A NOTE ON MEYERS' THEOREM IN $W^{k,1}$

IRENE FONSECA, GIOVANNI LEONI, AND ROBERTO PARONI

Abstract. Lower semicontinuity properties of multiple-integrals

$$u \in W^{k,1}(\Omega;\mathbb{R}^d) \mapsto \int_\Omega f(x,u(x),\cdots,\nabla^k u(x))\,dx$$

are studied when f grows at most linearly with respect to the highest order derivative, $\nabla^k u$, and admissible $W^{k,1}(\Omega;\mathbb{R}^d)$ sequences converge strongly in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. It is shown that under certain continuity assumptions on f, convexity or 1-quasiconvexity of $\xi \longmapsto f(x_0,u(x_0),\cdots,\nabla^{k-1}u(x_0),\xi)$ ensure lower semicontinuity. The case where $f(x_0,u(x_0),\cdots,\nabla^{k-1}u(x_0),\cdot)$ is k-quasiconvex remains open except in some very particular cases, as an example when $f(x,u(x),\cdots,\nabla^k u(x)) = h(x)g(\nabla^k u(x))$.

1. Introduction

In a classical paper Meyers [23] proved that k-quasiconvexity is a necessary and sufficient condition for (sequential) lower semicontinuity of a functional

$$u \mapsto \int_\Omega f(x,u(x),\cdots,\nabla^k u(x))\,dx,$$

with respect to weak convergence (resp. weak * convergence if $p=\infty$) in $W^{k,p}(\Omega;\mathbb{R}^d)$ and under appropriate growth and continuity conditions on the integrand f, thus extending to the case $k>1$ the notion of quasiconvexity introduced by Morrey when $k=1$. Here Ω is an open, bounded subset of $\mathbb{R}^N$, with $N \geq 1$, and $k,d \in \mathbb{N}$, $1 \leq p \leq \infty$. Meyers' theorem uses results of Agmon, Douglis and Nirenberg [1] concerning Poisson kernels for elliptic equations. A different proof was later presented by Fusco in [21] using De Giorgi Slicing Lemma. These results have recently been improved by Braides, Fonseca and Leoni in [8], who obtained a general relaxation result in $W^{k,p}(\Omega;\mathbb{R}^d)$ with respect to weak convergence.

In most applications, the lower semicontinuity results mentioned above are completely satisfactory when $p>1$ since bounded sequences in $W^{k,p}(\Omega;\mathbb{R}^d)$ admit weakly convergent subsequences. However, when $p=1$ due to loss of reflexivity of the space $W^{k,1}(\Omega;\mathbb{R}^d)$ one can only conclude that an energy bounded sequence $\{u_n\} \subset W^{k,1}(\Omega;\mathbb{R}^d)$ with

$$\sup_n \|u_n\|_{W^{k,1}} < \infty$$

admits a subsequence (not relabelled) such that

$$u_n \to u \quad \text{in } W^{k-1,1}(\Omega;\mathbb{R}^d), \tag{1.1}$$

where $u \in W^{k-1,1}(\Omega;\mathbb{R}^d)$ and $\nabla^{k-1}u$ is a vector-valued function of bounded variation. In this paper we seek to establish lower semicontinuity in the space $W^{k,1}(\Omega;\mathbb{R}^d)$ under this natural notion of convergence.

When $k=1$ the scalar case $d=1$ has been extensively treated, while the vectorial case $d>1$ was first studied by Fonseca and Müller in [18] where it was proven (sequential) lower semicontinuity in $W^{1,1}(\Omega;\mathbb{R}^d)$

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of a functional

$$u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx,$$

with respect to strong convergence in $L^1(\Omega;\mathbb{R}^d)$ (see also [4], [19], [16], [17] and the references contained therein). The approach in [18] is based on blow–up and truncation methods.

Similar truncation techniques have been used quite successfully in the study of existence and qualitative properties of solutions of second order elliptic equations and systems (see e.g. the work of [7] and the references contained within). Their main drawback lies in the fact that they cannot be easily extended to truncate gradients or higher order derivatives. This may explain in part why several important results for second order elliptic equations have no analog for higher order equations.

The main result of this paper extends Meyers' Theorem to the case where weak convergence in $W^{k,1}(\Omega;\mathbb{R}^d)$ is replaced by (1.1) together with a weak form of coercivity of the convex or 1-quasiconvex density f (see Theorem 2 below). We start with the case where f depends essentially only on x and on the highest order derivatives, that is $\nabla^k u(x)$. This situation is significantly simpler than the general case, since it does not require to truncate the initial sequence $\{u_n\} \subset W^{k,1}(\Omega;\mathbb{R}^d)$. Using the notation and terminology introduced in Section 2, we state the following:

Theorem 1. *Let $f : \Omega \times E^d_{[k-1]} \times E^d_k \to [0,\infty)$ be a Borel integrand. Suppose that for all $(x_0, \mathbf{v}_0) \in \Omega \times E^d_{[k-1]}$ and $\varepsilon > 0$ there exist $\delta_0 > 0$ and a modulus of continuity ρ, with $\rho(s) \le C_0(1+s)$ for $s > 0$ and for some $C_0 > 0$, such that*

$$f(x_0, \mathbf{v}_0, \xi) - f(x, \mathbf{v}, \xi) \le \varepsilon(1 + f(x, \mathbf{v}, \xi)) + \rho(|\mathbf{v} - \mathbf{v}_0|) \tag{1.2}$$

for all $x \in \Omega$ with $|x - x_0| \le \delta_0$, and for all $(\mathbf{v}, \xi) \in E^d_{[k-1]} \times E^d_k$. Assume also that one of the following three conditions is satisfied:
(a) $f(x_0, \mathbf{v}_0, \cdot)$ is k-quasiconvex in E^d_k and

$$\frac{1}{C_1}|\xi| - C_1 \le f(x_0, \mathbf{v}_0, \xi) \le C_1(1 + |\xi|) \qquad \text{for all } \xi \in E^d_k, \tag{1.3}$$

where $C_1 > 0$;
(b) $f(x_0, \mathbf{v}_0, \cdot)$ is 1-quasiconvex in E^d_k and

$$0 \le f(x_0, \mathbf{v}_0, \xi) \le C_1(1 + |\xi|) \qquad \text{for all } \xi \in E^d_k, \tag{1.4}$$

where $C_1 > 0$;
(c) $f(x_0, \mathbf{v}_0, \cdot)$ is convex in E^d_k.

