# On Hessian Matrices in the Space $BH$


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**Abstract**

An extension of Alberti's result to second order derivatives is obtained. Precisely, if $\Omega$ is an open subset of $\mathbb{R}^N$ and if $f \in L^1\left(\Omega; \mathbb{R}^{N\times N}\right)$ is symmetric-valued, then there exist $u \in W^{1,1}(\Omega)$ with $\nabla u \in BV(\Omega; \mathbb{R}^N)$ and a constant $C > 0$ depending only on $N$ such that

$$D^2 u = f\,\mathcal{L}^N + [\nabla u] \otimes \nu_{\nabla u}\,\mathcal{H}^{N-1} \lfloor S(\nabla u),$$

and

$$\int_\Omega |u| + |\nabla u|\,dx + \int_{S(\nabla u)\cap\Omega} |[\nabla u]|\,d\mathcal{H}^{N-1} \le C \int_\Omega |f|\,dx.$$



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## 1 Introduction

The theory of *second order structured deformations* (SOSDs) has been recently introduced by Owen and Paroni [12], and it offers a generalization to the realm of second order derivatives of the previously developed class of (first order) *structured deformations* (SDs) due to Del Piero and Owen [8], and subsequently treated analytically by Choksi and Fonseca [6] (see also Del Piero and Truskinowsky [9]). These theories aim at describing certain singular or defective behavior of elastic and elasto-plastic materials. A SOSD is a quadruple $(\kappa, u, g, f)$ satisfying some technical conditions preventing interprenetation of matter, as well as regularity properties of the fields away from $\kappa$, where $\kappa$ is the *disarrangement site*, $u$ is the *transplacement*, and $g$ and $f$ are tensor fields with properties similar to those of $\nabla u$ and $\nabla^2 u$, respectively. A pair $(\kappa, u)$ is a simple deformation from a region $\Omega$ if $\kappa \subset \Omega$ has zero volume, $u$ is injective and is a "piece-wise" classical deformation from $\Omega \backslash \kappa$. Owen and Paroni [12] have shown that a SOSD $(\kappa, u, g, f)$ may be approximated by simple deformations $(\kappa_n, u_n)$, i.e., $\kappa = \liminf \kappa_n$, $u = \lim u_n$, $g = \lim \nabla u_n$ and $f = \lim \nabla^2 u_n$, where the latter three limits are taken in the $L^\infty$ sense, while $\liminf \kappa_n := \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \kappa_m$. In light of this approximation, we may view $g(x)$ as the local deformation at $x$ without including the effects of discontinuities of the transplacement $u_n$ at the disarrangement site $\kappa_n$. A similar interpretation holds for $f(x)$. Mechanically we may interpret $\nabla u$ and $\nabla^2 u$ as macroscopic deformation measures, while $g$ and $f$ are the corresponding local (or microscopic) quantities. We refer to the references quoted above for further details and examples.

The energy associated to a simple deformation is

$$\begin{aligned} E(u, \kappa) \quad := \quad & \int_\Omega W(x, u, \nabla u, \nabla^2 u)\, dx + \int_\kappa \psi(x, [u], \nu)\, dx \\ & + \int_\kappa \eta(x, [\nabla u], \nu)\, dH^{N-1}, \end{aligned} \tag{1.1}$$

where the first term is the bulk energy of the material in the placement $\Omega$, the second and third terms take into account the contribution of the surface energy due to slips and separation and to interfaces between two phases of material, respectively. In (1.1) $\nu$ represents the normal to $\kappa$ and $[\cdot]$ denotes the jump. We define the energy of a SOSD as the energetically most economical way to build it up from simple deformations, i.e.,

$$I(\kappa, u, g, f) := \inf\{\liminf E(u_n, \kappa_n) : (k_n, u_n) \text{ approximates } (\kappa, u, g, f)\}.$$

By analogy with the previous work of Choksi and Fonseca [6], where the study of SDs was recasted within the theory of $SBV$, here we are tempted to reformulate the theory of SOSDs either in $SBH$ or in $SBV^2$ (see [4, 5, 10, 11, 13]). Then again, the first question that naturally arises concerns the characterization of all macroscopic $(\kappa, u, g, f)$ that may be attained as limits of quadruples built from smoother deformations. As a first iteration towards this quest, we search for necessary and sufficient conditions ensuring that a pair $(g, f)$ admits an underlying function $u$, either in $SBH$ or in $SBV^2$, such that $K := S(\nabla u)$,

$$g = \nabla u, \quad f = \nabla^2 u.$$

The corresponding problem for first order derivatives was completely solved by Alberti [1] who proved the following result:

**Theorem 1.1** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $f \in L^1\left(\Omega; \mathbb{R}^N\right)$. Then there exists $u \in BV(\Omega)$ and a constant $C > 0$, depending only on $N$, such that*

$$Du = f\,\mathcal{L}^N + [u]\nu_u\,\mathcal{H}^{N-1}\lfloor\, S(u),$$

*and*

$$\int_\Omega |u|\;dx + \int_{S(u)\cap\Omega} |[u]|\;d\mathcal{H}^{N-1} \le C \int_\Omega |f|\;dx.$$

Here $\nu_u(x) \in S^{N-1}$ is the unit normal to the jump set $S(u)$ at $x$.

Note that this theorem confirms the striking gap existing between the Sobolev space $W^{1,1}(\Omega)$ and the space $BV(\Omega)$ of functions of bounded variation. Indeed, it is well known that in a simply connected domain $\Omega \subset \mathbb{R}^N$, a function $f \in L^1\left(\Omega; \mathbb{R}^N\right)$ is the gradient of a Sobolev function $u \in W^{1,1}(\Omega)$ if and only if it satisfies $\operatorname{curl} f = 0$ in the sense of distributions, that is if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \text{ for all } i, j = 1, \cdots, N. \tag{1.2}$$

However, if one allows the function $u$ to belong to a class larger than $W^{1,1}(\Omega)$, such as $SBV(\Omega)$, then the PDE constraint (1.2) may no longer be satisfied.

When $\Omega = \mathbb{R}^N$ the proof of Theorem 1.1 is based on the construction of an approximating sequence $\{u_\delta\}$ of the form

$$u_\delta(x) := \sum_{T_i} f_i(x - y_i)\,\chi_{T_i}, \tag{1.3}$$

where $\{T_i\}$ is the family of all open cubes of the form $Q(y_i, \delta)$ with centers $y_i$ belonging to the lattice $(\delta\mathbb{Z})^N$ and $f_i$ is the mean value of $f$ on $T_i$.

Alberti in [1] proved also the following Lusin-type result:

**Theorem 1.2** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure and let $f : \Omega \to \mathbb{R}^N$ be a Borel function. Then, for every $\varepsilon > 0$, there exists an open set $A \subset \Omega$ and a function $u \in C_0^1(\Omega)$ such that*

$$|A| \le \varepsilon|\Omega|, \quad f = Du \quad \text{in}\, \Omega\backslash A$$

*and*

$$\|Du\|_{L^p(\Omega)} \le C\varepsilon^{1/p-1}\|f\|_{L^p(\Omega)} \quad \textit{for all } p \in [0, \infty]$$

*where $C$ is a constant which depends on $N$ only.*

The argument of the proof of Theorem 1.2 is very similar to that of Theorem 1.1, with the sequence (1.3) replaced by

$$v_\delta(x) := \sum_{T_i} f_i(x-y_i)\,\psi_i(x),$$

where now the functions $\psi_i \in C_0^1(T_i)$ are smooth cut-off functions rather than characteristic functions $\chi_{T_i}$.

