A global method for relaxation in $W^{1,p}$ and in SBV_p

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Abstract

An integral representation formula for a class of functionals defined on $W^{1,p}$ and in SBV_p is obtained without requiring the regularity conditions usually imposed in the literature. The approach is based on the general results of [17] and on a Poincaré–Wirtinger's type inequality introduced by De Giorgi, Carriero and Leaci [34]. Applications to relaxation problems and dimension reduction problems in brittle thin films are presented.

Key words. Relaxation, lower semicontinuity, special functions of bounded variation, brittle thin films

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Contents

1 Introduction

In a recent paper Bouchitté, Fonseca and Mascarenhas [17] introduced a general method for the identification of the integral representation of a certain class of functionals defined on the space of functions of bounded variation and with linear growth with respect to the total variation of the derivative. This method has important applications, including the integral representation of the relaxed energy in $BV(\Omega;\mathbb{R}^d)$ corresponding to a functional defined in $W^{1,1}(\Omega;\mathbb{R}^d)$ with a discontinuous integrand and growing linearly at infinity, the homogenization of energy functionals involving bulk and surface terms, the integral representation of relaxed energies when no apriori bounds are available, and the characterization of effective energies under Dirichlet boundary conditions (see [18]; see also [16]).

The purpose of this paper is to extend the work in [17] to functionals defined on $SBV_p(\Omega;\mathbb{R}^d)$, $p>1$, and with superlinear growth. More precisely, we consider a functional

$$\mathcal{F} : SBV_p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$$

satisfying for every $(u, A) \in SBV_p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega)$

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(H_1) $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;
(H_2) $\mathcal{F}(u;A)=\mathcal{F}(v;A)$ whenever $u=v$ $\mathcal{L}^N$ a.e. on $A\in\mathcal{A}(\Omega)$;
(H_3) $\mathcal{F}(\cdot;A)$ is $L^1(\Omega;\mathbb{R}^d)$- lower semicontinuous;
(H_4) there exists $C>0$ such that

$$\frac{1}{C}\left(\int_A|\nabla u|^p dx+\int_{S(u)\cap A}(1+|u^+-u^-|)d\mathcal{H}^{N-1}\right)\le\mathcal{F}(u;A)$$
$$\le C\left(\int_A(1+|\nabla u|^p)dx+\int_{S(u)\cap A}(1+|u^+-u^-|)d\mathcal{H}^{N-1}\right).$$

Here Ω is an open, bounded subset of $\mathbb{R}^N$, $\mathcal{A}(\Omega)$ is the class of all open subsets of Ω, and $SBV_p(\Omega;\mathbb{R}^d)$ is the space of functions $u\in SBV(\Omega;\mathbb{R}^d)$ such that $\nabla u\in L^p(\Omega;\mathbb{R}^{d\times N})$ and $\mathcal{H}^{N-1}(S(u)\cap\Omega)<\infty$.

For every $u\in SBV_p(\Omega;\mathbb{R}^d)$ and $A\in\mathcal{A}_\infty(\Omega)$, where $\mathcal{A}_\infty(\Omega)$ is the class of Lipschitz subdomains of Ω, define

$$\mathbf{m}(u;A):=\inf\{\mathcal{F}(w;A):\,w\in SBV_p(A;\mathbb{R}^d)\text{ such that }w=u\text{ in a neighborhood of }\partial A\}. \tag{1.0.1}$$

The main result of the paper is the following:

Theorem 1.0.1. *Under hypotheses $(H_1)-(H_4)$, for every $u\in SBV_p(\Omega;\mathbb{R}^d)$ and $A\in\mathcal{A}(\Omega)$ we have*

$$\mathcal{F}(u;A)=\int_A f(x,u,\nabla u)\,dx+\int_{S(u)\cap A}g(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1},$$

where

$$f(x_0,u_0,\xi):=\limsup_{\varepsilon\to0^+}\frac{\mathbf{m}(u_0+\xi(\cdot-x_0);Q_\nu(x_0,\varepsilon))}{\varepsilon^N}, \tag{1.0.2}$$
$$g(x_0,a,b,\nu):=\limsup_{\varepsilon\to0^+}\frac{\mathbf{m}(u_{x_0,a,b,\nu};Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}, \tag{1.0.3}$$

for all $x_0\in\Omega$, u_0, a, $b\in\mathbb{R}^d$, $\xi\in\mathbb{R}^{d\times N}$, $\nu\in S^{d-1}$, and where

$$u_{x_0,a,b,\nu}(x):=\begin{cases}a & \text{if } (x-x_0)\cdot\nu>0,\\ b & \text{if } (x-x_0)\cdot\nu\le0.\end{cases}$$

Theorem 1.0.1 was proved by Braides and Chiadó Piat [20] under the additional Lipschitz-type assumptions that there exists a sequence $\{\omega_k\}$ of integrable moduli of continuity such that

$$|\mathcal{F}(u+s;A)-\mathcal{F}(u;A)|\le\int_A\omega_k(x,|s|)\,dx \tag{1.0.4}$$

for every $k\in\mathbb{N}$, $A\in\mathcal{A}(\Omega)$, $s\in\mathbb{R}^d$, and for all functions u of class C^1 such that $||u||_\infty\le k$, $||u+s||_\infty\le k$, $||\nabla u||_\infty\le k$; and that there exists a modulus of continuity ω such that

$$|\mathcal{F}(u;S)-\mathcal{F}(v;S)|\le\int_S\omega(|u^+-v^+|+|u^--v^-|)\,d\mathcal{H}^{N-1} \tag{1.0.5}$$

for every u, $v\in SBV_p(\Omega;\mathbb{R}^d)$ and $S\subset S(u)\cap S(v)$.

Condition (1.0.4) was introduced by Buttazzo and Dal Maso [26] in the Sobolev setting (see Theorem 1.0.2 below) and is *equivalent* to requiring f to be Carathéodory, while condition (1.0.5) implies, in particular, that $g(x,\cdot,\cdot,\nu)$ is continuous, uniformly with respect to x and ν. Our approach, which is based on the general results of [17] and on a Poincaré–Wirtinger's type inequality introduced by De Giorgi, Carriero and Leaci [34], does not require the extra assumptions (1.0.4) and (1.0.5).

Although hypothesis (1.0.4) would slightly simplify the proofs of Lemmas 2.3.1 and 2.3.2 below, we prefer to give Theorem 1.0.1 in its full generality, since it is known that the continuity of f in the u variable is by no means a necessary condition for lower semicontinuity and relaxation results, and we refer the reader to [37] (and the references contained therein) where $f(x,\cdot,\xi)$ is assumed to be lower semicontinuous, or, in some cases, only measurable (see also Theorem 3.1.2 below). Moreover, conditions of the type (1.0.4) and (1.0.5) are, in general, not easy to verify in the applications.

We remark that Theorem 1.0.1 entails new results even in the Sobolev setting. Indeed,

Theorem 1.0.2. *Let $\mathcal{F}_1 : W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$ be a functional satisfying hypotheses $(H_1) - (H_3)$ and the growth condition*

$$\frac{1}{C}\int_A |\nabla u|^p dx \leq \mathcal{F}_1(u;A) \leq C\int_A (1+|\nabla u|^p)dx, \tag{1.0.6}$$

for some $C > 0$. Then, for every $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}_1(u;A) = \int_A f_1(x,u,\nabla u)\,dx, \qquad \text{where} \quad f_1(x_0,u_0,\xi) := \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_1(u_0+\xi(\cdot - x_0); Q_\nu(x_0,\varepsilon))}{\varepsilon^N}$$

for all $x_0 \in \Omega$, $u_0 \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d\times N}$, and where, for $(v,A) \in W^{1,p}(\Omega;\mathbb{R}^d) \times \mathcal{A}_\infty(\Omega)$,

$$\mathbf{m}_1(v;A) := \inf\{\mathcal{F}_1(w;A) : \, w \in W^{1,p}(A;\mathbb{R}^d) \text{ with } w = v \text{ in a neighborhood of } \partial A\}. \tag{1.0.7}$$

Theorem 1.0.2 is closely related to the result of Buttazzo and Dal Maso [26] mentioned above (see also [24], [25]), where the integral representation formula is obtained under the additional hypothesis (1.0.4).

Theorem 1.0.1 also allows us to recover the integral representation for functionals defined on partitions of sets of finite perimeter previously obtained by Ambrosio and Braides [8] under an additional continuity hypothesis. A similar result was proved by Braides and Piat [20] in the case where the ambiance space $BV(\Omega;T)$ is replaced by

$$SBV_0(\Omega;\mathbb{R}^d) := \left\{w \in SBV(\Omega;\mathbb{R}^d) : \, \nabla w = 0 \text{ a.e. in } \Omega \text{ and } \mathcal{H}^{N-1}(S(w)\cap\Omega) < \infty\right\}.$$

Theorem 1.0.3. *Let T be a finite subset of $\mathbb{R}^d$ and let $\mathcal{F}_2 : BV(\Omega;T) \times \mathcal{A}(\Omega) \to [0,\infty]$ be a functional satisfying hypotheses $(H_1) - (H_3)$ and the growth condition*

$$\frac{1}{C} H^{N-1}(S(u)\cap A) \leq \mathcal{F}_2(u;A) \leq C\, H^{N-1}(S(u)\cap A).$$

Then for every $u \in BV(\Omega;T)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{F}_2(u;A) = \int_{S(u)\cap A} g_2(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1},$$

where

$$g_2(x_0,a,b,\nu) := \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_2(u_{x_0,a,b,\nu}; Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}},$$

for all $x_0 \in \Omega$, a, $b \in \mathbb{R}^d$, $\nu \in S^{d-1}$, and where, for $(v,A) \in BV(\Omega;T) \times \mathcal{A}_\infty(\Omega)$,

$$\mathbf{m}_2(v;A) := \inf\{\mathcal{F}_2(w;A) : \, w \in BV(A;v(A)) \text{ with } w = v \text{ in a neighborhood of } \partial A\}. \tag{1.0.8}$$

An important example of the kind of functionals considered in Theorem 1.0.1 is given by the relaxed energy

$$\mathcal{H}(u;A) := \inf_{\{u_n\}}\left\{\liminf_{n\to\infty}\int_A f_0(x,u_n,\nabla u_n)\,dx + \int_{S(u_n)\cap A} g_0(x,u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1} : \right.$$
$$\left. u_n \in SBV_p(A;\mathbb{R}^d),\, u_n \to u \quad \text{in} \quad L^1(A;\mathbb{R}^d)\right\},$$

where both bulk and surface energies are present.

Theorem 1.0.4. *Assume that the densities*

$$f_0 \ : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty), \qquad g_0 : \Omega \times (\mathbb{R}^d)^2 \times S^{N-1} \to [0,+\infty)$$

are, respectively, a Carathéodory and a continuous function that satisfy

$$\begin{aligned} \frac{1}{C}|\xi|^p \ &\leq f_0(x,u,\xi) \ \leq C(1+|\xi|^p), \\ \frac{1}{C}(1+|\lambda-\theta|) \leq \ &g_0(x,\lambda,\theta,\nu) \ \leq C(1+|\lambda-\theta|) \quad \textit{and} \quad g_0(x,\lambda,\theta,\nu) = g_0(x,\lambda,\theta,-\nu) \end{aligned} \tag{1.0.9}$$

for all $x \in \Omega$, u, λ, $\theta \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$, and for some $C > 0$. Then for every $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{H}(u;A) = \int_A \mathcal{Q}f_0(x,u,\nabla u)\,dx + \int_{S(u)\cap A} \mathcal{R}g_0(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1},$$

where $\mathcal{Q}f_0$ and $\mathcal{R}g_0$ are the quasiconvex and the BV-elliptic envelopes of f_0 and g_0, respectively.

The *quasiconvex envelope* $\mathcal{Q}f$ of a function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is defined as

$$\mathcal{Q}f := \sup\{G : \; G \text{ is quasiconvex}, \; G \le f\},$$

and the *BV-elliptic envelope* $\mathcal{R}g$ of a function $g : \mathbb{R}^{2d} \times S^{N-1} \to \mathbb{R}$ is the greatest BV-elliptic function less than g. We recall that a locally bounded, Borel measurable function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be *quasiconvex* if

$$\int_{(0,1)^N} f\left(\xi + \nabla\varphi(y)\right) \, dy \ge f(\xi)$$

for all $\xi \in \mathbb{R}^{d\times N}$, $\varphi \in W_0^{1,\infty}((0,1)^N;\mathbb{R}^d)$. Similarly, we say that a locally bounded, Borel measurable function $g : \mathbb{R}^{2d} \times S^{N-1} \to \mathbb{R}$ is *BV-elliptic* if for all $(a,b,\nu) \in \mathbb{R}^{2d} \times S^{d-1}$ and for any *finite* subset T of $\mathbb{R}^d$ we have

$$\int_{S(w)\cap Q_\nu} g(w^+, w^-, \nu_w) \, d\mathcal{H}^{N-1} \ge g(a,b,\nu)$$

for all $w \in BV(Q_\nu;T)$ such that $w = u_{a,b,\nu}$ on ∂Q_ν, where $u_{a,b,\nu} := u_{0,a,b,\nu}$, i.e.

$$u_{a,b,\nu}(x) := \begin{cases} a & \text{if } x\cdot\nu > 0, \\ b & \text{if } x\cdot\nu \le 0. \end{cases}$$

Theorem 1.0.1 has also important applications in homogenization theory and brittle thin films (see Section 3.3).

As a final remark, it is worth noting that the proofs of Lemma 2.3.1 and 2.3.2 could be simplified if given a function $u \in SBV_p(\Omega;\mathbb{R}^d)$ one could show that for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$

$$\lim_{\varepsilon\to0^+} \frac{1}{\varepsilon^{N+p}} \int_{Q_\nu(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x-x_0)|^p \, dx = 0.$$

This result, which would be of independent interest, is true if $p \le N/(N-1)$ or in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^d)$, but we have been unable to prove it (or disprove it) in the general case where $u \in SBV_p(\Omega;\mathbb{R}^d)$.

Similarly, the proofs of Lemma 2.4.1 and 2.4.2 could be significantly shortened if we knew that for $\mathcal{H}^{N-1}$ a.e $x_0 \in S(u)$

$$\lim_{\varepsilon\to0^+} \frac{1}{\varepsilon^{N-1+p}} \int_{Q_\nu(x_0,\varepsilon)} |u(x) - u_{x_0,\nu}(x)|^p \, dx = 0,$$

where $u_{x_0,\nu} := u_{x_0,u^+(x_0),u^-(x_0),\nu(x_0)}$.

To overcome these difficulties we will make use of the approximation procedure described in Lemma 2.1.7 and Lemma 2.1.8, respectively.

2 Part I: The General Method

2.1 Preliminaries.

We start with some notation. Here $\Omega \subset \mathbb{R}^N$ is an open, bounded subset, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the N dimensional Lebesgue measure and the $N-1$ dimensional Hausdorff measure in $\mathbb{R}^N$. For any $\nu \in S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ let $\{\nu_1, \cdots, \nu_{N-1}, \nu\}$ be an orthonormal basis of $\mathbb{R}^N$ varying continuously with ν, and let $Q_\nu := \{x \in \mathbb{R}^N : |x\cdot\nu_i| < 1/2, \, |x\cdot\nu| < 1/2, \, i = 1,\cdots,N-1\}$ be a unit cube centered at the origin with two of its faces orthogonal to the direction ν. We set $Q_\nu(x_0,\varepsilon) := x_0 + \varepsilon Q_\nu$.

We recall briefly some facts about functions of bounded variation which will be useful in the sequel. A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation* if for all $i = 1,\cdots d$, and $j = 1,\cdots N$, there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x) \, \frac{\partial\varphi}{\partial x_j}(x) \, dx = -\int_\Omega \varphi(x) \, d\mu_{ij}$$

for every $\varphi \in C_0^1(\Omega;\mathbb{R})$. The distributional derivative Du is the matrix–valued measure with components μ_{ij}. Given $u \in BV(\Omega;\mathbb{R}^d)$ the *approximate upper* and *lower limit* of each component u_i, $i = 1,\cdots d$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon\to0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N\left(\{y \in \Omega\cap Q(x,\varepsilon) : u_i(y) > t\}\right) = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \mathcal{L}^N\left(\{y \in \Omega \cap Q(x,\varepsilon) : u_i(y) < t\}\right) = 0\right\},$$

while the *jump set* of u, or *singular set*, is defined by

$$S(u) := \bigcup_{i=1}^{d} \{x \in \Omega : u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega \backslash S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x), \cdots, u_d^+(x))$ and $(u_1^-(x), \cdots, u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$ a.e. $x \in \Omega \backslash S(u)$ and that $\mathcal{H}^{N-1}(S(u) \triangle L(u)) = 0$, where $L(u)$ stands for the set of Lebesgue points of u. Furthermore, for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ there exist a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at x, and two vectors $u^-(x)$, $u^+(x) \in \mathbb{R}^d$ (the traces of u on $S(u)$ at the point x) such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q(x_0,\varepsilon) :\, (y-x)\cdot \nu_u(x) > 0\}} |u(y) - u^+(x)|^{N/(N-1)} dy = 0$$

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q(x_0,\varepsilon) :\, (y-x)\cdot \nu_u(x) < 0\}} |u(y) - u^-(x)|^{N/(N-1)} dy = 0.$$

Note that, in general, $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. We denote the *jump of* u *across* $S(u)$ by $[u] := u^+ - u^-$. The distributional derivative Du may be decomposed as

$$Du = \nabla u \, \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^{N-1} \lfloor S(u) + C(u),$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$ and $C(u)$ is the Cantor part of Du. These three measures are mutually singular.

As it turns out, the Sobolev inequality

$$\left(\int_\Omega |u(x)|^{N/(N-1)} dx\right)^{(N-1)/N} \leq C(N) ||u||_{BV}$$

holds in $BV(\Omega; \mathbb{R}^d)$ when $N > 1$.

The space $SBV(\Omega; \mathbb{R}^d)$ of *special functions of bounded variation*, introduced by De Giorgi and Ambrosio [33], is the space of all functions $u \in BV(\Omega; \mathbb{R}^d)$ such that

$$Du = \nabla u \, \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^{N-1} \lfloor S(u).$$

The following compactness result for sequences bounded in $SBV(\Omega; \mathbb{R}^d)$ is due to Ambrosio [5].

Theorem 2.1.1. *Let $\Phi : [0, +\infty) \to [0, +\infty)$, $\Theta : (0, +\infty] \to (0, +\infty]$ be two functions, respectively convex and concave, and such that*

$$\lim_{t \to +\infty} \frac{\Phi(t)}{t} = +\infty, \quad \Phi \textit{ is nondecreasing},$$

$$\Theta(+\infty) = \lim_{t \to +\infty} \Theta(t), \quad \lim_{t \to 0^+} \frac{\Theta(t)}{t} = +\infty, \quad \Theta \textit{ is nondecreasing}.$$

Let $\{u_n\}$ be a sequence of functions in $SBV(\Omega; \mathbb{R}^d)$ such that

$$\sup_n \left\{\int_\Omega \Phi(|\nabla u_n|)\, dx + \int_{S(u_n)} \Theta(|[u_n]|)\, d\mathcal{H}^{N-1} + \int_\Omega |u_n|\, dx\right\} < +\infty.$$

Then there exists a subsequence $\{u_{n_k}\}$ converging in $L^1(\Omega; \mathbb{R}^d)$ to a function $u \in SBV(\Omega; \mathbb{R}^d)$, and

$$\nabla u_{n_k} \rightharpoonup \nabla u \quad \textit{in } L^1(\Omega; \mathbb{R}^{N \times d}), \quad [u_{n_k}] \otimes \nu_{u_{n_k}} \mathcal{H}^{N-1} \lfloor S(u_{n_k}) \overset{\star}{\rightharpoonup} [u] \otimes \nu_u \mathcal{H}^{N-1} \lfloor S(u),$$

$$\int_{S(u) \cap \Omega} \Theta(|[u]|)\, d\mathcal{H}^{N-1} \leq \liminf_{n \to +\infty} \int_{S(u_n) \cap \Omega} \Theta(|[u_n]|)\, d\mathcal{H}^{N-1}.$$

We refer to [19] and to [12] for more details on SBV functions.

For $p>1$, we say that a function $u \in SBV(\Omega;\mathbb{R}^d)$ belongs to $SBV_p(\Omega;\mathbb{R}^d)$ if

$$\nabla u \in L^p(\Omega;\mathbb{R}^{d\times N}) \quad \text{and} \quad \mathcal{H}^{N-1}(S(u)\cap\Omega)<\infty.$$

In the remaining of the section we present some theorems due to De Giorgi, Carriero and Leaci [34] (see also [27]) which will be useful in the sequel. We also prove some new related results, Lemma 2.1.7 and 2.1.8, which, we believe, are of independent interest.

Following the notation in [27] and [34], given $a=(a_1,\dots,a_d)$, $b=(b_1,\dots,b_d)\in\mathbb{R}^d$ we will denote $a\wedge b := (\min(a_1,b_1),\dots,\min(a_d,b_d))$ and $a\vee b := (\max(a_1,b_1),\dots,\max(a_d,b_d))$. Let Q be a cube in $\mathbb{R}^N$. For every measurable function $u: Q\to\mathbb{R}^d$, with $u=(u_1,\dots,u_d)$, we set

$$u_*(s;Q) := ((u_1)_*(s;Q),\dots,(u_d)_*(s;Q)), \qquad \operatorname{med}(u;Q) := u_*(\tfrac{1}{2}\,|Q|;Q),$$

where, with $i=1,\dots,d$,

$$(u_i)_*(s;Q) := \inf\{t\in\mathbb{R} : |\{u_i<t\}\cap Q|\geq s\} \qquad \text{for } 0\leq s\leq |Q|.$$

Moreover, for every $u\in SBV_p(B;\mathbb{R}^d)$ such that $(2\gamma_N\mathcal{H}^{N-1}(S(u)\cap Q))^{N/(N-1)} < \frac{1}{2}\,|Q|$ we define

$$\begin{aligned}
\tau'(u;Q) :=& u_*\left((2\gamma_N\mathcal{H}^{N-1}(S(u)\cap Q))^{N/(N-1)};Q\right),\\
\tau''(u;Q) :=& u_*\left(|Q|-(2\gamma_N\mathcal{H}^{N-1}(S(u)\cap Q))^{N/(N-1)};Q\right),\\
\bar{u}(x) :=& (u(x)\wedge\tau''(u,Q))\vee\tau'(u,Q),
\end{aligned}$$

where γ_N is the isoperimetric constant relative to cubes of $\mathbb{R}^N$, that is γ_N is the smallest number such that for every Borel set E

$$\min\{|E\cap Q|^{N/(N-1)}, |Q\backslash E|^{N/(N-1)}\} \leq \gamma_N \operatorname{Per}_Q(E),$$

where $\operatorname{Per}_Q(E)$ denotes the perimeter of E in Ω.

Remark 2.1.2. Note that when $S(u)=\emptyset$ then $\tau'(u_i;Q)=-\infty$, $\tau''(u_i;Q)=+\infty$ for all $i=1,\dots,d$, and so $\bar{u}(x)$ coincides with $u(x)$.

Lemma 2.1.3. ([34], Lemma 2.6) *Let $p\geq 1$ and let $u\in SBV(\Omega;\mathbb{R})$ be such that*

$$\int_K |\nabla u(x)|^p\,dx + \mathcal{H}^{N-1}(S(u)\cap K)<\infty$$

for every compact set $K\subset\Omega$. Then

$$\lim_{\varepsilon\to 0^+}\frac{\int_{Q(x_0,\varepsilon)}|\nabla u(x)|^p\,dx + \mathcal{H}^{N-1}(S(u)\cap Q(x_0,\varepsilon))}{\varepsilon^{N-1}} = 0$$

for $\mathcal{H}^{N-1}$ a.e. $x_0\in\Omega\setminus S(u)$.

Theorem 2.1.4. ([34], Thm. 3.1) *Assume that $u\in SBV(Q;\mathbb{R})$ and that*

$$\left(2\gamma_N\,\mathcal{H}^{N-1}(S(u)\cap Q)\right)^{N/(N-1)} < \frac{|Q|}{2}. \tag{2.1.1}$$

If $1\leq p<N$ then

$$\left(\int_Q|\bar{u}-\operatorname{med}(u;Q)|^{p^*}dx\right)^{1/p^*} \leq \frac{2\gamma_N p(N-1)}{N-p}\left(\int_Q|\nabla u|^p\,dx\right)^{1/p}, \tag{2.1.2}$$

$$|\{\bar{u}\neq u\}\cap Q| \leq 2\left(2\gamma_N\mathcal{H}^{N-1}(S(u)\cap Q)\right)^{N/(N-1)}, \tag{2.1.3}$$

where $p^*:=pN/(N-p)$. If $p\geq N$ then for every $q\geq 1$

$$||\bar{u}-\operatorname{med}(u;Q)||_{L^q(Q;\mathbb{R})} \leq \frac{2\gamma_N q(N-1)}{N}\,|Q|^{\frac{1}{N}+\frac{1}{q}-\frac{1}{p}}\,||\nabla u||_{L^p(Q;\mathbb{R})}. \tag{2.1.4}$$

Theorem 2.1.5. ([34], Thm. 3.5) *Assume that* $u_n \in SBV(Q;\mathbb{R})$, $p>1$, *and that*

$$\sup_n \int_Q |\nabla u_n|^p dx < \infty, \qquad \lim_{n\to\infty} \mathcal{H}^{N-1}(S(u_n)\cap Q) = 0.$$

Then (up to a subsequence) the sequences $\{\bar{u}_n - \operatorname{med}(u_n;Q)\}$ *and* $\{u_n - \operatorname{med}(u_n;Q)\}$ *converge to a function* $u_0 \in W^{1,p}(Q;\mathbb{R})$ *in* L^p *and pointwise almost everywhere, respectively.*

Theorem 2.1.6. ([34], Thm. 3.6) *Let* $u \in SBV_{\rm loc}(\Omega;\mathbb{R})$ *and* $p>1$, *and let* $x_0 \in \Omega$. *If*

$$\lim_{\varepsilon\to 0^+} \frac{\int_{Q(x_0,\varepsilon)} |\nabla u(x)|^p\, dx + \mathcal{H}^{N-1}(S(u)\cap Q(x_0,\varepsilon))}{\varepsilon^{N-1}} = 0$$

then $x_0 \notin S(u)$ *and*

$$\lim_{\varepsilon\to 0^+} \operatorname{med}(u, Q(x_0,\varepsilon)) = \operatorname*{aplim}_{x\to x_0} u(x) \in \mathbb{R}. \tag{2.1.5}$$

Moreover there exists $\varepsilon_0 > 0$ *such that for* $\alpha \in (\frac{3}{4},1)$ *and* ε, $\rho>0$, *with* $0<\alpha\varepsilon\le\rho\le\varepsilon<\varepsilon_0$, *we have*

$$\begin{aligned} \tau'(u,Q(x_0,\rho)) &\le \operatorname{med}(u,Q(x_0,\varepsilon)) \le \tau''(u,Q(x_0,\rho)),\\ \tau'(u,Q(x_0,\varepsilon)) &\le \operatorname{med}(u,Q(x_0,\rho)) \le \tau''(u,Q(x_0,\varepsilon)). \end{aligned} \tag{2.1.6}$$

The extensions of Theorems 2.1.4–2.1.6 to the vectorial case may be found in [27].

Lemma 2.1.7. *Assume that* $u \in SBV_p(\Omega;\mathbb{R}^d)$, *and let* $\nu \in S^{N-1}$. *Then for* $\mathcal{L}^N$ *a.e.* $x_0 \in \Omega$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} \left| \bar{u}_\varepsilon(x) - \nabla u(x_0)\, \frac{(x-x_0)}{\varepsilon} \right|^p dx = 0, \tag{2.1.7}$$

where

$$u_\varepsilon(x) := \frac{u(x)-u(x_0)}{\varepsilon}, \qquad \bar{u}_\varepsilon(x) := (u_\varepsilon(x) \wedge \tau''(u_\varepsilon, Q_\nu(x_0,\varepsilon))) \vee \tau'(u_\varepsilon, Q_\nu(x_0,\varepsilon)).$$

Proof. Note that if $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ then by Remark 2.1.2 $\bar{u}_\varepsilon(x) = u_\varepsilon(x)$, and (2.1.7) is a well known fact within the realm of Sobolev spaces.

Given $u \in SBV_p(\Omega;\mathbb{R}^d)$, it is clearly enough to show the result for $d=1$. Consider $x_0\in\Omega$ satisfying

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} |\nabla u(x) - \nabla u(x_0)|^p\, dx = 0, \tag{2.1.8}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)\cap S(u)} (1+|[u]|)\, d\mathcal{H}^{N-1} = 0, \tag{2.1.9}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N+1}} \int_{Q_\nu(x_0,\varepsilon)} |u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|\, dx = 0. \tag{2.1.10}$$

Define $v_\varepsilon(y) := u_\varepsilon(x_0+\varepsilon\, y)$. For $0 \le s \le \varepsilon^N$ we have $(u_\varepsilon)_*(s, Q_\nu(x_0,\varepsilon)) = (v_\varepsilon)_*(s/\varepsilon^N, Q_\nu)$ and, in turn,

$$\tau'(v_\varepsilon,Q_\nu) = \tau'(u_\varepsilon,Q_\nu(x_0,\varepsilon)), \quad \tau''(v_\varepsilon,Q_\nu) = \tau''(u_\varepsilon,Q_\nu(x_0,\varepsilon)), \quad \operatorname{med}(v_\varepsilon;Q_\nu) = \operatorname{med}(u_\varepsilon,Q_\nu(x_0,\varepsilon)).$$

It follows that $\bar{v}_\varepsilon(y) = \bar{u}_\varepsilon(x_0+\varepsilon\, y)$, where $\bar{v}_\varepsilon(y) := (v_\varepsilon(y)\wedge\tau''(v_\varepsilon,Q_\nu)) \vee \tau'(v_\varepsilon(y),Q_\nu)$. Thus proving (2.1.7) is equivalent to showing that

$$\{\bar{v}_\varepsilon(y)\} \text{ converges to } \nabla u(x_0)\cdot y \text{ in } L^p(Q_\nu;\mathbb{R}^d). \tag{2.1.11}$$

By (2.1.8) and (2.1.9)

$$\int_{Q_\nu} |\nabla v_\varepsilon(y)|^p\, dy \le C, \qquad \lim_{\varepsilon\to 0^+} \mathcal{H}^{N-1}(Q_\nu \cap S(v_\varepsilon)) = 0.$$

Hence we can apply Theorem 2.1.5 to conclude that there exists a subsequence $\{v_{\varepsilon_j}\}$ and a function $v_0 \in W^{1,p}(Q_\nu;\mathbb{R}^d)$ such that $\{v_{\varepsilon_j} - \operatorname{med}(v_{\varepsilon_j};Q_\nu)\}$ and $\{\bar{v}_{\varepsilon_j} - \operatorname{med}(v_{\varepsilon_j};Q_\nu)\}$ converge to v_0 pointwise almost everywhere and in L^p, respectively. We claim that

$$\lim_{\varepsilon\to 0^+} \operatorname{med}(v_\varepsilon;Q_\nu) = 0. \tag{2.1.12}$$

If (2.1.12) holds then (2.1.11) follows easily. Indeed, (2.1.12) implies that $\{v_{\varepsilon_j}\}$ and $\{\bar{v}_{\varepsilon_j}\}$ converge to v_0 pointwise almost everywhere and in L^p, respectively. Moreover, by (2.1.10) we have that $v_0(y) = \nabla u(x_0) \cdot y$. Consequently the whole sequence $\{\bar{v}_\varepsilon\}$ converges to $\nabla u(x_0) \cdot y$ in L^p and (2.1.11) is verified.

