Higher Order Variational Problems and Phase Transitions in Nonlinear Elasticity

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Abstract. Higher order variational problems ask for new techniques which do not reduce to mere generalizations of their counterparts for first order problems. These challenges will be illustrated within the context of phase transitions for nonlinear elastic materials, and of higher order lower semicontinuity results recently obtained.

1. Introduction

Higher order variational problems appear often in the engineering literature — in connection with the so-called gradient theories of phase transitions within elasticity regimes, and where higher order terms give rise to surface energies; in the study of equilibria of micromagnetic materials where mastery of second order energies (here accounting for the exchange energy) is required (see [57], [101]; see also [33], [42], [48], [49], [67], [87], [88], [89], [120]); in the Blake-Zisserman model for image segmentation in computer vision (see [38], [39], [40]; see also [54]), which seats squarely among second-order free discontinuity models that may be recasted as higher order Griffiths' models for fracture mechanics (see [9], [26], [30],[64], [76], [77], [78], [79]); in several mechanics of thin films, delamination and blistering,etc., with bending energy appearing as the higher order term; in the analysis of many other Landau theories with nonconvex variational problems regularized by higher order singular perturbations; etc.

Energy functionals may include lower dimensional order terms to take into account interfacial energies and discontinuities of underlying fields (see [12]), although in what follows we will neglect the role played by these terms and we will focus on the added difficulties inherent to the presence of derivatives of order two or more.

The main difficulty encountered with the handling of the higher order energies is that one would be tempted to treat them as first order problems. However, a lesson to be learned is that techniques for higher order variational problems do not reduce to mere generalizations of their counterparts for first order problems (e.g., by assuming k-quasiconvexity in place of quasiconvexity, see [46], [97], [100]). Indeed, although functionals depending *uniquely* on the highest order derivatives can be treated easily, those where lower order terms are present require new ideas and new tools to handle the localization and truncation of lower order terms.

To illustrate, consider an energy functional

$$I(u) := \int_\Omega f(x, u, \nabla u, \dots, \nabla^k u)\, dx \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded domain, $u : \Omega \to \mathbb{R}^d$, $N, d \geq 1$, $u := (u_1, \dots, u_d)$, $\nabla u \in \mathbb{R}^{d\times N}$, and $(\nabla u)_{ij} := \frac{\partial u_i}{\partial x_j}$ for $i \in \{1, \dots, d\}$, $j \in \{1, \dots, N\}$.

What assumptions on f guarantee that if $\{u_n\}$ is a sequence bounded in $W^{k,1}(\Omega; \mathbb{R}^d)$ and if $u_n \to u$ in $W^{k-1}(\Omega; \mathbb{R}^d)$ then

$$I(u) \leq \liminf_{n\to+\infty} I(u_n)?$$

The usual technique amounts to keep unchanged the higher order, oscillating terms, while freezing the lower order, strongly converging terms, by means of suitable truncations and the use of Scorza-Dragoni-type uniform continuity theorems. Truncating gradients so that they remain gradients may be achieved through the techniques of maximal functions and of Fourier multipliers (see [1], [109], [110]) in those cases where the bulk energy density f has superlinear growth (see the proof of Lemma 2.15 in [74]). In fact, the success of this approach relies heavily on p-equi-integrability, and thus cannot be extended to the case $p = 1$ where one replaces weak convergence in $W^{k,1}(\Omega; \mathbb{R}^n)$ with the natural convergence, i.e., strong convergence in $W^{k-1,1}(\Omega; \mathbb{R}^n)$ and bounds on the L^1 norms of the k-th order derivatives. As it turns out, when f grows at most linearly at infinity many seemingly simple questions, long ago answered within the realm of first order problems, still defy all attempts when we deal with order two or more. As an example, a standing open problem is the following (in the case where $k = 1$ this question was first answered by Fonseca and Müller in [72] and [73] and later improved by Fonseca and Leoni in [65] Theorem 1.8):

is it true that if $f : \Omega \times \mathbb{R}^s \times \mathbb{R}^{d\times N^k} \to [0, \infty)$ is a Borel integrand, with $s := d + d \times N + \ldots + d \times N^{k-1}$, if $f(x, \mathbf{v}, \cdot)$ is k-quasiconvex, with $k \geq 2$ and $\mathbf{v} := (u, \nabla u, \dots, \nabla^{k-1} u)$, if f satisfies "reasonable" continuity properties with respect to x and $\mathbf{v}$, if

$$C|\xi| - \frac{1}{C} \leq f(x, \mathbf{v}, \xi) \leq C(1 + |\xi|)$$

for all $(x, \mathbf{v}) \in \Omega \times \mathbb{R}^s$, if $u \in W^{k-1,1}(\Omega; \mathbb{R}^d)$, $\nabla^k u \in BV(\Omega; \mathbb{R}^{d\times N^k})$, and if $\{u_n\}$ is a sequence of functions in $W^{k,1}(\Omega; \mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$, then

$$\int_\Omega f(x, u, \dots, \nabla^k u)\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx?$$

2. Phase transitions

The asymptotic behavior of functionals of the type

$$J_\varepsilon(v;\Omega) := \int_\Omega \frac{1}{\varepsilon} W(v) + \varepsilon |\nabla v|^2 \, dx \tag{2}$$

has been extensively studied within the context of phase transitions. Adopting a Gibbs' criterium for equilibria, the energy of a stable fluid in a 2-phase mode may be identified with minima of the problem

$$(P_0) \qquad \text{minimize} \quad \int_\Omega W(u)\, dx \qquad \text{with} \quad \int_\Omega u \, dx = m$$

where Ω is an open, bounded set in $\mathbb{R}^N$ and W is a nonnegative bulk energy density with $\{W = 0\} = \{a, b\}$, $a, b \in \mathbb{R}$, $a < b$. Clearly, if $m < a$ or $m > b$ then the minimizer of (P_0) is simply $u \equiv m$ and no transitions are energetically favorable. However, if $m \in (a, b)$ then this problem admits infinitely many solutions, those of the form

$$u(x) := \begin{cases} a & \text{if } x \in D, \\ b & \text{if } x \in \Omega \setminus D, \end{cases}$$

where D is any measurable subset of Ω with measure $\frac{\mathcal{L}^N(D)}{\mathcal{L}^N(\Omega)} = \frac{b-m}{b-a} =: \theta$. This lack of uniqueness is due to the fact that interfaces are allowed to form without a concomitant increase of energy.