Let $u \in BV^k(\Omega;\mathbb{R}^d)$ and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x, u, \dots, \nabla^k u)\, dx \le \liminf_{n\to\infty} \int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx.$$

Here $\nabla^k u$ is the Radon–Nikodym derivative of the distributional derivative $D^k u$ of $\nabla^{k-1} u$, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$. An important class of integrands which satisfy (1.2) of Theorem 1 is given by

$$f = f(x, \xi) := h(x) g(\xi),$$

where $h(x)$ is a nonnegative lower semicontinuous function and g is a nonnegative function which satisfies either (a) or (b) or (c). The case where $h(x) \equiv 1$ and g satisfies condition (a) was proved by Amar and De Cicco [2]. Theorem 1 extends a result of Fonseca and Leoni [17] to higher order derivatives. Related results when $k = 1$ where obtained previously by Serrin [25] in the scalar case $d = 1$ and by Ambrosio and Dal Maso [4] in the vectorial case $d > 1$ (see also Fonseca and Müller [18], [19]). Even in the simple case $f = f(\xi)$ it is not known if Theorem 1(a) still holds without the coercivity condition

$$f(\xi) \ge \frac{1}{C_1}|\xi| - C_1.$$

When the integrand f depends on the full set of variables in an essential way, the situation becomes significantly more complicated since one needs to truncate gradients and higher order derivatives in order to localize lower order terms. The main result of the paper is given by the following theorem:

Theorem 2. *Let $f:\Omega\times E^d_{[k-1]}\times E^d_k\to[0,\infty)$ be a Borel integrand, with $f(x,\mathbf{v},\cdot)$ 1-quasiconvex in E^d_k, such that*

$$0\le f(x,\mathbf{v},\xi)\le C(1+|\xi|)\qquad \text{for all } (x,\mathbf{v},\xi)\in\Omega\times E^d_{[k-1]}\times E^d_k, \tag{1.5}$$

where $C>0$. Suppose that for all $(x_0,\mathbf{v}_0)\in\Omega\times E^d_{[k-1]}$ either $f(x_0,\mathbf{v}_0,\xi)\equiv 0$ for all $\xi\in E^d_k$, or for every $\varepsilon>0$ there exist C_1, $\delta_0>0$ such that

$$f(x_0,\mathbf{v}_0,\xi)-f(x,\mathbf{v},\xi)\le\varepsilon(1+f(x,\mathbf{v},\xi)), \tag{1.6}$$

$$f(x,\mathbf{v},\xi)\ge C_1|\xi|-\frac{1}{C_1} \tag{1.7}$$

for all $(x,\mathbf{v})\in\Omega\times E^d_{[k-1]}$ with $|x-x_0|+|\mathbf{v}-\mathbf{v}_0|\le\delta_0$ and for all $\xi\in E^d_k$. Let $u\in BV^k(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x,u,\dots,\nabla^k u)\,dx\le\liminf_{n\to\infty}\int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx.$$

A standing open problem is to decide whether Theorem 2 continues to hold under the weaker assumption that $f(x,\mathbf{v},\cdot)$ is k-quasiconvex, which is the natural assumption in this context. The main tool in the proof of Theorems 1-3 is the *blow–up method* introduced by Fonseca and Müller [18], [19], which reduces the domain Ω to a ball and the target function u to a polynomial. Rather than using a smooth truncation of the sequence $\{u_n\}$ within the space $W^{k,1}(\Omega;\mathbb{R}^d)$, we consider one of the type $u_n\,1_{E_n}$ where 1_{E_n} denotes the characteristic function of some set E_n, and thus we need to enlarge the class of admissible functions to include *special functions of bounded variation of order k*. A truncation of this type has been introduced by Carriero, Leaci and Tomarelli in [10].

As in Theorem 1, conditions (1.5) and (1.6) can be considerably weakened if we assume that $f(x,\mathbf{v},\cdot)$ is convex rather than 1-quasiconvex. Indeed we have the following result:

Theorem 3. *Let $f:\Omega\times E^d_{[k-1]}\times E^d_k\to[0,\infty]$ be a lower semicontinuous function, with $f(x,\mathbf{v},\cdot)$ convex in E^d_k. Suppose that for all $(x_0,\mathbf{v}_0)\in\Omega\times E^d_{[k-1]}$ either $f(x_0,\mathbf{v}_0,\xi)\equiv 0$ for all $\xi\in E^d_k$, or there exist C_1, $\delta_0>0$, and a continuous function $g:B(x_0,\delta_0)\times B(\mathbf{v}_0,\delta_0)\to E^d_k$ such that*

$$f(x,\mathbf{v},g(x,\mathbf{v}))\in L^\infty\left(B(x_0,\delta_0)\times B(\mathbf{v}_0,\delta_0);\mathbb{R}\right), \tag{1.8}$$

$$f(x,\mathbf{v},\xi)\ge C_1|\xi|-\frac{1}{C_1} \tag{1.9}$$

for all $(x,\mathbf{v})\in\Omega\times E^d_{[k-1]}$ with $|x-x_0|+|\mathbf{v}-\mathbf{v}_0|\le\delta_0$ and for all $\xi\in E^d_k$. Let $u\in BV^k(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{k,1}(\Omega;\mathbb{R}^d)$ converging to u in $W^{k-1,1}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x,u,\dots,\nabla^k u)\,dx\le\liminf_{n\to\infty}\int_\Omega f(x,u_n,\dots,\nabla^k u_n)\,dx.$$

It is interesting to observe that without a condition of the type (1.8) Theorem 3 is false in general. This has been recently proved by Černý and Malý in [12].

