RELAXATION OF CONVEX FUNCTIONALS: THE GAP PROBLEM

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ABSTRACT. It is shown that the relaxed energy

$$\mathcal{F}(u,A) := \inf\Big\{\liminf_{n\to+\infty}\int_A f(x,\nabla u_n)dx : \ \{u_n\}\subset W^{1,\beta}_{\rm loc}(A)\,,\ u_n\to u \text{ in } L^1(A)\Big\},$$

admits the representation

$$\mathcal{F}(u,A) = \int_A f(x,\nabla u)dx + \mu_s(A)\ ,$$

where f is a convex, Carathéodory integrand satisfying a non standard "α-β"growth hypothesis, $\beta\in[\alpha, N\alpha/(N-1))$. Sufficient conditions guaranteeing that $\mu_s = 0$ are discussed. An example asserting that this representation may fail in the quasiconvex case is provided.

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CONTENTS

1. INTRODUCTION

The Lavrentiev phenomenon, or gap problem, has stirred renewed interest in recent years as it challenges traditional theories in the Calculus of Variations. A prototype model, relevant to the study of cavitation in rubber-like materials (see [3], [20], [27], [28], among others), assigns to each deformation $u\in W^{1,N}(\Omega;\mathbb{R}^N)$ the total energy

$$F(u,\Omega) := \int_\Omega |\nabla u|^{N-1} + |\det\nabla u|\,dx, \tag{1.1}$$

where Ω is an open, bounded domain in $\mathbb{R}^N$. Clearly sequences of deformations in $W^{1,N}(\Omega;\mathbb{R}^N)$ with bounded energy will be weakly compact in $W^{1,N-1}(\Omega;\mathbb{R}^N)$ but not necessarily in $W^{1,N}$, so it may be possible to approach energetically functions $u\in W^{1,N-1}(\Omega;\mathbb{R}^N)\setminus W^{1,N}(\Omega;\mathbb{R}^N)$. We then seek to characterize the limiting, effective energy associated to u. This example has been studied at length, and in particular we refer to [1], [9], [16]. More generally, consider a bulk energy density $f:\Omega\times\mathbb{R}^{d\times N}\to[0,+\infty)$ satisfying the following standing hypotheses:

(H1) f is Carathéodory;
(H3) ["α-β"growth condition] $|z|^\alpha\le f(x,z)\le C(1+|z|^\beta)$ for all $z\in\mathbb{R}^{d\times N}$, $\mathcal{L}^N$ a.e. $x\in\Omega$, and for some $C>0$, where $1<\alpha\le\beta<\frac{N\alpha}{N-1}$.

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For every Borel set $E \subset \Omega$ and every $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ we set

$$F(u,E) := \int_E f\big(x, \nabla u(x)\big)\,dx,$$

and we introduce the *relaxed functional*

$$\mathcal{F}(u,A) := \inf\Big\{\liminf_{n\to+\infty} F(u_n,A) : \ \{u_n\} \subset W^{1,\beta}_{\mathrm{loc}}(A), u_n \to u \text{ in } L^1(A)\Big\}.$$

We search for an integral representation for $\mathcal{F}(u,A)$.

When $\alpha = \beta$ and (H1), (H3) hold, then it is well known that (see [AF], [B], [D], [M])

$$\mathcal{F}(u,A) = \int_A Qf(x,\nabla u)\,dx,$$

where the *quasiconvex envelope* Qf of f is defined by

$$Qf(\xi) := \inf\left\{\int_{(0,1)^N} f(\xi + \nabla\varphi(x))\,dx \ : \ \varphi \in W^{1,\infty}_0(Q;\mathbb{R}^d)\right\}.$$

If $\alpha < \beta - \beta/N$ then one may have $\mathcal{F}(u,\Omega) = 0$ (see [4]), and in the case where $\alpha = \beta - \beta/N$ it may happen that $\mathcal{F}(u,\cdot)$ is not even subadditive (see [11]). When f does not depend on the position vector $x \in \Omega$ these degeneracies cannot occur if $\beta \in [\alpha, \alpha N/(N-1))$. Within this range, and using the global method for relaxation introduced by Bouchitté, Fonseca and Mascarenhas (see [7]), together with an extension operator from $W^{1,p}$ into $W^{1,q}$ obtained by Fonseca and Malý (see [15], Lemma 2.2), it was proven in [6], Theorem 3.1, that if $f = f(\nabla u)$ satisfies (H3) and if $\mathcal{F}(u,\Omega) < +\infty$ then

$$\mathcal{F}(u,A) = \int_A Qf(\nabla u)\,dx + \mu_s(A) \tag{1.2}$$

for all Borel sets $A \subset \Omega$ and for some finite, Radon measure μ_s, singular with respect to $\mathcal{L}^N \llcorner \Omega$, the N-dimensional Lebesgue measure in Ω. Earlier results on lower semicontinuity for certain ranges $\alpha < \beta$ and with quasiconvex integrands were obtained by [24], [25], and for polyconvex energy densities with $\alpha \geq N-1$, $\beta = N$, we refer to [1], [8], [9], [11], [12], [17], [18], [21], [22], [23].

In this paper we address the effect of considering an inhomogeneous density, i.e. $f = f(x,\xi)$. In the main theorem of this paper we show that the analogue to (1.2),

$$\mathcal{F}(u,A) = \int_A Qf(x,\nabla u)\,dx + \mu_s(A) \tag{1.3}$$

still holds provided f satisfies (H1), (H3) and

(H2) $f(x,\cdot)$ is convex for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Precisely, we prove

Theorem 1.1. *If* (H1)–(H3) *hold, if A is an open subset of Ω, if $u \in W^{1,\alpha}(\Omega)$, and if $\mathcal{F}(u,A) < +\infty$, then for every open set $B \subset A$*

$$\mathcal{F}(u,B) = \int_B f(x,\nabla u)\,dx + \mu_s(B)$$

where μ_s is a nonnegative Radon measure singular with respect to $\mathcal{L}^N$.

Again the proof of this result is strongly hinged on the global method of relaxation introduced in [7] (see also [5]), although some of the arguments may now be greatly simplified by exploiting the convexity assumption. In particular, the uplifting operator $\mathcal{P}$ introduced in [15] (see also [6]) is no longer needed. The proof presented in Section 3 concerns the scalar case where $d = 1$ and the argument is entirely similar in the vector valued-scalar case. Therefore and for simplicity, we leave the obvious adaptations to the reader.

Section 4 is devoted to the study of several sufficient conditions ensuring that the Lavrentiev phenomenon does not occur. In view of Ioffe's Theorem (see Theorem 2.4), the functional F is lower semicontinuous and thus $F \leq \mathcal{F}$. We then say that the Lavrentiev phenomenon occurs when at some point we have the strict inequality $F < \mathcal{F}$.

The first result asserts that the Lavrentiev phenomenon is local, precisely,

Proposition 1.2. *If for every $x \in \Omega$ there is a ball $B = B_{\rho(x)}(x) \subset \Omega$ such that*

$$\mathcal{F}\left(u, B_{\rho(x)}(x)\right) = F\left(u, B_{\rho(x)}(x)\right) \tag{1.4}$$

then $\mathcal{F}(u,\cdot) = F(u,\cdot)$.

An interesting model example of convex functionals depending on x and with a growth-coercivity gap as in (H3) is provided by $\int |Du|^{p(x)}\,dx$ for some function $p : \Omega \to \mathbb{R}$ (see [29], [30]), or, more generally,

$$u \mapsto \int_\Omega f(x, \nabla u(x))\,dx$$

where $f : \Omega \times \mathbb{R}^N \to [0,+\infty)$ satisfies

$$|z|^{p(x)} \le f(x,z) \le C(1 + |z|^{p(x)}) \quad \text{for all } z \in \mathbb{R}^N, \text{ a.e. } x \in \Omega, \tag{1.5}$$

and for some $C > 0$. The proof of Theorem 1.3 below follows closely the original argument of Zhikov and Fan (see [29]), although some of the technical difficulties encountered by those authors may now be avoided using the theory developed in this paper.

Let $p : \Omega \mapsto \mathbb{R}$ be a continuous function such that

$$1 < \alpha \le p(x) \le \beta$$

and

$$|p(x) - p(y)| \le \frac{\gamma}{|\log|x-y||} \quad \text{whenever } 0 < |x-y| \le \frac{1}{2}, \tag{1.6}$$

for some $\alpha, \beta, \gamma > 0$.

Theorem 1.3. *Let $f : \Omega \times \mathbb{R}^N \to [0,+\infty)$ satisfy* (H1), (H2), (1.5), *and assume that* (1.6) *holds. If $u \in W^{1,\alpha}_{\mathrm{loc}}(\Omega)$ is such that $f(\cdot, \nabla u) \in L^1_{\mathrm{loc}}(\Omega)$ then*

$$\mathcal{F}(u,\cdot) = F(u,\cdot).$$

The last section of this work, Section 5, is dedicated in its entirety to the treatment of an example falling into the class of prototype energies (1.1) which satisfy (H1), (H3), are polyconvex rather than convex, and for which a genuine measurable dependence on x prevents (1.3) to hold. This fact provides one more evidence that quasiconvex energies and convex energies do share quite different properties, and that care must be taken when generalizing results from the convex to the non-convex setting. With Proposition 1.4 we show that Theorem 1.1 may be false if stated for integrands $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$, $d, N > 1$, satisfying (H1), (H3), and

(H2′) $f(x,\cdot)$ is quasiconvex for $\mathcal{L}^N$ a.e. $x \in \Omega$.

We recall that a Borel function $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be *quasiconvex* (see [Morrey]) if

$$\int_{(0,1)^N} g(\xi + \nabla\varphi(y))\,dy \ge g(\xi)$$

for all $\xi \in \mathbb{R}^{d\times N}$ and all $\varphi \in W^{1,\infty}_0((0,1)^N;\mathbb{R}^d)$. Clearly convex functions are quasiconvex, and a simple example of a quasiconvex function which is not convex is given by $\xi \mapsto |\det \xi|$ (for a detailed study of quasiconvexity we refer the reader to [1], [2], [9], [11], [12], [16], [17], [23], [24], [25], [26]). In the absence of growth-coercivity gap, i.e. when $\alpha = \beta$ in (H3), it was proven by Acerbi and Fusco (see [2]), that $F(\cdot,\Omega)$ is sequentially weakly lower semicontinuous in $W^{1,\alpha}(\Omega;\mathbb{R}^d)$, and, in addition, it can be shown that (see also [10])

$$\mathcal{F}(u,\Omega) = \int_\Omega f(x, \nabla u(x))\,dx.$$

If there is a gap and under (H1), (H3), then Theorem 3.1 in [15] still holds, and so for $u \in W^{1,\alpha}(\Omega;\mathbb{R}^d)$ such that $\mathcal{F}(u,\Omega) < +\infty$ we have

$$\mathcal{F}(u,A) = \int_A f^u(x)\,dx + \mu_s(A) \quad \text{for all } A \text{ open subset of } \Omega\ ,$$

where μ_s is a non-negative finite Radon measure on Ω, singular with respect to $\mathcal{L}^N \llcorner \Omega$, and for some density f^u.

The next natural question is: under which conditions can we guarantee that

$$f^u(x) = f(x, \nabla u(x)) \quad \text{for a.e. } x \in \Omega \; ? \tag{1.7}$$

Theorem 1.1 tells us that (H2), i.e. the convexity of $f(x, \cdot)$ for a.e. $x \in \Omega$, implies the identity in (1.7), and Theorem 3.2 in [6] ensures (1.7) if (H2′) holds and if f does not depend on x.

Here we show that this last statement is close to being optimal, in that if f has a genuine measurable dependence on x, and even though (H1), (H2′), (H3) still hold, then it may happen that

$$f^u(x) < f(x, \nabla u(x)) \quad \text{for all } x \in E,$$

where E is some set with $|E| > 0$. The construction of such function f is motivated by the result of Gangbo (see [18], Theorem 3.1) asserting that

$$u \in W^{1,N}(\Omega; \mathbb{R}^N) \mapsto \int_\Omega \chi_K(x) \; |\det(\nabla u(x))| \, dx$$

is lower semicontinuous in $W^{1,N}(\Omega; \mathbb{R}^N)$ for the weak topology of $W^{1,p}(\Omega; \mathbb{R}^N)$, with $N - 1 < p < N$, if and only if $|\partial K| = 0$.

