An Existence Result for a Nonconvex Variational Problem via Regularity

Irene Fonseca, Nicola Fusco, Paolo Marcellini

Abstract

Local Lipschitz continuity of minimizers of certain integrals of the Calculus of Variations is obtained when the integrands are convex with respect to the gradient variable, but are *not necessarily uniformly convex*. In turn, these regularity results entail existence of minimizers of variational problems with non-homogeneous integrands *nonconvex* with respect to the gradient variable. The x-dependence, explicitly appearing in the integrands, not only adds significant technical difficulties, but also yields some unexpected phenomena.

Keywords: nonconvex variational problems, uniform convexity, regularity, implicit differential equations.

1991 Mathematics Subject Classification: Primary 49J45, 49K20. Secondary 35F30, 35R70.

1 Introduction

In this paper we establish existence and regularity of minimizers of energy integrals of the type

$$\int_{\Omega} f\left(x, Du(x)\right)\, dx\ , \tag{1.1}$$

subject to Dirichlet boundary conditions. The main feature of our problem is the fact that the integrand $f = f(x, \xi)$ is not convex with respect to the gradient variable ξ.

In recent years the study of non convex variational problems has undergone remarkable developments, motivated in part by advances in the study of material stability and instability. Contemporary issues such as phase transitions in certain alloys (see [3], [4]), nucleation [20], the onset of microstructure, and optimal design problems for thin films [18], require a good understanding of existence of (classical or generalized) equilibrium solutions for nonconvex energies. In addition, qualitative information on quasistatic solutions (e.g. regularity, hysteresis, oscillatory behavior) are needed in order to develop the evolutionary framework, and, in particular, to search for the dynamical evolution of phase boundaries. These issues have challenged traditional theories.

Within this setting, a relevant example has been considered by Ball-James [3], [4], who studied the two "*potential wells*" problem:

$$\text{minimize} \int_{\Omega} f\left(Du(x)\right)\, dx\ ,$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, $f : \mathbb{R}^{n\times n} \to [0,+\infty)$ is identically zero on two distinct potential wells $SO(n)\xi$, $SO(n)\eta$ and $f > 0$ elsewhere. Here $\xi,\eta \in \mathbb{R}^{n\times n}$ and $SO(n)$ stands for the special orthogonal group. The existence of minimizers for the two potential wells problem has been obtained in two dimensions (i.e., $n = 2$) by Dacorogna–Marcellini [10] and by Müller–Sverak [26] (for the case $n = 3$ see also Dolzmann–Kirchheim–Müller–Sverak [15]). Nothing is known in higher dimension or for general integrands f as in (1.1).

In this paper we restrict ourselves to the scalar-valued case, as a starting point to approach the vectorial setting. Also, the scalar-valued case is still far from being completely understood, unless the integrand f depends only on the gradient variable ξ and some special assumptions are made on the boundary data (see the references quoted below). Here we consider general boundary data $u_0 \in W^{1,p}(\Omega)$, $p > 1$, and we allow the nonconvex integrands f to explicitly depend on x as in (1.1). In the proofs of the attainment results presented below the x-dependence not only introduces substantial technical difficulties, but also seems to yield some unexpected phenomena.

The proof of the existence results for nonconvex variational problems considered in this paper hinges on the local Lipschitz continuity of minimizers of the relaxed problem associated to the *bipolar* f^{**} of f. These regularity results are presented in Section 2, and they apply to minimizers of some integrals of the Calculus of Variations with integrands $f^{**}(x,\xi)$ convex with respect to $\xi \in \mathbb{R}^n$, but not everywhere *uniformly convex*; hence, we believe that the regularity results presented in Section 2 should be of interest by themselves.

In Section 3 we consider the variational problem

$$\inf\left\{\int_{\Omega} f\left(x, Du(x)\right)\, dx : u \in u_0 + W_0^{1,p}(\Omega)\right\}, \tag{1.2}$$

where $u_0 \in W^{1,p}(\Omega)$ is a given boundary datum and $f = f(x,\xi)$ is a continuous function satisfying some growth conditions similar to the ones considered in the previous Section 2, so as to ensure Lipschitz continuity of minimizers of the relaxed problem. Here the most relevant fact is that f may be *nonconvex* with respect to the variable $\xi \in \mathbb{R}^n$.

It is known that the variational problem (1.2) may lack a minimizer (see Marcellini [22]; see also [6], [21], [12]). In the examples of nonexistence the following condition, expressed in terms of the *bipolar* f^{**} of f, is violated: for every $x \in \Omega$ the function $f^{**}(x,\cdot)$ is *affine* on the set

$$A(x) = \{\xi \in \mathbb{R}^n :\ f(x,\xi) > f^{**}(x,\xi)\}\,,$$

i.e., there exist a continuous function q and a vector field m of class C^1, defined in the open set $\Omega_A := \{x \in \Omega : A(x) \neq \emptyset\}$, such that

$$f^{**}(x,\xi) = q(x) + \langle m(x), \xi \rangle \,, \quad \forall\, x \in \Omega_A, \;\; \xi \in A(x) \,. \tag{1.3}$$

We also assume that the *boundary* (more precisely, the part of the boundary in Ω) of the set

$$\{x \in \Omega_A : \; \operatorname{div} m(x) = 0\} \tag{1.4}$$

has zero (n-dimensional) measure.

In this paper we prove that (1.3), (1.4) (see also the more general assumptions made in Section 3.2) are sufficient conditions for existence of minimizers to the variational problem (1.2). We emphasize that we do not require any other condition on the vector field m other than (1.4); in particular, we do not assume that the vector field m has null divergence. We notice that, while condition (1.3) is necessary for guaranteeing the existence of minimizers (see [22], [6], [21]), we do not know whether condition (1.4) may be removed.

Existence theorems without convexity assumptions have been widely investigated in the one-dimensional case $n = 1$ (see [12] for an extensive list of references). Theorem 3.1 in Section 3 is specific to the case $n \geq 2$, and it is an extension of some analogous results, obtained under more restrictive assumptions, by Marcellini [22], Mascolo-Schianchi [23], Cellina [7] and Friesecke [21]. In particular, Theorem 3.1 is an extension of related results recently proved by Sychev [28] and Zagatti [29] for integrands independent of x and under a strong assumption on the growth of f which ensures the almost everywhere differentiability of minimizers, i.e., $p > n$, by Celada-Perrotta [5] for $p > 1$, and by Dacorogna-Marcellini in [13], [12].

Finally we recall that Marcellini [22] pointed out the necessity of the condition of affinity (1.3) of the function f^{**} on the set where $f \neq f^{**}$ to guarantee existence of minimizers. Cellina [6], [7] and Friesecke [21] proved the necessity and sufficiency of the condition of affinity for linear boundary data u_0. The explicit dependence of the integrand on the variable x was first considered by Mascolo-Schianchi in [24], assuming that the divergence of the vector field m in (1.3) is identically equal to zero in Ω, in addition to other strong assumptions on the boundary data u_0. Also, in [27] Raymond studied a case where the divergence of the vector field m in (1.3) is always different from zero in Ω, and some type of explicit dependence on u is allowed.

2 Local Lipschitz Continuity

2.1 Preliminary results

Let $f : \mathbb{R}^n \to [0, +\infty)$ be a continuous function such that

$$0 \leq f(\xi) \leq L(1 + |\xi|^p) \,, \tag{2.1}$$

where $L > 0$, $p > 1$. We say that f is *uniformly convex at infinity* if there exist $R, \nu > 0$ such that, if the segment with endpoints ξ_1, ξ_2 (that we will denote by $[\xi_1, \xi_2]$) is contained in the complement of the closed ball $\overline{B}_R$, then

$$f\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{2}f(\xi_1) + \frac{1}{2}f(\xi_2) - \nu(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}|\xi_1-\xi_2|^2 . \qquad (2.2)$$

Note that (2.2) is equivalent to

$$f\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{2}f(\xi_1) + \frac{1}{2}f(\xi_2) - \nu'(|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}|\xi_1-\xi_2|^2 ,$$

for some $\nu' > 0$, since $|\xi_1|, |\xi_2| > R > 0$.

If the above inequality (2.2) is satisfied for any $\xi_1, \xi_2 \in \mathbb{R}^n$, then we say that f is *uniformly convex* in $\mathbb{R}^n$. A form of uniform convexity at infinity was also considered by Mascolo and Schianchi in [25].

The following result is proved in [1].

Lemma 2.1 *If $\gamma > -1/2$ then there exist positive constants $c_1 = c_1(\gamma)$, $c_2 = c_2(\gamma)$ such that, for all $\xi, \eta \in \mathbb{R}^n$,*

$$c_1(1+|\xi|^2+|\eta|^2)^\gamma \leq \int_0^1 t(1+|t\xi+(1-t)\eta|^2)^\gamma \, dt \leq c_2(1+|\xi|^2+|\eta|^2)^\gamma .$$

The following result provides two conditions which are equivalent to uniform convexity in $\mathbb{R}^n$.

Proposition 2.2 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a continuous function satisfying (2.1). The following conditions are equivalent:*

(i) f is uniformly convex in $\mathbb{R}^n$;

(ii) $f(\xi) = c_1\nu(1+|\xi|^2)^{p/2} + g(\xi)$, for some $c_1 = c_1(p) > 0$, where $g(\xi)$ is a convex function such that $0 \leq g(\xi) \leq L(1+|\xi|^p)$ for all ξ;

(iii) $\int_Q [f(\xi + D\varphi(x)) - f(\xi)]\, dx \geq c_2\nu \int_Q (1+|\xi|^2+|D\varphi|^2)^{\frac{p-2}{2}}|D\varphi(x)|^2\, dx$, for all $\xi \in \mathbb{R}^n$, $\varphi \in C_0^1(Q)$, where $Q = (0,1)^n$ and $c_2 = c_2(p)$ is a suitable constant.

Remark 2.3 The condition (iii) in Proposition 2.2 is related to the notion of *uniform quasiconvexity*, introduced by Evans [16] and later studied by Evans-Gariepy [17].

Proof of Proposition 2.2. (i) $\Longrightarrow$ (ii). We define $g(\xi) := f(\xi) - c_1\nu(1+|\xi|^2)^{p/2}$, where c_1 will be chosen later, and we show that g is convex. Given $\xi_1, \xi_2 \in \mathbb{R}^n$ we set $\xi := (\xi_1+\xi_2)/2$. From (i) we easily get that

$$\frac{1}{2}g(\xi_1) + \frac{1}{2}g(\xi_2) \geq g(\xi) + \nu(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}|\xi_1-\xi_2|^2$$
$$+\frac{c_1\nu}{2}\left[2(1+|\xi|^2)^{p/2} - (1+|\xi_1|^2)^{p/2} - (1+|\xi_2|^2)^{p/2}\right].$$

Thus the assertion follows immediately from the fact that there exists a constant $c = c(p)$ such that

$$(1+|\xi_1|^2)^{p/2} + (1+|\xi_2|^2)^{p/2} - 2(1+|\xi|^2)^{p/2} \le c(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}|\xi_1-\xi_2|^2\,,$$

and by setting $c_1 := c^{-1}$. To establish this inequality we write, for $i = 1, 2$,

$$h(\xi_i) = h(\xi) + \langle Dh(\xi), \xi_i - \xi\rangle$$
$$+\int_0^1 (1-t)\left\langle D^2h(\xi + t(\xi_i-\xi))(\xi_i-\xi), \xi_i-\xi\right\rangle\, dt$$

where $h(\xi) := (1+|\xi|^2)^{p/2}$, yielding

$$(1+|\xi_i|^2)^{p/2} - (1+|\xi|^2)^{p/2} \le p(1+|\xi|^2)^{\frac{p-2}{2}}\langle \xi_i-\xi, \xi\rangle$$
$$+c(p)|\xi_i-\xi|^2\int_0^1 (1-t)(1+|\xi+t(\xi_i-\xi)|^2)^{\frac{p-2}{2}}\, dt\,.$$

It suffices now to sum the above inequalities for $i = 1, 2$ and apply Lemma 2.1.

(ii) $\Longrightarrow$ (iii). From Lemma 2.1 we easily get that (iii) holds for the function $\xi \mapsto (1+|\xi|^2)^{p/2}$. Hence the general case follows from Jensen's inequality applied to g.