2. Preliminaries

We start with some notation. Here $\Omega\subset\mathbb{R}^N$ is an open, bounded subset, $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ are, respectively, the N dimensional Lebesgue measure and the $N-1$ dimensional Hausdorff measure in $\mathbb{R}^N$. Given $\nu\in S^{N-1}:=\{x\in\mathbb{R}^N:|x|=1\}$ let $\{\nu_1,\cdots,\nu_{N-1},\nu\}$ be an orthonormal basis of $\mathbb{R}^N$ varying continuously with ν, and let $Q_\nu:=\{x\in\mathbb{R}^N:|x\cdot\nu_i|<1/2,\,|x\cdot\nu|<1/2,\,i=1,\cdots,N-1\}$ be a unit cube centered at the origin with two of its faces orthogonal to the direction ν. We set $Q_\nu(x_0,\varepsilon):=x_0+\varepsilon Q_\nu$.

We recall briefly some facts about functions of bounded variation which will be useful in the sequel. A function $u \in L^1(\Omega;\mathbb{R}^d)$ is said to be of *bounded variation*, and we write $u \in BV\left(\Omega;\mathbb{R}^d\right)$, if for all $i = 1,\cdots d$, and $j = 1,\cdots N$, there exists a Radon measure μ_{ij} such that

$$\int_\Omega u_i(x)\,\frac{\partial \varphi}{\partial x_j}(x)\,dx = -\int_\Omega \varphi(x)\,d\mu_{ij}$$

for every $\varphi \in C_0^1(\Omega;\mathbb{R})$. The distributional derivative Du is the matrix–valued measure with components μ_{ij}. Given $u \in BV(\Omega;\mathbb{R}^d)$ the *approximate upper* and *lower limit* of each component u_i, $i = 1,\cdots d$, are given by

$$u_i^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\;|(\{y \in \Omega\cap Q(x,\varepsilon) :\; u_i(y) > t\})| = 0\right\}$$

and

$$u_i^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^N}\;|(\{y \in \Omega\cap Q(x,\varepsilon) :\; u_i(y) < t\})| = 0\right\},$$

while the *jump set* of u, or *singular set*, is defined by

$$S(u) := \bigcup_{i=1}^{d}\{x \in \Omega :\; u_i^-(x) < u_i^+(x)\}.$$

It is well known that $S(u)$ is $N-1$ rectifiable, i.e.

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup E,$$

where $\mathcal{H}^{N-1}(E) = 0$ and K_n is a compact subset of a C^1 hypersurface. If $x \in \Omega\backslash S(u)$ then $u(x)$ is taken to be the common value of $(u_1^+(x),\cdots,u_d^+(x))$ and $(u_1^-(x),\cdots,u_d^-(x))$. It can be shown that $u(x) \in \mathbb{R}^d$ for $\mathcal{H}^{N-1}$ a.e. $x \in \Omega\backslash S(u)$. Furthermore, for $\mathcal{H}^{N-1}$ a.e. $x \in S(u)$ there exist a unit vector $\nu_u(x) \in S^{N-1}$, normal to $S(u)$ at x, and two vectors $u^-(x)$, $u^+(x) \in \mathbb{R}^d$ (the traces of u on $S(u)$ at the point x) such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{\{y\in Q(x_0,\varepsilon):\,(y-x)\cdot\nu_u(x)>0\}} |u(y) - u^+(x)|^{N/(N-1)}dy = 0$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N}\int_{\{y\in Q(x_0,\varepsilon):\,(y-x)\cdot\nu_u(x)<0\}} |u(y) - u^-(x)|^{N/(N-1)}dy = 0.$$

Note that, in general, $(u_i)^+ \neq (u^+)_i$ and $(u_i)^- \neq (u^-)_i$. We denote the *jump of u across $S(u)$* by $[u] := u^+ - u^-$. The distributional derivative Du may be decomposed as

$$Du = \nabla u\,\mathcal{L}^N + (u^+ - u^-)\otimes\nu_u\,\mathcal{H}^{N-1}\lfloor S(u) + C(u),$$

where ∇u is the density of the absolutely continuous part of Du with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$ and $C(u)$ is the Cantor part of Du. These three measures are mutually singular.

The space $SBV(\Omega;\mathbb{R}^d)$ of *special functions of bounded variation*, introduced by De Giorgi and Ambrosio [14], is the space of all functions $u \in BV(\Omega;\mathbb{R}^d)$ such that

$$Du = \nabla u\,\mathcal{L}^N + (u^+ - u^-)\otimes\nu_u\,\mathcal{H}^{N-1}\lfloor S(u).$$

For any multi-index $\alpha = (\alpha_1,\ldots,\alpha_N) \in \mathbb{N}^N$, we use the notation

$$\nabla^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_N^{\alpha_N}}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_N$$

and for each $j \in \mathbb{N}$ the symbol $\nabla^j u$ stands for the vector-valued function whose components are all the components of the $\nabla^\alpha u$ for $|\alpha| = j$. If u is C^∞ then for $j \geq 2$ we have that $\nabla^j u(x) \in E_j^d$, where E_j^d stands for the space of symmetric j-linear maps from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}^d$. We set $E_1^d := \mathbb{R}^{d\times N}$ and

$$E_{[j-1]}^d := \mathbb{R}^d \times E_2^d \times \cdots \times E_{j-1}^d.$$

For any integer $k \geq 2$ we define

$$BV^k(\Omega;\mathbb{R}^d) := \left\{u \in W^{k-1,1}(\Omega;\mathbb{R}^d) : \nabla^{k-1}u \in BV(\Omega;E^d_{k-1})\right\},$$
$$SBV^k(\Omega;\mathbb{R}^d) := \left\{u \in SBV(\Omega;\mathbb{R}^d) : \nabla^j u \in SBV(\Omega;E^d_j) \quad j = 1,\dots,k-1\right\},$$

where $\nabla^j u$ is the Radon–Nikodym derivative of the distributional derivative $D^j u$ of $\nabla^{j-1}u$, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$.

We recall that a function $f : E^d_k \to \mathbb{R}$ is said to be *k-quasiconvex* if

$$f(\xi) \leq \int_Q f(\xi + \nabla^k w(y))\, dy$$

for all $\xi \in E^d_k$ and all $w \in C^\infty_0(Q;\mathbb{R}^d)$.

The following theorem was proved in the case $k = 1$ by Ambrosio and Dal Maso [4], while Fonseca and Müller [18] treated general integrands of the form $f = f(x, u, \nabla u)$, but their argument requires coercivity. The case $k \geq 2$ is due to Amar and De Cicco [2]. For completeness we give a proof for all $k \geq 1$.

