int_{Y_1}\left(\frac{\partial U^{(j)}}{\partial y_i}(x,y) - \frac{\partial U^{(i)}}{\partial y_j}(x,y)\right)\psi(y)\,dy + \int_{Y_1}\psi(y)\left(\int_{Y_2}\left(\frac{\partial W^{(i)}}{\partial z_j}(x,y,z) - \frac{\partial W^{(j)}}{\partial z_i}(x,y,z)\right)dz\right)dy \\ &= \int_{Y_1}\left(\frac{\partial U^{(j)}}{\partial y_i}(x,y) - \frac{\partial U^{(i)}}{\partial y_j}(x,y)\right)\psi(y)dy \end{aligned}$$

for a.e. $x \in \Omega$, in light of the periodicity of $W^{(i)}(x,y,\cdot)$ and $W^{(j)}(x,y,\cdot)$. We conclude that

$$\frac{\partial U^{(j)}}{\partial y_i}(x,y) = \frac{\partial U^{(i)}}{\partial y_j}(x,y) \quad \text{for a.e. } x \in \Omega \text{ and } y \in Y_1. \tag{3.3}$$

Now (3.2) reduces to

$$\int_{Y_1}\int_{Y_2}\left(\frac{\partial W^{(i)}}{\partial z_j} - \frac{\partial W^{(j)}}{\partial z_i}\right)\psi(y,z)dydz = 0,$$

and choosing as test function $\psi(y,z) := \psi_1(y)\psi_2(z)$, with $\psi_1$ and $\psi_2$ smooth and periodic, we obtain

$$\frac{\partial W^{(j)}}{\partial z_i}(x,y,z) = \frac{\partial W^{(i)}}{\partial z_j}(x,y,z) \quad \text{for a.e. } (x,y,z) \in \Omega \times Y_1 \times Y_2. \tag{3.4}$$

Note that in (3.2), (3.3) and (3.4) we have used the fact the tensor product space $C_{\mathrm{per}}^\infty(X_1)\otimes \ldots \otimes C_{\mathrm{per}}^\infty(X_n)$ is dense in $C_{\mathrm{per}}^\infty(X_1 \times \ldots \times X_n)$, where $X_1, \ldots X_n$ are arbitrary Banach spaces.

We claim that there exists $U \in L^p(\Omega; W^{2,p}(Y_1))$ such that

$$\frac{\partial U}{\partial y_i}(x,y) = U^{(i)}(x,y) \tag{3.5}$$

for a.e. $x \in \Omega, y \in Y_1$. We define

$$\overline{U} := \left(U^{(1)}, \ldots, U^{(N)}\right) \in L^p\left(\Omega; W_{\mathrm{per}}^{1,p}(Y_1)^N\right).$$

Although it is clear from (3.3) that for a.e. $x \in \Omega$ there exists $g_x \in W^{2,p}(Y_1)$ such that $D_y g_x(y) = \overline{U}(x,y)$ a.e. $y \in Y_1$, we need to ensure the measurability and integrability in $x$ of $(x,y) \mapsto g_x(y)$. Choose piecewise affine functions $\theta_k \in L^p\left(\Omega; W_{\mathrm{per}}^{1,p}(Y_1)^N\right)$ such that

$$\lim_{k\to\infty} \|\theta_k - \overline{U}\|_{L^p\left(\Omega; W_{\mathrm{per}}^{1,p}(Y_1)^N\right)} = 0. \tag{3.6}$$

Consider the first order linear system of PDEs given by $\mathcal{A}v := \operatorname{curl} v$ for $v \in L^p\left(Y_1; \mathbb{R}^N\right)$. We show that

$$\|\mathcal{A}(\theta_k)\|_{L^p(\Omega; L^p(Y_1))} \to 0 \tag{3.7}$$

where here $\mathcal{A}(\theta_k) := \operatorname{curl}_y \theta_k(x,y)$ for a.e. $x \in \Omega$. Indeed, for every $i,j = 1, \ldots, N$, we have

$$\begin{aligned} &\int_\Omega \left\|\frac{\partial \theta_k{}^{(i)}}{\partial y_j}(x,\cdot) - \frac{\partial \theta_k{}^{(j)}}{\partial y_i}(x,\cdot)\right\|_{L^p(Y_1)}^p dx \le C\int_\Omega \left\|\frac{\partial \theta_k{}^{(i)}}{\partial y_j}(x,\cdot) - \frac{\partial U^{(i)}}{\partial y_j}(x,\cdot)\right\|_{L^p(Y_1)}^p dx \\ &\quad + C\int_\Omega \left\|\frac{\partial U^{(i)}}{\partial y_j}(x,\cdot) - \frac{\partial U^{(j)}}{\partial y_i}(x,\cdot)\right\|_{L^p(Y_1)}^p dx + C\int_\Omega \left\|\frac{\partial \theta_k{}^{(j)}}{\partial y_i}(x,\cdot) - \frac{\partial U^{(j)}}{\partial y_i}(x,\cdot)\right\|_{L^p(Y_1)}^p dx \\ &\le C\int_\Omega \|\theta_k(x,\cdot) - \overline{U}(x,\cdot)\|_{W^{1,p}(Y_1)}^p\, dx \to 0, \end{aligned}$$

where we have used (3.3) and (3.6).

We are now in position to apply Theorem 2.3, where here $(X, \mathcal{M}) := (\Omega, \mathcal{L})$, $Z := W^{1,p}_{\mathrm{per}}(Y_1)$, $\mu$ is the Lebesgue measure and $\mathcal{L}$ is the $\sigma$-algebra of Lebesgue measurable sets in $\mathbb{R}^N$. For each $k \in \mathbb{N}$ consider

$$\begin{aligned} F_k : \Omega &\to \{C \subset W^{1,p}_{\mathrm{per}}(Y_1) : C \neq \emptyset,\ C \text{ is closed}\} \\ x \mapsto & F_k(x) := \left\{ v \in W^{1,p}_{\mathrm{per}}(Y_1) : \int_{Y_1} v \, dy = 0,\ D_y v = \mathcal{T}(\theta_k(x,y)) \right\} \end{aligned}$$

where $\mathcal{T}$ is the operator introduced in (2.2). Note that since $\mathcal{T}(\theta_k)$ is curl free, then $F_k(x)$ is nonempty for a.e. $x \in \Omega$, and it is trivially closed. Note also that, with $\theta_k(x,y) = \sum_{i=1}^k \chi_{A_i}(x) {\lambda_k}^{(i)}(y)$ for some $k \in \mathbb{R}^N$, $A_i \subset \Omega$ mutually disjoint and ${\lambda_k}^{(i)} \in W^{1,p}_{\mathrm{per}}(Y_1)$, we have

$$\{(x,v) : x \in \Omega, v \in F_k(x)\} = \cup_{i=1}^k A_i \times \left\{ v \in W^{1,p}(Y_1) : \int_{Y_1} v \, dy = 0,\ D_y v = \mathcal{T}\left({\lambda_k}^{(i)}\right) \right\},$$

which belongs to $\mathcal{L} \times \beta\left(W^{1,p}(Y_1)\right)$ because the sets

$$\left\{ v \in W^{1,p}(Y_1) : \int_{Y_1} v dy = 0, \quad D_y v = \mathcal{T}\left({\lambda_k}^{(i)}\right) \right\}$$

are closed. By Theorem 2.3 we may find measurable functions $f_{n,k} : \Omega \to W^{1,p}(Y_1)$ such that

$$F_k(x) = \overline{\{f_{n,k}(x)\}}.$$

Now set $v_k(x) := f_{1,k}(x)$. By Poincaré-Friedrichs inequality, and in view of Proposition 2.4, we have

$$\begin{aligned} \int_\Omega \|v_k(x,\cdot)\|^p_{W^{1,p}_{\mathrm{per}}(Y_1)} \, dx &\leq C \int_\Omega \|D_y v_k(x,\cdot)\|^p_{L^p_{\mathrm{per}}(Y_1)} \, dx \\ &\leq C \int_\Omega \|\theta_k(x,\cdot)\|^p_{(L^p_{\mathrm{per}}(Y_1))^N} \, dx \to \int_\Omega \|\overline{U}(x,\cdot)\|^p_{(L^p_{\mathrm{per}}(Y_1))^N} \, dx, \end{aligned}$$

and so there exists a subsequence (not relabelled), and there exists $\tilde{U} \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y_1))$ such that

$$v_k \rightharpoonup \tilde{U} \quad \text{in } L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y_1)).$$