(iii) $\Longrightarrow$ (i). Given $\xi_1, \xi_2 \in \mathbb{R}^n$ we set $\xi := (\xi_1+\xi_2)/2$, $\lambda := \xi_2-\xi_1$ and denote by χ the characteristic function of the interval $(0, 1/2)$ extended periodically to $\mathbb{R}$ with periodicity 1. Define

$$u_h(x) := |\lambda|\int_0^{\langle x, \lambda/|\lambda|\rangle} (\chi(ht) - 1/2)\, dt\,.$$

Then $u_h \to 0$ uniformly in Q, as $h \to +\infty$, and since

$$Du_h(x) = (\chi(\langle hx, \lambda/|\lambda|\rangle) - 1/2)\,\lambda, \qquad \|Du_h\|_\infty = |\lambda|/2\,,$$

we have that

$$\frac{1}{2}f(\xi_1) + \frac{1}{2}f(\xi_2) = \lim_{h\to+\infty}\int_Q f(\xi + Du_h(x))\, dx\,. \tag{2.3}$$

Fix $t \in (0, 1/2)$, set $Q_t = (t, 1-t)^n$, and let $\varphi \in C_0^1(Q)$ be a function such that $0 \le \varphi(x) \le 1$ for all $x \in Q$, $\varphi(x) = 1$ on Q_t. By (2.3), (2.1), since (iii) ensures the convexity of f, we have

$$\frac{1}{2}f(\xi_1) + \frac{1}{2}f(\xi_2) = \lim_{h\to+\infty}\int_Q f(\xi + Du_h(x))\, dx$$
$$\ge \lim_{h\to+\infty}\left[\int_Q f(\xi + D(\varphi u_h))\, dx - \int_{Q\setminus Q_t} c(1+|\xi_1|^p+|\xi_2|^p+|u_hD\varphi|^p)\right]$$
$$\ge f(\xi) + c_2\,\nu\,\liminf_{h\to+\infty}\int_Q (1+|\xi|^2+|D(\varphi u_h)|^2)^{\frac{p-2}{2}}|D(\varphi u_h)|^2\, dx\ - ct$$
$$\ge f(\xi) + c_2'\,\nu\,|Q_t|\,(1+|\xi|^2+|\lambda|^2)^{\frac{p-2}{2}}|\xi_1-\xi_2|^2\ - ct\,.$$

The result now follows by letting $t \to 0$. ■

Lemma 2.4 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a C^2 function. Then f satisfies (2.2) if and only if there exists a constant c_0 such that for all $\xi \in \mathbb{R}^n \setminus \overline{B}_R$*

$$\langle D^2 f(\xi)\lambda, \lambda\rangle \geq c_0\, \nu\, (1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \qquad \forall \lambda \in \mathbb{R}^n \,. \tag{2.4}$$

Proof. If f satisfies (2.2) and $|\xi| > R$ then, for any λ we have for $t > 0$ small enough,

$$\frac{1}{2}[f(\xi + t\lambda) - f(\xi)] + \frac{1}{2}[f(\xi - t\lambda) - f(\xi)] \geq 4t^2 \nu\, (1 + |\xi + t\lambda|^2 + |\xi - t\lambda|^2)^{\frac{p-2}{2}} |\lambda|^2 \,.$$

Hence, by Taylor's formula, we get

$$\frac{t^2}{2}\langle D^2 f(\xi)\lambda, \lambda\rangle + o(t^2) \geq 4t^2 \nu\, (1 + |\xi + t\lambda|^2 + |\xi - t\lambda|^2)^{\frac{p-2}{2}} |\lambda|^2 \,;$$

thus, dividing by t^2 and letting $t \to 0$, we obtain

$$\langle D^2 f(\xi)\lambda, \lambda\rangle \geq 8\nu\, (1 + 2|\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \,.$$

This inequality implies (2.4) with $c_0 = 4\sqrt{2}$.

Conversely, if (2.4) holds and $[\xi_1, \xi_2] \subset \mathbb{R}^n \setminus \overline{B}_R$, setting $\xi := (\xi_1 + \xi_2)/2$ by Lemma 2.1 we have for $i = 1, 2$

$$\begin{aligned} f(\xi_i) &= f(\xi) + \langle Df(\xi), \xi_i - \xi\rangle + \int_0^1 (1-t)\langle D^2 f(\xi + t(\xi_i - \xi))(\xi_i - \xi), \xi_i - \xi\rangle \, dt \\ &\geq f(\xi) + \langle Df(\xi), \xi_i - \xi\rangle + c_0 \nu \int_0^1 (1-t)(1 + |\xi + t(\xi_i - \xi)|^2)^{\frac{p-2}{2}} |\xi_i - \xi|^2 \, dt \\ &\geq f(\xi) + \langle Df(\xi), \xi_i - \xi\rangle + c(p) c_0 \nu\, (1 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2 \,. \end{aligned}$$

The result then follows by adding up this inequality with $i = 1, 2$. ■

Lemma 2.5 *Let $f : \mathbb{R}^n \to [0, +\infty)$ be a continuous function satisfying (2.1) and (2.2). Then there exist $R_0, \nu_0, C_0 > 0$, depending only on R, ν and L, such that for all $\xi \in \mathbb{R}^n \setminus \overline{B}_{R_0}$ there exists $q_\xi \in \mathbb{R}^n$ such that $|q_\xi| \leq C_0(1 + |\xi|^{p-1})$ and*

$$f(\eta) \geq f(\xi) + \langle q_\xi, \eta - \xi\rangle + \nu_0 (1 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} |\xi - \eta|^2 \quad \forall \eta \in \mathbb{R}^n \,. \tag{2.5}$$

Moreover, if $|\xi| > R_0$ then $f^{**}(\xi) = f(\xi)$.

Proof. For $0 < \varepsilon < 1$ set $f_\varepsilon := \rho_\varepsilon * f$, where $\rho_\varepsilon(\eta) := \varepsilon^{-n}\rho(\eta/\varepsilon)$ and $\rho(\eta) = \hat{\rho}(|\eta|)$ is a positive radially symmetric mollifier with support equal to $\overline{B}$, with $B := B_1$, $\rho(\eta) > 0$ if $|\eta| < 1$ and $\int_B \rho(\eta) d\eta = 1$. From (2.2) it follows easily that if $[\xi_1, \xi_2] \subset \mathbb{R}^n \setminus \overline{B}_{R+1}$, $\xi := (\xi_1 + \xi_2)/2$, then

$$\frac{1}{2}[f_\varepsilon(\xi_1) + f_\varepsilon(\xi_2)] \geq f_\varepsilon(\xi) + \nu |\xi_1 - \xi_2|^2 \int_B \rho(\eta)(1 + |\xi_1 + \varepsilon\eta|^2 + |\xi_2 + \varepsilon\eta|^2)^{\frac{p-2}{2}} \, d\eta \,.$$

The integral above can be estimated from below by

$$\int_{B\cap\{\langle\xi,\eta\rangle\geq 0\}}\rho(\eta)(1+|\xi_1|^2+|\xi_2|^2+2\varepsilon^2|\eta|^2+4\varepsilon\langle\xi,\eta\rangle)^{\frac{p-2}{2}}\,d\eta\geq\frac{1}{2}(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}$$

if $p\geq 2$, and by

$$\int_B\rho(\eta)(1+2|\xi_1|^2+2|\xi_2|^2+4\varepsilon^2|\eta|^2)^{\frac{p-2}{2}}\,d\eta\geq 5^{(p-2)/2}(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}$$

when $1<p<2$. In both cases

$$\frac{1}{2}[f_\varepsilon(\xi_1)+f_\varepsilon(\xi_2)]\geq f_\varepsilon(\xi)+c\nu(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}}|\xi_1-\xi_2|^2\,;$$

hence, by Lemma 2.4, if $|\xi|\geq R+1$, then

$$\langle D^2f_\varepsilon(\xi)\lambda,\lambda\rangle\geq c\nu\,(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2,\qquad\forall\,\lambda\in\mathbb{R}^n. \tag{2.6}$$

Moreover, it can be easily checked that $0\leq f_\varepsilon(\xi)\leq C(L)(1+|\xi|^p)$ for all ξ, and a simple argument based on the convexity of f_ε in $\mathbb{R}^n\setminus\overline{B}_{R+1}$ shows that there exists a constant $C_1(L,R)$ such that

$$|Df_\varepsilon(\xi)|\leq C_1(1+|\xi|^{p-1}),\qquad\forall\,\xi\in\mathbb{R}^n\setminus\overline{B}_{R+2}\,. \tag{2.7}$$

We claim that there exists $R_0>>1$ such that, if $|\xi|>R_0$, then

$$f_\varepsilon(\eta)\geq f_\varepsilon(\xi)+\langle Df_\varepsilon(\xi),\eta-\xi\rangle+c(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2 \tag{2.8}$$

for all $\eta\in\mathbb{R}^n$. Assume that (2.8) holds. Notice that, if $|\xi|>R_0>R+2$, then by (2.7) there exists a sequence (ε_h) converging to 0 such that $Df_{\varepsilon_h}(\xi)\to q_\xi$ for some $q_\xi\in\mathbb{R}^n$ such that $|q_\xi|\leq C_1(1+|\xi|^{p-1})$. Hence (2.5) follows from (2.8), letting ε go to 0^+. The equality $f^{**}(\xi)=f(\xi)$ for $|\xi|>R_0$ then follows at once from (2.5).

The remaining of the proof concerns the assertion of the claim (2.8). Fix ξ such that $|\xi|>R_0$, with $R_0>2(R+3)$ to be chosen later, and denote by C_ξ the open cone with vertex at ξ, tangent to the ball $\overline{B}_{R+3}$.

Case 1: If $\eta\in\mathbb{R}^n\setminus C_\xi$ or $\eta\in C_\xi\setminus B_{R+3}$ and $\langle\eta,\xi\rangle\geq 0$, then f_ε is convex along the line $t\to\xi+t(\eta-\xi)$ provided R_0 is sufficiently large, and (2.8) follows from (2.6).

Case 2: If $\eta\in B_{R+3}$ we consider $\overline{\xi}:=\xi(R+3)/|\xi|$ and a constant $M:=C(L)(1+(R+3)^p)$ such that

$$0\leq f_\varepsilon(\eta)\leq M$$

for all $\eta\in B_{R+3}$ and all $\varepsilon\in(0,1)$; by Case 1 we may apply (2.8) to $\overline{\xi}$ (notice that $\overline{\xi}\in\partial B_{R+3}$, $\langle\overline{\xi},\xi\rangle>0$), thus getting

$$f_\varepsilon(\eta)=f_\varepsilon(\overline{\xi})+f_\varepsilon(\eta)-f_\varepsilon(\overline{\xi})$$

$$\geq f_\varepsilon(\xi) + \langle Df_\varepsilon(\xi), \overline{\xi} - \xi\rangle - M + c\,(1 + (R+3)^2 + |\xi|^2)^{\frac{p-2}{2}}(|\xi| - R - 3)^2$$
$$\geq f_\varepsilon(\xi) + \langle Df_\varepsilon(\xi), \eta - \xi\rangle - |Df_\varepsilon(\xi)||\eta - \overline{\xi}| - M + \tilde{c}\,(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \eta|^2.$$

The estimate (2.8) now follows for η and ξ from the previous inequalities, together with the estimate

$$|Df_\varepsilon(\xi)||\eta - \overline{\xi}| + M \leq \frac{1}{2}\tilde{c}(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \eta|^2,$$

and the latter holds by virtue of (2.7), and provided $\xi > R_0$ and $R_0 > 2\,(R+3)$ sufficiently large.

Case 3: Finally, let us assume that $\eta \in C_\xi \setminus B_{R+3}$, $\langle \eta, \xi\rangle < 0$. In this case we have $|\xi - \eta|^2 > |\xi|^2 + |\eta|^2$ and, denoting by $\tilde{\eta}$ the projection of η on the boundary of the cone, and by α_ξ the half angle at the vertex of C_ξ,

$$|\eta - \tilde{\eta}| \leq |\xi - \eta| \sin\alpha_\xi = \frac{R+3}{|\xi|}|\xi - \eta| \leq \frac{1}{2}|\xi - \eta|\,. \tag{2.9}$$

Notice that, if R_0 is sufficiently large, then $[\eta, \tilde{\eta}] \subset \mathbb{R}^n \setminus \overline{B}_{R+2}$; therefore we may use (2.7) to estimate $f_\varepsilon(\eta) - f_\varepsilon(\tilde{\eta})$. This, together with (2.8) applied to $\tilde{\eta} \in \partial C_\xi$, yields

$$f_\varepsilon(\eta) = f_\varepsilon(\tilde{\eta}) + f_\varepsilon(\eta) - f_\varepsilon(\tilde{\eta})$$
$$\geq f_\varepsilon(\xi) + \langle Df_\varepsilon(\xi), \tilde{\eta} - \xi\rangle - C_1(1 + |\eta|^{p-1} + |\tilde{\eta}|^{p-1})|\eta - \tilde{\eta}| + c(1 + |\tilde{\eta}|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \tilde{\eta}|^2.$$

Since, by (2.9), $|\xi - \eta| \leq 2|\xi - \tilde{\eta}| \leq 3|\xi - \eta|$, for any $p > 1$ we have easily

$$c(1 + |\tilde{\eta}|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \tilde{\eta}|^2 \geq \overline{c}(p)(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \eta|^2\,,$$

and, using (2.7) once more, we obtain

$$\begin{aligned} f_\varepsilon(\eta) \quad \geq \quad & f_\varepsilon(\xi) + \langle Df_\varepsilon(\xi), \eta - \xi\rangle \\ & - C_1|\eta - \tilde{\eta}|\left[(1 + |\xi|^{p-1}) + (1 + |\eta|^{p-1} + |\tilde{\eta}|^{p-1})\right] \\ & + \overline{c}(p)(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \eta|^2\,. \end{aligned} \tag{2.10}$$

By virtue of (2.9), and recalling that $|\xi|^2 + |\eta|^2 < |\xi - \eta|^2$, we have

$$\begin{aligned} & C_1|\eta - \tilde{\eta}|\left[(1 + |\xi|^{p-1}) + (1 + |\eta|^{p-1} + |\tilde{\eta}|^{p-1})\right] \\ \leq \quad & c\frac{R+3}{|\xi|}|\xi - \eta|(1 + |\eta|^{p-1} + |\xi|^{p-1}) \leq c'\frac{R+3}{|\xi|}|\xi - \eta|^2(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}} \\ \leq \quad & (\overline{c}(p)/2)(1 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\xi - \eta|^2 \end{aligned}$$

if $|\xi| > R_0$, with R_0 large enough. This, together with (2.10), concludes the proof of (2.8). ■

Remark 2.6 Let f satisfy (2.1) and (2.2), and fix a point ξ_0 such that $R_0 < |\xi_0| < 2R_0$. Applying (2.5) with $\xi = \xi_0$, and recalling that $|q_\xi| \leq C_0(1 + |\xi|^{p-1})$, for all η such that $|\eta| > 2R_0$ it holds

$$f(\eta) \geq \tilde{c}_1(R_0, \nu_0, C_0)\,|\eta|^p - \tilde{c}_2(R_0, \nu_0, C_0)\,.$$

Hence, $f(\xi) \geq c_1|\xi|^p - c_2$ for all $\xi \in \mathbb{R}^n$, with c_1, c_2 depending only on R, ν, L.