Later, Alberti [2] extended Theorem 1.2 to higher order derivatives, and he obtained

**Theorem 1.3** *Let $k$ be a positive integer and let $\lambda$ be a positive finite measure on the open set $\Omega \subset \mathbb{R}^N$. Let $f : \Omega \to (\mathbb{R}^M)^{I(k)}$ be a continuous function and let $\varepsilon$ be a positive real number. Then there exists a compact set $K \subset \Omega$ and a function $u \in C_0^k\left(\Omega, \mathbb{R}^M\right)$ such that*

$$\lambda(\Omega \backslash K) \leq \varepsilon \lambda(\Omega), \quad f = D^k u \quad \text{in}\, K$$

*and*

$$\|D^k u\|_{L^\infty(\Omega)} \leq C\varepsilon^{-k}\|f\|_{L^\infty(\Omega)}$$

*where $C$ is a constant which depends only on $N$ and $k$.*

The approximating sequence in this case has the form

$$w_\delta(x) =: \sum_{T_i} \left( \sum_{|\alpha|\leq k} \frac{1}{\alpha!}(f_i)_\alpha (x-y_i)^\alpha \right) \psi_i(x),$$

where we are using the standard multi-index notation.

Unlike the generalization of Theorem 1.2 to higher order derivatives, i.e. Theorem 1.3, the analog of Theorem 1.1 for second order derivatives does not follow by a straightforward adaptation of its proof when we ask the potential $u$ to be in the Sobolev space $W^{1,1}$. Moreover, while Theorem 1.1, as originally proved by Alberti [1], relies on a careful modification of the argument of Theorem 1.2, the same strategy will no longer work when attempting to establish Theorem 1.4 below from the corresponding second order Lusin result in Theorem 1.3. Indeed, it is well known that gradients of continuous maps can only jump across hyperplanes, and that the gradient jump must be a rank-one matrix oriented following the normal to the hyperplane (see Ball and James [3]). Therefore, a careful geometric construction is required, and here we achieve this via finite element techniques of triangulations of $\mathbb{R}^N$ for polynomials of degree 2. Our main theorem is

**Theorem 1.4** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $f \in L^1\left(\Omega; \mathbb{R}^{N\times N}_{\text{sym}}\right)$. Then there exists $u \in BH(\Omega)$ and a constant $C > 0$ depending only on $N$ such that*

$$D^2 u = f\,\mathcal{L}^N + [\nabla u] \otimes \nu_{\nabla u}\,\mathcal{H}^{N-1}\lfloor\, S(\nabla u),$$

*and*

$$\int_\Omega |u| + |\nabla u| \; dx + \int_{S(\nabla u)\cap\Omega} |[\nabla u]| \; d\mathcal{H}^{N-1} \leq C \int_\Omega |f| \; dx.$$

In the passage from the Sobolev space $W^{2,1}$ to the space $BH$ the Hessian matrix $D^2u$ remains symmetric but it may loose in general the PDE constraint curl= 0. By going to the larger space $SBV^2$ also symmetry may be lost. Indeed, applying Theorem 1.1 twice we prove that

**Theorem 1.5** *Let $\Omega$ be an open subset of $\mathbb{R}^N$ and let $f \in L^1\left(\Omega; \mathbb{R}^{N\times N}\right)$. Then there exist $u \in SBV^2(\Omega)$ and a constant $C > 0$, depending only on $N$, such that*

$$\nabla^2 u(x) = f(x) \;\; \textit{for} \;\; \mathcal{L}^N \;\; \textit{a.e.} \;\; x \in \mathbb{R}^N,$$

*and*

$$\int_\Omega |u| + |\nabla u| \; dx + \int_{(S(u)\cup S(\nabla u))\cap\Omega} |[u]| + |[\nabla u]| \; d\mathcal{H}^{N-1} \leq C \int_\Omega |f| \; dx.$$

Note that, by virtue of the very definition of $SBV^2$, it can be easily established that, given $(g, f) \in L^1(\Omega; \mathbb{R}^N) \times L^1(\Omega; \mathbb{R}^{N\times N})$, there exists $u \in SBV^2(\Omega)$ satisfying

$$\nabla u = g, \quad \nabla^2 u = f \quad \mathcal{L}^N \text{ a.e. } x \in \Omega,$$

if and only if

$$g \in SBV(\Omega; \mathbb{R}^N) \quad \text{and } f = \nabla g \quad \mathcal{L}^N \text{ a.e. } x \in \Omega.$$

## 2 Preliminaries

In what follows $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the $N$ dimensional Lebesgue measure and the $N-1$ dimensional Hausdorff measure in $\mathbb{R}^N$. When there is no possibility of confusion, if $E \subset \mathbb{R}^N$ is a Lebesgue measurable set then we abbreviate $|E| := \mathcal{L}^N(E)$. Let $Q$ be the the unit cube $(-1/2, 1/2)^N$ and set $Q(x_0, \varepsilon) := x_0 + \varepsilon Q$ for $x_0 \in \mathbb{R}^N$, $\varepsilon > 0$. We denote by $\mathbb{R}^{N\times N}_{\text{sym}}$ the space of all symmetric $N \times N$ matrices endowed with the norm

$$|f| := \sum_{i,i=1}^N |f_{ij}| \quad \text{where } f = (f_{ij}) \quad i,j = 1, \cdots, N,$$

i.e., $f_{ij} = f\mathbf{e}_i \cdot \mathbf{e}_j$ where $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ is the standard orthonormal basis of $\mathbb{R}^N$.

We review briefly some facts about functions of bounded variation which will be useful in the sequel. A function $u \in L^1(\Omega; \mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV(\Omega; \mathbb{R}^d)$, if for all $i = 1, \cdots d$, and $j = 1, \cdots N$, there exists a finite Radon measure $\mu_{ij}$ such that

$$\int_\Omega u_i(x) \frac{\partial v}{\partial x_j}(x) \, dx = -\int_\Omega v(x) \, d\mu_{ij}$$

for every $v \in C_c^1(\Omega;\mathbb{R})$. The distributional derivative $Du$ is the matrix–valued measure with $(i,j)$-th entry $\mu_{ij}$. Given $u \in BV(\Omega;\mathbb{R}^d)$ the *approximate upper* and *lower limit* of each component $u_i$, $i = 1,\cdots d$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N\left(\{y \in \Omega \cap Q(x,\varepsilon) : u_i(y) > t\}\right) = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N\left(\{y \in \Omega \cap Q(x,\varepsilon) : u_i(y) < t\}\right) = 0\right\},$$

while the *jump set* of $u$, or *singular set*, is defined by

$$S(u) := \bigcup_{i=1}^{d} \{x \in \Omega : u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and $K_n$ is a compact subset of a $C^1$ hypersurface. If $x \in \Omega\backslash S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x),\cdots,u_d^+(x))$ and $(u_1^-(x),\cdots,u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$ a.e. $x \in \Omega\backslash S(u)$. Furthermore, for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ there exist a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at $x$, and two vectors $u^-(x),\ u^+(x) \in \mathbb{R}^d$ (the traces of $u$ on $S(u)$ at the point $x$) such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{\{y\in Q(x_0,\varepsilon):\,(y-x)\cdot\nu_u(x)>0\}} |u(y) - u^+(x)|^{N/(N-1)} dy = 0$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{\{y\in Q(x_0,\varepsilon):\,(y-x)\cdot\nu_u(x)<0\}} |u(y) - u^-(x)|^{N/(N-1)} dy = 0.$$