Now since the norm is weakly lower semicontinuous, by Poincaré-Friedrichs inequality we have

$$\begin{aligned} &\int_\Omega \left\| \frac{\partial \tilde{U}}{\partial y_i}(x,\cdot) - \left( U^{(i)}(x,\cdot) - \int_{Y_1} U^{(i)}(x,z)\,dz \right) \right\|^p_{L^p_{\mathrm{per}}(Y_1)} dx \\ &\leq \liminf_{k\to\infty} \int_\Omega C \left\| \frac{\partial v_k}{\partial y_i}(x,\cdot) - \left( U^{(i)}(x,\cdot) - \int_{Y_1} U^{(i)}(x,z)\,dz \right) \right\|^p_{L^p_{\mathrm{per}}(Y_1)} dx \\ &\leq \limsup_{k\to\infty} \int_\Omega C \left\| \frac{\partial v_k}{\partial y_i}(x,\cdot) - (\mathcal{T}(\theta_k(x,\cdot)))^{(i)} \right\|^p_{L^p_{\mathrm{per}}(Y_1)} dx \\ &\quad + \limsup_{k\to\infty} \int_\Omega C \left\| (\mathcal{T}(\theta_k(x,\cdot)))^{(i)} - \left( \theta_k^{(i)}(x,\cdot) - \int_{Y_1} \theta_k^{(i)}((x,z)\,dz \right) \right\|^p_{L^p_{\mathrm{per}}(Y_1)} dx \\ &\quad + \limsup_{k\to\infty} \int_\Omega C \left\| \theta_k^{(i)}(x,\cdot) - \int_{Y_1} \theta_k^{(i)}(x,z)\,dz - \left( U^{(i)}(x,\cdot) - \int_{Y_1} U^{(i)}(x,z)\,dz \right) \right\|^p_{L^p_{\mathrm{per}}(Y_1)} dx \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where $I_1 = 0$ by definition of $F_k$, (3.6) implies that $I_3 \to 0$, and by Proposition 2.4(ii) and (3.7) it follows that

$$I_2 \leq C \limsup_{k\to\infty} \|\mathcal{A}(\theta_k)\|_{L^p(\Omega; W^{-1,p}(Y_1))} \leq C \limsup_{k\to\infty} \|\mathcal{A}(\theta_k)\|_{L^p(\Omega; L^p(Y_1))} \to 0.$$

Therefore, setting $A(x) := -\int_{Y_1} \overline{U}(x,z)\,dz$ then $A \in L^p(\Omega;\mathbb{R}^N)$ and

$$\frac{\partial U}{\partial y_i}(x,y) = U^{(i)}(x,y)$$

for a.e. $x \in \Omega$, $y \in Y_1$, where

$$U := \tilde{U} + A(x)\cdot y$$

clearly satisfies condition (i) of Theorem 1.11, and this concludes the proof of (3.5). In a similar way, it can be shown that there exists $W$ verifying (ii) in the statement of Theorem 1.11, and such that

$$\frac{\partial W}{\partial z_i}(x,y,z) = W^{(i)}(x,y,z)$$

for a.e. $x \in \Omega, y \in Y_1, z \in Y_2$. These, together with (3.1), assert the 3-scale compactness of $W^{2,p}$ bounded sequences.

Finally we show that any such triple $(u, U, W)$ may be attained as a 3-scale limit. Assume first that we have the extra regularity $\hat{U}(x,y) := U(x,y) - A(x)y \in C_c^\infty(\Omega; C_{\mathrm{per}}^\infty(Y_1))$ and $\hat{W}(x,y,z) := W(x,y,z) - C(x,y)z \in C_c^\infty(\Omega\times Y_1; C_{\mathrm{per}}^\infty(Y_2))$. Extend $\hat{W}(x,\cdot,z)$ to $\mathbb{R}^N$ periodically with period $Y_1$.

Using (1.4) it is easy to verify that the sequence

$$u_\varepsilon(x) := u(x) + \varepsilon^2 \hat{U}\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^4 \hat{W}\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)$$

satisfies

$$u_\varepsilon \overset{3\text{-s}}{\rightharpoonup} u, \quad Du_\varepsilon \overset{3\text{-s}}{\rightharpoonup} Du, \quad \text{and } D^2_{xx}u_\varepsilon \overset{3\text{-s}}{\rightharpoonup} D^2_{xx}u + D^2_{yy}\hat{U}(x,y) + D^2_{zz}\hat{W}(x,y,z).$$

Since $D^2_{yy}\hat{U} = D^2_{yy}U$ and $D^2_{zz}\hat{W} = D^2_{zz}W$, we have obtained (1.10).

The result for arbitrary $U \in L^p(\Omega; W^{2,p}(Y_1))$ with $U - A(x)y \in L^p(\Omega; W^{2,p}_{\mathrm{per}}(Q;\mathbb{R}^d))$ for some $A \in L^p(\Omega;\mathbb{R}^{d\times N})$, and $W \in L^p(\Omega\times Y_1; W^{2,p}(Y_2))$ with $W - C(x,y)z \in L^p(\Omega\times Q; W^{2,p}_{\mathrm{per}}(Q;\mathbb{R}^d))$ for some $C \in L^p(\Omega\times Q;\mathbb{R}^{d\times N})$, will now follow via a standard density and diagonalization argument. □

# 4 Proof of Theorem 1.9: The Case $s = 1$

We divide the proof of Theorem 1.9 into a series of lemmas, the first of which establishes the regularity of $f_{\mathrm{hom}}$.

**Lemma 4.1.** *Under the hypotheses (A1), (A2) and (H4), $f_{\mathrm{hom}} : \Omega \times \mathbb{R}^{d\times N} \to [0, +\infty)$ is continuous.*

*Proof.* We recall that (see (1.9))

$$f_{\mathrm{hom}}(x,\xi) := \inf\left\{ \int_Q f(x,y,\xi + D\psi(y))\,dy : \psi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d) \right\}.$$

Fix $(x,\xi) \in \Omega \times \mathbb{R}^{d\times N}$ and consider a sequence $\{(x_n,\xi_n)\}$ converging to $(x,\xi)$. Let $\varepsilon > 0$ and choose $\psi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d)$ such that

$$f_{\mathrm{hom}}(x,\xi) + \varepsilon \geq \int_Q f(x,y,\xi + D\psi(y))\,dy.$$

We then have

$$\begin{aligned}\limsup_{n\to\infty} f_{\mathrm{hom}}(x_n,\xi_n) &\leq \limsup_{n\to\infty} \int_Q f(x_n,y,\xi_n + D\psi(y))\,dy \\ &= \int_Q f(x,y,\xi + D\psi(y))\,dy \leq f_{\mathrm{hom}}(x,\xi) + \varepsilon, \end{aligned} \tag{4.1}$$

where we have used (A1), (A2) and (H4).

Conversely, let $\psi_n \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d)$ be such that

$$f_{\mathrm{hom}}(x_n,\xi_n)+\varepsilon \geq \int_Q f(x_n,y,\xi+D\psi_n(y))\,dy.$$

By the coercivity condition (H4), using Poincaré-Friedrichs inequality we deduce that $\left\{\psi_n - \int_Q \psi_n(z)\,dz\right\}$ is a sequence bounded in $W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d)$, and thus, up to a subsequence not relabelled,

$$\psi_n - \int_Q \psi_n(z)\,dz \rightharpoonup \psi \quad \text{in } W^{1,p}$$

for some function $\psi \in W^{1,p}_{\mathrm{per}}(Q;\mathbb{R}^d)$. In view of (A1), (A2), and by Theorem 2.1, we conclude that

$$f_{\mathrm{hom}}(x,\xi) \leq \int_Q f(x,y,\xi+D\psi(y))\,dy \leq \liminf_{n\to\infty}\int_Q f(x_n,y,\xi+D\psi_n(y))\,dy \leq \liminf_{n\to\infty} f_{\mathrm{hom}}(x_n,\xi_n)+\varepsilon. \tag{4.2}$$

Letting $\varepsilon \to 0^+$ in (4.1) and in (4.2) yields

$$\lim_{n\to\infty} f_{\mathrm{hom}}(x_n,\xi_n) = f_{\mathrm{hom}}(x,\xi).$$

□

Consider a sequence $\{\varepsilon_n\}$ converging to $0^+$ and set

$$\mathcal{F}_{\{\varepsilon_n\}}(u,A) := \inf\left\{\liminf_{n\to\infty} F_{\varepsilon_n}(u,A) : u_n \to u \text{ in } L^p(A;\mathbb{R}^d)\right\}$$

where

$$F_\varepsilon(u,A) := \int_A f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du\right)\,dx$$

for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$.