2.2 A regularity result

In this section we assume that $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a continuous function satisfying the *growth condition*

$$0 \le f(x,\xi) \le L(1+|\xi|^p), \qquad \forall\,(x,\xi) \in \Omega \times \mathbb{R}^n \tag{2.11}$$

and for some $L > 0$. Let us denote by $f^{**} := f^{**}(x,\xi)$ the *bipolar* of f, that is the *convex envelope* of $f(x,\cdot)$. We assume that f^{**} is continuous and that f is *uniformly convex at infinity* with respect to ξ (see (2.2)), i.e., there exist $R, \nu > 0$ such that if the segment $\overline{\xi_1, \xi_2}$ is contained in the complement of the closed ball $\overline{B}_R$, then for all $x \in \Omega$

$$f\left(x, \frac{\xi_1+\xi_2}{2}\right) \le \frac{1}{2} f(x,\xi_1) + \frac{1}{2} f(x,\xi_2) - \nu(1+|\xi_1|^2+|\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2. \tag{2.12}$$

We assume further that, if $|\xi| > R$ then the vector field $x \in \Omega \mapsto f_\xi(x,\xi)$ is weakly differentiable and

$$|D_x f_\xi(x,\xi)| \le L(1+|\xi|^{p-1}), \quad \forall\,(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \overline{B}_R)\,. \tag{2.13}$$

If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and $A \subset \Omega$ is open, then we set

$$F(u,A) := \int_A f(x, Du(x))\,dx\,.$$

The main result of this section is Theorem 2.7 below. We recall that u is said to be a *local minimizer* of F in Ω if

$$F(u, B_R(x_0)) \le F(v, B_R(x_0))$$

whenever $B_R(x_0) \subset\subset \Omega$ and $v \in u + W^{1,p}_0(B_R(x_0))$.

Theorem 2.7 *Let $f : \Omega \times \mathbb{R}^n \to [0,+\infty)$ be a continuous function satisfying (2.11), (2.12) and (2.13). If $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a local minimizer of the functional F, then u is locally Lipschitz continuous in Ω. Moreover, there exists a constant C_0, depending on L, p, ν, R, such that, if $B_r(x_0) \subset \Omega$ then*

$$\sup_{B_{r/2}(x_0)} |Du|^p \le C_0 \left(1 + \fint_{B_r(x_0)} |Du|^p\,dx \right)\,. \tag{2.14}$$

We first show in Lemma 2.8 that, provided we know already that u is locally Lipschitz, (2.14) holds with a constant C_0 depending only on L, p, ν, R. Once the *a priori* estimate (2.14) is established, the regularity result is obtained using an approximation argument.

Lemma 2.8 *Let f satisfy the assumptions of Theorem 2.7. Assume, in addition, that f is C^2 and that, for all $x \in \Omega$, $\xi, \lambda \in \mathbb{R}^n$,*

$$D_{\xi_i \xi_j} f(x,\xi) \lambda_i \lambda_j \ge \varepsilon_0 (1+|\xi|^2)^{\frac{p-2}{2}} |\lambda|^2, \tag{2.15}$$

and that $u \in W^{1,p}_{\rm loc}(\Omega)$ is a locally Lipschitz local minimizer of F in Ω. Then (2.14) holds with a constant depending only on L, p, ν, R and, in particular, independent on ε_0.

Proof. Step 1: From Lemma 2.4 we have that for every $x \in \Omega$, $\xi, \lambda \in \mathbb{R}^n$, with $|\xi| > R$,

$$D_{\xi_i\xi_j} f(x,\xi)\lambda_i\lambda_j \geq c\nu(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \,. \tag{2.16}$$

Since $\tilde{u}(y) := u(x_0+ry)/r$ is a local minimizer in $(\Omega - x_0)/r$ of the functional $\tilde{F}$, where $\tilde{F}(v) := \int_{(\Omega-x_0)/r} f(x_0+ry, Dv(y))dy$ still satisfies the assumptions of Theorem 2.7, it is clear that in order to prove (2.14) we may always assume, with no loss of generality, that $B := B_1 \subset \Omega$. Since u satisfies the Euler equation for F,

$$\int_B D_{\xi_i} f(x,Du) D_i\phi \, dx = 0 \,, \qquad \forall\, \phi \in C^1_0(B),$$

using (2.15) and the fact that $D_{\xi_i\xi_j} f(x, Du(x))$ are locally bounded in B (which follows from the C^2 regularity of f, together with the fact that Du is locally bounded) we have that $u \in W^{2,2}_{\rm loc}(B)$, by a standard different quotient argument.

We fix $s \in \{1,\dots,n\}$, $\eta \in C^1_0(B)$, $0 \leq \eta \leq 1$, $\psi \in C^2(B)$, and in the above Euler equation we take $\phi = \eta^2 D_s\psi$ to obtain

$$\int_B D_{\xi_i} f(x,Du) D_s(D_i\psi)\eta^2 \, dx = -2\int_B \eta D_{\xi_i} f(x,Du) D_s\psi D_i\eta \, dx \,.$$

Integrating by parts the first integral, we have

$$\int_B D_{\xi_i\xi_j} f(x,Du) D_j(D_s u) D_i\psi\eta^2 \, dx = 2\int_B \eta D_{\xi_i} f(x,Du) D_s\psi D_i\eta \, dx$$

$$-2\int_B D_{x_s\xi_i} f(x,Du) D_i\psi\eta^2 \, dx - 2\int_B \eta D_{\xi_i} f(x,Du) D_i\psi D_s\eta \, dx \tag{2.17}$$

for all functions $\psi \in W^{1,2}(B)$. Set

$$V_+(x) := 1+R^2+\sum_{h=1}^n [(D_h u(x)-R)^+]^2, \;\; V_-(x) := 1+R^2+\sum_{h=1}^n [(D_h u(x)+R)^-]^2,$$

and notice that there exist constants c_1, c_2, depending only on n, such that

$$c_1(V_+(x)+V_-(x)) \leq 1+R^2+|Du(x)|^2 \leq c_2(V_+(x)+V_-(x)) \,. \tag{2.18}$$

Let $\psi := V_+^\beta (D_s u - R)^+$, where $\beta \geq 0$. By (2.11) and the convexity of $f(x,\cdot)$, $|D_\xi f(x,\xi)| \leq c(1+|\xi|^2)^{\frac{p-1}{2}}$, and (2.17) yields

$$\int_B D_{\xi_i\xi_j} f(x,Du) D_j(D_s u - R)^+ D_i(D_s u - R)^+ V_+^\beta\eta^2 \, dx$$

$$+\beta\int_B D_{\xi_i\xi_j} f(x,Du) D_j(D_s u - R)^+ (D_s u - R)^+ D_i\left(\sum_{h=1}^n [(D_h u - R)^+]^2\right) V_+^{\beta-1}\eta^2 \, dx$$

$$\leq c\int_B \eta(\eta+|D\eta|)(1+|Du|^{p-1})|D(D_su-R)^+|V_+^\beta\,dx$$

$$+c\beta\int_B \eta(\eta+|D\eta|)(1+|Du|^{p-1})\left|D\left(\sum_{h=1}^n[(D_hu-R)^+]^2\right)\right|(D_su-R)^+V_+^{\beta-1}\,dx\,.$$

Since all the integrals are evaluated in the set where $|Du|>R$, summing up on s, using (2.16), the fact that $(D_su-R)^+\leq V_+^{1/2}$ and Young's inequality, it follows easily that

$$\begin{aligned}&\int_B(1+|Du|^2)^{\frac{p-2}{2}}\left|D\left(\sum_{h=1}^n[(D_iu-R)^+]^2\right)\right|^2V_+^{\beta-1}\eta^2\,dx\\ \leq\ &\frac{c}{\nu^2}\int_B(1+|Du|^p)\,V_+^\beta\,(\eta^2+|D\eta|^2)\,dx\,,\end{aligned}$$

where the constant c depends only on n, p, L. Since the integral on the left hand side is evaluated in the set where $|Du|>R$, in turn this last inequality is equivalent to the following one:

$$\begin{aligned}&\int_B(1+R^2+|Du|^2)^{\frac{p-2}{2}}\left|D\left(\sum_{h=1}^n[(D_iu-R)^+]^2\right)\right|^2V_+^{\beta-1}\eta^2\,dx\\ \leq\ &\frac{c}{\nu^2}\int_B(1+R^2+|Du|^2)^{p/2}\,V_+^\beta\,(\eta^2+|D\eta|^2)\,dx\,.\end{aligned}$$

Inserting $\psi=V_-^\beta(D_su+R)^-$ in (2.17), and using a similar argument, we get also

$$\begin{aligned}&\int_B(1+R^2+|Du|^2)^{\frac{p-2}{2}}\left|D\left(\sum_{h=1}^n[(D_iu+R)^-]^2\right)\right|^2V_-^{\beta-1}\eta^2\,dx\\ \leq\ &\frac{c}{\nu^2}\int_B(1+R^2+|Du|^2)^{p/2}\,V_-^\beta(\eta^2+|D\eta|^2)\,dx\,.\end{aligned}$$

Therefore, adding the last two inequalities and using (2.18), we arrive to

$$\begin{aligned}&\int_B V^{\frac{p-2}{2}}\left[|DV_+|^2V_+^{\beta-1}+|DV_-|^2V_-^{\beta-1}\right]\eta^2\,dx\\ \leq\ &\frac{c}{\nu^2}\int_B V^{p/2+\beta}(\eta^2+|D\eta|^2)\,dx\,,\end{aligned}$$

where $V:=\max\{V_+;V_-\}$.

Step 2: From the inequality above we deduce that

$$\int_B|DV^{p/4+\beta/2}|^2\eta^2\,dx\leq c(p+2\beta)^2\int_B V^{p/2+\beta}(\eta^2+|D\eta|^2)\,dx\,.$$

In turn, this implies that

$$\int_B|D(V^{p/4+\beta/2}\eta)|^2\,dx\leq c(p+2\beta)^2\int_B V^{p/2+\beta}(\eta^2+|D\eta|^2)\,dx\,,$$

where the constant c depends only on L, p, n, R, ν. Setting $\gamma := p/4 + \beta/2 \geq p/4$, using the Sobolev–Poincaré inequality, and the arbitrariness of $\beta \geq 0$, we get that, for any $\gamma \geq p/4$,

$$\|V^\gamma \eta\|_{L^{2\chi}(B)} \leq c\gamma \, \|V^\gamma (\eta + |D\eta|)\|_{L^2(B)} ,$$

where $\chi := n/(n-2)$ if $n \geq 3$, or any number > 1 if $n = 2$. Considering the sequence of radii $r_i := 1/2 + 1/2^i$ for $i = 1, \ldots$, we apply the inequality above to $\gamma = \gamma_i := (p/4)\chi^{i-1}$, and choose $\eta \in C^1_0(B)$ such that $\eta = 1$ on $B_{r_{i+1}}$, $0 \leq \eta \leq 1$, $|D\eta| \leq c2^i$. We obtain

$$\|V\|_{L^{2\gamma_{i+1}}(B_{r_{i+1}})} \leq (c2^i\gamma_i)^{1/\gamma_i} \, \|V\|_{L^{2\gamma_i}(B_{r_i})} .$$

Iterating the above formula yields, for every i,

$$\|V\|_{L^{2\gamma_{i+1}}(B_{1/2})} \leq C \, \|V\|_{L^{p/2}(B)} ,$$

where $C = \prod_{i=1}^{\infty}(c2^i\gamma_i)^{1/\gamma_i} < +\infty$. Therefore, letting i go to $+\infty$ and using (2.18), we obtain (2.14). ∎

Remark 2.9 It follows immediately from the proof that the estimate (2.14) may be generalized to read

$$\sup_{B_\rho(x_0)} |Du|^p \leq C(\rho)\left(1 + \fint_{B_r(x_0)} |Du|^p \, dx\right),$$

for all $0 < \rho < r$, where $C(\rho)$ depends only on L, p, ν, R and ρ.

We are now in position to prove Theorem 2.7, by using the following approximation lemma.