In 1893 Van der Waals proposed a gradient theory as a selection criteria for physically preferred solutions which takes into account interfacial energy. Important developments of this theory concerning the interfacial energy between phases were later obtained by Cahn and Hilliard [35]. Upon rescaling, the new penalized energy now reads as (2), and the minimization problem becomes

$$(P_\varepsilon) \qquad \text{minimize} \quad J_\varepsilon(u;\Omega) \qquad \text{with} \quad \int_\Omega u \, dx = m.$$

In 1983 Gurtin [85] conjectured that minimizers for (P_ε) converge to minimizers of (P_0) with minimal interfacial energy. This conjecture was proved in 1984 in the scalar case (where $N = 1$) by Carr, Gurtin and Slemrod [37]. Precisely, if $\Omega = (0, 1)$ then (P_ε) admits only two minimizers ($u_\varepsilon(x)$ and $u_\varepsilon(1-x)$), and moreover

$$J_\varepsilon(u_\varepsilon) = \varepsilon K_0 + O\left(\mathrm{e}^{-C/\varepsilon}\right)$$

where $K_0 := 2\int_a^b \sqrt{W(s)}\, ds$ stands for the surface energy per unit area, and C is a positive constant.

In the higher dimensional case ($N > 1$) Gurtin's conjecture was proved independently by Modica [98] and by Sternberg [111]. The approach in both [98] and [111] uses De Giorgi's notion of Γ-convergence ([53]; see also [4], [29], [50]) and follows closely ideas of Modica and Mortola [99] who studied a similar functional proposed by De Giorgi in a completely different physical context. In particular, the result of Modica [98] only holds for minimizers of (P_ε), as it relies on results on the nature of minimizers of (P_0) obtained by Gonzalez, Massari and Tamanini [83], but it requires no regularity hypotheses on W beyond continuity, while Sternberg [111], under stronger regularity hypotheses on W, proved that

$$\Gamma - \lim_{\varepsilon \to 0^+} J_\varepsilon\left(u_0; \Omega\right) = \begin{cases} K_0 \mathrm{Per}_\Omega(E) & \text{if} \quad u = \chi_E a + (1 - \chi_E) b,\ |E| = \theta\,|\Omega|\,, \\ & \qquad u \in BV\left(\Omega; \{a, b\}\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

The theory of Γ-convergence guarantees, in particular, that preferred designs are those which exhibit minimal interfacial area for the given volume fraction θ.

Generalizations of (2)-(3) were obtained by Bouchitté [25] and by Owen and Sternberg [104] for the undecoupled problem, in which the integrand in J_ε has the form $\varepsilon^{-1} f(x, v(x), \varepsilon \nabla v(x))$. For the study of local minimizers we refer to Kohn and Sternberg [92].

The vector-valued setting, where $u : \Omega \to \mathbb{R}^d$, $\Omega \subset \mathbb{R}^N$, $d, N > 1$, was considered in [20], [75] and [111], where K_0 is replaced by

$$K_1 := \inf\left\{ \int_{-L}^{L} W\left(g\left(s\right)\right) + \left|g'\left(s\right)\right|^2 ds : L > 0,\ g \text{ piecewise } C^1,\ \ g(-L) = a, \quad g(L) = b \right\}.$$

The case where W has more than two wells was addressed by Baldo [16] (see also Sternberg [112]), and later generalized by Ambrosio [7].

Motivated by questions within the realm of elastic solid-to-solid phase transitions (see [19], [43] , [91]) we now consider the corresponding problem for gradient vector fields where $u : \Omega \to \mathbb{R}^d$ stands for the deformation, and in place of J_ε we introduce

$$I_\varepsilon\left(u; \Omega\right) := \begin{cases} \displaystyle\int_\Omega \frac{1}{\varepsilon} W(\nabla u) + \varepsilon |\nabla^2 u|^2 \, dx & \text{if } u \in W^{2,2}\left(\Omega; \mathbb{R}^d\right), \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

The analysis of this model has defied considerable mathematical efforts during the past decade, and a significant contribution for two distinct classes of constitutive assumptions was made recently by Conti, Fonseca and Leoni [44].

An intermediate case between (2) and (4), where the nonconvex potential depends on u and the singular perturbation on $\nabla^2 u$, has been recently studied by Fonseca and Mantegazza [71] (for other generalizations see [63]). Also, in the two-dimensional case and when W vanishes on the unit circle (4) reduces to the so-called Eikonal functional which arises in the study of liquid crystals (see [13]) as well as in blistering of delaminated thin films (see [103]). Recently, the Eikonal problem has received considerable mathematical attention, but in spite of substantial partial progress (see [11], [14], [59], [90]) its Γ-limit remains to be identified.

Going back to the results obtained in [44] concerning (4), we first notice that frame-indifference requires that $W(\xi) = W(R\xi)$ for all ξ and all $R \in SO(N)$, where

$SO(N)$ is the set of rotations in $\mathbb{R}^N$. Therefore, and by analogy with the hypotheses initially placed on (2), if we assume that $W(A) = 0 = W(B)$ then $\{W = 0\} \subset SO(N)A \cup SO(N)B$. Also, in order to guarantee the existence of "classical" (as opposed to measure-valued) non affine solutions for the limiting problem, and in view of Hadamard's compatibility condition for layered deformations (see also Ball and James [19]), the two wells must be rank-one connected. Hence, so as to be able to construct gradients taking values only on $\{A, B\}$ and layered perpendicularly to ν, we assume that

$$A - B = a \otimes \nu$$

for some $a \in \mathbb{R}^N$ and $\nu \in S^{N-1} := \partial B(0,1) \subset \mathbb{R}^N$, and we simplify (greatly!) the problem by removing the frame-indifference constraint, and setting simply

$$\{W = 0\} = \{A, B\}.$$

Without loss of generality we may assume that

$$A = -B = a \otimes e_N.$$

Since now interfaces of minimizers must be planar with normal e_N (see [19]), at first glance the analysis may seem to be significantly less involved as compared with the initial problem (2) which requires the handling of minimal surfaces. However, it turns out that the PDE constraint curl $= 0$ imposed on the admissible fields presents numerous difficulties to the characterization of the Γ-limsup. In particular, if, say, ∇u has a layered structure with two interfaces then it is possible to construct a "realizing" (effective, or recovering) sequence nearby each interface, but the task of gluing together the two sequences on a suitable low-energy intermediate layer is very delicate. In order to illustrate the difficulties encountered here, we explain briefly how we would "normally" undertake the heuristic argument to glue together two optimal sequences $\{(u_n, \varepsilon_n)\}$, corresponding to an interface of a cylindrical body Ω at a given height h, and $\{(v_n, \delta_n)\}$, corresponding to an interface at a height h', $h' > h$.

First we must convince ourselves that the sequences $\{\varepsilon_n\}$ and $\{\delta_n\}$ (related to the "periodicity" of the ripples of the optimal fine structure near each interface) may be taken to be the same. This is by no means trivial (although true)! But let us take this for granted, and, as it is usual, we consider as a candidate for the two-interface situation a convex combination

$$w_{k,n} := \varphi_k u_n + (1 - \varphi_k) v_n$$

where φ_k is a smooth cut-off function, with $\{0 < \varphi_k < 1\} \subset L_{k,n}$ and $L_{k,n}$ is a horizontal layer intermediate between heights h and h'. The crux of the problem is to choose $L_{k,n}$ in a judicious way so that no extra energy is added to the system by the new sequence $\{w_{k,n}\}$.