Proposition 1.4. *Fix $\theta \in (0, 1)$, let $p \in (N - 1, N)$, let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain, let K be a compact subset of Ω with $|\partial K| > 0$, and let $f : \Omega \times \mathbb{R}^{N\times N} \to [0, +\infty)$ be given by*

$$f(x, \xi) := \theta|\xi|^p + \chi_K \, |\det \xi| \quad \text{for a.e. } x \in \Omega, \text{ and all } \xi \in \mathbb{R}^{N\times N}.$$

If $\mathrm{id}(\cdot)$ is the identity function and $\mathbb{I}$ is the identity matrix, then there exists $\theta_0 \in (0, 1)$ such that if $\theta < \theta_0$ then for all $x \in \partial K$

$$f^{\mathrm{id}}(x) < f(x, \mathbb{I}).$$

It is clear that f satisfies (H1), (H3) with $\alpha := p$, $\beta := N$, and $f(x, \cdot)$ is quasiconvex. In light of Proposition 1.4, we have

$$\mathcal{F}(\mathrm{id}, A) = \int_A f^{\mathrm{id}}(x) \, dx + \mu_s(A) \; ,$$

where there holds, for θ sufficiently small, $f^{\mathrm{id}}(x) < f(x, \mathbb{I})$ for all $x \in \partial K$.

2. Preliminaries

Let Ω be an open, bounded subset of $\mathbb{R}^N$, and consider a function $f : \Omega \times \mathbb{R}^N \to [0, +\infty)$ satisfying the following standing hypotheses:

(H1) f is Carathéodory;
(H2) $f(x, \cdot)$ is convex for $\mathcal{L}^N$ a.e. $x \in \Omega$;
(H3) $|z|^\alpha \le f(x, z) \le C(1 + |z|^\beta)$ for all $z \in \mathbb{R}^N$, $\mathcal{L}^N$ a.e. $x \in \Omega$, and for some $C > 0$, where

$$1 < \alpha \le \beta < \frac{N\alpha}{N-1}. \tag{2.1}$$

We recall that f is said to be Carathéodory if $f(\cdot, z)$ is $\mathcal{L}^N$-measurable for all z and $f(x, \cdot)$ is continuous for $\mathcal{L}^N$ a.e. $x \in \Omega$, where $\mathcal{L}^N$ stands for the N-dimensional Lebesgue measure on $\mathbb{R}^N$. Often we will write $|E|$ in place of $\mathcal{L}^N(E)$. Also, throughout this paper the letter C will denote a generic positive constant which may vary from line to line and within the same formula. Note that in view of (2.1), if $A \subset \mathbb{R}^N$ is an open set with Lipschitz boundary then the inclusion $W^{1,\alpha}(\partial A) \subset W^{1-1/\beta,\beta}(\partial A)$ is continuous and compact, where $W^{1-1/\beta,\beta}(\partial A)$ is the space of traces on ∂A of functions in $W^{1,\beta}(A)$. More generally standard results in the theory of Sobolev spaces yield the lifting property presented below.

Lemma 2.1. *If $A \subset \mathbb{R}^N$ is an open, bounded, Lipschitz domain and if (2.1) holds, then there exists a continuous, linear, compact mapping*

$$\mathrm{E}_A : W^{1,\alpha}(\partial A) \to W^{1,\beta}(A) \qquad \textit{such that} \qquad \mathrm{tr}\, \mathrm{E}_A(g) = g \quad \text{on } \partial A.$$

Remark 2.2. Since the "uplift" operator E_A of Lemma 2.1 is constructed with symmetric convolution kernels, it is readily verified that affine functions remain unchanged, i.e., if $u(x) = a + b \cdot x$ then $\mathrm{E}_A \, u = u$.

Remark 2.3. (i) The growth hypothesis (H3) together with the convexity of f imply that f is β-locally Lipschitz continuous with respect to z, i.e., for a.e. $x \in \Omega$ and all $z, w \in \mathbb{R}^N$ (see [10], [24], [25])

$$|f(x,z) - f(x,w)| \leq C(1 + |z|^{\beta-1} + |w|^{\beta-1})|z - w|. \tag{2.2}$$

(ii) Let $\{z_i\}$ be a countable dense subset of $\mathbb{R}^N$, and for all i let K_i be the set of Lebesgue points of the function $f(\cdot, z_i)$. Setting $\mathbb{K}(f) := \cap K_i$, then $|\Omega \setminus \mathbb{K}(f)| = 0$, and if $x_0 \in \mathbb{K}(f)$ then it is a Lebesgue point for $f(\cdot, z)$ for all $z \in \mathbb{R}^N$. Indeed, fix $0 < \varepsilon < 1$, let $M := 1 + |z|$, and choose z_i with $|z - z_i| < \varepsilon$. By (2.2)

$$|f(x,z) - f(x_0,z)| \leq CM^{\beta-1}|z - z_i| + |f(x,z_i) - f(x_0,z_i)|,$$

thus for $\rho > 0$,

$$\fint_{B_\rho(x_0)} |f(x,z) - f(x_0,z)|\,dx \leq CM^{\beta-1}\varepsilon + \fint_{B_\rho(x_0)} |f(x,z_i) - f(x_0,z_i)|\,dx$$

and since $x_0 \in \cap K_i$, we now have

$$\limsup_{\rho \to 0} \fint_{B_\rho(x_0)} |f(x,z) - f(x_0,z)|\,dx \leq CM^{\beta-1}\varepsilon.$$

The assertion follows by letting $\varepsilon \to 0$.

For every Borel set $E \subset \Omega$ and every $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, we set

$$F(u,E) := \int_E f\big(x, \nabla u(x)\big)\,dx.$$

Given an open set $A \subset \Omega$ and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, the *set of admissible sequences for u in A* is defined by

$$\mathcal{A}(u;A) := \Big\{ \{u_n\} \subset W^{1,\beta}_{\mathrm{loc}}(A) \ : \ u_n \to u \text{ in } L^1(A) \Big\},$$

and we introduce the *relaxed functional*

$$\mathcal{F}(u,A) := \inf\Big\{ \liminf_{n \to +\infty} F(u_n, A) : \ \{u_n\} \in \mathcal{A}(u,A) \Big\}. \tag{2.3}$$

We define the *set of good sequences for u in A* to be

$$\mathcal{G}(u,A) := \left\{ \{u_n\} \in \mathcal{A}(u,A) \ : \ \mathcal{F}(u,A) = \lim_{n \to +\infty} F(u_n, A) \right\}. \tag{2.4}$$

It is clear that $\mathcal{G}(u,A)$ is nonempty. If $F(u,A) < +\infty$ then by (H3) we have that $\nabla u \in L^\alpha(A)$, and so for almost all balls $B_\rho \subset A$ the trace of u on ∂B_ρ belongs to $W^{1,\alpha}(\partial B_\rho)$. In view of Lemma 2.1, we may define for every such ball

$$\mathbf{m}(u, B_\rho) := \inf\Big\{ F(v, B_\rho) : \ v \in W^{1,\beta}(B_\rho), \ v - u \in W^{1,\alpha}_0(B_\rho) \Big\},$$

and we also set

$$\mathcal{F}_{\mathrm{av}}(u,A) := \frac{1}{|A|}\mathcal{F}(u,A), \qquad \mathbf{m}_{\mathrm{av}}(u, B_\rho) = \frac{1}{|B_\rho|}\mathbf{m}(u, B_\rho),$$

where B_ρ stands for a ball $B(x,\rho)$ of center $x \in \Omega$ and radius $\rho > 0$.

Next we recall a semicontinuity result by Ioffe (see [19]).

Theorem 2.4. *Let $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a Carathéodory function such that $g(x,u,\cdot)$ is convex for every $u \in \mathbb{R}^d$ and for $\mathcal{L}^N$ a.e. $x \in \Omega$. Then the functional*

$$G(u) := \int_\Omega g(x, u, \nabla u)\,dx$$

is lower semicontinuous on $W^{1,1}(\Omega;\mathbb{R}^d)$ with respect to the weak convergence in $W^{1,1}(\Omega;\mathbb{R}^d)$.

By Ioffe's Theorem, the functional F is lower semicontinuous, thus

$$F \leq \mathcal{F}.$$

On the other hand, if $u \in W^{1,\beta}_{\mathrm{loc}}(A) \cap W^{1,\alpha}(A)$ then the constant sequence $\{u\}$ is admissible and thus $\mathcal{F}(u,A) \leq F(u,A)$. This, together with the previous inequality, yields

$$\mathcal{F}(u,A) = F(u,A) \quad \text{whenever } u \in W^{1,\beta}_{\mathrm{loc}}(A) \cap W^{1,\alpha}(A). \tag{2.5}$$

Using a good sequence in $\mathcal{G}(u,A)$ to approach $\mathcal{F}(u,A)$, and in view of (2.5), it now follows that

$$\mathcal{F} = \sup\left\{H \ : \ H \leq \overline{F},\ H \text{ is } L^1 \text{ lower semicontinuous}\right\}$$

where

$$\overline{F}(u,A) := \begin{cases} F(u,A) & \text{if } u \in W^{1,\beta}_{\mathrm{loc}}(A), \\ +\infty & \text{otherwise.} \end{cases}$$

The following lemma will be instrumental to show that $\mathcal{F}(u,\cdot)$ is the trace in $\mathbb{A}(\Omega)$ of a positive Radon measure, where $\mathbb{A}(\Omega)$ denotes the class of open subsets of Ω.

Lemma 2.5. *Let $\lambda : \mathbb{A}(\Omega) \to [0,+\infty)$ and μ be such that*

(i) *μ is a finite Radon measure on Ω;*
(ii) *$\lambda(\Omega) \geq \mu(\Omega)$;*
(iii) *$\lambda(A) \leq \mu(\overline{A})$ for all $A \in \mathbb{A}(\Omega)$;*
(iv) *(subadditivity) $\lambda(A) \leq \lambda(A \setminus \overline{C}) + \lambda(B)$ for all $A, B, C \in \mathbb{A}(\Omega)$ such that $C \subset\subset B \subset A$;*
(v) *(inner regularity) for all $A \in \mathbb{A}(\Omega)$, $\varepsilon > 0$, there exists $C \in \mathbb{A}(\Omega)$ such that $C \subset\subset A$ and $\lambda(A \setminus \overline{C}) < \varepsilon$.*

Then $\lambda = \mu$ on $\mathbb{A}(\Omega)$.

Proof. We start by showing that

$$\lambda(A) \leq \mu(A) \quad \text{for all } A \in \mathbb{A}(\Omega). \tag{2.6}$$

Fix $A \in \mathbb{A}(\Omega)$, $\varepsilon > 0$. By (iv) and (v) we may find open sets $C \subset\subset B \subset\subset A$ such that $\lambda(A \setminus \overline{C}) < \varepsilon$ and

$$\lambda(A) \leq \lambda(A \setminus \overline{C}) + \lambda(B) \leq \varepsilon + \mu(\overline{B}) \leq \varepsilon + \mu(A),$$

where we used (iii). Clearly (2.6) now follows by letting $\varepsilon \to 0^+$.