Thus, to conclude the proof of the lemma it remains to show (2.1.12). Fix $t > 0$ small. If $\nabla u(x_0) = 0$, and since by (2.1.10) we have $v_\varepsilon \to 0$ in $L^1(Q)$, we may find $\bar{\varepsilon} > 0$ such that if $0 < \varepsilon \leq \bar{\varepsilon}$ then

$$|\{y \in Q_\nu : v_\varepsilon(y) < 2t\}| \geq \frac{1}{2}. \tag{2.1.13}$$

Suppose now that $\nabla u(x_0) \neq 0$. Since

$$|\{y \in Q_\nu : \nabla u(x_0) \cdot y < t\}| \leq |\{y \in Q_\nu : |v_\varepsilon(y) - \nabla u(x_0) \cdot y| > t\}| + |\{y \in Q_\nu : v_\varepsilon(y) < 2t\}|,$$

with $|\{y \in Q_\nu : \nabla u(x_0) \cdot y < t\}| = \frac{1}{2} + \rho$, where $\rho = \rho(t) > 0$, and as by (2.1.10) $\{v_\varepsilon\}$ converges to $\nabla u(x_0) \cdot y$ in L^1, and hence in measure, we can find $\bar{\varepsilon} > 0$ such that $|\{y \in Q_\nu : |v_\varepsilon(y) - \nabla u(x_0) \cdot y| > t\}| < \rho$ for all $0 < \varepsilon \leq \bar{\varepsilon}$, and so (2.1.13) still holds. We conclude that $\mathrm{med}(v_\varepsilon; Q_\nu) \leq 2t$, and, in turn, this implies that

$$\limsup_{\varepsilon \to 0^+} \mathrm{med}(v_\varepsilon; Q_\nu) \leq 2t.$$

By letting $t \to 0^+$ we get

$$\limsup_{\varepsilon \to 0^+} \mathrm{med}(v_\varepsilon; Q_\nu) \leq 0.$$

Conversely, for $t < 0$ small and if $\nabla u(x_0) = 0$, we may find $\bar{\varepsilon} > 0$ such that for $\varepsilon \leq \bar{\varepsilon}$

$$|\{y \in Q_\nu : v_\varepsilon(y) < t\}| < \frac{1}{2}. \tag{2.1.14}$$

Also, if $\nabla u(x_0) \neq 0$ then we have

$$\begin{aligned}|\{y \in Q_\nu : v_\varepsilon(y) < t\}| &\leq |\{y \in Q_\nu : \nabla u(x_0) \cdot y < t/2\}| + |\{y \in Q_\nu : |v_\varepsilon(y) - \nabla u(x_0) \cdot y| > -t/2\}| \\ &= \frac{1}{2} + \rho + |\{y \in Q_\nu : |v_\varepsilon(y) - \nabla u(x_0) \cdot y| > -t/2\}|,\end{aligned}$$

where $\rho = \rho(t) < 0$. Again by (2.1.10) there exits $\tilde{\varepsilon} > 0$ such that $|\{y \in Q_\nu : |v_\varepsilon(y) - \nabla u(x_0)\, y| > -t/2\}| < -\rho$ for all $0 < \varepsilon \leq \tilde{\varepsilon}$. Hence (2.1.14) holds and thus $\mathrm{med}(v_\varepsilon; Q_\nu) \geq t$. By first letting $\varepsilon \to 0^+$ and then $t \to 0^-$ we obtain (2.1.12).

□

Lemma 2.1.8. *Assume that $u \in SBV_p(\Omega; \mathbb{R}^d)$. Then for $\mathcal{H}^{N-1}$ a.e $x_0 \in S(u)$ and for every $\varepsilon > 0$ there exists a function $v_\varepsilon \in SBV_p(Q_\nu(x_0, \varepsilon)); \mathbb{R}^d)$ such that $\mathcal{H}^{N-1}((S(v_\varepsilon) \setminus S(u)) \cap Q_\nu(x_0, \varepsilon)) = 0$ with $\nu = \nu_u(x_0)$,*

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0,\varepsilon)} |\nabla v_\varepsilon(x)|^p\, dx = 0, \qquad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1+p}} \int_{Q_\nu(x_0,\varepsilon)} |v_\varepsilon(x) - u_{x_0,\nu}(x)|^p\, dx = 0 \tag{2.1.15}$$

and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} |\{x \in Q_\nu(x_0, \varepsilon) : u \neq v_\varepsilon\}| = 0, \qquad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} |u(x) - v_\varepsilon(x)|\, dx = 0. \tag{2.1.16}$$

Proof. Clearly it is enough to show the result for $d = 1$. Since $S(u)$ is a countably $(N-1)$-rectifiable set in the sense of Federer, there exists a sequence of compact sets K_j such that

$$S(u) = \cup_{j=1}^\infty K_j \cup N_0, \qquad \mathcal{H}^{N-1}(N_0) = 0,$$

and each set K_j is a subset of a C^1 hypersurface. For each $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ set $x' := (x_1, \ldots, x_{N-1})$. In a neighborhood $Q(y, \varepsilon_0) \subset \Omega$ of each point $y \in K_j$, after a rotation, we may find a C^1 function $h_j : \mathbb{R}^{N-1} \to \mathbb{R}$ such that

$$K_j \cap Q(y, \varepsilon_0) \subset \{(x', x_N) \in Q(y, \varepsilon_0) : x_N = h_j(x')\}.$$

The function u restricted to the Lipschitz set $\Omega_j := \{(x', x_N) \in \Omega : x_N > h_j(x')\}$ has a unique trace on its boundary. Thus, by taking ε_0 smaller, if necessary, we obtain that the functions

$$w(x) := \begin{cases} u(x', x_N) & \text{if } x_N > h_j(x') \\ u(x', -x_N + 2h_j(x')) & \text{if } x_N < h_j(x') \end{cases}, \quad z(x) := \begin{cases} u(x', x_N) & \text{if } x_N < h_j(x') \\ u(x', -x_N + 2h_j(x')) & \text{if } x_N > h_j(x') \end{cases}$$

belong to the space $SBV_p(Q(y,\varepsilon_0);\mathbb{R}^d)$ and that $S(w)\cap Q(y,\varepsilon_0)$, $S(z)\cap Q(y,\varepsilon_0)\subset Q(y,\varepsilon_0)\setminus K_j$. Thus by Lemma 2.1.3 for $\mathcal{H}^{N-1}$ a.e $x_0\in K_j\cap Q(y,\varepsilon_0)$ we have that

$$\lim_{\varepsilon\to 0^+}\frac{\int_{Q_\nu(x_0,\varepsilon)}|\nabla w(x)|^p\,dx+\mathcal{H}^{N-1}(S(w)\cap Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}=0. \tag{2.1.17}$$

We claim that for $\mathcal{H}^{N-1}$ a.e $x\in K_j\cap Q(y,\varepsilon_0)$ we have that $w(x)$ is either $u^+(x)$ or $u^-(x)$. Indeed fix $x_0=(x_0',h_j(x_0'))\in K_j\cap Q(y,\varepsilon_0)$ for which the following hold:

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)}|\nabla u(x)|^p\,dx=0,\qquad \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N}}\int_{Q_\nu(x_0,\varepsilon)}|u(x)-u_{x_0,\nu}(x)|\,dx=0, \tag{2.1.18}$$

where

$$\nu=\nu_u(x_0)=\pm\frac{1}{\sqrt{1+|\nabla_{x'}h_j(x_0')|^2}}\left(\nabla_{x'}h_j(x_0'),-1\right), \tag{2.1.19}$$

and such that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N}}\int_{Q_\nu(x_0,\varepsilon)}|w(x)-w(x_0)|\,dx=0. \tag{2.1.20}$$

Without loss of generality we may assume that (2.1.19) is satisfied with the plus sign (the other case being treated in an analogous way), and, after a change of coordinates, we may take $\nu:=(0,\dots,0,-1)$. Then

$$\begin{aligned}\int_{Q_\nu(x_0,\varepsilon)\cap\Omega_j}|w(x)-u^+(x_0)|\,dx&=\int_{Q_\nu(x_0,\varepsilon)\cap\{x_N>h_j(x')\}}|u(x)-u^+(x_0)|\,dx\\ \le\int_{Q_\nu(x_0,\varepsilon)}|u(x)-u_{x_0,\nu}(x)|\,dx&+|u^-(x_0)-u^+(x_0)|\,\left|Q_\nu(x_0,\varepsilon)\cap\{h_j(x')<x_N\le h_j(x_0')\}\right|.\end{aligned} \tag{2.1.21}$$

Since h_j is C^1 and $\nabla_{x'}h_j(x_0')=0$, we have that $|h_j(x')-h_j(x_0')|\le C|x'-x_0'|^2$ for $\varepsilon>0$ sufficiently small. Thus

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\left|Q_\nu(x_0,\varepsilon)\cap\{h_j(x')<x_N\le h_j(x_0')\}\right|=0,$$

and, in turn, dividing (2.1.21) by ε^N and letting $\varepsilon\to 0^+$ we get, also by (2.1.18),

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^N}\int_{Q_\nu(x_0,\varepsilon)\cap\Omega_j}|w(x)-u^+(x_0)|\,dx=0.$$

This, together with (2.1.20), implies that $w(x_0)=u^+(x_0)$ and proves the claim.

Fix $x_0=(x_0',x_{0N})\in K_j\cap Q(y,\varepsilon_0)$ for which (2.1.18), (2.1.17) and (2.1.20) hold with, say, $w(x_0)=u^+(x_0)$ and

$$\nu=\nu_u(x_0)=\frac{1}{\sqrt{1+|\nabla_{x'}h_j(x_0')|^2}}\left(\nabla_{x'}h_j(x_0'),-1\right)$$

and let $0<\eta<1/(2\gamma_N 4^{(N-1)/N})$. By (2.1.17) there exists $\varepsilon_0>0$ such that

$$\int_{Q_\nu(x_0,\varepsilon)}|\nabla w(x)|^p\,dx+\mathcal{H}^{N-1}(S(w)\cap Q_\nu(x_0,\varepsilon))<\eta\,\varepsilon^{N-1} \tag{2.1.22}$$

for all $0<\varepsilon<\varepsilon_0$, and by Hölder's inequality, (2.1.2) and (2.1.4), we have

$$\begin{aligned}\int_{Q_\nu(x_0,\varepsilon)}|\bar w_\varepsilon(x)-\mathrm{med}(w,Q_\nu(x_0,\varepsilon))|^p\,dx&\le\left(\int_{Q_\nu(x_0,\varepsilon)}|\bar w_\varepsilon-\mathrm{med}(w,Q_\nu(x_0,\varepsilon))|^{p^*}\,dx\right)^{1/q}\varepsilon^{N/q'}\\ &\le C\varepsilon^p\int_{Q_\nu(x_0,\varepsilon)}|\nabla w(x)|^p\,dx,\end{aligned}$$

where $q:=p^*/p$, q' is its Hölder conjugate, with $N/q'=p$, and $\bar w_\varepsilon(x):=(w(x)\wedge\tau''(w,Q_\nu(x_0,\varepsilon))\vee\tau'(w,Q_\nu(x_0,\varepsilon))$. Thus, by (2.1.17), we get

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1+p}}\int_{Q_\nu(x_0,\varepsilon)}|\bar w_\varepsilon(x)-\mathrm{med}(w,Q_\nu(x_0,\varepsilon))|^p\,dx=0. \tag{2.1.23}$$

We claim that

$$\operatorname{med}(w, Q_\nu(x_0,\varepsilon)) = u^+(x_0) + o\left(\varepsilon^{(p-1)/p}\right) \qquad \text{as } \varepsilon \to 0^+. \tag{2.1.24}$$

Conditions (2.1.17) and Theorem 2.1.6 imply that w is approximately continuous at x_0, with $w(x_0) = u^+(x_0)$, and by (2.1.5) we also have that

$$\lim_{\varepsilon\to 0^+} \operatorname{med}(w, Q_\nu(x_0,\varepsilon)) = u^+(x_0). \tag{2.1.25}$$

Fix $\alpha \in (\frac{3}{4}, 1)$ and ε, $\rho > 0$ such that $\alpha\,\varepsilon \le \rho \le \varepsilon < \varepsilon_0$. Setting $w_\varepsilon := \operatorname{med}(w, Q_\nu(x_0,\varepsilon))$ and

$$\hat{w} := \Big(w \wedge \big(\tau''(w, Q_\nu(x_0,\varepsilon)) \wedge \tau''(w, Q_\nu(x_0,\rho))\big)\Big) \vee \Big(\tau'(w, Q_\nu(x_0,\varepsilon)) \vee \tau'(w, Q_\nu(x_0,\rho))\Big),$$

then, again by Theorem 2.1.4 and (2.1.22),

$$\begin{aligned}
|w_\varepsilon - w_\rho| &= \left(\frac{1}{\rho^N}\int_{Q_\nu(x_0,\rho)} |w_\varepsilon - w_\rho|^{p^*}\,dx\right)^{1/p^*} \\
&\le C\rho^{-N/p^*}\left[\left(\int_{Q_\nu(x_0,\rho)} |\hat{w} - w_\rho|^{p^*}\,dx\right)^{1/p^*} + \left(\int_{Q_\nu(x_0,\rho)} |\hat{w} - w_\varepsilon|^{p^*}\,dx\right)^{1/p^*}\right] \\
&\le C\rho^{-N/p^*}\left[\left(\int_{Q_\nu(x_0,\rho)} |\bar{w}_\rho - w_\rho|^{p^*}\,dx\right)^{1/p^*} + \left(\int_{Q_\nu(x_0,\rho)} |\bar{w}_\varepsilon - w_\varepsilon|^{p^*}\,dx\right)^{1/p^*}\right] \\
&\le C(\alpha\varepsilon)^{-N/p^*}\left[\left(\int_{Q_\nu(x_0,\rho)} |\nabla u(x)|^p\,dx\right)^{1/p} + \left(\int_{Q_\nu(x_0,\varepsilon)} |\nabla u(x)|^p\,dx\right)^{1/p}\right] \\
&\le C\,\eta^{1/p}\alpha^{-N/p*}\,\varepsilon^{(p-1)/p},
\end{aligned}$$

where we have used the inequalities

$$|\hat{w} - w_\rho| \le |\bar{w}_\rho - w_\rho|, \qquad |\hat{w} - w_\epsilon| \le |\bar{w}_\epsilon - w_\epsilon|,$$

which follow from (2.1.6). Hence for $k \ge 0$ and with ρ_k such that $\alpha^{k+1}\varepsilon \le \rho_k \le \alpha^k\varepsilon$ we obtain

$$|w_\varepsilon - w_{\rho_k}| \le |w_{\varepsilon\,\alpha^k} - w_{\rho_k}| + \sum_{j=0}^{k-1} |w_{\varepsilon\,\alpha^{j+1}} - w_{\varepsilon\,\alpha^j}| \le C\,\eta^{1/p}\,\alpha^{-N/p*}\,\varepsilon^{(p-1)/p}\sum_{j=0}^{k}(\alpha^{(p-1)/p})^j.$$

If we now let $k \to \infty$ and use (2.1.25) we get

$$|w_\varepsilon - u^+(x_0)| \le C\,\eta^{1/p}\,\alpha^{-N/p*}\,\varepsilon^{(p-1)/p}\sum_{j=0}^{\infty}(\alpha^{(p-1)/p})^j = C\,\eta^{1/p}\,\varepsilon^{(p-1)/p}\frac{1}{1-\alpha^{(p-1)/p}}.$$

Hence

$$\limsup_{\varepsilon\to 0^+} \frac{|\operatorname{med}(w, Q_\nu(x_0,\varepsilon)) - u^+(x_0)|}{\varepsilon^{(p-1)/p}} \le C\,\eta^{1/p}.$$

To obtain (2.1.24) it now suffices to let $\eta \to 0^+$.

By (2.1.23) and (2.1.24) we obtain that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1+p}}\int_{Q_\nu(x_0,\varepsilon)} |\bar{w}_\varepsilon(x) - u^+(x_0)|^p\,dx = 0. \tag{2.1.26}$$

With a similar proof we can show that

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1+p}}\int_{Q_\nu(x_0,\varepsilon)} |\bar{z}_\varepsilon(x) - u^-(x_0)|^p\,dx = 0. \tag{2.1.27}$$

We can now define

$$v_\varepsilon(x) := \begin{cases} \bar{w}_\varepsilon(x) & \text{if } x_N > h_j(x'), \\ \bar{z}_\varepsilon(x) & \text{if } x_N < h_j(x'). \end{cases}$$

Then (2.1.15) follows from (2.1.17), (2.1.26) and (2.1.27). Note that

$$\mathcal{H}^{N-1}((S(v_\varepsilon) \setminus S(u)) \cap Q_\nu(x_0, \varepsilon)) = 0$$

and, by (2.1.3) and (2.1.17),

$$\begin{aligned}
\frac{1}{\varepsilon^N} |\{x \in Q_\nu(x_0,\varepsilon) : u \neq v_\varepsilon\}| &= \frac{1}{\varepsilon^N} |\{x \in Q_\nu(x_0,\varepsilon) : x_N > h_j(x'),\, w(x) \neq \bar{w}_\varepsilon(x)\}| \\
&+ \frac{1}{\varepsilon^N} |\{x \in Q_\nu(x_0,\varepsilon) : x_N < h_j(x'),\, z(x) \neq \bar{z}_\varepsilon(x)\}| \\
&\leq C \left(\frac{\mathcal{H}^{N-1}(S(w) \cap Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \right)^{N/(N-1)} + C \left(\frac{\mathcal{H}^{N-1}(S(z) \cap Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \right)^{N/(N-1)} \to 0,
\end{aligned}$$

as $\varepsilon \to 0^+$, so that $(2.1.16)_1$ is satisfied. To prove $(2.1.16)_2$ we use Hölder's inequality, (2.1.18) and (2.1.15) to get

$$\begin{aligned}
\frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} |u(x) - v_\varepsilon(x)|\, dx \leq & \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} |u(x) - u_{x_0,\nu}(x)|\, dx \\
&+ \left(\frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} |v_\varepsilon(x) - u_{x_0,\nu}(x)|^p\, dx \right)^{1/p} \to 0 \quad \text{as } \varepsilon \to 0^+.
\end{aligned}$$

□

2.2 The General Method.

In the remaining of this section we assume that hypotheses $(H_1) - (H_4)$ hold.

Let $u \in SBV_p(\Omega; \mathbb{R}^d)$, $\nu \in S^{N-1}$, and define $\mu := \mathcal{L}^N \lfloor \Omega + (1 + |u^+ - u^-|)\mathcal{H}^{N-1} \lfloor (S(u) \cap \Omega)$. Let $\mathcal{A}^* := \{Q_\nu(x, \varepsilon) : x \in \Omega,\, \varepsilon > 0\}$, and for $\delta > 0$, $A \in \mathcal{A}(\Omega)$, set

$$\begin{aligned}
\mathbf{m}^\delta(u; A) := \inf \Bigg\{ \sum_{i=1}^{\infty} m(u; Q_i) : Q_i \in \mathcal{A}^*,\, Q_i \cap Q_j = \emptyset,\, Q_i \subset A, \\
\operatorname{diam}(Q_i) < \delta,\, \mu(A \setminus \cup_{i=1}^{\infty} Q_i) = 0 \Bigg\}.
\end{aligned}$$

Note that the existence of coverings as those needed in the definition of $\mathbf{m}^\delta$ follows from a generalization of Besicovitch's Covering Theorem (see [43], Th. 5.11) provided we take closed cubes. However, since μ is a Radon measure then $\mu(\partial Q_\nu(x, \varepsilon)) = 0$ for a.e. $\varepsilon > 0$, and thus we may consider open cubes instead. Since $\mathbf{m}^\delta(u; A)$ is decreasing as a function of δ, we define

$$\mathbf{m}^*(u; A) := \lim_{\delta \to 0^+} \mathbf{m}^\delta(u; A).$$

Lemma 2.2.1.

$$\mathcal{F}(u; A) = \mathbf{m}^*(u; A).$$

Proof. The proof of Lemma 2.2.1 is entirely identical to the proof of Lemma 3.3 of [17], with the exception that the function v^δ defined there now belongs to $SBV_p(\Omega; \mathbb{R}^d)$ due to the coercitivity hypothesis (H_4) and by virtue of Ambrosio's Compactness Theorem 2.1.1.

□

Lemma 2.2.2. *For μ a.e. $x_0 \in \Omega$ and for all $\nu \in S^{N-1}$*

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))} = \lim_{\varepsilon \to 0^+} \frac{\mathbf{m}(u; Q_\nu(x_0, \varepsilon))}{\mu(Q_\nu(x_0, \varepsilon))}.$$

The proof of this result may be carried out exactly as in Lemma 3.5 in [17]. Since the latter relied on Remark 3.2 1) in [17], we now prove that this remark still applies in our present context, precisely,

Lemma 2.2.3. *If $A \in \mathcal{A}_\infty(\Omega)$, $u \in SBV_p(\Omega;\mathbb{R}^d)$, and if*

$$A_\delta := \{x \in A : \operatorname{dist}(x, \partial A) > \delta\}$$

then

$$\lim_{\delta\to 0} \mathbf{m}(u; A_\delta) = \mathbf{m}(u; A).$$

Proof. Given δ, $\eta > 0$ choose $v \in SBV_p(\Omega;\mathbb{R}^d)$ such that $v = u$ on a neighborhood of ∂A_δ and

$$\mathbf{m}(u; A_\delta) \geq \mathcal{F}(v; A_\delta) - \eta.$$

Define

$$w := \begin{cases} v & \text{in } A_\delta, \\ u & \text{in } \Omega \setminus \overline{A_\delta}. \end{cases} \tag{2.2.1}$$

Let $\varepsilon > 0$ be small enough so that $v = u$ on $A_\delta \setminus A_{\delta+\varepsilon}$. We have

$$\begin{aligned} \mathbf{m}(u; A) &\leq \mathcal{F}(w; A) \\ &\leq \mathcal{F}(v; A_\delta) + C\int_{A\setminus A_{\delta+\varepsilon}} (1 + |\nabla u|^p)\,dx + C\int_{(A\setminus A_{\delta+\varepsilon})\cap S(u)} (1 + |[u]|)\,d\mathcal{H}^{N-1} \\ &\leq \mathbf{m}(u; A_\delta) + \eta \\ &\quad + C\int_{A\setminus A_{\delta+\varepsilon}} (1 + |\nabla u|^p)\,dx + C\int_{(A\setminus A_{\delta+\varepsilon})\cap S(u)} (1 + |[u]|)\,d\mathcal{H}^{N-1}. \end{aligned}$$

Thus, letting $\varepsilon \to 0^+$ and then $\eta \to 0^+$ we conclude that

$$\mathbf{m}(u; A) \leq \mathbf{m}(u; A_\delta) + C\int_{A\setminus A_\delta} (1 + |\nabla u|^p)\,dx + C\int_{(A\setminus A_\delta)\cap S(u)} (1 + |[u]|)\,d\mathcal{H}^{N-1}, \tag{2.2.2}$$

and so

$$\liminf_{\delta\to 0^+} \mathbf{m}(u; A_\delta) \geq \mathbf{m}(u; A).$$

Conversely, fix $\varepsilon > 0$ and choose $v \in SBV_p(\Omega;\mathbb{R}^d)$ such that $v = u$ on a neighborhood of ∂A and

$$\mathbf{m}(u; A) \geq \mathcal{F}(v; A) - \varepsilon.$$

Let $\delta_0 > 0$ be small enough so that $v = u$ on $A \setminus A_\delta$ for all $0 < \delta < \delta_0$. Then

$$\mathbf{m}(u; A_\delta) \leq \mathcal{F}(v; A_\delta) \leq \mathcal{F}(v; A) \leq \mathbf{m}(u; A) + \varepsilon.$$

Letting $\delta \to 0^+$ and then $\varepsilon \to 0^+$ we deduce that

$$\limsup_{\delta\to 0^+} \mathbf{m}(u; A_\delta) \leq \mathbf{m}(u; A).$$

□

Remark 2.2.4. We can show that if $A' \subset\subset A$, with A', $A \in \mathcal{A}(\Omega)$, then

$$\mathbf{m}(u; A) \leq \mathbf{m}(u; A') + C\int_{A\setminus A'} (1 + |\nabla u|^p)\,dx + C\int_{(A\setminus A')\cap S(u)} (1 + |[u]|)\,d\mathcal{H}^{N-1}.$$

This inequality follows immediately from (2.2.2) with A' in place of A_δ. Similarly, it can be shown that

$$\mathcal{F}(u; A) \leq \mathcal{F}(u; A') + C\int_{A\setminus A'} (1 + |\nabla u|^p)\,dx + C\int_{(A\setminus A')\cap S(u)} (1 + |[u]|)\,d\mathcal{H}^{N-1}.$$

2.3 Proof of Theorem 1.0.1: Lebesgue Points.

Lemma 2.3.1. *For $\mathcal{L}^N$ a.e. $x_0 \in \Omega$*

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0)),$$

where f is the function defined in Theorem 1.0.1.

Proof. Let u_ε and $\bar{u}_\varepsilon$ be defined as in Lemma 2.1.7, and let x_0 satisfy (2.1.8)–(2.1.10). By Lemma 2.2.2 for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{F}(u(x_0)+\varepsilon\, u_\varepsilon; Q_\nu(x_0,\varepsilon))}{\varepsilon^N} = \lim_{\varepsilon\to 0^+} \frac{\mathbf{m}(u(x_0)+\varepsilon\, u_\varepsilon; Q_\nu(x_0,\varepsilon))}{\varepsilon^N} < \infty. \tag{2.3.1}$$

Thus, by the coercitivity hypothesis (H_4), there exists $K > 0$ such that

$$\frac{\mathcal{H}^{N-1}(S(u_\varepsilon)\cap Q_\nu(x_0,\varepsilon))}{\varepsilon^N} \leq K$$

for all $\varepsilon > 0$ sufficiently small. Hence (2.1.1) is satisfied for $0 < \varepsilon << 1$ and by (2.1.3)

$$\begin{aligned} 0 &\leq \frac{1}{\varepsilon^N}\int_0^1 \mathcal{H}^{N-1}\left(\{u_\varepsilon \neq \bar{u}_\varepsilon\}\cap \partial Q_\nu(x_0,\lambda\varepsilon)\right) d\,\lambda = \frac{1}{\varepsilon^{N+1}}\left|\{x\in Q_\nu(x_0,\varepsilon) :\, u_\varepsilon \neq \bar{u}_\varepsilon\}\right| \\ &\leq \frac{C}{\varepsilon^{N+1}}\left(\mathcal{H}^{N-1}(S(u_\varepsilon)\cap Q_\nu(x_0,\varepsilon))\right)^{N/(N-1)} \leq C\,K^{N/(N-1)}\,\varepsilon^{1/(N-1)} \to 0 \quad \text{as } \varepsilon\to 0^+, \end{aligned}$$

and by (2.1.10) and (2.1.7)

$$\begin{aligned} 0 &\leq \frac{1}{\varepsilon^N}\int_0^1\int_{\partial Q_\nu(x_0,\lambda\varepsilon)} |\varepsilon\, u_\varepsilon - \varepsilon\,\bar{u}_\varepsilon|\, d\,\mathcal{H}^{N-1}\, d\,\lambda \leq \frac{1}{\varepsilon^{N+1}}\int_{Q_\nu(x_0,\varepsilon)} |u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|\,dx \\ &+ \frac{1}{\varepsilon^{N+1}}\int_{Q_\nu(x_0,\varepsilon)} |\varepsilon\,\bar{u}_\varepsilon(x) - \nabla u(x_0)(x-x_0)|\,dx \to 0 \quad \text{as } \varepsilon\to 0^+. \end{aligned}$$

Thus for any sequence $\varepsilon_h \to 0^+$ there exists a subsequence (not relabeled) such that for $\mathcal{L}^1$ a.e $\lambda\in(0,1)$

$$\begin{aligned} &\mu(\partial Q_\nu(x_0,\lambda\varepsilon_h)) = \mathcal{H}^{N-1}(\partial Q_\nu(x_0,\lambda\varepsilon_h)\cap S(\bar{u}_{\varepsilon_h})) = 0, \\ &\lim_{\varepsilon_h\to 0^+}\frac{1}{\varepsilon_h{}^N}\int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u_{\varepsilon_h}\neq\bar{u}_{\varepsilon_h}\}} (1+|\varepsilon_h\,u_{\varepsilon_h} - \varepsilon_h\,\bar{u}_{\varepsilon_h}|)\,d\,\mathcal{H}^{N-1} = 0. \end{aligned} \tag{2.3.2}$$

Fix a sequence $\varepsilon_h \to 0^+$, and consider a subsequence (not relabeled) and $\lambda\in(0,1)$ for which (2.3.2) holds. Given $v_{\varepsilon_h} \in SBV_p(Q_\nu(x_0,\varepsilon_h);\mathbb{R}^d)$ such that $v_{\varepsilon_h} = u(x_0)+\varepsilon_h\,\bar{u}_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0,\lambda\varepsilon_h)$ and

$$\mathcal{F}(v_{\varepsilon_h}; Q_\nu(x_0,\lambda\varepsilon_h)) \leq \mathbf{m}(u(x_0)+\varepsilon_h\,\bar{u}_{\varepsilon_h}; Q_\nu(x_0,\lambda\varepsilon_h)) + \varepsilon_h{}^{N+1},$$

define

$$w_{\varepsilon_h} := \begin{cases} v_{\varepsilon_h}(x) & \text{in } Q_\nu(x_0,\lambda\varepsilon_h), \\ u(x_0)+\varepsilon_h\,u_{\varepsilon_h}(x) & \text{in } Q_\nu(x_0,\varepsilon_h)\backslash\overline{Q_\nu(x_0,\lambda\varepsilon_h)}. \end{cases}$$

Since $w_{\varepsilon_h} = u(x_0)+\varepsilon_h\,u_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0,\varepsilon_h)$, by Remark 2.2.4 we have

$$\begin{aligned} &\mathbf{m}(u(x_0)+\varepsilon_h\,u_{\varepsilon_h}; Q_\nu(x_0,\varepsilon_h)) \leq \mathcal{F}(w_{\varepsilon_h}; Q_\nu(x_0,\varepsilon_h)) \\ &\leq \mathcal{F}(v_{\varepsilon_h}; Q_\nu(x_0,\lambda\,\varepsilon_h)) + C\int_{Q_\nu(x_0,\varepsilon_h)\backslash Q_\nu(x_0,\lambda\,\varepsilon_h)} (1+|\nabla u|^p)\,dx \\ &\quad + \int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u_{\varepsilon_h}\neq\bar{u}_{\varepsilon_h}\}} (1+|\varepsilon_h\,u_{\varepsilon_h} - \varepsilon_h\,\bar{u}_{\varepsilon_h}|)\,d\,\mathcal{H}^{N-1} + C\int_{Q_\nu(x_0,\varepsilon_h)\cap S(u)} (1+|[u]|)\,d\mathcal{H}^{N-1} \\ &\leq \mathbf{m}(u(x_0)+\varepsilon_h\,\bar{u}_{\varepsilon_h}; Q_\nu(x_0,\lambda\varepsilon_h)) + \varepsilon_h{}^{N+1} + C\left((1-\lambda^N)\varepsilon_h{}^N + \int_{Q_\nu(x_0,\varepsilon_h)\backslash Q_\nu(x_0,\lambda\,\varepsilon_h)} |\nabla u|^p\,dx\right) \\ &\quad + \int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u_{\varepsilon_h}\neq\bar{u}_{\varepsilon_h}\}} (1+|\varepsilon_h\,u_{\varepsilon_h} - \varepsilon_h\,\bar{u}_{\varepsilon_h}|)\,d\,\mathcal{H}^{N-1} + C\int_{Q_\nu(x_0,\varepsilon_h)\cap S(u)} (1+|[u]|)\,d\mathcal{H}^{N-1}. \end{aligned}$$

Dividing by ${\varepsilon_h}^N$ yields

$$\frac{\mathbf{m}(u(x_0)+\varepsilon_h\, u_{\varepsilon_h}; Q_\nu(x_0,\varepsilon_h))}{{\varepsilon_h}^N} \le \frac{\mathbf{m}(u(x_0)+\varepsilon_h\, \bar u_{\varepsilon_h}; Q_\nu(x_0,\lambda\varepsilon_h))}{{\varepsilon_h}^N} + \varepsilon_h + C(1-\lambda^N)$$
$$+\frac{C}{{\varepsilon_h}^N}\int_{Q_\nu(x_0,\varepsilon_h)\setminus Q_\nu(x_0,\lambda\,\varepsilon_h)} |\nabla u|^p dx + \frac{C}{{\varepsilon_h}^N}\int_{Q_\nu(x_0,\varepsilon_h)\cap S(u)} (1+|[u]|)\, d\mathcal{H}^{N-1}$$
$$+\frac{1}{{\varepsilon_h}^N}\int_{\partial Q_\nu(x_0,\lambda\varepsilon_h)\cap\{u_{\varepsilon_h}\neq \bar u_{\varepsilon_h}\}} (1+|\varepsilon_h\, u_{\varepsilon_h}-\varepsilon_h\,\bar u_{\varepsilon_h}|)\, d\,\mathcal{H}^{N-1}.$$

We claim that the last three terms on the right hand side approach zero if we first let $\varepsilon_h \to 0^+$ and then $\lambda \to 1^-$. Indeed, for the last two integrals this follows from (2.3.2) and from (2.1.9), respectively. Moreover

$$\frac{1}{{\varepsilon_h}^N}\int_{Q_\nu(x_0,\varepsilon_h)\setminus Q_\nu(x_0,\lambda\,\varepsilon_h)} |\nabla u|^p dx = \frac{1}{{\varepsilon_h}^N}\int_{Q_\nu(x_0,\varepsilon_h)} |\nabla u|^p dx - \lambda^N \frac{1}{(\lambda\,\varepsilon_h)^N}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} |\nabla u|^p dx. \tag{2.3.3}$$

Hence by (2.1.8)

$$\lim_{\varepsilon_h\to 0^+}\frac{1}{{\varepsilon_h}^N}\int_{Q_\nu(x_0,\varepsilon_h)\setminus Q_\nu(x_0,\lambda\,\varepsilon_h)} |\nabla u|^p dx = |\nabla u(x_0)|^p - \lambda^N\,|\nabla u(x_0)|^p.$$

The claim is proved if we let $\lambda \to 1^-$.