Using De Giorgi's Slicing Method, we slice horizontally the layer between heights h and h' into M horizontal sub-layers L_k of width $(h' - h)/M$. In view of the fact that $||\nabla\varphi_k||_\infty = O(M)$, we then have

$$\sum_{k=1}^{M}\int_{L_k}\frac{1}{\varepsilon_n}W(w_{k,n})+\varepsilon_n|\nabla w_{k,n}|^2 \leq O\left(\frac{1}{\varepsilon_n}\right)+\varepsilon_n M^2||u_n-v_n||^2. \tag{5}$$

Choosing $k = k(n)$ such that

$$\int_{L_{k(n)}}\frac{1}{\varepsilon_n}W(w_{k(n),n})+\varepsilon_n|\nabla w_{k(n),n}|^2 \leq \frac{1}{M}\sum_{k=1}^{M}\int_{L_k}\frac{1}{\varepsilon_n}W(w_{k,n})+\varepsilon_n|\nabla w_{k,n}|^2, \tag{6}$$

it is clear that by setting $M = O\left(\frac{1}{\varepsilon_n\sqrt{||u_n-v_n||_{L^2}}}\right)$, and using the fact that the admissible sequences $\{u_n\}$ and $\{v_n\}$ satisfy $||u_n - v_n||_{L^2} \to 0$ in the intermediate layer between heights h and h', we may conclude that

$$\lim_{n\to\infty}\int_{L_{k(n)}}\frac{1}{\varepsilon_n}W(w_{k(n),n})+\varepsilon_n|\nabla w_{k(n),n}|^2 = 0.$$

Suppose now that we want to extend this argument to the present setting involving second order derivatives. Since $||\nabla^2\varphi_k||_\infty = O(M^2)$, the estimate (5) now becomes

$$\sum_{k=1}^{M}\int_{L_k}\frac{1}{\varepsilon_n}W(\nabla w_{k,n})+\varepsilon_n|\nabla^2 w_{k,n}|^2 \leq O\left(\frac{1}{\varepsilon_n}\right)+\varepsilon_n M^4||u_n-v_n||^2,$$

and, seeking *equi-partition of energy* as we have done above, we are led to

$$\varepsilon_n M^4||u_n-v_n||^2 = O\left(\frac{1}{\varepsilon_n}\right).$$

This would entail $M = O\left(\frac{1}{\sqrt{\varepsilon_n}\sqrt{||u_n-v_n||_{L^2}}}\right)$, and thus the upper bound in (6) would be

$$\int_{L_{k(n)}}\frac{1}{\varepsilon_n}W(\nabla w_{k,n})+\varepsilon_n|\nabla^2 w_{k,n}|^2 \leq \frac{1}{M}O\left(\frac{1}{\varepsilon_n}\right) = O\left(\sqrt{\frac{||u_n-v_n||_{L^2}}{\varepsilon_n}}.\right)$$

Therefore, in order to ensure that the extra energy in the layer $L_{k(n)}$ does not affect the optimality of the sequence, we would have to guarantee that $\{u_n - v_n\}$ goes to zero in L^2 faster than ε_n, and whether or not this holds it remains an open question!

The restrictive constitutive hypotheses placed on W in Theorems 2.1 and 2.2 below will allow us to find alternative ways in which the gluing is successful. We must, however, accept the fact that matching in one single swift step works well for first order problems but it is simply too abrupt when dealing with higher order derivatives. Indeed, the proofs of Theorems 2.1 and 2.2 are strongly hinged on a two-step matching technique where the control of Poincaré and Poincaré-Freidrichs' constants is carefully kept.

Theorem 2.1 ([44], Theorem 1.3). *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions*

(H_1) *W is continuous, $W(\xi) = 0$ if and only if $\xi \in \{A, B\}$, where $A = -B = a \otimes e_N$ for some $a \in \mathbb{R}^d \setminus \{0\}$;*

(H_2) *there exists $C > 0$ such that*

$$W(\xi) \geq C|\xi| - \frac{1}{C}$$

for all $\xi \in \mathbb{R}^{d\times N}$;

(H_3) *$W(\xi) \geq W(0, \xi_N)$ where $\xi = (\xi', \xi_N) \in \mathbb{R}^{d\times(N-1)} \times \mathbb{R}^d$.*

Suppose, in addition, that W is differentiable at A and B. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, with $\nabla u \in BV(\Omega; \{A, B\})$. Then there exists $K > 0$ such that

$$\Gamma - \lim_{\varepsilon \to 0^+} I_\varepsilon(u; \Omega) = K \operatorname{Per}_\Omega(E),$$

where $\nabla u = (1 - \chi_E(x)) A + \chi_E(x) B$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

The hypothesis (H_3) entails a one dimensional (geodesic) character to the asymptotic problem. Indeed, it can be shown that K reduces to the analog of the constant K_1 introduced in (2), precisely,

$$K := \inf\left\{\int_{-L}^{L} W(0, g(s)) + |g'(s)|^2\, ds \,:\, L > 0,\ g \text{ piecewise } C^1,\right.$$
$$\left. g(-L) = -a,\ g(L) = a\right\}.$$

Theorem 2.2 ([44], Theorem 1.4). *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, simply connected domain with Lipschitz boundary. Assume that W satisfies the conditions (H_1),*

(H_4) *W is even in each variable ξ_i, $i = 1, \cdots, N-1$, that is*
$W(\xi_1, \cdots, -\xi_i, \cdots, \xi_N) = W(\xi_1, \cdots, \xi_i, \cdots, \xi_N)$ for each $i = 1, \cdots, N-1$,
and that there exist an exponent $p \geq 2$, constants $c, C, \rho > 0$ and a convex function $g : [0, \infty) \to [0, \infty)$, with $g(s) = 0$ if and only if $s = 0$, such that g is derivable in $s = 0$, it satisfies the doubling condition $g(2t) \leq cg(t)$ for all $0 \leq t \leq \rho$,

$$g(|\xi - A|) \leq W(\xi) \leq cg(|\xi - A|) \ \text{ if } |\xi - A| \leq \rho,$$
$$g(|\xi - B|) \leq W(\xi) \leq cg(|\xi - B|) \ \text{ if } |\xi - B| \leq \rho,$$

and

$$\frac{1}{C}|\xi|^p - C \leq W(\xi) \leq C(|\xi|^p + 1)$$

for all $\xi \in \mathbb{R}^{d\times N}$. Let $u \in W^{1,1}(\Omega; \mathbb{R}^d)$, with $\nabla u \in BV(\Omega; \{A, B\})$. Then

$$\Gamma - \lim_{\varepsilon \to 0^+} I_\varepsilon(u; \Omega) = K_{\text{per}} \operatorname{Per}_\Omega(E),$$

where $\nabla u(x) = (1 - \chi_E(x))\, A + \chi_E(x)\; B$ *for* $\mathcal{L}^N$ *a.e.* $x \in \Omega$, *and*

$$K_{\text{per}} := \inf\left\{ \int_Q L\, W(\nabla v) + \frac{1}{L}\left|\nabla^2 v\right|^2 \, dx : L > 0, v \in W^{2,\infty}\left(Q;\mathbb{R}^d\right), \right.$$
$$\left. \nabla v = \pm a \otimes e_N \text{ nearby } x_N = \pm\frac{1}{2},\ v \text{ periodic of period one in } x' \right\}.$$