Lemma 2.10 *Let $g : \mathbb{R}^n \to [0, +\infty)$ be a C^2 convex function such that for all $\xi \in \mathbb{R}^n$*

$$0 \leq g(\xi) \leq L(1 + |\xi|^p) ,$$

where $p > 1$, $L > 0$, and assume that there exist $R, \nu > 0$, such that if $|\xi| > R$, $\lambda \in \mathbb{R}^n$,

$$D_{ij}g(\xi)\lambda_i\lambda_j \geq \nu(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 .$$

Then there exists a constant $c = c(n, p)$ and a sequence g_h of $C^2(\mathbb{R}^n)$ convex functions such that

(a) $0 \leq g_h(\xi) \leq cL(1 + |\xi|^p) \qquad \forall\, \xi \in \mathbb{R}^n$;

(b) for any h there exists $\varepsilon_h > 0$ such that, for all $\xi, \lambda \in \mathbb{R}^n$,

$$\varepsilon_h(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq D_{ij}g_h(\xi)\lambda_i\lambda_j \leq \varepsilon_h^{-1}(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2;$$

(c) $D_{ij}g_h(\xi)\lambda_i\lambda_j \geq c\nu(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2$, $\forall\, \lambda \in \mathbb{R}^n$, $|\xi| > R + 1$;

(d) $g_h \to g$ uniformly on compact subset of $\mathbb{R}^n$.

Proof. The proof of this lemma can be obtained arguing as in the Step 2 and Step 3 of the proof of Lemma 3.4 of [19], with the obvious simple modifications needed in the present case. Therefore we omit the details. ■

Proof of Theorem 2.7. Notice that if u is a local minimizer F, then u is also a local minimizer of the *relaxed functional* $v \mapsto \int_\Omega f^{**}(x, Dv)dx$, where, for all $x \in \Omega$, $f^{**}(x,\xi)$ is the bipolar of $\xi \mapsto f(x,\xi)$. Indeed, if $v \in u + W_0^{1,p}(B_r(x_0))$, then

$$\int_{B_r(x_0)} f^{**}(x, Dv)\, dx$$
$$= \inf\left\{ \liminf_{h\to+\infty} \int_{B_r(x_0)} f(x, Dv_h)\, dx : \, v_h \rightharpoonup v \text{ in } W_0^{1,p}(B_r(x_0)) \right\}.$$

Also, by virtue of Lemma 2.5, the function f^{**} satisfies the assumptions of Theorem 2.7. Therefore, with no loss of generality, we may assume that f is convex in ξ.

Step 1: Let us assume that

$$f(x,\xi) = \sum_{i=1}^{N} a_i(x) g_i(\xi)\,,$$

where, for $i = 1, \dots, N$, the function $g_i \in C^2(\mathbb{R}^n)$ satisfies the assumptions of Lemma 2.10 for some $L, R, \nu > 0$, and $\langle D^2 g_i(\xi)\lambda, \lambda\rangle \geq \varepsilon_0(1+|\xi|^2)^{(p-2)/2}|\lambda|^2$ for all $\xi, \lambda \in \mathbb{R}^n$ and for some $\varepsilon_0 > 0$. Moreover, let us assume that, for all i, the function a_i is a nonnegative C^2 function such that $\|Da_i\|_\infty \leq M$ and that $\gamma^{-1} < \sum_{i=1}^N a_i(x) < \gamma$ for all $x \in \Omega$ and for some positive constant γ.

For every i let us denote by $g_{i,h}$ a sequence of $C^2(\mathbb{R}^n)$ functions such that $g_{i,h} \to g_i$ uniformly on the compact sets of $\mathbb{R}^n$, satisfying the conditions (a), (b) and (c) of Lemma 2.10, and let us set for all $(x,\xi) \in \Omega \times \mathbb{R}^n$

$$f_h(x,\xi) := \sum_{i=1}^{N} a_i(x)\, g_{i,h}(\xi)\,.$$

From Remark 2.6 it follows easily that there exist constants c_1, c_2, depending only on L, R, ν, γ, such that, for all (x,ξ) and for any h,

$$f(x,\xi),\ f_h(x,\xi) \geq c_1|\xi|^p - c_2\,. \tag{2.19}$$

Given $B_r(x_0) \subset\subset \Omega$, we denote by u_h the solution of the problem

$$\min\left\{ \int_{B_r(x_0)} f_h(x, Dv)\, dx : \, v \in u + W_0^{1,p}(B_r(x_0)) \right\}. \tag{2.20}$$

Since the functions $g_{i,h}$ satisfy condition (b) of Lemma 2.10, standard elliptic regularity theory implies that $u_h \in C^{1,\alpha}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$ for any h. From the assumptions on f, from the approximation provided by Lemma 2.10, and from

(2.19), it follows that the sequence u_h is bounded in $W^{1,p}(B_r(x_0))$. Moreover, by Lemma 2.8 (see Remark 2.9), for all $\rho < r$ we obtain

$$\sup_{B_\rho(x_0)} |Du_h|^p \le C \left(1 + \fint_{B_r(x_0)} |Du_h|^p \, dx\right), \tag{2.21}$$

where the constant C depends ultimately only on L, p, R, ν, γ, M and ρ, but not on h. Hence we may assume, up to a subsequence, that $u_h \rightharpoonup u_\infty$ weakly* in $W^{1,\infty}(B_\rho(x_0))$ for any $\rho < r$. Since $f_h \to f$ uniformly on compact sets of $\mathbb{R}^n$ and the integrand f is convex, for any $\rho < r$ we have

$$\int_{B_\rho(x_0)} f(x, Du_\infty)\, dx \le \liminf_{h\to+\infty} \int_{B_\rho(x_0)} f(x, Du_h)\, dx = \liminf_{h\to+\infty} \int_{B_\rho(x_0)} f_h(x, Du_h)\, dx$$

$$\le \liminf_{h\to+\infty} \int_{B_r(x_0)} f_h(x, Du)\, dx = \int_{B_r(x_0)} f(x, Du)\, dx\,,$$

where we used the fact that u_h is a solution for (2.20). Letting $\rho \to r$ and recalling that u is a local minimizer of the functional F, we deduce that u_∞ is also a minimizer of F in $B_r(x_0)$. Since the functional F is strictly convex, we have that $u = u_\infty$. Passing to the limit as $h \to +\infty$ in (2.21), we conclude that u is also locally Lipschitz. Indeed, using the minimality of u_h and (2.21), we get

$$\sup_{B_{r/2}(x_0)} |Du|^p \le \liminf_{h\to+\infty} \left(\sup_{B_{r/2}(x_0)} |Du_h|^p \right)$$

$$\le c \liminf_{h\to+\infty} \left(1 + \fint_{B_r(x_0)} |Du_h|^p\, dx\right) \le c \liminf_{h\to+\infty} \left(1 + \fint_{B_r(x_0)} f_h(x, Du_h)\, dx\right)$$

$$\le c \liminf_{h\to+\infty} \left(1 + \fint_{B_r(x_0)} f_h(x, Du)\, dx\right) \le c \left(1 + \fint_{B_r(x_0)} |Du|^p\, dx\right).$$

Step 2: Let us now assume that $f \in C^2(\Omega \times \mathbb{R}^n)$ and that there exists $\varepsilon_0 > 0$ such that $D_{\xi_i \xi_j} f(x,\xi)\lambda_i\lambda_j \ge \varepsilon_0(1 + |\xi|^2)^{(p-2)/2}|\lambda|^2$ for all $(x,\xi) \in \Omega \times \mathbb{R}^n$ and for any $\lambda \in \mathbb{R}^n$. Fix an open set $A \subset\subset \Omega$ and let us prove that (2.14) holds for any ball $B_r(x_0) \subset A$ (with a constant C_0 not depending on A).

To this aim let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $0 \le \psi(x) \le 1$ for all x, $\mathrm{supp}\psi \subset (-3,3)^n$ and such that $\psi(x) \equiv 1$ if $x \in [-1,1]^n$. For any $h \in \mathbb{N}$ we denote by $Q_{i,h}(x_{i,h})$ the *standard* covering of $\mathbb{R}^n$ with closed cubes, centered at $x_{i,h}$, with sides parallel to the coordinates axes, side length equal to $2/h$ and having pairwise disjoint interiors. Then, for any i, h, we set $\psi_{i,h}(x) := \psi\left(h\left(x - x_{i,h}\right)\right)$ and

$$\sigma_h(x) := \sum_{i=1}^{\infty} \psi_{i,h}(x), \qquad \varphi_{i,h}(x) := \frac{\psi_{i,h}(x)}{\sigma_h(x)}\,.$$

Finally, for all h such that $12\sqrt{n}/h <$dist$(A;\partial\Omega)$, and for every $x \in A$, $\xi \in \mathbb{R}^n$, we set

$$f_h(x,\xi) := \sum_{i=1}^{\infty} \varphi_{i,h}(x) f(x_{i,h},\xi) \, .$$

Notice that the above sum is actually finite (indeed it consists of at most 3^n terms), and that each function f_h is of the type considered in Step 1. Moreover, we claim that the functions f_h satisfy the assumptions of Lemma 2.8 with suitable constants L, ε_0, R, ν not depending on h. The verification of our claim in the case of assumptions (2.11), (2.12) (or equivalently (2.16)) and (2.15) follows easily by the corresponding assumptions on f. We limit ourselves to show that for any h

$$|D_{x\xi} f_h(x,\xi)| \le cL(1+|\xi|^{p-1}), \quad \forall\, (x,\xi) \in A \times (\mathbb{R}^n \setminus \overline{B}_R) \, , \tag{2.22}$$

where L is the constant appearing in (2.13) (relative to f) and c is a constant depending only on n, ψ, but not on h. Let us fix $x_0 \in A$ and $\xi \in \mathbb{R}^n \setminus \overline{B}_R$. By construction there exist at most 3^n cubes, $Q_{j_1,h}(x_{j_1,h}), \dots, Q_{j_{3^n},h}(x_{j_{3^n},h})$, such that for any x in a suitable neighborhood U of x_0

$$f_h(x,\xi) = \sum_{l=1}^{3^n} \varphi_{j_l,h}(x)\, f(x_{j_l,h},\xi), \qquad \sum_{l=1}^{3^n} \varphi_{j_l,h}(x) = 1 \, .$$

Therefore for all $x \in U$ we have that

$$\begin{aligned} D_{x\xi} f_h(x,\xi) &= \sum_{l=1}^{3^n} D_x \varphi_{j_l,h}(x)\, D_\xi f(x_{j_l,h},\xi) \\ &= \sum_{l=2}^{3^n} D_x \varphi_{j_l,h}(x)\, [D_\xi f(x_{j_l,h},\xi) - D_\xi f(x_{j_1,h},\xi)] \, . \end{aligned} \tag{2.23}$$

In view of assumption (2.13), we have that for all l,

$$|D_\xi f(x_{j_l,h},\xi) - D_\xi f(x_{j_1,h},\xi)| \le \frac{c(n)L}{h}(1+|\xi|^{p-1}) \, .$$

On the other hand, for any j, there exists a set of indices I_j containing j, with $\#(I_j) = 3^n$, such that, for all $x \in \mathbb{R}^n$,

$$D_x \varphi_{j,h}(x) = \frac{D_x \psi_{j,h}(x)}{\sigma_h(x)} - \frac{\psi_{j,h}(x)}{\sigma_h^2(x)} \sum_{k \in I_j} D_x \psi_{k,h}(x) \, .$$

Therefore, since by construction $\sigma_h(x) \ge 1$ for all x, we have that, for all $x \in \mathbb{R}^n$ and any j,

$$|D_x \varphi_{j,h}(x)| \le (3^n+1)h(\|D_x\psi\|_\infty) \, .$$

In view of the above estimates and from (2.23), we may conclude that for all $(x,\xi) \in A \times (\mathbb{R}^n \setminus \overline{B}_R)$ and for any h

$$|D_{x\xi} f_h(x,\xi)| \le c(n)L\, \|D_x\psi\|_\infty (1+|\xi|^{p-1}) \, ,$$

and thus (2.22) follows. Finally, notice that $f_h(x,\xi) \to f(x,\xi)$ uniformly in $A \times K$ for every K compact subset of $\mathbb{R}^n$. Hence the rest of the proof goes as in Step 1, since also in this case the function $\xi \to f(x,\xi)$ is strictly convex for all $x \in A$.

Step 3: Let f satisfy the assumptions of Theorem 2.7. Fix an open set $A \subset\subset \Omega$, an infinitesimal sequence ε_h of positive numbers and a positive symmetric mollifier ρ. For h large enough we set for all $(x,\xi) \in A \times \mathbb{R}^n$

$$f_h(x,\xi) := \int_{B\times B} \rho(y)\rho(\eta) f(x+\varepsilon_h y, \xi+\varepsilon_h \eta)\, dy d\eta + \varepsilon_h(1+|\xi|^p)\,,$$

where B is the unit open ball in $\mathbb{R}^n$. Notice that each function f_h is of the type considered in Step 2 and that $f_h(x,\xi) \to f(x,\xi)$ uniformly on any set of the type $A \times K$, with $K \subset \mathbb{R}^n$ compact. Moreover, the functions f_h satisfy the assumptions of Theorem 2.7 with the corresponding constants L, R, ν bounded from above and away from zero.