In conclusion, by (2.3.1), we have shown that

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0) \le \liminf_{\lambda\to 1^-}\liminf_{\varepsilon_h\to 0^+}\frac{\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h}; Q_\nu(x_0,\lambda\varepsilon_h))}{{\varepsilon_h}^N}, \tag{2.3.4}$$

where the liminf in λ is taken only over those λ for which (2.3.2) holds.

To complete the proof of the lemma it remains to show that the right hand side of (2.3.4) is less than or equal to $f(x_0,u(x_0),\nabla u(x_0))$. Fix $0<s<1$ and let $v_{\varepsilon_h}\in SBV_p(Q_\nu(x_0,s\,\lambda\,\varepsilon_h);\mathbb{R}^d)$ be such that $v_{\varepsilon_h}(x) = u(x_0)+\nabla u(x_0)\,(x-x_0)$ on a neighborhood of $\partial Q_\nu(x_0,s\lambda\varepsilon_h)$ and

$$\mathcal{F}(v_{\varepsilon_h}; Q_\nu(x_0,s\lambda\varepsilon_h)) \le \mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0); Q_\nu(x_0,s\lambda\varepsilon_h)) + (\lambda\varepsilon_h)^{N+1}.$$

Extend v_{ε_h} to be equal to $u(x_0)+\nabla u(x_0)\,(x-x_0)$ outside $Q_\nu(x_0,s\,\lambda\,\varepsilon_h)$ and consider a layer L of width $(1-s)\lambda\varepsilon_h/2$ and a smooth cut-off function φ such that $0<\varphi<1$ in L, $\varphi\equiv 1$ in $Q_\nu(x_0,s\lambda\varepsilon_h)$, $\varphi=0$ on $\partial Q_\nu(x_0,\lambda\,\varepsilon_h)$ and $||\nabla\varphi||_\infty \le 3/(1-s)\lambda\varepsilon_h$. Define $w_{\varepsilon_h}(x) := \varphi\, v_{\varepsilon_h} + (1-\varphi)(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h}(x))$. Since $w_{\varepsilon_h} = u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0,\lambda\,\varepsilon_h)$ we have, in view of Remark 2.2.4,

$$\begin{aligned}
&\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h}; Q_\nu(x_0,\lambda\,\varepsilon_h)) \le \mathcal{F}(w_{\varepsilon_h}; Q_\nu(x_0,\lambda\,\varepsilon_h))\\
&\le \mathcal{F}(v_{\varepsilon_h}; Q_\nu(x_0,s\lambda\varepsilon_h)) + C\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h)} (1+|\nabla u|^p+|\nabla u(x_0)|^p)\,dx\\
&\quad+\frac{C}{(1-s)^p(\lambda\varepsilon_h)^p}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h)} |\varepsilon_h\,\bar u_{\varepsilon_h}(x)-\nabla u(x_0)\,(x-x_0)|^p dx\\
&\quad+C\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\cap S(u)} (1+|[u]|)\,d\mathcal{H}^{N-1}\\
&\le \mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0); Q_\nu(x_0,s\lambda\varepsilon_h))\\
&\quad+(\lambda\varepsilon_h)^{N+1} + C(1+|\nabla u(x_0)|^p)(1-s^N)(\lambda\varepsilon_h)^N + C\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h)} |\nabla u|^p\,dx\\
&\quad+\frac{C}{(1-s)^p(\lambda\varepsilon_h)^p}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} |\varepsilon_h\,\bar u_{\varepsilon_h}(x)-\nabla u(x_0)\,(x-x_0)|^p dx + C\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\cap S(u)} (1+|[u]|)\,d\mathcal{H}^{N-1},
\end{aligned}$$

where we have used the facts that $||\nabla\varphi||_\infty \le 3/(1-s)\lambda\varepsilon_h$, and that

$$\begin{gathered}
|\nabla \bar u_{\varepsilon_h}| \le |\nabla u_{\varepsilon_h}| \le |\nabla u|/\varepsilon_h \quad \mathcal{L}^N \text{ a.e.},\\
\mathcal{H}^{N-1}(S(\bar u_{\varepsilon_h})\setminus S(u)) = 0, \quad |[\bar u_{\varepsilon_h}]| \le |[u_{\varepsilon_h}]| \le |[u]|/\varepsilon_h \quad \mathcal{H}^{N-1} \text{ a.e.}
\end{gathered} \tag{2.3.5}$$

Dividing by $(\lambda\,\varepsilon_h)^N$ yields

$$\begin{aligned}
&\frac{\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h};Q_\nu(x_0,\lambda\,\varepsilon_h))}{(\lambda\,\varepsilon_h)^N}\le\frac{\mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0);Q_\nu(x_0,s\lambda\varepsilon_h))}{(\lambda\,\varepsilon_h)^N}+\lambda\varepsilon_h\\
&+C(1+|\nabla u(x_0)|^p)(1-s^N)+\frac{C}{(\lambda\,\varepsilon_h)^N}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h)}|\nabla u|^p\,dx\\
&+\frac{C}{(1-s)^p(\lambda\varepsilon_h)^{N+p}}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)}|\varepsilon_h\,\bar u_{\varepsilon_h}(x)-\nabla u(x_0)\,(x-x_0)|^p dx\\
&+\frac{C}{(\lambda\,\varepsilon_h)^N}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\cap S(u)}(1+|[u]|)\,d\mathcal{H}^{N-1}.
\end{aligned}$$

We claim that the last three terms on the right hand side approach zero if we first let $\varepsilon_h\to 0^+$, then $s\to 1^-$ and finally $\lambda\to 1^-$. Indeed, for the second integral this follows from (2.1.7), while the first and the third integrals may be treated as before (see (2.1.9) and (2.3.3)). Consequently, and also by (2.3.4),

$$\begin{aligned}
\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0)&\le\liminf_{\lambda\to1^-}\liminf_{s\to1^-}\liminf_{\varepsilon_h\to0^+}\frac{\mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0);Q_\nu(x_0,s\lambda\varepsilon_h))}{(\lambda\,\varepsilon_h)^N}\\
&\le\limsup_{\varepsilon\to0^+}\frac{\mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0);Q_\nu(x_0,\varepsilon))}{\varepsilon^N}=f(x_0,u(x_0),\nabla u(x_0)).
\end{aligned}$$

□

In order to assert the representation (1.0.2) it suffices now to establish the reverse inequality, precisely

Lemma 2.3.2. *For $\mathcal{L}^N$ a.e. $x_0\in\Omega$*

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0)\ge f(x_0,u(x_0),\nabla u(x_0)).$$

Proof. The proof of this lemma is very similar to the previous one and thus we will indicate only the main changes. Let $\varepsilon_h\to 0^+$ be such that

$$\limsup_{\varepsilon\to0^+}\frac{\mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0);Q_\nu(x_0,\varepsilon))}{\varepsilon^N}=\lim_{\varepsilon_h\to0^+}\frac{\mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0);Q_\nu(x_0,\varepsilon_h))}{{\varepsilon_h}^N}. \tag{2.3.6}$$

Fix $0<s<1$ and let $v_{\varepsilon_h}\in SBV_p(Q_\nu(x_0,s\,\varepsilon_h);\mathbb{R}^d)$ be such that $v_{\varepsilon_h}(x)=u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0,s\varepsilon_h)$ and

$$\mathcal{F}(v_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))\le\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))+{\varepsilon_h}^{N+1}.$$

Extend v_{ε_h} to be $u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h}$ outside $Q_\nu(x_0,s\,\varepsilon_h)$ and consider a layer L of width $(1-s)\varepsilon_h/2$ and a cut-off function φ such that $0<\varphi<1$ in L, $\varphi\equiv1$ on $Q_\nu(x_0,s\varepsilon_h)$, $\varphi=0$ on $\partial Q_\nu(x_0,\varepsilon_h)$ and $||\nabla\varphi||_\infty\le 3/(1-s)\varepsilon_h$. Define $w_{\varepsilon_h}(x):=\varphi\,v_{\varepsilon_h}+(1-\varphi)(u(x_0)+\nabla u(x_0)\,(x-x_0))$. Since $w_{\varepsilon_h}=u(x_0)+\nabla u(x_0)\,(x-x_0)$ on a neighborhood of $\partial Q_\nu(x_0,\varepsilon_h)$ then, reasoning as in the second part of the proof of Lemma 2.3.1 and by (2.3.5), we have by Remark 2.2.4,

$$\begin{aligned}
&\frac{\mathbf{m}(u(x_0)+\nabla u(x_0)\,(\cdot-x_0);Q_\nu(x_0,\varepsilon_h))}{\varepsilon_h^N}\le\frac{\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))}{\varepsilon_h^N}+\varepsilon_h\\
&+C(1+|\nabla u(x_0)|^p)(1-s^N)+\frac{C}{\varepsilon_h^N}\int_{Q_\nu(x_0,\varepsilon_h)\setminus Q_\nu(x_0,s\varepsilon_h)}|\nabla u|^p\,dx\\
&+\frac{C}{(1-s)^p\varepsilon_h^{N+p}}\int_{Q_\nu(x_0,\varepsilon_h)}|\varepsilon_h\,\bar u_{\varepsilon_h}(x)-\nabla u(x_0)\,(x-x_0)|^p dx+\frac{C}{\varepsilon_h^N}\int_{Q_\nu(x_0,\varepsilon_h)\cap S(u)}(1+|[u]|)\,d\mathcal{H}^{N-1}.
\end{aligned}$$

As before, and invoking (2.3.6), we obtain

$$f(x_0,u(x_0),\nabla u(x_0))\le\liminf_{s\to1^-}\liminf_{\varepsilon_h\to0^+}\frac{\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))}{\varepsilon_h^N}. \tag{2.3.7}$$

Consider a sequence $s_j\to 1^-$ such that

$$\liminf_{s\to1^-}\liminf_{\varepsilon_h\to0^+}\frac{\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))}{\varepsilon_h^N}=\lim_{s_j\to1^-}\liminf_{\varepsilon_h\to0^+}\frac{\mathbf{m}(u(x_0)+\varepsilon_h\,\bar u_{\varepsilon_h};Q_\nu(x_0,s_j\varepsilon_h))}{\varepsilon_h^N}, \tag{2.3.8}$$

and consider a subsequence of $\{\varepsilon_h\}$ (not relabeled) and $\lambda \in (0,1)$ such that, for all $j \in \mathbb{N}$, and in light of (2.3.2),

$$\mu(\partial Q_\nu(x_0, s_j\lambda\varepsilon_h)) = \mathcal{H}^{N-1}(\partial Q_\nu(x_0, s_j\lambda\varepsilon_h) \cap S(\bar{u}_{\varepsilon_h})) = 0,$$
$$\lim_{\varepsilon_h \to 0^+} \frac{1}{\varepsilon_h{}^N} \int_{\partial Q_\nu(x_0, s_j\lambda\varepsilon_h) \cap \{u_{\varepsilon_h} \neq \bar{u}_{\varepsilon_h}\}} (1 + |\varepsilon_h\, u_{\varepsilon_h} - \varepsilon_h\, \bar{u}_{\varepsilon_h}|) d\,\mathcal{H}^{N-1} = 0.$$

Given $v_{\varepsilon_h} \in SBV_p(Q_\nu(x_0, s_j\lambda\varepsilon_h); \mathbb{R}^d)$ such that $v_{\varepsilon_h} = u(x_0) + \varepsilon_h\, u_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0, s_j\lambda\varepsilon_h)$ and

$$\mathcal{F}(v_{\varepsilon_h}; Q_\nu(x_0, s_j\lambda\varepsilon_h)) \leq \mathbf{m}(u(x_0) + \varepsilon_h\, u_{\varepsilon_h}; Q_\nu(x_0, s_j\lambda\varepsilon_h)) + (s_j\, \varepsilon_h)^{N+1},$$

define

$$w_{\varepsilon_h} := \begin{cases} v_{\varepsilon_h}(x) & \text{in } Q_\nu(x_0, s_j\lambda\varepsilon_h), \\ u(x_0) + \varepsilon_h\, \bar{u}_{\varepsilon_h}(x) & \text{in } Q_\nu(x_0, s_j\varepsilon_h) \backslash \overline{Q_\nu(x_0, s_j\lambda\varepsilon_h)}. \end{cases}$$

Since $w_{\varepsilon_h} = u(x_0) + \varepsilon_h\, \bar{u}_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0, s_j\varepsilon_h)$ we have, as before, and in view of Remark 2.2.4

$$\begin{aligned} \frac{\mathbf{m}(u(x_0) + \varepsilon_h\, \bar{u}_{\varepsilon_h}; Q_\nu(x_0, s_j\varepsilon_h))}{(s_j\varepsilon_h)^N} &\leq \frac{\mathbf{m}(u(x_0) + \varepsilon_h\, u_{\varepsilon_h}; Q_\nu(x_0, s_j\lambda\varepsilon_h))}{(s_j\varepsilon_h)^N} + s_j\varepsilon_h + C(1-\lambda^N) \\ + \frac{C}{(s_j\varepsilon_h)^N} \int_{Q_\nu(x_0, s_j\varepsilon_h) \backslash Q_\nu(x_0, s_j\lambda\varepsilon_h)} &|\nabla u|^p dx + \frac{C}{(s_j\varepsilon_h)^N} \int_{Q_\nu(x_0, s_j\varepsilon_h) \cap S(u)} (1 + |[u]|)\, d\mathcal{H}^{N-1} \\ &+ \frac{1}{(s_j\varepsilon_h)^N} \int_{\partial Q_\nu(x_0, s_j\lambda\varepsilon_h) \cap \{u_{\varepsilon_h} \neq \bar{u}_{\varepsilon_h}\}} (1 + |\varepsilon_h\, u_{\varepsilon_h} - \varepsilon_h\, \bar{u}_{\varepsilon_h}|) d\,\mathcal{H}^{N-1}, \end{aligned} \tag{2.3.9}$$

where we have used (2.3.5). We can proceed as in the first part of the proof of Lemma 2.3.1 to conclude from (2.3.7)–(2.3.8), (2.3.9), that

$$f(x_0, u(x_0), \nabla u(x_0)) \leq \liminf_{s_j \to 1^-} \liminf_{\lambda \to 1^-} \liminf_{\varepsilon_h \to 0^+} \frac{\mathbf{m}(u(x_0) + \varepsilon_h\, u_{\varepsilon_h}; Q_\nu(x_0, s_j\lambda\varepsilon_h))}{(s_j\varepsilon_h)^N} = \frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N}(x_0).$$

□

2.4 Proof of Theorem 1.0.1: Jump Points.

In this section we prove that the jump energy density g is given by (1.0.3).

Lemma 2.4.1.

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \leq g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)), \tag{2.4.1}$$

where g is the function defined in Theorem 1.0.1.

Proof. Let v_ε be the function constructed according to Lemma 2.1.8, let x_0 satisfy (2.1.18), and set $\nu := \nu_u(x_0)$. By Lemma 2.2.1 for $\mathcal{H}^{N-1}$ a.e. $x_0 \in \Omega \cap S(u)$

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}(u; Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}} = \lim_{\varepsilon \to 0^+} \frac{\mathbf{m}(u; Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}} < \infty. \tag{2.4.2}$$

By (2.1.16)

$$0 \leq \frac{1}{\varepsilon^{N-1}} \int_0^1 \mathcal{H}^{N-1}\left(\{u \neq v_\varepsilon\} \cap \partial Q_\nu(x_0, \lambda\varepsilon)\right) d\,\lambda = \frac{1}{\varepsilon^N} \left|\{x \in Q_\nu(x_0, \varepsilon) : \, u \neq v_\varepsilon\}\right| \to 0 \quad \text{as } \varepsilon \to 0^+,$$

and

$$0 \leq \frac{1}{\varepsilon^{N-1}} \int_0^1 \int_{\partial Q_\nu(x_0, \lambda\varepsilon)} |u - v_\varepsilon| d\,\mathcal{H}^{N-1}\, d\,\lambda \leq \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0, \varepsilon)} |u - v_\varepsilon|\, dx \to 0 \quad \text{as } \varepsilon \to 0^+.$$

Thus for any sequence $\varepsilon_h \to 0^+$ we may find a subsequence (not relabeled) such that for $\mathcal{L}^1$ a.e $\lambda \in (0,1)$

$$\begin{aligned} &\mu(\partial Q_\nu(x_0, \lambda\varepsilon_h)) = \mathcal{H}^{N-1}(\partial Q_\nu(x_0, \lambda\varepsilon_h) \cap S(v_{\varepsilon_h})) = 0, \\ &\lim_{\varepsilon_h \to 0^+} \frac{1}{\varepsilon_h^{N-1}} \int_{\partial Q_\nu(x_0, \lambda\, \varepsilon_h) \cap \{u \neq v_{\varepsilon_h}\}} (1 + |u - v_{\varepsilon_h}|) d\, \mathcal{H}^{N-1} = 0. \end{aligned} \tag{2.4.3}$$

Fix a subsequence $\varepsilon_h \to 0^+$ and $\lambda \in (0,1)$ for which (2.4.3) holds. Given $z_{\varepsilon_h} \in SBV_p(Q_\nu(x_0, \varepsilon_h); \mathbb{R}^d)$ such that $z_{\varepsilon_h} = v_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0, \lambda\varepsilon_h)$ and

$$\mathcal{F}(z_{\varepsilon_h}; Q_\nu(x_0, \lambda\varepsilon_h)) \leq \mathbf{m}(v_{\varepsilon_h}; Q_\nu(x_0, \lambda\varepsilon_h)) + {\varepsilon_h}^N,$$

define

$$w_{\varepsilon_h} := \begin{cases} z_{\varepsilon_h} & \text{in } Q_\nu(x_0, \lambda\varepsilon_h), \\ u & \text{in } Q_\nu(x_0, \varepsilon_h) \backslash \overline{Q_\nu(x_0, \lambda\varepsilon_h)}. \end{cases}$$

Since $w_{\varepsilon_h} = u$ on $\partial Q_\nu(x_0, \varepsilon_h)$ we have by Remark 2.2.4,

$$\begin{aligned} &\mathbf{m}(u; Q_\nu(x_0, \varepsilon_h)) \leq \mathcal{F}(w_{\varepsilon_h}; Q_\nu(x_0, \varepsilon_h)) \\ &\leq \mathcal{F}(z_{\varepsilon_h}; Q_\nu(x_0, \lambda\, \varepsilon_h)) + C \int_{Q_\nu(x_0, \varepsilon_h)} (1 + |\nabla v_{\varepsilon_h}|^p) dx \\ &\quad + \int_{\partial Q_\nu(x_0, \lambda\, \varepsilon_h) \cap \{u \neq v_{\varepsilon_h}\}} (1 + |u - v_{\varepsilon_h}|) d\, \mathcal{H}^{N-1} + C \int_{(Q_\nu(x_0, \varepsilon_h) \backslash Q_\nu(x_0, \lambda\, \varepsilon_h)) \cap S(u)} (1 + |[u]|) d\mathcal{H}^{N-1} \\ &\leq \mathbf{m}(v_{\varepsilon_h}(x); Q_\nu(x_0, \lambda\varepsilon_h)) + {\varepsilon_h}^N + C \int_{Q_\nu(x_0, \varepsilon_h)} (1 + |\nabla v_{\varepsilon_h}|^p)\, dx \\ &\quad + \int_{\partial Q_\nu(x_0, \lambda\, \varepsilon_h) \cap \{u \neq v_{\varepsilon_h}\}} (1 + |u - v_{\varepsilon_h}|) d\mathcal{H}^{N-1} + C \int_{(Q_\nu(x_0, \varepsilon_h) \backslash Q_\nu(x_0, \lambda\, \varepsilon_h)) \cap S(u)} (1 + |[u]|)\, d\mathcal{H}^{N-1}. \end{aligned}$$

Dividing by ${\varepsilon_h}^{N-1}$ yields

$$\begin{aligned} \frac{\mathbf{m}(u; Q_\nu(x_0, \varepsilon_h))}{{\varepsilon_h}^{N-1}} &\leq \frac{\mathbf{m}(v_{\varepsilon_h}; Q_\nu(x_0, \lambda\varepsilon_h))}{{\varepsilon_h}^{N-1}} + \varepsilon_h + \frac{C}{{\varepsilon_h}^{N-1}} \int_{Q_\nu(x_0, \varepsilon_h)} (1 + |\nabla v_{\varepsilon_h}|^p) dx \\ &\quad + \frac{C}{{\varepsilon_h}^{N-1}} \int_{\partial Q_\nu(x_0, \lambda\, \varepsilon_h) \cap \{u \neq v_{\varepsilon_h}\}} (1 + |u - v_{\varepsilon_h}|) d\mathcal{H}^{N-1} \\ &\quad + \frac{C}{{\varepsilon_h}^{N-1}} \int_{(Q_\nu(x_0, \varepsilon_h) \backslash Q_\nu(x_0, \lambda\, \varepsilon_h)) \cap S(u)} (1 + |[u]|) d\mathcal{H}^{N-1}. \end{aligned}$$

We claim that the last three terms on the right hand side approach zero if we first let $\varepsilon_h \to 0^+$ and then $\lambda \to 1^-$. Indeed, for the first two integrals this follows from $(2.1.15)_1$ and from (2.4.3), respectively. Moreover

$$\begin{aligned} &\frac{1}{\varepsilon_h^{N-1}} \int_{(Q_\nu(x_0, \varepsilon_h) \backslash Q_\nu(x_0, \lambda\, \varepsilon_h)) \cap S(u)} (1 + |[u]|)\, d\mathcal{H}^{N-1} \\ &= \frac{(1 + |[u]|)\mathcal{H}^{N-1} \lfloor S(u)(Q_\nu(x_0, \varepsilon_h))}{\mathcal{H}^{N-1} \lfloor S(u)(Q_\nu(x_0, \varepsilon_h))} - \lambda^{N-1} \frac{(1 + |[u]|)\mathcal{H}^{N-1} \lfloor S(u)(Q_\nu(x_0, \lambda\, \varepsilon_h))}{\mathcal{H}^{N-1} \lfloor S(u)(Q_\nu(x_0, \lambda\, \varepsilon_h))}. \end{aligned}$$

Hence

$$\begin{aligned} &\lim_{\varepsilon_h \to 0^+} \frac{1}{\varepsilon_h^{N-1}} \int_{(Q_\nu(x_0, \varepsilon_h) \backslash Q_\nu(x_0, \lambda\, \varepsilon_h)) \cap S(u)} (1 + |[u]|)\, d\mathcal{H}^{N-1} \\ &\quad = \frac{d((1 + |[u]|)\mathcal{H}^{N-1} \lfloor S(u))}{d\mathcal{H}^{N-1} \lfloor S(u)} (x_0) - \lambda^{N-1} \frac{d((1 + |[u]|)\mathcal{H}^{N-1} \lfloor S(u))}{d\mathcal{H}^{N-1} \lfloor S(u)} (x_0). \end{aligned} \tag{2.4.4}$$

It now suffices to let $\lambda \to 1^-$. In conclusion, by (2.4.2), we have shown that

$$\frac{d\mathcal{F}(u; \cdot)}{d\mathcal{H}^{N-1} \lfloor S(u)} (x_0) \leq \liminf_{\lambda \to 1^-} \liminf_{\varepsilon_h \to 0^+} \frac{\mathbf{m}(v_{\varepsilon_h}; Q_\nu(x_0, \lambda\varepsilon_h))}{{\varepsilon_h}^{N-1}}. \tag{2.4.5}$$

To complete the proof of the lemma it remains to show that the right hand side of (2.4.5) is less than or equal to $g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0))$. Fix $0 < s < 1$ and let $z_{\varepsilon_h} \in SBV_p(Q_\nu(x_0, s\, \lambda\, \varepsilon_h); \mathbb{R}^d)$ be such that $z_{\varepsilon_h} = u_{x_0, \nu}$ on a neighborhood of $\partial Q_\nu(x_0, s\lambda\varepsilon_h)$ and

$$\mathcal{F}(z_{\varepsilon_h}; Q_\nu(x_0, s\lambda\varepsilon_h)) \leq \mathbf{m}(u_{x_0, \nu}; Q_\nu(x_0, s\lambda\varepsilon_h)) + (\lambda\varepsilon_h)^N.$$

Extend z_{ε_h} to be equal to $u_{x_0,\nu}$ outside $Q_\nu(x_0, s\,\lambda\,\varepsilon_h)$ and consider a layer L of width $(1-s)\lambda\varepsilon_h/2$ and a smooth cut-off function φ such that $0 < \varphi < 1$ in L, $\varphi \equiv 1$ on $Q_\nu(x_0, s\lambda\varepsilon_h)$, $\varphi = 0$ on $\partial Q_\nu(x_0, \lambda\,\varepsilon_h)$ and $||\nabla\varphi||_\infty \leq 3/(1-s)\lambda\varepsilon_h$. Define $w_{\varepsilon_h} := \varphi\, z_{\varepsilon_h} + (1-\varphi)v_{\varepsilon_h}$. Since $w_{\varepsilon_h} = v_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0, \lambda\,\varepsilon_h)$ we have by Remark 2.2.4

$$
\begin{aligned}
&\mathbf{m}(v_{\varepsilon_h}; Q_\nu(x_0, \lambda\,\varepsilon_h)) \leq \mathcal{F}(w_{\varepsilon_h}; Q_\nu(x_0, \lambda\,\varepsilon_h)) \leq \mathcal{F}(z_{\varepsilon_h}; Q_\nu(x_0, s\lambda\varepsilon_h)) \\
&+ C\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} (1 + |\nabla v_{\varepsilon_h}|^p)\, dx + \frac{C}{(1-s)^p(\lambda\varepsilon_h)^p}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h)} |v_{\varepsilon_h}(x) - u_{x_0,\nu}(x)|^p dx \\
&+ C\int_{(Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h))\cap S(w_{\varepsilon_h})} (1 + |[w_{\varepsilon_h}]|)\, d\mathcal{H}^{N-1} \\
&\leq \mathbf{m}(u_{x_0,\nu}; Q_\nu(x_0, s\lambda\varepsilon_h)) \\
&+ (\lambda\varepsilon_h)^N + C\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} (1 + |\nabla v_{\varepsilon_h}|^p)\, dx + \frac{C}{(1-s)^p(\lambda\varepsilon_h)^p}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} |v_{\varepsilon_h}(x) - u_{x_0,\nu}(x)|^p dx \\
&+ C\int_{(Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h))\cap S(w_{\varepsilon_h})} (1 + |[w_{\varepsilon_h}]|)\, d\mathcal{H}^{N-1},
\end{aligned}
$$

where we have used the facts that $||\nabla\varphi||_\infty \leq 3/(1-s)\lambda\varepsilon_h$. Dividing by $(\lambda\,\varepsilon_h)^{N-1}$ yields

$$
\begin{aligned}
\frac{\mathbf{m}(v_{\varepsilon_h}(x); Q_\nu(x_0, \lambda\,\varepsilon_h))}{(\lambda\,\varepsilon_h)^{N-1}} &\leq \frac{\mathbf{m}(u_{x_0,\nu}; Q_\nu(x_0, s\lambda\varepsilon_h))}{(\lambda\,\varepsilon_h)^{N-1}} \\
&+ \lambda\varepsilon_h + \frac{C}{(\lambda\,\varepsilon_h)^{N-1}}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} (1 + |\nabla v_{\varepsilon_h}|^p)\, dx \\
&+ \frac{C}{(1-s)^p(\lambda\varepsilon_h)^{N-1+p}}\int_{Q_\nu(x_0,\lambda\,\varepsilon_h)} |v_{\varepsilon_h}(x) - u_{x_0,\nu}(x)|^p dx \\
&+ \frac{C}{(\lambda\,\varepsilon_h)^{N-1}}\int_{(Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h))\cap S(w_{\varepsilon_h})} (1 + |[w_{\varepsilon_h}]|)\, d\mathcal{H}^{N-1}.
\end{aligned}
$$

We claim that the last three terms on the right hand side approach zero if we first let $\varepsilon_h \to 0^+$, then $s \to 1^-$ and finally $\lambda \to 1^-$. Indeed, for the first two integrals this follows from (2.1.15). To estimate the third integral, and since

$$
\mathcal{H}^{N-1}(S(v_{\varepsilon_h}) \setminus S(u)) = 0, \quad |[v_{\varepsilon_h}]| \leq |[u]| \quad \mathcal{H}^{N-1} \text{ a.e.},
$$

we have that if $x \in (S(w_{\varepsilon_h}) \cap S(u_{x_0,\nu}))\setminus S(u)$ then $|[w_{\varepsilon_h}]| \leq |[u_{x_0,\nu}]|$, and if $x \in S(w_{\varepsilon_h}) \cap S(u)$ then

$$
|[w_{\varepsilon_h}]| \leq |[u_{x_0,\nu}]| + |[u]| \leq C(1 + |[u]|).
$$

Here we have used the fact that (see (3.8) in [13]) if $u, v \in BV(\Omega)$ then for $\mathcal{H}^{N-1}$ a.e. $x \in S(u) \cap S(v)$ we have $\nu_u = \nu_v$ or $\nu_u = -\nu_v$, and, consequently, $|[u+v]| \leq |[u]| + |[v]|$. Hence

$$
\begin{aligned}
&\frac{1}{(\lambda\,\varepsilon_h)^{N-1}}\int_{(Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h))\cap S(w_{\varepsilon_h})} (1 + |[w_{\varepsilon_h}]|)\, d\mathcal{H}^{N-1} \\
&\leq \frac{C}{(\lambda\,\varepsilon_h)^{N-1}}\int_{(Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h))\cap S(u_{x_0,\nu})} (1 + |[u_{x_0,\nu}]|)\, d\mathcal{H}^{N-1} \\
&\quad + \frac{C}{(\lambda\,\varepsilon_h)^{N-1}}\int_{(Q_\nu(x_0,\lambda\,\varepsilon_h)\setminus Q_\nu(x_0,s\lambda\varepsilon_h))\cap S(u)} (1 + |[u]|)\, d\mathcal{H}^{N-1}.
\end{aligned}
$$

Both these terms can be treated as in (2.4.4). Finally, by (2.4.1), $(2.4.3)_1$, and (2.4.5),

$$
\begin{aligned}
\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\leq \liminf_{\lambda\to 1^-}\liminf_{s\to 1^-}\liminf_{\varepsilon_h\to 0^+} \frac{\mathbf{m}(u_{x_0,\nu}; Q_\nu(x_0, s\lambda\varepsilon_h))}{(\lambda\,\varepsilon_h)^{N-1}} \\
&\leq \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}(u_{x_0,\nu}; Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}} = g(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)).
\end{aligned}
$$

$\square$

We conclude this section by proving that equality holds in (2.4.1).