Note that the hypotheses of Theorems 2.1 and 2.2 are satisfied by the prototype bulk energy density

$$W(\xi) := \min\left\{ |\xi - A|^2, |\xi - B|^2 \right\}.$$

As asserted above, under hypothesis (H_3) one has $K_{\text{per}} = K$, hence one-dimensional interface profiles are energetically preferred. Building upon the example by Jin and Kohn [90], in [44] we show that without hypothesis (H_3) we may have

$$K_{\text{per}} < K$$

thereby proving that optimal interface profiles are, at least in some cases, not one-dimensional. This happens because generating finite in–plane gradients (i.e. having a dependence on the coordinates parallel to the interface) reduces the energy in the regions far away from the two potential wells. The zero-curl constraint leads then to an oscillatory pattern. In elasticity, this multidimensional behavior has been predicted in [36]. Similar mechanisms are at play in the theory of micromagnetism, where indeed various non-one-dimensional wall structures are known, such as cross-tie domain walls and charged zigzag walls in ferromagnetic thin films (see e.g. [87] and references therein). It would be interesting to know if K_{per} is smaller or equal to K for realistic ferroelastic potentials obtained from the Landau theory of phase transitions.

3. Lower Semicontinuity for Higher Order Variational Problems in $\mathbf{W^{k,p}}$, $\mathbf{p > 1}$

Morrey's notion of *quasiconvexity* was extended by Meyers [97] to the realm of higher-order variational problems. We recall that $f : \Omega \to [0, +\infty)$ is said to be *quasiconvex* if (see [46], [100])

$$f(\xi) \leq \int_Q f(\xi + \nabla\varphi(x))\, dx \tag{7}$$

for all $\varphi \in C_c^\infty(Q;\mathbb{R}^d)$, and a function $F : E_k^d \to \mathbb{R}$ is said to be *k-quasiconvex* if

$$F(\xi) \leq \int_Q F(\xi + \nabla^k w(y))\, dy \tag{8}$$

for all $\xi \in E_k^d$ and all $w \in C_c^\infty(Q;\mathbb{R}^d)$.

To fix notation, here and in what follows, Ω is an open, bounded domain in $\mathbb{R}^N$, $Q := (-1/2, 1/2)^N$, $C_c^\infty(\mathbb{R}^N; \mathbb{R}^d)$ is the space of infinitely differentiable $\mathbb{R}^d$-valued functions in Ω with compact support, and $C_{\mathrm{per}}^\infty(\mathbb{R}^N; \mathbb{R}^d)$ stands for the space of Q–periodic functions in $C^\infty(\mathbb{R}^N; \mathbb{R}^d)$. Recall that f is said to be *Q-periodic* if $f(x+k\mathbf{e}_i) = f(x)$ for all x, all $k \in \mathbb{Z}^k$, and for all $i = 1, \ldots, N$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$. For each $j \in \mathbb{N}$ the symbol $\nabla^j u$ stands for the vector-valued function whose components are all the jth order derivatives of u. If u is C^∞ then for $j \geq 2$ we have that $\nabla^j u(x) \in E_j^d$, where E_j^d denotes the space of symmetric j-linear maps from $\mathbb{R}^N$ into $\mathbb{R}^d$. We set $E_0^d := \mathbb{R}^d$, $E_1^d := \mathbb{R}^{d\times N}$ and

$$E_{[j-1]}^d := E_0^d \times \cdots \times E_{j-1}^d, \quad E_{[0]}^d := E_0^d.$$

For any integer $k \geq 2$ we define

$$BV^k(\Omega; \mathbb{R}^d) := \left\{u \in W^{k-1,1}(\Omega; \mathbb{R}^d) : \nabla^{k-1} u \in BV(\Omega; E_{k-1}^d)\right\},$$

where here $\nabla^j u$ is the Radon–Nikodym derivative of the distributional derivative $D^j u$ of $\nabla^{j-1} u$, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$ (see [12]).

Meyers [97] proved that *k-quasiconvexity* is a necessary and sufficient condition for (sequential) lower semicontinuity of (1) with respect to weak convergence (resp. weak* convergence if $p = \infty$) in $W^{k,p}(\Omega; \mathbb{R}^d)$ under appropriate growth and continuity conditions on the integrand f. Meyers' argument uses results of Agmon, Douglis and Nirenberg [2] concerning Poisson kernels for elliptic equations, and later Fusco [80] gave a simpler proof using De Giorgi's Slicing Lemma. He also extended the result to Carathéodory integrands when $p = 1$, while the case $p > 1$ has been recently established by Guidorzi and Poggiolini [84], who relied heavily on a p-Lipschitz assumption, i.e.,

$$|f(x, \mathbf{v}, \xi) - f(x, \mathbf{v}, \xi_1)| \leq C(1 + |\xi|^{p-1} + |\xi_1|^{p-1})|\xi - \xi_1|.$$

As it turns out, k-quasiconvex integrands with p-growth are p-Lipschitz. This assertion was established by Marcellini [94] for $k = 1$, the case $k = 2$ was proven in [84], and recently Santos and Zappale [107] extended it to arbitrary k.

The first integral representations for the relaxed energies when the integrand depends on the full set of variables, that is $f = f(x, u, \ldots, \nabla^k u)$, were obtained by Braides, Fonseca and Leoni in [32], where this question may be seen as a corollary of very broad results casted for variational problems under PDE constraints (here curl$= 0$), the *$\mathcal{A}$-quasiconvexity theory.* In [74] Fonseca and Müller proved that $\mathcal{A}$-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$(u, v) \mapsto \int_\Omega f(x, u(x), v(x))\, dx,$$

whenever $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty)$ is a Carathéodory integrand satisfying

$$0 \leq f(x, u, v) \leq a(x, u)\,(1 + |v|^q),$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^d \times \mathbb{R}^m$, where $1 \le q < \infty$, $a \in L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R}; [0,\infty))$, $u_n \to u$ in measure, $v_n \rightharpoonup v$ in $L^q(\Omega;\mathbb{R}^m)$ and $\mathcal{A}v_n \to 0$ in $W^{-1,q}(\Omega;\mathbb{R}^l)$ (see also [47]). In the sequel $\mathcal{A} : L^q(\Omega;\mathbb{R}^m) \to W^{-1,q}(\Omega;\mathbb{R}^l)$, $\mathcal{A}v := \sum_{i=1}^N A^{(i)} \frac{\partial v}{\partial x_i}$ is a constant–rank, first order linear partial differential operator, with $A^{(i)} : \mathbb{R}^m \to \mathbb{R}^l$ linear transformations, $i = 1, \dots, N$. We recall that $\mathcal{A}$ satisfies the *constant-rank* property if there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}w = r \quad \text{for all } w \in S^{N-1}, \tag{9}$$

where

$$\mathbb{A}w := \sum_{i=1}^N w_i \, A^{(i)}, \qquad \text{for} \quad \text{for} \quad w \in \mathbb{R}^N.$$

A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be $\mathcal{A}$-*quasiconvex* if

$$f(v) \le \int_Q f(v + w(y))\, dy$$

for all $v \in \mathbb{R}^m$ and all $w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^m)$ such that $\mathcal{A}w = 0$ and $\int_Q w(y)\, dy = 0$.