We define

$$\begin{aligned} F(u,A) := \inf\Bigg\{ &\int_\Omega\int_{Y_1}\int_{Y_2} f\left(x,y,Du(x)+D_yU(x,y)+D_zW(x,y,z)\right)\,dx\,dy\,dz : \\ & U \in L^p\left(\Omega;W^{1,p}_{\mathrm{per}}(Y_1)\right), W \in L^p\left(\Omega\times Y_1;W^{1,p}_{\mathrm{per}}(Y_2)\right)\Bigg\}. \end{aligned} \tag{4.3}$$

**Lemma 4.2.** *If $f$ satisfies hypotheses (A1), (A2), (H2), (H3) and (H4), then*

$$\mathcal{F}_{\{\varepsilon\}}(u,\Omega) \leq F(u,\Omega) \tag{4.4}$$

*for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$.*

*Proof.* In view of (H4) it suffices to prove (4.4) when we have the extra regularity $U \in C^\infty_c\left(\Omega;W^{1,p}_{\mathrm{per}}(Y_1)\right)$ and $W \in C^\infty_c\left(\Omega\times Y_1;W^{1,p}_{\mathrm{per}}(Y_2)\right)$. Extend $W(y,\cdot,\xi)$ to $\mathbb{R}^N$ as a $Y_1$-periodic function, and set $u_\varepsilon(x) := u(x)+\varepsilon U\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2 W\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)$. By (1.4) we have that

$$u_\varepsilon \rightharpoonup u \in W^{1,p}(\Omega;\mathbb{R}^d),$$

and so

$$\begin{aligned} \mathcal{F}_{\{\varepsilon\}}(u,\Omega) \leq \liminf_{\varepsilon\to 0^+}\int_\Omega f\Bigg(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},&Du(x)+\varepsilon D_xU\left(x,\frac{x}{\varepsilon}\right)+D_yU\left(x,\frac{x}{\varepsilon}\right) \\ &+\varepsilon^2 D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)+\varepsilon D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)+D_zW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\Bigg)\,dx. \end{aligned}$$

Since by (H3) and (H4) the function $f(x,y,\cdot)$ is convex and has a growth of order $p$, it is $p$-Lipschitz continuous, precisely (see [Ma2])

$$|f(x,y,\xi)-f(x,y,\xi')|\le C\left(1+|\xi|^{p-1}+|\xi'|^{p-1}\right)|\xi-\xi'| \tag{4.5}$$

for some constant $C>0$, all $x\in\mathbb{R}^N$, a.e. $y\in\mathbb{R}^N$, and all $\xi,\xi'\in\mathbb{R}^{d\times N}$. Thus

$$\begin{aligned}
\mathcal{F}_{\{\varepsilon\}}(u,\Omega) &\le \liminf_{\varepsilon\to0+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du(x)+D_yU\left(x,\frac{x}{\varepsilon}\right)+D_zW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)dx\\
&+\limsup_{\varepsilon\to0^+}\int_\Omega C\left(1+|Du(x)|^{p-1}+\left|\varepsilon D_xU\left(x,\frac{x}{\varepsilon}\right)\right|^{p-1}+\left|D_yU\left(x,\frac{x}{\varepsilon}\right)\right|^{p-1}\right.\\
&+\left.\left|\varepsilon^2D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p-1}+\left|\varepsilon D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p-1}+\left|D_zW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p-1}\right)\\
&\cdot\left|\varepsilon D_xU\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)+\varepsilon D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|dx\\
&=\int_\Omega\int_{Y_1}\int_{Y_2} f\left(y,z,Du(x)+D_yU(x,y)+D_zW(x,y,z)\right)dx\,dy\,dz,
\end{aligned}$$

where we have used (1.4), (A1), (A2), and the fact that, by Hölder inequality,

$$\begin{aligned}
&\limsup_{\varepsilon\to0^+}\int_\Omega\left(1+|Du(x)|^{p-1}+\left|\varepsilon D_xU\left(x,\frac{x}{\varepsilon}\right)\right|^{p-1}+\left|D_yU\left(x,\frac{x}{\varepsilon}\right)\right|^{p-1}\right.\\
&+\left.\left|\varepsilon^2D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p-1}+\left|\varepsilon D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p-1}+\left|D_zW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p-1}\right)\\
&\cdot\left|\varepsilon D_xU\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)+\varepsilon D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|dx\\
&\le\limsup_{\varepsilon\to0^+}C\varepsilon\left[\int_\Omega\left(1+|Du(x)|^p+\left|D_xU\left(x,\frac{x}{\varepsilon}\right)\right|^p+\left|D_yU\left(x,\frac{x}{\varepsilon}\right)\right|^p\right.\right.\\
&+\left.\left.\left|D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p+\left|D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p+\left|D_zW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p\right)dx\right]^{\frac{p}{p-1}}\\
&\cdot\left[\int_\Omega\left(\left|D_xU\left(x,\frac{x}{\varepsilon}\right)\right|^p+\left|D_xW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p+\left|D_yW\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p\right)dx\right]^{\frac{1}{p}}
\end{aligned}$$

which converges to zero by virtue of (1.4). This concludes the proof. □

Next we want to establish

**Lemma 4.3.** *If $f$ satisfies hypotheses (H1) [or (A1), (A2), (A3)], (H2), (H3) and (H4), then*

$$\mathcal{F}_{\{\varepsilon\}}(u,\Omega)\ge F(u,\Omega) \tag{4.6}$$

*for every $u\in W^{1,p}(\Omega;\mathbb{R}^d)$.*

The proof of Lemma 4.3 is hinged on the two general multiscale lowersemicontinuity results below, the first of which is proven following step by step Allaire's argument in the proof of Theorem 1.8 in [A]).

**Lemma 4.4.** *Let $1\le p\le+\infty$. If $f$ satisfies hypotheses (A1), (A2), (A3), (H2), (H3), and (H4) if $1\le p<+\infty$, and if a sequence $\{w_e\}\subset L^p(\Omega;\mathbb{R}^d)$ 3-scale converges to a function $w_0\in L^p(\Omega\times Y_1\times Y_2)$, then*

$$\liminf_{\varepsilon\to0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\dots,\frac{x}{\varepsilon^2},w_\varepsilon(x)\right)dx=\int_\Omega\int_{Y_1}\int_{Y_2}f(y,z,w_0(x,y,z))\,dx\,dy\,dz. \tag{4.7}$$

*Proof.* As it is well known, if $1\le p<+\infty$ and if (H4) holds then we have

$$\left|\frac{\partial f}{\partial\xi}(x,y,\xi)\right|\le C(1+|\xi|^{p-1}) \tag{4.8}$$

for all $x \in \mathbb{R}^N$, a.e. $y \in \mathbb{R}^N$, and all $\xi \in \mathbb{R}^{d\times N}$. Indeed,

$$f(x,y,\xi') \geq f(x,y,\xi) + \frac{\partial f}{\partial \xi}(x,y,\xi)\cdot(\xi'-\xi)$$

and taking $\xi' := \xi \pm \mathbf{E}_i$, with $i = 1, \dots, d\times N$, and $\{\mathbf{E_1}, \dots, \mathbf{E}_{d\times N}\}$ an orthonormal basis of $\mathbb{R}^{d\times N}$, we have by (4.5) and in view of (H4)

$$\frac{\partial f}{\partial \xi}(x,y,\xi)\cdot(\pm\mathbf{E}_i) \leq f(x,y,\xi\pm\mathbf{E}_i) - f(x,y,\xi) \leq C\left(1 + |\xi\pm\mathbf{E}_i|^{p-1} + |\xi|^{p-1}\right),$$

and (4.8) follows.