As in Step 1, given a ball $B_r(x_0) \subset A$ we denote by u_h the solution of the problem

$$\min \left\{ \int_{B_r(x_0)} f_h(x, Dv)\, dx : \; v \in u + W^{1,p}(B_r(x_0)) \right\}.$$

From the assumptions on f and the construction of the functions f_h it follows easily that the sequence u_h is bounded in $W^{1,p}(B_r(x_0))$. Moreover, by Step 2, for all $\rho < r$ we have

$$\sup_{B_\rho(x_0)} |Du_h|^p \le C \left(1 + \fint_{B_r(x_0)} |Du_h|^p\, dx \right),$$

where the constant C depends only on L, p, R, ν and ρ, but not on h. Hence we may assume that, up to a subsequence, $u_h \rightharpoonup u_\infty$ weakly* in $W^{1,\infty}(B_\rho(x_0))$ for any $\rho < r$. As in Step 1 we have again that also u_∞ is a minimizer of F in $B_r(x_0)$. However, in the present case the functional F is not necessarily strictly convex, hence we may not conclude as before that $u_\infty = u$ in $B_r(x_0)$.

Set $E_0 := \{x \in B_r(x_0) : |Du_\infty(x)+Du(x)| > 2R_0\}$, where R_0 is the constant provided by Lemma 2.5. If E_0 has positive measure, then from the convexity of $f(x,\cdot)$ and Remark 2.6 we have, setting $\tilde{u} := (u+u_\infty)/2$,

$$\int_{B_\rho(x_0)\setminus E_0} f(x, D\tilde{u})\, dx \le \frac{1}{2}\int_{B_\rho(x_0)\setminus E_0} f(x,Du)\, dx + \frac{1}{2}\int_{B_\rho(x_0)\setminus E_0} f(x, Du_\infty)\, dx\,. \tag{2.24}$$

Also, applying twice (2.5) in Lemma 2.5, first with $\xi := D\tilde{u}$ and $\eta := Du$, then with $\xi := D\tilde{u}$ and $\eta := Du_\infty$, and adding up the corresponding two inequalities, yields

$$\int_{E_0\cap B_\rho(x_0)} f(x, D\tilde{u})\, dx < \frac{1}{2}\int_{E_0\cap B_\rho(x_0)} f(x,Du)\, dx + \frac{1}{2}\int_{E_0\cap B_\rho(x_0)} f(x, Du_\infty)\, dx\,. \tag{2.25}$$

Adding (2.24) and (2.25) we get a contradiction with the minimality of u and u_∞. Therefore E_0 has zero measure. Applying Step 2 to the functions u_h, (2.5) to the functions f_h, and using the minimality of u_h, we deduce that

$$\sup_{B_{r/2}(x_0)} |Du_\infty|^p \le \liminf_{h\to+\infty} \left(\sup_{B_{r/2}(x_0)} |Du_h|^p \right)$$

$$\le \tilde{C} \liminf_{h\to+\infty} \left(1 + \fint_{B_r(x_0)} |Du_h|^p \, dx \right) \le c \liminf_{h\to+\infty} \left(1 + \fint_{B_r(x_0)} f_h(x, Du_h) \, dx \right)$$

$$\le c \liminf_{h\to+\infty} \left(1 + \fint_{B_r(x_0)} f_h(x, Du_\infty) \, dx \right) \le c \left(1 + \fint_{B_r(x_0)} |Du_\infty|^p \, dx \right).$$

Then the result follows, since $|Du(x) + Du_\infty(x)| \le 2R_0$ for a.e. x. ■

3 Attainment of minima for nonconvex problems

Here we give an existence result for the variational problem

$$\inf \left\{ \int_\Omega f\left(x, Du(x)\right) \, dx : \; u \in u_0 + W_0^{1,p}(\Omega) \right\}, \tag{3.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$ and $u_0 \in W^{1,p}(\Omega)$, $p > 1$. Throughout this section we assume that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying the *growth condition*

$$c_1 |\xi|^p - c_2 \le f(x,\xi) \le L(1 + |\xi|^p), \qquad \forall\, (x,\xi) \in \Omega \times \mathbb{R}^n, \tag{3.2}$$

for some constants $c_1, c_2, L > 0$, and is *uniformly convex at infinity with respect to* ξ, i.e., there exist $R, \nu > 0$ such that, if the segment $[\xi_1, \xi_2]$ is contained in $\mathbb{R}^n \setminus \overline{B}_R$, then

$$f\left(x, \frac{\xi_1 + \xi_2}{2}\right) \le \frac{1}{2} f(x,\xi_1) + \frac{1}{2} f(x,\xi_2) - \nu (1 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2 \tag{3.3}$$

for every $x \in \Omega$ (see (2.2)). Notice that, if $0 \le f(x,\xi) \le L(1 + |\xi|^p)$, then condition (3.3) implies the coercivity inequality in the left hand side of (3.2) (see Remark 2.6). In addition, we assume that there exists the distributional derivative $D_{x\xi} f(x,\xi)$ and

$$|D_{x\xi} f(x,\xi)| \le L(1 + |\xi|^{p-1}), \quad \forall\, x \in \Omega,\, |\xi| > R, \tag{3.4}$$

provided L in (3.2) is chosen to be sufficiently large (see (2.13)). Let us denote by $f^{**} := f^{**}(x,\xi)$ the *bipolar* of f, that is the *convex envelope* of $f(x,\cdot)$ (i.e., the largest convex function in ξ which is less than or equal to $f(x,\cdot)$ on $\mathbb{R}^n$). We assume that f^{**} is continuous; hence, for any $x \in \Omega$, the set

$$A(x) := \{\xi \in \mathbb{R}^n : \ f(x,\xi) > f^{**}(x,\xi)\} \tag{3.5}$$

is open.

We shall prove the existence of a minimizer for the problem (3.1) under the main assumption that $f^{**}(x,\cdot)$ is *affine on each connected component* of $A(x)$. However, in order to present the argument of the proof in a simplified setting, we shall treat first the case where $f^{**}(x,\cdot)$ is affine (with the same slope) on the whole set $A(x)$. We will refer to this situation as the *model case*. The proof of this case contains all the ideas and technical tools which are needed to treat the general situation in which $f^{**}(x,\cdot)$ is affine (with possibly different slopes) on each connected component of $A(x)$.

3.1 The model case

As before $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function satisfying (3.2)–(3.4). We assume that $f^{**} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is continuous and we denote by $A(x)$ the set defined in (3.5) and by $\Omega_A := \{x \in \Omega : A(x) \neq \emptyset\}$. Since f and f^{**} are continuous functions, Ω_A is open. Here we consider the case where $f^{**}(x,\cdot)$ is *affine* in $A(x)$; more precisely, we assume that there exist a function $q \in C^0(\Omega_A)$ and a vector field $m \in C^1(\Omega_A;\mathbb{R}^n)$ such that

$$f^{**}(x,\xi) = q(x) + \langle m(x), \xi \rangle\,, \qquad \forall\, x \in \Omega_A,\ \xi \in A(x)\,. \tag{3.6}$$

We also assume that the *boundary* of the set where the divergence of m is equal to zero is negligible, i.e.,

$$\operatorname{meas}\left(\Omega \cap \partial \{x \in \Omega_A : \ \operatorname{div} m(x) = 0\}\right) = 0\,. \tag{3.7}$$

Finally, for every $x \in \Omega_A$, we set

$$E(x) := \{\xi \in \mathbb{R}^n : \ f^{**}(x,\xi) = q(x) + \langle m(x), \xi \rangle\}\ ;$$

note that, by the growth conditions in (3.2) and by the assumption that f is uniformly convex at infinity with respect to ξ, then the set $E(x)$ is bounded uniformly for $x \in \Omega_A$ (see also Lemma 2.5). We assume there exists an increasing function $\omega : [0,+\infty) \to [0,+\infty)$, with $\omega(t) = 0$ if and only if $t = 0$, such that, if $x \in \Omega_A$, $\xi \in \partial E(x)$, $\eta \in \mathbb{R}^n \setminus E(x)$, then

$$f^{**}\left(x, \frac{\xi+\eta}{2}\right) \leq \frac{1}{2} f^{**}(x,\xi) + \frac{1}{2} f^{**}(x,\eta) - \omega\left(|\xi - \eta|\right). \tag{3.8}$$

The main result of this section is the following existence theorem.

Theorem 3.1 *Let f, $f^{**} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be continuous functions (f not necessarily convex with respect to $\xi \in \mathbb{R}^n$). Under the above assumptions on f and f^{**} ((3.2)–(3.4) and (3.6)–(3.8)), for any given boundary datum $u_0 \in W^{1,p}(\Omega)$ the variational problem (3.1) attains its minimum. Moreover, any minimizers is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*

The proof of Theorem 3.1 is obtained using the same method as in Dacorogna–Marcellini [13], [12], who considered integrands independent of x. Our proof however follows from some new lemmas. The first one concerns the *relaxed variational problem*

$$\inf\left\{\int_{\Omega} f^{**}\left(x, Du(x)\right)\, dx : u \in u_0 + W_0^{1,p}\left(\Omega\right)\right\}. \tag{3.9}$$

Lemma 3.2 *The minimum of the relaxed variational problem (3.9) is attained. Moreover, there exist a minimizer* $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap \left(u_0 + W_0^{1,p}(\Omega)\right)$ *of (3.9) and an open set* $\Omega' \subset \Omega$ *(possibly empty) such that*

$$\begin{cases} Dv(x) \in A(x) & a.e. \quad x \in \Omega' \\ Dv(x) \notin A(x) & a.e. \quad x \in \Omega \setminus \Omega' \end{cases} \tag{3.10}$$

and $\operatorname{div} m = 0$ *in* Ω'.

Remark 3.3 Formally, if $Dv(x) \in A(x)$ then by (3.6)

$$f^{**}(x, Dv(x)) = q(x) + \langle m(x), Dv(x)\rangle .$$

Therefore, the Euler's equation for v gives

$$\sum_{s=1}^{n} \frac{\partial}{\partial x_s} f^{**}_{\xi_s}(x, Dv(x)) = \operatorname{div} m(x) = 0, \quad \text{a.e. } x \text{ such that } Dv(x) \in A(x).$$

Thus (3.10) would follow if we could define $\Omega' := \{Dv(x) \in A(x)\}$. A striking feature of Lemma 3.2 is that the set Ω' may be chosen to be open, and so Lemma 3.2 may be considered to be a strong form of Euler's first variation for the minimizers. The property (3.10) is a regularity result, and in fact it follows from the regularity results obtained in Section 2.

The proof of Lemma 3.2 follows the argument by Dacorogna–Marcellini [12] in Theorem 10.9. Previous arguments related to Lemma 3.2 are due to De Blasi–Pianigiani [14], Sychev [28], and Zagatti [29].

Proof of Lemma 3.2. As before we denote by Ω_A the open subset of Ω consisting of those points x such that $A(x) \neq \emptyset$. We split Ω_A into three sets (possibly empty),

$$\Omega_A^+ := \{x \in \Omega_A : \operatorname{div} m(x) > 0\}, \quad \Omega_A^- := \{x \in \Omega_A : \operatorname{div} m(x) < 0\}, \tag{3.11}$$

$$\Omega_A^0 := \{x \in \Omega_A : \operatorname{div} m(x) = 0\}. \tag{3.12}$$

Since $\operatorname{div} m$ is a continuous function, $\Omega_A^+ \cup \Omega_A^- \cup \operatorname{int}\Omega_A^0$ is an open set and, by (3.7),

$$\operatorname{meas}\left(\Omega_A \setminus \left(\Omega_A^+ \cup \Omega_A^- \cup \operatorname{int}\Omega_A^0\right)\right) \leq \operatorname{meas}\left(\Omega \cap \partial\Omega_A^0\right) = 0 .$$

By (3.2), (3.3), (3.4), Lemma 2.5, and Theorem 2.7 the relaxed variational problem (3.9) has a minimizer u^{**} in the Sobolev class $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap \left(u_0 + W^{1,p}(\Omega)\right)$. Thus, by Rademacher theorem (see, for example, Theorem 2.2.1 of [30], or Theorem 2.14 of [2]) u^{**} is classically differentiable for almost every $x \in \Omega$.