Lemma 2.4.2.
$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)).$$

Proof. The proof of this lemma is very similar to the previous one, and thus we will indicate only the main changes. Take $\varepsilon_h \to 0^+$ such that

$$g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)) = \limsup_{\varepsilon\to 0^+}\frac{\mathbf{m}(u_{x_0,\nu};Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} = \lim_{\varepsilon_h\to 0^+}\frac{\mathbf{m}(u_{x_0,\nu};Q_\nu(x_0,\varepsilon_h))}{\varepsilon_h^{N-1}}, \tag{2.4.6}$$

fix $0<s<1$, and let $z_{\varepsilon_h}\in SBV_p(Q_\nu(x_0,s\,\varepsilon_h);\mathbb{R}^d)$ be such that $z_{\varepsilon_h}=v_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0,s\varepsilon_h)$ and

$$\mathcal{F}(z_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h)) \leq \mathbf{m}(v_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h)) + \varepsilon_h^N.$$

Extend z_{ε_h} to be v_{ε_h} outside $Q_\nu(x_0,s\,\varepsilon_h)$ and consider a layer L of width $(1-s)\varepsilon_h/2$ and a smooth cut-off function φ such that $0<\varphi<1$ in L, $\varphi\equiv 1$ on $Q_\nu(x_0,s\varepsilon_h)$, $\varphi=0$ on $\partial Q_\nu(x_0,\varepsilon_h)$ and $||\nabla\varphi||_\infty \leq 3/(1-s)\varepsilon_h$. Define $w_{\varepsilon_h} := \varphi\, z_{\varepsilon_h} + (1-\varphi)u_{x_0,\nu}$. Since $w_{\varepsilon_h}=u_{x_0,\nu}$ on a neighborhood of $\partial Q_\nu(x_0,\varepsilon_h)$, we obtain, as in the second part of the proof of the previous lemma, and in view of Remark 2.2.4,

$$\begin{aligned}
&\frac{\mathbf{m}(u_{x_0,\nu}(x);Q_\nu(x_0,\varepsilon_h))}{\varepsilon_h^{N-1}} \leq \frac{\mathbf{m}(v_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))}{\varepsilon_h^{N-1}}\\
&+\varepsilon_h + \frac{C}{\varepsilon_h^{N-1}}\int_{Q_\nu(x_0,\varepsilon_h)}(1+|\nabla v_{\varepsilon_h}|^p)\,dx + \frac{C}{(1-s)^p\varepsilon_h^{N-1+p}}\int_{Q_\nu(x_0,\varepsilon_h)}|v_{\varepsilon_h}(x)-u_{x_0,\nu}(x)|^p dx\\
&+\frac{C}{\varepsilon_h^{N-1}}\int_{(Q_\nu(x_0,\varepsilon_h)\setminus Q_\nu(x_0,s\varepsilon_h))\cap S(w_{\varepsilon_h})}(1+|[w_{\varepsilon_h}]|)\,d\mathcal{H}^{N-1}.
\end{aligned}$$

As before, and also by (2.4.6), we may conclude that

$$g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)) \leq \liminf_{s\to 1^-}\liminf_{\varepsilon_h\to 0^+}\frac{\mathbf{m}(v_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))}{\varepsilon_h^{N-1}}. \tag{2.4.7}$$

Let $s_j\to 1^-$ be such that

$$\liminf_{s\to 1^-}\liminf_{\varepsilon_h\to 0^+}\frac{\mathbf{m}(v_{\varepsilon_h};Q_\nu(x_0,s\varepsilon_h))}{\varepsilon_h^{N-1}} = \lim_{s_j\to 1^-}\liminf_{\varepsilon_h\to 0^+}\frac{\mathbf{m}(v_{\varepsilon_h};Q_\nu(x_0,s_j\varepsilon_h))}{\varepsilon_h^{N-1}}, \tag{2.4.8}$$

and, by virtue of (2.4.3), choose a subsequence of $\{\varepsilon_h\}$ (not relabeled) and $\lambda\in(0,1)$ such that, for every $j\in\mathbb{N}$,

$$\begin{gathered}
\mu(\partial Q_\nu(x_0,s_j\lambda\varepsilon_h)) = \mathcal{H}^{N-1}(\partial Q_\nu(x_0,s_j\lambda\varepsilon_h)\cap S(\bar v_{\varepsilon_h}))=0,\\
\lim_{\varepsilon_h\to 0^+}\frac{1}{(s_j\varepsilon)^{N-1}}\int_{\partial Q_\nu(x_0,s_j\lambda\,\varepsilon_h)\cap\{u\neq v_{\varepsilon_h}\}}(1+|u-v_{\varepsilon_h}|)d\,\mathcal{H}^{N-1}=0.
\end{gathered} \tag{2.4.9}$$

Given $z_{\varepsilon_h}\in SBV_p(Q_\nu(x_0,s_j\varepsilon_h);\mathbb{R}^d)$ such that $z_{\varepsilon_h}=u$ on a neighborhood of $\partial Q_\nu(x_0,s_j\lambda\varepsilon_h)$ and

$$\mathcal{F}(z_{\varepsilon_h};Q_\nu(x_0,s_j\lambda\varepsilon_h)) \leq \mathbf{m}(u;Q_\nu(x_0,s_j\lambda\varepsilon_h)) + (s_j\varepsilon_h)^N,$$

define

$$w_{\varepsilon_h} := \begin{cases} z_{\varepsilon_h} & \text{in } Q_\nu(x_0,s_j\lambda\varepsilon_h),\\ v_{\varepsilon_h} & \text{in } Q_\nu(x_0,s_j\varepsilon_h)\setminus\overline{Q_\nu(x_0,s_j\lambda\varepsilon_h)}.\end{cases}$$

Since $w_{\varepsilon_h}=v_{\varepsilon_h}$ on a neighborhood of $\partial Q_\nu(x_0,s_j\varepsilon_h)$, we have by Remark 2.2.4,

$$\begin{aligned}
&\frac{\mathbf{m}(v_{\varepsilon_h};Q_\nu(x_0,s_j\varepsilon_h))}{(s_j\varepsilon_h)^{N-1}} \leq \frac{\mathbf{m}(u;Q_\nu(x_0,s_j\lambda\varepsilon_h))}{(s_j\varepsilon_h)^{N-1}} + s_j\varepsilon_h\\
&+\frac{C}{(s_j\varepsilon_h)^{N-1}}\int_{Q_\nu(x_0,s_j\varepsilon_h)}(1+|\nabla v_{\varepsilon_h}|^p)\,dx + \frac{C}{(s_j\varepsilon_h)^{N-1}}\int_{\partial Q_\nu(x_0,s_j\lambda\,\varepsilon_h)\cap\{u\neq v_{\varepsilon_h}\}}(1+|u-v_{\varepsilon_h}|)d\,\mathcal{H}^{N-1}\\
&+\frac{C}{(s_j\varepsilon_h)^{N-1}}\int_{(Q_\nu(x_0,s_j\varepsilon_h)\setminus Q_\nu(x_0,s_j\lambda\,\varepsilon_h))\cap S(v_{\varepsilon_h})}(1+|[v_{\varepsilon_h}]|)\,d\mathcal{H}^{N-1}.
\end{aligned}$$

As before, and by (2.4.7)–(2.4.9), we conclude that

$$g(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)) \leq \liminf_{s_j\to 1^-}\liminf_{\lambda\to 1^-}\liminf_{\varepsilon_h\to 0^+}\frac{\mathbf{m}(u;Q_\nu(x_0,s_j\lambda\varepsilon_h))}{(s_j\varepsilon_h)^{N-1}} = \frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0).$$

□

Remark 2.4.3. We remark that Lemma 2.4.1 and Lemma 2.4.2 are strongly hinged on the construction of the sequence $\{v_\varepsilon\}$ in Lemma 2.1.8, which, in turn, depends on the choice of sign for $\nu_u(x_0)$ (see 2.1.19). Had we taken the normal with opposite sign, we would have obtained (see 2.1.15)

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^{N-1+p}} \int_{Q_\nu(x_0,\varepsilon)} |v_\varepsilon(x) - u_{x_0,u^-(x_0),u^+(x_0),-\nu}(x)|^p\, dx = 0.$$

Lemmas 2.4.1 and 2.4.2 would now yield

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = g(x_0, u^+(x_0), u^-(x_0), -\nu_u(x_0)).$$

We conclude, therefore, that $g(x_0,\lambda,\theta,\nu) = g(x_0,\lambda,\theta,-\nu)$ for $\mathcal{L}^N$ a.e. $x\in\Omega$ and all λ, $\theta\in\mathbb{R}^d$, $\nu\in S^{N-1}$.

2.5 Proofs of Theorems 1.0.2 and 1.0.3.

Proof of Theorem 1.0.2. To prove Theorem 1.0.2 we replace $\mathbf{m}(u;A)$, as defined in (1.0.1), by $\mathbf{m}_1(u;A)$, which is defined in (1.0.7). Here we take $\mu := \mathcal{L}^N\lfloor\Omega$. Lemma 2.2.1 then continues to hold, with the exception that we need to show that the function v^δ defined in Lemma 3.3 of [17] now belongs to $W^{1,p}(\Omega;\mathbb{R}^d)$. Fix $\delta>0$ and let $\{Q_i^\delta\}$ be an admissible sequence, in the sense of the definition of $m_1^\delta(u;A)$, such that

$$\sum_{i=1}^\infty \mathbf{m}_1(u;Q_i^\delta) \le \mathbf{m}_1^\delta(u;A) + \delta. \tag{2.5.1}$$

By the definition of m_1 we can choose $v_i^\delta\in W^{1,p}(\Omega;\mathbb{R}^d)$ such that

$$v_i^\delta = u \quad \text{on } \partial Q_i^\delta, \qquad \mathcal{F}_1(v_i^\delta;Q_i^\delta) \le \mathbf{m}_1(u;Q_i^\delta) + \delta\,\mathcal{L}^N(Q_i^\delta). \tag{2.5.2}$$

For each $k\in\mathbb{N}$ set

$$v^{\delta,k} := \sum_{i=1}^k v_i^\delta \chi_{Q_i^\delta} + u\,\chi_{\Omega\setminus\cup_{j=1}^k Q_j^\delta}. \tag{2.5.3}$$

Then $v^{\delta,k}\in W^{1,p}(\Omega;\mathbb{R}^d)$, and by Poincaré's inequality, and since $v_i^\delta = u$ on ∂Q_i^δ, we deduce that

$$||v^{\delta,k} - u||^p_{W^{1,p}(\Omega)} \le \sum_{i=1}^k ||v_i^\delta - u||^p_{W^{1,p}(Q_i^\delta)} \le C(N,\delta)\sum_{i=1}^k ||\nabla v_i^\delta - \nabla u||^p_{L^p(Q_i^\delta)}$$

$$\le C(N,\delta)(\mathbf{m}_1^\delta(u;A) + \delta + ||\nabla u||^p_{L^p(A)}),$$

where we have used the coercitivity hypothesis (1.0.6) for $\mathcal{F}_1$, (2.5.1) and (2.5.2). Hence, for fixed δ, the sequence $\{v^{\delta,k}\}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^d)$, and thus it converges weakly to some function $v^\delta\in W^{1,p}(\Omega;\mathbb{R}^d)$. Clearly

$$v^\delta := \sum_{i=1}^\infty v_i^\delta\chi_{Q_i^\delta} + u\,\chi_{\Omega\setminus\cup_{j=1}^\infty Q_j^\delta}.$$

We may now proceed as in Lemma 3.3 of [17].

Lemma 2.2.3 of [17] continues to hold in the present context, since the function w defined in (2.2.1) clearly belongs to $W^{1,p}(\Omega;\mathbb{R}^d)$, as u and v coincide on a neighborhood of ∂A_δ (contained in A_δ). In turn, also Lemma 2.2.1 continues to hold.

The proofs of Lemmas 2.3.1 and 2.3.2 become significantly simpler in the Sobolev case. Indeed, since $\mathcal{H}^{N-1}(S(u)\cap\Omega) = 0$ then by Remark 2.1.2 the function $\bar{u}_\varepsilon$ defined in Lemma 2.1.7 coincides with u_ε. Consequently the first part of Lemma 2.3.1, up to (2.3.4), and the second part of Lemma 2.3.2, after (2.3.7), are not needed in this case.

□

Proof of Theorem 1.0.3. This time we replace $\mathbf{m}(u;A)$ by $\mathbf{m}_2(u;A)$, which is defined in (1.0.8). Clearly the functions $v^{\delta,k}$ defined in (2.5.3) belong to $BV(\Omega;T)$, and so does their limit v^δ as $k\to\infty$.

Lemmas 2.2.1 and 2.2.2 continue to hold, while Lemmas 2.4.1 and 2.4.2 should be replaced by Lemma 2.5.1 below. $\square$

Lemma 2.5.1.

$$\frac{d\mathcal{F}_2(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)=g_2(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)),$$

where g_2 is the function defined in Theorem 1.0.3.

Proof. Let x_0 satisfy (2.1.18), and set $\nu:=\nu_u(x_0)$. By Lemma 2.2.2 for $\mathcal{H}^{N-1}$ a.e. $x_0\in\Omega\cap S(u)$

$$\frac{d\mathcal{F}_2(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mathcal{F}_2(u;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}=\lim_{\varepsilon\to0^+}\frac{\mathbf{m}_2(u;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}<\infty.$$

Since T is finite,

$$d:=\min\{|u_1-u_2|:\,u_1,\,u_2\in T,\,u_1\neq u_2\}>0.$$

Hence, by (2.1.18),

$$\begin{aligned}0&\le\frac{1}{\varepsilon^{N-1}}\int_0^1\int_{\partial Q_\nu(x_0,\lambda\,\varepsilon)\cap\{u\neq u_{x_0,\nu}\}}(1+|u-u_{x_0,\nu}|)d\,\mathcal{H}^{N-1}d\,\lambda\\&\le\frac{1/d+1}{\varepsilon^N}\int_{Q_\nu(x_0,\varepsilon)}|u-u_{x_0,\nu}|\,dx\to0\quad\text{as }\varepsilon\to0^+.\end{aligned}$$

Thus for any sequence $\varepsilon_h\to0^+$ we may find a subsequence (not relabeled) such that for $\mathcal{L}^1$ a.e $\lambda\in(0,1)$

$$\mu(\partial Q_\nu(x_0,\lambda\varepsilon_h))=0,\qquad\lim_{\varepsilon_h\to0^+}\frac{1}{\varepsilon_h^{N-1}}\int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u\neq u_{x_0,\nu}\}}(1+|u-u_{x_0,\nu}|)d\,\mathcal{H}^{N-1}=0.\tag{2.5.4}$$

Fix a sequence $\varepsilon_h\to0^+$ and $\lambda\in(0,1)$ for which (2.5.4) holds. Given $z_{\varepsilon_h}\in BV(Q_\nu(x_0,\varepsilon_h);T)$ such that $z_{\varepsilon_h}=u_{x_0,\nu}$ on a neighborhood of $\partial Q_\nu(x_0,\lambda\varepsilon_h)$ and

$$\mathcal{F}_2(z_{\varepsilon_h};Q_\nu(x_0,\lambda\varepsilon_h))\le\mathbf{m}_2(u_{x_0,\nu};Q_\nu(x_0,\lambda\varepsilon_h))+{\varepsilon_h}^N,$$

define

$$w_{\varepsilon_h}:=\begin{cases}z_{\varepsilon_h}&\text{in }Q_\nu(x_0,\lambda\varepsilon_h),\\u&\text{in }Q_\nu(x_0,\varepsilon_h)\backslash\overline{Q_\nu(x_0,\lambda\varepsilon_h)}.\end{cases}$$

Since $w_{\varepsilon_h}=u$ on $\partial Q_\nu(x_0,\varepsilon_h)$ we have by Remark 2.2.4

$$\begin{aligned}&\mathbf{m}_2(u;Q_\nu(x_0,\varepsilon_h))\le\mathcal{F}_2(w_{\varepsilon_h};Q_\nu(x_0,\varepsilon_h))\le\mathcal{F}_2(z_{\varepsilon_h};Q_\nu(x_0,\lambda\,\varepsilon_h))\\&\quad+C\int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u\neq u_{x_0,\nu}\}}(1+|u-u_{x_0,\nu}|)d\,\mathcal{H}^{N-1}+C\int_{(Q_\nu(x_0,\varepsilon_h)\backslash Q_\nu(x_0,\lambda\,\varepsilon_h))\cap S(u)}(1+|[u]|)d\mathcal{H}^{N-1}\\&\le\mathbf{m}_2(u_{x_0,\nu};Q_\nu(x_0,\lambda\varepsilon_h))+{\varepsilon_h}^N\\&\quad+C\int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u\neq u_{x_0,\nu}\}}(1+|u-u_{x_0,\nu}|)d\mathcal{H}^{N-1}+C\int_{(Q_\nu(x_0,\varepsilon_h)\backslash Q_\nu(x_0,\lambda\,\varepsilon_h))\cap S(u)}(1+|[u]|)\,d\mathcal{H}^{N-1}.\end{aligned}$$

Dividing by ${\varepsilon_h}^{N-1}$ yields

$$\begin{aligned}\frac{\mathbf{m}_2(u;Q_\nu(x_0,\varepsilon_h))}{{\varepsilon_h}^{N-1}}\le&\frac{\mathbf{m}_2(u_{x_0,\nu};Q_\nu(x_0,\lambda\varepsilon_h))}{{\varepsilon_h}^{N-1}}+\varepsilon_h\\&+\frac{C}{{\varepsilon_h}^{N-1}}\int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u\neq u_{x_0,\nu}\}}(1+|u-u_{x_0,\nu}|)d\mathcal{H}^{N-1}\\&+\frac{C}{{\varepsilon_h}^{N-1}}\int_{(Q_\nu(x_0,\varepsilon_h)\backslash Q_\nu(x_0,\lambda\,\varepsilon_h))\cap S(u)}(1+|[u]|)d\mathcal{H}^{N-1}.\end{aligned}$$

We can now proceed as in the first part of Lemma 2.4.1 to conclude that

$$\frac{d\mathcal{F}_2(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)\le\liminf_{\lambda\to1^-}\liminf_{\varepsilon_h\to0^+}\frac{\mathbf{m}_2(u_{x_0,\nu};Q_\nu(x_0,\lambda\varepsilon_h))}{{\varepsilon_h}^{N-1}}\le g_2(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)).$$

To prove the converse inequality take $\varepsilon_h \to 0^+$ such that

$$g_2(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) = \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}_2(u_{x_0,\nu}; Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} = \lim_{\varepsilon_h\to 0^+} \frac{\mathbf{m}_2(u_{x_0,\nu}; Q_\nu(x_0,\varepsilon_h))}{\varepsilon_h^{N-1}},$$

Extract a subsequence of $\{\varepsilon_h\}$ (not relabeled) and fix $\lambda \in (0,1)$ for which (2.5.4) holds. Choosing admissible test functions $z_{\varepsilon_h} \in BV(Q_\nu(x_0,\varepsilon_h); T)$ such that $z_{\varepsilon_h} = u$ on a neighborhood of $\partial Q_\nu(x_0, \lambda\varepsilon_h)$ and

$$\mathcal{F}_2(z_{\varepsilon_h}; Q_\nu(x_0,\lambda\varepsilon_h)) \le \mathbf{m}_2(u; Q_\nu(x_0,\lambda\varepsilon_h)) + \varepsilon_h^N,$$

define

$$w_{\varepsilon_h} := \begin{cases} z_{\varepsilon_h} & \text{in } Q_\nu(x_0,\lambda\varepsilon_h), \\ u_{x_0,\nu} & \text{in } Q_\nu(x_0,\varepsilon_h)\backslash\overline{Q_\nu(x_0,\lambda\varepsilon_h)}. \end{cases}$$

Since $w_{\varepsilon_h} = u_{x_0,\nu}$ on $\partial Q_\nu(x_0,\varepsilon_h)$ we have by Remark 2.2.4

$$\begin{aligned} \frac{\mathbf{m}_2(u_{x_0,\nu}; Q_\nu(x_0,\varepsilon_h))}{\varepsilon_h^{N-1}} \le & \frac{\mathbf{m}_2(u; Q_\nu(x_0,\lambda\varepsilon_h))}{\varepsilon_h^{N-1}} + \varepsilon_h \\ & + \frac{C}{\varepsilon_h^{N-1}} \int_{\partial Q_\nu(x_0,\lambda\,\varepsilon_h)\cap\{u\neq u_{x_0,\nu}\}} (1 + |u - u_{x_0,\nu}|)\, d\mathcal{H}^{N-1} \\ & + \frac{C}{\varepsilon_h^{N-1}} \int_{(Q_\nu(x_0,\varepsilon_h)\backslash Q_\nu(x_0,\lambda\,\varepsilon_h))\cap S(u_{x_0,\nu})} (1 + |[u_{x_0,\nu}]|)\, d\mathcal{H}^{N-1}. \end{aligned}$$

As before, we conclude that

$$g_2(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) \le \liminf_{\lambda\to 1^-} \liminf_{\varepsilon_h\to 0^+} \frac{\mathbf{m}_2(u; Q_\nu(x_0,\lambda\varepsilon_h))}{\varepsilon_h^{N-1}} = \frac{d\mathcal{F}_2(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0).$$

□

3 Part II: Applications

3.1 Preliminary Results.

In the next theorem we use the notation $x = (x', x_N) \in \mathbb{R}^N$ and $\nu = (\nu', \nu_N) \in S^{N-1}$, where x', $\nu' \in \mathbb{R}^{N-1}$ and x_N, $\nu_N \in \mathbb{R}$.

Theorem 3.1.1 (Slicing Theorem). *Let $u \in SBV_p(\mathbb{R}^N; \mathbb{R}^d)$. Then for $\mathcal{H}^1$ a.e. $t \in \mathbb{R}$ the function $u^t(x') := u(x', t) \in SBV_p(\mathbb{R}^{N-1}; \mathbb{R}^d)$ with $\nabla_{x'} u^t(x') = \nabla_{x'} u(x', t)$,*

$$S(u^t) = \left\{ x' \in \mathbb{R}^{N-1} : (x', t) \in S(u) \right\},$$

and if $\mathcal{H}^{N-2}(S(u^t)) > 0$ then for $\mathcal{H}^{N-2}$ a.e. $x' \in S(u^t)$ we have $(u^t(x'))^\pm = u^\pm(x', t)$ and

$$\nu'_{u^t(x')} = \frac{\nu'_{u(x',t)}}{|\nu'_{u(x',t)}|}.$$

Moreover, if $g : \mathbb{R}^N \to [0, +\infty)$ is a Borel measurable function, then for every open set $A \subset \mathbb{R}^N$ we have

$$\int_A g(x)\, d\mu(x) \ge \int_{\mathbb{R}} \left(\int_{A_t} g(x', t) d\mu_t(x') \right) dt, \tag{3.1.1}$$

where

$$A_t := \left\{ x' \in \mathbb{R}^{N-1} : (x', t) \in A \right\},$$

$\mu := C_1\,|Du| + C_2\mathcal{H}^{N-1}\lfloor S(u)$, $\mu_t := C_1\,|D_{x'} u_t| + C_2\mathcal{H}^{N-2}\lfloor S(u_t)$, *with C_1, $C_2 \ge 0$. In particular*

$$\int_{S(u)\cap A} g(x)\, d\mathcal{H}^{N-1} \ge \int_{\mathbb{R}} \left(\int_{S(u^t)\cap A_t} g(x', t) d\mathcal{H}^{N-2}(x') \right) dt. \tag{3.1.2}$$

Proof. A detailed proof of the first part of the theorem in the BD (Bounded Deformation) setting may be found in Proposition 3.4 and in the proof of Theorem 4.5 of [10]. The BV case is analogous and rests on Theorem 3.3 of [44] (see also page 204 of [12]). To prove the second part note that, following the proof of Theorem 3.3 of [44], it may be shown that

$$\int_A |Du| \geq \int_A |D_{x'}u| = \int_{\mathbb{R}} \left(\int_{A_t} |D_{x'}u_t| \right) dt. \tag{3.1.3}$$

This inequality is straightforward if $u \in C^1(\mathbb{R}^N; \mathbb{R}^d)$, and the general case where $u \in BV(\mathbb{R}^N; \mathbb{R}^d)$ follows by a regularization argument.

By rewriting (3.1.3) as

$$\int_{\mathbb{R}^N} \chi_A(x)\, d|Du|(x) \geq \int_{\mathbb{R}} \left(\int_{\{x_n=t\}} \chi_A(x) d|D_{x'}u_t|(x') \right) dt,$$

one may extend the validity of (3.1.3) to characteristic functions of Borel sets $B \subset \mathbb{R}^N$, to simple functions and then to any non negative Borel measurable function h. Hence we have

$$\int_A h(x)\, d|Du|(x) \geq \int_{\mathbb{R}} \left(\int_{A_t} h(x', t) d|D_{x'}u_t|(x') \right) dt. \tag{3.1.4}$$

Let now $g : \mathbb{R}^N \to [0, +\infty)$ be a Borel measurable function, and set

$$h(x) := \frac{g(x)}{|[u(x)]|} \chi_{S(u)\cap A}(x).$$

Applying (3.1.4) to the function g, we can see easily that the left-hand side of the inequality reduces to

$$\int_{S(u)\cap A} g(x)\, d\mathcal{H}^{N-1},$$

while the right-hand side reads

$$\int_{\mathbb{R}} \left(\int_{A_t} \frac{g(x', t)}{|[u(x', t)]|} \chi_{S(u)\cap A}(x', t) d|D_{x'}u_t|(x') \right) dt.$$

Recalling that

$$\chi_{S(u)\cap A}(x', t) = \chi_{S(u^t)\cap A_t}(x'),$$
$$d|D_{x'}u_t| = \left|\nabla_{x'}u^t(x')\right| \mathcal{L}^{N-1} + |[u^t(x')]| \mathcal{H}^{N-2} \lfloor S(u^t),$$

and that

$$|[u(x', t)]| = |[u^t(x')]| \,\text{for}\, \mathcal{H}^{N-2} \text{ a.e. } x' \in S(u^t),$$

we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{A_t} \frac{g(x', t)}{|[u(x', t)]|} \chi_{S(u)\cap A}(x', t) d|D_{x'}u_t|(x') \right) dt \\
&\qquad = \int_{\mathbb{R}} \left(\int_{S(u^t)\cap A_t} \frac{g(x', t)}{|[u^t(x')]|} \left(\left|\nabla_{x'}u^t(x')\right| dx' + |[u^t(x')]|\, d\mathcal{H}^{N-2} \right) \right) dt \\
&\qquad = \int_{\mathbb{R}} \left(\int_{S(u^t)\cap A_t} g(x', t) d\mathcal{H}^{N-2}(x') \right) dt.
\end{aligned}$$

This establishes (3.1.2), which, together with (3.1.4) now yields (3.1.1).