Let $\{\theta_j\} \subset C_c(\Omega; C_{\mathrm{per}}(Y_1\times Y_2))$ be such that

$$\theta_j \to u_0 \text{ in } L^s(\Omega\times Y_1\times Y_2), \tag{4.9}$$

where $s := p'$ if $1 \leq p < +\infty$ and $s := 1$ if $p = +\infty$ (see Remark 1.5). By the convexity of $f$ we now have for fixed $j \in \mathbb{N}$

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},w_\varepsilon\right)dx &\geq \lim_{\varepsilon\to0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\theta_j\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)dx\\ &\quad -\limsup_{\varepsilon\to0^+}\int_\Omega \frac{\partial f}{\partial\xi}\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\theta_j\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\cdot\left[w_\varepsilon(x)-\theta_j\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right]dx\\ &\geq \int_\Omega\int_{Y_1}\int_{Y_2} f(y,z,\theta_j(x,y,z))\,dx\,dy\,dz - \limsup_{\varepsilon\to0^+} I_{j,\varepsilon},\end{aligned}$$

where we have used (A1), (A2), (A3), (H2) and (1.4) to obtain the iterated integral (note that here, and in light of Remark 1.10, the continuity of $\frac{\partial f}{\partial\xi}$ is crucial), and where

$$I_{j,\varepsilon} := \int_\Omega \frac{\partial f}{\partial\xi}\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\theta_j\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\cdot w_\varepsilon(x)\,dx - \int_\Omega \frac{\partial f}{\partial\xi}\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\theta_j\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\cdot\theta_j\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)dx.$$

By the definition of 3-scale convergence (see Definition (1.2)) and in light of Remark 1.3, the first term in this expression converges, as $\varepsilon\to0^+$, to

$$\int_\Omega\int_{Y_1}\int_{Y_2}\frac{\partial f}{\partial\xi}(y,z,\theta_j(x,y,z))\cdot w_0(x,y,z)\,dx\,dy\,dz,$$

and by (1.4) the second term tends to

$$\int_\Omega\int_{Y_1}\int_{Y_2}\frac{\partial f}{\partial\xi}(y,z,\theta_j(x,y,z))\cdot\theta_j(x,y,z)\,dx\,dy\,dz.$$

Given the arbitrariness of $j$, we conclude that

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},w_\varepsilon\right)dx &\geq \lim_{j\to\infty}\int_\Omega\int_{Y_1}\int_{Y_2} f(y,z,\theta_j(x,y,z))\,dx\,dy\,dz\\ &\quad + \lim_{j\to\infty}\int_\Omega\int_{Y_1}\int_{Y_2}\frac{\partial f}{\partial\xi}(y,z,\theta_j(x,y,z))\cdot(w_0(x,y,z)-\theta_j(x,y,z))\,dx\\ &= \int_\Omega\int_{Y_1}\int_{Y_2} f(y,z,w_0(x,y,z))\,dx\,dy\,dz,\end{aligned}$$

where to assert the equality above we have used (H4), Hölder inequality with respect to $s$, (4.8), (4.9), and this entails (4.7). □

**Lemma 4.5.** *If $f$ satisfies hypotheses (H1), (H2), (H3), and if a sequence $\{w_\varepsilon\} \subset L^\infty(\Omega;\mathbb{R}^d)$ 3-scale converges to a function $w_0 \in L^\infty\left(\Omega \times Y_1 \times Y_2;\mathbb{R}^d\right)$, then*

$$\liminf_{\varepsilon\to 0^+} \int_\Omega f\left(\frac{x}{\varepsilon},\dots,\frac{x}{\varepsilon^2}, w_\varepsilon(x)\right)\, dx \geq \int_\Omega \int_{Y_1}\int_{Y_2} f(y,z,w_0(x,y,z))\, dx\, dy\, dz.$$

*Proof.* Note that in order to be in a position to apply Lemma 4.4 we need to modify $f$ so as to guarantee the differentiability with respect to $\xi$ requested in (A3). Let $\rho_k$, $k \in \mathbb{R}^N$, denote the standard radially symmetric mollifiers. Define for $k \in \mathbb{R}^N$

$$f_k(x,y,\xi) := \int_{B_{1/k}(0)} \rho_k(\xi')f(x,y,\xi-\xi')\, d\xi', \quad (x,y,\xi) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{d\times N}.$$

It is clear that $f_k$ satisfies (H1), (H2), (H3), and (A3). Fix $M > 0$ such that $\|w_\varepsilon\|_{L^\infty(\Omega)}, \|w_0\|_{L^\infty(\Omega\times Y_1\times Y_2)} \leq M$ for all $\varepsilon > 0$. Fix $\delta > 0$ and choose $k \in \mathbb{N}$ such that

$$\max_{x\in Y_1, y\in Y_2, \xi,\xi'\in \overline{B(0,M)}} |f(x,y,\xi) - f(x,y,\xi')| < \delta \text{ whenever } |\xi-\xi'| \leq \frac{1}{k}.$$

Since $f(\cdot,\cdot,\xi)$ is periodic, we then have

$$\sup_{x,y\in\mathbb{R}^N, \xi,\xi'\in \overline{B(0,M)}} |f(x,y,\xi) - f(x,y,\xi')| < \delta \text{ whenever } |\xi-\xi'| \leq \frac{1}{k},$$

and thus

$$\sup_{x,y\in\mathbb{R}^N, \xi\in \overline{B(0,M)}} |f_k(x,y,\xi) - f(x,y,\xi')| \leq \delta.$$

By Lemma 4.4 we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}, w_\varepsilon\right)\, dx &\geq \lim_{\varepsilon\to 0^+} \int_\Omega f_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}, w_\varepsilon\right)\, dx - \delta \\ &\geq \int_\Omega\int_{Y_1}\int_{Y_2} f_k(y,z,w_0(x,y,z))\, dx\, dy\, dz - \delta \\ &\geq \int_\Omega\int_{Y_1}\int_{Y_2} f(y,z,w_0(x,y,z))\, dx\, dy\, dz - 2\delta.\end{aligned}$$

Now it suffices to let $\delta \to 0^+$. □

*Proof of Lemma 4.3. Step 1.* We assume first that $f$ satisfies (A1), (A2), (A3), (H2), (H3) and (H4). Let $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^d)$. Extracting a subsequence if necessary, without loss of generality and by Theorem 1.7 we may assume that

$$u_\varepsilon \overset{3\text{-}s}{\rightharpoonup} u, \quad Du_\varepsilon \overset{3\text{-}s}{\rightharpoonup} Du + D_yU(x,y) + D_zW(x,y,z)$$

for some $U \in L^p\left(\Omega; W^{1,p}_{\mathrm{per}}(Y_1)\right)$ and $W \in L^p\left(\Omega\times Y_1; W^{1,p}_{\mathrm{per}}(Y_2)\right)$. We are in a position to apply Lemma 4.4, where (4.7) becomes

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} \int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}, Du_\varepsilon\right)\, dx &\geq \int_\Omega\int_{Y_1}\int_{Y_2} f\left(y,z,Du(x) + D_yU(x,y) + D_zW(x,y,z)\right)\, dx\, dy\, dz \\ &\geq F(u,\Omega).\end{aligned}$$

*Step 2.* Now assume that $f$ satisfies (H1), (H2), (H3) and (H4) (but not necessarily (A3)), and $u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^d)$. By Proposition 2.5 we may find a subsequence (not relabelled) and a new sequence $\{v_\varepsilon\}$

(if $p = 1$ or $p = +\infty$ we set $v_\varepsilon := u_\varepsilon$), which still converges weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$ to $u$ and $\{|Dv_\varepsilon|^p\}$ is equi-integrable. If $1 < p < +\infty$ we have by (H4) and

$$\liminf_{\varepsilon\to 0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon\right)dx \geq \liminf_{\varepsilon\to 0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Dv_\varepsilon\right)dx \tag{4.10}$$

$$- C\limsup_{\varepsilon\to 0^+}\int_{\{u_\varepsilon\neq v_\varepsilon\}}(1+|Dv_\varepsilon|^p)\,dx \tag{4.11}$$

$$= \liminf_{\varepsilon\to 0^+}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Dv_\varepsilon\right)dx, \tag{4.12}$$

where we have used the equi-integrability property and (2.3).