In order to prove the converse inequality, since μ is inner regular for $\varepsilon > 0$ we may find an open set $A' \subset\subset A$ such that $\mu(A) \leq \varepsilon + \mu(\overline{A'})$, and by virtue of (ii), (2.6), and (iv),

$$\mu(A) \leq \varepsilon + \mu(\overline{A'}) = \varepsilon + \mu(\Omega) - \mu(\Omega \setminus \overline{A'}) \leq \varepsilon + \lambda(\Omega) - \lambda(\Omega \setminus \overline{A'}) \leq \varepsilon + \lambda(A).$$

By the arbitrariness of $\varepsilon > 0$, we now conclude that

$$\mu(A) \leq \lambda(A).$$

□

Remark 2.6. If (i) and (iii) in the statement of Lemma 2.5 hold, then the inner regularity property (v) is satisfied provided we can show that there exists $C > 0$ such that if $U \in \mathbb{A}(\Omega)$, $U = \cup_{i=1}^{\infty} U_i$, $U_i \in \mathbb{A}(\Omega)$, $U_i \subset\subset U_{i+1} \subset\subset U$, then

$$\lambda\left(U \setminus \overline{U_1}\right) \leq C \sum_{i=1}^{\infty} \lambda\left(U_{i+3} \setminus \overline{U_i}\right). \tag{2.7}$$

Indeed, if $U \in \mathbb{A}(\Omega)$, take $U_i := \{x \in U : \operatorname{dist}(x, \partial U) > 1/i\}$, and choose k large enough so that

$$\mu(U \setminus U_k) < \frac{\varepsilon}{4C}.$$

We have

$$\begin{aligned} \lambda(U \setminus \overline{U_k}) \quad &\leq \quad C \sum_{i=k}^{\infty} \lambda\left(U_{i+3} \setminus \overline{U_i}\right) \leq C \sum_{i=k}^{\infty} \mu\left(\overline{U_{i+3} \setminus \overline{U_i}}\right) \\ &\leq \quad C \sum_{i=k}^{\infty} \mu(U_{i+4} \setminus U_i) \leq 4C\, \mu(U \setminus U_k) < \varepsilon. \end{aligned}$$

3. A Representation Theorem

In this section we prove Theorem 1.1. We start by showing that $\mathcal{F}(u,\cdot)$ is the trace in $\mathbb{A}(\Omega)$ of a Radon measure. For every $A \in \mathbb{A}(\Omega)$ set

$$\lambda(A) := \mathcal{F}(u, A) .$$

Consider a sequence $\{u_n\} \in \mathcal{G}(u, A)$ such that

$$\mathcal{F}(u,\Omega) = \lim_{n\to+\infty} F(u_n,\Omega) = \lim_{n\to+\infty} \int_\Omega f(x, \nabla u_n)\, dx,$$

and define the Radon measures on $\mathbb{R}^N$

$$\lambda_n := f(x, \nabla u_n)\mathcal{L}^N \llcorner \Omega .$$

Since these measures are equibounded, and up to the extraction of a subsequence, we have

$$\lambda_n \overset{*}{\rightharpoonup} \mu \qquad \text{in the sense of measures}$$

for some finite Radon measure μ, i.e.,

$$\int_\Omega \varphi\, d\lambda_n \to \int_\Omega \varphi\, d\mu \quad \text{for all } \varphi \in C_c(\Omega).$$

Note that

$$\mu(\Omega) \le \liminf_{n\to+\infty} \lambda_n(\Omega) = \lim_{n\to+\infty} \int_\Omega f(x, \nabla u_n)\, dx = \lambda(\Omega), \tag{3.1}$$

and if $A \in \mathbb{A}(\Omega)$ then

$$\lambda(A) \le \liminf_{n\to+\infty} \int_A f(x, \nabla u_n)\, dx = \liminf_{n\to+\infty} \lambda_n(A) \le \mu(\overline{A}). \tag{3.2}$$

Therefore, by virtue of Lemma 2.5 we have

$$\lambda = \mu \quad \text{on} \quad \mathbb{A}(\Omega)$$

provided λ satisfies the subadditivity and inner regularity properties (iv) and (v). These depend mostly on the growth condition (H3) and on (2.1), and were asserted in [15] (see Lemma 2.4) by means of an uplifting operator $\mathcal{P}$ more refined and complex than the operator E_A introduced in Lemma 2.1. Below we give an alternative proof which relies heavily on the convexity assumption (H2), and it bypasses the need to exploit the operator $\mathcal{P}$.

Proposition 3.1. *If $A, B, C \in \mathbb{A}(\Omega)$, $C \subset\subset B \subset A$, then*

$$\mathcal{F}(u, A) \le \mathcal{F}(u, B) + \mathcal{F}(u, A \setminus \overline{C}) .$$

Proof. Fix $\{v_n\} \in \mathcal{G}(u, B)$ and $\{w_n\} \in \mathcal{G}(u, A\setminus\overline{C})$, let $\varphi \in C_c^\infty(B)$ be such that $0 \le \varphi \le 1$, $\{\varphi < 1\} \subset\subset A\setminus\overline{C}$, and set

$$u_n := \varphi v_n + (1 - \varphi)w_n ,$$

where φv_n is taken to be zero there where $\varphi = 0$, even though v_n may be not defined on that set, and analogously $(1 - \varphi)w_n$ is to be equal to zero on $\{\varphi = 1\}$. Clearly $\{u_n\} \in \mathcal{A}(u, A)$, and for all $0 < t < 1$ also $\{tu_n\} \in \mathcal{A}(tu, A)$, thus

$$\mathcal{F}(tu, A) \le \liminf_{n\to+\infty} F(tu_n, A) . \tag{3.3}$$

By the convexity of f, and since $0 < t < 1$, $0 \leq \varphi \leq 1$, we may find some positive constant M such that

$$\begin{aligned}
F(tu_n, A) &= \int_A f\Big(x, t[\varphi\nabla v_n + (1-\varphi)\nabla w_n] + (1-t)\frac{v_n - w_n}{1-t}t\nabla\varphi\Big)\,dx \\
&\leq t\int_A f\big(x, \varphi\nabla v_n + (1-\varphi)\nabla w_n\big)\,dx + (1-t)\int_A M\left(1 + \left|\frac{v_n - w_n}{1-t}t\nabla\varphi\right|^\beta\right)dx \\
&\leq \int_A \big[\varphi(x) f(x, \nabla v_n) + \big(1-\varphi(x)\big) f(x, \nabla w_n)\big]\,dx + M(1-t)|A| \\
&\quad + M\frac{\|\nabla\varphi\|^\beta_{L^\infty}}{(1-t)^{\beta-1}}\int_{B\setminus\overline{C}} |v_n - w_n|^\beta\,dx \\
&\leq F(v_n, B) + F(w_n, A\setminus\overline{C}) + M(1-t)|A| + M\frac{\|\nabla\varphi\|^\beta_{L^\infty}}{(1-t)^{\beta-1}}\|v_n - w_n\|^\beta_{L^\beta(B\setminus\overline{C})},
\end{aligned}$$

where we have used the growth condition in (H3). Also, by the coercivity assumption in (H3), we deduce that $\{\|\nabla v_n\|_{L^\alpha(B\setminus\overline{C})} + \|\nabla w_n\|_{L^\alpha(B\setminus\overline{C})}\}$ is bounded, and since $v_n - w_n \to u - u = 0$ in $L^1(B\setminus\overline{C})$, we have that $v_n - w_n \rightharpoonup 0$ weakly in $W^{1,\alpha}(B\setminus\overline{C})$, and therefore strongly in $L^\beta(B\setminus\overline{C})$ by (2.1) and Rellich Theorem. This, together with (3.3), yields

$$\mathcal{F}(tu, A) \leq \mathcal{F}(u, B) + \mathcal{F}(u, A\setminus\overline{C}) + C(1-t)|A|,$$

and the result now follows by letting $t \nearrow 1$ and using the L^1 lower semicontinuity property of $\mathcal{F}(\cdot, A)$. □

Finally, and in light of Remark 2.6, in order to assert inner regularity for λ it suffices to establish (2.7).

Proposition 3.2. *If $U \in \mathbb{A}(\Omega)$, $U = \cup_{i=1}^\infty U_i$, $U_i \in \mathbb{A}(\Omega)$, $U_i \subset\subset U_{i+1} \subset\subset U$, then*

$$\mathcal{F}\left(u, U\setminus\overline{U_1}\right) \leq 2\sum_{i=1}^\infty \mathcal{F}\left(u, U_{i+3}\setminus\overline{U_i}\right).$$

Proof. Fix $U \in \mathbb{A}(\Omega)$ and suppose that $U = \cup_{i=1}^\infty U_i$, $U_i \in \mathbb{A}(\Omega)$, $U_i \subset\subset U_{i+1} \subset\subset U$. Choose $\{u_n^{(i)}\} \in \mathcal{G}(u, U_{i+2}\setminus\overline{U_i})$ and $\{u_n^\infty\} \in \mathcal{G}(u, U)$, so that, recalling that $\lambda = \mathcal{F}(u, \cdot)$,

$$\lambda(U_{i+2}\setminus\overline{U_i}) = \lim_{n\to+\infty}\int_{U_{i+2}\setminus\overline{U_i}} f(x, \nabla u_n^{(i)})\,dx, \quad \lambda(U) = \lim_{n\to+\infty}\int_U f(x, \nabla u_n^\infty(x))\,dx.$$

Let φ_i be a smooth cut-off function such that $\varphi_i = 1$ in U_i, $\varphi_i = 0$ outside U_{i+1}, and $\{0 < \varphi_i < 1\} \subset\subset U_{i+1}\setminus\overline{U_i}$.

Up to the extraction of a subsequence, we may assume that $f(\cdot, \nabla u_n^\infty)\lfloor U \overset{*}{\rightharpoonup} \eta$, where η is a finite Radon measure on U. Fix $\varepsilon > 0$ and choose $M \in \mathbb{N}$ large enough so that $\eta(U\setminus\overline{U_M}) < \varepsilon$. Define

$$\begin{aligned}
\bar{u}_n := {}& \chi_{U_2\setminus\overline{U_1}}\, u_n^{(1)} + \sum_{i=2}^M \chi_{U_{i+1}\setminus U_i}\left[\varphi_i u_n^{(i-1)} + (1-\varphi_i)u_n^{(i)}\right] \\
& + \chi_{U\setminus U_{M+1}}\left[\varphi_{M+1}u_n^{(M)} + (1-\varphi_{M+1})u_n^\infty\right].
\end{aligned}$$

Clearly $\bar{u}_n \in \mathcal{A}(u, U\setminus\overline{U_1})$, and we have

$$\begin{aligned}
\lambda(U\setminus\overline{U_1}) &\leq \liminf_{t\to1}\liminf_{n\to+\infty}\int_{U\setminus\overline{U_1}} f(x, t\nabla\bar{u}_n)\,dx \\
&\leq \limsup_{t\to1}\limsup_{n\to+\infty} t\int_{U_2\setminus\overline{U_1}} f(x, \nabla u_n^{(1)})\,dx + \limsup_{t\to1}(1-t)\int_{U_2\setminus\overline{U_1}} f(x, 0)\,dx \\
&\quad + \limsup_{t\to1}\limsup_{n\to+\infty}\sum_{i=2}^M\int_{U_{i+1}\setminus U_i} f(x, t\nabla\bar{u}_n)\,dx \\
&\quad + \limsup_{t\to1}\limsup_{n\to+\infty}\int_{U\setminus U_{M+1}} f(x, t\nabla\bar{u}_n).
\end{aligned} \tag{3.4}$$

Now

$$\limsup_{t\to 1}\limsup_{n\to+\infty} t\int_{U_2\setminus\overline{U_1}} f(x,\nabla u_n^{(1)})\,dx + \limsup_{t\to 1}(1-t)\int_{U_2\setminus\overline{U_1}} f(x,0)\,dx \le \lambda(U_3\setminus\overline{U_1}), \tag{3.5}$$

and

$$\begin{aligned}
&\limsup_{t\to 1}\limsup_{n\to+\infty}\sum_{i=2}^{M}\int_{U_{i+1}\setminus U_i} f(x,t\nabla\bar u_n)\,dx\\
&\quad=\limsup_{t\to 1}\limsup_{n\to+\infty}\sum_{i=2}^{M}\int_{U_{i+1}\setminus U_i} f\Big(x,t\left[\varphi_i\nabla u_n^{(i-1)}+(1-\varphi_i)\nabla u_n^{(i)}\right]\\
&\qquad\qquad+(1-t)\frac{u_n^{(i-1)}-u_n^{(i)}}{1-t}\,t\,\nabla\varphi_i\Big)\,dx\\
&\quad\le\limsup_{t\to 1}\limsup_{n\to+\infty} t\,\sum_{i=2}^{M}\left\{\int_{U_{i+1}\setminus\overline{U_{i-1}}}\varphi_i f(x,\nabla u_n^{(i-1)})\,dx\right.\\
&\qquad\qquad\left.+\int_{U_{i+2}\setminus\overline{U_i}}(1-\varphi_i)f(x,\nabla u_n^{(i)})\,dx\right\}\\
&\qquad+\limsup_{t\to 1}\limsup_{n\to+\infty}(1-t)\,\sum_{i=2}^{M}\int_{U_{i+1}\setminus\overline{U_i}} C\left(1+\frac{|u_n^{(i-1)}-u_n^{(i)}|^\beta}{(1-t)^\beta}\,t^\beta\,|\nabla\varphi_i|^\beta\right)dx\\
&\quad\le\sum_{i=2}^{M}\left[\lambda(U_{i+1}\setminus\overline{U_{i-1}})+\lambda(U_{i+2}\setminus\overline{U_i})\right], \qquad (3.6)
\end{aligned}$$

where we have used the fact that $u_n^{(i-1)}-u_n^{(i)}\to 0$ in $L^\beta(U_{i+1}\setminus\overline{U_i})$.