Proposition 1. *Let $f : E^d_k \to [0,\infty)$ be a k-quasiconvex function, such that*

$$0 \leq f(\xi) \leq C\,(1 + |\xi|)\,, \tag{2.1}$$

for all $\xi \in E^d_k$. Moreover, when $k \geq 2$ assume that

$$f(\xi) \geq C_1\,|\xi| \quad \textit{for } |\xi| \textit{ large.} \tag{2.2}$$

Let $\{u_n\}$ be a sequence of functions in $W^{k,1}(Q;\mathbb{R}^d)$ converging to 0 in $W^{k-1,1}(Q;\mathbb{R}^d)$. Then

$$f(0) \leq \liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx.$$

Proof. We start with the case $k \geq 2$. Without loss of generality we may assume that

$$\liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx = \lim_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx < \infty$$

so that by condition (2.2)

$$K := \sup_n \int_Q |\nabla^k u_n|\, dx < \infty.$$

Let $\varepsilon > 0$, $M \in \mathbb{N}$, and decompose $L := Q \setminus (1-\varepsilon)\,Q$ into M layers with mutually disjoint interiors, $L_i := \alpha_{i+1}Q \setminus \alpha_i Q$, so that $1 - \varepsilon = \alpha_1 < \alpha_2 < \ldots < \alpha_M < 1 =: \alpha_{M+1}$. Since

$$\sum_{i=1}^{M} \int_{L_i} \left(1 + |\nabla^k u_n|\right)\, dx \leq K + 1$$

for all $n \in \mathbb{N}$, there exists $i_\varepsilon \in \{1,\dots,M\}$ and a subsequence of $\{u_n\}$ (not relabelled) such that

$$\int_{L_{i_\varepsilon}} \left(1 + |\nabla^k u_n|\right)\, dx \leq \frac{K+1}{M} \qquad \text{for all } n \in \mathbb{N}. \tag{2.3}$$

Let $\varphi \in C^\infty_c(Q;[0,1])$ with $\varphi(x) = 1$ in $\alpha_{i_\varepsilon}Q$, $\varphi(x) = 0$ if $x \notin \alpha_{i_\varepsilon+1}Q$. Since f is k-quasiconvex

$$\begin{aligned} f(0) &\leq \liminf_{n\to\infty} \int_Q f(\nabla^k(\varphi u_n))\, dx \\ &\leq \liminf_{n\to\infty} \int_Q f(\nabla^k u_n)\, dx + \int_{Q \setminus \alpha_{i_\varepsilon+1}Q} f(0)\, dx \\ &\quad + C \limsup_{n\to\infty} \int_{L_{i_\varepsilon}} \left(1 + |\nabla^k(\varphi u_n)|\right)\, dx, \end{aligned}$$

where we have used (2.1). As $u_n \to 0$ in $W^{k-1,1}(Q;\mathbb{R}^d)$ strongly, we have

$$\limsup_{n\to\infty} \int_{L_{i_\varepsilon}} \left(1 + |\nabla^k(\varphi u_n)|\right)\, dx \leq \limsup_{n\to\infty} \int_{L_{i_\varepsilon}} \left(1 + |\nabla^k u_n|\right)\, dx \leq \frac{K+1}{M}$$

by (2.3). We conclude that

$$\alpha_{i_\varepsilon+1}f(0) \leq \liminf_{n\to\infty}\int_Q f(\nabla^k u_n)\,dx + \frac{K+1}{M},$$

and the result now follows by letting first $\varepsilon \to 0^+$ and then $M \to \infty$.

Next we prove the result when $k = 1$. As before, let $\varepsilon > 0$, $M \in \mathbb{N}$ and decompose $L := Q \setminus (1-\varepsilon)\,Q$ into M layers with mutually disjoint interiors, $L_i := \alpha_{i+1}Q \setminus \alpha_i Q$, so that $1-\varepsilon = \alpha_1 < \alpha_2 < \ldots < \alpha_M < 1 =: \alpha_{M+1}$ and, in addition $\alpha_{i+1} - \alpha_i = \frac{\varepsilon}{M}$, $i = 1, \ldots, M$. Fix

$$M = M_n := \left[n\int_Q (1 + |\nabla u_n|)\ dx\right] + 1,$$

where $[\cdot]$ denotes the integer part, and let $\varphi_i \in C_c^\infty(Q;[0,1])$ with $\varphi_i(x) = 1$ in $\alpha_i Q$, $\varphi_i(x) = 0$ if $x \notin \alpha_{i+1}Q$, $\|\nabla\varphi_i\|_\infty \leq \frac{2M}{\varepsilon}$, $i = 1, \ldots, M$. We have

$$\begin{aligned}\int_Q f(\nabla(\varphi_i u_n))\,dx \leq \int_Q f(\nabla u_n)\,dx + \int_{Q\setminus\alpha_{i+1}Q} f(0)\,dx \\ + C\int_{L_i}(1+|\nabla u_n|)\ dx + C\frac{2M}{\varepsilon}\int_{L_i}|u_n|\,dx.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{M}\sum_{i=1}^M \int_Q f(\nabla(\varphi_i u_n))\,dx &\leq \int_Q f(\nabla u_n)\,dx + \int_{Q\setminus\alpha_1 Q} f(0)\,dx \\ &\quad + \frac{C}{M}\int_{Q\setminus\alpha_1 Q}(1+|\nabla u_n|)\ dx + \frac{C}{\varepsilon}\int_{Q\setminus\alpha_1 Q}|u_n|\,dx \\ &\leq \int_Q f(\nabla u_n)\,dx + \mathrm{O}(\varepsilon) + \frac{1}{n} + \frac{C}{\varepsilon}\int_{Q\setminus\alpha_1 Q}|u_n|\,dx.\end{aligned}$$

We may, therefore, find $i = i(n,\varepsilon) \in \{1, \ldots, M\}$ such that

$$f(0) \leq \int_Q f(\nabla(\varphi_i u_n))\,dx \leq \int_Q f(\nabla u_n)\,dx + \mathrm{O}(\varepsilon) + \frac{1}{n} + \frac{C}{\varepsilon}\int_{Q\setminus\alpha_1 Q}|u_n|\,dx,$$

and the conclusion follows by letting $n \to \infty$ and then $\varepsilon \to 0^+$. ☐

Next we present two approximation results for convex functions.