The relevance of this general framework, as emphasized by Tartar (see [113], [114], [115], [116], [117], [118]; see also [46], [47], [102]) lies on the fact that in continuum mechanics and electromagnetism PDEs other than $\operatorname{curl} v = 0$ arise naturally, and this calls for a relaxation theory which encompasses PDE constraints of the type $\mathcal{A}v = 0$. Important examples include the cases where $\mathcal{A}v = 0$ if and only if $curlv = 0$, as well as k-th order gradients, where, by replacing the target space $\mathbb{R}^m$ by the finite dimensional vector space E^d_k, it is possible to find a first order linear partial differential operator $\mathcal{A}$ such that $v \in L^q(\Omega; E^d_k)$ and $\mathcal{A}v = 0$ if and only if there exists $\varphi \in W^{k,q}(\Omega;\mathbb{R}^d)$ such that $v = \nabla^k \varphi$ (see Corollary 3.2). Here A-quasiconvexity reduces to k-quasiconvexity (see (8)) when the energy density is continuous.

Let $1 \le p < \infty$ and $1 < q < \infty$, and consider the functional

$$F : L^p(\Omega;\mathbb{R}^d) \times L^q(\Omega;\mathbb{R}^m) \times \mathcal{O}(\Omega) \to [0,\infty)$$

defined by

$$F((u,v);D) := \int_D f(x, u(x), v(x))\, dx,$$

where $\mathcal{O}(\Omega)$ is the collection of all open subsets of Ω, and the density f satisfies the following hypothesis:

(H) $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0,\infty)$ is a Carathéodory function, and

$$0 \le f(x,u,v) \le C\,(1 + |u|^p + |v|^q)$$

for a.e. $x \in \Omega$ and all $(u,v) \in \mathbb{R}^d \times \mathbb{R}^m$, and for some constant $C > 0$.

For $D \in \mathcal{O}(\Omega)$ and $(u,v) \in L^p(\Omega;\mathbb{R}^d) \times (L^q(\Omega;\mathbb{R}^m) \cap \operatorname{Ker}\mathcal{A})$ define

$$\mathcal{F}((u,v);D) := \inf\Big\{ \liminf_{n\to\infty} F((u_n,v_n);D) :\ (u_n,v_n) \in L^p(D;\mathbb{R}^d) \times L^q(D;\mathbb{R}^m), \\ u_n \to u \text{ in } L^p(D;\mathbb{R}^d), \quad v_n \rightharpoonup v \text{ in } L^q(D;\mathbb{R}^m), \\ \mathcal{A}v_n \to 0 \quad \text{in } W^{-1,q}(D;\mathbb{R}^l) \Big\}. \tag{10}$$

It can be shown that the condition $\mathcal{A}v_n \to 0$ imposed in (10) may be equivalently replaced by requiring that v_n satisfy the homogeneous PDE $\mathcal{A}v = 0$. Precisely,

$$\mathcal{F}((u,v);D) = \inf\Big\{ \liminf_{n\to\infty} F((u_n,v_n);D) :\ (u_n,v_n) \in L^p(D;\mathbb{R}^d) \times L^q(D;\mathbb{R}^m), \\ u_n \to u \text{ in } L^p(D;\mathbb{R}^d), \quad v_n \rightharpoonup v \text{ in } L^q(D;\mathbb{R}^m), \quad \mathcal{A}v_n = 0 \Big\}. \tag{11}$$

The following integral representation for the relaxed energy $\mathcal{F}$ was obtained in [32].

Theorem 3.1 ([32], Theorem 1.1). *Under condition (H) and the constant-rank hypothesis (9), for all $D \in \mathcal{O}(\Omega)$, $u \in L^p(\Omega;\mathbb{R}^d)$, and $v \in L^q(\Omega;\mathbb{R}^m) \cap \operatorname{Ker}\mathcal{A}$, we have*

$$\mathcal{F}((u,v);D) = \int_D \mathrm{Q}_{\mathcal{A}} f(x,u(x),v(x))\, dx$$

where, for each fixed $(x,u) \in \Omega \times \mathbb{R}^d$, the function $\mathrm{Q}_{\mathcal{A}} f(x,u,\cdot)$ is the $\mathcal{A}$- quasiconvexification of $f(x,u,\cdot)$, namely

$$\mathrm{Q}_{\mathcal{A}} f(x,u,v) := \inf\left\{ \int_Q f(x,u,v+w(y))\, dy : w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^m) \cap \operatorname{Ker}\mathcal{A}, \right. \\ \left. \int_Q w(y)\, dy = 0 \right\}$$

for all $v \in \mathbb{R}^m$.

The proof of this theorem relies heavily on the use of Young measures (see [17], [121]), together with the blow-up method introduced by Fonseca and Müller in [72], and the arguments developed in [74] (see also [15], [93]) .

Corollary 3.2 ([32], Theorem 1.3). *Let $1 \le p \le \infty$, $s \in \mathbb{N}$, and suppose that $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty)$ is a Carathéodory function satisfying*

$$0 \le f(x,\mathbf{u},v) \le C\left(1 + |\mathbf{u}|^p + |v|^p\right), \qquad 1 \le p < \infty,$$

for a.e. $x \in \Omega$ and all $(\mathbf{u},v) \in E^d_{[k-1]} \times E^d_k$, where $C > 0$, and

$$f \in L^\infty_{\mathrm{loc}}(\overline{\Omega} \times E^d_{[k-1]} \times E^d_k;[0,\infty)) \qquad \text{if } p = \infty.$$

Then for every $u \in W^{k,p}(\Omega;\mathbb{R}^d)$ *we have*

$$\int_\Omega \mathcal{Q}^k f(x,u,\dots,\nabla^k u)\,dx = \inf\left\{ \liminf_{n\to\infty} \int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx : \right.$$

$$\left. \{u_n\} \subset W^{k,p}(\Omega;\mathbb{R}^d), u_n \rightharpoonup u \text{ in } W^{k,p}(\Omega;\mathbb{R}^d) \quad (\overset{*}{\rightharpoonup} \text{ if } p=\infty) \right\},$$

where, for a.e. $x \in \Omega$ *and all* $(\mathbf{u}, v) \in E^d_{[k-1]} \times E^d_k$,

$$\mathcal{Q}^k f(x,\mathbf{u},v) := \inf\left\{ \int_Q f(x,\mathbf{u},v+\nabla^k w(y))\,dy : w \in C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d) \right\}.$$