In a similar way, we can show that

$$\begin{aligned}
\limsup_{t\to 1}\limsup_{n\to+\infty}\int_{U\setminus U_{M+1}} f(x,t\nabla\bar u_n) &\le \limsup_{n\to+\infty}\int_{U_{M+2}\setminus U_{M+1}} f(x,\nabla u_n^{(M)})\\
&\qquad+\limsup_{n\to+\infty}\int_{U\setminus U_{M+1}} f(x,\nabla u_n^\infty)\,dx\\
&\le \lambda(U_{M+3}\setminus\overline{U_M})+\eta(U\setminus\overline{U_M})\\
&\le \lambda(U_{M+3}\setminus\overline{U_M})+\varepsilon. \qquad (3.7)
\end{aligned}$$

In view of (3.4), (3.5), (3.6), and (3.7), we obtain

$$\lambda\left(U\setminus\overline{U_1}\right)\le 2\sum_{i=1}^{\infty}\lambda\left(U_{i+3}\setminus\overline{U_i}\right)+\varepsilon.$$

Letting $\varepsilon\to 0$ yields property (2.7). □

Proposition 3.3. *If* (H1)–(H3) *hold and if $\mathcal{F}(u,\Omega)<+\infty$ then there exists a finite Radon measure μ on Ω such that $\mathcal{F}(u,\cdot)$ is the trace of μ on $\mathbb{A}(\Omega)$.*

Proof. The proof follows immediately from Lemma 2.5, Remark 2.6, (3.1), (3.2), Propositions 3.1 and 3.2. □

In the sequel we will often write μ to designate the measure $\mathcal{F}(u,\cdot)$.

Lemma 3.4. *If* (H1)–(H3) *hold and if $\mathcal{F}(u,\Omega)<+\infty$ then there exists a sequence $\{u_n\}$ with the following properties:*

(a) $u_n\in W^{1,\beta}_{\mathrm{loc}}(\Omega)$;
(b) $u_n\rightharpoonup u$ *in* $W^{1,\alpha}(\Omega)$;
(c) $\mathcal{F}(u,\Omega)=\lim_{n\to+\infty}F(u_n,\Omega)$;
(d) $\mathcal{F}(u,A)=\lim_{n\to+\infty}F(u_n,A)$ *for all open sets* $A\subset\Omega$ *such that* $\mu(\partial A)=0$;
(e) *for all* $x_0\in\Omega$ *and for almost all* $0<t<R:=\operatorname{dist}(x_0,\partial\Omega)$, *there exists a subsequence* $\{u_{k_n}\}$ *such that* $\mathcal{F}\big(u,B_t(x_0)\big)=\lim_{n\to+\infty}F\big(u_{k_n},B_t(x_0)\big)$ *and* $u_{k_n}\rightharpoonup u$ *in* $W^{1,\alpha}\big(\partial B_t(x_0)\big)$.

Proof. By the definition of $\mathcal{F}$ there exists a sequence $\{u_n\}$ satisfying properties (a), (b), (c). If $A \in \mathbb{A}(\Omega)$ is such that $\mu(\partial A) = 0$ then

$$\begin{aligned}\limsup_{n\to+\infty} F(u_n, A) &= \lim_{n\to+\infty} F(u_n, \Omega) - \liminf_{n\to+\infty} F(u_n, \Omega \setminus \overline{A}) \\ &\le \mathcal{F}(u, \Omega) - \mathcal{F}(u, \Omega \setminus \overline{A}) \\ &= \mathcal{F}(u, A) \\ &\le \liminf_{n\to+\infty} F(u_n, A)\end{aligned}$$

and property (d) follows. To prove (e), fix $x_0 \in \Omega$ and remark that due to the finiteness of the measure μ, property (d) holds for $B_t(x_0)$ for $\mathcal{L}^1$ a.e. $t \in (0, R)$. With

$$M := \sup_n \int_{B_R(x_0)} |\nabla u_n|^\alpha \, dx < +\infty,$$

define for $K > 0$

$$E_{K,n} = \bigcap_{m\ge n} \Big\{ t \in (0, R) : \mu_s(\partial B_t(x_0)) = 0, \ \int_{\partial B_t(x_0)} |\nabla u_m|^\alpha \, dH^{N-1} \ge K \Big\}.$$

Clearly $|E_{K,n}| \le M/K$, and since $E_{K,n+1} \supset E_{K,n}$, we still have $|E_K| \le M/K$ where $E_K := \cup_n E_{K,n}$. As a consequence, if $E := \cap_K E_K$ then $|E| = 0$, and if $t \notin E$ then

$$\liminf_{n\to+\infty} \int_{\partial B_t(x_0)} |\nabla u_n|^\alpha \, dH^{N-1} < +\infty,$$

which, together with (d), asserts the existence of a subsequence fulfilling (e). □

Definition 3.5. *If $\mathcal{F}(u, \Omega) < +\infty$ then we say that $B_\rho(x_0)$ is an uplift ball for u if*

(i) $\mu\big(\partial B_\rho(x_0)\big) = 0$;
(ii) *there exists a sequence $\{u_n\}$ satisfying properties (a), (b), and (c) of Lemma 3.4 and $u_n \rightharpoonup u$ in $W^{1,\alpha}\big(\partial B_\rho(x_0)\big)$.*

Remark 3.6. By Lemma 3.4, if $\mathcal{F}(u, \Omega) < +\infty$ then for all $x_0 \in \Omega$ almost all balls centered at x_0 are uplift balls for u. Also, by means of a diagonal argument we may assume that the sequence $\{u_n\}$ satisfying (ii) in Definition 3.5 is the same for a countable set of uplift balls.

Lemma 3.7. *Assume that $\mathcal{F}(u, \Omega) < +\infty$ and let B be an uplift ball for u. Let $v \in W^{1,\beta}(B)$ satisfy $v = u$ on ∂B, set*

$$\bar{u}(x) := \begin{cases} v(x) & \text{if } x \in B, \\ u(x) & \text{otherwise,} \end{cases}$$

and let $\bar{\mu}$ be the measure which extends $\mathcal{F}(\bar{u}, \cdot)$ to all Borel sets of Ω. Then

$$\bar{\mu}(\partial B) = 0.$$

Proof. Let $\{u_n\}$ be a sequence as in Definition 3.5 (ii), and let $B' \subset\subset B \subset\subset B''$ be uplift balls concentric with B, so close together that if we denote by A the annulus $B'' \setminus \overline{B'}$, and using the fact that $\mathcal{F}(u, \partial B) = 0$, then

$$\mathcal{F}(u, A) < \varepsilon \,, \qquad F(v, A \cap B) < \varepsilon.$$

Since the trace on ∂B of $u_n - u$ converges to zero weakly on $W^{1,\alpha}(\partial B)$, by Lemma 2.1 if we take $v_n := \mathrm{E}_B(u_n - u)$ then

$$v_n \in W^{1,\beta}(B) \,, \qquad v_n = u_n - u \quad \text{on } \partial B \,, \qquad v_n \to 0 \quad \text{in } W^{1,\beta}(B).$$

We define in A

$$\bar{u}_n(x) := \begin{cases} u_n(x) & \text{if } x \in A \setminus B, \\ v_n(x) + v(x) & \text{if } x \in A \cap B. \end{cases}$$

Clearly $\bar{u}_n \in W^{1,\beta}(A)$, and

$$F(\bar{u}_n, A) = F(u_n, A \setminus B) + F(\bar{u}_n, A \cap B) \le F(u_n, A) + F(\bar{u}_n, A \cap B).$$

Since $\bar{u}_n \to \bar{u}$ in $L^1(A)$ and $\bar{u}_n \to v$ in $W^{1,\beta}(A \cap B)$, by the $W^{1,\beta}$-continuity of F and the property (d) satisfied by $\{u_n\}$, in view of the fact that $\mu(\partial A) = 0$, we deduce that

$$\begin{aligned}\mathcal{F}(\bar{u}, A) &\leq \liminf_{n\to+\infty} F(\bar{u}_n, A)\\ &\leq \limsup_{n\to+\infty} F(\bar{u}_n, A \setminus B) + \limsup_{n\to+\infty} F(\bar{u}_n, A \cap B)\\ &\leq \lim_{n\to+\infty} F(u_n, A) + F(v, A \cap B)\\ &= \mathcal{F}(u, A) + F(v, A \cap B) < 2\varepsilon\end{aligned}$$

and the assertion follows. □

Proposition 3.8. *If $\mathcal{F}(u, \Omega) < +\infty$ then for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$*

$$\liminf_{\rho\to 0} \mathbf{m}_{\mathrm{av}}\big(u, B_\rho(x_0)\big) \leq f\big(x_0, \nabla u(x_0)\big).$$

Proof. Using the notation introduced in Remark 2.3, let $x_0 \in \mathbb{K}(f)$ be a Lebesgue point for u and ∇u, and let $\rho_j \searrow 0$ be any sequence such that $B_{\rho_1}(x_0) \subset \Omega$. We recall that, since $u \in W^{1,\alpha}(B_{\rho_1})$, for $\mathcal{L}^N$-a.e. such point x_0 we have (see [13], Theorem 2, p.230)

$$\fint_{B_{\rho_j}(x_0)} \left[|\nabla u(x) - \nabla u(x_0)|^\alpha + \frac{1}{\rho_j^\alpha}|u(x) - u(x_0) - \nabla u(x_0)(x - x_0)|^\alpha\right] dx \to 0.$$

On $B := B_1(0)$ we define

$$u_j(y) := \frac{1}{\rho_j}[u(x_0 + \rho_j y) - u(x_0)]\,, \qquad u_0(y) := \nabla u(x_0) y. \tag{3.8}$$