Proposition 2. *Let M be a closed set of $\mathbb{R}^p$, let V be a reflexive and separable Banach space. Let $f : M \times V \to (-\infty, +\infty]$ be a $M \times (w - V)$ sequentially lower semicontinuous function, convex in the last variable and such that there exists a continuous function $v_0 : M \to V$ with*

$$(f(\cdot, v_0(\cdot)))^+ \in L^\infty(M;\mathbb{R}). \tag{2.4}$$

Then there exist two sequences of continuous functions

$$a_j : M \to \mathbb{R}, \qquad b_j : M \to V^*,$$

where V^* is the dual space of V, such that

$$f(t,v) = \sup_j\,(a_j(t) + < b_j(t), v >)$$

for all $t \in M$ and $v \in V$. Moreover if f is bounded from below, then (2.4) *can be weakened to*

$$(f(\cdot, v_0(\cdot)))^+ \in L^\infty_{\mathrm{loc}}(M;\mathbb{R}). \tag{2.5}$$

Proposition 2 was proved by Fonseca and Leoni in [17], following closely the argument of Ambrosio in [3], who studied the case where (2.4) is replaced by the assumption that $f(\cdot, v_0(\cdot))$ is continuous.

The following result is due to Serrin (cf. [25]).

Proposition 3. *Let A be an open set of $\mathbb{R}^p$ and let $f : A \times \mathbb{R}^q \to [0, +\infty)$ be a continuous function, convex in the last variable. Then for every pair of positive numbers L, ε, and every compact set C of A, there exists a function $g(t, v)$ with compact support in A, satisfying the same hypotheses as f, and such that*

(i) $g(t, v) \leq f(t, v) + \varepsilon(1 + |v|)$.
(ii) $|g(t, v) - f(t, v)| \leq \varepsilon$ *for* $t \in C$ *and* $|v| \leq L$.
(iii) *There exist constants C_1 and C_2 such that*

$$0 \leq g(t, v) \leq C_1(1 + |v|), \qquad |g(s, v) - g(t, v)| \leq C_2|s - t|(1 + |v|).$$

The following result is due to Fonseca and Müller (see Lemma 2.6 in [18]; see also [22]).

Proposition 4. *Let $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^d)$, let $0 < \alpha < \beta < L$, and let $K > 0$ be such that*

$$\int_{\{|v| \leq L\} \cap Q} |\nabla v(y)|\, dy \leq K. \tag{2.6}$$

Then

$$\operatorname*{essinf}_{t \in [\alpha, \beta]} t\, \mathcal{H}^{N-1}(\{y \in Q : \, |v(y)| = t\}) \leq \frac{K}{\log(\beta/\alpha)}.$$

3. Proof of Theorems 1-3

Proof of Theorem 1. Without loss of generality we may assume that

$$\liminf_{n \to \infty} \int_\Omega f(x, u_n(x), \dots, \nabla^k u_n(x))\, dx = \lim_{n \to \infty} \int_\Omega f(x, u_n(x), \dots, \nabla^k u_n(x))\, dx < \infty.$$

Passing to a subsequence, if necessary, there exists a nonnegative Radon measure μ such that

$$f(x, u_n(x), \dots, \nabla^k u_n(x))\, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly $\star$ in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} \geq f(x_0, u(x_0), \dots, \nabla^k u(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega, \tag{3.1}$$

where $Q_\nu(x_0, \varepsilon) := x_0 + \varepsilon\, Q_\nu$. If (3.1) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega; \mathbb{R})$, $0 \leq \varphi \leq 1$. We have

$$\begin{aligned}\int_\Omega f(x, u_n, \dots, \nabla^k u_n)\, dx &\geq \liminf_{n \to \infty} \int_\Omega \varphi\, f(x, u_n, \dots, \nabla^k u_n)\, dx \\ &= \int_\Omega \varphi\, d\mu \geq \int_\Omega \varphi\, \frac{d\mu}{d\mathcal{L}^N}\, dx \geq \int_\Omega \varphi\, f(x, u, \dots, \nabla^k u)\, dx.\end{aligned}$$

By letting $\varphi \to 1$, and using Lebesgue Dominated Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem, it suffices to show (3.1).

Take $x_0 \in \Omega$ such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} < \infty, \qquad \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{Q(x_0, \varepsilon)} \frac{|u(x) - T_k(x)|}{|x - x_0|^k} dx = 0, \tag{3.2}$$

where

$$T_k(x) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \nabla^\alpha u(x_0)(x - x_0)^\alpha,$$

and set

$$\mathbf{v}_0 := \left(u(x_0), \dots, \nabla^{k-1} u(x_0)\right).$$

Choosing $\varepsilon_h \searrow 0$ such that $\mu(\partial Q(x_0,\varepsilon_h)) = 0$, then

$$\begin{aligned}\lim_{h\to\infty} \frac{\mu(Q(x_0,\varepsilon_h))}{\varepsilon_h^N} &= \lim_{h\to\infty}\lim_{n\to\infty} \frac{1}{\varepsilon_h^N}\int_{Q(x_0,\varepsilon_h)} f(x,u_n,\dots,\nabla^k u_n)\,dx \\ &= \lim_{h\to\infty}\lim_{n\to\infty}\int_Q f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_{n,h}(y), \nabla T_{k-1}(x_0+\varepsilon_h y) \\ &\qquad + \varepsilon_h^{k-1}\nabla w_{n,h}(y), \nabla^2 T_{k-1}(x_0+\varepsilon_h y) + \varepsilon_h^{k-2}\nabla^2 w_{n,h}(y),\dots,\nabla^k w_{n,h}(y))\,dy,\end{aligned}$$

where

$$w_{n,h}(y) := \frac{u_n(x_0+\varepsilon_h y) - T_{k-1}(x_0+\varepsilon_h y)}{\varepsilon_h^k}.$$