Let x_0 be a point of $\Omega_A \setminus \partial\Omega^0_A$ where u^{**} is differentiable. Then

$$u^{**}(x) = u^{**}(x_0) + \langle Du^{**}(x_0), x - x_0\rangle + o\left(|x - x_0|\right), \; x \in \Omega. \tag{3.13}$$

Also, assume that

$$Du^{**}(x_0) \in A(x_0) = \left\{\xi \in \mathbb{R}^n : \; f(x_0,\xi) > f^{**}(x_0,\xi)\right\} .$$

Since Ω_A and $A(x)$ are open sets, there exists $\gamma \in (0,1)$ (depending on u^{**} and x_0) such that

$$x \in \Omega_A, \;\; \xi \in A(x), \tag{3.14}$$

for all $(x,\xi) \in \Omega \times \mathbb{R}^n$ such that

$$|x - x_0| \le \gamma, \;\; |\xi - Du^{**}(x_0)| \le 2\gamma\,. \tag{3.15}$$

Recall that $x_0 \in \Omega_A \setminus \partial\Omega^0_A$; thus we can also assume that γ is sufficiently small so that

$$\begin{cases} x_0 \in \Omega^{\pm}_A, \;\; |x - x_0| \le \gamma & \Longrightarrow \quad x \in \Omega^{\pm}_A\,, \\ x_0 \in \operatorname{int}\Omega^0_A, \;\; |x - x_0| \le \gamma & \Longrightarrow \quad x \in \operatorname{int}\Omega^0_A. \end{cases} \tag{3.16}$$

Choose $\delta \in (0,\gamma]$ (depending on x_0) sufficiently small such that

$$\frac{|o\left(|x - x_0|\right)|}{|x - x_0|} \le \gamma, \qquad \forall\, x \in \overline{B}_\delta(x_0), \; x \ne x_0\,, \tag{3.17}$$

and

$$\delta \le \frac{\gamma}{2\,|Du^{**}(x_0)| + 4\gamma}, \qquad \overline{B}_\delta(x_0) \subset \Omega\,. \tag{3.18}$$

By (3.16) and by the definition of Ω^+_A, Ω^-_A, Ω^0_A in (3.11), (3.12), we have

$$\begin{cases} x_0 \in \Omega^{\pm}_A & \Longrightarrow \quad \operatorname{div} m(x) \gtrless 0 \;\; \forall x \in \overline{B}_\delta(x_0), \\ x_0 \in \operatorname{int}\Omega^0_A & \Longrightarrow \quad \operatorname{div} m(x) = 0 \;\; \forall x \in \overline{B}_\delta(x_0). \end{cases} \tag{3.19}$$

For every $r \in (0,\delta]$, let us define in Ω the function $v^r_{x_0}$ by

$$v^r_{x_0}(x) := u^{**}(x_0) + \langle Du^{**}(x_0), x - x_0\rangle \pm \gamma \cdot \left(r - 2\,|x - x_0|\right), \;\; x \in \Omega,$$

the sign $+$ being chosen if $x_0 \in \Omega^+_A$, while the sign $-$ is selected if $x_0 \in \Omega^-_A$. If $x_0 \in \operatorname{int}\Omega^0_A$ then any sign in the definition of $v^r_{x_0}(x)$ is a good choice; in order to fix the ideas, we choose the sign $+$ if $x_0 \in \operatorname{int}\Omega^0_A$. Since $|D\,|x|| = 1$ for every $x \in \mathbb{R}^n \setminus \{0\}$, we obtain

$$\left|Dv^r_{x_0}(x) - Du^{**}(x_0)\right| = 2\gamma \cdot |\mp D\,|x - x_0|| = 2\gamma \quad \text{a.e. } x \in \Omega,$$

and thus by (3.14), (3.15),

$$Dv^r_{x_0}(x) \in A(x), \quad \text{a.e. } x \in \overline{B}_\delta(x_0), \ \forall\, r \in (0,\delta]\ . \tag{3.20}$$

If $x_0 \in \Omega_A^+ \cup \operatorname{int}\Omega_A^0$ we set

$$G(x_0,r) := \left\{ x \in \overline{B}_\delta(x_0) \subset \Omega :\ v^r_{x_0}(x) \geq u^{**}(x) \right\}, \tag{3.21}$$

and if $x_0 \in \Omega_A^-$ we define

$$G(x_0,r) := \left\{ x \in \overline{B}_\delta(x_0) \subset \Omega :\ v^r_{x_0}(x) \leq u^{**}(x) \right\}\ . \tag{3.22}$$

We claim that $G(x_0,r)$ is a closed set satisfying

$$\overline{B}_{r/3}(x_0) \subset G(x_0,r) \subset \overline{B}_r(x_0). \tag{3.23}$$

Let us verify (3.23) when $x_0 \in \Omega_A^+ \cup \operatorname{int}\Omega_A^0$. If $x \in \overline{B}_\delta(x_0)$ but $x \notin \overline{B}_r(x_0)$ (that is $r < |x-x_0| \leq \delta$) then by (3.13) and (3.17) we get

$$v^r_{x_0}(x) - u^{**}(x) = \gamma\cdot(r - 2\,|x-x_0|) - o\left(|x-x_0|\right)$$

$$< -\gamma\,|x-x_0| - o\left(|x-x_0|\right) = -\,|x-x_0| \cdot \left(\gamma + \frac{o\left(|x-x_0|\right)}{|x-x_0|} \right) \leq 0.$$

Thus $v^r_{x_0}(x) - u^{**}(x) < 0$ and $x \notin G(x_0,r)$. On the other hand if $x \in \overline{B}_{r/3}(x_0)$, then $r/3 \geq |x-x_0|$, and again by (3.13), (3.17), we obtain

$$v^r_{x_0}(x) - u^{**}(x) = \gamma\cdot(r - 2\,|x-x_0|) - o\left(|x-x_0|\right) \geq$$

$$\gamma\,|x-x_0| - o\left(|x-x_0|\right) = |x-x_0| \left(\gamma - \frac{o\left(|x-x_0|\right)}{|x-x_0|} \right) \geq 0$$

and $x \in G(x_0,r)$. Thus (3.23) is proved.

By (3.21) and the continuity of u^{**} and $v^r_{x_0}$

$$\partial G(x_0,r) = \left\{ x \in \overline{B}_\delta(x_0) :\ v^r_{x_0}(x) = u^{**}(x) \right\}\ ,$$

and setting

$$w(x) := u^{**}(x) - u^{**}(x_0) - \langle Du^{**}(x_0), x - x_0\rangle \pm 2\gamma\,|x-x_0|\,, \ \ x \in \Omega,$$

then

$$\partial G(x_0,r) = \left\{ x \in \overline{B}_\delta(x_0) :\ w(x) = \gamma r \right\},$$

i.e., for every $r \in (0,\delta)$ $\partial G(x_0,r)$ is a level set of a $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ function. Therefore we can always choose a sequence r_h of real numbers such that

$$\begin{cases} r_h \to 0 \ \text{ as } h \to +\infty, \\ 0 < r_h \leq \delta, \ \ \forall\, h \in \mathbb{N}, \\ \operatorname{meas}\left(\partial G(x_0,r_h)\right) = 0, \ \ \forall\, h \in \mathbb{N}. \end{cases} \tag{3.24}$$

Let us consider the measurable subset of Ω

$$M := \left\{x_0 \in \Omega_A^+ \cup \Omega_A^- \cup \operatorname{int}\Omega_A^0 : \, u^{**} \text{ differentiable at } x_0, \quad Du^{**}(x_0) \in A(x_0)\right\}$$

and consider the family of open sets

$$\mathcal{G} := \{\operatorname{int} G(x, r_h) : x \in M, \; r_h \text{ as in (3.24)}\} \; .$$

Since $\mathcal{G}$ is a fine covering of M and each set $G(x,r) \in \mathcal{G}$ satisfies (3.23), by Vitali's covering theorem (see, for example, Chapter 10 of [12]), there exists in $\mathcal{G}$ a (at most) countable subcollection $\mathcal{G}'$ of sets with disjoint closures such that the open set

$$\Omega' = \bigcup_{\operatorname{int} G(x_k, r_k) \in \mathcal{G}'} \operatorname{int} G(x_k, r_k) \, , \tag{3.25}$$

covers M up to a set of zero measure, i.e. $\operatorname{meas}(M \setminus \Omega') = 0$. Let us define the function v in Ω by

$$v(x) := \begin{cases} u^{**}(x) & \text{if } x \in \Omega \setminus \Omega', \\ v_{x_k}^{r_k}(x) & \text{if } x \in G(x_k, r_k), \end{cases} \tag{3.26}$$

and introduce the functions

$$u_h(x) := \begin{cases} u^{**}(x) & \text{if } x \in \Omega \setminus \cup_{k=1}^h G(x_k, r_k), \\ v_{x_k}^{r_k}(x) & \text{if } x \in G(x_k, r_k), \text{ for some } 1 \le k \le h. \end{cases}$$

By (3.21) and (3.22) the functions u_h belong to $u^{**} + W_0^{1,p}(\Omega) \cap W_{\text{loc}}^{1,\infty}(\Omega)$ for all h, since u_h is locally maximum or minimum of two $W^{1,p}(\Omega) \cap W_{\text{loc}}^{1,\infty}(\Omega)$ functions. Moreover, we claim that each u_h is a minimizer of (3.9). To this aim, notice that by (3.19), (3.21), (3.22), we have for all h

$$\operatorname{div} m(x) \; (u_h(x) - u^{**}(x)) \ge 0 \quad \text{a.e. } \; x \in \cup_{k=1}^h G(x_k, r_k) \, . \tag{3.27}$$

Moreover, by (3.20)

$$Du_h(x) \in A(x), \quad \text{a.e. } \; x \in \cup_{k=1}^h G(x_k, r_k) \, . \tag{3.28}$$

and $Du_h(x) = Du^{**}(x)$ a.e. in $\Omega \setminus \cup_{k=1}^h G(x_k, r_k)$. By the convexity inequality $f^{**}(x, \xi) \ge q(x) + \langle m(x), \xi \rangle$, valid for every $x \in \Omega_A$ and every $\xi \in \mathbb{R}^n$, we have

$$\inf \left\{ \int_\Omega f^{**}\left(x, Du(x)\right) \, dx : u \in u_0 + W_0^{1,p}\left(\Omega\right) \right\}$$

$$= \int_{\Omega \setminus \cup_{k=1}^h G(x_k, r_k)} f^{**}\left(x, Du^{**}(x)\right) \, dx + \int_{\cup_{k=1}^h G(x_k, r_k)} f^{**}\left(x, Du^{**}(x)\right) \, dx$$

$$\ge \int_{\Omega \setminus \cup_{k=1}^h G(x_k, r_k)} f^{**}\left(x, Du^{**}(x)\right) \, dx + \int_{\cup_{k=1}^h G(x_k, r_k)} \left\{ q(x) + \langle m(x), Du^{**}(x) \rangle \right\} \, dx,$$

where, in view of (3.14), (3.15) and (3.23), we have used the fact that $\cup_{k=1}^h G(x_k, r_k) \subset \Omega_A$.

Since $u_h \in u^{**} + W_0^{1,p}(\operatorname{int} G(x_k, r_k))$ for all $1 \le k \le h$, using (3.27) we obtain

$$\begin{aligned} &\int_{G(x_k,r_k)} \{\langle m(x), Du^{**}(x)\rangle - \langle m(x), Du_h(x)\rangle\}\, dx \\ = \; & -\int_{G(x_k,r_k)} \operatorname{div} m(x)\,(u^{**}(x) - u_h(x))\; dx \ge 0\,. \end{aligned}$$

Therefore we may conclude, from the inequality above and by (3.28), that

$$\inf\left\{\int_\Omega f^{**}\left(x, Du(x)\right)\, dx : u \in u_0 + W_0^{1,p}\left(\Omega\right)\right\}$$

$$\ge \int_{\Omega\setminus\cup_{k=1}^h G(x_k,r_k)} f^{**}\left(x, Du^{**}(x)\right)\, dx + \int_{\cup_{k=1}^h G(x_k,r_k)} \{q(x) + \langle m(x), Du_h(x)\rangle\}\, dx$$

$$= \int_\Omega f^{**}\left(x, Du_h(x)\right)\, dx\,.$$

This proves the minimality of each u_h. By (3.2) the sequence u_h is bounded in $W^{1,p}(\Omega)$ and since $u_h(x)$ converges to $v(x)$ a.e. in Ω, it follows that u_h converges weakly to v in $W^{1,p}(\Omega)$. In particular, $v \in u_0 + W_0^{1,p}(\Omega)$ and, by lower semicontinuity, v is also a minimizer of (3.9). Theorem 2.7 now yields $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. By (3.20) we have

$$Dv(x) \in A(x), \quad \text{a.e. } \; x \in \Omega'.$$

On the other hand, since $v = u^{**}$ in $\Omega \setminus \Omega'$, there exists a negligible set E_0 such that, for all $x \in (\Omega \setminus \Omega') \setminus E_0$, v and u^{**} are differentiable at x and $Dv(x) = Du^{**}(x)$. Thus, if $x \notin E_0 \cup (M \setminus \Omega') \cup \left(\Omega \cap \partial\Omega_A^0\right)$, which is a set of measure zero, and if $x \notin \Omega \setminus \Omega'$, then we have

$$Dv(x) \notin A(x)\,.$$

Finally, let us prove that $\operatorname{div} m = 0$ in Ω'. To this end notice that, since

$$f^{**}(x,\xi) = q(x) + \langle m(x), \xi\rangle\,, \quad \text{whenever } x \in \Omega_A \text{ and } \xi \in A(x),$$

the function $\xi \to f^{**}(x,\xi)$ is differentiable in $A(x)$ and $D_\xi f^{**}(x,\xi) = m(x)$ for every $x \in \Omega'$, $\xi \in A(x)$. Therefore the standard argument used to derive the Euler equation of an integral functional still applies in the open set Ω' and the Euler's equation in weak form gives

$$\int_{\Omega'} \operatorname{div} m(x)\, \varphi(x)\, dx = 0\,, \quad \forall\, \varphi \in C_0^1\left(\Omega'\right),$$

which proves that $\operatorname{div} m = 0$ in Ω'. ■

The second lemma that we consider in this section uses the notion of *Kuratowski convergence*, or *convergence in the Hausdorff metric* of a sequence of compact sets of $\mathbb{R}^n$. We recall that a sequence E_h of compact sets of $\mathbb{R}^n$ converges, as $h \to +\infty$, *in the sense of Kuratowski* (or *in the Hausdorff metric*) to a compact set $E \subset \mathbb{R}^n$ if, for every $\varepsilon > 0$, there exists h_0 such that

$$E_h \subset I_\varepsilon(E), \quad E \subset I_\varepsilon(E_h) \quad \forall\, h > h_0 ,$$

where $I_\varepsilon(\cdot)$ denotes the neighborhood of radius ε of the set under consideration. We recall that, if the sequence E_h is bounded in $\mathbb{R}^n$ uniformly with respect to $h \in \mathbb{N}$, then $E_h \to E$ in the sense of Kuratowski if and only if the following two properties hold:

(i) for every $\xi \in E$ there exists a sequence ξ_h, with $\xi_h \in E_h$ for every $h \in \mathbb{N}$, such that ξ_h converges to ξ as $h \to +\infty$;

(ii) if a sequence ξ_h, with $\xi_h \in E_h$ for every $h \in \mathbb{N}$, admits a subsequence ξ_{h_k} converging to a point ξ, then $\xi \in E$.