Then

$$\fint_B [|\nabla u_j - \nabla u_0|^\alpha + |u_j - u_0|^\alpha]\, dy \to 0,$$

i.e.,

$$\int_0^1 dt \int_{\partial B_t} [|\nabla u(x_0 + \rho_j y) - \nabla u(x_0)|^\alpha + |u_j(y) - \nabla u(x_0) y|^\alpha]\; dH^{N-1} \to 0,$$

where $B_t := B_t(0)$, and so, up to a subsequence,

$$\int_{\partial B_t} [|\nabla u(x_0 + \rho_j y) - \nabla u(x_0)|^\alpha + |u_j(y) - \nabla u(x_0) y|^\alpha]\; dH^{N-1} \to 0 \tag{3.9}$$

for $\mathcal{L}^1$ a.e. $t \in (0,1)$. As for almost all t

$$\mu(\partial B_{t\rho_j}) = 0 \quad \text{and } u_j \in W^{1,\alpha}(\partial B_t) \qquad \text{for all } j, \tag{3.10}$$

we may find $\bar{t} \in (0,1)$ satisfying (3.9), (3.10), and we relabel as $\{\rho_j\}$ the sequence $\{\bar{t}\rho_j\}$. We then have that the sequence defined in (3.8) with the new ρ_j satisfies

$$\int_{\partial B} [|\nabla u(x_0 + \rho_j y) - \nabla u(x_0)|^\alpha + |u_j(y) - \nabla u(x_0) y|^\alpha]\; dH^{N-1} \to 0,$$

$$\mu(\partial B_{\rho_j}) = 0, \quad u_j \in W^{1,\alpha}(\partial B) \quad \text{for all } j, \tag{3.11}$$

and u_j still converges to u_0 in $W^{1,\alpha}(B)$. Setting $v_j := \mathrm{E}_B\, u_j$, and as $\mathrm{E}_B\, u_0 = u_0$ (see Remark 2.2), from Lemma 2.1 and (3.11) we deduce that

$$v_j \to u_0 \qquad \text{in } W^{1,\beta}(B),$$

and

$$\begin{aligned}\mathbf{m}_{\mathrm{av}}\big(u, B_{\rho_j}(x_0)\big) &\leq \fint_{B_{\rho_j}} f\left(x, \nabla\left(\rho_j v_j\left(\frac{x - x_0}{\rho_j}\right) + u(x_0)\right)\right) dy\\ &= \fint_B f\big(x_0 + \rho_j y, \nabla v_j(y)\big)\, dy.\end{aligned}$$

Since f is β-locally Lipschitz (see (2.2)), we obtain

$$\begin{aligned}
&\int_B \left|f\big(x_0+\rho_j y, \nabla v_j(y)\big) - f\big(x_0+\rho_j y, \nabla u(x_0)\big)\right| dy \\
&\qquad \le C \int_B (1+|\nabla v_j(y)|^{\beta-1}+|\nabla u(x_0)|^{\beta-1})|\nabla v_j(y)-\nabla u(x_0)|\, dy \\
&\qquad \le C\|\nabla v_j - \nabla u_0\|_{L^\beta(B)}
\end{aligned}$$

which vanishes as $j \to +\infty$. Therefore

$$\begin{aligned}
\liminf_{\rho\to 0} \mathbf{m}_{\rm av}\big(u, B_\rho(x_0)\big) &\le \limsup_{j\to+\infty} \mathbf{m}_{\rm av}\big(u, B_{\rho_j}(x_0)\big) \\
&\le \limsup_{j\to+\infty} \fint_B f\big(x_0+\rho_j y, \nabla u(x_0)\big)\, dy \\
&= \limsup_{j\to+\infty} \fint_{B_{\rho_j}(x_0)} f\big(x, \nabla u(x_0)\big)\, dx. \qquad (3.12)
\end{aligned}$$

By (3.12) and the definition of the set $\mathbb{K}(f)$ (see Remark 2.3), we conclude that

$$\liminf_{\rho\to 0} \mathbf{m}_{\rm av}\big(u, B_\rho(x_0)\big) \le f\big(x_0, \nabla u(x_0)\big).$$

□

Remark 3.9. Recall that Ioffe's Theorem (Theorem 2.4) ensures that $F \le \mathcal{F}$. In particular, if $\mathcal{F}(u,\Omega) < +\infty$ then $f(\cdot, \nabla u) \in L^1(\Omega)$. Moreover, if the point x_0 in the previous proof is also a Lebesgue point for $f\big(\cdot, \nabla\big)$, which holds $\mathcal{L}^N$-a.e. in Ω, then the sequences $\{\rho_j\}$ and $\{u_j\}$ we found above satisfy

$$\begin{cases}
\mu_s(\partial B_{\rho_j}) = 0, \\
u \in W^{1,\alpha}(\partial B_{\rho_j}), \\
u_j \to u \text{ in } W^{1,\alpha}(B_{\rho_j}) \ \text{ and in } W^{1,\alpha}(\partial B_{\rho_j}), \\
\limsup\limits_{j\to+\infty} \mathbf{m}_{\rm av}\big(u, B_{\rho_j}(x_0)\big) \le \lim\limits_{j\to+\infty} \fint_{B_{\rho_j}(x_0)} f\big(x, \nabla u(x)\big)\, dx\ ,
\end{cases}$$

and from the last inequality we deduce, in particular, that for all $t > 0$ there exists $\bar{j}$ such that for all $j \ge \bar{j}$

$$\mathbf{m}\big(u, B_{\rho_j}(x_0)\big) \le \int_{B_{\rho_j}(x_0)} f\big(x, \nabla u(x)\big)\, dx + t|B_{\rho_j}(x_0)|.$$

Proposition 3.10. *For $\mathcal{L}^N$-a.e. $x_0 \in \Omega$*

$$\liminf_{\rho\to 0} \mathcal{F}_{\rm av}\big(u, B_\rho(x_0)\big) \le f\big(x_0, \nabla u(x_0)\big).$$

Proof. Let x_0 be a Lebesgue point for $f(\cdot, \nabla u(\cdot))$ such that

$$\frac{d\mu^s}{d\mathcal{L}^N}(x_0) = 0 \qquad (3.13)$$

where μ_s is the singular part of μ with respect to $\mathcal{L}^N \llcorner \Omega$. Fix an open set $\Omega' \subset \Omega$ and define for every $t \in (0,1)$

$$E_t := \Big\{x \in \Omega' : \exists \rho_j \searrow 0 \text{ such that } \mathbf{m}\big(u, B_{\rho_j}(x_0)\big) \le \int_{B_{\rho_j}(x_0)} f\big(x, \nabla u(x)\big)\, dx + t|B_{\rho_j}(x_0)|\Big\}.$$

By Remark 3.9 we have that $|\Omega' \setminus E_t| = 0$, thus, in particular, E_t is $\mathcal{L}^N$-measurable. Fix $\varepsilon > 0$ and choose an open set ω and a compact set K such that

$$K \subset E_t \subset \omega \subset \Omega'\ ,$$

with

$$\mu_s(\omega) \le \varepsilon + \mu_s(E_t)\ , \qquad \mu_s(\omega \setminus K) \le \varepsilon + \mu^s(E_t). \qquad (3.14)$$

Fix $\delta > 0$ and set

$$\begin{aligned}\mathcal{X}^\delta \quad &:= \quad \Big\{B_\rho(x) : x \in E_t,\ \rho < \delta,\ \overline{B_\rho(x)} \subset \omega,\ \mu_s\big(\partial B_\rho(x)\big) = 0,\ u \in W^{1,\alpha}\big(\partial B_\rho(x)\big),\\ &\qquad \mathbf{m}\big(u, B_\rho(x)\big) \le \int_{B_\rho(x)} f\big(y, \nabla u(y)\big)\, dy + t|B_\rho(x)|\Big\}\\ \mathcal{Y}^\delta \quad &:= \quad \Big\{B_\rho(x) : \rho < \delta,\ \overline{B_\rho(x)} \subset \omega \setminus K,\ \mu_s\big(\partial B_\rho(x)\big) = 0\Big\}.\end{aligned}$$

The balls of $\mathcal{X}^\delta$ and $\mathcal{Y}^\delta$ form a fine cover of ω, and so Besicovitch Covering Theorem yields a countable subcover

$$\omega = N \cup \Big(\bigcup_{i=1}^{+\infty} B_i^{\mathcal{Y}}\Big) \cup \Big(\bigcup_{i=1}^{+\infty} B_i^{\mathcal{X}}\Big) \tag{3.15}$$

with $\mu(N) = 0$, so that the balls $B_i^{\mathcal{Y}} \in \mathcal{Y}^\delta$ and $B_i^{\mathcal{X}} \in \mathcal{X}^\delta$ are all disjoint. By (3.14) we may take n large enough such that

$$\mu[N \cup (\cup_i B_i^{\mathcal{Y}}) \cup (\cup_{i>n} B_i^{\mathcal{X}})] < \varepsilon + \mu^s(E_t), \tag{3.16}$$

and for all $i \le n$ choose $v_i \in W^{1,\beta}(B_i^{\mathcal{X}})$ such that $v_i - u \in W_0^{1,\alpha}(B_i^{\mathcal{X}})$ and

$$F(v_i, B_i^{\mathcal{X}}) \le \mathbf{m}(u, B_i^{\mathcal{X}}) + \delta|B_i^{\mathcal{X}}|. \tag{3.17}$$

Due to the choice of $\mathcal{X}^\delta$ and by (H3) we have

$$\begin{aligned}\int_{B_i^{\mathcal{X}}} |Dv_i|^\alpha\, dx \quad &\le \quad \int_{B_i^{\mathcal{X}}} f(x, \nabla v_i(x))\, dx \le \mathbf{m}(u, B_i^{\mathcal{X}}) + \delta|B_i^{\mathcal{X}}|\\ &\le \quad \int_{B_i^{\mathcal{X}}} f(x, \nabla u(x))\, dx + (\delta + t)|B_i^{\mathcal{X}}|,\end{aligned}$$

and in view of the boundary assumption on v_i and Poincaré inequality, using again (H3),

$$\|v_i - u\|^\alpha_{L^\alpha(B_i^{\mathcal{X}})} \le C\delta^\alpha\Big(\int_{B_i^{\mathcal{X}}} f\big(x, \nabla u(x)\big)\, dx + (\delta + t)|B_i^{\mathcal{X}}|\Big),$$

where we have invoked the fact that the radius of $B_i^{\mathcal{X}}$ is less than δ. Set

$$u_\delta(x) := \begin{cases} v_i(x) & \text{if } x \in B_i^{\mathcal{X}},\ i \le n,\\ u(x) & \text{otherwise.}\end{cases}$$

The previous inequality yields

$$\|u_\delta - u\|^\alpha_{L^\alpha(\omega)} \le C\delta^\alpha\Big(\int_\omega f\big(x, \nabla u(x)\big)\, dx + |\omega|\Big)\ ,$$

thus $u_\delta \to u$ in $L^\alpha(\omega)$.

By Lemma 3.7, (3.15), and (3.16), we have

$$\mathcal{F}(u_\delta, \omega) = \mathcal{F}(u, \omega \setminus \cup_{i\le n} \overline{B_i^{\mathcal{X}}}) + \sum_{i\le n} \mathcal{F}(u_\delta, B_i^{\mathcal{X}}) \le \varepsilon + \mu^s(E_t) + \sum_{i\le n} \mathcal{F}(v_i, B_i^{\mathcal{X}}).$$

Since $v_i \in W^{1,\beta}(B_i^{\mathcal{X}})$, by (2.5), (3.17), and in light of the choice of $\mathcal{X}^\delta$,

$$\begin{aligned}\mathcal{F}(u_\delta, \omega) \quad &\le \quad \varepsilon + \mu^s(E_t) + \sum_{i\le n} \mathbf{m}(u, B_i^{\mathcal{X}}) + \delta|\cup_{i\le n} B_i^{\mathcal{X}}|\\ &\le \quad \varepsilon + \mu^s(E_t) + \int_\omega f\big(x, \nabla u(x)\big)\, dx + (\delta + t)|\omega|.\end{aligned}$$

The L^1 lower semicontinuity of $\mathcal{F}(\cdot, \omega)$ now yields

$$\mathcal{F}(u, \omega) \le \liminf_{\delta\to 0} \mathcal{F}(u_\delta, \omega) \le \varepsilon + \mu^s(E_t) + \int_\omega f\big(x, \nabla u(x)\big)\, dx + t|\omega|,$$

so letting $\omega \searrow E_t$ and then $\varepsilon \to 0$ we get

$$\mu(E_t) \le \mu^s(E_t) + t|E_t| + \int_{E_t} f\big(x, \nabla u(x)\big)\, dx.$$

Since $|\Omega' \setminus E_t| = 0$ this implies

$$\mathcal{F}(u, \Omega') \le \mu(E_t) + \mu^s(\Omega') \le 2\mu^s(\Omega') + t|\Omega'| + \int_{\Omega'} f\big(x, \nabla u(x)\big)\, dx.$$

Taking $\Omega' := B_\rho(x_0)$ in the previous inequality, dividing through by $|B_\rho(x_0)|$, and letting $\rho \to 0$ we obtain

$$\limsup_{\rho \to 0} \mathcal{F}_{\mathrm{av}}\big(u, B_\rho(x_0)\big) \le t + f\big(x_0, \nabla u(x_0)\big),$$

and by (3.13) the result now follows by the arbitrariness of t. □

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. The statement of Theorem 1.1 is now easily asserted by appealing to Proposition 3.3, Proposition 3.10, and to the fact that (see Theorem 2.4)

$$\int_A f(x, \nabla u)\, dx \le \mathcal{F}(u, A)$$

for all $A \in \Omega$. □

In the remaining of this section we show that, near $x_0 \in \Omega$, $\mathcal{F}(u \cdot)$ may be identified with a Dirichlet problem. Precisely,

Theorem 3.11. *If $\mathcal{F}(u, \Omega) < +\infty$ then for μ-a.e. $x_0 \in \Omega$*

$$\lim_{\rho \to 0} \frac{\mathbf{m}\big(u, B_\rho(x_0)\big)}{\mu\big(B_\rho(x_0)\big)} = 1.$$

We first prove an auxiliary result.