4. Lower Semicontinuity for Second Order Variational Problems in $\mathbf{BH^p}$, $\mathbf{p > 1}$

There are now several lower semicontinuity results in spaces involving jumps as well as higher order derivatives. In particular, in [70] we considered functionals defined on the space of *functions with bounded Hessian*

$$\begin{aligned} BH(\Omega;\mathbb{R}^d) &:= \{u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^d) : D^2u \text{ is a finite Radon measure}\} \\ &= \{u \in \mathrm{L}^1(\Omega;\mathbb{R}^d) : Du \in BV(\Omega;\mathbb{R}^{d\times N}\} \end{aligned}$$

where D^2u denotes the distributional Hessian of u. For $1 < p < +\infty$ we also define

$$BH^p(\Omega;\mathbb{R}^d) := \{u \in BH(\Omega;\mathbb{R}^d) : \nabla^2 u \in L^p(\Omega;E^d_2)\}.$$

For various properties of the space BH, we refer to Demengel [55], [56], Carriero, Leaci and Tomarelli [38] and Temam [119]. We recall that if $u \in BH(\Omega;\mathbb{R}^d))$ then $Du = \nabla u$ and $[\nabla u](x) = 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$. We obtained the following BH generalization of Ambrosio's Theorem 4.3 in [8] originally stated within the framework of SBV spaces.

Theorem 4.1 ([70], Theorem 1.2). *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \times E^d_2 \to [0,+\infty)$$

be a normal integrand, 2−quasiconvex in Λ, *and such that*

$$|\Lambda|^p \le f(x,u,\xi,\Lambda) \ \le a\,(x,u,\xi)\,(1+|\Lambda|^p) \tag{12}$$

for $\mathcal{L}^N$ *a.e.* $x \in \Omega$ *and all* $(u,\xi,\Lambda) \in \mathbb{R}^N \times \mathbb{R}^{d\times N} \times E^d_2$, *where* $a(x,u,\xi)$ *is a nonnegative constant, and* $p > 1$. *Then for every* $u \in W^{1,1}(\Omega;\mathbb{R}^d)$ *and any sequence* $\{u_n\} \subset BH(\Omega;\mathbb{R}^d)$ *converging to* u *in* $W^{1,1}(\Omega;\mathbb{R}^d)$ *and such that*

$$\left|D^2_s u_n\right|(\Omega) \to 0 \tag{13}$$

we have

$$\int_\Omega f(x,u,\nabla u,\nabla^2 u)\,dx \le \liminf_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n,\nabla^2 u_n)\,dx.$$

We remark that no smoothness nor integrability properties are required from the function $(x,u,\xi) \mapsto a(x,u,\xi)$ that appears in the upper bound (12). All that is needed is that $a(x,u,\xi)$ be defined, finite, and non-negative for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(u,\xi,\Lambda) \in \mathbb{R}^N \times \mathbb{R}^{d\times N} \times E_2^d$. In this generality, Theorem 4.1 is new even in the Sobolev setting, considerably improving our Corollary 3.2 above. Indeed, we have

Corollary 4.2 ([70], Corollary 1.3). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \times E_2^d \to [0,+\infty)$$

be a normal integrand, $2-$quasiconvex in Λ, and such that

$$0 \le f(x,u,\xi,\Lambda) \ \le a\,(x,u,\xi)\,(1+|\Lambda|^p)$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(u,\xi,\Lambda) \in \mathbb{R}^N \times \mathbb{R}^{d\times N} \times E_2^d$, where $a(x,u,\xi)$ is a non-negative constant, and $p > 1$. Then for every $u \in W^{2,p}(\Omega;\mathbb{R}^d)$ and any sequence $\{u_n\} \subset W^{2,p}(\Omega;\mathbb{R}^d)$ weakly converging to u in $W^{2,p}(\Omega;\mathbb{R}^d)$ we have

$$\int_\Omega f(x,u,\nabla u,\nabla^2 u)\,dx \le \liminf_{n\to\infty} \int_\Omega f\,(x,u_n,\nabla u_n,\nabla^2 u_n)\,dx.$$

The approach in Theorem 4.1 is quite different from the one of Theorem 3.1 and uses instead some approximation results of BH^p functions by $W^{2,\infty}$ functions obtained following Ambrosio [8] lead and Acerbi and Fusco [1] ideas, via maximal functions.

Indeed, the $A-$quasiconvexity method used in [32] strongly relies on the underlying PDEs which characterize the space $W^{2,p}(\Omega;\mathbb{R}^d)$. However, in a recent paper [69] extending a result of Alberti [3] we have shown that in the passage from the Sobolev spaces to the space BH the Hessian matrix D^2u remains symmetric but it may loose, in general, the PDE constraint curl$=0$. More precisely, we have proved that

Theorem 4.3. *([69], Theorem 1.4) Let Ω be an open subset of $\mathbb{R}^N$ and let f be a function in $L^1\left(\Omega;E_2^d\right)$. Then there exists $u \in BH(\Omega)$ and a constant $C > 0$ depending only on N such that*

$$D^2u = f\,\mathcal{L}^N + [\nabla u] \otimes \nu_{\nabla u}\,\mathcal{H}^{N-1} \lfloor\, S(\nabla u),$$

and

$$\int_\Omega |u| + |\nabla u|\ dx + \int_{S(\nabla u)\cap\Omega} |[\nabla u]|\ d\mathcal{H}^{N-1} \le C \int_\Omega |f|\ dx.$$

5. Lower Semicontinuity for Higher Order Variational Problems in $\mathbf{W^{k,1}}$

As mentioned before, classical truncation methods for $k=1$ cannot be extended in a simple way to truncate higher order derivatives, and successful techniques often rely on p-equi-integrability, and thus cannot work in the linear growth case. Indeed, when $p=1$ due to loss of reflexivity of the space $W^{k,1}(\Omega;\mathbb{R}^d)$ one can only conclude that an energy bounded sequence $\{u_n\}\subset W^{k,1}(\Omega;\mathbb{R}^d)$ with $\sup_n\|u_n\|_{W^{k,1}}<\infty$ admits a subsequence (not relabeled) such that

$$u_n\to u \quad \text{in } W^{k-1,1}(\Omega;\mathbb{R}^d),$$

where $u\in W^{k-1,1}(\Omega;\mathbb{R}^d)$ and $\nabla^{k-1}u$ is a vector-valued function of bounded variation. Consequently, we now seek to establish lower semicontinuity in the space $W^{k,1}(\Omega;\mathbb{R}^d)$ under this natural notion of convergence, and when $u\in BV^k(\Omega;\mathbb{R}^d)$ (see [61], [122]). When $k=1$ the scalar case $d=1$ has been extensively treated, while the vectorial case $d>1$ was first studied by Fonseca and Müller in [72] where it was proven (sequential) lower semicontinuity in $W^{1,1}(\Omega;\mathbb{R}^d)$ of a functional

$$u\mapsto\int_\Omega f(x,u(x),\nabla u(x))\,dx,$$

with respect to strong convergence in $L^1(\Omega;\mathbb{R}^d)$ (see also [10], [65], [66], [73], and the references contained therein). The approach in [72] is based on blow–up and truncation methods.