□

We recall that a locally bounded, Borel measurable function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is said to be *quasiconvex* if

$$\int_{(0,1)^N} f\left(\xi + \nabla\varphi(y)\right)\, dy \geq f(\xi)$$

for all $\xi \in \mathbb{R}^{d\times N}$, $\varphi \in W_0^{1,\infty}((0,1)^N; \mathbb{R}^d)$. The *quasiconvex envelope* $\mathcal{Q}f$ of a function $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is defined as

$$\mathcal{Q}f := \sup\{G : \ G \text{ is quasiconvex, } G \leq f\}.$$

If f is Borel measurable and $0 \le f(\xi) \le C(1+|\xi|^p)$ for some $p \ge 1$ then it can be shown that

$$\mathcal{Q}f(\xi) = \inf\left\{ \int_{(0,1)^N} f\left(\xi + \nabla\varphi(y)\right)\,dy : \ \varphi \in W_0^{1,\infty}((0,1)^N;\mathbb{R}^d) \right\}.$$

For a detailed study of quasiconvexity we refer the reader to [30].

We say that a locally bounded, Borel function $g : \mathbb{R}^{2d} \times S^{N-1} \to \mathbb{R}$ is *BV-elliptic* if for all $(a,b,\nu) \in \mathbb{R}^{2d} \times S^{d-1}$ and for any *finite* subset T of $\mathbb{R}^d$ we have

$$\int_{S(w)\cap Q_\nu} g(w^+, w^-, \nu_w)\,d\mathcal{H}^{N-1} \ge g(a,b,\nu)$$

for all $w \in BV(Q_\nu;T)$ such that $w = u_{a,b,\nu}$ on ∂Q_ν, where $u_{a,b,\nu} := u_{0,a,b,\nu}$, i.e.

$$u_{a,b,\nu}(x) := \begin{cases} a & \text{if } x\cdot\nu > 0, \\ b & \text{if } x\cdot\nu \le 0. \end{cases}$$

Given a function $g : \mathbb{R}^{2d} \times S^{N-1} \to \mathbb{R}$ we define $\mathcal{R}g$ as the greatest BV-elliptic function less than g. If g is continuous and satisfies a growth condition of the type (1.0.9) then it can be shown that (see [22], [29])

$$\mathcal{R}g(a,b,\nu) = \inf\left\{ \int_{S(w)\cap Q_\nu} g(w^+, w^-, \nu_w)\,d\mathcal{H}^{N-1} : \ w \in SBV_0(Q_\nu;\mathbb{R}^d) \cap L^\infty(Q_\nu;\mathbb{R}^d), \ w = u_{a,b,\nu} \text{ on } \partial Q_\nu \right\},$$

where, we recall, the space $SBV_0(\Omega;\mathbb{R}^d)$ is defined by

$$SBV_0(\Omega;\mathbb{R}^d) := \left\{ w \in SBV(\Omega;\mathbb{R}^d) : \ \nabla w = 0 \text{ a.e. in } \Omega \text{ and } \mathcal{H}^{N-1}(S(w)\cap\Omega) < \infty \right\}.$$

We say that $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty]$ is a *normal integrand* if for $\mathcal{L}^N$ a.e. $x \in \Omega$ the function $f(x,\cdot,\cdot)$ is lower semicontinuous and there exists a Borel integrand $\tilde{f} : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty]$ such that $f(x,\cdot,\cdot) = \tilde{f}(x,\cdot,\cdot)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$.

The following lower semicontinuity result was proved by Ambrosio in [6] and [7] in the case where the integrand f_0 is Carathéodory and g_0 continuous, and by Kristensen [42] when f_0 is a normal integrand. For the convenience of the reader, we present a slightly modified version of Ambrosio's proof in the Appendix.

Theorem 3.1.2. *Assume that $f_0 : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ is a normal integrand, quasiconvex in ξ, and that $g_0 : \Omega \times (\mathbb{R}^d)^2 \times S^{N-1} \to [0,+\infty)$ is a BV-elliptic function, lower semicontinuous in x and continuous in the remaining variables. Suppose also that the following growth conditions are satisfied:*

$$0 \le f_0(x,u,\xi) \le C(1+|\xi|^p) \quad \textit{and} \quad 0 \le g_0(x,\lambda,\theta,\nu) \le C\left(1+|\lambda-\theta|\right) \tag{3.1.5}$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$, all u, λ, $\theta \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$, and for some $C > 0$. Then for every $u \in SBV_p(\Omega;\mathbb{R}^d)$ and any sequence $\{u_n\} \subset SBV_p(\Omega;\mathbb{R}^d)$ converging to u in $L^1(\Omega;\mathbb{R}^d)$ and such that

$$\sup_n \left(\int_\Omega |\nabla u_n|^p\,dx + \int_{S(u_n)\cap\Omega} \left(1+|[u_n]|\right)\,d\mathcal{H}^{N-1} \right) < \infty, \tag{3.1.6}$$

we have

$$\int_\Omega f_0(x,u,\nabla u)\,dx + \int_{S(u)\cap\Omega} g_0(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} \le \liminf_{n\to\infty} \left(\int_\Omega f_0(x,u_n,\nabla u_n)\,dx + \int_{S(u_n)\cap\Omega} g_0(x,u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1} \right).$$

Consider a functional $F : SBV_p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$. For any $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ define

$$\begin{aligned} \mathcal{J}(u;A) &:= \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F(u_n;A) : \ u_n \in SBV_p(A;\mathbb{R}^d), \ u_n \to u \text{ in } L^1(A;\mathbb{R}^d) \right\}, \\ \mathcal{J}_p(u;A) &:= \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F(u_n;A) : \ u_n \in SBV_p(A;\mathbb{R}^d), \ u_n \to u \text{ in } L^p(A;\mathbb{R}^d) \right\}. \end{aligned} \tag{3.1.7}$$

Lemma 3.1.3. *Let $F : SBV_p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$ be a functional satisfying conditions (H_1), (H_2) and (H_4). Then for any $u \in SBV_p(\Omega;\mathbb{R}^d) \cap L^p(\Omega;\mathbb{R}^d)$*

$$\mathcal{J}_p(u;A) = \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F(u_n;A) : \; u_n \in SBV_p(A;\mathbb{R}^d),\, u_n \to u \text{ in } L^p(A;\mathbb{R}^d), \;\; u_n = u \text{ on a neighborhood of } \partial A \right\}.$$

Proof. Fix $u \in SBV_p(\Omega;\mathbb{R}^d)$ and write

$$\mathcal{J}_p^*(u;A) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F(u_n;A) : \; u_n \in SBV_p(A;\mathbb{R}^d),\, u_n \to u \text{ in } L^p(A;\mathbb{R}^d), \;\; u_n = u \text{ on a neighborhood of } \partial A \right\}.$$

It is clear that $\mathcal{J}_p(u;A) \leq \mathcal{J}_p^*(u;A)$. We now prove the converse inequality. For every $v \in SBV_p(\Omega;\mathbb{R}^d)$ define

$$\mathcal{G}_p(v;A) := \int_A (1 + |\nabla v|^p)\, dx + \int_{A\cap S(v)} (1 + |[v]|)\, d\mathcal{H}^{N-1}.$$

Given $\varepsilon > 0$ and consider a sequence $\{u_n\}$ in $SBV_p(\Omega;\mathbb{R}^d)$ such that $u_n \to u$ in $L^p(A;\mathbb{R}^d)$ and

$$\liminf_{n\to+\infty} F(u_n;A) \leq \mathcal{J}_p(u;A) + \varepsilon.$$

Due to the coercivity hypothesis (H_4), we may extract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\lim_{k\to+\infty} F(u_{n_k};A) = \liminf_{n\to+\infty} F(u_n;A) \tag{3.1.8}$$

and the sequence of measures $\nu_k := \mathcal{G}_p(u;\cdot) + \mathcal{G}_p(u_{n_k};\cdot)$ converges $\star$-weakly to some Radon measure ν. Denoting, for every $t > 0$, $A_t := \{x \in A |\ \mathrm{dist}(x,\partial A) > t\}$, we fix some $\eta > 0$ and for every $0 < \delta < \eta$ we define the subsets $L_\delta := A_{\eta-2\delta} \setminus \overline{A_{\eta+\delta}}$. Consider a smooth cut-off function $\varphi_\delta \in \mathcal{C}_0^\infty(A_{\eta-\delta};[0,1])$ such that $\varphi_\delta = 1$ on A_η. As the thickness of the strip L_δ is of order δ, we have an upper bound of the form $\|\nabla\varphi_\delta\|_{L^\infty(A_{\eta-\delta})} \leq C/\delta$. Define

$$w_n := u_n\varphi_\delta + u(1-\varphi_\delta).$$

Clearly this sequence converges to u in $L^p(A;\mathbb{R}^d)$ and satisfies $w_n = u$ on ∂A. On the other hand, by (H_1), the locality hypothesis (H_2), and the growth condition (H_4), we have the estimate

$$\begin{aligned} F(w_{n_k};A) &\leq F(w_{n_k};A_\eta) + F(w_{n_k};A\setminus\overline{A_{\eta-\delta}}) + F(w_{n_k};A_{\eta-2\delta}\setminus\overline{A_{\eta+\delta}}) \\ &\leq F(u_{n_k};A_\eta) + F(u;A\setminus\overline{A_{\eta-\delta}}) + C\left(\mathcal{G}_p(u_{n_k};L_\delta) + \mathcal{G}_p(u;L_\delta) + \frac{1}{\delta^p}\int_{L_\delta} |v_n - u|^p\ dx\right). \end{aligned} \tag{3.1.9}$$

Then by (3.1.8)

$$\begin{aligned} \liminf_{k\to\infty} F(w_{n_k};A) &\leq \lim_{k\to\infty} F(u_{n_k};A) + C\nu(A\setminus\overline{A_{\eta-\delta}}) + C\nu(\overline{L_\delta}) \\ &\leq \mathcal{J}_p(u;A) + \varepsilon + C\nu(u;A\setminus\overline{A_{\eta-\delta}}) + C\nu(\overline{L_\delta}). \end{aligned}$$

Letting δ go to zero we obtain

$$\mathcal{J}_p^*(u;A) \leq \mathcal{J}_p(u;A) + \varepsilon + C\nu(A\setminus A_\eta) + C\nu(\partial A_\eta).$$

Choose a subsequence $\{\eta_k\}$ such that $\eta_k \to 0^+$ and $\nu(\partial A_{\eta_k}) = 0$. By letting first $k \to \infty$ and then $\varepsilon \to 0^+$ we conclude that $\mathcal{J}_p^*(u;A) \leq \mathcal{J}_p(u;A)$. ☐

Remark 3.1.4. Given a sequence of functionals $F_n : SBV_p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$, for any $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ define Γ^- and Γ_p^- as the Γ-lower limits of the sequence $\{F_n\}$ with respect to the $L^1(A;\mathbb{R}^d)$ and $L^p(A;\mathbb{R}^d)$ topology, respectively, that is

$$\begin{aligned} \Gamma^-(u;A) &:= \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F_n(u_n;A) : \; u_n \in SBV_p(A;\mathbb{R}^d),\, u_n \to u \text{ in } L^1(A;\mathbb{R}^d) \right\}, \\ \Gamma_p^-(u;A) &:= \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} F_n(u_n;A) : \; u_n \in SBV_p(A;\mathbb{R}^d),\, u_n \to u \text{ in } L^p(A;\mathbb{R}^d) \right\}. \end{aligned} \tag{3.1.10}$$

Assume that the sequence $\{F_n\}$ satisfies conditions (H_1), (H_2) and (H_4) uniformly in n. Then Lemma 3.1.3 still holds for $\mathcal{H}_p$.

The next lemma is essentially due to Braides, Defranceschi and Vitali [22] and is based on De Giorgi's Slicing Lemma (see also [15] and [28]). In what follows we use the notation.

$$\partial_l u := \frac{\partial u}{\partial x_l}, \quad l = 1, \dots, N.$$

Lemma 3.1.5. *Let $F : SBV_p(\Omega;\mathbb{R}^d) \times \mathcal{A}(\Omega) \to [0,\infty]$ be a functional satisfying condition (H_1). Assume that there exist C, $C_1, \dots C_N$, $D_1, \dots D_N > 0$ such that*

$$0 \le F(u;A) \le C\Big(|A| + \sum_{l=1}^N C_l \int_A |\partial_l u|^p\,dx + \sum_{l=1}^N D_l \int_{S(u)\cap A} (1+|[u]|)|\nu_l(u)|\,d\mathcal{H}^{N-1}\Big)$$

for every $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \times\mathcal{A}(\Omega)$, and for every u, $v \in SBV_p(\Omega;\mathbb{R}^d)$

$$F(u;B) = F(v;B) \quad \text{whenever } u = v \ \mathcal{H}^{N-1} \text{ a.e. on a Borel set } B \subset \Omega. \tag{3.1.11}$$

Given j integers $\{\lambda_1, \dots, \lambda_j\}$, with $0 < \lambda_1 < \dots < \lambda_j$, for any $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we may find an integer $i \in \{1, \dots, j-1\}$ such that

$$\begin{aligned} &F(\tau_i(u);A) \le F(u;A) \\ &\quad + \frac{C}{j-1}\Big(|A| + \sum_{l=1}^N C_l \int_A |\partial_l u|^p\,dx + \sum_{l=1}^N D_l \int_{S(u)\cap A} (1+|[u]|)|\nu_l(u)|\,d\mathcal{H}^{N-1}\Big) + \frac{C}{e^{\lambda_1}}\int_A |u|\,dx, \end{aligned}$$

where $\tau_i : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth truncation function such that

$$\tau_i(z) := \begin{cases} z & \text{if } |z| \le e^{\lambda_i}, \\ 0 & \text{if } |z| > e^{\lambda_{i+1}}. \end{cases}$$

and $|\nabla \tau_i(z)| \le C$ for all $z \in \mathbb{R}^d$.

Proof. Given $v \in SBV_p(\Omega;\mathbb{R}^d)$ define

$$\mathcal{G}(v;A) := |A| + \sum_{l=1}^N C_l \int_A |\partial_l v|^p\,dx + \sum_{l=1}^N D_l \int_{S(u)\cap A} (1+|[u]|)|\nu_l(u)|\,d\mathcal{H}^{N-1}. \tag{3.1.12}$$

For each integer $i \in \{1, \dots, j-1\}$ set $w_i(x) := \tau_i(u(x))$; then the following formula holds (see e.g. [11])

$$Dw_i = \nabla\tau_i(u)\,\nabla u\,\mathcal{L}^N + \big(\tau_i(u^+) - \tau_i(u^-)\big) \otimes \mathcal{H}^{N-1}\lfloor S(u).$$

Since $|\nabla\tau_i(z)| \le C$, for every $l = 1, \dots, N$ we have that $|\partial_l w_i(x)| \le C|\partial_l u|\ \mathcal{L}^N$ a.e. on Ω and $|\tau_i(u^+) - \tau_i(u^-)| \le C|u^+ - u^-|$. Using the facts that $\tau_i(u(x)) = u(x)\ \mathcal{H}^{N-1}$ a.e. in the Borel set $\{x \in A : |u| \le e^{\lambda_i}\}$ and that in the Borel set $\{x \in A : |u| \ge e^{\lambda_{i+1}}\}$ we have $\nabla\tau_i(u(x)) = 0\ \mathcal{L}^N$ a.e. and $\tau_i(u(x)) = 0\ \mathcal{H}^{N-1}$ a.e., we get

$$\begin{aligned} F(w_i;A) &= F(w_i;\{x \in A : |u| \le e^{\lambda_i}\}) + F(w_i;\{x \in A : e^{\lambda_i} < |u| < e^{\lambda_{i+1}}\}) \\ &\quad + F(w_i;\{x \in A : |u| \ge e^{\lambda_{i+1}}\}) \\ &\le F(u;A) + C\mathcal{G}(u;\{x \in A : e^{\lambda_i} < |u| < e^{\lambda_{i+1}}\}) + C\Big|\{x \in A : |u| \ge e^{\lambda_{i+1}}\}\Big|. \end{aligned}$$

Since

$$\Big|\{x \in A : |u| \ge e^{\lambda_{i+1}}\}\Big| \le \frac{1}{e^{\lambda_{i+1}}}\int_A |u|\,dx$$

we have

$$\frac{1}{j-1}\sum_{i=1}^{j-1} F(w_i;A) \le F(u;A) + \frac{C}{j-1}\mathcal{G}(u;A) + \frac{C}{e^{\lambda_j}}\int_A |u|\,dx.$$

It now suffices to choose $i_j \in \{1, \dots, j\}$ such that

$$F(w_{i_j};A) \le \frac{1}{j-1}\sum_{i=1}^{j-1} F(w_i;A).$$

$\square$

Remark 3.1.6. The previous proof continues to work if instead of positive integers we consider a set of positive real numbers $\{\lambda_1, \dots, \lambda_j\}$, with $0 < \lambda_1 < \dots < \lambda_j$ and such that $\lambda_{i+1} - \lambda_i \geq \alpha > 0$ for all $i \in \{1, \dots, j-1\}$. In this case $||\nabla \tau_i(z)||_\infty \leq C(\alpha)$.

Lemma 3.1.7. (cf. [15]). *Let F satisfy conditions (H_1), (3.1.11). Further assume that*

$$\begin{aligned}\frac{1}{C}\bigg(\sum_{l=1}^{N} C_l \int_A |\partial_l u|^p \,dx + \sum_{l=1}^{N} D_l \int_{S(u)\cap A} (1+|[u]|)|\nu_l(u)|\, d\mathcal{H}^{N-1}\bigg) \\ \leq F(u;A) \leq C\bigg(|A| + \sum_{l=1}^{N} C_l \int_A |\partial_l u|^p \,dx + \sum_{l=1}^{N} D_l \int_{S(u)\cap A} (1+|[u]|)|\nu_l(u)|\, d\mathcal{H}^{N-1}\bigg)\end{aligned} \tag{3.1.13}$$

for every $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \times\mathcal{A}(\Omega)$, and for some positive constants C, $C_1, \dots C_N$, $D_1, \dots D_N > 0$. Then for any $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$ we have

$$\mathcal{J}(u;A) = \lim_{j\to\infty} \mathcal{J}(\tau_j(u);A),$$

where

$$\tau_j(z) := \begin{cases} z & \text{if } \ |z| \leq e^j, \\ 0 & \text{if } \ |z| > e^{j+1}. \end{cases}$$

Moreover, for any $u \in SBV_p(\Omega;\mathbb{R}^d) \cap L^\infty(\Omega;\mathbb{R}^d)$ we have

$$\mathcal{J}(u;A) = \mathcal{J}_p(u;A) = \inf_{\{v_n\}}\bigg\{\liminf_{n\to\infty} F(v_n;A) : \, v_n \in SBV_p(A;\mathbb{R}^d),\, \sup_n \|v_n\|_{L^\infty(A;\mathbb{R}^d)} < \infty,\, v_n \to u \text{ in } L^p(A;\mathbb{R}^d)\bigg\}.$$

Proof. Fix an integer $j \in \mathbb{N}$ and let $\{u_m\} \subset SBV_p(A;\mathbb{R}^d)$ be a sequence converging to u in $L^1(A;\mathbb{R}^d)$ and such that

$$\liminf_{m\to+\infty} F(u_m;A) \leq \mathcal{J}(u;A) + \frac{1}{j}.$$

Upon extracting a subsequence $\{u_{m_n}\}$ of $\{u_m\}$, if necessary, we may assume that

$$\lim_{n\to+\infty} F(u_{m_n};A) = \liminf_{m\to+\infty} F(u_m;A) < \infty. \tag{3.1.14}$$

For simplicity in the notation we write u_n instead of u_{m_n}. By (3.1.13) and the fact that $u_n \to u$ in $L^1(A;\mathbb{R}^d)$, we may find a constant $M > 0$ (independent of j) such that

$$\sup_n \bigg(\int_A |u_n|\,dx + \mathcal{G}(u_n;A)\bigg) \leq M,$$

where $\mathcal{G}$ has been introduced in (3.1.12). Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that $\lambda_n \to \infty$. Applying Lemma 3.1.5 with the integers $\{\lambda_{j+1}, \dots, \lambda_{2j}\}$ to each u_n, we may find $i_n \in \{j+1, \dots, 2j-1\}$ such that

$$F(\tau_{i_n}(u_n);A) \leq F(u_n;A) + \frac{C}{j-1}\mathcal{G}(u_n;A) + \frac{C}{e^{\lambda_{j+1}}}\int_A |u_n|\,dx \leq F(u_n;A) + \frac{CM}{j-1} + \frac{CM}{e^{\lambda_{j+1}}},$$

where

$$\tau_{i_n}(z) := \begin{cases} z & \text{if } \ |z| \leq e^{\lambda_{i_n}}, \\ 0 & \text{if } \ |z| > e^{\lambda_{i_n+1}}. \end{cases}$$

Since $i_n \in \{j+1, \dots, 2j-1\}$ for all $n \in \mathbb{N}$, there must exist an integer $l_j \in \{j+1, \dots, 2j-1\}$ such that $i_n = l_j$ for infinitely indices $n \in \mathbb{N}$. Select, therefore, an increasing sequence $\{n_k\}$ of integers such that $i_{n_k} = l_j$ for all k. Then by (3.1.14)

$$\begin{aligned}\liminf_{n\to\infty} F(\tau_{l_j}(u_n);A) &\leq \liminf_{k\to\infty} F(\tau_{l_j}(u_{n_k});A) \\ &\leq \lim_{k\to\infty} F(u_{n_k};A) + \frac{CM}{j-1} + \frac{CM}{e^{\lambda_{j+1}}} \\ &\leq \mathcal{J}(u;A) + \frac{1}{j} + \frac{CM}{j-1} + \frac{CM}{e^{\lambda_{j+1}}}.\end{aligned} \tag{3.1.15}$$

Clearly $\{\tau_{l_j}(u_n)\}$ converges in $L^1(A;\mathbb{R}^d)$ to $\tau_{l_j}(u)$, and a standard diagonalization argument shows that $\mathcal{J}(\cdot;A)$ is L^1-lower semicontinuous, and so, by (3.1.15),

$$\mathcal{J}(\tau_{l_j}(u);A) \leq \liminf_{n\to\infty} \mathcal{J}(\tau_{l_j}(u_n);A) \leq \liminf_{n\to\infty} F(\tau_{l_j}(u_n);A) \leq \mathcal{J}(u;A) + \frac{1}{j} + \frac{CM}{j-1} + \frac{CM}{e^{\lambda_{j+1}}}.$$

Since $\{\tau_{l_j}(u)\}$ converges to u in $L^1(A;\mathbb{R}^d)$ as $j\to\infty$, we get

$$\mathcal{J}(u;A) \leq \liminf_{j\to\infty} \mathcal{J}(\tau_{l_j}(u);A) \leq \limsup_{j\to\infty} \mathcal{J}(\tau_{l_j}(u);A) \leq \mathcal{J}(u;A).$$

Since $\{\lambda_n\}$ was arbitrary, this proves the first part of the Lemma.

To prove the second part of the lemma, assume that $u \in SBV_p(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$, consider an increasing sequence of positive integers $\{\lambda_n\}$ such that $\lambda_n \to \infty$ and fix $j \in \mathbb{N}$ such that $e^{\lambda_j} > \|u\|_{L^\infty(A;\mathbb{R}^d)} + 1$. Note that $\tau_{i_n}(u(x)) \equiv u(x)$ for all $x \in A$ and that, by construnction, $\|\tau_{i_n}(u_n)\|_{L^\infty(A;\mathbb{R}^d)} \leq e^{\lambda_{2j}}$. Using the fact that $|\nabla \tau_{i_n}(z)| \leq C$ for all $z \in \mathbb{R}^d$, we get

$$\begin{aligned}|\tau_{i_n}(u_n) - u|^p = |\tau_{i_n}(u_n) - \tau_{i_n}(u)|^p &\leq \left(1 + |\tau_{i_n}(u_n)|^{p-1} + |\tau_{i_n}(u)|^{p-1}\right)|\tau_{i_n}(u_n) - \tau_{i_n}(u)| \\ &\leq C\left(1 + 2e^{(p-1)\lambda_{2j}}\right)|u_n - u|,\end{aligned}$$

and so $\|\tau_{i_n}(u_n) - u\|_{L^p(A;\mathbb{R}^d)} \to 0$. Thus, by (3.1.15),

$$\begin{aligned}\inf_{\{v_n\}}\left\{\liminf_{n\to\infty} F(v_n;A) : v_n \in SBV_p(A;\mathbb{R}^d),\ \sup_n \|v_n\|_{L^\infty(A;\mathbb{R}^d)} < \infty,\ v_n \to u \quad \text{in} \quad L^p(A;\mathbb{R}^d)\right\} \\ \leq \mathcal{J}(u;A) + \frac{1}{j} + \frac{CM}{j-1} + \frac{CM}{e^{\lambda_{j+1}}}.\end{aligned}$$

By letting $j \to \infty$ we obtain

$$\mathcal{J}_\infty(u;A) := \inf_{\{v_n\}}\left\{\liminf_{n\to\infty} F(v_n;A) : v_n \in SBV_p(A;\mathbb{R}^d),\ \sup_n \|v_n\|_{L^\infty(A;\mathbb{R}^d)} < \infty,\ v_n \to u \quad \text{in} \quad L^p(A;\mathbb{R}^d)\right\} \leq \mathcal{J}(u;A).$$

The converse inequality is trivial, since

$$\mathcal{J}(u;A) \leq \mathcal{J}_p(u;A) \leq \mathcal{J}_\infty(u;A).$$

$\square$

Remark 3.1.8. Since (H_4) and (3.1.13) are clearly equivalent, it is easy to see from Lemmas 3.1.3 and 3.1.7 that for any $u \in SBV_p(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$ we have

$$\begin{aligned}\mathcal{J}(u;A) = \mathcal{J}_p(u;A) = \inf_{\{u_n\}}\Bigg\{\liminf_{n\to\infty} F(u_n;A) : u_n \in SBV_p(A;\mathbb{R}^d),\ \sup_n \|u_n\|_{L^\infty(A;\mathbb{R}^d)} < \infty, \\ u_n \to u \quad \text{in} \quad L^p(A;\mathbb{R}^d),\ \ u_n = u \text{ on a neighborhood of } \partial A\Bigg\}.\end{aligned}$$

Remark 3.1.9. Given a sequence of functionals $F_n : SBV_p(\Omega;\mathbb{R}^d)\times\mathcal{A}(\Omega) \to [0,\infty]$ satisfying conditions (H_1), (H_2) and

$$\begin{aligned}&\frac{1}{C}\left(\sum_{l=1}^N C_{l,n}\int_A |\partial_l u|^p\,dx + \sum_{l=1}^N D_{l,n}\int_{S(u)\cap A}(1+|[u]|)|\nu_l(u)|\,d\mathcal{H}^{N-1}\right) \\ &\qquad \leq F_n(u;A) \leq C\left(|A| + \sum_{l=1}^N C_{l,n}\int_A |\partial_l u|^p\,dx + \sum_{l=1}^N D_{l,n}\int_{S(u)\cap A}(1+|[u]|)|\nu_l(u)|\,d\mathcal{H}^{N-1}\right)\end{aligned}$$

for every $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \times\mathcal{A}(\Omega)$, and for some constants C, $C_{l,n}$, $C_{l,n} > 0$, $l = 1,\dots,N$, $n \in \mathbb{N}$, then Lemma 3.1.7 still holds for Γ^- and Γ_p^- (see (3.1.10)) for those u for which

$$\sup_n\left(\sum_{l=1}^N C_{l,n}\int_A |\partial_l u|^p\,dx + \sum_{l=1}^N D_{l,n}\int_{S(u)\cap A}(1+|[u]|)|\nu_l(u)|\,d\mathcal{H}^{N-1}\right) < \infty.$$

3.2 Relaxed Energy: Proof of Theorem 1.0.4

Proof of Theorem 1.0.4. Let $\{u_n\} \subset SBV_p(\Omega;\mathbb{R}^d)$ be any sequence converging to $u \in SBV_p(\Omega;\mathbb{R}^d)$ in $L^1(\Omega;\mathbb{R}^d)$ and such that

$$\sup_n \left(\int_A \mathcal{Q}f_0(x,u_n,\nabla u_n)\,dx + \int_{S(u_n)\cap A} \mathcal{R}g_0(x,u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1} \right) < \infty.$$

By (1.0.9) we may apply Ambrosio's lower semicontinuity results (see Theorem 3.1.2) to obtain

$$\begin{aligned}\int_A \mathcal{Q}f_0(x,u,\nabla u)\,dx &+ \int_{S(u)\cap A} \mathcal{R}g_0(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} \\ &\leq \liminf_{n\to\infty} \left(\int_A \mathcal{Q}f_0(x,u_n,\nabla u_n)\,dx + \int_{S(u_n)\cap A} \mathcal{R}g_0(x,u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1} \right)\end{aligned}$$

and taking the infimum over all such sequences $\{u_n\}$ we get

$$\int_A \mathcal{Q}f_0(x,u,\nabla u)\,dx + \int_{S(u)\cap A} \mathcal{R}g_0(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1} \leq \mathcal{H}(u;A).$$

Conversely, define, for every $u \in SBV_p(\Omega;\mathbb{R}^d)$ and $A \in \mathcal{A}(\Omega)$,

$$F(u;A) := \int_A f_0(x,u,\nabla u)\,dx + \int_{S(u)\cap A} g_0(x,u^+,u^-,\nu_u)\,d\mathcal{H}^{N-1}.$$

We claim that the functional $\mathcal{H}$ defined by (3.1.7) satisfies conditions (H_1)–(H_4). Indeed hypotheses (H_2) and (H_3) are immediate consequences of the definition of $\mathcal{H}$. The upper bound of $\mathcal{H}$ in (H_4) is obvious, since $\mathcal{H}(u;A) \leq F(u;A)$. The lower bound is a consequence of the lower semicontinuity of

$$u \mapsto \int_A |\nabla u|^p\,dx + \int_{A\cap S(u)} (1+|[u]|)\,d\mathcal{H}^{N-1}$$

with respect to the $L^1(A;\mathbb{R}^d)$ topology, as it is guaranteed by Theorem 3.1.2.