Note that, in general, $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. We denote the *jump of $u$ across $S(u)$* by $[u] := u^+ - u^-$. The distributional derivative $Du$ may be decomposed as

$$Du = \nabla u\, \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u\, \mathcal{H}^{N-1} \lfloor S(u) + C(u), \tag{2.1}$$

where $\nabla u$ is the density of the absolutely continuous part of $Du$ with respect to $\mathcal{L}^N$ and $C(u)$ is the Cantor part of $Du$. These three measures are mutually singular. We designate by $||u||_{BV}$ the norm of $u$ in the Banach space $BV(\Omega)$ defined by

$$||u||_{BV(\Omega)} := \int_\Omega |\nabla u|\, dx + \int_{S(u)} |[u]| dH^{N-1} + |C(u)|(\Omega).$$

The space $SBV(\Omega)$ of *special functions with bounded variation* is

$$SBV(\Omega) := \{u \in BV(\Omega) : C(u) = 0\}.$$

We now introduce the space of *functions of bounded Hessian*

$$\begin{aligned} BH(\Omega) : &= \big\{u \in W^{1,1}(\Omega) : D^2 u \text{ is a finite Radon measure}\big\} \\ &= \big\{u \in W^{1,1}(\Omega) : \nabla u \in BV(\Omega; \mathbb{R}^N)\big\}, \end{aligned}$$

where $D^2 u$ denotes the distributional Hessian of $u$.

In view of (2.1) now applied to $\nabla u$ in place of $u$, we may decompose $D^2 u$ as

$$D^2 u = \nabla^2 u \, \mathcal{L}^N \lfloor \Omega + [\nabla u] \otimes \nu_{\nabla u} \, \mathcal{H}^{N-1} \lfloor S(\nabla u) + C(\nabla u),$$

where $\nabla^2 u$ is the density of the absolutely continuous part of $D^2 u$ with respect to $\mathcal{L}^N$, i.e., $\nabla^2 u = \nabla(\nabla u)$, $[\nabla u] := (\nabla u)^+ - (\nabla u)^-$, and $C(\nabla u)$ is the Cantor part of $D^2 u$.

Since $D^2 u$ is a symmetric distribution, we then have that

$$\nabla^2 u(x) \in \mathbb{R}^{N \times N}_{\text{sym}}, \quad (\nabla u)^+ - (\nabla u)^- = a \, \nu_{\nabla u} \tag{2.2}$$

for some $a \in L^1_{H^{N-1} \lfloor S(\nabla u)}(\Omega)$. The latter equality results immediately from the fact that $\Big((\nabla u)^+ - (\nabla u)^-\Big) \otimes \nu_{\nabla u}$ must a symmetric rank-one matrix.

Note also that if $u$ is $W^{2,1}(\Omega)$ then $D^2 u = \nabla^2 u \, \mathcal{L}^N \lfloor \Omega$ with

$$\nabla^2 u = \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right), \quad i, j = 1, \cdots, N.$$

The space $SBH(\Omega)$ of *special functions of bounded Hessian*, cf. [13, 10, 11, 4], is the space of all functions of bounded Hessian such that $\nabla u \in SBV(\Omega; \mathbb{R}^N)$, i.e., $u \in BH(\Omega)$ and

$$D^2 u = \nabla^2 u \, \mathcal{L}^N \lfloor \Omega + [\nabla u] \otimes \nu_{\nabla u} \, \mathcal{H}^{N-1} \lfloor S(\nabla u).$$

Finally we define the space $SBV^2(\Omega)$, cf. [5], as

$$SBV^2(\Omega) := \big\{u \in SBV(\Omega) : \nabla u \in SBV(\Omega; \mathbb{R}^N)\big\}.$$

Clearly $SBH(\Omega) \subset SBV^2(\Omega)$, and if $u \in SBH(\Omega)$ then $Du = \nabla u$, while if $u \in SBV^2(\Omega)$ then $Du$ is given by (2.1) and, again writing $\nabla^2 u = \nabla(\nabla u)$,

$$D \nabla u = \nabla^2 \, \mathcal{L}^N \lfloor \Omega + [\nabla u] \otimes \nu_{\nabla u} \, \mathcal{H}^{N-1} \lfloor S(\nabla u).$$

As shown in Theorem 1.5 below, if $u \in SBV^2(\Omega)$ then the symmetry of $\nabla^2 u$ is no longer guaranteed.

# 3 The $SBH$ setting

Here we improve Theorem 1.5 by obtaining the potential $u$ in $W^{1,1}(\Omega)$ provided $f$ is symmetric valued, as dictated by (2.2). Guided by Alberti's work [1], this property will be a direct consequence of a Lusin's-type result for second order derivatives. Precisely,

**Lemma 3.1** *Let $f \in L^1\left(\mathbb{R}^N;\mathbb{R}^{N\times N}_{\mathrm{sym}}\right)$ and let $\eta > 0$. Then there exists $u \in SBH\left(\mathbb{R}^N\right)$ and a constant $C > 0$, depending only on $N$, such that*

$$\int_{\mathbb{R}^N} |u| + |\nabla u| + \left|f - \nabla^2 u\right| \, dx \leq \eta, \tag{3.1}$$

$$\int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} \leq C \int_{\mathbb{R}^N} |f| \, dx. \tag{3.2}$$

Assuming that the lemma holds, we now prove our main result.

**Proof of Theorem 1.4.** We proceed as in [1] to construct by an induction argument two sequences $\{u_n\} \subset SBH\left(\mathbb{R}^N\right)$ and $\{f_n\} \subset L^1\left(\mathbb{R}^N;\mathbb{R}^{N\times N}_{\mathrm{sym}}\right)$ as follows:

set $u_0 := 0$ and $f_0 := f$, and for given $n \in N$ suppose that $u_{n-1} \in SBH\left(\mathbb{R}^N\right)$ and $f_{n-1} \in L^1\left(\mathbb{R}^N;\mathbb{R}^{N\times N}_{\mathrm{sym}}\right)$ have been selected. By Lemma 3.1 there exists $u_n \in SBH\left(\mathbb{R}^N\right)$ such that

$$\int_{\mathbb{R}^N} |u_n| + |\nabla u_n| + \left|f_{n-1} - \nabla^2 u_n\right| \, dx \leq \frac{1}{2^n} \int_{\mathbb{R}^N} |f_{n-1}| \, dx,$$

$$\int_{S(\nabla u_n)} |[\nabla u_n]| \, d\mathcal{H}^{N-1} \leq C \int_{\mathbb{R}^N} |f_{n-1}| \, dx.$$