Extracting a subsequence if necessary, without loss of generality and by Theorem 1.7 we may assume that

$$v_\varepsilon \overset{3\text{-s}}{\rightharpoonup} u, \quad Dv_\varepsilon \overset{3\text{-s}}{\rightharpoonup} T := Du + D_yU(x,y) + D_zW(x,y,z),$$

and

$$\tau_M\left(Dv_\varepsilon\right) \overset{3\text{-s}}{\rightharpoonup} T_M$$

for all $M \in \mathbb{N}$, and for some $U \in L^p\left(\Omega; W^{1,p}_{\mathrm{per}}(Y_1)\right)$, $W \in L^p\left(\Omega\times Y_1; W^{1,p}_{\mathrm{per}}(Y_2)\right)$, and $T_M \in L^p(\Omega\times Y_1\times Y_2;\mathbb{R}^{d\times N})$. Here $\tau_M(\cdot)$ stands for the usual truncation function

$$\tau_M(\xi) := \begin{cases} \xi & \text{if } |\xi| \leq M, \\ M\frac{\xi}{|\xi|} & \text{otherwise.}\end{cases}$$

Note that

$$\sup_{M\in\mathbb{N}}\|T_M\|_{L^p(\Omega\times Y_1\times Y_2)} \leq \sup_{M\in\mathbb{N}}\liminf_{\varepsilon>0}\|\tau_M\left(Dv_\varepsilon\right)\|_{L^p(\Omega)} \leq \sup_{\varepsilon>0}\|Dv_\varepsilon\|_{L^p(\Omega)} < +\infty,$$

and thus every subsequence of $\{T_M\}$ admits a further subsequence converging weakly in $L^p(\Omega\times Y_1\times Y_2;\mathbb{R}^{d\times N})$. We claim that the weak limit is $T$, thus asserting that the full sequence $\{T_M\}$ converges to $T$ weakly in $L^p(\Omega\times Y_1\times Y_2)$. Indeed, fix $\varphi\in C^\infty_c(\Omega; C^\infty_{\mathrm{per}}(Y_1\times Y_2))$. We have

$$\begin{aligned}\lim_{M\to+\infty}\int_\Omega\int_{Y_1}\int_{Y_2}\varphi(x,y,z)\,T_M(x,y,z)\,dx &= \lim_{M\to+\infty}\lim_{\varepsilon\to 0^+}\int_\Omega\left[\int_\Omega\varphi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)Dv_\varepsilon(x)\,dx\right.\\ &\qquad\left.-\int_{\{|Dv_e|>M\}}\varphi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\left(Dv_e-\tau_M(Dv_e)\right)dx\right]\\ &= \int_\Omega\int_{Y_1}\int_{Y_2}\varphi(x,y,\xi)T(x,y,\xi)\,dx\,dy\,dz,\end{aligned}$$

where we have used (1.2), and where due to (1.4) and to the equi-integrability of $\{|Dv_\varepsilon|^p\}$, by Hoölder inequality we deduce

$$\begin{aligned}&\limsup_{M\to+\infty}\limsup_{\varepsilon\to 0^+}\int_{\{|Dv_\varepsilon|>M\}}\left|\varphi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|\,|Dv_\varepsilon-\tau_M(Dv_\varepsilon)|\,dx\\ &\leq C\left[\lim_{\varepsilon\to 0^+}\left(\int_\Omega\left|\varphi\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^{p'}dx\right)^{1/p'}\limsup_{M\to+\infty}\sup_{\varepsilon>0^+}\left(\int_{\{|Dv_\varepsilon|>M\}}|Dv_\varepsilon|^p\,dx\right)^{1/p}\right]\\ &= \left(\int_\Omega\int_{Y_1}\int_{Y_2}|\varphi(x,y,z)|\,dx\,dy\,dz\right)\left(\limsup_{M\to+\infty}\sup_{\varepsilon>0^+}\left(\int_{\{|Dv_\varepsilon|>M\}}|Dv_\varepsilon|^p\,dx\right)^{1/p}\right) = 0.\end{aligned}$$

By Lemma 4.5 and for every fixed $M \in \mathbb{N}$ we have

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Dv_\varepsilon\right) dx \geq & \liminf_{\varepsilon\to 0^+} \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \tau_M(Dv_\varepsilon)\right) dx \\
& - \limsup_{\varepsilon\to 0^+} \int_{\{|Dv_\varepsilon|>M\}} f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \tau_M(Dv_\varepsilon)\right) dx \\
& \geq \int_\Omega \int_{Y_1} \int_{Y_2} f(y,z,T_M(x,y,z))\, dx\, dy\, dz - \sup_{\varepsilon>0} C \int_{\{|Dv_\varepsilon|>M\}} (1+|Dv_\varepsilon|^p)\, dx,
\end{aligned}$$

where we have used (H4). Letting $M \to +\infty$, since $T_M \rightharpoonup T$ in $L^p$ by Theorem 2.1 and in view of the equi-integrability of $\{|Dv_\varepsilon|^p\}$ we obtain

$$\liminf_{\varepsilon\to 0^+} \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Dv_\varepsilon\right) dx \geq \int_\Omega \int_{Y_1} \int_{Y_2} f(y,z,T(x,y,z))\, dx\, dy\, dz,$$

which, together with (4.10), concludes the proof. □

**Lemma 4.6.** *If $f$ satisfies hypotheses (H1) [or (A1), (A2)], (H2), (H3) and (H4), then*

$$\mathcal{F}_{\{\varepsilon\}}(u,\Omega) \leq \int_\Omega \overline{f_{\text{hom}}}(Du(x))\, dx \tag{4.13}$$

*for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$.*

*Proof.* Fix $u \in W^{1,p}(\Omega;\mathbb{R}^d)$. In view of (H4) it can be shown easily that $\mathcal{F}_{\{\varepsilon\}}(u,\cdot)$ is the trace on $\mathcal{A}(\Omega)$ of a Radon measure absolutely continuous with respect to the $N$-dimensional Lebesgue measure $\mathcal{L}^N$ (for proofs of similar results see [BFLM], [BrFF]). Therefore, proving (4.13) is equivalent to showing that

$$\lim_{\delta\to 0^+} \frac{\mathcal{F}_{\{\varepsilon\}}(u, Q(x_0,\delta))}{\delta^N} \leq \overline{f_{\text{hom}}}(Du(x_0))$$

for a.e. $x_0 \in \Omega$. Choose $x_0$ to be a $p$-Lebesgue point for $Du$, i.e.

$$\lim_{\delta\to 0^+} \frac{1}{\delta^N} \int_{Q(x_0,\delta)} |Du(x) - Du(x_0)|^p\, dx = 0, \tag{4.14}$$

fix $\alpha > 0$, and using the definition of $\overline{f_{\text{hom}}}(Du(x_0))$ choose $\varphi \in C^\infty_{\text{per}}(Q;\mathbb{R}^d)$ such that

$$\overline{f_{\text{hom}}}(Du(x_0)) + \alpha \geq \int_Q f_{\text{hom}}(x, Du(x_0) + D\varphi(x))\, dx. \tag{4.15}$$

In order to apply Theorem 2.3, with $(X,\mathcal{M}) := (\Omega,\mathcal{L})$, $Z := W^{1,p}_{\text{per}}(Q;\mathbb{R}^d)$, $\mu$ is the Lebesgue measure, and $\mathcal{L}$ is the $\sigma$-algebra of Lebesgue measurable sets in $\mathbb{R}^N$, we introduce the multi-valued map

$$\begin{aligned}
H : \Omega \to & \left\{C \subset W^{1,p}_{\text{per}}(Q) : C \neq \emptyset,\ C \text{ is closed}\right\} \\
x \mapsto & H(x) := \left\{\psi \in W^{1,p}_{\text{per}}(Q;\mathbb{R}^d) : \int_Q \psi(y)\, dy = 0,\right. \\
& \left. f_{\text{hom}}(x, Du(x_0) + D\varphi(x)) + \alpha \geq \int_Q f(x,y,Du(x_0) + D\varphi(x) + D\psi(y))\, dy\right\}.
\end{aligned}$$