Clearly $w_{n,h} \in W^{k,1}(Q;\mathbb{R}^d)$, and by (3.2), $\lim_{h\to\infty}\lim_{n\to\infty} ||w_{n,h} - w_0||_{W^{k-1,1}(Q;\mathbb{R}^d)} = 0$, where

$$w_0(y) := \sum_{|\alpha|=k} \frac{1}{\alpha!}\nabla^\alpha u(x_0) y^\alpha.$$

By a standard diagonalization argument, we may extract a subsequence $w_h := w_{n_h,h}$ which converges to w_0 in $W^{k-1,1}(Q;\mathbb{R}^d)$, and such that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{h\to\infty}\int_Q f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^k w_h(y))\,dy. \tag{3.3}$$

By condition (1.2) for all $\varepsilon > 0$ and for h large enough

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{h\to\infty}\left(\int_Q f(x_0,u(x_0),\dots,\nabla^{k-1}u(x_0),\nabla^k w_h(y))\;dy - \int_Q \rho(|z_h(y)|)\,dy\right),$$

where

$$z_h(y) := \left(T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^{k-1}T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h\nabla^{k-1}w_h(y)\right) - \mathbf{v}_0.$$

By Fatou's Lemma, and since ρ is continuous with $\rho(0)=0$, we have

$$\begin{aligned}C_0 - \limsup_{h\to\infty}\int_Q \rho(\,|z_h(y)|)\,dy &= \liminf_{h\to\infty}\int_Q [C_0(1+\,|z_h(y)|) - \rho(|z_h(y)|)]\,dy \\ &\geq \int_Q \liminf_{h\to\infty}[C_0(1+|z_h(y)|) - \rho(\,|z_h(y)|)]\,dy = C_0,\end{aligned}$$

and so

$$\int_Q \rho(|z_h(y)|)\,dy \to 0 \quad \text{as } h\to\infty.$$

Thus

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{h\to\infty}\int_Q f(x_0,\mathbf{v}_0,\nabla^k w_h(y))\;dy. \tag{3.4}$$

If $g(\xi) := f(x_0,\mathbf{v}_0,\xi)$ satisfies either condition (a) or (b) then we may apply Proposition 1 to conclude that

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq f(x_0,u(x_0),\dots,\nabla^k u(x_0)),$$

and it suffices to let $\varepsilon \to 0^+$. If g is convex then we can write

$$g(\xi) = \sup_j g_j(\xi),$$

where $g_j(\xi)$ is convex, $0 \leq g_j(\xi) \leq g_{j+1}(\xi) \leq C_{j+1}(1+|\xi|)$. From (3.4) and for any fixed j, we have

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \liminf_{h\to\infty}\int_Q g_j(\nabla^k w_h(y))\,dy \geq g_j(\nabla^k u(x_0)),$$

where we have used Proposition 1. By letting $j\to\infty$ we obtain as before that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0,u(x_0),\dots,\nabla^k u(x_0)).$$

□

Proof of Theorem 2. We proceed as in Theorem 1 until (3.3). By (1.5), without loss of generality we may assume that $w_h \in C_c^\infty(Q;\mathbb{R}^d)$. If $f(x_0,\mathbf{v}_0,\xi)\equiv 0$ for all ξ then there is nothing to prove. Otherwise, fix $\varepsilon>0$ and let $0<\delta<\delta_0$, where δ_0 is given by (1.6) and (1.7). For h sufficiently large, we have that

$$\left|\left(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y),\dots,\nabla^{k-1}T_{k-1}(x_0+\varepsilon_h y)\right)-(x_0,\mathbf{v}_0)\right|\le \delta/2. \tag{3.5}$$

Let

$$Q_h := \left\{y\in Q:\ \left|\left(w_h(y),\dots,\nabla^{k-1}w_h(y)\right)\right|\le \delta/2\varepsilon_h\right\}.$$

From (3.3) and (1.7), and for h large,

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)+1\ge\int_{Q_h} f(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k w_h(y),\dots,\nabla^k w_h(y))\,dy \tag{3.6}$$

$$\ge C_1\int_{Q_h}|\nabla^k w_h(y)|\,dy-1/C_1,$$

and so there exists a constant $K>0$ such that

$$\int_{Q_h}\left|\left(\nabla w_h(y),\dots,\nabla^k w_h(y)\right)\right|\,dy\le K \qquad \text{for all } h\in\mathbb{N}. \tag{3.7}$$

Set $\alpha:=\delta/2\varepsilon_h^{1/2}$ and $\beta:=\delta/2\varepsilon_h$. By Proposition 4 we may find $L_h\in(\delta/2\varepsilon_h^{1/2},\delta/2\varepsilon_h)$ such that

$$L_h\mathcal{H}^{N-1}\left(\left\{y\in Q:\left|\left(w_h(y),\dots,\nabla^{k-1}w_h(y)\right)\right|=L_h\right\}\right)\le\frac{2K}{\log(1/\varepsilon_h^{1/2})}. \tag{3.8}$$

Define

$$D_h^+ := \left\{y\in Q:\ \left|\left(w_h(y),\dots,\nabla^{k-1}w_h(y)\right)\right|>L_h\right\};\quad D_h^- := \left\{y\in Q:\ \left|\left(w_h(y),\dots,\nabla^{k-1}w_h(y)\right)\right|\le L_h\right\}.$$

Since $w_h\in C_c^\infty(Q;\mathbb{R}^d)$, the C^∞open set D_h^+ is compactly contained in Q. Define

$$v_h(y):=w_h(y)\,1_{D_h^-}(y),$$

where $1_{D_h^-}$ denotes the characteristic function of the set D_h^-. It is easy to see that $v_h\in SBV^k(\Omega;\mathbb{R}^d)$, with

$$\nabla^j v_h(x)=\begin{cases}\nabla^j w_h(x) & \mathcal{L}^N \text{ a.e. in } D_h^-\\ 0 & \mathcal{L}^N \text{ a.e. in } D_h^+,\end{cases}$$

for $j=1,\dots,k$, so that

$$\int_Q\left|\left(\nabla v_h(y),\dots,\nabla^k v_h(y)\right)\right|\,dy\le K \qquad \text{for all } h\in\mathbb{N}, \tag{3.9}$$

where we have used (3.7) and the fact that $D_h^-\subset Q_h$, while for $j=0,\dots,k-1$, we have that