To prove (H_1), and as it is usual (see e.g. [13], [39]), it suffices to prove that

$$\mathcal{H}(u;A) \leq \mathcal{H}(u;B) + \mathcal{H}(u;A\setminus\overline{C})$$

for all A, B, $C \in \mathcal{A}(\Omega)$, with $C \subset\subset B \subset\subset A$. Assume that $u \in SBV_p(\Omega;\mathbb{R}^d)\cap L^\infty(\Omega;\mathbb{R}^d)$, and fix $\eta > 0$ and $B_0 \in \mathcal{A}_\infty(\Omega)$ such that $C \subset\subset B_0 \subset\subset B$ and $\mu(\partial B_0) = 0$, where

$$\mu := \mathcal{L}^N\lfloor\Omega + (1+|[u]|)\mathcal{H}^{N-1}\lfloor(S(u)\cap\Omega).$$

By Remark 3.1.8 there exist two sequences $\{v_k\}$, $\{u_k\} \subset SBV_p(\Omega;\mathbb{R}^d)$ such that $v_k \to u$ in $L^p(A\setminus\overline{B_0};\mathbb{R}^d)$, $u_k \to u$ in $L^p(B_0;\mathbb{R}^d)$, $v_k = u_k = u$ on ∂B_0 and

$$\lim_{k\to\infty} F(v_k;A\setminus\overline{B_0}) \leq \mathcal{H}(u;A\setminus\overline{B_0}) + \eta, \qquad \lim_{k\to\infty} F(u_k;B_0) \leq \mathcal{H}(u;B_0) + \eta. \tag{3.2.1}$$

Define $w_k = u_k$ in B_0 and $w_k = v_k$ in $A\setminus\overline{B_0}$. Then $\{w_k\} \subset SBV_p(\Omega;\mathbb{R}^d)$ and $w_k \to u$ in $L^p(A;\mathbb{R}^d)$. Moreover, by the growth conditions (1.0.9) and the fact that $\mu(\partial B_0) = 0$

$$\begin{aligned}F(w_k;A) &= F(w_k;B_0) + F(w_k;A\setminus\overline{B_0}) \\ &= F(u_k;B_0) + F(v_k;A\setminus\overline{B_0}).\end{aligned}$$

Thus, passing to the limit in k we have

$$\begin{aligned}\mathcal{H}(u;A) &\leq \mathcal{H}(u;B_0) + \mathcal{H}(u;A\setminus\overline{B_0}) + 2\eta \\ &\leq \mathcal{H}(u;B) + \mathcal{H}(u;A\setminus\overline{C}) + 2\eta,\end{aligned}$$

where we have used (3.2.1) and the fact that $\mathcal{H}(u;\cdot)$ is an increasing set function. It now suffices to let $\eta \to 0^+$.

In the general case, by Lemma 3.1.7

$$\mathcal{H}(u;B) + \mathcal{H}(u;A\setminus\overline{C}) = \lim_{j\to\infty} \left(\mathcal{H}(\tau_j(u);B) + \mathcal{H}(\tau_j(u);A\setminus\overline{C})\right).$$

Since $\tau_j(u)$ is bounded by the first part we have

$$\mathcal{H}(\tau_j(u); A) \leq \mathcal{H}(\tau_j(u); B) + \mathcal{H}(\tau_j(u); A \setminus \overline{C}),$$

and letting $j \to \infty$ we get, again by Lemma 3.1.7,

$$\mathcal{H}(u; A) = \lim_{j\to\infty} \mathcal{H}(\tau_j(u); A) \leq \lim_{j\to\infty} \big(\mathcal{H}(\tau_j(u); B) + \mathcal{H}(\tau_j(u); A \setminus \overline{C})\big) = \mathcal{H}(u; B) + \mathcal{H}(u; A \setminus \overline{C}).$$

By Theorem 1.0.1 we now have

$$\mathcal{H}(u; A) = \int_A f(x, u, \nabla u)\, dx + \int_{S(u)\cap A} g(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1},$$

where f and g are given by (1.0.2) and (1.0.3).

Step 1: We claim that

$$f(x_0, u_0, \xi_0) \leq \mathcal{Q} f_0(x_0, u_0, \xi_0).$$

for $\mathcal{L}^N$ a.e. $x_0 \in \Omega$ and all $(u_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^{d\times N}$. Since f_0 is a Carathéodory function, by the Scorza-Dragoni Theorem for each $j \in \mathbb{N}$ there exists a compact set $K_j \subset \Omega$, with $|\Omega \setminus K_j| \leq 1/j$, such that

$$f_0 \;\; : K_j \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0, +\infty)$$

is continuous. Let K_j^* be the set of Lebesgue points of χ_{K_j}, and set

$$\omega := \bigcup_j^{\infty} (K_j \cap K_j^*). \tag{3.2.2}$$

Then

$$|\Omega \setminus \omega| \leq |\Omega \setminus K_j| \leq \frac{1}{j} \to 0 \quad \text{as } j \to \infty.$$

Fix $(x_0, u_0, \xi_0) \in \omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ and $\eta > 0$, and let $w \in C_0^\infty(Q; \mathbb{R}^d)$ be such that

$$\int_Q f_0(x_0, u_0, \xi_0 + \nabla w(y))\, dy \leq \mathcal{Q} f_0(x_0, u_0, \xi_0) + \eta. \tag{3.2.3}$$

For any fixed $\varepsilon > 0$ set $v_\varepsilon(x) := \varepsilon\, w((x - x_0)/\varepsilon)$. Then $v_\varepsilon \in W_0^{1,p}(Q(x_0, \varepsilon); \mathbb{R}^d)$, and by (1.0.2) and the definitions of $\mathbf{m}$ and $\mathcal{H}$

$$\begin{aligned} f(x_0, u_0, \xi_0) :=& \limsup_{\varepsilon\to 0^+} \frac{\mathbf{m}(u_0 + \xi_0(\cdot - x_0); Q(x_0, \varepsilon))}{\varepsilon^N} \leq \limsup_{\varepsilon\to 0^+} \frac{\mathcal{H}(u_0 + \xi_0(\cdot - x_0) + v_\varepsilon; Q(x_0, \varepsilon))}{\varepsilon^N} \\ &\leq \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)} f_0(x, u_0 + \xi_0(x - x_0) + v_\varepsilon(x), \xi_0 + \nabla v_\varepsilon(x))\, dx. \end{aligned} \tag{3.2.4}$$

Fix $L > \max\{|u_0|, |\xi_0| + ||\nabla w||_{L^\infty}\} + 1$, and let j be such that $x_0 \in K_j \cap K_j^*$, where we are using the notation introduced in (3.2.2). Since

$$f_0 : K_j \times \overline{B_d(0, L)} \times \overline{B_{d\times N}(0, L)} \to [0, \infty)$$

is uniformly continuous, there exists $0 < \rho < 1$ such that

$$|f_0(x, u, \xi) - f_0(x_1, u_1, \xi_1)| \leq \eta \tag{3.2.5}$$

for all (x, u, ξ), $(x_1, u_1, \xi_1) \in K_j \times \overline{B_d(0, L)} \times \overline{B_{d\times N}(0, L)}$, with $|x - x_1| \leq \rho$, $|u - u_1| \leq \rho$ and $|\xi - \xi_1| \leq \rho$. For $\varepsilon > 0$ sufficiently small

$$(x, u_0 + \xi_0(x - x_0) + v_\varepsilon(x)) \in B_N(x_0, \rho) \times B_d(u_0, \rho)$$

for all $x \in Q(x_0, \varepsilon)$, and thus by (3.2.4), (1.0.9), (3.2.5), (3.2.3),

$$\begin{aligned} f(x_0, u_0, \xi_0) \leq& \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\cap K_j} f_0(x, u_0 + \xi_0(x - x_0) + v_\varepsilon(x), \xi_0 + \nabla v_\varepsilon(x))\, dx \\ &+ \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\setminus K_j} C(1 + |\xi_0|^q + ||\nabla w||_{L^\infty}^q)\, dx \\ \leq& \limsup_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0,\varepsilon)\cap K_j} f_0(x_0, u_0, \xi_0 + \nabla v_\varepsilon(x))\, dx + \eta \\ \leq& \int_Q f_0(x_0, u_0, \xi_0 + \nabla w(y))\, dy + \eta \leq \mathcal{Q} f(x_0, u(x_0), v(x_0)) + 2\eta, \end{aligned}$$

where we have used the fact that
$$\frac{|Q(x_0,\varepsilon)\setminus K_j|}{\varepsilon^N}\to 0 \quad \text{as } \varepsilon\to 0^+,$$
since x_0 is a Lebesgue point of χ_{K_j}. It now suffices to let $\eta\to 0^+$.

Step 2: To conclude the proof it remains to show that
$$g(x_0,a,b,\nu)\le \mathcal{R}g_0(x_0,a,b,\nu)$$
for all $(x_0,a,b,\nu)\in\Omega\times\mathbb{R}^{2d}\times S^{d-1}$.

Fix $(x_0,a,b,\nu)\in\Omega\times\mathbb{R}^{2d}\times S^{d-1}$ and $\eta>0$, and let $w\in SBV_p(Q_\nu;\mathbb{R}^d)\cap L^\infty(Q_\nu;\mathbb{R}^d)$ be such that $\nabla w=0$ $\mathcal{L}^N$ a.e in Q_ν, $w=u_{x_0,a,b,\nu}$ in a neighborhood of ∂Q_ν and
$$\int_{S(w)\cap Q_\nu} g(x_0,w^+(y),w^-(y),\nu_w(y))\,d\mathcal{H}^{N-1}(y)\le \mathcal{R}g_0(x_0,a,b,\nu)+\eta, \tag{3.2.6}$$
where, we recall,
$$u_{x_0,a,b,\nu}(x):=\begin{cases} a & \text{if } (x-x_0)\cdot\nu>0,\\ b & \text{if } (x-x_0)\cdot\nu\le 0.\end{cases}$$
For any fixed $\varepsilon>0$ set $v_\varepsilon(x):=w((x-x_0)/\varepsilon)$. Then by (1.0.3) and the definition of $\mathbf{m}$
$$\begin{aligned} g(x_0,a,b,\nu) &:= \limsup_{\varepsilon\to 0^+}\frac{\mathbf{m}(u_{x_0,a,b,\nu};Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}\le \limsup_{\varepsilon\to 0^+}\frac{\mathcal{H}(v_\varepsilon;Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}\\ &\le \limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)\cap S(v_\varepsilon)} g(x,v_\varepsilon^+(x),v_\varepsilon^-(x),\nu_{v_\varepsilon}(x))\,d\mathcal{H}^{N-1}(x). \end{aligned}\tag{3.2.7}$$
Fix $L>||w||_{L^\infty}+1$. Since
$$g_0:\overline{B_N(x_0,1)}\times\overline{B_d(0,L)}\times\overline{B_d(0,L)}\times S^{d-1}\to[0,\infty)$$
is uniformly continuous, there exists $0<\rho<1$ such that
$$|g_0(x,u,v,\nu)-g_0(x_1,u_1,v_1,\nu_1)|\le\eta \tag{3.2.8}$$
for all (x,u,v,ν), $(x_1,u_1,v_1,\nu_1)\in\overline{B_N(x_0,1)}\times\overline{B_d(0,L)}\times\overline{B_d(0,L)}\times S^{d-1}$, with $|x-x_1|\le\rho$, $|u-u_1|\le\rho$, $|v-v_1|\le\rho$ and $|\nu-\nu_1|\le\rho$. Thus by (3.2.7), (3.2.8), (3.2.6),
$$\begin{aligned} g(x_0,a,b,\nu) &\le \limsup_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_\nu(x_0,\varepsilon)\cap S(v_\varepsilon)} g(x_0,v_\varepsilon^+(x),v_\varepsilon^-(x),\nu_{v_\varepsilon}(x))\,d\mathcal{H}^{N-1}(x)+\eta\\ &=\int_{S(w)\cap Q_\nu} g(x_0,w^+(y),w^-(y),\nu_w(y))\,d\mathcal{H}^{N-1}(y)+\eta\le\mathcal{R}g_0(x_0,a,b,\nu)+2\eta. \end{aligned}$$
It now suffices to let $\eta\to 0^+$.

□

3.3 Brittle Thin Films.

In this section we deduce, via Γ-convergence, the 2-dimensional model for brittle thin films from the 3-dimensional model allowing fracture. Using the global method introduced in Section 2.2, we recover easily the result recently obtained by Braides and Fonseca in [23], now holding without having to assume some technical Lipschitz regularity hypotheses imposed by those authors.

Considering a sequence of positive real numbers $\varepsilon\to 0^+$ corresponding to the thickness of a 3-dimensional cilinder of fixed 2-dimensional cross section $\omega\subset\mathbb{R}^2$, we analyze the asymptotic behavior of the 3-dimensional energy
$$v\in SBV_p(\Omega_\varepsilon;\mathbb{R}^3)\mapsto\int_{\Omega_\varepsilon} f_0(\nabla v)\,dx+\int_{S(u)\cap\Omega_\varepsilon} g_0([v],\nu(v))\,d\mathcal{H}^2, \tag{3.3.1}$$

where $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$ and

$$f_0 : \mathbb{R}^{3\times 3} \to [0,\infty), \qquad g_0 : \mathbb{R}^3 \times S^2 \to [0,\infty),$$

are the bulk and the surface energy density, respectively. In order to set the problem in a fixed domain, we use the rescaling (see [23])

$$u(x_\alpha, x_3) := v(x_\alpha, \varepsilon\, x_3), \qquad x_\alpha := (x_1, x_2),$$

so that

$$\frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} f_0(\nabla v)\,dx + \int_{S(u)\cap\Omega_\varepsilon} g_0([v],\nu(v))\,d\mathcal{H}^2\right)$$
$$= \int_\Omega f_0\left(\nabla_\alpha u, \frac{1}{\varepsilon}\nabla_3 u\right)\,dx + \int_{S(u)\cap\Omega} g_0\left([u], \frac{(\nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u))}{\left|(\nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u))\right|}\right)\left|\left(\nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u)\right)\right|\,d\mathcal{H}^2,$$

where $\Omega := \omega \times (-1/2, 1/2)$. Here, and in what follows, (ξ_α, ξ_3) denotes the 3×3 matrix ξ with columns ξ_1, ξ_2 and ξ_3. To simplify the notation we extend g_0 by homogeneity of degree one to $\mathbb{R}^3 \times \mathbb{R}^3$ by setting

$$g_0(z,\nu) := \begin{cases} |\nu| g_0\left(z, \frac{\nu}{|\nu|}\right) & \text{if } \nu \neq 0, \\ 0 & \text{if } \nu = 0, \end{cases}$$

and thus we are led to the problem of Γ-convergence, as ε goes to zero, of the sequence of functionals

$$F_\varepsilon : SBV_p(\Omega;\mathbb{R}^3) \times \mathcal{A}(\omega) \to [0,\infty]$$

defined by

$$F_\varepsilon(u;A) := \int_{A\times(-\frac{1}{2},\frac{1}{2})} f_0\left(\nabla_\alpha u, \frac{1}{\varepsilon}\nabla_3 u\right)\,dx + \int_{S(u)\cap\left(A\times(-\frac{1}{2},\frac{1}{2})\right)} g_0\left([u], \nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u)\right)\,d\mathcal{H}^2. \tag{3.3.2}$$

Given a family of functions $\{u_\varepsilon\} \subset SBV_p(\Omega;\mathbb{R}^3)$ such that (cf. [23] Prop. 2.3)

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon;\omega) < \infty \quad \text{and } u_\varepsilon \to u \text{ in } L^1(\Omega;\mathbb{R}^3),$$

by the growth conditions (3.3.6) we obtain

$$\sup_\varepsilon \left(\int_\Omega \left|\left(\nabla_\alpha u_\varepsilon, \frac{1}{\varepsilon}\nabla_3 u_\varepsilon\right)\right|^p\,dx + \int_{S(u_\varepsilon)\cap\Omega} (1+|[u_\varepsilon]|)\left|\left(\nu_\alpha(u_\varepsilon), \frac{1}{\varepsilon}\nu_3(u_\varepsilon)\right)\right|\,d\mathcal{H}^2\right) \leq M < \infty$$

for some $M > 0$. Thus for $0 < \varepsilon < 1$ we have

$$\sup_\varepsilon \left(\int_\Omega |\nabla u_\varepsilon|^p\,dx + \int_{S(u_\varepsilon)\cap\Omega} (1+|[u_\varepsilon]|)\,d\mathcal{H}^2\right) \leq M < \infty,$$

and we may apply Theorem 3.1.2 to the functional

$$v \mapsto \int_\Omega |\nabla_3 v|^p\,dx + \int_{S(v)\cap\Omega} |\nu_3(v)|\,d\mathcal{H}^2$$

to obtain

$$\int_\Omega |\nabla_3 u|^p\,dx + \int_{S(u)\cap\Omega} |\nu_3(u)|\,d\mathcal{H}^2 \leq \liminf_{\varepsilon\to 0^+}\left(\int_\Omega |\nabla_3 u_\varepsilon|^p\,dx + \int_{S(u_\varepsilon)\cap\Omega} |\nu_3(u_\varepsilon)|\,d\mathcal{H}^2\right) \leq \liminf_{\varepsilon\to 0^+} \varepsilon F_\varepsilon(u_\varepsilon;\omega) = 0,$$

which implies that $u \in \mathcal{V}$, where

$$\mathcal{V} := \left\{u \in SBV_p(\Omega;\mathbb{R}^3) : \nabla_3 u = 0 \quad \mathcal{L}^3 \text{ a.e. and } \nu_3(u) = 0 \quad \mathcal{H}^2 \text{ a.e. in } \Omega\right\}.$$

Since functions in $\mathcal{V}$ are independent of x_3, in what follows $\mathcal{V}$ is identified with $SBV_p(\omega;\mathbb{R}^3)$.

Set

$$\bar{f}_0(\xi_\alpha) := \inf\{f_0(\xi) : \xi := (\xi_\alpha, \xi_3),\ \xi_3 \in \mathbb{R}\} \tag{3.3.3}$$

where $\xi_\alpha \in \mathbb{R}^{3\times 2}$, and

$$\bar{g}_0(z,\mu) := \inf\{g_0(z,\mu,t) : t \in \mathbb{R}\}, \tag{3.3.4}$$

where $z \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^2 \setminus \{0\}$. Note that $\bar{g}_0(z,\cdot)$ is homogeneous of degree one, and thus we may define $\bar{g}_0(z,0) := 0$.

Given $u \in SBV_p(\Omega;\mathbb{R}^3)$, by Theorem 3.1.1 for $\mathcal{L}^1$ a.e. $t \in (-1/2,1/2)$ the function $u^t(x_\alpha) := u(x_\alpha,t) \in SBV_p(\omega;\mathbb{R}^3)$ and

$$\begin{aligned}
F_\varepsilon(u;A) &= \int_{A\times(-\frac{1}{2},\frac{1}{2})} f_0\left(\nabla_\alpha u, \frac{1}{\varepsilon}\nabla_3 u\right)dx + \int_{S(u)\cap\left(A\times(-\frac{1}{2},\frac{1}{2})\right)} g_0\left([u],\nu_\alpha(u),\frac{1}{\varepsilon}\nu_3(u)\right)d\mathcal{H}^2\\
&\geq \int_{A\times(-\frac{1}{2},\frac{1}{2})} \bar{f}_0(\nabla_\alpha u)\,dx + \int_{S(u)\cap\left(A\times(-\frac{1}{2},\frac{1}{2})\right)} \bar{g}_0([u],\nu_\alpha(u))\,d\mathcal{H}^2\\
&\geq \int_{-1/2}^{1/2}\left(\int_A \bar{f}_0\left(\nabla_\alpha u^t\right)dx_\alpha + \int_{S(u^t)\cap A} \bar{g}_0\left([u^t],\bar{\nu}(u^t)\right)|\nu_\alpha(u)|\,d\mathcal{H}^1\right)dt \\
&\geq \int_{-1/2}^{1/2}\left(\int_A \mathcal{Q}_2\,\bar{f}_0\left(\nabla_\alpha u^t\right)dx_\alpha + \int_{S(u^t)\cap A} \mathcal{R}_2\,\bar{g}_0\left([u^t],\bar{\nu}(u^t)\right)|\nu_\alpha(u)|\,d\mathcal{H}^1\right)dt\\
&= \int_{-1/2}^{1/2}\left(\int_A \mathcal{Q}_2\,\bar{f}_0\left(\nabla_\alpha u^t\right)dx_\alpha + \int_{S(u^t)\cap A} \mathcal{R}_2\,\bar{g}_0\left([u^t],\bar{\nu}(u^t)\right)\sqrt{1-|\nu_3(u)|^2}\,d\mathcal{H}^1\right)dt,
\end{aligned} \tag{3.3.5}$$

where we have used Fubini's Theorem and the fact that $\bar{g}_0(z,\cdot)$ is homogeneous of degree one. Here $\mathcal{Q}_2\,\bar{f}_0$ and $\mathcal{R}_2\,\bar{g}_0$ are, respectively, the quasiconvex envelope of $\bar{f}_0$ and the BV-elliptic envelope of $\bar{g}_0(z,\mu)$ corresponding to the ambient space $SBV_p(\omega;\mathbb{R}^3)$, precisely for $\xi_\alpha \in \mathbb{R}^{3\times 2}$

$$\mathcal{Q}_2\,\bar{f}_0(\xi_\alpha) := \inf\left\{\int_{(0,1)^2} \bar{f}_0\left(\xi_\alpha + \nabla_\alpha\varphi(y_\alpha)\right)dy_\alpha : \varphi \in W_0^{1,\infty}((0,1)^2;\mathbb{R}^3)\right\},$$

while for $z \in \mathbb{R}^3$ and $\mu \in S^1$

$$\mathcal{R}_2\,\bar{g}_0(z,\mu) := \inf\left\{\int_{S(v)\cap Q'_\mu} \bar{g}_0([v],\nu(v))\,d\mathcal{H}^1 : v \in SBV_0(Q'_\mu;\mathbb{R}^3)\cap L^\infty(Q'_\mu;\mathbb{R}^3),\ v = u_{z,\mu} \text{ on } \partial Q'_\mu\right\},$$

where Q'_μ is the unit square in $\mathbb{R}^2$ centered at zero with two of its faces orthogonal to μ, and

$$u_{z,\mu}(x_\alpha) := \begin{cases} z & \text{if } x_\alpha\cdot\mu > 0,\\ 0 & \text{if } x_\alpha\cdot\mu \leq 0.\end{cases}$$

These considerations now lead us to the following result.

Theorem 3.3.1. *Assume that*

$$f_0 : \mathbb{R}^{3\times 3} \to [0,\infty), \qquad g_0 : \mathbb{R}^3\times\mathbb{R}^3 \to [0,\infty)$$

are continuous functions, $g_0(z,\cdot)$ is even and positively homogeneous of degree one, and

$$\frac{1}{C}|\xi|^p \leq f_0(\xi) \leq C(1+|\xi|^p), \qquad \frac{1}{C}(1+|z|) \leq g_0(z,\nu) \leq C(1+|z|), \qquad g_0(z,\nu) = g_0(-z,-\nu), \tag{3.3.6}$$

for all $\xi \in \mathbb{R}^{3\times 3}$, $z \in \mathbb{R}^3$, $\nu \in S^2$, and for some $C > 0$. Then for every $u \in \mathcal{V}$ and $A \in \mathcal{A}(\omega)$ the functionals

$$v \in SBV_p(\Omega;\mathbb{R}^3) \mapsto F_\varepsilon(v;A)$$

Γ-converge with respect to the $L^1(\Omega;\mathbb{R}^3)$ convergence as $\varepsilon \to 0^+$ to the functional

$$\mathcal{F}_0(u;A) := \int_A \mathcal{Q}_2\,\bar{f}_0(\nabla_\alpha u)\,dx_\alpha + \int_{S(u)\cap A} \mathcal{R}_2\,\bar{g}_0([u],\nu_\alpha(u))\,d\mathcal{H}^1.$$

Proof. In order to prove that the sequence $\{F_\varepsilon(u;A)\}$ Γ-converges to $\mathcal{F}_0(u;A)$ for all $u \in \mathcal{V}$ and $A \in \mathcal{A}(\omega)$, it is enough to show that for any given sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, there exists a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that the Γ-lower limit defined by

$$\mathcal{F}^-(u;A) := \inf\left\{\liminf_{k\to\infty} F_{\varepsilon_{n_k}}(u_k;A) : \{u_k\} \subset SBV_p(\Omega;\mathbb{R}^3),\ u_k \to u \quad \text{in } L^1(\Omega;\mathbb{R}^3)\right\} \tag{3.3.7}$$

coincides with $\mathcal{F}_0(u;A)$ for all $u \in \mathcal{V}$ and $A \in \mathcal{A}(\omega)$.

Fix a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$, and let $\mathcal{R}(\omega)$ be the countable subfamily of $\mathcal{A}(\omega)$ obtained by taking all finite unions of open squares in ω with faces parallel to the axes, centered at $x_\alpha \in \omega \cap \mathbb{Q}^2$ and with rational edge length. Since $L^1(\Omega;\mathbb{R}^3)$ is a separable metric space, using Kuratowski's Compactness Theorem and a diagonal argument (see Prop. 7.9 in [21]), we may find a subsequence $\{\varepsilon_{n_k}\}$ of $\{\varepsilon_n\}$ such that

$$\Gamma - \lim_{k\to\infty} F_{\varepsilon_{n_k}}(u;A') \text{ exists for all } u \in \mathcal{V} \text{ and for all } A' \in \mathcal{R}(\omega). \tag{3.3.8}$$

In order to apply Theorem 1.0.1 we need to show that the functional defined in (3.3.7) satisfies (H_1)-(H_4) for all $A \in \mathcal{A}(\omega)$ and $u \in \mathcal{V}$. For simplicity of notation we write ε_k in place of ε_{n_k}.

Hypotheses (H_2) and (H_3) are immediate consequences of the definition, and in view of (3.3.6) the upper bound in (H_4) is obtained by taking $u_k \equiv u \in \mathcal{V}$. In order to establish the lower bound in (H_4), let $\{u_j\} \subset SBV_p(\Omega;\mathbb{R}^3)$ be such that $u_j \to u$ in $L^1(\Omega;\mathbb{R}^3)$ and $\mathcal{F}^-(u;A) = \lim_{j\to\infty} F_{\varepsilon_{k_j}}(u_j;A)$ for a suitable subsequence $\{\varepsilon_{k_j}\}$ of $\{\varepsilon_k\}$. By (3.3.6) and using the lower semicontinuity result of Theorem 3.1.2 we have

$$\begin{aligned}
\mathcal{F}^-(u;A) = \lim_{j\to\infty} F_{\varepsilon_{k_j}}(u_j;A) &\geq \frac{1}{C}\liminf_{j\to\infty}\left(\int_{A\times\left(-\frac12,\frac12\right)} |\nabla u_j|^p\,dx + \int_{S(u_j)\cap\left(A\times\left(-\frac12,\frac12\right)\right)} (1+|[u_j]|)\,d\mathcal{H}^2\right)\\
&\geq \frac{1}{C}\left(\int_{A\times\left(-\frac12,\frac12\right)} |\nabla u|^p\,dx + \int_{S(u)\cap\left(A\times\left(-\frac12,\frac12\right)\right)} (1+|[u]|)\,d\mathcal{H}^2\right)\\
&= \frac{1}{C}\left(\int_A |\nabla_\alpha u|^p\,dx_\alpha + \int_{S(u)\cap A} (1+|[u]|)\,d\mathcal{H}^1\right).
\end{aligned}$$

To prove that $\mathcal{F}^-(u;\cdot)$ is the trace of a Borel measure, we first observe that in view of Remark 3.1.9, and as in the proof of Theorem 1.0.4, it is enough to show that

$$\mathcal{F}^-(u;A) \leq \mathcal{F}^-(u;B) + \mathcal{F}^-(u;A\setminus\overline{C})$$

for all A, B, $C \in \mathcal{A}(\omega)$, with $C \subset\subset B \subset\subset A$, and for all $u \in \mathcal{V}\cap\mathcal{L}^\infty(\Omega;\mathbb{R}^3)$, and with convergence in $L^1(\Omega;\mathbb{R}^3)$ replaced by convergence in $L^p(\Omega;\mathbb{R}^3)$ in the definition of $\mathcal{F}^-(u;A)$. Fix $\eta > 0$ and find $\{v_k\} \subset SBV_p(\Omega;\mathbb{R}^3)$ such that $v_k \to u$ in $L^p((A\setminus\overline{C})\times(-1/2,1/2);\mathbb{R}^3)$ and

$$\liminf_{k\to\infty} F_{\varepsilon_k}(v_k;A\setminus\overline{C}) \leq \mathcal{F}^-(u;A\setminus\overline{C}) + \eta.$$

Extract a subsequence $\{k_j\}$ for which

$$\lim_{j\to+\infty} F_{\varepsilon_{k_j}}(v_{k_j};A\setminus\overline{C}) \leq \mathcal{F}^-(u;A\setminus\overline{C}) + \eta. \tag{3.3.9}$$

Let $B_0 \in \mathcal{R}(\omega)$ be such that $C \subset\subset B_0 \subset\subset B$. By (3.3.8) $\mathcal{F}^-(u;B_0) = \Gamma - \lim_{j\to\infty} F_{\varepsilon_{k_j}}(u;B_0)$ and thus, by Remark 3.1.8, there exists a sequence $\{u_j\} \subset SBV_p(\Omega;\mathbb{R}^3)$ such that $u_j \to u$ in $L^p(B_0\times(-1/2,1/2);\mathbb{R}^3)$ and

$$\lim_{j\to\infty} F_{\varepsilon_{k_j}}(u_j;B_0) = \mathcal{F}^-(u;B_0). \tag{3.3.10}$$

For every $t > 0$ let $B_t := \{x_\alpha \in B_0 |\ \mathrm{dist}(x_\alpha,\partial B_0) > t\}$, and

$$\mathcal{G}_\varepsilon(v;A) := \int_{A\times\left(-\frac12,\frac12\right)} \left|\left(\nabla_\alpha v, \frac{1}{\varepsilon}\nabla_3 v\right)\right|^p dx + \int_{S(v)\cap\left(A\times\left(-\frac12,\frac12\right)\right)} (1+|[v]|)\left|\left(\nu_\alpha(v), \frac{1}{\varepsilon}\nu_3(v)\right)\right|^2 d\mathcal{H}^2,$$

$v \in SBV_p(\Omega;\mathbb{R}^3)$. Due to the coercivity hypothesis (3.3.6), we may extract a bounded subsequence from the sequences of measures $\nu_j := \mathcal{G}_{\varepsilon_{k_j}}(v_{k_j};\cdot) + \mathcal{G}_{\varepsilon_{k_j}}(u_j;\cdot)$ restricted to $B_0\setminus\overline{C}$ converging $\star$-weakly to some Radon measure ν defined on $B_0\setminus\overline{C}$. For every $0 < \delta < \eta$ we define the subsets $L_\delta := B_{\eta-2\delta}\setminus\overline{B_{\eta+\delta}}$. Consider a smooth cut-off function $\varphi_\delta \in \mathcal{C}_0^\infty(B_{\eta-\delta};[0,1])$ such that $\varphi_\delta(x_\alpha) = 1$ on B_η. As the thickness of the strip L_δ is of order δ, we have an upper bound of the form $\|\nabla\varphi_\delta\|_{L^\infty(B_{\eta-\delta})} \leq C/\delta$. Define

$$w_j(x) := u_j(x)\varphi_\delta(x_\alpha) + v_{k_j}(x)(1-\varphi_\delta(x_\alpha)).$$

Clearly this sequence converges to u in $L^p(A\times(-1/2,1/2);\mathbb{R}^3)$ as $j\to\infty$, and satisfies

$$\nabla_\alpha w_j = \varphi_\delta \nabla_\alpha u_j + (1-\varphi_\delta)\nabla_\alpha v_{k_j} + \nabla_\alpha \varphi_\delta \otimes (u_j - v_{k_j}), \qquad \nabla_3 w_j = \varphi_\delta \nabla_3 u_j + (1-\varphi_\delta)\nabla_3 v_{k_j}.$$

By the growth condition (3.3.6), we have the estimate

$$\begin{aligned} F_{\varepsilon_{k_j}}(w_j;A) \le & F_{\varepsilon_{k_j}}(u_j;B_\eta) + F_{\varepsilon_{k_j}}(v_{k_j};A\setminus\overline{B_{\eta-\delta}}) \\ &+ C\left(\mathcal{G}_{\varepsilon_{k_j}}(u_j;L_\delta) + \mathcal{G}_{\varepsilon_{k_j}}(u_j;L_\delta) + \frac{1}{\delta^p}\int_{L_\delta\times(-\frac12,\frac12)} |u_j - v_{k_j}|^p\,dx\right) \\ \le & F_{\varepsilon_{k_j}}(u_j;B_0) + F_{\varepsilon_{k_j}}(v_{k_j};A\setminus\overline{C}) \\ &+ C\left(\mathcal{G}_{\varepsilon_{k_j}}(u_j;L_\delta) + \mathcal{G}_{\varepsilon_{k_j}}(u_j;L_\delta) + \frac{1}{\delta^p}\int_{L_\delta\times(-\frac12,\frac12)} |u_j - v_{k_j}|^p\,dx\right). \end{aligned}$$

Thus, passing to the limit in k and using (3.3.9) and (3.3.10), we have

$$\begin{aligned} \mathcal{F}^-(u;A) &\le \mathcal{F}^-(u;B_0) + \mathcal{F}^-(u;A\setminus\overline{C}) + \eta + C\nu(\overline{L_\delta}) \\ &\le \mathcal{F}^-(u;B) + \mathcal{F}^-(u;A\setminus\overline{C}) + \eta + C\nu(\overline{L_\delta}), \end{aligned}$$

where we have used the fact that $\mathcal{F}^-(u;\cdot)$ is clearly an increasing set function. We let δ go to zero to obtain

$$\mathcal{F}^-(u;A) \le \mathcal{F}^-(u;B) + \mathcal{F}^-(u;A\setminus\overline{C}) + \eta + C\nu(\partial B_\eta).$$

It suffices to choose a subsequence $\{\eta_k\}$ such that $\eta_k\to 0^+$ and $\nu(\partial B_{\eta_k})=0$, to conclude that for $u\in\mathcal{V}\cap\mathcal{L}^\infty(\Omega;\mathbb{R}^3)$ we have

$$\mathcal{F}^-(u;A) \le \mathcal{F}^-(u;B) + \mathcal{F}^-(u;A\setminus\overline{C}).$$

We are now in position to apply Theorem 1.0.1 to the functional $\mathcal{F}^- : \mathcal{V}\times\mathcal{A}(\omega)\to[0,\infty)$ to obtain the existence of two Borel functions

$$f:\omega\times\mathbb{R}^3\times\mathbb{R}^{3\times 2}\to[0,\infty) \quad\text{and}\quad g:\omega\times\mathbb{R}^3\times\mathbb{R}^3\times S^1\to[0,\infty)$$

such that

$$\mathcal{F}^-(u;A) := \int_A f(x_\alpha,u,\nabla_\alpha u)\,dx_\alpha + \int_{S(u)\cap A} g\left(x_\alpha,u^+,u^-,\nu_\alpha(u)\right)\,d\mathcal{H}^1$$

for all $u\in\mathcal{V}$ and $A\in\mathcal{A}(\omega)$. Since $\mathcal{F}^-(u+c;A)=\mathcal{F}^-(u;A)$ for all $c\in\mathbb{R}^3$, and f_0 and g_0 are independent of x, it follows immediately from (1.0.2) and (1.0.3) that $f=f(\xi_\alpha)$ and $g=g(\lambda-\theta,\nu_\alpha)$. We prove that $f=\mathcal{Q}_2\bar{f}_0$ and $g=\mathcal{R}_2\bar{g}_0$, where $\bar{f}_0$ and $\bar{g}_0$ are given by (3.3.3) and (3.3.4), respectively.