Proposition 3.12. *If $\mathcal{F}(u, \Omega) < +\infty$ and if B is an uplift ball for u then*

$$\mathcal{F}(u, B) \ge \mathbf{m}(u, B).$$

Proof. By the definition of uplift ball we have that $u \in W^{1,\alpha}(\partial B)$, and consider a sequence $\{u_n\}$ satisfying the conditions of Lemma 3.5 (ii). Set

$$\bar{u} := \mathrm{E}_B\, u\ , \qquad w_n := \mathrm{E}_B(u_n - u),$$

so that $\bar{u}, w_n \in W^{1,\beta}(B)$, $\bar{u} = u$ on ∂B. Since $u_n \rightharpoonup u$ in $W^{1,\alpha}\big(\partial B\big)$, by Lemma 2.1

$$w_n \to 0 \qquad \text{in } W^{1,\beta}(B). \tag{3.18}$$

Define for all $w \in W^{1,\beta}(B)$

$$h(w) := \inf\Big\{F(v, B) :\ v \in W^{1,\beta}(B),\ v = \bar{u} + w \text{ on } \partial B\Big\}\ ,$$

and remark that

$$h(0) = \mathbf{m}(u, B)\ , \qquad h(w_n) = \mathbf{m}(u_n, B). \tag{3.19}$$

In order to show that h is lower semicontinuous at 0 with respect to the strong topology of $W^{1,\beta}(B)$, we prove that it is convex and that it is locally bounded at 0. To establish convexity we fix $w_1, w_2 \in W^{1,\beta}(B)$, $0 < t < 1$, $\varepsilon > 0$, and choose $v_1, v_2 \in W^{1,\beta}(B)$ such that

$$v_i = \bar{u} + w_i \text{ on } \partial B\ , \quad F(v_i, B) \le h(w_i) + \varepsilon.$$

Then $tv_1 + (1-t)v_2 = \bar{u} + tw_1 + (1-t)w_2$ on ∂B, and by the convexity of f

$$\begin{aligned} h\big(tw_1 + (1-t)w_2\big) &\le F\big(tv_1 + (1-t)v_2, B\big) \le tF(v_1, B) + (1-t)F(v_2, B) \\ &\le t\, h(w_1) + (1-t)\, h(w_2) + 2\varepsilon. \end{aligned}$$

It suffices now to let $\varepsilon \to 0$.

On the other hand, for all $w \in W^{1,\beta}(B)$

$$h(w) \leq F(\bar{u}+w, B) \leq C\left(|B| + \|\nabla(\bar{u}+w)\|^{\beta}_{L^{\beta}(B)}\right) \leq C\left(|B| + \|\bar{u}+w\|^{\beta}_{W^{1,\beta}(B)}\right),$$

from which we conclude that h is bounded in the unit ball of $W^{1,\beta}(B)$.

In view of Lemma 3.4 (c), (3.18), and (3.19) we now have

$$\mathcal{F}(u,B) = \lim_{n\to+\infty} F(u_n, B) \geq \liminf_{n\to+\infty} \mathbf{m}(u_n, B) = \liminf_{n\to+\infty} h(w_n) \geq h(0) = \mathbf{m}(u,B).$$

□

Proof of Theorem 3.11. By Proposition 3.12 we have

$$\limsup_{\rho\to 0} \frac{\mathbf{m}\big(u, B_\rho(x_0)\big)}{\mu\big(B_\rho(x_0)\big)} \leq 1, \tag{3.20}$$

where we have used the fact that the limit on the left hand side of the previous inequality exists μ-a.e.

For all $t > 0$ set

$$G_t := \{B_\rho \subset \Omega : \ B_\rho \text{ uplift ball}, \ \mathcal{F}(u, B_\rho) > \mathbf{m}(u, B_\rho) + t\mu(B_\rho)\}$$

and

$$E_t := \{x \in \Omega : \exists \rho_h \to 0 \text{ with } B_{\rho_h}(x) \in G_t\}.$$

We claim that $\mu(E_t) = 0$, therefore establishing the converse of (3.20). Indeed, if $\mu(E_t) = 0$ then for μ-a.e. $x_0 \in \Omega$ there exists $\delta_0 > 0$ such that

$$B_\delta(x_0) \notin G_t \text{ for all } 0 < \delta < \delta_0. \tag{3.21}$$

Consider any sequence $\{r_n\}$ of positive numbers with $r_n \to 0$, fix $t > 0$, and choose $0 < \delta_n < r_n$ such that B_{δ_n} is an uplift ball and $\mathcal{F}(u, B_{r_n} \backslash B_{\delta_n}) < t$, where here we have used the fact that $\mu(\partial B_{\delta_n}) = 0$ (see Remark 3.6). By (3.21) we have

$$\begin{aligned} \mathcal{F}(u, B_{r_n}) &= \mathcal{F}(u, B_{\delta_n}) + \mathcal{F}(u, B_{r_n} \setminus B_{\delta_n}) \leq \mathcal{F}(u, B_{\delta_n}) + t \\ &\leq \mathbf{m}(u, B_{\delta_n}) + t\mu(B_{\delta_n}) + t \leq \mathbf{m}(u, B_{r_n}) + t\mu(B_{r_n}) + t. \end{aligned}$$

The arbitrariness of t yields

$$\mathcal{F}(u, B_{r_n}) \leq \mathbf{m}(u, B_{r_n}),$$

and, in turn, the arbitrariness of $\{r_n\}$ allows us to conclude that

$$\liminf_{\rho\to 0} \frac{\mathbf{m}\big(u, B_\rho(x_0)\big)}{\mu\big(B_\rho(x_0)\big)} \geq 1.$$

We now establish the claim. Note that at this point there is no guarantee that E_t is even measurable, so we must proceed with care. For all $\delta > 0$ define

$$U_\delta := \bigcup_{B_\rho(x)\in G_t, \ x\in E_t, \ \rho<\delta} B_\rho(x), \quad U_0 := \bigcap_{\delta>0} U_\delta.$$

The sets U_δ are open, thus U_0 is a Borel set, and $U_0 \supset E_t$. It suffices, therefore, to prove that $\mu(U_0) = 0$. Suppose that, on the contrary, $\mu(U_0) = \gamma > 0$. Let $\varepsilon < \gamma/2$, fix $\delta_0 > 0$, and choose a compact set $K \subset U_0$ such that

$$\mu(U_{\delta_0} \setminus K) < \varepsilon. \tag{3.22}$$

We now define for all $\delta < \delta_0$

$$\mathcal{X}^\delta := \{B_\rho(x) : x \in E_t, \ \rho < \delta, \ B_\rho(x) \in G_t\}$$

and

$$\mathcal{Y}^\delta := \{B_\rho(x) : \rho < \delta, \ \overline{B_\rho(x)} \subset U_\delta \setminus K, \ \mu(\partial B_\rho(x)) = 0\}.$$

These sets form a fine covering of U_0, so by Besicovitch Covering Theorem we may find a countable subcover such that

$$U_0 \subset N \cup \left(\bigcup_{i=1}^{+\infty} B_i^{\mathcal{Y}}\right) \cup \left(\bigcup_{i=1}^{+\infty} B_i^{\mathcal{X}}\right)$$

with $\mu(N)=0$ and for suitable mutually disjoint balls $B_i^{\mathcal{Y}} \in \mathcal{Y}^\delta$ and $B_i^{\mathcal{X}} \in \mathcal{X}^\delta$. By (3.22) we may find n large enough such that

$$\mu[N \cup (\cup_i B_i^{\mathcal{Y}}) \cup (\cup_{i>n} B_i^{\mathcal{X}})] < \varepsilon,$$

and thus

$$\mu\left(\cup_{i\le n} B_i^{\mathcal{X}}\right) \ge \mu(U_0) - \varepsilon \ge \frac{\gamma}{2}. \tag{3.23}$$

For all $i \le n$ we choose $v_i \in W^{1,\beta}(B_i^{\mathcal{X}})$ such that $v_i - u \in W_0^{1,\alpha}(B_i^{\mathcal{X}})$ and

$$F(v_i, B_i^{\mathcal{X}}) \le \mathbf{m}(u, B_i^{\mathcal{X}}) + \delta\mu(B_i^{\mathcal{X}}).$$

By (H3) and Proposition 3.12 we then get

$$\int_{B_i^{\mathcal{X}}} |\nabla v_i|^\alpha\, dx \le F(v_i, B_i^{\mathcal{X}}) \le \mathbf{m}(u, B_i^{\mathcal{X}}) + \delta\mu(B_i^{\mathcal{X}}) \le \mathcal{F}(u, B_i^{\mathcal{X}}) + \delta\mu(B_i^{\mathcal{X}})\ ,$$

hence

$$\sum_{i\le n} \int_{B_i^{\mathcal{X}}} |Dv_i|^\alpha\, dx \le \mathcal{F}(u,\Omega) + \delta\mu(\Omega) < +\infty.$$

Poincaré inequality now implies that

$$\sum_{i\le n} \|v_i - u\|_{L^\alpha(B_i^{\mathcal{X}})} \le C\,\delta^\alpha\ ,$$

where we used the fact that the radii of the balls $B_i^{\mathcal{X}}$ are less than δ. We define

$$u_\delta(x) := \begin{cases} v_i(x) & \text{if } x \in B_i^{\mathcal{X}},\ i \le n, \\ u(x) & \text{otherwise.} \end{cases}$$

Clearly $u_\delta \to u$ in $L^\alpha(U_{\delta_0})$, and by our choice of the functions v_i and by definition of the set E_t we deduce that

$$\begin{aligned}
\mathcal{F}(u_\delta, U_{\delta_0}) &= \mathcal{F}(u, U_{\delta_0} \setminus \cup_{i\le n} \overline{B_i^{\mathcal{X}}}) + \sum_{i\le n} F(v_i, B_i^{\mathcal{X}}) \\
&\le \mathcal{F}(u, U_{\delta_0} \setminus \cup_{i\le n} B_i^{\mathcal{X}}) + \sum_{i\le n} [\mathbf{m}(u, B_i^{\mathcal{X}}) + \delta\mu(B_i^{\mathcal{X}})] \\
&\le \mathcal{F}(u, U_{\delta_0} \setminus \cup_{i\le n} B_i^{\mathcal{X}}) + \sum_{i\le n} [\mathcal{F}(u, B_i^{\mathcal{X}}) + (\delta - t)\mu(B_i^{\mathcal{X}})] \\
&\le \mathcal{F}(u, U_{\delta_0}) + (\delta - t)\mu(\cup_{i\le n} B_i^{\mathcal{X}}).
\end{aligned}$$

By (3.23) and in view of the L^1 lower semicontinuity of $\mathcal{F}$, the latter inequality implies that

$$\frac{t\gamma}{2} \le \liminf_{\delta\to 0} [\mathcal{F}(u, U_{\delta_0}) - \mathcal{F}(u_\delta, U_{\delta_0})] \le 0$$

and we have reached a contradiction. We conclude that $\mu(U_0) = 0$. □

Remark 3.13. If $\mathcal{F}(u,\Omega) < +\infty$ then

$$f(x, \nabla u(x)) = \lim_{\rho\to 0} \frac{\mathcal{F}\big(u, B_r(x)\big)}{|B_r(x)|} = \lim_{\rho\to 0} \frac{\mathbf{m}\big(u, B_r(x)\big)}{|B_r(x)|} \qquad \text{for } \mathcal{L}^N\text{-a.e. } x.$$

Indeed, if $f(x,\nabla u(x)) > 0$ then this follows from Theorems 1.1 and 3.11, and if $f(x,\nabla u(x)) = 0$ then it suffices to use Proposition 3.12 to deduce that

$$\limsup_{\rho\to 0} \frac{\mathbf{m}\big(u, B_r(x)\big)}{|B_r(x)|} \le \lim_{\rho\to 0} \frac{\mathcal{F}\big(u, B_r(x)\big)}{|B_r(x)|} = f(x, \nabla u(x))$$

where the last equality is asserted by Theorem 1.1. Moreover

$$\mu_s(\{x\}) = \lim_{\rho\to 0} \mathbf{m}\big(u, B_r(x)\big).$$

Indeed, if $\mu_s(\{x\}) = 0$ then this equality is a direct consequence of Proposition 3.12, while if $\mu_s(\{x\}) > 0$ then Theorem 3.11 yields

$$1 = \lim_{r\to 0} \frac{\mathbf{m}(u, B_r(x))}{\mu(B(x))} = \frac{\lim_{r\to 0} \mathbf{m}(u, B_r(x))}{\mu_s(\{x\})}.$$

4. Remarks on the Lavrentiev Phenomenon

In this section we prove Proposition 1.2 and Theorem 1.3.