The following theorem was proved in the case $k=1$ by Ambrosio and Dal Maso [10], while Fonseca and Müller [72] treated general integrands of the form $f=f(x,u,\nabla u)$, but their argument requires coercivity (see also [81]). The case $k\geq 2$ is due to Amar and De Cicco [5] (see [68] for a proof for all $k\geq 1$).

Proposition 5.1 ([68], Proposition 2.1). *Let $f: E_k^d\to[0,\infty)$ be a function k-quasiconvex, such that*

$$0\leq f(\xi)\leq C\,(1+|\xi|)\,,$$

for all $\xi\in E_k^d$. Moreover, when $k\geq 2$ assume that

$$f(\xi)\geq C_1\,|\xi| \quad \textit{for } |\xi| \textit{ large.}$$

Let $\{u_n\}$ be a sequence of functions in $W^{k,1}(Q;\mathbb{R}^d)$ converging to 0 in the space $W^{k-1,1}(Q;\mathbb{R}^d)$. Then

$$f(0)\ \leq\liminf_{n\to\infty}\int_Q f(\nabla^k u_n)\,dx.$$

More generally we consider the case where f depends essentially only on x and on the highest order derivatives, that is $\nabla^k u(x)$. This situation is significantly simpler than the general case, since it does not require to truncate the initial sequence $\{u_n\}\subset W^{k,1}(\Omega;\mathbb{R}^d)$.

Theorem 5.2 ([68], Theorem 1.1). *Let $f:\Omega\times E_{[k-1]}^d\times E_k^d\to[0,\infty)$ be a Borel integrand. Suppose that for all $(x_0,\mathbf{v}_0)\in\Omega\times E_{[k-1]}^d$ and $\varepsilon>0$ there exist $\delta_0>0$ and a modulus of continuity ρ, with $\rho(s)\leq C_0(1+s)$ for $s>0$ and for some $C_0>0$, such that*

$$f(x_0,\mathbf{v}_0,\xi)-f(x,\mathbf{v},\xi)\leq\varepsilon(1+f(x,\mathbf{v},\xi))+\rho(|\mathbf{v}-\mathbf{v}_0|) \tag{14}$$

for all $x \in \Omega$ *with* $|x - x_0| \leq \delta_0$, *and for all* $(\mathbf{v}, \xi) \in E^d_{[k-1]} \times E^d_k$. *Assume also that one of the following three conditions is satisfied:*
(a) $f(x_0, \mathbf{v}_0, \cdot)$ *is* k*-quasiconvex in* E^d_k *and*

$$\frac{1}{C_1}|\xi| - C_1 \leq f(x_0, \mathbf{v}_0, \xi) \leq C_1(1 + |\xi|) \qquad \textit{for all } \xi \in E^d_k,$$

where $C_1 > 0$;
(b) $f(x_0, \mathbf{v}_0, \cdot)$ *is* 1*-quasiconvex in* E^d_k *and*

$$0 \leq f(x_0, \mathbf{v}_0, \xi) \leq C_1(1 + |\xi|) \qquad \textit{for all } \xi \in E^d_k,$$

where $C_1 > 0$;
(c) $f(x_0, \mathbf{v}_0, \cdot)$ *is convex in* E^d_k.
Let $u \in BV^k(\Omega; \mathbb{R}^d)$ *and let* $\{u_n\}$ *be a sequence of functions in* $W^{k,1}(\Omega; \mathbb{R}^d)$ *converging to* u *in* $W^{k-1,1}(\Omega; \mathbb{R}^d)$. *Then*

$$\int_\Omega f(x, u, \ldots, \nabla^k u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \ldots, \nabla^k u_n)\, dx.$$

Here $f(x_0, \mathbf{v}_0, \cdot)$ is said to be 1-quasiconvex if $f(x_0, \mathbf{v}_0, \cdot)$ is the trace on E^d_k of a 1-quasiconvex function $\bar{f}$ defined on $\mathbb{R}^{(d \times N^{k-1}) \times N}$.

An important class of integrands which satisfy (14) of Theorem 5.2 is given by

$$f = f(x, \xi) := h(x) g(\xi),$$

where $h(x)$ is a nonnegative lower semicontinuous function and g is a nonnegative function which satisfies either (a) or (b) or (c). The case where $h(x) \equiv 1$ and g satisfies condition (a) was proved by Amar and De Cicco [5]. Theorem 5.2 extends a result of Fonseca and Leoni (Theorem 1.7 in [65]) to higher order derivatives, where the statement is exactly that of Theorem 5.2 setting $k = 1$ and excluding part (a). Related results when $k = 1$ where obtained previously by Serrin [108] in the scalar case $d = 1$ and by Ambrosio and Dal Maso [10] in the vectorial case $d > 1$ (see also Fonseca and Müller [72], [73]). Even in the simple case where $f = f(\xi)$ it is not known if Theorem 5.2(a) still holds without the coercivity condition

$$f(\xi) \geq \frac{1}{C_1}|\xi| - C_1.$$

The main tool in the proof of Theorem 5.2, used also in an essential way in subsequent results, is the *blow–up method* introduced by Fonseca and Müller [72], [73], which reduces the domain Ω to a ball and the target function u to a polynomial.

When the integrand f depends on the full set of variables in an essential way, the situation becomes significantly more complicated since one needs to truncate gradients and higher order derivatives in order to localize lower order terms. The following theorem was proved for $k = 1$ by Fonseca and Leoni in [65], Theorem 1.8, and extended to the higher order case in [68].

Theorem 5.3 ([68], Theorem 1.2). *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty)$ be a Borel integrand, with $f(x,\mathbf{v},\cdot)$ 1-quasiconvex in E^d_k. Suppose that for all $(x_0,\mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ either $f(x_0,\mathbf{v}_0,\cdot) \equiv 0$, or for every $\varepsilon > 0$ there exist C, $\delta_0 > 0$ such that*

$$f(x_0,\mathbf{v}_0,\xi) - f(x,\mathbf{v},\xi) \leq \varepsilon(1 + f(x,\mathbf{v},\xi)), \tag{15}$$

$$C|\xi| - \frac{1}{C} \leq f(x_0,\mathbf{v}_0,\xi) \leq C(1+|\xi|) \tag{16}$$

for all $(x,\mathbf{v}) \in \Omega \times E^d_{[k-1]}$ with $|x - x_0| + |\mathbf{v} - \mathbf{v}_0| \leq \delta_0$ and for all $\xi \in E^d_k$. Let $u \in BV^k(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x,u,\dots,\nabla^k u)\,dx \leq \liminf_{n\to\infty} \int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx.$$

A standing open problem is to decide whether Theorem 5.3 continues to hold under the weaker assumption that $f(x,\mathbf{v},\cdot)$ is k-quasiconvex, which is the natural assumption in this context.