Given $\xi_\alpha\in\mathbb{R}^{3\times 2}$ let $\xi_3\in\mathbb{R}^3$ be such that $\bar{f}_0(\xi_\alpha)=\min\{f_0(\xi) : \xi:=(\xi_\alpha,\xi_3),\ \xi_3\in\mathbb{R}\}$. Note that the minimum exists by (3.3.6) and the continuity of f_0. Consider the sequence $u_k(x):=\xi_\alpha x_\alpha+\varepsilon_k\,\xi_3 x_3$, and fix $A\in\mathcal{A}(\omega)$. Clearly $u_k\to u$ in $L^1\left(A\times\left(-\frac12,\frac12\right);\mathbb{R}^3\right)$, where $u(x)=u(x_\alpha)=\xi_\alpha x_\alpha$, and so

$$\mathcal{F}^-(u;A) = f(\xi_\alpha)\mathcal{L}^2(A) \le \liminf_{k\to\infty} F_{1/k}(u_k;A) = f_0(\xi_\alpha,\xi_3)\mathcal{L}^2(A) = \bar{f}_0(\xi_\alpha)\mathcal{L}^2(A),$$

that is

$$f(\xi_\alpha)\le\bar{f}_0(\xi_\alpha). \tag{3.3.11}$$

Similarly, for $\mu\in S^1$, $z\in\mathbb{R}^3$, let $t\in\mathbb{R}$ be such that $\bar{g}_0(z,\mu)=g_0(z,\mu,t)$. Consider the sequence

$$v_k(x) := \begin{cases} z & \text{if } x_\alpha\cdot\mu+\varepsilon_k\,x_3 t>0, \\ 0 & \text{if } x_\alpha\cdot\mu+\varepsilon_k\,x_3 t\le 0. \end{cases}$$

Then $v_k\to v$ in $L^1\left(A\times\left(-\frac12,\frac12\right);\mathbb{R}^3\right)$, where $A\in\mathcal{A}(\omega)$, and

$$v = v(x_\alpha) := \begin{cases} z & \text{if } x_\alpha\cdot\nu>0, \\ 0 & \text{if } x_\alpha\cdot\nu\le 0. \end{cases}$$

We have

$$\begin{aligned}
\mathcal{F}^-(v;A) &= f(0)\mathcal{L}^2(A) + g(z,\mu)\mathcal{H}^1\left(\{x_\alpha \in A : x_\alpha \cdot \mu = 0\}\right)\\
&\le \liminf_{k\to\infty} F_{\varepsilon_k}(v_k;A)\\
&= f_0(0)\mathcal{L}^2(A) + \liminf_{k\to\infty} g_0\left(z, \frac{\mu}{\sqrt{1+\varepsilon_k^2 t^2}}, \frac{t}{\sqrt{1+\varepsilon_k^2 t^2}}\right)\mathcal{H}^2\left(\left\{x \in A \times \left(-\frac{1}{2},\frac{1}{2}\right) : x_\alpha\cdot\mu + \varepsilon_k\, x_3 t = 0\right\}\right)\\
&= f_0(0)\mathcal{L}^2(A) + g_0(z,\mu,t)\,\mathcal{H}^1\left(\{x_\alpha \in A : x_\alpha\cdot\mu = 0\}\right).
\end{aligned}$$

We conclude that

$$f(0)\mathcal{L}^2(A) + g(z,\mu)\mathcal{H}^1\left(A \cap \{x_\alpha\cdot\mu = 0\}\right) \le f_0(0)\mathcal{L}^2(A) + \bar{g}_0(z,\mu)\,\mathcal{H}^1\left(A\cap\{x_\alpha\cdot\mu=0\}\right). \tag{3.3.12}$$

Taking $A = A_\eta := \{x_\alpha \in \omega : |x_\alpha\cdot\mu| < \eta\}$, $\eta > 0$, letting $\eta \to 0^+$ in (3.3.12), yields

$$g(z,\mu) \le \bar{g}_0(z,\mu). \tag{3.3.13}$$

By (3.3.11) and (3.3.13) we now have

$$\mathcal{F}^-(u;A) \le \int_A \bar{f}_0(\nabla_\alpha u)\,dx_\alpha + \int_{S(u)\cap A} \bar{g}_0\left([u];\nu_\alpha(u)\right)\,d\mathcal{H}^1,$$

for every $u \in SBV_p(\omega;\mathbb{R}^3)$ and all $A \in \mathcal{A}(\omega)$. Since $\bar{f}_0$ and $\bar{g}_0$ satisfy the hypotheses of Theorem 1.0.4, and in view of the $L^1(\omega)$-lower semicontinuity of $\mathcal{F}^-(\cdot;A)$, we conclude that

$$f \le \mathcal{Q}_2\bar{f}_0 \quad\text{and}\quad g \le \mathcal{R}_2\bar{g}_0. \tag{3.3.14}$$

Conversely, let $A \in \mathcal{A}(\omega)$, $u \in \mathcal{V}$, and consider any sequence $\{u_k\} \subset SBV_p(\Omega;\mathbb{R}^3)$ such that

$$\sup_k F_{\varepsilon_k}(u_k;A) < \infty \quad\text{and } u_k \to u \text{ in } L^1\left(A\times\left(-\frac{1}{2},\frac{1}{2}\right);\mathbb{R}^3\right).$$

In light of the coercivity hypotheses (3.3.6) we obtain

$$\sup_k\left(\int_{A\times\left(-\frac{1}{2},\frac{1}{2}\right)}\left|\left(\nabla_\alpha u_k, \frac{1}{\varepsilon}\nabla_3 u_k\right)\right|^p dx + \int_{S(u_k)\cap\left(A\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)}(1+|[u_k]|)\left|\left(\nu_\alpha(u_k), \frac{1}{\varepsilon_k}\nu_3(u_k)\right)\right| d\mathcal{H}^2\right) < \infty.$$

Thus for $0 < \varepsilon_k < 1$ condition (3.1.6) is satisfied, and

$$\lim_{k\to\infty}\int_{S(u_k)\cap\left(A\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)}(1+|[u_k]|)\,|\nu_3(u_k)|\,d\mathcal{H}^2 = 0. \tag{3.3.15}$$

By (3.3.5) and Theorem 3.1.1

$$\begin{aligned}
F_{\varepsilon_k}(u_k;A) &\ge \int_{-1/2}^{1/2}\left(\int_A \mathcal{Q}_2\,\bar{f}_0\left(\nabla_\alpha u_k^t\right)\,dx_\alpha + \int_{S(u_k^t)\cap A}\mathcal{R}_2\,\bar{g}_0\left([u_k^t],\bar{\nu}(u_k^t)\right)\sqrt{1-|\nu_3(u_k)|^2}\,d\mathcal{H}^1\right)dt\\
&\ge \int_{-1/2}^{1/2}\left(\int_A \mathcal{Q}_2\,\bar{f}_0\left(\nabla_\alpha u_k^t\right)\,dx_\alpha + \int_{S(u_k^t)\cap A}\mathcal{R}_2\,\bar{g}_0\left([u_k^t],\bar{\nu}(u_k^t)\right)\,d\mathcal{H}^1\right)dt\\
&\quad - C\int_{S(u_k)\cap\left(A\times\left(-\frac{1}{2},\frac{1}{2}\right)\right)}(1+|[u_k]|)\,|\nu_3(u_k)|\,d\mathcal{H}^2,
\end{aligned}$$

where we have used (3.3.6) and the fact that

$$1 - |\nu_3(u_k)| \le 1 - \sqrt{1-|\nu_3(u_k)|^2}.$$

Letting $k\to\infty$ yields, by Fatou's Lemma and (3.3.15)

$$\begin{aligned}
\liminf_{k\to\infty} F_{\varepsilon_k}(u_k;A) &\ge \int_{-1/2}^{1/2}\liminf_{k\to\infty}\left(\int_A \mathcal{Q}_2\,\bar{f}_0(\nabla_\alpha u_k^t)\,dx_\alpha + \int_{S(u_k^t)\cap A}\mathcal{R}_2\,\bar{g}_0([u_k^t],\bar{\nu}(u_k^t))\,d\mathcal{H}^1\right)dt\\
&\ge \int_{-1/2}^{1/2}\left(\int_A \mathcal{Q}_2\,\bar{f}_0\left(\nabla_\alpha u^t\right)\,dx_\alpha + \int_{S(u^t)\cap A}\mathcal{R}_2\,\bar{g}_0\left([u^t],\bar{\nu}(u^t)\right)\,d\mathcal{H}^1\right)dt\\
&= \int_A \mathcal{Q}_2\,\bar{f}_0(\nabla_\alpha u)\,dx_\alpha + \int_{S(u)\cap A}\mathcal{R}_2\,\bar{g}_0\left([u];\nu_\alpha(u)\right)\,d\mathcal{H}^1,
\end{aligned}$$

where we have used Theorem 3.1.2. Given the arbitrariness of the sequence $\{u_k\}$, u, and $A \in \mathcal{A}(\omega)$, we deduce that

$$f(\xi_\alpha) \geq \mathcal{Q}_2\, \bar{f}_0(\xi_\alpha) \quad \text{and} \quad g(z,\mu) \geq \mathcal{R}_2\, \bar{g}_0(z,\mu).$$

This, together with (3.3.14), establishes the identity

$$f(\xi_\alpha) = \mathcal{Q}_2\, \bar{f}_0(\xi_\alpha) \quad \text{and} \quad g(z,\mu) = \mathcal{R}_2\, \bar{g}_0(z,\mu).$$

□

A Appendix

In this Appendix, and for the convenience of the reader, we provide a proof of Theorem 3.1.2 which departs slightly from the original proof of Ambrosio.

Lemma A.0.2. *Let $f\ : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ be a normal integrand. Then for each $\varepsilon_0 > 0$ there exists a compact set $K \subset \Omega$ such that $|\Omega \setminus K| \leq \varepsilon_0$ and $f\ : K \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$ is lower semicontinuous. Moreover, if $(x_0,u_0) \in K \times \mathbb{R}^d$ and if $f(x_0,u_0,\cdot)$ is continuous, then for each $\varepsilon > 0$ and $L > 0$ there exists $\delta = \delta(x_0,u_0,\varepsilon,K,L) \in (0,1)$ such that*

$$f(x_0,u_0,\xi) \leq f(x,u,\xi) + \varepsilon$$

for all $(x,u) \in K \times \mathbb{R}^d$, with $|x-x_0| + |u-u_0| \leq \delta$ and all $|\xi| \leq L$.

Proof. The first part of the lemma follows from Theorem 1.1 in [35]. To prove the second claim assume, for contradiction, that there exist $(x_0,u_0) \in K \times \mathbb{R}^d$, $\bar{L} > 0$, $\bar{\varepsilon} > 0$ and a sequence

$$\{(x_n,u_n,\xi_n)\} \subset K \times \overline{B_d(u_0,1)} \times \overline{B_{d\times N}(0,\bar{L})},$$

such that

$$\bar{\varepsilon} + f(x_n,u_n,\xi_n) < f(x_0,u_0,\xi_n) \tag{A.0.16}$$

and $(x_n,u_n,\xi_n) \to (x_0,u_0,\xi_0)$ as $n \to \infty$, for some $\xi_0 \in \overline{B_{d\times N}(0,\bar{L})}$. Since $f(x_0,u_0,\cdot)$ is continuous and f restricted to $K \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ is lower semicontinuous, for any $\varepsilon < \frac{1}{2}\bar{\varepsilon}$ there exists $\delta > 0$ such that

$$|f(x_0,u_0,\xi) - f(x_0,u_0,\xi_0)| \leq \varepsilon, \qquad f(x_0,u_0,\xi_0) \leq f(x,u,\xi) + \varepsilon$$

for all $(x,u,\xi) \in K \times \overline{B_d(u_0,1)} \times \overline{B_{d\times N}(0,\bar{L})}$ with $|x-x_0| + |u-u_0| \leq \delta$ and $|\xi - \xi_0| \leq \delta$. Thus for all n sufficiently large, and in view of (A.0.16), we have

$$\bar{\varepsilon} + f(x_n,u_n,\xi_n) < f(x_0,u_0,\xi_n) \leq f(x_0,u_0,\xi_0) + \varepsilon \leq f(x_n,u_n,\xi_n) + 2\varepsilon$$

which is a contradiction. □

Remark A.0.3. With a similar proof we can show that if $g : \Omega \times (\mathbb{R}^d)^2 \times S^{N-1} \to [0,+\infty)$ is lower semicontinuous in x and continuous in the remaining variables, then for every $x_0 \in \Omega$, $\varepsilon > 0$, and $L > 0$, there exists $\delta = \delta(x_0,\varepsilon,L) \in (0,1)$ such that

$$g(x_0,\lambda,\theta,\nu) \leq g(x,\lambda,\theta,\nu) + \varepsilon$$

for all $x \in \Omega$, with $|x - x_0| \leq \delta$ and all $(\lambda,\theta,\nu) \in (\mathbb{R}^d)^2 \times S^{N-1}$ with $|\lambda|$, $|\theta| \leq L$.

Given a non negative, finite Radon measure μ on $\mathbb{R}^N$, the *maximal function* $M(\mu)$ of μ is defined by

$$M(\mu)(x) := \sup_{r>0} \frac{\mu(B(x,r))}{|B(x,r)|}.$$

It can be shown that (see [7]) if $u \in SBV_p(\mathbb{R}^N;\mathbb{R}^d)$ then for any $\lambda > 0$ and for any Borel set $E \subset \mathbb{R}^N$ we have that

$$\begin{aligned} |\{x \in E :\ M(|Du|) > 2\lambda\}| &\leq \frac{1}{\lambda^p} \int_{E\cap\{M(|\nabla u|)>\lambda\}} |M(|\nabla u|)|^p\, dx + C(N,d)\frac{1}{\lambda}|D^s u(\mathbb{R}^N)|, \\ \int_{\mathbb{R}^N} |M(|\nabla u|)|^p\, dx &\leq C(N,d,p) \int_{\mathbb{R}^N} |\nabla u|^p\, dx. \end{aligned} \tag{A.0.17}$$

For a proof of the following theorem see [12].

Theorem A.0.4. *Let $u \in BV(\mathbb{R}^N;\mathbb{R}^d)$. Given $\lambda > 1$ there exists a Lipschitz function v such that $u = v$ $\mathcal{L}^N$ a.e. on $\{x \in \mathbb{R}^N : M(|Du|) \le \lambda\}$ and the Lipschitz constant for v is bounded by $C\lambda$ where C depends only on N and d.*

The following lemma was proved by Fonseca and Müller [38] using Young measures.

Lemma A.0.5. *Let $1 < p < +\infty$, let $\{u_n\}$ be a bounded sequence in $L^p(\Omega;\mathbb{R}^d)$. For $\lambda > 0$ consider the truncation $\tau_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ given by*

$$\tau_\lambda(z) := \begin{cases} z & \text{if } |z| \le \lambda, \\ \lambda \frac{z}{|z|} & \text{if } |z| > \lambda. \end{cases}$$

Then there exists a subsequence of $\{u_n\}$ (not relabeled) and an increasing sequence $\lambda_n \to +\infty$ such that the truncated sequence $\{\tau_{\lambda_n} \circ u_n\}$ is p-equi-integrable and

$$||\tau_{\lambda_n} \circ u_n - u_n||_{L^q(\Omega)} \to 0 \quad \text{for all } 1 \le q < p.$$

Remark A.0.6. If $\varepsilon_n > 0$ is any decreasing sequence approaching zero, without loss of generality we may assume that

$$\lim_{n\to\infty} \lambda_n \varepsilon_n = 0.$$

The following result was obtained by Kristensen in [42]. We present here a different proof based on the approach of Fonseca, Müller and Pedregal [41]. In what follows we denote by $\mathcal{T}_\lambda$ and τ_λ truncations in $\mathbb{R}^d$ and $\mathbb{R}$, respectively.

Theorem A.0.7. *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded subset with Lipschitz boundary, and let $\{u_n\} \subset SBV_p(\Omega;\mathbb{R}^d)$ be a sequence converging in $L^1(\Omega;\mathbb{R}^N)$ to some function u_0 and such that*

$$\sup_n \left(\int_\Omega |\nabla u_n|^p \, dx + \int_{S(u_n)\cap\Omega} |[u_n]| \, d\mathcal{H}^{N-1} \right) < \infty$$

and

$$\mathcal{H}^{N-1}(S(u_n) \cap \Omega) \to 0 \quad \text{as } n \to \infty.$$

Then there exists a subsequence $\{u_{n_k}\}$ and a sequence $\{v_k\} \subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that $\{v_k\}$ converges to u_0 weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$, $v_k = u_0$ on $\partial\Omega$,

$$|\{x \in \Omega : v_k(x) \ne u_{n_k}(x)\}| \to 0 \quad \text{as } k \to \infty$$

and $\{|\nabla v_k|^p\}$ is equi-integrable.

Proof. Since Ω is Lipschitz we can extend the functions u_n outside Ω (see e.g. [12]) in such a way that the new functions $U_n \in SBV_p(\mathbb{R}^N;\mathbb{R}^d)$ and

$$\begin{gathered} \int_{\mathbb{R}^N} (|U_n| + |\nabla U_n|^p) \, dx + \int_{S(U_n)} |[U_n]| \, d\mathcal{H}^{N-1} \le C(N,d) \left(\int_\Omega (|u_n| + |\nabla u_n|^p) \, dx + \int_{S(u_n)\cap\Omega} |[u_n]| \, d\mathcal{H}^{N-1} \right), \\ \mathcal{H}^{N-1}(S(U_n)) \to 0 \quad \text{as } n \to \infty. \end{gathered} \tag{A.0.18}$$

By (A.0.17) we have that

$$\sup_n \int_{\mathbb{R}^N} |M(|\nabla U_n|)|^p \, dx < \infty, \tag{A.0.19}$$

and thus by Lemma A.0.5 and Remark A.0.6 there exists an increasing sequence $\lambda_n \to +\infty$ such that

$$\lim_{n\to\infty} \lambda_n^p \mathcal{H}^{N-1}(S(U_n)) = 0, \tag{A.0.20}$$

and the truncated sequence $\{\tau_{\lambda_n} \circ M(|\nabla U_n|)\}$ is p-equi-integrable. Define $V_n(x) := \mathcal{T}_{\lambda_n}(U_n)$. Then (see [11]) $S(V_n) \subset S(U_n)$,

$$|\nabla V_n(x)| \le |\nabla U_n(x)| \quad \mathcal{L}^N \text{ a.e.}, \tag{A.0.21}$$

and, in turn $M(|\nabla V_n|) \le M(|\nabla U_n|)$, so that

$$\{\tau_{\lambda_n} \circ M(|\nabla V_n|)\} \text{ is } p\text{-equi-integrable.} \tag{A.0.22}$$

Set

$$E_n := \{x \in \mathbb{R}^N : M(|DV_n|) \le 2\lambda_n\}.$$

By Theorem A.0.4 there exists a Lipschitz function v_n such that $v_n = V_n$ $\mathcal{L}^N$ a.e. on E_n and $|\nabla v_n| \le C\lambda_n$, where C depends only on N and d. Hence $\mathcal{L}^N$ a.e. on E_n

$$|\nabla v_n| = |\nabla V_n| \le |M(|\nabla V_n|)| \le \begin{cases} |\tau_{\lambda_n} \circ M(|\nabla V_n|)| & \text{if } M(|\nabla V_n|) < \lambda_n, \\ 2|\tau_{\lambda_n} \circ M(|\nabla V_n|)| & \text{if } \lambda_n \le M(|\nabla V_n|) < 2\lambda_n, \end{cases} \tag{A.0.23}$$

while if $M(|\nabla V_n|) \ge 2\lambda_n$ then

$$|\nabla v_n| \le C\lambda_n = 2C|\tau_{\lambda_n} \circ M(|\nabla V_n|)|. \tag{A.0.24}$$

In view of (A.0.22), (A.0.23) and (A.0.24) to prove the equi-integrability of $\{|\nabla v_n|^p\}$ it remains to study the sequence $\{|\nabla v_n|^p \mathcal{L}^N \lfloor F_n\}$ where

$$F_n := \{x \in \Omega \setminus E_n : M(|\nabla V_n|) < 2\lambda_n\}.$$

By (A.0.17) and (A.0.20)

$$\int_{F_n} |\nabla v_n|^p \, dx \le C\lambda_n^p |F_n| + C\lambda_n^{p-1} \int_{S(U_n)} |[V_n]| \, d\mathcal{H}^{N-1} \le C\lambda_n^p \mathcal{H}^{N-1}(S(U_n)) \to 0$$

as $n \to \infty$. Moreover by (A.0.17)

$$\begin{aligned} |\{x \in \mathbb{R}^N : v_n \ne U_n\}| &\le |\{x \in \mathbb{R}^N : |U_n| > \lambda_n\}| + |\{x \in \mathbb{R}^N : M(|DV_n|) > 2\lambda_n\}| \\ &\le |\{x \in \mathbb{R}^N : |U_n| > \lambda_n\}| + \frac{1}{\lambda_n^p} \int_{\{M(|\nabla V_n|) > \lambda_n\}} |M(|\nabla V_n|)|^p \, dx + \frac{C}{\lambda_n} \int_{S(U_n)} |[V_n]| \, d\mathcal{H}^{N-1} \\ &\le \frac{1}{\lambda_n} \int_{\mathbb{R}^N} |U_n| \, dx + \frac{1}{\lambda_n^p} \int_{\{M(|\nabla V_n|) > \lambda_n\}} |M(|\nabla V_n|)|^p \, dx + C\mathcal{H}^{N-1}(S(U_n)) \to 0 \end{aligned}$$

as $k \to \infty$, where we have used the fact that the sequence $\{U_n\}$ is uniformly bounded in $L^1(\mathbb{R}^N; \mathbb{R}^d)$ by construction, (A.0.19) and (A.0.21), as well as (A.0.18).

To prove that $\{v_n\}$ still converges to u_0 in $L^1(\Omega; \mathbb{R}^N)$, we fix n and a ball $B(0, R)$ such that $\Omega \subset B(0, R)$. Define $z_n := v_n - V_n$ and consider a sequence $\{z_{n,j}\} \subset BV(B(0, 4R); \mathbb{R}^d) \cap C^\infty(B(0, 4R); \mathbb{R}^d)$ such that

$$z_{n,j} \to z_n \quad \text{in } L^1(B(0, 4R); \mathbb{R}^d), \qquad ||Dz_{n,j}||(B(0, 4R)) \to ||Dz_n||(B(0, 4R))$$

as $j \to \infty$. Without loss of generality we may assume that $\{z_{n,j}\}$ converges to z_n pointwise $\mathcal{L}^N$ a.e. on $B(0, 4R)$. Since $z_n(x) = 0$ $\mathcal{L}^N$ a.e. on $E_n \cap B(0, 4R)$, we may find a point $x_0 \in E_n \cap \Omega$ such that $z_n(x_0) = 0$ and $z_{n,j}(x_0)$ converges to $z_n(x_0)$. Then $B(x_0, 2R) \subset B(0, 4R)$, and by Poincaré's inequality on balls

$$\int_{B(x_0, 2R)} |z_{n,j}(x) - z_{n,j}(x_0)| \, dx \le C(N, R) \int_{B(x_0, 2R)} |\nabla z_{n,j}(x)| \, dx.$$

Hence

$$\int_{B(x_0, 2R)} |z_{n,j}(x)| \le |z_{n,j}(x_0)| \, |B(x_0, 2R)| + C(N, R) \int_{B(x_0, 2R)} |\nabla z_{n,j}(x)| \, dx,$$

and if we now let $j \to \infty$ we get

$$\int_{B(x_0, 2R)} |z_n(x)| \, dx \le C(N, R) ||Dz_n||(B(0, 4R)),$$

and, in turn,

$$\int_\Omega |v_n(x)| \, dx \le \int_\Omega |U_n(x)| \, dx + C(N, R) ||Dz_n||(B(0, 4R)).$$

Thus the sequence $\{v_n\}$ is uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^N)$, and, up to a subsequence, it converges weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$ to some function $v_0 \in W^{1,p}(\Omega;\mathbb{R}^N)$. On the other hand, since

$$|\{x\in\Omega:\, v_n(x)\neq U_n(x)\}|\to 0$$

we conclude that $v_0 = u_0$ $\mathcal{L}^N$ a.e. on Ω.

Finally, in order to match the boundary conditions for $k\in\mathbb{N}$ let

$$\Omega_k := \left\{x\in\Omega:\, \operatorname{dist}(x,\partial\Omega) < \frac{1}{k}\right\},$$

and consider smooth cut-off functions $\theta_k \in C_c^\infty(\Omega;[0,1])$ such that $\theta_k\equiv 1$ in $\Omega\setminus\Omega_k$. Set $v_{k,n} := \theta_k v_n + (1-\theta_k)u_0$. Then $v_{k,n} = v_n$ on $\Omega\setminus\Omega_k$ and

$$\nabla v_{k,n} = (v_n - u_0)\otimes\nabla\theta_k + \theta_k(\nabla v_n - \nabla u_0),$$

thus

$$\begin{aligned}
&\lim_{k\to\infty}\lim_{n\to\infty}|\{x\in\Omega:\, v_{k,n}(x)\neq u_n(x),\,\nabla v_{k,n}(x)\neq\nabla u_n(x)\}| \le \lim_{k\to\infty}|\Omega_k| = 0,\\
&\lim_{k\to\infty}\lim_{n\to\infty}||v_{k,n}-u_0||_{L^p(\Omega;\mathbb{R}^d)} \le \lim_{n\to\infty}||v_n-u_0||_{L^p(\Omega;\mathbb{R}^d)} = 0,\\
&\lim_{k\to\infty}\lim_{n\to\infty}||\nabla v_{k,n}-(\nabla v_n-\nabla u_0)||_{L^p(\Omega;\mathbb{R}^{d\times N})} \le \lim_{k\to\infty}\lim_{n\to\infty}||\nabla\theta_k||_{L^\infty(\Omega;\mathbb{R}^N}||v_n-u_0||_{L^p(\Omega;\mathbb{R}^d)}\\
&\qquad\qquad + \lim_{k\to\infty}\lim_{n\to\infty}||\nabla v_n-\nabla u_0||_{L^p(\Omega_k;\mathbb{R}^d)} = 0,
\end{aligned}$$

where we have used the equi-integrability of $\{|\nabla v_n|^p\}$. Thus we may extract a subsequence with all the required properties.

$\square$

Next we present a result due to Braides, Defranceschi and Vitali [22] (see also [6]).

Lemma A.0.8. *Let $\{u_n\}\subset SBV_p(Q_\nu;\mathbb{R}^d)\cap L^\infty(Q_\nu;\mathbb{R}^d)$ be a sequence satisfying*

$$\|u_n\|_{L^\infty(Q_\nu;\mathbb{R}^d)} + \mathcal{H}^{N-1}(Q_\nu\cap S(u_n)) \le C, \quad \lim_{n\to\infty}\int_{Q_\nu}|\nabla u_n|^p\,dx = 0,$$

and $u_n(y) = u_{a,b,\nu}(y)$ on ∂Q_ν. Then there exists a sequence $\{v_n\}\subset SBV_0(Q_\nu;\mathbb{R}^d)$ such that

$$\lim_{n\to\infty}\|v_n-u_n\|_{L^\infty(Q_\nu;\mathbb{R}^d)} = 0, \qquad \lim_{n\to\infty}\mathcal{H}^{N-1}(S(v_n)\setminus S(u_n)) = 0$$

and $v_n(y) = u_{a,b,\nu}(y)$ on ∂Q_ν.