Proof of Proposition 1.2. In view of Theorem 1.1 it suffices to prove that $\mu_s(\Omega) = 0$. For every $x \in \Omega$ let $B_{\rho(x)}(x)$ be a ball satisfying (1.4). By Theorem 1.1 we have that $\mu_s\left(B_{\rho(x)}(x)\right) = 0$, and so $\mu_s\left(\overline{B_r(x)}\right) = 0$ for all $0 < r < \rho(x)$. Since

$$\Omega = \bigcup_{x\in\Omega,\, 0<r<\rho(x)} B_r(x),$$

using Besicovitch Covering Theorem we may find a countable family of disjoint closed balls and a μ_s-null set N such that

$$\Omega = \bigcup_{i=1}^{\infty} \overline{B_{r_i}(x_i)} \cup N.$$

We conclude that

$$\mu_s(\Omega) = \sum_{i=1}^{\infty} \mu_s\left(\overline{B_{r_i}(x_i)}\right) = 0.$$

□

Remark 4.1. It can be easily verified that condition (1.4) is equivalent to assuming that there exists a sequence $\{u_n\} \subset W^{1,\beta}(B)$ such that

$$u_n \rightharpoonup u \text{ in } W^{1,\alpha}(B), \quad F(u_n, B) \to F(u, B). \tag{4.1}$$

That (1.4) implies (4.1) follows immediately from the definition of $\mathcal{F}(u, B)$: it suffices to choose any good sequence for u in B (see (2.4)). Conversely, by Ioffe's Theorem (see Theorem 2.4) we have that $\mathcal{F}\left(u, B_{\rho(x)}(x)\right) \geq F\left(u, B_{\rho(x)}(x)\right)$, while (4.1) yields the converse inequality; hence

$$\mathcal{F}\left(u, B_{\rho(x)}(x)\right) = F\left(u, B_{\rho(x)}(x)\right) \quad \text{for all } x \in \Omega.$$

Theorem 1.3 follows immediately from Proposition 1.2, Remark 4.1, and the lemma below.

Lemma 4.2. *If f is a Carathéodory function satisfying* (1.5) *and* (1.6) *then there exists a sequence $\{u_n\}$ such that*

$$u_n \in C_0^\infty(\Omega), \quad u_n \to u \text{ in } L^1_{\mathrm{loc}}(\Omega), \quad f(x, \nabla u_n) \to f(x, \nabla u) \text{ in } L^1_{\mathrm{loc}}(\Omega).$$

Moreover for every $x_0 \in \Omega$ and for almost all $\rho < \operatorname{dist}(x_0, \partial\Omega)$

$$u_n \to u \quad \text{strongly in } W^{1,\alpha}\big(B_\rho(x_0)\big) \text{ and weakly in } W^{1,\alpha}\big(\partial B_\rho(x_0)\big).$$

Proof. Let $\rho(x)$ be a standard symmetric mollifier with support in $B_1(0)$, for every $n \in \mathbb{N}$ set $\rho_n(x) := n^N \rho(nx)$, and set

$$u_n := \bar{u}_n * \rho_n, \quad \bar{u}_n(x) := \begin{cases} u(x) & \text{if } \operatorname{dist}(x, \partial\Omega) > 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $u_n \in C_0^\infty(\Omega)$, $u_n \to u$ in $W^{1,\alpha}_{\mathrm{loc}}(\Omega)$, thus $u_n \to u$ strongly in $W^{1,\alpha}\big(B_r(x_0)\big)$ for any ball $B_r(x_0)$ with $x_0 \in \Omega$ and $r < \operatorname{dist}(x_0, \partial\Omega)$. In particular, for almost all such r we have $u_n \rightharpoonup u$ in $W^{1,\alpha}\big(\partial B_r(x_0)\big)$. We only have to show that

$$f(x, \nabla u_n) \to f(x, \nabla u) \text{ in } L^1_{\mathrm{loc}}(\Omega). \tag{4.2}$$

Fix $A \subset\subset \Omega$, and let $\operatorname{dist}(A, \partial\Omega) \geq 2\varepsilon_0$; without loss of generality we may always take $n > 1/\varepsilon_0$, thus in A we have $Du_n = Du * \rho_n$. Set $A' := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \varepsilon_0\}$, and define

$$g(x) := f\big(x, Du(x)\big), \quad p_n(x) = \min\{p(y) : |x - y| \leq 1/n\}, \quad \varphi_n(x, z) = |z|^{p_n(x)}.$$

Clearly φ_n is a Carathéodory function, $\varphi(x,\cdot)$ is convex for all $x\in\Omega$, and if $|x-y|<1/n$ then $p_n(x)\le p(y)$ and so

$$\varphi_n(x,z)=|z|^{p_n(x)}\le 1+|z|^{p(y)}\le 1+f(y,z).$$

In view of this inequality and by Jensen's inequality, if $x\in A$ then

$$\begin{aligned}\varphi_n\big(x,\nabla u_n(x)\big) &= \varphi_n\Big(x,\int_{B(x,1/n)}\nabla u(y)\big)\rho_n(x-y)\,dy\Big)\\ &\le \int_{B(x,1/n)}\varphi_n\big(x,\nabla u(y)\big)\rho_n(x-y)\,dy\\ &\le \int_{B(x,1/n)}[1+f(y,\nabla u(y))]\,\rho_n(x-y)\,dy\\ &= 1+(g*\rho_n)(x).\end{aligned}\tag{4.3}$$

Now remark that if $x\in A$ then, with $\alpha':=1-1/\alpha$, and using the fact that $u\in W^{1,\alpha}_{\mathrm{loc}}$ (see (1.5)), we have

$$\begin{aligned}|\nabla u_n(x)| &= \Big|\int_{A'}\nabla u(y)\rho_n(x-y)\,dy\Big|\\ &\le C(\alpha)\|\nabla u\|_{L^\alpha(A')}\,n^N\Big|\int_{A'}\rho^{\alpha'}(n(x-y))\,dy\Big|^{1/\alpha'}\\ &\le C(\alpha)\|\nabla u\|_{L^\alpha(A')}\,n^N n^{-N/\alpha'}\Big|\int_{\mathbb{R}^N}\rho^{\alpha'}(y)\,dy\Big|^{1/\alpha'}\\ &= C'\|\nabla u\|_{L^\alpha(A')}\,n^{N/\alpha},\end{aligned}$$

where $C'=C'(\alpha)$ is a positive constant. By (1.5) we deduce that

$$\begin{aligned}\frac{1}{C}f\big(x,\nabla u_n(x)\big)-1 &\le |\nabla u_n(x)|^{p(x)}=|\nabla u_n(x)|^{p(x)-p_n(x)}\varphi_n\big(x,\nabla u_n(x)\big)\\ &\le\Big(1+{C'}^{\beta-\alpha}\,\|\nabla u\|^{\beta-\alpha}_{L^\alpha(A')}\Big)\,\big(n^{p(x)-p_n(x)}\big)^{N/\alpha}\varphi_n\big(x,\nabla u_n(x)\big)\\ &\le C\,\mathrm{e}^{N\gamma/\alpha}\,\varphi_n\big(x,\nabla u_n(x)\big),\end{aligned}$$

where we have used the fact that

$$0\le p(x)-p_n(x)\le\gamma/\log n\ ,$$

so that

$$\big(n^{p(x)-p_n(x)}\big)^{N/\alpha}\le\mathrm{e}^{N\gamma/\alpha}.$$

In view of (4.3) we now have for all $x\in A$

$$0\le f\big(x,\nabla u_n(x)\big)\le C(1+(g*\rho_n)(x)).$$

Since $\nabla u_n\to\nabla u$ a.e. in A, by the continuity of $f(x,\cdot)$ and Fatou's Lemma we have

$$\liminf_{n\to+\infty}\int_A f(x,\nabla u_n(x))\,dx\ge\int_A f(x,\nabla u(x))\,dx.\tag{4.4}$$

Moreover, once more by Fatou's Lemma and using the fact that $g*\rho_n\to g$ in $L^1(A)$,

$$\begin{aligned}\int_A C(1+g(x))-\limsup_{n\to+\infty}\int_A f(x,\nabla u_n(x))\,dx &= \liminf_{n\to+\infty}\int_A[C(1+(g*\rho_n)(x))-f(x,\nabla u_n(x))]\,dx\\ &\ge \int_A[C(1+g(x))-f(x,\nabla u(x))]\,dx,\end{aligned}$$

and we conclude that

$$\limsup_{n\to+\infty}\int_A f(x,\nabla u_n(x))\,dx\le\int_A f(x,\nabla u(x))\,dx.$$

This, together with (4.4), yields the convergence of $f(\cdot,\nabla u_n(\cdot))$ to $f(\cdot,\nabla u(\cdot))$ in $L^1(A)$. □

The last result of this section shows that under some additional assumptions on the convergence of the traces of approximating sequences, it is possible to impose Dirichlet boundary conditions on the admissible sequences without increasing the overall energy. Since *a priori* the infimum taken over the class of functions restricted under the Dirichlet boundary condition is greater than the infimum over the unconstrained class, we may interpret this result as a situation where the Lavrentiev phenomenon is avoided.

Proposition 4.3. *Let $\mathcal{F}(u,\Omega)<+\infty$, let $A\subset\Omega$ be an open set with Lipschitz boundary, and assume that $u_n\in W^{1,\beta}(A)$ satisfies*

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,\alpha}(A) \text{ and weakly in } W^{1,\alpha}(\partial A),$$

and

$$\mathcal{F}(u,A)=\lim_{n\to+\infty}F(u_n,A).$$

Then there exists a sequence $v_n\in W^{1,\beta}(A)$ such that

$$v_n=u \quad \text{on}\,\partial A, \quad v_n\rightharpoonup u \quad \text{weakly in } W^{1,\alpha}(A), \quad \mathcal{F}(u,A)=\lim_{n\to+\infty}F(v_n,A).$$

Moreover if the sequence $\{u_n\}$ converges to u strongly in $W^{1,\alpha}(A)$ then so does $\{v_n\}$.