By definition of $f_{\text{hom}}$, the set $H(x)$ is nonnempty and by (H4) it is closed. Also, the set

$$\left\{(x,\psi) \in \Omega \times W^{1,p}_{\text{per}}(Q;\mathbb{R}^d) : \psi \in H(x)\right\}$$

is closed (hence Borel) because it coincides with $\mathcal{H}^{-1}([0,+\infty) \times \{0\})$ where

$$\mathcal{H}(x,\psi) := \left(f_{\text{hom}}(x, Du(x_0) + D\varphi(x)) + \alpha - \int_Q f(x,y,Du(x_0) + D\varphi(x) + D\psi(y))\, dy,\ \int_Q \psi(y)\, dy\right),$$

and by (H4), and in view of the continuity of $D\varphi$ and of $f_{\text{hom}}$ (see Lemma 4.1), the function $\mathcal{H}$ is continuous. By Theorem 2.3 we may find a measurable selection

$$x \mapsto \psi(x,\cdot), \quad \psi(x,\cdot) \in W^{1,p}_{\text{per}}(Q;\mathbb{R}^d), \ \int_Q \psi(x,y)\,dy = 0,$$

and

$$f_{\text{hom}}(x, Du(x_0) + D\varphi(x)) + \alpha \geq \int_Q f(x, y, Du(x_0) + D\varphi(x) + D_y\psi(x,y))\,dy. \tag{4.16}$$

By (H4) and (4.16) we now have

$$\begin{aligned}
&+\infty > \int_Q [f_{\text{hom}}(x, Du(x_0) + D\varphi(x)) + \alpha]\ dx \\
&\geq C\int_Q \left\{\int_Q [|D_y\psi(x,y)|^p - |Du(x_0)|^p - |D\varphi(x)|^p]\ dy\right\}\ dx \geq C'\int_Q \|\psi(x,\cdot)\|^p_{W^{1,p}_{\text{per}}(Q)}\,dx - \frac{1}{C'},
\end{aligned}$$

where we have used Poincaré-Friedrichs inequality. We conclude that $\psi \in L^p\left(Q; W^{1,p}_{\text{per}}(Q;\mathbb{R}^d)\right)$. Now let $\psi_k \in C^\infty_c\left(Q; W^{1,p}_{\text{per}}(Q;\mathbb{R}^d)\right)$ be such that

$$\|\psi_k - \psi\|_{L^p\left(Q;W^{1,p}_{\text{per}}(Q;\mathbb{R}^d)\right)} \to 0, \tag{4.17}$$

and define

$$u_{k,\varepsilon} := u(x) + \varepsilon\varphi\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right).$$

For fixed $\delta > 0$ it is clear that $u_{k,\varepsilon} \to u$ in $L^p(Q(x_0,\delta))$, and so

$$\begin{aligned}
&\lim_{\delta\to 0^+} \frac{\mathcal{F}_{\{\varepsilon\}}(u, Q(x_0,\delta))}{\delta^N} \\
&\leq \liminf_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\frac{1}{\delta^N}\int_{Q(x_0,\delta)} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}, Du(x) + D\varphi\left(\frac{x}{\varepsilon}\right) + \varepsilon D_x\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right) + D_y\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\,dx \\
&\leq \lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\frac{1}{\delta^N}\int_{Q(x_0,\delta)} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}, Du(x_0) + D\varphi\left(\frac{x}{\varepsilon}\right) + D_y\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)\,dx \\
&\quad + C\limsup_{\delta\to 0^+}\limsup_{\varepsilon\to 0^+}\frac{1}{\delta^N}\left[\int_{Q(x_0,\delta)}\left(1 + |Du(x)|^p + \left|D\varphi\left(\frac{x}{\varepsilon}\right)\right|^p\right.\right. \\
&\qquad\qquad \left.\left. + \left|\varepsilon D_x\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p + \left|D_y\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p\right)\,dx\right]^{\frac{p-1}{p}} \\
&\quad \cdot\left[\int_{Q(x_0,\delta)}\left(|Du(x) - Du(x_0)|^p + \left|\varepsilon D_x\psi_k\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|^p\right)\,dx\right]^{\frac{1}{p}} \\
&= \int_{Y_1}\int_{Y2} f(x,y,Du(x_0) + D\varphi(x) + D_y\psi_k(x,y))\,dx\,dy
\end{aligned}$$

where we have used (4.5), (1.4), (4.14) and Hölder inequality. We deduce that

$$\lim_{\delta\to 0^+}\frac{\mathcal{F}_{\{\varepsilon\}}(u, Q(x_0,\delta))}{\delta^N} \leq \int_{Y_1}\int_{Y2} f(x,y,Du(x_0) + D\varphi(x) + D_y\psi_k(x,y))\,dx\,dy,$$

and in view of (H4) and by (4.15), (4.16) and (4.17), letting $k \to +\infty$ we obtain

$$\begin{aligned}
\lim_{\delta\to 0^+}\frac{\mathcal{F}_{\{\varepsilon\}}(u, Q(x_0,\delta))}{\delta^N} &\leq \int_{Y_1}\int_{Y2} f(x,y,Du(x_0) + D\varphi(x) + D_y\psi(x,y))\,dx\,dy \\
&\leq \overline{f_{\text{hom}}}(Du(x_0)) + 2\alpha.
\end{aligned}$$

Letting $\alpha \to 0^+$ we conclude (4.13). $\square$

*Proof of Theorem 1.9.* Fix $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and let $\{\varepsilon_n\}$ be a sequence converging to $0^+$. By Theorem 2.2 we may extract a subsequence $\{\varepsilon_j\}$ such that the $\Gamma(L^p(\Omega)) - \lim_{j\to\infty} F_{\varepsilon_j}(u,A)$ exists and is given by $\mathcal{F}_{\{\varepsilon_j\}}$. As before, for simplicity of notation we write simple $\varepsilon$ in place of $\varepsilon_j$. We claim that

$$\mathcal{F}_{\{\varepsilon_j\}}(u,A) = \int_A \overline{f_{\hom}}(Du(x))\,dx.$$

If the claim is asserted, then due to the arbitrariness of the initial sequence $\{\varepsilon_n\}$ we conclude the proof of Theorem 1.9. Now Lemma 4.6 establishes that

$$\mathcal{F}_{\{\varepsilon_j\}}(u,A) \le \int_A \overline{f_{\hom}}(Du(x))\,dx,$$

and in view of Lemma 4.3, it suffices to prove that

$$F(u,A) \ge \int_A \overline{f_{\hom}}(Du(x))\,dx.$$

Let $U \in L^p\left(\Omega; W^{1,p}_{\mathrm{per}}(Y_1)\right)$ and $W \in L^p\left(\Omega\times Y_1; W^{1,p}_{\mathrm{per}}(Y_2)\right)$. By the very definitions of $f_{\hom}$ and $\overline{f_{\hom}}$ (see (1.9) and (1.8)) we have by Fubini's Theorem

$$\begin{aligned}
&\int_A\int_{Y_1}\int_{Y_2} f(y,z,Du(x)+D_yU(x,y)+D_zW(x,y,z))\,dx\,dy\,dz\\
&\ge \int_A\int_{Y_1}\left[\int_{Y_2} f(y,z,Du(x)+D_yU(x,y)+D_zW(x,y,z)dz\right]dy\,dx\\
&\ge \int_A\int_{Y_1} f_{\hom}(y,Du(x)+D_yU(x,y))\,dx\,dy \ge \int_A \overline{f_{\hom}}(Du(x))dx,
\end{aligned}$$

and so

$$F(u,A) \ge \int_\Omega \overline{f_{\hom}}(Du(x))\,dx.$$

□

# 5 Proof of Theorem 1.9: The Case $s = 2$

The proof of Theorem 1.9 for second order derivatives follows closely that of the case where $s = 1$ treated in the previous section, with the obvious adaptations. In particular, in Lemma 4.3 the compactness with respect to 3-scale convergence should now invoke Theorem 1.11 in place of Theorem 1.7, and the function $F$ introduced in (4.3) should be replaced by

$$\begin{aligned}
F(u,A) := \inf\Bigg\{&\int_\Omega\int_{Y_1}\int_{Y_2} f(x,y,D^2_{xx}u(x)+D^2_{yy}U(x,y)+D^2_{zz}W(x,y,z))\,dx\,dy\,dz :\\
&U\in L^p(\Omega;W^{2,p}_{\mathrm{per}}(Y_1)), W\in L^p(\Omega\times Y_1;W^{2,p}_{\mathrm{per}}(Y_2)),\\
&U-A(x)y\in L^p(\Omega;W^{2,p}_{\mathrm{per}}(Q;\mathbb{R}^d)) \;\text{ for some } A\in L^p(\Omega;\mathbb{R}^{d\times N}),\\
&W-C(x,y)z\in L^p(\Omega\times Q;W^{2,p}_{\mathrm{per}}(Q;\mathbb{R}^d)) \;\text{ for some } C\in L^p(\Omega\times Q;\mathbb{R}^{d\times N})\Bigg\}.
\end{aligned}$$

We leave the details to the reader.