Proof. Let k_n be the integer part of

$$\left(\int_{Q_\nu}|\nabla u_n|\,dx\right)^{-1/2},$$

so that $k_n\to\infty$ and $k_n\int_{Q_\nu}|\nabla u_n|\,dx\to 0$ as $n\to\infty$. Take n large enough so that $2/k_n < |b^{(i)}-a^{(i)}|$, whenever $b^{(i)}\neq a^{(i)}$, $i=1,\dots,d$, where $a^{(i)}$ and $b^{(i)}$ are the components of the vectors a and b, respectively. By the Coarea Formula for each $i=1,\dots,d$ we have

$$\begin{aligned}
\int_{Q_\nu}|\nabla u_n^{(i)}|\,dx &= \int_{Q_\nu\setminus S(u_n)}|\nabla u_n^{(i)}|\,dx = \int_{Q_\nu\setminus S(u_n)}|Du_n^{(i)}|\\
&= \sum_{j=0}^{k_n-1}\int_{\alpha_j}^{\alpha_{j+1}}\mathcal{H}^{N-1}\left((Q_\nu\setminus S(u_n))\cap\partial^*\{x\in Q_\nu:\, u_n^{(i)}>t\}\right)dt,
\end{aligned}$$

where $\alpha_0 := -C$, $\alpha_1 := -C+\frac{1}{k_n}$, $\dots,\alpha_{k_n} := C$. It is clear that for each $j\in\{0,\dots,k_n-1\}$ we may find $t_j^{(i)}\in(\alpha_j,\alpha_{j+1})$ such that

$$\mathcal{H}^{N-1}\left(Q_\nu\setminus S(u_n)\cap\partial^*\{x\in Q_\nu:\, u_n^{(i)}>t_j^{(i)}\}\right) < 2\int_{Q_\nu}|\nabla u_n^{(i)}|\,dx.$$

Let $t^{(i)}_{-1} := -C$, $t^{(i)}_{k_n} := C$ and for $j \in \{-1, \dots, k_n - 1\}$ set

$$E^{(i)}_j := \left\{x \in Q_\nu : t^{(i)}_j < u^{(i)}_n \le t^{(i)}_{j+1}\right\}, \qquad v^{(i)}_n(x) := \begin{cases} a^{(i)} & \text{if } x \in E^{(i)}_j \text{ and } t^{(i)}_j < a^{(i)} \le t^{(i)}_{j+1} \\ b^{(i)} & \text{if } x \in E^{(i)}_j \text{ and } t^{(i)}_j < b^{(i)} \le t^{(i)}_{j+1} \\ t^{(i)}_j & \text{otherwise in } E^{(i)}_j, \end{cases}$$

where we have used the fact that $2/k_n < |b^{(i)} - a^{(i)}|$ whenever $b^{(i)} \neq a^{(i)}$. Clearly the sets $E^{(i)}_j = \{x \in Q_\nu : u^{(i)}_n > t^{(i)}_j\} \setminus \{x \in Q_\nu : u^{(i)}_n > t^{(i)}_{j+1}\}$ are of finite perimeter, with $Q_\nu = \cup_{j=-1}^{k_n-1} E^{(i)}_j$. Moreover $v_n(y) = u_{a,b,\nu}(y)$ on ∂Q_ν, $\|v_n - u_n\|_\infty \le 1/k_n$,

$$S(v_n) \subset \bigcup_{i=1}^{d} \bigcup_{j=-1}^{k_n-1} \partial^* E^{(i)}_j \subset \bigcup_{i=1}^{d} \bigcup_{j=-1}^{k_n-1} \partial^*\{x \in Q_\nu : u^{(i)}_n > t^{(i)}_j\} \cup \partial^*\{x \in Q_\nu : u^{(i)}_n > t^{(i)}_{j+1}\},$$

so that

$$\mathcal{H}^{N-1}(S(v_n) \setminus S(u_n)) \le \sum_{i=1}^{d} \sum_{j=-1}^{k_n-1} \Big[\mathcal{H}^{N-1}\left((Q_\nu \setminus S(u_n)) \cap \partial^*\{x \in Q_\nu : u^{(i)}_n > t^{(i)}_j\}\right)$$
$$+\mathcal{H}^{N-1}\left((Q_\nu \setminus S(u_n)) \cap \partial^*\{x \in Q_\nu : u^{(i)}_n > t^{(i)}_{j+1}\}\right)\Big] \le C(k_n + 1) \int_{Q_\nu} |\nabla u_n| \, dx \to 0.$$

□

Proof of Theorem 3.1.2. **Step 1.** We first prove Theorem 3.1.2 under the additional hypotheses

$$f_0(x, u, \xi) \ \ge C_1 |\xi|^p \quad \text{and} \quad g_0(x, \lambda, \theta, \nu) \ \ge C_1 \left(1 + |\lambda - \theta|\right) \tag{A.0.25}$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$, all $u, \lambda, \theta \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$, and for some $C_1 > 0$.

Without loss of generality we may assume that

$$\liminf_{n\to\infty} \left(\int_\Omega f_0(x, u_n, \nabla u_n)\, dx + \int_{S(u_n)\cap\Omega} g_0(x, u^+_n, u^-_n, \nu_{u_n})\, d\mathcal{H}^{N-1}\right)$$
$$= \lim_{n\to\infty} \int_\Omega f_0(x, u_n, \nabla u_n)\, dx + \lim_{n\to\infty} \int_{S(u_n)\cap\Omega} g_0(x, u^+_n, u^-_n, \nu_{u_n})\, d\mathcal{H}^{N-1} < \infty.$$

Passing to a subsequence, if necessary, there exists a nonnegative Radon measure μ such that

$$f_0(x, u_n(x), \nabla u_n(x))\, \mathcal{L}^N \lfloor \Omega + g_0(x, u^+_n, u^-_n, \nu_{u_n})\, \mathcal{H}^{N-1} \lfloor S(u_n) \cap \Omega \overset{\star}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly $\star$ in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge f(x_0, u_n(x_0), \nabla u_n(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega, \tag{A.0.26}$$

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \ge g_0(x_0, u^+(x_0), u^-(x_0), \nu_u(x_0)) \qquad \text{for } \mathcal{H}^{N-1} \text{ a.e. } x_0 \in \Omega \cap S(u). \tag{A.0.27}$$

If (A.0.26) and (A.0.27) hold, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega; \mathbb{R})$, $0 \le \varphi \le 1$. We have

$$\liminf_{n\to\infty} \left(\int_\Omega f_0(x, u_n, \nabla u_n)\, dx + \int_{S(u_n)\cap\Omega} g_0(x, u^+_n, u^-_n, \nu_{u_n})\, d\mathcal{H}^{N-1}\right) \ge \liminf_{n\to\infty} \Big(\int_\Omega \varphi(x)\, f_0(x, u_n, \nabla u_n)\, dx$$
$$+ \int_{S(u_n)\cap\Omega} \varphi(x)\, g_0(x, u^+_n, u^-_n, \nu_{u_n})\, d\mathcal{H}^{N-1}\Big) = \int_\Omega \varphi\, d\mu \ge \int_\Omega \varphi\, \frac{d\mu}{d\mathcal{L}^N}\, dx + \int_{S(u)\cap\Omega} \varphi\, \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}\, d\mathcal{H}^{N-1}$$
$$\ge \int_\Omega \varphi(x)\, f_0(x, u, \nabla u)\, dx + \int_{S(u)\cap\Omega} \varphi(x)\, g_0(x, u^+, u^-, \nu_u)\, d\mathcal{H}^{N-1}.$$

By letting $\varphi \to 1$, and using Monotone Convergence Theorem, we obtain the desired result.

To show (A.0.26) for each $j \in \mathbb{N}$ we apply Lemma A.0.2 to obtain a compact set $K_j \subset \Omega$, with $|\Omega \setminus K_j| \le 1/j$, such that

$$f_0 \ : K_j \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0, +\infty)$$

is lower semicontinuous. Let K_j^* be the set of Lebesgue points of χ_{K_j}, and set

$$\omega := \bigcup_j^{\infty} (K_j \cap K_j^*). \tag{A.0.28}$$

Then

$$|\Omega \setminus \omega| \leq |\Omega \setminus K_j| \leq \frac{1}{j} \to 0 \quad \text{as } j \to \infty.$$

Fix $x_0 \in \omega$ such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N} < \infty, \quad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N+1}} \int_{Q(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x - x_0)| dx = 0. \tag{A.0.29}$$

Choosing $\varepsilon_k \searrow 0$ such that $\mu(\partial Q(x_0, \varepsilon_k)) = 0$, we have

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k \to \infty} \frac{\mu(Q(x_0,\varepsilon_k))}{\varepsilon_k^N} \\
&= \lim_{k \to \infty} \lim_{n \to \infty} \frac{1}{\varepsilon_k^N} \left(\int_{Q(x_0,\varepsilon_k)} f_0(x, u_n, \nabla u_n)\, dx + \int_{S(u_n) \cap Q(x_0,\varepsilon_k)} g_0(x, u_n^+, u_n^-, \nu_{u_n})\, d\mathcal{H}^{N-1} \right) \\
&= \lim_{k \to \infty} \lim_{n \to \infty} \left(\int_Q f_0(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_{n,k}(y), \nabla w_{n,k}(y))\, dy \right. \\
&\left. + \frac{1}{\varepsilon_k} \int_{S(w_{n,k}) \cap Q} g_0(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_{n,k}^+(y), u(x_0) + \varepsilon_k w_{n,k}^-(y), \nu_{w_{n,k}})\, d\mathcal{H}^{N-1} \right),
\end{aligned} \tag{A.0.30}$$

where

$$w_{n,k}(y) := \frac{u_n(x_0 + \varepsilon_k y) - u(x_0)}{\varepsilon_k}.$$

Clearly $w_{n,k} \in SBV_p(Q;\mathbb{R}^d)$, and by $(\text{A.0.29})_2$, $\lim\limits_{k \to \infty} \lim\limits_{n \to \infty} ||w_{n,k} - w_0||_{L^1(Q;\mathbb{R}^d)} = 0$, where $w_0(y) := \nabla u(x_0) y$. Moreover, by $(\text{A.0.29})_1$, (A.0.30) and the coercitivity hypotheses (A.0.25 on f_0 and g_0, we have that

$$\sup_k \int_Q |\nabla w_{n,k}|^p\, dy < \infty, \qquad \lim_{k \to \infty} \lim_{n \to \infty} \int_{S(w_{n,k}) \cap Q} (1/\varepsilon_k + |[w_{n,k}]|)\, d\mathcal{H}^{N-1} < \infty.$$

By a standard diagonalization argument, we may extract a subsequence $w_k := w_{n_k,k}$ which converges to w_0 in $L^1(Q;\mathbb{R}^d)$ and such that

$$\lim_{k \to \infty} \mathcal{H}^{N-1}(S(w_k) \cap Q) = 0, \qquad \sup_k \left(\int_Q |\nabla w_k|^p\, dy + \int_{S(w_k) \cap Q} |[w_k]|\, d\mathcal{H}^{N-1} \right) < \infty,$$

and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \lim_{k \to \infty} \int_Q f_0(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k(y), \nabla w_k(y))\, dy.$$

By Theorem A.0.7 we may find a subsequence (not relabeled) of $\{w_k\}$ and a sequence $\{v_k\} \subset W^{1,\infty}(Q;\mathbb{R}^d)$ weakly converging to w_0 in $W^{1,p}(Q;\mathbb{R}^d)$ such that $v_k = w_0$ on ∂Q,

$$|\{y \in Q : \, v_k(y) \neq w_k(y), \, \nabla v_k(y) \neq \nabla w_k(y)\}| \to 0 \quad \text{as } k \to \infty, \tag{A.0.31}$$

and $\{|\nabla v_k|^p\}$ is equi-integrable. Hence

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{k \to \infty} \int_{\{v_k = w_k\}} f_0(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k(y), \nabla v_k(y))\, dy \\
&= \lim_{k \to \infty} \int_Q f_0(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k v_k(y), \nabla v_k(y))\, dy,
\end{aligned}$$

where we have also used the growth condition on f_0, the p-equi-integrability of $\{v_k\}$ and (A.0.31). As $x_0 \in \omega$ there exists an integer j_0 such that $x_0 \in K_{j_0} \cap K_{j_0}^*$. By Lemma A.0.2 for any fixed integer $j \in \mathbb{N}$ there exists $0 < \rho_j < 1$ such that

$$f_0(x_0, u(x_0), \xi) \leq f_0(x, u, \xi) + \frac{1}{j} \tag{A.0.32}$$

for all $(x,u,\xi) \in K_{j_0} \times \overline{B_d(u(x_0),1)} \times \overline{B_{d\times N}(0,j)}$, with $|x-x_0|+|u-u(x_0)| \le \rho_j$. Set

$$E_{k,j} := \{y \in Q : |\varepsilon_k v_k(y)| \le \rho_j,\ |\nabla v_k(y)| \le j\}.$$

Since $\{v_k\}$ is bounded in $W^{1,1}(Q;\mathbb{R}^d)$, we have that

$$\lim_{j\to\infty} \lim_{k\to\infty} |Q \setminus E_{k,j}| = 0. \tag{A.0.33}$$

Thus

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \lim_{k\to\infty} \int_Q f_0(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_k(y), \nabla v_k(y))\,dy \\
&\ge \lim_{j\to\infty} \lim_{k\to\infty} \int_{E_{k,j}} f_0(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k v_k(y), \nabla v_k(y))\,dy, \\
&= \lim_{j\to\infty} \lim_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{x_0+\varepsilon_k E_{k,j}} f_0(x, u(x_0)+\varepsilon_k v_k((x-x_0)/\varepsilon_k), \nabla v_k((x-x_0)/\varepsilon_k))\,dx.
\end{aligned}$$

By (A.0.32)

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \liminf_{j\to\infty} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{K_{j_0}\cap(x_0+\varepsilon_k E_{k,j})} f_0(x, u(x_0)+\varepsilon_k v_k((x-x_0)/\varepsilon_k), \nabla v_k((x-x_0)/\varepsilon_k))\,dx \\
&\ge \liminf_{j\to\infty} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{K_{j_0}\cap(x_0+\varepsilon_k E_{k,j})} f_0(x_0, u(x_0), \nabla v_k((x-x_0)/\varepsilon_k))\,dx.
\end{aligned} \tag{A.0.34}$$

Using the fact that $|\nabla v_k| \le j$ in $E_{k,j}$ and the growth condition on f_0, we have that for each fixed j

$$\frac{1}{\varepsilon_k^N} \int_{(x_0+\varepsilon_k E_{k,j})\setminus K_{j_0}} f_0(x_0, u(x_0), \nabla v_k((x-x_0)/\varepsilon_k))\,dx \le C(1+j^p)\frac{|Q(x_0,\varepsilon_k)\setminus K_{j_0}|}{\varepsilon_k^N} \to 0$$

as $k\to\infty$, because x_0 is a Lebesgue point of $\chi_{K_{j_0}}$. Consequently, from (A.0.34) we get

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \liminf_{j\to\infty} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{x_0+\varepsilon_k E_{k,j}} f_0(x_0, u(x_0), \nabla v_k((x-x_0)/\varepsilon_k))\,dx \\
&= \lim_{j\to\infty} \lim_{k\to\infty} \int_{E_{k,j}} f_0(x_0, u(x_0), \nabla v_k(y))\,dy = \lim_{k\to\infty} \int_Q f_0(x_0, u(x_0), \nabla v_k(y))\,dy,
\end{aligned}$$

where we have used, once again, the growth condition on f_0, the equi-integrability of $\{|\nabla v_k|^p\}$ and (A.0.33). Since $v_k = w_0$ on ∂Q it now follows from the quasiconvexity of f_0 that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge f_0(x_0, u(x_0), \nabla u(x_0)),$$

thus establishing (A.0.26).

In order to prove (A.0.27) fix $x_0 \in \Omega$ such that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} < \infty, \qquad \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N} \int_{Q_\nu(x_0,\varepsilon)} |u(x)-u_{x_0,\nu}(x)|\,dx = 0. \tag{A.0.35}$$

Choosing $\varepsilon_k \searrow 0$ such that $\mu(\partial Q_\nu(x_0,\varepsilon_k)) = 0$, we have

$$\begin{aligned}
\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &= \lim_{k\to\infty} \frac{\mu(Q_\nu(x_0,\varepsilon_k))}{\varepsilon_k^{N-1}} \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{\varepsilon_k^{N-1}} \left(\int_{Q_\nu(x_0,\varepsilon_k)} f_0(x,u_n,\nabla u_n)\,dx + \int_{S(u_n)\cap Q_\nu(x_0,\varepsilon_k)} g_0(x,u_n^+,u_n^-,\nu_{u_n})\,d\mathcal{H}^{N-1} \right) \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \left(\int_{Q_\nu} \varepsilon_k\, f_0\left(x_0+\varepsilon_k y, w_{n,k}(y), \frac{1}{\varepsilon_k}\nabla w_{n,k}(y)\right)\,dy \right. \\
&\qquad \left. + \int_{S(w_{n,k})\cap Q_\nu} g_0(x_0+\varepsilon_k y, w_{n,k}^+(y), w_{n,k}^-(y), \nu_{w_{n,k}})\,d\mathcal{H}^{N-1} \right),
\end{aligned} \tag{A.0.36}$$

where

$$w_{n,k}(y) := u_n(x_0+\varepsilon_k y).$$

Clearly $w_{n,k} \in SBV_p(Q_\nu;\mathbb{R}^d)$, and by $(A.0.35)_2$, $\lim_{k\to\infty}\lim_{n\to\infty} ||w_{n,k} - u_{x_0,\nu}||_{L^1(Q_\nu;\mathbb{R}^d)} = 0$. Moreover by $(A.0.35)_1$, (A.0.36) and the coercitivity hypotheses on f_0 and g_0 we have that

$$\lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\varepsilon_k^{p-1}} \int_{Q_\nu} |\nabla w_{n,k}|^p \, dy < \infty, \qquad \sup_k \int_{S(w_{n,k})\cap Q_\nu} (1 + |[w_{n,k}]|) \, d\mathcal{H}^{N-1} < \infty.$$

By a standard diagonalization argument, we may extract a subsequence $w_k := w_{n_k,k}$ which converges to $u_{x_0,\nu}$ in $L^1(Q_\nu;\mathbb{R}^d)$ and such that

$$\lim_{k\to\infty} \int_{Q_\nu} |\nabla w_k|^p \, dy = 0, \qquad \sup_k \int_{S(w_k)\cap Q_\nu} (1 + |[w_k]|) \, d\mathcal{H}^{N-1} < \infty \tag{A.0.37}$$

and

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq \lim_{k\to\infty} \int_{S(w_k)\cap Q_\nu} g_0(x_0 + \varepsilon_k y, w_k^+(y), w_k^-(y), \nu_{w_k}) \, d\mathcal{H}^{N-1}.$$

Using Remarks 3.1.4 and 3.1.9 (with $C_{l,k} = D_{l,k} \equiv 1$), we may assume that $\sup_k \|w_k\|_{L^\infty(Q_\nu;\mathbb{R}^d)} \leq M$, for some M, and that $w_k = u_{x_0,\nu}$ on ∂Q_ν. Fix $\eta > 0$. By Remark A.0.3 there exists $\delta = \delta(x_0, \eta, M) \in (0, 1)$ such that

$$g(x_0, \lambda, \theta, \nu) \leq g(x, \lambda, \theta, \nu) + \eta$$

for all $x \in \Omega$, with $|x - x_0| \leq \delta$ and all $(\lambda, \theta, \nu) \in (\mathbb{R}^d)^2 \times S^{N-1}$ with $|\lambda|$, $|\theta| \leq M$. Hence

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq \limsup_{k\to\infty} \int_{S(w_k)\cap Q_\nu} g_0(x_0, w_k^+(y), w_k^-(y), \nu_{w_k}) \, d\mathcal{H}^{N-1} - \eta \sup_j \mathcal{H}^{N-1}(S(w_j) \cap Q_\nu).$$

By letting $\eta \to 0^+$, and in view of A.0.37, we conclude that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq \limsup_{k\to\infty} \int_{S(w_k)\cap Q_\nu} g_0(x_0, w_k^+(y), w_k^-(y), \nu_{w_k}) \, d\mathcal{H}^{N-1}.$$

By Lemma A.0.8, there exists a sequence $\{v_k\} \subset SBV_0(Q_\nu;\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|v_k - w_k\|_{L^\infty(Q_\nu;\mathbb{R}^d)} = 0, \qquad \lim_{n\to\infty} \mathcal{H}^{N-1}(S(v_k) \setminus S(w_k)) = 0$$

and $v_k = u_{x_0,\nu}$ on ∂Q_ν. Then

$$\begin{aligned} \frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) &\geq \limsup_{k\to\infty} \int_{Q_\nu\cap S(w_k)\cap S(v_k)} g_0(x_0, w_k^+, w_k^-, \nu_{w_k}) \; d\mathcal{H}^{N-1} \\ &\geq \limsup_{k\to\infty} \int_{Q_\nu\cap S(v_k)} g_0(x_0, v_k^+, v_k^-, \nu_{v_k}) \; d\mathcal{H}^{N-1} - I_1 - I_2, \end{aligned} \tag{A.0.38}$$

where

$$I_1 := \limsup_{k\to\infty} \int_{Q_\nu\cap(S(v_k)\setminus S(w_k))} g_0(x_0, v_k^+, v_k^-, \nu_{v_k}) \; d\mathcal{H}^{N-1},$$

$$I_2 := \limsup_{k\to\infty} \int_{Q_\nu\cap S(w_k)\cap S(v_k)} \left| g_0(x_0, w_k^+, w_k^-, \nu_{w_k}) - g_0(x_0, v_k^+, v_k^-, \nu_{v_k}) \right| d\mathcal{H}^{N-1}.$$

Since $\mathcal{H}^{N-1}(S(v_k) \setminus S(w_k)) \to 0$ and, by the growth condition (3.1.5), $\{g_0(x_0, v_k^+, v_k^-, \nu_{v_k})\}$ is bounded, we deduce that $I_1 = \mathcal{O}(1/k)$. Also, as $g_0(x_0, \cdot, \cdot, \cdot)$ is uniformly continuous in $B_d(0, M) \times B_d(0, M) \times S^{N-1}$, $\nu_{v_k} = \nu_{w_k}$ $\mathcal{H}^{N-1}$a.e. in $S(w_k) \cap S(v_k)$ and $\|v_k - u_k\|_{L^\infty(Q_\nu;\mathbb{R}^d)} \to 0$, we conclude that $I_2 = \mathcal{O}(1/k)$. Passing to the limit as $k \to +\infty$ in (A.0.38), we get

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0) \geq \limsup_{k\to\infty} \int_{S(v_k)\cap Q_\nu} g_0(x_0, v_k^+, v_k^-, \nu_{v_k}) \, d\mathcal{H}^{N-1} \geq g_0(x_0, u^+(x_0)), u^-(x_0), \nu_u(x_0))$$

due to the BV-ellipticity of g_0.

Step 2. The proof of Theorem 3.1.2 in the general case is now standard. We fix $\varepsilon > 0$ and consider the new integrands

$$f_\varepsilon(x, u, \xi) := f_0(x, u, \xi) + \varepsilon\, |\xi|^p \quad \text{and} \quad g_\varepsilon(x, \lambda, \theta, \nu) := g_0(x, \lambda, \theta, \nu) + \varepsilon\, (1 + |\lambda - \theta|).$$

We may now apply the Step 1 and then by (3.1.6) we may let $\varepsilon \to 0^+$. We omit the details.

□

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References

[1] E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations*, Arch. Rat. Mech. Anal., 86 (1984), pp. 125 –145.

[2] Alberti G., *Integral representation of local functionals*, Ann. Mat. Pura et Appl., (IV) vol. CLXV (1993) 239-274.

[3] Alberti G. and G. Buttazzo, *Integral representation of local functionals* , Manuscr. Math. , 79 (1993) 81-97 .

[4] Alberti G. and C. Mantegazza, *A note on the theory of SBV functions*, Boll. Un. Mat. Ital. B , 11 (1997) 375-382 .

[5] Ambrosio L., *A compactness theorem for a special class of functions of bounded variation*, Boll. Un. Mat. Ital. , 3B 7 (1989) 857-881.

[6] Ambrosio L., *Existence theory for a new class of variational problems*, Arch. Rat. Mech. Anal., 111 (1990) 291-322.

[7] Ambrosio L., *On the lower semicontinuity of quasi–convex integrals in SBV*, Nonlinear Anal., 23 (1994) 405-425.

[8] Ambrosio L. and A. Braides, *Functionals defined on partitions in sets of finite perimeter I: integral representation and* Γ*–convergence*, J. Math. pures et appl., 69 (1990) 285-305.

[9] Ambrosio L. and A. Braides, *Functionals defined on partitions in sets of finite perimeter II: semicontinuity, relaxation and homogenization*, J. Math. pures et appl., 69 (1990) 307-333.

[10] Ambrosio L., Coscia A. and G. Dal Maso, *Fine Properties of functions with bounded deformation*, Arch. Rat. Mech. Anal., 139 (1997) 201-238.

[11] Ambrosio L. and G. Dal Maso, *A general chain–rule for distributional derivatives*, Proc. Am. Math. Soc., 108 (1990) 691-702.

[12] Ambrosio L., Fusco N. and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Mathematical Monographs, Oxford University Press, 2000.

[13] L. Ambrosio, S. Mortola and V.M. Tortorelli, *Functionals with linear growth defined on vector valued BV functions*, J. Math. Pures et Appl., 70 (1991), pp. 269–323.

[14] J. M. Ball and F. Murat, *Remarks on Chacon's biting lemma*, Proc. AMS, 107 (1989), pp.655–663.

[15] Barroso A. C. Bouchitté G., Buttazzo G. and I. Fonseca, *Relaxation of bulk and interfacial energies*, Arch. Ration. Mech. Anal., 135 (1996) 107-173.

[16] Bellieud and G. Bouchitté, *Regularization of a set function. Application to integral representation*, to appear in the special volume II-De Giorgi- Ricerche Mat.

[17] Bouchitté G., Fonseca I. and L. Mascarenhas, *A global method for relaxation* , Arch. Rat. Mech. Anal., 145 (1998) 51-98.

[18] Bouchitté G., Fonseca I. and L. Mascarenhas, *Relaxation of variational problems under trace constraints*, to appear in Nonlinear Analysis.

[19] Braides A., *Approximation of free–discontinuity problems*, Lectures Notes in Math. 1694, Springer-Verlag, 1999.

[20] Braides A. and V. Chiadó Piat, *Integral representation results for functionals defined on* $SBV(\Omega;\mathbb{R}^m)$, J. Math. Pures Appl., 75 (1996) 595-626.

[21] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Clarendon Press, Oxford, 1998.

[22] Braides A., Defranceschi A. and E. Vitali, *Homogenization of free–discontinuity problems*, Arch. Rat. Mech. Anal., 135 (1996) 297-356.

[23] Braides A. and I. Fonseca, *Brittle thin films*, to appear.

[24] Buttazzo G., *Semicontinuity, relaxation and integral representation in the calculus of variations*, Pitman Research Notes in Mathematics Series, 207, John Wiley & Sons, 1989.

[25] Buttazzo G. and G. Dal Maso, *Integral representation and relaxation of local functionals*, Nonlinear Anal. TMA , 9 (1985) 515-532.

[26] Buttazzo G. and G. Dal Maso, *A characterization of nonlinear functionals on Sobolev spaces which admit an integral representation with a Carathéodory integrand*, J. Math. Pures Appl., IX. Ser. 64 (1985) 337-361 .

[27] Carriero M. and A. Leaci , *S^k–valued maps minimizing the L^p norm of the gradient with free discontinuities*, Ann. Scuola Norm. Sup. Pisa, 18 (1991) 321-352.

[28] Celada P. and G. Dal Maso, *Further remarks on the lower semicontinuity of polyconvex integrals*, Ann. Inst. Henri Poincaré, Ann. Non Lin., 11 (1994) 661-691.

[29] Choksi R. and I. Fonseca, *Bulk and interfacial energy densities for structured deformations of continua*, Arch. Ration. Mech. Anal., 138 (1997) 37-103.

[30] Dacorogna B. *Direct Methods in the Calculus of Variations*, Applied Math. Sciences 78, Springer-Verlag (1989).

[31] Dal Maso G., *An Introduction to Γ-Convergence*, Birkhäuser, Boston, 1993.

[32] Dal Maso G. and L. Modica, *Integral functionals determined by their minima*, Rend. Semin. Mat. Univ. Padova , 76 (1986) 255-267.

[33] De Giorgi E. and L. Ambrosio, *Un nuovo tipo di funzionale del Calcolo delle Variazioni*, Atti Accad. Naz. Lincei, 82 (1988) 199-210.

[34] De Giorgi E., Carriero M. and A. Leaci, *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rat. Mech. Anal., 108 (1989) 195-218.

[35] Ekeland I. and R. Temam, *Convex Analysis and Variational Problems*, North-Holland Publishing Company, (1976).

[36] Fonseca I., *The lower quasiconvex envelope of the stored energy function for an elastic crystal*, J. Math. Pures et Appl., 67 (1988), pp. 175–195.

[37] Fonseca I. and G. Leoni, *On lower semicontinuity and relaxation*, to appear in the Proc. Royal Soc. Edinburgh.

[38] Fonseca I. and S. Müller, *Quasi–convex integrands and lower semicontinuity in L^1*, SIAM J. Math. Anal., 23 (1992) 1081-1098.

[39] Fonseca I. and S. Müller, *Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^d)$ for integrands $f(x, u, \nabla u)$*, Arch. Rat. Mech. Anal., 123 (1993) 1-49.

[40] Fonseca I. and S. Müller, *$\mathcal{A}$-quasiconvexity, lower semicontinuity and Young measures*, SIAM J. Math. Anal., 30 (1999) 1355-1390.

[41] Fonseca I., Müller S. and P. Pedregal, *Analysis of concentration and oscillation effects generated by gradients*, SIAM J. Math. Anal. 29, (1998) 736-756.

[42] Kristensen J., *Lower semicontinuity in spaces of weakly differentiable functions*, Math. Ann. 313 (1999) 653-710.

[43] Morse A.P., *Perfect blankets*, Trans. Amer. Math. Soc., 61 (1974) 418-442.

[44] Miranda, M., *Superfici cartesiane generalizzate ed insiemi di perimetro localmente finito sui prodotti cartesiani* (Italian), Ann. Scuola Norm. Sup. Pisa, Sci. Fis. Mat., III. Ser. 18 (1964) 515-542.