In the sequel we need some properties of the distance function to a convex set. For the statements and the proofs, we refer the reader to the Appendix.

Let $E(x) \neq \emptyset$ be a compact subset of $\mathbb{R}^n$ for every value of a parameter x in an open set $\Omega_A \subset \mathbb{R}^n$. We say that the set function $E(x)$ is *continuous in the sense of Kuratowski* if $E(x_k)$ converges in the sense of Kuratowski to $E(x)$ for every $x, x_k \in \Omega_A$ such that $x_k \to x$.

Lemma 3.4 *Let $f : \Omega_A \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that the bipolar f^{**} is also continuous and satisfies (3.6) and (3.8). Let $E(x) \subset \mathbb{R}^n$ be defined by*

$$E(x) := \{\xi \in \mathbb{R}^n : \ f^{**}(x,\xi) - [q(x) + \langle m(x), \xi\rangle] = 0\} .$$

Then the map $x \in \Omega_A \mapsto E(x)$ is continuous in the sense of Kuratowski.

Proof. Let x_h be a sequence in Ω_A converging to $x \in \Omega_A$. We claim that, if $\xi_{h_k} \in E(x_{h_k})$ and $\xi_{h_k} \to \xi$, then $\xi \in E(x)$. In fact, since

$$f^{**}(x_{h_k}, \xi_{h_k}) = q(x_{h_k}) + \left\langle m(x_{h_k}), \xi_{h_k} \right\rangle ,$$

the continuity of f^{**}, m, and q yields $\xi \in E(x)$.

Let us now show that, given $\xi \in E(x)$, there exist $\xi_h \in E(x_h)$ such that $\xi_h \to \xi$. Indeed, if this is not true, then we can find $\varepsilon > 0$ and a sequence h_k such that $\operatorname{dist}(\xi; E(x_{h_k})) > \varepsilon$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$ choose $\xi_{h_k} \in \partial E(x_{h_k})$ such that $\left|\xi - \xi_{h_k}\right| = \operatorname{dist}(\xi; E(x_{h_k})) > \varepsilon$. Extracting, if necessary, a further subsequence, we may assume that $\xi_{h_k} \to \eta$, and $\eta \in E(x)$ by the first part of the proof. Since $\xi \notin E(x_{h_k})$ and $\xi_{h_k} \in \partial E(x_{h_k})$, from (3.8) we get, for all $k \in \mathbb{N}$,

$$f^{**}\left(x_{h_k}, \frac{\xi + \xi_{h_k}}{2}\right) \leq \frac{1}{2} f^{**}(x_{h_k}, \xi) + \frac{1}{2} f^{**}(x_{h_k}, \xi_{h_k}) - \omega\left(|\xi - \xi_{h_k}|\right) .$$

Passing to the limit as $k \to +\infty$, we get

$$f^{**}\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2} f^{**}(x, \xi) + \frac{1}{2} f^{**}(x, \eta) - \omega(\varepsilon) .$$

Since $f^{**}(x,\cdot)$ is affine on $E(x)$ and $\xi,\eta \in E(x)$, the inequality above implies $\omega(\varepsilon) \le 0$, which gives a contradiction since $\omega(t) > 0$ for all $t > 0$. $\blacksquare$

Lemma 3.5 *Under the same assumptions of Lemma 3.4 the function $F : \Omega_A \times \mathbb{R}^n \to \mathbb{R}$, defined by*

$$F(x,\xi) = \begin{cases} \operatorname{dist}(\xi; \partial E(x)) & \text{if } \xi \in \mathbb{R}^n \setminus E(x), \\ -\operatorname{dist}(\xi; \partial E(x)) & \text{if } \xi \in E(x), \end{cases}$$

is continuous in $\Omega_A \times \mathbb{R}^n$ and convex with respect to $\xi \in \mathbb{R}^n$.

Proof. The convexity of $F(x,\cdot)$ follows from Lemma 4.4 and the convexity of $E(x)$. The continuity of F with respect to $x \in \Omega_A$ is a direct consequence of Lemma 4.2, asserting the continuity of the distance function with respect to the Kuratowski convergence, and of Lemma 3.4, yielding the continuity of the set function $E(x)$ with respect to x. The continuity of F with respect to (x,ξ) then follows from the fact that for all $x \in \Omega_A$, $\xi,\eta \in \mathbb{R}^n$ we have that $|F(x,\xi) - F(x,\eta)| \le |\xi - \eta|$. $\blacksquare$

We are now ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2 the minimum of the relaxed variational problem (3.9) is attained and there exist a minimizer v and an *open* set $\Omega' \subset \Omega$ (possibly empty) such that

$$\begin{cases} Dv(x) \in A(x) & \text{a.e. } x \in \Omega' \\ Dv(x) \notin A(x) & \text{a.e. } x \in \Omega \setminus \Omega', \end{cases}$$

with $\operatorname{div} m = 0$ in Ω'.

If Ω' is an empty set then $Dv(x) \notin A(x)$ a.e. $x \in \Omega$, and so $f^{**}(x, Dv(x)) = f(x, Dv(x))$ a.e. $x \in \Omega$, and v is also a minimizer for the original problem (3.1).

Otherwise, recalling (3.25), we consider for all k the Dirichlet problem for the implicit differential equation

$$\begin{cases} F(x, Du(x)) = 0, & \text{a.e. } x \in \operatorname{int} G(x_k, r_k), \\ u(x) = v(x), & x \in \partial G(x_k, r_k), \end{cases} \tag{3.29}$$

where $F : \Omega_A \times \mathbb{R}^n \to \mathbb{R}$ is the signed distance function defined in Lemma 3.5. Notice that v is Lipschitz continuous on $G(x_k, r_k)$, and since $A(x) \subset E(x)$ we deduce that

$$F(x, Dv(x)) \le 0 \quad \text{a.e. } x \in \operatorname{int} G(x_k, r_k). \tag{3.30}$$

Therefore, we may apply Theorem 2.3 by Dacorogna-Marcellini [12] (note that (3.30) is exactly the compatibility condition (2.6) required in Theorem 2.3 in [12]) to obtain the existence of a function $v_k \in W^{1,\infty}(\operatorname{int} G(x_k, r_k))$ satisfying (3.29). Notice that $F(x,\xi) = 0$ if and only if $\xi \in \partial E(x)$, and thus $Dv_k(x) \in \partial E(x) \subset \mathbb{R}^n \setminus A(x)$ almost everywhere in $\operatorname{int} G(x_k, r_k)$. Therefore

$$f^{**}(x, Dv_k(x)) = f(x, Dv_k(x)) \quad \text{a.e. } x \in \operatorname{int} G(x_k, r_k). \tag{3.31}$$

Let us now prove that the functions

$$w_h(x) := \begin{cases} v_k(x) & \text{if } x \in G(x_k, r_k),\ 1 \le k \le h\,, \\ v(x) & \text{if } x \in \Omega \setminus \cup_{k=1}^{h} G(x_k, r_k)\,, \end{cases}$$

are all minimizers in the class $u_0 + W_0^{1,p}(\Omega)$ of the integral in (3.9). In fact, by the affinity assumption (3.6), for any h we get

$$\begin{aligned} &\int_\Omega f^{**}(x, Dv(x))\, dx - \int_\Omega f^{**}(x, Dw_h(x))\, dx \qquad (3.32) \\ = &\sum_{k=1}^{h} \int_{G(x_k, r_k)} \{f^{**}(x, Dv(x)) - f^{**}(x, Dv_k(x))\}\, dx \\ = &\sum_{k=1}^{h} \int_{G(x_k, r_k)} \langle m(x), Dv(x) - Dv_k(x)\rangle\, dx = 0\,, \end{aligned}$$

where we have used the fact that $v = v_k$ on $\partial G(x_k, r_k)$, together with the property that $\operatorname{div} m = 0$ in Ω'. The minimality of the function w_h follows immediately.

The proof will be over once we show that w_h converges strongly in $W^{1,p}(\Omega)$ to a function w. Indeed, the limiting function w will then be the required minimizer because by (3.2), (3.32), and (3.31),

$$\int_\Omega f(x, Dw(x))\, dx = \lim_{h \to +\infty} \int_\Omega f(x, Dw_h(x))\, dx$$

$$= \int_\Omega f^{**}(x, Dv(x))\, dx + \lim_{h \to +\infty} \left[\int_\Omega f(x, Dw_h(x))\, dx - \int_\Omega f^{**}(x, Dw_h(x))\, dx \right]$$

$$= \int_\Omega f^{**}(x, Dv(x))\, dx + \lim_{h \to +\infty} \int_{\Omega' \setminus (\cup_{k \le h} G(x_k, r_k))} [f(x, Dv(x)) - f^{**}(x, Dv(x))]\, dx$$

$$= \int_\Omega f^{**}(x, Dv(x))\, dx\,,$$

where we have used the fact that $f(x, Dv(x)) = f^{**}(x, Dv(x))$ for a.e. $x \notin \Omega'$. In order to prove that w_h converges in $W^{1,p}(\Omega)$, it suffices to show that Dw_h is a Cauchy sequence in L^p. Since v_k minimizes the integral

$$\int_{G(x_k, r_k)} f^{**}(x, Du(x))\, dx$$

in the class $v + W_0^{1,p}(\operatorname{int} G(x_k, r_k))$, by (3.2) there exists a constant c such that for all k

$$\int_{G(x_k, r_k)} |Dv_k|^p\, dx \le c \int_{G(x_k, r_k)} (1 + |Dv|^p)\, dx\,.$$

Hence, if $h > k$ we get

$$\int_\Omega |Dw_h - Dw_k|^p \, dx = \sum_{i=k+1}^{h} \int_{G(x_i,r_i)} |Dv_i - Dv|^p \, dx$$

$$\leq c \int_{\Omega' \setminus (\cup_{i \leq k} G(x_i,r_i))} (1 + |Dv|^p) \, dx \, ,$$

and the integral on the right hand side converges to zero as $k \to \infty$. ■

3.2 The general case

In the previous section we have proved Theorem 3.1 under the assumption that the bipolar function f^{**} coincides with the affine function $\langle m(\cdot), \xi \rangle + q(\cdot)$ on the set $A(x) = \{\xi \in \mathbb{R}^n : \ f(x,\xi) > f^{**}(x,\xi)\}$ defined in (3.5). Here we consider the more general situation where the set $A(x)$ can be split into an union of a (at most countable) family of pairwise disjoint open sets $A_j(x)$, and that in each $A_j(x)$ the function $f^{**}(x,\xi)$ coincides with an affine function $\langle m_j(x), \xi \rangle + q(x)$, where the slopes m_j and the functions q_j may vary with j.

Precisely, let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function, satisfying the growth condition (3.2), the uniform convexity condition (3.3), and (3.4). As before, we assume that f^{**} is continuous and we denote by Ω_A the set of points $x \in \Omega$ such that $A(x) \neq \emptyset$. Moreover, we assume that, for all $x \in \Omega_A$, there exists a sequence $A_j(x)$ of pairwise disjoint open sets such that

$$A(x) = \cup_j A_j(x) \, , \tag{3.33}$$

and that, for every j, the set $\Omega_{A_j} := \{x \in \Omega : \ A_j(x) \neq \emptyset\}$ is open. Further, assume that, for all j, there exist a function $q_j \in C^0\left(\Omega_{A_j}\right)$ and a vector field $m_j \in C^1\left(\Omega_{A_j}; \mathbb{R}^n\right)$ such that

$$f^{**}(x,\xi) = q_j(x) + \langle m_j(x), \xi \rangle \, , \qquad \forall \, x \in \Omega_{A_j} \, , \ \xi \in A_j(x) \, , \tag{3.34}$$

and the *boundary* of the set where the divergence of m_j is equal to zero is negligible; i.e., for all j,

$$\operatorname{meas}\left(\Omega \cap \partial \left\{x \in \Omega_{A_j} : \ \operatorname{div} m_j(x) = 0\right\}\right) = 0 \, .$$

For all $x \in \Omega_{A_j}$ set

$$E_j(x) := \{\xi \in \mathbb{R}^n : \ f^{**}(x,\xi) = q_j(x) + \langle m_j(x), \xi \rangle\}$$

and we assume that, for every $j \in \mathbb{N}$, there exists an increasing function $\omega_j : [0,+\infty) \to [0,+\infty)$, with $\omega_j(t) = 0$ if and only if $t = 0$, such that, if $x \in \Omega_A$, $\xi \in \partial E_j(x)$, $\eta \in \mathbb{R}^n \setminus E_j(x)$, then

$$f^{**}\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2} f^{**}(x,\xi) + \frac{1}{2} f^{**}(x,\eta) - \omega_j\left(|\xi - \eta|\right) . \tag{3.35}$$

As in the previous section we have the following existence result.