# 6 Appendix

Here we present the proof of Proposition 2.2, which, in turn, uses the following result (see Theorem 2.2. in [BhFF] and Theorem 2.5 in [BrFF]).

**Proposition 6.1.** *Let $F_\varepsilon : W^{1,p}(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega) \to [0,+\infty)$ be a sequence of functionals verifying the assumptions (i)–(iii) of Proposition 2.2. Then for any $u \in W^{1,p}(\Omega,\mathbb{R}^d)$*

$$\Gamma^p(u,A) := \Gamma(L^p) - \liminf_{\varepsilon\to 0} F_\varepsilon(u,A) = \inf\left\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,A) : u_\varepsilon \to u \text{ in } L^p(A)\right\}$$

*coincides with*

$$\Gamma_0^p(u,A) := \inf\left\{\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,A) : u_\varepsilon \to u \text{ in } L^p(A), u_\varepsilon = u \text{ on a neighborhood of } \partial A\right\}.$$

*Proof of Proposition 6.1.* Clearly $\Gamma^p(u,A) \le \Gamma_0^p(u,A)$, so it remains to prove the opposite inequality. Define for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ the functional $\mathcal{G}(u,A) := \int_A (1+|Du|^p)\,dx$. For every $\gamma > 0$ there exists a sequence $\{u_\varepsilon\} \subset W^{1,p}(A,\mathbb{R}^m)$ such that $u_\varepsilon \to u$ in $L^p(A,\mathbb{R}^d)$ and $\liminf_\varepsilon F_\varepsilon(u_\varepsilon,A) \le \Gamma^p(u,A)+\gamma$. By the coercivity assumption (2.1) there exists a subsequence $\{u_{\varepsilon_k}\}$ of $\{u_\varepsilon\}$ such that $\lim_{k\to\infty} F_{\varepsilon_k}(u_{\varepsilon_k},A) = \liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon,A)$ and the sequence of measures $\nu_k := \mathcal{G}(u,\cdot)+\mathcal{G}(u_{\varepsilon_k},\cdot) \overset{*}{\rightharpoonup} \nu$, where $\nu$ is a finite Radon measure.

For every $t > 0$ let $A_t := \{x \in A : \operatorname{dist}(x,\partial A) > t\}$, fix $\eta > 0$ and for any $0 < \delta < \eta$ define $L_\delta := A_{\eta-2\delta}\backslash\overline{A_{\eta+\delta}}$. Next, Consider a smooth cut off function $\varphi_\delta \in C_c^\infty(A_{\eta-\delta};[0,1])$ such that $\varphi_\delta = 1$ on $A_\eta$, $\|D\varphi_\delta\|_\infty \le \frac{C}{\delta}$ and set $w_\varepsilon := \varphi_\delta u_\varepsilon + (1-\varphi_\delta)u$. Then $w_\varepsilon \to u$ in $L^p(A,\mathbb{R}^d)$ as $\varepsilon \to 0^+$ and $w_\varepsilon = u$ on a neighborhood of $\partial A$. Thus,

$$\begin{aligned} F_{\varepsilon_k}(w_{\varepsilon_k},A) &\le F_{\varepsilon_k}(w_{\varepsilon_k},A_\eta) + F_{\varepsilon_k}(w_{\varepsilon_k},A\backslash A_{\eta-\delta}) + F_{\varepsilon_k}(w_{\varepsilon_k},A_{\eta-2\delta}\backslash\overline{A_{\eta+\delta}}) \\ &\le F_{\varepsilon_k}(u_{\varepsilon_k},A_\eta) + F_{\varepsilon_k}(u,A\backslash\overline{A_{\eta-\delta}}) + C\left[\mathcal{G}(u_{\varepsilon_k},L_\delta)+\mathcal{G}(u,L_\delta)+\frac{1}{\delta^p}\int_{L_\delta}|u_{\varepsilon_k}-u|^p\,dx\right]. \end{aligned}$$

Letting $\varepsilon_k \to 0^+$ we get

$$\limsup_{\varepsilon_k\to 0^+} F_{\varepsilon_k}(w_{\varepsilon_k},A) \le \Gamma^p(u,A) + C\nu(A\backslash\overline{A_{\eta-\delta}}) + C\nu(\overline{L_\delta}) + \gamma,$$

and so

$$\Gamma_0^p(u,A) \le \Gamma^p(u,A) + \gamma + C\nu(A\backslash\overline{A_{\eta-\delta}}) + C\nu(\overline{L_\delta}).$$

Letting $\delta$ go to $0^+$ we have

$$\Gamma_0^p(u,A) \le \Gamma^p(u,A) + \gamma + C\nu(A\backslash A_\eta) + C\nu(\partial A_\eta).$$

Choosing a sequence $\{\eta_j\}$ such that $\eta_j \to 0^+$ and $\nu(\partial A_{\eta_j}) = 0$, letting first $j \to +\infty$ and finally $\gamma \to 0^+$, now yields

$$\Gamma_0^p(u,A) \le \Gamma^p(u,A).$$

□

*Proof of Proposition 2.2.* We consider a countable collection $\mathcal{C}$ of open subsets of $\Omega$ such that for all $\delta > 0$ and $A \in \mathcal{A}(\Omega)$ there exists a finite union $C_A$ of disjoint elements of $\mathcal{C}$ such that

$$\begin{cases} \overline{C_A} \subset A \\ \mathcal{L}^N(A) \le \mathcal{L}^N(C_A) + \delta. \end{cases}$$

Denote by $\mathcal{R}$ the countable collection of all finite unions of elements of $\mathcal{C}$, i.e. $\mathcal{R} := \left\{\cup_{i=1}^k C_i : k \in \mathbb{N}, C_i \in \mathcal{C}\right\}$. Since $L^p$ is a separable metric space, by using a diagonalization argument we can assert that there exists a subsequence $\{\varepsilon_\mathcal{R}\} \subset \{\varepsilon\}$, $\varepsilon_\mathcal{R} \to 0$, such that for every $C \in \mathcal{R}$ and for every $v \in W^{1,p}(C;\mathbb{R}^d)$ there exists $\{u^C_{\varepsilon_\mathcal{R}}\}$ in $W^{1,p}(C;\mathbb{R}^d)$ such that $u^C_{\varepsilon_\mathcal{R}} \to v$ in $L^p(C;\mathbb{R}^d)$ and

$$\Gamma(L^p(C)) - \lim_{\varepsilon_\mathcal{R}\to 0^+} F_{\varepsilon_\mathcal{R}}(v,C) = \lim_{\varepsilon_R\to 0^+} F_{\varepsilon_\mathcal{R}}(u^C_{\varepsilon_\mathcal{R}},C). \tag{6.1}$$

In other words, we get the existence of the $\Gamma-$limit of a fixed subsequence and for every $C \in \mathcal{R}$. We seek to extend (6.1) from the elements of $\mathcal{R}$ to any open subset $A$ of $\Omega$ and every $u \in W^{1,p}(A;\mathbb{R}^d)$. Let $A$ be an open subset of $\Omega$ and $u \in W^{1,p}(A;\mathbb{R}^d)$. Fix $\delta > 0$ and choose a subset $C^\delta$ of $A$ in $\mathcal{R}$ such that

$$\begin{cases} \overline{C^\delta} \subset A, \\ \int_{A \setminus C^\delta} (1 + |Du|^p)\, dx \leq \frac{\delta}{C}, \end{cases}$$

where $C$ is the constant of the growth condition (2.1). Consider a sequence $\{v^{C^\delta}_{\varepsilon^{\mathcal{R}}}\}$ in $W^{1,p}(C^\delta;\mathbb{R}^d)$ such that $v^{C^\delta}_{\varepsilon^{\mathcal{R}}} \to u$ in $L^p(C^\delta;\mathbb{R}^d)$ satisfying

$$\lim_{\varepsilon^{\mathcal{R}} \to 0^+} F_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}}, C^\delta) = \mathcal{F}^-_{\varepsilon^{\mathcal{R}}}(u, C^\delta) = \Gamma(L^p(C)) - \lim_{\varepsilon^{\mathcal{R}} \to 0^+} F_{\varepsilon^{\mathcal{R}}}(u, C^\delta).$$