Set $f_n := f_{n-1} - \nabla^2 u_n$. We claim that the function

$$u := \sum_{n=1}^{\infty} u_n$$

satisfies the requirements. Indeed, note that

$$\int_{\mathbb{R}^N} |f_n| \, dx \leq \frac{1}{2^{\alpha_n}} \int_{\mathbb{R}^N} |f| \, dx \quad \text{for all} \quad n \in \mathbb{N},$$

where $\alpha_1 := 1$, $\alpha_n := \alpha_{n-1} + n$ for $n \geq 2$. Thus

$$\begin{aligned}
\int_{\mathbb{R}^N} |u| + |\nabla u| \, dx &\leq \sum_{n=1}^{\infty} \int_{\mathbb{R}^N} |u_n| + |\nabla u_n| \, dx \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{\mathbb{R}^N} |f_{n-1}| \, dx \\
&\leq \int_{\mathbb{R}^N} |f| \, dx,
\end{aligned}$$

similarly

$$\int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} \leq C \sum_{n=1}^{\infty} \int_{\mathbb{R}^N} |f_{n-1}| \, dx \leq C \int_{\mathbb{R}^N} |f| \, dx,$$

also $\nabla^2 u(x) = f(x)$ for $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$, and this concludes the proof. ■

The remaining two subsections are dedicated to the proof of Lemma 3.1. In Subsection 3.1 we restrict our attention to the case $N = 2$, where the main ideas may be easily introduced with the help of figures. In Subsection 3.2 we will prove Lemma 3.1 in its full generality.

## 3.1 The $2-$dimensional case

We prove Lemma 3.1 in the case $N = 2$. Fix $h > 0$, $i, j \in \mathbb{Z}$, and denote by $A^h_{i,j}$ the lattice points

$$A^h_{i,j} := ih\mathbf{e_1} + jh\mathbf{e_2} = (ih, jh) \quad \text{with } i, j \in \mathbb{Z},$$

$L^h_{i,j}$ is the closed triangle whose vertices are the points $A^h_{i,j}, A^h_{i+1,j}$ and $A^h_{i,j+1}$, and $U^h_{i,j}$ denotes the closed triangle whose vertices are $A^h_{i,j}, A^h_{i-1,j}$ and $A^h_{i,j-1}$. Clearly the union of all triangles $L^h_{i,j}$ and $U^h_{i,j}$ forms a triangulation $\tau^h$ of $\mathbb{R}^2$ (see Figure 1). Let

$$\mathbf{t} := \frac{1}{\sqrt{2}}(\mathbf{e_2} - \mathbf{e_1}) = \frac{(-1,1)}{\sqrt{2}}$$

where $\{\mathbf{e_1}, \mathbf{e_2}\}$ denotes an orthonormal basis of $\mathbb{R}^2$.

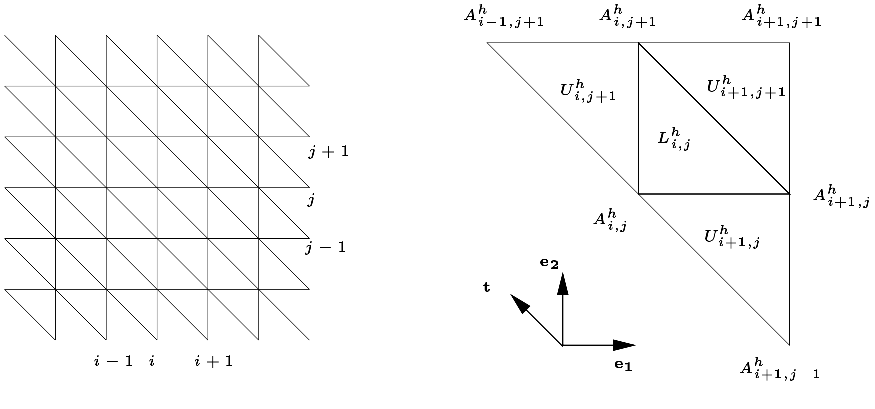


Figure 1: Triangulation for $N = 2$

**Proof of Lemma 3.1.**
**Step 1.** Assume first that $f \in C_c\left(\mathbb{R}^2; \mathbb{R}^{2\times 2}_{\text{sym}}\right)$. Let $\ell > 0$ be so large that $f(x) = 0$ outside the square $(-\ell/2, \ell/2)^2$. Fix $\eta > 0$ and let $h > 0$ be sufficiently

small such that

$$|z - z'| < 2h \Rightarrow |f(z) - f(z')| < \frac{\eta}{\ell^2}. \tag{3.3}$$

Consider a triangulation $\tau^h$ as described above, and define

$$\begin{aligned}
F^h_{A^h_{i,j},A^h_{i+1,j}} &:= \frac{1}{h^2}\left(\int_{L^h_{i,j}\cup U^h_{i+1,j}} f\,dx\right)\mathbf{e_1}\cdot\mathbf{e_1} = \frac{1}{h^2}\int_{L^h_{i,j}\cup U^h_{i+1,j}} f_{11}\,dx,\\
F^h_{A^h_{i,j},A^h_{i,j+1}} &:= \frac{1}{h^2}\left(\int_{L^h_{i,j}\cup U^h_{i,j+1}} f\,dx\right)\mathbf{e_2}\cdot\mathbf{e_2} = \frac{1}{h^2}\int_{L^h_{i,j}\cup U^h_{i,j+1}} f_{22}\,dx,\\
F^h_{A^h_{i,j+1},A^h_{i+1,j}} &:= \frac{1}{h^2}\left(\int_{L^h_{i,j}\cup U^h_{i+1,j+1}} f\,dx\right)\mathbf{t}\cdot\mathbf{t}\\
&= \frac{1}{h^2}\int_{L^h_{i,j}\cup U^h_{i+1,j+1}}\left(\frac{f_{11}+f_{22}}{2} - f_{12}\right)dx.
\end{aligned}$$

If the function $f$ was the Hessian of a function $v$, i.e., $f = \nabla^2 v$, then $F^h_{A^h_{i,j},A^h_{i+1,j}}$ would be the average over $L^h_{i,j} \cup U^h_{i+1,j}$ of the second derivative of $v$ calculated along the direction of the segment $\overrightarrow{A^h_{i,j}A^h_{i+1,j}}$, i.e., $\partial^2 v/\partial x^2$ ( note that the two triangles have as a common side the segment $\overrightarrow{A^h_{i,j}A^h_{i+1,j}}$). A similar interpretation holds for $F^h_{A^h_{i,j},A^h_{i,j+1}}$ and $F^h_{A^h_{i,j+1},A^h_{i+1,j}}$. We now define a function $u$ which is continuous, piecewise quadratic ($u$ is quadratic on any triangle $L^h_{i,j}$ and $U^h_{i,j}$), and whose Hessian is close to $f$ in $L^1$ norm.