As in Theorem 5.2, conditions (15) and (16) can be considerably weakened if we assume that $f(x,\mathbf{v},\cdot)$ is convex rather than 1-quasiconvex. Indeed we have the following result:

Theorem 5.4 ([68], Theorem 1.5). *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty]$ be a lower semicontinuous function, with $f(x,\mathbf{v},\cdot)$ convex in E^d_k. Suppose that for all $(x_0,\mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ either $f(x_0,\mathbf{v}_0,\cdot) \equiv 0$, or there exist C_1, $\delta_0 > 0$, and a continuous function $g : B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0) \to E^d_k$ such that*

$$f(x,\mathbf{v},g(x,\mathbf{v})) \in L^\infty\left(B(x_0,\delta_0) \times B(\mathbf{v}_0,\delta_0);\mathbb{R}\right), \tag{17}$$

$$f(x,\mathbf{v},\xi) \geq C_1|\xi| - \frac{1}{C_1}$$

for all $(x,\mathbf{v}) \in \Omega \times E^d_{[k-1]}$ with $|x - x_0| + |\mathbf{v} - \mathbf{v}_0| \leq \delta_0$ and for all $\xi \in E^d_k$. Let $u \in BV^k(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x,u,\dots,\nabla^k u)\,dx \leq \liminf_{n\to\infty} \int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx.$$

Theorem 5.4 was obtained by Fonseca and Leoni (see [66], Theorem 1.1) in the case $k = 1$. It is interesting to observe that without a condition of the type (17) Theorem 5.4 is false in general. This has been recently proved by Černý and Malý in [41].

The proofs of Theorems 5.2(b) and (c), 5.3 and 5.4 can be deduced easily from the corresponding ones in [65], [66], where $k = 1$. It suffices to write

$$\int_\Omega f(x,u(x),\dots,\nabla^k u(x))\,dx =: \int_\Omega F(x,\mathbf{v}(x),\nabla\mathbf{v}(x))\,dx$$

with $\mathbf{v} := \left(u, \dots, \nabla^{k-1}u\right)$, and then to perturb the new integrand F in order to recover the full coercivity conditions necessary to apply the results in [65], [66]. This approach cannot be used for k-polyconvex integrands and a new proof is needed to treat this case (see [52]). Thus Theorem 5.2(a) and Theorem 5.5 below are the only truly genuine higher order results, in that they cannot be reduced in a trivial way to a first order problem.

For each $\xi \in E^d_k$ let $\mathcal{M}(\xi) \in \mathbb{R}^\tau$ be the vector whose components are all the minors of ξ.

Theorem 5.5 ([65], Theorem 1.6). *Let $h : \Omega \times E^d_{[k-1]} \times \mathbb{R}^\tau \to [0, \infty]$ be a lower semicontinuous function, with $h(x, \mathbf{v}, \cdot)$ convex in $\mathbb{R}^\tau$. Suppose that for all points $(x_0, \mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ either $h(x_0, \mathbf{v}_0, \cdot) \equiv 0$, or there exist C, $\delta_0 > 0$, and a continuous function $g : B(x_0, \delta_0) \times B(\mathbf{v}_0, \delta_0) \to \mathbb{R}^\tau$ such that*

$$h(x, \mathbf{v}, g(x, \mathbf{v})) \in L^\infty \left(B(x_0, \delta_0) \times B(\mathbf{v}_0, \delta_0); \mathbb{R}\right),$$
$$h(x, \mathbf{v}, v) \geq C|v| - \frac{1}{C}$$

for all $(x, \mathbf{v}) \in \Omega \times E^d_{[k-1]}$ with $|x - x_0| + |\mathbf{v} - \mathbf{v}_0| \leq \delta_0$ and for all $v \in \mathbb{R}^\tau$. Let $u \in BV^k(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,p}(\Omega; \mathbb{R}^d)$ which converges to u in $W^{k-1,1}(\Omega; \mathbb{R}^d)$, where p is the minimum between N and the dimension of the vectorial space E^d_{k-1}. Then

$$\int_\Omega h\left(x, u, \dots, \nabla^{k-1}u, \mathcal{M}(\nabla^k u)\right) dx$$
$$\leq \liminf_{n\to\infty} \int_\Omega h(x, u_n, \dots, \nabla^{k-1}u_n, \mathcal{M}(\nabla^k u_n))\, dx.$$

Theorem 5.5 is closely related to a result of Ball, Currie and Olver [18], where it was assumed that

$$h(x, \mathbf{v}, v) \geq \gamma\left(|v|\right) - \frac{1}{C},$$

where

$$\frac{\gamma\left(s\right)}{s} \to \infty \text{ as } s \to \infty.$$

Also, as stated above and with $k = 1$, Theorem 5.5 was proved by Fonseca and Leoni in [66], Theorem 1.4.

In the scalar case $d = 1$, that is when u is an $\mathbb{R}$-valued function, and for first order gradients, i.e. $k = 1$, condition (16) can be eliminated, see Theorem 1.1 in [65]. In particular in [65] Fonseca and Leoni have shown the following result

Proposition 5.6 ([65], Corollary 1.2). *Let $g : \mathbb{R}^N \to [0, \infty)$ be a convex function, and let $h : \Omega \times \mathbb{R} \to [0, \infty)$ be a lower semicontinuous function. If $u \in BV(\Omega; \mathbb{R})$*

and $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R})$ *converges to* u *in* $L^1(\Omega;\mathbb{R})$, *then*

$$\int_\Omega h(x,u)g(\nabla u)\,dx \le \liminf_{n\to\infty} \int_\Omega h(x,u_n)g(\nabla u_n)\,dx.$$

It is interesting to observe that an analog of this result is false when $k \ge 2$.

Theorem 5.7 ([68], Theorem 1.4). *Let* $\Omega := (0,1)^N$, $N \ge 3$, *and let* h *be a smooth cut-off function on* $\mathbb{R}$ *with* $0 \le h \le 1$, $h(u) = 1$ *for* $u \le \frac{1}{2}$, $h(u) = 0$ *for* $u \ge 1$. *There exists a sequence of functions* $\{u_n\}$ *in* $W^{2,1}(\Omega;\mathbb{R})$ *converging to zero in* $W^{1,1}(\Omega;\mathbb{R})$ *such that* $\{\|\Delta u_n\|_{L^1(\Omega;\mathbb{R})}\}$ *is uniformly bounded and*

$$\limsup_{n\to\infty} \int_\Omega h(u_n)(1-\Delta u_n)^+\,dx < \int_\Omega h(0)\,dx.$$

Once again, we are confronted here with new challenges that are present on variational problems involving higher order derivatives.

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