$$S\left(\nabla^j v_h\right)\subseteq\partial D_h^-$$

so that, from the definition of the sets D_h^- and (3.8),

$$\begin{aligned}&\int_{S(\nabla^j v_h)\cap Q}\left(1+\left|\left[\nabla^j v_h\right]\right|\right)d\mathcal{H}^{N-1} \\ &\quad\le(1+L_h)\,\mathcal{H}^{N-1}\left(\left\{y\in Q:\left|\left(w_h(y),\dots,\nabla^{k-1}w_h(y)\right)\right|=L_h\right\}\right)\le\frac{4K}{\log(1/\varepsilon_h^{1/2})}\to 0\end{aligned} \tag{3.10}$$

as $h\to\infty$. Moreover for $j=0,\dots,k-1$,

$$\begin{aligned}\|\nabla^j v_h-\nabla^j w_0\|_{L^1(Q;E_j^d)}&=\|1_{D_h^-}\nabla^j w_h-\nabla^j w_0\|_{L^1(Q;E_j^d)}\\&\le\|\nabla^j w_h-\nabla^j w_0\|_{L^1(Q;E_j^d)}+\left\|\nabla^j w_0\right\|_{L^1(D_h^+;E_j^d)}\\&\le\|\nabla^j w_h-\nabla^j w_0\|_{L^1(Q;E_j^d)}+\|\nabla^j w_0\|_\infty\left|D_h^+\right|\to 0 \qquad \text{as } h\to\infty,\end{aligned}$$

because

$$
\begin{aligned}
0 \le \left|D_h^+\right| &= \left|\left\{y \in Q : \left|\left(w_h(y), \dots, \nabla^{k-1} w_h(y)\right)\right| > L_h\right\}\right| \\
&\le \left|\left\{y \in Q : \left|\left(w_h(y), \dots, \nabla^{k-1} w_h(y)\right) - \left(w_0(y), \dots, \nabla^{k-1} w_0(y)\right)\right| \ge 1\right\}\right| \\
&\le ||w_h - w_0||_{W^{k-1,1}(Q;\mathbb{R}^d)} \to 0 \qquad \text{as } h \to \infty,
\end{aligned} \tag{3.11}
$$

where we have used the fact that $L_h > 1 + || \left(w_0, \dots, \nabla^{k-1} w_0\right) ||_\infty$ for h large enough. By (3.5), the definition of the set D_h^- and the fact that $\varepsilon_h \to 0$, we have that

$$
\left|\left(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y), \dots, \nabla^{k-1} T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h \nabla^{k-1} w_h(y)\right) - (x_0, \mathbf{v}_0)\right| \le \delta,
$$

for $y \in D_h^-$, and thus by (3.3), (1.5) and (1.6),

$$
\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \lim_{h\to\infty} \int_{D_h^-} f(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k w_h(y), \dots, \nabla^k w_h(y))\, dy \\
&\ge \lim_{h\to\infty} \left(\int_{D_h^-} f(x_0, \mathbf{v}_0, \nabla^k w_h(y))\, dy - \varepsilon C \int_{D_h^-} (1 + |\nabla^k w_h(y)|)\, dy \right) \\
&\ge \lim_{h\to\infty} \int_{D_h^-} f(x_0, \mathbf{v}_0, \nabla^k v_h(y))\, dy - \varepsilon C(1 + K),
\end{aligned} \tag{3.12}
$$

where we have used (3.7). Moreover by (3.11)

$$
\int_{D_h^+} f(x_0, \mathbf{v}_0, \nabla^k v_h(y))\, dy = \int_{D_h^+} f(x_0, \mathbf{v}_0, 0)\, dy \to 0 \quad \text{as } h \to \infty,
$$

and so, from (3.12), we deduce that

$$
\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \liminf_{h\to\infty} \int_Q f(x_0, \mathbf{v}_0, \nabla^k v_h(y))\ dy - C\varepsilon(1 + K). \tag{3.13}
$$

Let $z_h(y) := \nabla^{k-1} v_h(y)$. Then $z_h \in SBV(Q; E_{k-1}^d)$, with $||z_h - \nabla^{k-1} w_0||_{L^1(Q;E_{k-1}^d)} \to 0$. Moreover, since $D_h^+ \Subset Q$ we have that $z_h(y) \equiv 0$ in a neighborhood of ∂Q, so that we may extend z_h to be zero outside Q, without introducing further jumps in z_h nor in its derivatives. Let $\{\varphi_\delta\}$, $\delta > 0$, be a family of C^∞ mollifiers and define

$$
Z_{h,\delta}(y) := (\varphi_\delta * z_h)(y) := \int_{\mathbb{R}^N} \varphi_\delta(y - x)\, z_h(x)\, dx.
$$

Then

$$
\lim_{h\to\infty} \lim_{\delta\to 0^+} ||Z_{h,\delta} - \nabla^{k-1} w_0||_{L^1(Q;E_{k-1}^d)} = 0. \tag{3.14}
$$

Since $\nabla Z_{h,\delta} = \varphi_\delta * Dz_h = \varphi_\delta * \nabla z_h + \varphi_\delta * D^s z_h$, by the Lipschitz continuity of $f(x_0, \mathbf{v}_0, \cdot)$, which follows from 1-quasiconvexity (see e.g. [13]), we obtain

$$
\begin{aligned}
\int_Q f(x_0, \mathbf{v}_0, \nabla Z_{h,\delta}(y))\ dy &\le \int_Q f(x_0, \mathbf{v}_0, \varphi_\delta * \nabla z_h(y))\ dy + C\,|D^s z_h|\,((1 + \delta)\,Q) \\
&= \int_Q f(x_0, \mathbf{v}_0, \varphi_\delta * \nabla^k v_h(y))\ dy + C \int_{S(\nabla^{k-1} v_h)\cap Q} \left(1 + \left|\left[\nabla^{k-1} v_h\right]\right|\right) d\mathcal{H}^{N-1},
\end{aligned} \tag{3.15}
$$

where we have used the fact that z_h has compact support in Q. Since for each fixed h

$$
\lim_{\delta\to 0^+} \int_Q f(x_0, \mathbf{v}_0, \varphi_\delta * \nabla^k v_h(y))\ dy = \int_Q f(x_0, \mathbf{v}_0, \nabla^k v_h(y))\ dy
$$