On each triangle $L^h_{i,j}$ and $U^h_{i,j}$ the function $u$ shall have the form

$$u(x,y) = \frac{1}{2}a_1x^2 + a_2xy + \frac{1}{2}a_3y^2 + a_4x + a_5y + a_6,$$

where the constants $a_1, \dots, a_6$ may vary from triangle to triangle. Since we want the function $u$ to be small, continuous, and with its Hessian close to $f$, we require that $u(A^h_{i,j}) = 0$ for every $i, j \in \mathbb{Z}$, and that the second derivative of $u$ calculated along a side of a triangle be equal to the mean value of $f$ along the same side. Precisely, in the triangle $L^h_{i,j}$ we prescribe

$$u(A^h_{i,j}) = u(A^h_{i+1,j}) = u(A^h_{i,j+1}) = 0 \tag{3.4}$$

and

$$\nabla^2 u\,\mathbf{e_1}\cdot\mathbf{e_1} = F^h_{A^h_{i,j},A^h_{i+1,j}},\quad \nabla^2 u\,\mathbf{e_2}\cdot\mathbf{e_2} = F^h_{A^h_{i,j},A^h_{i,j+1}}\quad \nabla^2 u\,\mathbf{t}\cdot\mathbf{t} = F^h_{A^h_{i,j+1},A^h_{i+1,j}}.$$

A simple computation then shows that

$$\begin{aligned}
u\,|_{L^h_{i,j}}(x,y) &= \frac{1}{2}a_1^{i,j}(x-ih)(x-(i+1)h) + a_2^{i,j}(x-ih)(y-jh)\\
&\quad + \frac{1}{2}a_3^{i,j}(y-jh)(y-(j+1)h),
\end{aligned} \tag{3.5}$$

where the constants $a_i$, $i = 1, \ldots, 6$, are given by

$$\begin{cases} a_1^{i,j} := F^h_{A^h_{i,j}, A^h_{i+1,j}}, \\ a_2^{i,j} := \frac{1}{2}\left(a_1^{i,j} + a_3^{i,j}\right) - F^h_{A^h_{i,j+1}, A^h_{i+1,j}}, \\ a_3^{i,j} := F^h_{A^h_{i,j}, A^h_{i,j+1}}. \end{cases}$$

Clearly the function $u$ is continuous, since it is continuous on each triangle and its values on the boundary of each triangle are assigned via (3.4) and by fixing the second order derivative of $u$ along the sides. Moreover, using (3.5) we have

$$\int_{L^h_{i,j}} |u|\, dx \leq Ch^2 |L^h_{i,j}| (|a_1^{i,j}| + |a_2^{i,j}| + |a_3^{i,j}|) \leq Ch^2 \int_{B^h_{i,j}} |f|\, dx,$$

where $B^h_{i,j} := L^h_{i,j} \cup U^h_{i,j+1} \cup U^h_{i+1,j+1} \cup U^h_{i+1,j}$. Similarly, we find

$$\int_{L^h_{i,j}} |\nabla u|\, dx \leq Ch \int_{B^h_{i,j}} |f|\, dx.$$

Since

$$\begin{aligned}
(\nabla^2 u)_{12} - f_{12} &= \left(\frac{f_{11} + f_{22}}{2} - f_{12}\right) - \left(\frac{(\nabla^2 u)_{11} + (\nabla^2 u)_{22}}{2} - (\nabla^2 u)_{12}\right) \\
&\quad + \frac{(\nabla^2 u)_{11} - f_{11}}{2} + \frac{(\nabla^2 u)_{22} - f_{22}}{2} \\
&= f\mathbf{t} \cdot \mathbf{t} - \nabla^2 u\, \mathbf{t} \cdot \mathbf{t} + \frac{(\nabla^2 u)_{11} - f_{11}}{2} + \frac{(\nabla^2 u)_{22} - f_{22}}{2} \\
&= f\mathbf{t} \cdot \mathbf{t} - F^h_{A^h_{i,j+1}, A^h_{i+1,j}} + \frac{F^h_{A^h_{i,j}, A^h_{i+1,j}} - f_{11}}{2} + \frac{F^h_{A^h_{i,j}, A^h_{i,j+1}} - f_{22}}{2},
\end{aligned}$$

by virtue of (3.3) we have

$$\begin{aligned}
&\int_{L^h_{i,j}} |f - \nabla^2 u|\, dx \\
&= \int_{L^h_{i,j}} \big(|f_{11} - (\nabla^2 u)_{11}| + |f_{22} - (\nabla^2 u)_{22}| + 2|f_{12} - (\nabla^2 u)_{12}|\big)\, dx \\
&\leq C \int_{L^h_{i,j}} \Bigg(\left|f\mathbf{e_1} \cdot \mathbf{e_1} - F^h_{A^h_{i,j}, A^h_{i+1,j}}\right| + \left|f\mathbf{e_2} \cdot \mathbf{e_2} - F^h_{A^h_{i,j}, A^h_{i,j+1}}\right| \\
&\quad + \left|f\mathbf{t} \cdot \mathbf{t} - F^h_{A^h_{i,j+1}, A^h_{i+1,j}}\right|\Bigg)\, dx \\
&\leq C\frac{\eta}{\ell^2} |L^h_{i,j}|.
\end{aligned}$$

Clearly the estimates above hold also for the triangles $U_{ij}$. Summing over all the triangles of $\tau^h$ contained in $(-\ell, \ell)^2$ yields

$$\int_{\mathbb{R}^2} |f - \nabla^2 u|\, dx \leq C\eta,$$

where we have used the fact that for $h$ sufficiently small both $u$ and $f$ vanish outside $(-\ell,\ell)^2$ and hence (3.1) is established.

To obtain (3.2) notice that $S(\nabla u)$ is contained in the union of the sides of the triangles, and thus

$$\int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} \leq Ch^2 \sum_{i,j} (|a_1^{i,j}| + |a_2^{i,j}| + |a_3^{i,j}|) \leq C \int_{\mathbb{R}^N} |f| \, dx.$$

This concludes Step 1.

**Step 2.** Given $f \in L^1\left(\mathbb{R}^2; \mathbb{R}^{2\times 2}_{\text{sym}}\right)$ and $\eta > 0$, by density we may find $f_\eta \in C_c\left(\mathbb{R}^2; \mathbb{R}^{2\times 2}_{\text{sym}}\right)$ such that

$$\int_{\mathbb{R}^N} |f - f_\eta| \, dx \leq \eta \int_{\mathbb{R}^N} |f| \, dx. \tag{3.6}$$

By Step 1 applied to $f_\eta$, there exists $u \in SBH\left(\mathbb{R}^2\right)$ such that

$$\begin{gathered}\int_{\mathbb{R}^N} |u| + |\nabla u| + \left|f_\eta - \nabla^2 u\right| \, dx \leq \eta, \\ \int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} \leq C \int_{\mathbb{R}^N} |f_\eta| \, dx,\end{gathered}$$

which, together with (3.6), yields

$$\begin{gathered}\int_{\mathbb{R}^N} |u| + |\nabla u| + \left|f - \nabla^2 u\right| \, dx \leq \eta \left(1 + \int_{\mathbb{R}^N} |f| \, dx\right), \\ \int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} \leq C\,(1+\eta) \int_{\mathbb{R}^N} |f| \, dx\end{gathered}$$

and the proof is complete. ∎

## 3.2 The $N-$dimensional case

In what follows we will exploit well-known facts of triangulations of $\mathbb{R}^N$ commonly used in finite element methods (see Ciarlet [7]). An *$N-$simplex* is the convex hull $K$ of $N+1$ points $A_j \in \mathbb{R}^N$ (the *vertices of* $K$) which are not contained in a hyperplane, namely

$$K := \left\{ x \in \mathbb{R}^N \,:\, x = \sum_{j=1}^{N+1} \lambda_j A_j,\, 0 \leq \lambda_j \leq 1,\, \sum_{j=1}^{N+1} \lambda_j = 1 \right\},$$

and for all $i \in \{1, \ldots, N+1\}$

$$\{A_j - A_i : j \in \{1, \ldots, N+1\} \setminus \{i\}\} \quad \text{is a basis of } \mathbb{R}^N. \tag{3.7}$$