Proof of Proposition 4.3. Set $w_n=\mathrm{E}_A(u_n-u)$, so that by Lemma 2.1 $w_n\to 0$ strongly in $W^{1,\beta}(A)$ and $w_n=u_n-u$ on ∂A. For $k\in\mathbb{N}$ let $\varepsilon=1/k$ and define $h_\varepsilon: W^{1,\beta}(A)\to[0,+\infty]$ by

$$h_\varepsilon(w):=\inf\Big\{F(v,A):\ v\in W^{1,\beta}(A),\ v=u+w \text{ on } \partial A,\ \|v-u\|_{L^1(A)}<\varepsilon\Big\}.$$

Since for $n\in\mathbb{N}$ sufficiently large

$$u_n=u+w_n \text{ on } \partial A \quad \text{and} \quad \|u_n-u\|_{L^1(A)}<\varepsilon,$$

it follows that

$$h_\varepsilon(w_n)\le F(u_n,A). \tag{4.5}$$

A similar proof to that of Proposition 3.12 yields the convexity of h_ε. Also, let $C(\varepsilon)>0$ be such that

$$\|w\|_{W^{1,\beta}(A)}<C(\varepsilon)\Rightarrow\|w\|_{L^1(A)}<\varepsilon/2.$$

We claim that h_ε is bounded in the ball in $W^{1,\beta}(A)$ of center zero and radius $C(\varepsilon)$. Indeed, as $u\in W^{1,\alpha}(\partial A)$ there exists $\bar u\in W^{1,\beta}(A)$ such that $u-\bar u\in W_0^{1,\alpha}(A)$, but then there exists $\varphi_\varepsilon\in C_0^\infty(A)$ such that $\|(u-\bar u)-\varphi_\varepsilon\|_{W^{1,\alpha}(A)}<\varepsilon/2$. Setting $u_\varepsilon:=\bar u+\varphi_\varepsilon$ we have $u_\varepsilon\in W^{1,\beta}(A)$, $u_\varepsilon=u$ on ∂A, $\|u_\varepsilon-u\|_{W^{1,\alpha}(A)}<\varepsilon/2$, and thus for all $w\in W^{1,\beta}(A)$ with $\|w\|_{W^{1,\beta}(A)}<C(\varepsilon)$, and in view of (H3)

$$h_\varepsilon(w)\le F(u_\varepsilon+w,A)\le C\left(|A|+\|(u_\varepsilon+w)\|^\beta_{W^{1,\beta}(A)}\right)\le C'(\varepsilon)$$

for some constant $C'(\varepsilon)>0$. Therefore the convex function h_ε is lower semicontinuous at the origin, and by (4.5) we have

$$\mathcal{F}(u,A)=\lim_{n\to+\infty}F(u_n,A)\ge\lim_{n\to+\infty}h_\varepsilon(w_n)\ge h_\varepsilon(0),$$

that is

$$\mathcal{F}(u,A)\ge\inf\Big\{F(v,A):\ v\in W^{1,\beta}(A),\ v=u \text{ on } \partial A,\ \|v-u\|_{L^1(A)}<\varepsilon\Big\}.$$

In particular, there exists $v_\varepsilon\in W^{1,\beta}(A)$ such that $v_\varepsilon=u$ on ∂A and

$$\|v_\varepsilon-u\|_{L^1(A)}<\varepsilon\,,\quad F(v_\varepsilon,A)\le\mathcal{F}(u,A)+\varepsilon \tag{4.6}$$

and the result now follows by letting $\varepsilon\to 0$.

If $u_n\to u$ strongly in $W^{1,\alpha}(A)$ then one may replace the L^1 norm in the definition of h_ε by the norm in $W^{1,\alpha}$, thus obtaining in (4.6) that $\|v_\varepsilon-u\|_{W^{1,\alpha}(A)}<\varepsilon$. □

5. A Counterexample for Carathéodory Quasiconvex Integrands

Here we present the proof of Proposition 1.4.

Recall that under (H1), (H3), by Theorem 3.1 in [15] if $u \in W^{1,\alpha}(\Omega;\mathbb{R}^d)$ is such that $\mathcal{F}(u,\Omega) < +\infty$ then

$$\mathcal{F}(u,A) = \int_A f^u(x)\,dx + \mu_s(A) \quad \text{for all } A \in \mathbb{A}(\Omega),$$

where μ_s is a non-negative finite Radon measure on Ω, singular with respect to $\mathcal{L}^N \llcorner \Omega$, and (see Theorem 3.2 in [6] for a.e. $x_0 \in \Omega$

$$f^u(x_0) = \lim_{\rho\to 0,\, \rho\in\mathcal{R}(u,x_0)} \frac{\mathbf{m}\big(u, B_\rho(x_0)\big)}{|B_\rho(x_0)|}. \tag{5.1}$$

Here $\mathcal{R}(u,x_0)$ stands for the *set of good radii* of the function u at the point $x_0 \in \Omega$ (see [6]), Definition 3.4), and it can be shown that almost all $r \in (0, \operatorname{dist}(x_0,\partial\Omega))$ belong to $\mathcal{R}(u,x_0)$.

The following lemma may be found in [6], Lemma 3.5.

Lemma 5.1. *Under assumptions* (H1) *and* (H3)*, if $R \in \mathcal{R}(u,x_0)$ and if $\{u_n\} \subset W^{1,p}(\Omega;\mathbb{R}^d)$ is such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\partial B_R(x_0);\mathbb{R}^d)$ then*

$$\mathbf{m}(u, B_R(x_0)) = \lim_{n\to+\infty} \mathbf{m}(u_n, B_R(x)).$$

Next we simplify the representation formula for f^u in the general quasiconvex context and in the case where u is affine.

Lemma 5.2. *If* (H1) *and* (H3) *hold, and if u is affine, then for $\mathcal{L}^N$-a.e. $x_0 \in \Omega$*

$$f^u(x_0) = \lim_{n\to+\infty} \left\{ \inf \fint_B f(x_0+\varepsilon_n y, \nabla w(y))\,dy : w \in W^{1,\beta}(B;\mathbb{R}^d),\ w-u \in W_0^{1,\alpha}(B;\mathbb{R}^d) \right\},$$

where B is the unit ball centered at the origin and $\{\varepsilon_n\} \subset \mathcal{R}(u,x_0)$ is any sequence of positive numbers converging to 0*.*

Proof. Consider $\{\varepsilon_n\} \subset \mathcal{R}(u,x_0)$ with $\varepsilon_n \to 0^+$. By (5.1) we have

$$\begin{aligned} f^u(x_0) &= \lim_{n\to+\infty} \Big\{ \inf \fint_{B_{\varepsilon_n}(x_0)} f(x,\nabla v(x))\,dx : v \in W^{1,\beta}(B_{\varepsilon_n}(x_0);\mathbb{R}^d),\ v-u \in W_0^{1,\alpha}(B_{\varepsilon_n}(x_0);\mathbb{R}^d) \Big\} \\ &= \lim_{n\to+\infty} \Big\{ \inf \fint_B f(x_0+\varepsilon_n y, \nabla v(x_0+\varepsilon_n y))\,dy : v \in W^{1,\beta}(B_{\varepsilon_n}(x_0);\mathbb{R}^d),\ v-u \in W_0^{1,\alpha}(B_{\varepsilon_n}(x_0);\mathbb{R}^d) \Big\} \\ &= \lim_{n\to+\infty} \Big\{ \inf \fint_B f(x_0+\varepsilon_n y, \nabla w(y))\,dy : w \in W^{1,\beta}(B;\mathbb{R}^d),\ w-u \in W_0^{1,\alpha}(B;\mathbb{R}^d) \Big\}, \end{aligned}$$

where we used the substitution

$$w(y) := \frac{v(x_0+\varepsilon_n y) - u(x_0) + \varepsilon_n u(0)}{\varepsilon_n}$$

for $y \in B$, and we invoked the fact that u is affine. □

Proof of Proposition 1.4. Let K, p and f be as stated. We claim that for all $\delta \in (0,1)$ there exists $\theta_0 \in (0,1)$ such that for all $\theta < \theta_0$ and for all $x_0 \in \partial K$

$$f^{\mathrm{id}}(x_0) < f(x_0,\mathbb{I}) - 1 + \delta. \tag{5.2}$$

Fix $x_0 \in \partial K$ and let $\varepsilon \in (0,1)$. Since $0 \in \partial(K-x_0)$ we may find $y_\varepsilon \in B_{\varepsilon^2}(0)$ such that $y_\varepsilon \notin K - x_0$. Define $x_\varepsilon := y_\varepsilon/\varepsilon$. Then $|x_\varepsilon| < \varepsilon$ and $x_\varepsilon \notin (K-x_0)/\varepsilon$. Denoting by B the unit ball $B_1(0)$, we clearly have

$$B(x_\varepsilon, 1-|x_\varepsilon|) \subset B, \quad |B \setminus B(x_\varepsilon, 1-|x_\varepsilon|)| \to 0 \quad \text{as } \varepsilon \to 0.$$

Since $(K-x_0)/\varepsilon$ is closed, there exists $r_\varepsilon \in (0,1)$ such that

$$B(x_\varepsilon, r_\varepsilon) \subset B(x_\varepsilon, 1-|x_\varepsilon|) \setminus \frac{K-x_0}{\varepsilon}, \tag{5.3}$$

and we define in B the function

$$v_\varepsilon(x) := \begin{cases} x & \text{if } x \notin B(x_\varepsilon, 1-|x_\varepsilon|), \\ x_\varepsilon + (1-|x_\varepsilon|)\,\frac{x-x_\varepsilon}{|x-x_\varepsilon|} & \text{if } r_\varepsilon < |x-x_\varepsilon| < 1-|x_\varepsilon|, \\ x_\varepsilon + \frac{1-|x_\varepsilon|}{r_\varepsilon}\,(x-x_\varepsilon) & \text{if } |x-x_\varepsilon| < r_\varepsilon. \end{cases}$$

Clearly $v_\varepsilon \in W^{1,\infty}(B;\mathbb{R}^N)$, $v_\varepsilon = \text{id}$ on ∂B, and

$$\begin{aligned} ⨍_B |\nabla v_\varepsilon|^p\,dx &\le 1 + \int_{B(x_\varepsilon,1-|x_\varepsilon|)\setminus B(x_\varepsilon,r_\varepsilon)} \left|\frac{\mathbb{I}}{|x-x_\varepsilon|} - \frac{(x-x_\varepsilon)\otimes(x-x_\varepsilon)}{|x-x_\varepsilon|^3}\right|^p dx \\ &\quad + \int_{B_{r_\varepsilon}(x_\varepsilon)} \left|\frac{1-|x_\varepsilon|}{r_\varepsilon}\right|^p dx \\ &\le 1 + C\int_{r_\varepsilon}^{1-|x_\varepsilon|} r^{N-1-p}\,dr + C\,r_\varepsilon^{N-p} \le C_0 \end{aligned} \tag{5.4}$$

where C_0 is a constant independent of x_0, and where we have used the fact that $p < N$. In view of Lemma 5.2, and choosing a sequence $\{\varepsilon_n\} \subset \mathcal{R}(\text{id}, x_0)$ of positive numbers converging to 0, we have

$$\begin{aligned} f^{\text{id}}(x_0) &= \lim_{n\to+\infty} \left\{ \inf ⨍_B [\chi_K(x_0+\varepsilon_n x)\,|\det\,\nabla w(x)| + \theta|\nabla w(x)|^p]\,dx : w \in W^{1,N}(B;\mathbb{R}^N),\ w = \text{id on } \partial B \right\} \\ &\le \liminf_{n\to+\infty} ⨍_B \left[\chi_{\frac{K-x_0}{\varepsilon_n}}(x)\,|\det\,\nabla v_{\varepsilon_n}(x)| + \theta|\nabla v_{\varepsilon_n}(x)|^p\right] dx \\ &\le \liminf_{n\to+\infty} \frac{|B\setminus B(x_{\varepsilon_n}, 1-|x_{\varepsilon_n}|)|}{|B|} + \theta\,C_0 = \theta\,C_0, \end{aligned}$$

where we have used (5.3), (5.4), and the fact that $\det \nabla v_{\varepsilon_n}(x) = 0$ if $r_{\varepsilon_n} < |x - x_{\varepsilon_n}| < 1 - |x_{\varepsilon_n}|$. Since $f(x_0, \mathbb{I}) \ge 1$, we have now established the claim with $\theta_0 = \delta/C_0$. □

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