By Proposition 6.1 we may assume that $v^{C^\delta}_{\varepsilon^{\mathcal{R}}} = u$ nearby $\partial C^\delta$, and extend $v^{C^\delta}_{\varepsilon^{\mathcal{R}}}$ as u outside $C^\delta$, so that the extension (not relabeled) $v^{C^\delta}_{\varepsilon^{\mathcal{R}}}$ belongs to $W^{1,p}(A;\mathbb{R}^d)$. We have

$$\begin{aligned} \limsup_{\delta \to 0^+} \limsup_{\varepsilon^{\mathcal{R}} \to 0^+} F_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}}, A) &\leq \limsup_{\delta \to 0^+} \limsup_{\varepsilon^{\mathcal{R}} \to 0^+} \left\{ F_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}}, C^\delta) + C \int_{A \setminus C^\delta} (1 + |Du|^p)\, dx \right\} \\ &= \limsup_{\delta \to 0^+} \mathcal{F}_{\{\varepsilon^{\mathcal{R}}\}}(u, C^\delta) \leq \mathcal{F}_{\{\varepsilon^{\mathcal{R}}\}}(u, A) \leq \liminf_{\delta \to 0^+} \liminf_{\varepsilon^{\mathcal{R}} \to 0^+} F_{\varepsilon^{\mathcal{R}}}(v^{C^\delta}_{\varepsilon^{\mathcal{R}}}, A). \qquad (6.2) \end{aligned}$$

A diagonalization argument (cf. Lemma 7.1 [BrFF]) and (6.2) concludes the proof. □


## Acknowledgments

The research of I. Fonseca was partially funded by the National Science Foundation under Grant DMS–0103798, the research of E. Zappale by Università di Salerno through PGR Omogeneizzazione di materiali perforati and by GNAMPA. The authors thank the Center for Nonlinear Analysis (NSF Grant No. DMS–9803791), Carnegie Mellon University, Pittsburgh, PA, USA, for its support during the preparation of this paper. E. Zappale is indebted to the Department of Mathematical Sciences at Carnegie Mellon University for its hospitality during her residence at CMU in 2000/01.


## References


[A] G. Allaire, Homogenization and two scale convergence, *SIAM J. Math. Anal.* **23** (1992), 1482–1518.

[ABr] G. Allaire and M. Briane, Multiscale convergence and reiterated homogenization, *Proc. Royal Soc. Edin.* **126** (1996), 297–342.

[AHZ] Y. Amirat, K. Hamdache and A. Ziani, Homogénéisation non-locale pour des équations dégénérées à coéfficients périodiques, *C. R. Acad. Sci. Paris* **312** (1991), 963–966.

[BhFF] K. Bhattacharya, I. Fonseca and G.Francfort, An asymptotic study ot the debonding of thin films. To appear in *Arch. Rat. Mech. Anal.*

[BFLM] G. Bouchitté, I. Fonseca, G. Leoni and L. Mascarenhas, A global method for relaxation in $W^{1,p}$ and $SBV_p$, Preprint No. 00-CNA-017 (2000).

[BFM] G. Bouchitté, I. Fonseca and L. Mascarenhas, A global method for relaxation , *Arch. Rat. Mech. Anal.* **145** (1998), 51–98.

[BrD] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Clarendon Press, Oxford, 1998.

[BL] A. Braides and D. Lukkassen, Reiterated homogenization of integral functionals, *Math. Mod. Meth. Appl.Sc.* **10** (2000), 47–71..

[BrFF] A. Braides, I. Fonseca and G. Francfort, 3D-2D asymptotic analysis for inhomogeneous thin films, *Indiana Univ. Math. J.* **49** (2000), 1367–1404.

[BrFL] A. Braides, I. Fonseca and G. Leoni, A-quasiconvexity: relaxation and homogenization, *ESAIM:COCV* **5** (2000), 539–577.

[Da] G. Dal Maso, *An Introduction to $\Gamma$-Convergence*, Birkhäuser, Boston, 1993.

[D] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.

[Do] P. Donato, Alcune osservazioni sulla convergenza debole di funzioni non uniformemente oscillanti, *Richerche Mat.* **32** (1983), 203–219.

[Do2] P. Donato, Una stima per la differenza di $H$-limiti e qualche applicazione a problemi di omogeneizzazione, *Rendiconti di Matematica* **4** (1983), 623–640.

[FL] I. Fonseca and G. Leoni, Multiscale $\Gamma$-convergence: the non convex case. In preparation.

[FM1] I. Fonseca and S. Müller, Quasi-convex integrands and lower semicontinuity in $L^1$, *SIAM J: Math. Anal.* **23** (1992), 1081–1098.

[FM2] I. Fonseca and S. Müller, A-quasiconvexity, lower semicontinuity and Young measures, *SIAM J. Math. Anal.* **30** (1999), 1355–1390.

[FMP] I. Fonseca , S. Müller and P. Pedregal, Analysis of concentration and oscillation effects generated by gradients, *SIAM J. Math. Anal.* **29** (1998), 736–756.

[I1] A. D. Ioffe, On lower semicontinuity of integral functionals I, *SIAM J. Cont. Optim.* **15** (1977), 521–538.

[I2] A. D. Ioffe, On lower semicontinuity of integral functionals II, *SIAM J. Cont. Optim.* **15** (1977), 991–1000.

[K] J. Kristensen, Finite functionals and Young measures generated by gradients of Sobolev functions. Report 1994-34, Mathematical Institute, Tech. U. of Denmark, Lyngby, Denmark, 1994.

[Ma1] P.Marcellini, Periodic solutions and homogenization of non linear variational problems, *Ann.Mat. Pura Appl.* **117** (1978), 139–152.

[Ma2] P.Marcellini, Approximation of quasiconvex functions and semicontinuity of multiple integrals, *Manus. Math.* **51** (1985), 1–28.

[MT] L. Mascarenhas and A.M. Toader, Scale convergence in homogenization , *Num. Funct. Anal. Optim.* **22** (2001), 127–158.

[Mo] Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, Berlin, 1966.

[Mu] F. Murat, Compacité par compensation : condition necessaire et suffisante de continuité faible sous une hypothése de rang constant, *Ann. Sc. Norm. Sup. Pisa* bf 8 (1981), 68–102.

[N1] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989), 608–623.

[N2] G. Nguetseng, Asymptotic Analysis for a stiff variational problem arising in mechanichs, *SIAM J. Math. Anal.* **21** (1990), 1394–1414.

[T1] L. Tartar, Compensated Compactness and Applications to partial differential equations, *Nonlinear Analysis and Mechanics*, Heriott-Watt Symposium, ed. R.J.Knops, vol. IV, Research Notes in Matemathics, **39**, Pitman , (1979) 136–212.

[T2] L. Tartar, The compensated compactness method applied to systems of conservation laws, in *Systems of Nonlinear Partial Differential Eq.*, J. M. Ball, ed., Riedel, 1983.

[T3] L. TARTAR, Étude des oscillations dans les équations aux dérivées partielles nonlinéaires, Springer Lectures Notes in Physics, Springer-Verlag, Berlin, **195** (1984), 384–412.

[T4] L. TARTAR, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, *Proc. Roy. Soc. Edin.* **115A** (1990), 193–230.

[T5] L. TARTAR, On mathematical tools for studying partial differential equations of continuum physics: H-measures and Young measures, in *Developments in Partial Differential Equations and Applications to Mathematical Physics*, Buttazzo, Galdi, Zanghirati, eds., Plenum, New York, 1991.