The *barycentric coordinates* $\lambda_j = \lambda_j(x)$, $j = 1, \cdots, N+1$, of any point $x \in \mathbb{R}^N$ with respect to the points $A_j$, $j = 1, \cdots, N+1$, are the unique solutions of the linear system

$$\begin{cases} \sum_{j=1}^{N+1} \lambda_j a_{ij} = x_i \quad i = 1, \cdots, N, \\ \sum_{j=1}^{N+1} \lambda_j = 1, \end{cases} \tag{3.8}$$

where $A_j := (a_{1j}, \cdots, a_{Nj})$. Let $\mathcal{P}_2$ denote the set of polynomials of degree less than or equal to two in the $N$ variables $x_1, \cdots, x_N$. It can be shown that if $p \in \mathcal{P}_2$ then

$$p = \sum_{i=1}^{N+1} \lambda_i (2\lambda_i - 1) p(A_i) + \sum_{i<j} 4\lambda_i \lambda_j p(A_{ij}),$$

where $A_{ij} := \frac{A_i + A_j}{2}$ are the midpoints of the edges of the $N$–simplex $K$.

Let

$$\mathcal{Q}_K := \{p \in \mathcal{P}_2 : \; p(A_i) = 0 \text{ for all } i = 1, \cdots, N+1\}.$$

If $p \in \mathcal{Q}_K$ then

$$p = \sum_{i<j} 4\lambda_i \lambda_j p(A_{ij}). \tag{3.9}$$

The set

$$\Xi_K := \{p(A_{ij}) : \; 1 \le i < j \le N+1\}$$

is called the set of *degrees of freedom* in the space $\mathcal{Q}_K$.

**Lemma 3.2** *Let $p \in \mathcal{Q}_K$. Then*

(i) *for all $x \in \operatorname{int} K$ and $i, j \in \{1 \cdots, N+1\}$, $i \neq j$,*

$$\nabla p(x) \cdot (A_i - A_j) = 4(\lambda_j - \lambda_i) p(A_{ij});$$

(ii) *for all $x \in \operatorname{int} K$ and $i, j \in \{1 \cdots, N+1\}$, $i < j$,*

$$\left(\nabla^2 p(x)(A_i - A_j)\right) \cdot (A_i - A_j) = -8 p(A_{ij});$$

**Proof.** Fix $x \in \operatorname{int} K$ and choose $k \in \{1, \ldots N\}$. In view of (3.8), and for $t \in \mathbb{R}$ close to 0, we obtain

$$\begin{aligned} \sum_{r=1}^{N+1} \lambda_r(x + t(A_i - A_j)) \, a_{kr} &= (x + t(A_i - A_j))_k \\ &= x_k + t\, a_{ki} - t\, a_{kj} \\ &= \sum_{r=1}^{N+1} \lambda_r(x) \, a_{kr} + t\, a_{ki} - t\, a_{kj}. \end{aligned}$$

The uniqueness of solutions of (3.8) now yields

$$\begin{cases} \lambda_r(x+t(A_i-A_j)) = \lambda_r(x) & \text{if } r \neq i,j, \\ \lambda_i(x+t(A_i-A_j)) = \lambda_i(x)+t, & \\ \lambda_j(x+t(A_i-A_j)) = \lambda_j(x)-t. & \end{cases}$$

By (3.9) we have

$$\begin{aligned} \nabla p(x)\cdot(A_i-A_j) &= \frac{d}{dt}p(x+t(A_i-A_j))\Big|_{t=0} \\ &= \frac{d}{dt}\left[4\left(\lambda_i(x)+t\right)\left(\lambda_j(x)-t\right)p(A_{ij})\right]\Big|_{t=0} \\ &= 4\left(\lambda_j(x)-\lambda_i(x)\right)p\left(A_{ij}\right). \end{aligned}$$

and

$$\begin{aligned} \left(\nabla^2 p(x)\left(A_i-A_j\right)\right)\cdot\left(A_i-A_j\right) &= \frac{d^2}{dt^2}p(x+t(A_i-A_j))\Big|_{t=0} \\ &= \frac{d^2}{dt^2}\left[4\left(\lambda_i(x)+t\right)\left(\lambda_j(x)-t\right)p(A_{ij})\right]\Big|_{t=0} \\ &= -8p\left(A_{ij}\right), \end{aligned}$$

and the proof is complete. ■

Decompose the unit cube $\overline{Q} := [-1/2, 1/2]^N$ into a triangulation $\tau$ of a finite number of $N$–simplexes in such a way that any two distinct $N$–simplexes are either disjoint or they have in common a vertex, an edge, or a face. For any fixed $h > 0$ subdivide $\mathbb{R}^N$ into closed cubes of the form $\overline{Q(x,h)}$ with centers $x$ belonging to the lattice $(h\mathbb{Z})^N$. In each cube consider a triangulation $\tau^{x,h}$ of $N$–simplexes, self-similar to the triangulation $\tau$ of $\overline{Q}$. The family

$$\tau^h := \bigcup_{x\in(h\mathbb{Z})^N} \tau^{x,h}$$

is a triangulation of $\mathbb{R}^N$.

Note that by construction there exists a positive constants $C$ independent of $h$ such that

$$\frac{1}{C}h^N \leq \left|K^h\right| \leq Ch^N, \quad \frac{1}{C}h^{N-1} \leq \mathcal{H}^{N-1}\left(\partial K^h\right) \leq Ch^{N-1}, \tag{3.10}$$

for all $K^h \in \tau^h$.

If $A_{ij}^h \in \mathbb{R}^N$ is the middle point on an edge of an $N$–simplex in $\tau^h$, then we define $\tau^h_{A^h_{ij}}$ as the set of all $N$–simplexes $K^h \in \tau^h$ which contain the point $A^h_{ij}$, precisely

$$\tau^h_{A^h_{ij}} := \left\{K^h \in \tau^h : A^h_{ij} \in K^h\right\}.$$

Due to the self-similarity of the construction, it is clear that there exists an integer $M \in \mathbb{N}$, independent of $h$, such that

$$\operatorname{card}\left(\tau^h_{A^h_{ij}}\right) \leq M \tag{3.11}$$

for all middle points $A^h_{ij}$.