by (3.13), (3.15) we have

$$
\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{h\to\infty} \liminf_{\delta\to 0^+} \int_Q f(x_0, \mathbf{v}_0, \nabla Z_{h,\delta}(y))\; dy \\
&\quad - \limsup_{h\to\infty} C \int_{S(\nabla^{k-1}v_h)\cap Q} \left(1 + \left|\left[\nabla^{k-1}v_h\right]\right|\right) d\mathcal{H}^{N-1} - \varepsilon C(1+K) \qquad (3.16)\\
&= \liminf_{h\to\infty} \liminf_{\delta\to 0^+} \int_Q f(x_0, \mathbf{v}_0, \nabla Z_{h,\delta}(y))\; dy - \varepsilon C(1+K),
\end{aligned}
$$

by virtue of (3.10). By (3.14), the 1-quasiconvexity of $f(x_0, \mathbf{v}_0, \cdot)$, and applying Proposition 1 we have

$$
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \dots, \nabla^k u(x_0)) - \varepsilon C(1+K).
$$

Now we let $\varepsilon \to 0^+$. □

Proof of Theorem 3. We proceed as in Theorem 1 until (3.3). By Proposition 2, with $M = (x_0 + \varepsilon_1\overline{Q}) \times \overline{B(\mathbf{v}_0, \delta)}$ and $V = E^d_k$, there exist two sequences of continuous functions

$$
a_j : M \to \mathbb{R}, \qquad b_j : M \to E^d_k,
$$

such that

$$
f(x, \mathbf{v}, \xi) = \sup_j \left(a_j(x, \mathbf{v}) + b_j(x, \mathbf{v}) \cdot \xi\right)
$$

for all $(x, \mathbf{v}) \in M$ and $\xi \in E^d_k$. Define

$$
f_j(x, \mathbf{v}, \xi) := \sup_{i\leq j} \left(a_i(x, \mathbf{v}) + b_i(x, \mathbf{v}) \cdot \xi\right)^+ .
$$

Then $f_j(x, \mathbf{v}, \xi) \leq f(x, \mathbf{v}, \xi)$ and $f_j(x, \mathbf{v}, \xi) \to f(x, \mathbf{v}, \xi)$ as $j \to \infty$. Moreover, f_j is continuous, convex in ξ and

$$
0 \leq f_j(x, \mathbf{v}, \xi) \leq C_j(|\xi| + 1) \qquad (3.17)
$$

for all $(x, \mathbf{v}) \in M$ and $\xi \in E^d_k$, where

$$
C_j := \max\{A_j(x, \mathbf{v}) : (x, \mathbf{v}) \in M\}, \quad \text{with} \quad A_j(x, \mathbf{v}) = \sup_{i\leq j}\{|a_i(x, \mathbf{v})| + |b_i(x, \mathbf{v})|\}.
$$

By (3.3) for any fixed j, and with the notation introduced in the proof of Theorem 2,

$$
\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \lim_{h\to\infty} \int_{D_h^-} f(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k w_h(y), \dots, \nabla^k w_h(y))\, dy \\
&\geq \lim_{h\to\infty} \int_{D_h^-} f_j(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k w_h(y), \dots, \nabla^k w_h(y))\, dy \qquad (3.18)\\
&= \lim_{h\to\infty} \int_{D_h^-} f_j(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k v_h(y), \dots, \nabla^k v_h(y))\, dy.
\end{aligned}
$$

Moreover, by (3.17) and (3.11)

$$
\int_{D_h^+} f_j(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k v_h(y), \dots, \nabla^k v_h(y))\; dy \leq C_j \int_{D_h^+} (1 + |\nabla^k v_h|\,) dy = C_j \left|D_h^+\right| \to 0
$$

as $h \to \infty$, and so, from (3.18)

$$
\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{h\to\infty} \int_Q f_j(x_0 + \varepsilon_h y, T_{k-1}(x_0 + \varepsilon_h y) + \varepsilon_h^k v_h(y), \dots, \nabla^k v_h(y))\; dy. \qquad (3.19)
$$

We now fix $\varepsilon > 0$, and apply Proposition 3 with

$$
A := (x_0 + \varepsilon_1 Q) \times E^d_{[k-1]}, \quad C := (x_0 + \varepsilon_2\overline{Q}) \times \overline{B(\mathbf{v}_0, \delta)} \quad \text{and} \quad L := |\nabla^k u(x_0)|,
$$

to obtain a function g_j such that, by (3.19),

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &\geq \liminf_{h\to\infty}\left(\int_Q g_j(x_0+\varepsilon_h y, T_{k-1}(x_0+\varepsilon_h y)+\varepsilon_h^k v_h(y),\dots,\nabla^k v_h(y))\,dy - \varepsilon\int_Q (1+|\nabla^k v_h(y)|)\,dy\right)\\ &\geq \liminf_{h\to\infty}\left(\int_Q g_j(x_0,\mathbf{v}_0,\nabla^k v_h(y))\,dy - (C_2\delta+\varepsilon)\int_Q(1+|\nabla^k v_h(y)|)\,dy\right)\\ &\geq \liminf_{h\to\infty}\int_Q g_j(x_0,\mathbf{v}_0,\nabla^k v_h(y))\,dy-(C_2\delta+\varepsilon)(1+K)\end{aligned}$$

by Proposition 2(iii) and (3.9), and where we have used the fact that

$$\left|\left(v_h(y),\dots,\nabla^{k-1}v_h(y)\right)\right| \leq \delta/2\varepsilon_h.$$

We can now continue as in the previous theorem to conclude that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq g_j(x_0,u(x_0),\cdots,\nabla^k u(x_0)) - (C_2\delta+\varepsilon)(1+K).$$

By applying Proposition 3(ii) we have

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f_j(x_0,u(x_0),\cdots,\nabla^k u(x_0)) - (C_2\delta+\varepsilon)(2+K).$$

Now we let first $\delta\to 0^+$, then $\varepsilon\to 0^+$, and finally $j\to\infty$. ☐

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Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213
E-mail address: fonseca@cmu.edu

Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Alessandria, Italy 15100
E-mail address: leoni@unipmn.it

Mathematical Institute, University of Oxford, 24-29 St. Giles, Oxford OX1 3LB UK
E-mail address: paroni@maths.ox.ac.uk