**Proposition 3.3** *Given a triangulation $\tau^h$ constructed as above, there exists a constant $C > 0$, depending on the triangulation but not on $h$, such that for every $N$-simplex $K^h \in \tau^h$ with vertices $\{A^h_1, \dots, A^h_{N+1}\}$ we have*

(i) *for every fixed $i \in \{1, \dots, N+1\}$*

$$|v| \leq C \sum_{j \neq i} \left|v \cdot E^h_{ij}\right| \quad \textit{for all } v \in \mathbb{R}^N;$$

(ii)

$$|\xi| \leq C \sum_{1 \leq i < j \leq N+1} \left|\left(\xi\, E^h_{ij}\right) \cdot E^h_{ij}\right| \quad \textit{for all } \xi \in \mathbb{R}^{N\times N}_{\text{sym}}, \tag{3.12}$$

*where*

$$E_{ij} := \frac{A^h_i - A^h_j}{\left|A^h_i - A^h_j\right|}.$$

**Proof.** Since by construction each $N$–simplex $K^h \in \tau^h$ maybe obtained by a translation and a dilation of an $N$–simplex $K \in \tau$ in the unit cube, then $E^h_{ij} = E_{ij}$ where $E_{ij}$ are the corresponding vectors for $K$. Thus (i) follows immediately from (3.7). To prove (ii), without loss of generality, and via a translation of the axes, we may suppose that $A_{N+1} = 0$, and so (see (3.7))

$$\{A_1, \dots, A_N\} \quad \text{is a basis of } \mathbb{R}^N. \tag{3.13}$$

We claim that the matrices

$$e_{ij} := E_{ij} \otimes E_{ij}, \quad 1 \leq i < j \leq N,$$

form a basis for $\mathbb{R}^{N\times N}_{\text{sym}}$. For any two vectors $a, b \in \mathbb{R}^N$ set

$$a \odot b := a \otimes b + b \otimes a.$$

Since

$$\begin{aligned} e_{i(N+1)} &= \frac{1}{|A_i|^2} A_i \otimes A_i, \qquad i = 1, \dots, N, \\ e_{ij} &= \frac{1}{|A_i - A_j|^2}\left[A_i \otimes A_i + A_i \otimes A_i - A_i \odot A_j\right], \qquad 1 \leq i < j \leq N, \end{aligned}$$

to prove the claim it suffices to show that

$$\{A_i \odot A_j : 1 \leq i \leq j \leq N\} \quad \text{are linearly independent.}$$

Suppose, by contradiction, that for some $s, p \in \{1, \ldots, N\}$, with $s \geq p$, there exist constants $\alpha_{ij}$ such that

$$A_p \odot A_s = \sum_{(i,j) \neq (p,s), 1 \leq i \leq j \leq N} \alpha_{ij} \, A_i \odot A_j. \tag{3.14}$$

Let $c \in \mathbb{R}^N \setminus \{0\}$ be such that (see (3.7))

$$c \cdot A_k = 0 \quad \text{for all} \quad k \in \{1, \ldots, N\} \setminus \{s\}, \quad A_s \cdot c \neq 0. \tag{3.15}$$

By (3.14) and (3.15) we find

$$\begin{aligned} A_p(A_s \cdot c) &= A_p(A_s \cdot c) + A_s(A_p \cdot c) = (A_p \odot A_s)c \\ &= \sum_{i \leq s, \; i \neq p} \alpha_{is} A_i (A_s \cdot c) + \sum_{j > s} \alpha_{sj} A_j (A_s \cdot c), \end{aligned}$$

and we deduce that

$$A_p = \sum_{i \leq s, \; i \neq p} \alpha_{is} A_i + \sum_{j > s} \alpha_{sj} A_j,$$

i.e.,

$$A_p \in \operatorname{span}\{A_1, \ldots, A_{p-1}, A_{p+1}, \ldots, A_N, A_{N+1}\},$$

which is in contradiction with (3.13). This asserts the claim, from which (3.12) follows. ■

To the triangulation $\tau^h$ we associate the space $V_h$ of piecewise quadratic functions, i.e. $u|_{K^h} \in \mathcal{Q}_{K^h}$ for each $K^h \in \tau^h$, and $u$ is completely determined by the set of *degrees of freedom*

$$\Xi_h := \bigcup_{K^h \in \tau^h} \Xi_{K^h}.$$

Note that $V_h \subset C\left(\mathbb{R}^N\right)$. Indeed, if $u \in V_h$ then clearly $u$ is continuous in the interior of each $N$–simplex $K^h \in \tau^h$. If two $N$–simplexes $K_1, K_2 \in \tau^h$ have in common an edge of vertices $A_i^h$ and $A_j^h$, since the traces $u|_{K_1}$ and $u|_{K_2}$ are polynomials of degree two which agree on three points (the vertices and the middle point), then they must agree along the whole segment $\overrightarrow{A_i^h A_j^h}$, and we have established continuity across the $N$-simplexe boundaries.

**Proof of Lemma 3.1.**

**Step 1:** Assume first that $f \in C_c\left(\mathbb{R}^N;\mathbb{R}^{N\times N}_{\text{sym}}\right)$. Let $\ell > 0$ be so large that $f(x) = 0$ outside the cube $\ell Q = (-\ell/2, \ell/2)^N$, and for $\eta > 0$ choose $h = h(\eta) > 0$ such that

$$|x-y| \le Nh \Rightarrow |f(x) - f(y)| \le \frac{\eta}{\ell^N} \tag{3.16}$$

for all $x, y \in \mathbb{R}^N$ with $|x-y| \le Nh$. Let $\tau^h$ be a triangularization of $\mathbb{R}^N$ corresponding to the parameter $h$. Define a function $u_h \in V_h$ as follows:

if $K^h$ is a $N$-simplex of $\tau^h$ with vertices $\{A_1^h, \dots, A_{N+1}^h\}$, then

$$u\left(A_{ij}^h\right) := -\frac{1}{8\sum\limits_{T\in\tau^h_{A^h_{ij}}} |T|} \int_{\bigcup\limits_{T\in\tau^h_{A^h_{ij}}} T} \left(f(x)(A_i^h - A_j^h)\right)\cdot(A_i^h - A_j^h)dx. \tag{3.17}$$

Since $f$ is continuous, from the Integral Mean Value Theorem it follows that

$$u\left(A_{ij}^h\right) = -\frac{1}{8}\left(f\left(x_{i,j,h}\right)\left(A_i^h - A_j^h\right)\right)\cdot\left(A_i^h - A_j^h\right) \tag{3.18}$$

for some $x_{i,j,h} \in \bigcup\limits_{T\in\tau^h_{A^h_{ij}}} T$ (note that this set is connected). By Lemma 3.3 we have

$$\begin{aligned}\int_{\mathbb{R}^N} \left|f - \nabla^2 u\right| dx &= \sum_{K^h\in\tau_h} \int_{K^h} \left|f - \nabla^2 u\right| dx \\ &\le C \sum_{K^h\in\tau_h}\sum_{i<j} \int_{K^h} \left|\left(\left(f - \nabla^2 u\right) E_{ij}^h\right)\cdot E_{ij}^h\right| dx.\end{aligned}$$

In view of Lemma 3.2 and (3.18) we have

$$\left(\nabla^2 u\, E_{ij}^h\right)\cdot E_{ij}^h = -\frac{8}{|A_i^h - A_j^h|^2} u\left(A_{ij}^h\right) = \left(f\left(x_{i,j,h}\right) E_{ij}^h\right)\cdot E_{ij}^h.$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N} \left|f - \nabla^2 u\right| dx &\le C \sum_{K^h\in\tau^h}\sum_{i<j} \int_{K^h} \left|\left(\left(f(x) - f\left(x_{i,j,h}\right)\right) E_{ij}^h\right)\cdot E_{ij}^h\right| dx \\ &\le C \sum_{K^h\in\tau^h}\sum_{i<j} \int_{K^h} \left|f(x) - f\left(x_{i,j,h}\right)\right| dx \\ &= C \sum_{K^h\in\tau^h, K^h\subset\ell Q}\ \sum_{i<j} \int_{K^h} \left|f(x) - f\left(x_{i,j,h}\right)\right| dx \\ &\le C\,\eta,\end{aligned}$$

where we have used (3.16).

Next we show that there exists a constant $C$, independent of $h$, such that for every fixed $i \in \{1, \dots, N+1\}$ we have

$$||\nabla u||_{L^\infty(K^h;\mathbb{R}^N)} \le \frac{C}{h} \sum_{j \neq i} |u(A_{ij}^h)|. \tag{3.19}$$

Indeed, by Lemmas 3.2 and 3.3(i) for all $x \in K^h$ and for every fixed $i \in \{1, \dots, N+1\}$ we have

$$\begin{aligned} |\nabla u(x)| \le C \sum_{j \neq i} \left| \nabla u(x) \cdot E_{ij}^h \right| &\le C \sum_{j \neq i} \frac{1}{|A_i^h - A_j^h|} \left| u(A_{ij}^h) \right| \\ &\le \frac{C}{h} \sum_{j \neq i} \left| u(A_{ij}^h) \right|, \end{aligned}$$

where we have used the fact that $|A_i^h - A_j^h| \ge \alpha\, h$ for some $\alpha > 0$ that does not depend on $h$.

We estimate the jump term of the $BV$ norm of $\nabla u$. Since $u|_{K^h} \in \mathcal{P}_2$ for each $K^h \in \tau^h$, we have

$$S(\nabla u) \subset \sum_{K^h \in \tau^h} \partial K^h,$$

and so

$$\begin{aligned} \int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} &\le \sum_{K^h \in \tau^h} \int_{S(\nabla u) \cap \partial K^h} |[\nabla u]| \, d\mathcal{H}^{N-1} \\ &\le \frac{C}{h} \sum_{K^h \in \tau^h} \sum_{A_{ij}^h \in K^h} \left| u\left(A_{ij}^h\right) \right| \mathcal{H}^{N-1}\left(\partial K^h\right) \\ &\le C h^{N-2} \sum_{K^h \in \tau^h} \sum_{A_{ij}^h \in K^h} \left| u\left(A_{ij}^h\right) \right|, \end{aligned}$$

where we have used (3.10) and (3.19). By (3.17) and (3.11), we get

$$\begin{aligned} \int_{S(\nabla u)} |[\nabla u]| \, d\mathcal{H}^{N-1} &\le C h^N \sum_{K^h \in \tau^h} \sum_{A_{ij}^h \in K^h} \frac{1}{\sum\limits_{T \in \tau_{A_{ij}^h}^h} |T|} \int_{\bigcup\limits_{T \in \tau_{A_{ij}^h}^h} T} |f(x)| \, dx \\ &\le C \sum_{K^h \in \tau^h} \sum_{A_{ij}^h \in K^h} \int_{\bigcup\limits_{T \in \tau_{A_{ij}^h}^h} T} |f(x)| \, dx \\ &\le C \int_{\mathbb{R}^N} |f| \, dx. \end{aligned}$$

Finally, by (3.9) and (3.19),

$$\begin{aligned}
\int_{\mathbb{R}^N} |u| + |\nabla u| \; dx &\leq \sum_{K^h \in \tau^h} \int_{K^h} |u| + |\nabla u| \; dx \\
&\leq C \sum_{K^h \in \tau^h} \sum_{A^h_{ij} \in K^h} \left(1 + \frac{1}{h}\right) \left| u\left(A^h_{ij}\right)\right| h^N \\
&\leq C h^{N+1} \sum_{K^h \in \tau^h} \sum_{A^h_{ij} \in K^h} \frac{1}{\sum\limits_{T \in \tau^h_{A^h_{ij}}} |T|} \int_{\bigcup\limits_{T \in \tau^h_{A^h_{ij}}} T} |f(x)| \, dx \\
&\leq C h \sum_{K^h \in \tau^h} \sum_{A^h_{ij} \in K^h} \int_{\bigcup\limits_{T \in \tau^h_{A^h_{ij}}} T} |f(x)| \, dx \\
&\leq C h \int_{\mathbb{R}^N} |f| \; dx.
\end{aligned}$$

This completes the proof of Step 1.
**Step 2:**
The argument in the general case where $f \in L^1\left(\mathbb{R}^N; \mathbb{R}^{N\times N}_{\text{sym}}\right)$ is entirely identical to that of Step 2 in Subsection 4.1. ■

# 4 The $SBV^2$ setting

In this section we prove Theorem 1.5, precisely, we show that any function $f \in L^1\left(\Omega; \mathbb{R}^{N\times N}\right)$, not necessarily symmetric valued, is the the density with respect to the $N$-dimensional Lebesgue measure of the distributional Hessian of a $SBV^2$ function.

We obtain this result by applying Theorem 1.1 twice.

**Proof of Theorem 1.5.** For every $i = 1, \ldots, N$, define

$$F_i := \sum_{j=1}^{N} f_{ij} \mathbf{e}_j.$$

Since $F_i \in L^1\left(\Omega; \mathbb{R}^N\right)$, by Theorem 1.1 there exist $v_i \in SBV(\Omega)$ and a constant $C = C(N) > 0$ such that

$$Dv_i = F_i \, \mathcal{L}^N \lfloor \Omega + [v_i] \nu_{v_i} \, \mathcal{H}^{N-1} \lfloor \, S(v_i),$$

with

$$\int_{\Omega} |v_i| \; dx + \int_{S(v_i)\cap\Omega} |[v_i]| \; d\mathcal{H}^{N-1} \leq C \int_{\Omega} |F_i| \; dx.$$

Set $v := \sum_{i=1}^{N} v_i \mathbf{e}_i$. Clearly $v \in SBV\left(\Omega; \mathbb{R}^N\right)$, and we have

$$Dv = f\, \mathcal{L}^N \lfloor \Omega + [v] \otimes \nu_v\, \mathcal{H}^{N-1} \lfloor S(v), \tag{4.1}$$

with

$$\int_\Omega |v|\ dx + \int_{S(v)\cap\Omega} |[v]|\ d\mathcal{H}^{N-1} \leq C \int_\Omega |f|\ dx. \tag{4.2}$$

Now, since $v \in L^1\left(\Omega; \mathbb{R}^N\right)$, and invoking again Theorem 1.1, there exists a function $u \in SBV(\Omega)$ such that

$$Du = v\, \mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u\, \mathcal{H}^{N-1} \lfloor S(u), \tag{4.3}$$

and

$$\int_\Omega |u|\ dx + \int_{S(u)\cap\Omega} |[u]|\ d\mathcal{H}^{N-1} \leq C \int_\Omega |v|\ dx. \tag{4.4}$$

Therefore we conclude that $\nabla u = v\ \mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$, thus $u \in SBV^2(\Omega)$, and (4.1)-(4.4) assert that $u$ fulfills the statement of the theorem. ■

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