Theorem 3.6 *Let f, $f^{**} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be continuous functions (f not necessarily convex with respect to $\xi \in \mathbb{R}^n$). Under the above assumptions on f and f^{**}, for any given boundary datum $u_0 \in W^{1,p}(\Omega)$ the variational problem (3.1) attains its minimum. Moreover, any minimizers is of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*

Remark 3.7 Notice that the assumptions (3.33) and (3.34) are equivalent to the following local assumption: for any $(x_0, \xi_0) \in \Omega \times \mathbb{R}^n$ such that $f(x_0, \xi_0) > f^{**}(x_0, \xi_0)$ there exist $\delta > 0$, a function $q \in C^0(B_\delta(x_0))$, and a vector field $m \in C^1(B_\delta(x_0); \mathbb{R}^n)$ such that, if $|x - x_0| < \delta$, $|\xi - \xi_0| < \delta$, then

$$f^{**}(x, \xi) = q(x) + \langle m(x), \xi \rangle . \tag{3.36}$$

In fact, set $D := \{(x, \xi) \in \Omega \times \mathbb{R}^n : f(x, \xi) > f^{**}(x, \xi)\}$. The set D is open, hence we may consider its connected components D_j. For every $x \in \Omega$ and every j let us set $A_j(x) := \{\xi \in \mathbb{R}^n : (x, \xi) \in D_j\}$. For every x the family $\{A_j(x)\}$ is a partition of $A(x) := \{\xi \in \mathbb{R}^n : f(x, \xi) > f^{**}(x, \xi)\}$. Moreover, the set $\Omega_{A_j} := \{x \in \Omega : A_j(x) \neq \emptyset\}$ is open, since it is equal to the projection of D_j onto $\mathbb{R}^n$. Finally, the existence for every j of a function $q_j \in C^0(\Omega_{A_j})$ and a vector field $m_j \in C^1(\Omega_{A_j}; \mathbb{R}^n)$ satisfying (3.34) is an immediate consequence of the local assumption (3.36) and of the connectedness of D_j.

To prove Theorem 3.6 one may follow exactly the same argument used in the proof of Theorem 3.1, with the obvious changes due to the fact that now we have to deal separately with the different affine representations (3.34) of f^{**}. Therefore we shall limit ourselves to point out where the proof has to be modified.

The statement of Lemma 3.2 must be replaced by the following one.

Lemma 3.8 *The minimum of the relaxed variational problem (3.9) is attained. Moreover, there exists a minimizer $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap \left(u_0 + W^{1,p}(\Omega)\right)$ of (3.9) and there exist pairwise disjoint open sets $\Omega'_j \subset \Omega$ (possibly empty) such that*

$$\begin{cases} Dv(x) \in A_j(x) & a.e. \;\; x \in \Omega'_j \\ Dv(x) \notin A(x) & a.e. \;\; x \in \Omega \setminus \cup_j \Omega'_j \end{cases}$$

and $\operatorname{div} m_j = 0$ *in* Ω'_j.

Proof. To proof this lemma we argue as in the proof of Lemma 3.2, splitting each open set Ω_{A_j} into the three open sets

$$\Omega^+_{A_j} := \left\{x \in \Omega_{A_j} : \operatorname{div} m_j(x) > 0\right\}, \; \Omega^-_{A_j} := \left\{x \in \Omega_{A_j} : \operatorname{div} m_j(x) < 0\right\},$$

$$\Omega^0_{A_j} := \left\{x \in \Omega_{A_j} : \operatorname{div} m_j(x) = 0\right\}.$$

Taking a point $x_0 \in \Omega_{A_j} \setminus \partial\Omega^0_{A_j}$ such that u^{**} is differentiable in x_0 and $Du^{**}(x_0) \in A_j(x_0)$ for some j, we construct the function $v^r_{x_0}$ as before, noticing that the parameter γ can be always chosen so small (see (3.20)) that

$$Dv^r_{x_0}(x) \in A_j(x) \quad \text{a.e. } \; x \in \overline{B}_\delta(x_0), \;\; \forall\, r \in (0, \delta] \, . \tag{3.37}$$

Arguing as in the model case, we get again a sequence $G(x_k, r_k)$ of pairwise disjoint sets such that the open set

$$\Omega' := \bigcup_{\operatorname{int} G(x_k, r_k) \in \mathcal{G}'} \operatorname{int} G(x_k, r_k)$$

covers, up to a set of measure zero, the set of points x_0 where u^{**} is differentiable and $Du^{**}(x_0) \in A_j(x_0)$ for some j. Therefore, if we define v as in (3.26) we have again that $Dv(x) \notin A(x)$ for a.e. $x \in \Omega \setminus \Omega'$ and, by (3.37) that for any k there exists j_k such that

$$Dv(x) \in A_{j_k}(x) \quad \text{for a.e. } x \in G(x_k, r_k)\,.$$

In particular Ω' can be written as the disjoint union of the open sets

$$\Omega'_j = \bigcup \left\{ \operatorname{int} G(x_s, r_s) : \, Dv_{x_s}^{r_s}(x) \in A_j(x) \quad \forall\, x \in G(x_s, r_s) \right\}$$

and $Dv(x) \in A_j(x)$ a.e. $x \in \Omega'_j$, for all j. The rest of the proof follows with the obvious adaption. ■

Proof of Theorem 3.6. We claim that, for all j, the set function $x \in \Omega_{A_j} \mapsto E_j(x)$ is continuous in the sense of Kuratowski. It can be easily checked via the same argument used in Lemma 3.4, and by virtue of assumption (3.35) (this is the only point where this hypothesis is needed). Therefore, in view of Lemma 3.5, for every j the function F_j defined by

$$F_j\,(x, \xi) := \begin{cases} \operatorname{dist}\,(\xi; \partial E_j(x)) & \text{if } \xi \in \mathbb{R}^n \setminus E_j(x) \\ -\operatorname{dist}\,(\xi; \partial E_j(x)) & \text{if } \xi \in E_j(x), \end{cases}$$

is continuous in $\Omega_{A_j} \times \mathbb{R}^n$ and convex with respect to ξ. Then, replacing (3.20) with (3.37), the proof proceeds exactly as in the model case. ■

4 Appendix: some properties of the distance function

In this section we prove some properties of the signed distance function to a convex set, which has been used in Section 3 in order to establish Lemma 3.4 and Lemma 3.5. We recall that if $E \subset \mathbb{R}^n$ then the signed distance function to E is a function $d_E : \mathbb{R}^n \to \mathbb{R}$ defined as follows:

$$d_E(\xi) := \begin{cases} \operatorname{dist}\,(\xi; \partial E) & \text{if } \xi \notin E, \\ -\operatorname{dist}\,(\xi; \partial E) & \text{if } \xi \in E. \end{cases} \tag{4.1}$$

Lemma 4.1 *Let $E \subset \mathbb{R}^n$ be a closed convex set. If $\xi \in \mathbb{R}^n$ is such that*

$$\xi \in I_\delta\,(E)\,, \quad \operatorname{dist}\,(\xi; \partial I_\delta\,(E)) > \delta,$$

for some $\delta > 0$, then $\xi \in E$ (more precisely $\xi \in \operatorname{int} E$).

Proof. If $\xi \notin E$ then we consider the projection ξ_0 of ξ on E and a supporting hyperplane H to E through ξ_0. This hyperplane H separates ξ from E. Let us denote by η the point

$$\eta := \xi + \delta \frac{\xi - \xi_0}{|\xi - \xi_0|};$$

clearly $|\eta - \xi| = \delta$. Since by the assumptions $\overline{B}_\delta(\xi) \subset I_\delta(E)$, in particular $\eta \in I_\delta(E)$. On the other hand, $\operatorname{dist}(\eta; E) \geq \operatorname{dist}(\eta; H) = |\xi - \xi_0| + \delta > \delta$, which gives a contradiction. ■

Lemma 4.2 *Let E_h and E be convex, compact sets of $\mathbb{R}^n$, such that $E_h \to E$ in the sense of Kuratowski. Then*

$$d(\xi) = \lim_{h \to +\infty} d_h(\xi), \qquad \forall\, \xi \in \mathbb{R}^n,$$

where d_h and d are, respectively, the signed distance functions to E_h and E.

Proof. Note that we use the assumption of convexity of the sets E_h, E only in Steps 2 and 3 below.

Step 1: Let us consider the case where $\xi \notin E$. Then there exists $\varepsilon > 0$ such that $\operatorname{dist}(\xi; \partial E) > \varepsilon$ and, for large $h \in \mathbb{N}$, $\xi \notin E_h$. Let $\xi_0 \in E$ be a point such that $d(\xi) = |\xi - \xi_0|$, and let ξ_h be a sequence of points in E_h converging to ξ_0. Then

$$d(\xi) = \lim_{h \to +\infty} |\xi - \xi_h| \geq \limsup_{h \to +\infty} d_h(\xi).$$

Conversely, let $\xi_{h_k} \in E_{h_k}$ be such that

$$\liminf_{h \to +\infty} d_h(\xi) = \lim_{k \to +\infty} \left|\xi - \xi_{h_k}\right|.$$

By extracting, if necessary, a further subsequence, we may assume that $\xi_{h_k} \to \xi_0$, for some $\xi_0 \in E$. Thus

$$\liminf_{h \to +\infty} d_h(\xi) = \lim_{k \to +\infty} \left|\xi - \xi_{h_k}\right| = |\xi - \xi_0| \geq d(\xi).$$

Step 2: Let $\xi \in \partial E$. There exists a sequence h_k such that

$$\limsup_{h \to +\infty} d_h(\xi) = \lim_{k \to +\infty} d_{h_k}(\xi).$$

Let $\xi_{h_k} \in E_{h_k}$ be such that $\xi_{h_k} \to \xi$. Since $\left|\xi - \xi_{h_k}\right| \geq d_{h_k}(\xi)$, we obtain

$$\limsup_{h \to +\infty} d_h(\xi) \leq 0 = d(\xi).$$

Suppose now that $\liminf_{h \to +\infty} d_h(\xi) < 0$. As before, there exists a sequence h_k such that $\liminf_{h \to +\infty} d_h(\xi) = \lim_{k \to +\infty} d_{h_k}(\xi)$. Fix $\varepsilon > 0$. For k large enough $d_{h_k}(\xi) < 0$ and $\xi \in E_{h_k} \subset I_\varepsilon(E)$. Let us denote by ν a unit vector in $\mathbb{R}^n$, orthogonal to a supporting hyperplane to E at ξ, pointing to the exterior of E.

By the convexity of E, the vector $\xi + 2\varepsilon\nu \notin I_\varepsilon(E)$, and so $\xi + 2\varepsilon\nu \notin E_{h_k}$ for large k. Therefore, for such k, $\operatorname{dist}(\xi; \partial E_{h_k}) = |d_{h_k}(\xi)| < 2\varepsilon$. This implies that $\liminf_{h\to+\infty} d_h(\xi) \geq 0$.

Step 3: Let $\xi \in \operatorname{int} E$ and fix $\delta \in (0, \operatorname{dist}(\xi; \partial E))$ such that $B_\delta(\xi) \subset E$. For every $\varepsilon < \delta$ there exists h_ε such that $E \subset I_\varepsilon(E_h)$ for $h > h_\varepsilon$. If $\eta \in B_{\delta-\varepsilon}(\xi) \subset E \subset I_\varepsilon(E_h)$ then $\operatorname{dist}(\eta; \partial I_\varepsilon(E_h)) > \varepsilon$; therefore, by Lemma 4.1, we have that $B_{\delta-\varepsilon}(\xi) \subset E_h$ for $h > h_\varepsilon$ (in particular $\xi \in \operatorname{int} E_h$) and $\operatorname{dist}(\xi; \partial E_h) \geq \delta - \varepsilon$. From this it follows that $\liminf_{h\to+\infty} \operatorname{dist}(\xi; \partial E_h) = \liminf_{h\to+\infty} [-d_h(\xi)] \geq \delta$. Letting $\delta \to \operatorname{dist}(\xi; \partial E)$, we get

$$d(\xi) \geq \limsup_{h\to+\infty} d_h(\xi).$$

Conversely, let $\xi_0 \in \partial E$ be such that $|\xi - \xi_0| = \operatorname{dist}(\xi; \partial E)$. From Step 2 it follows that there exists $\xi_h \in \partial E_h$ such that $|\xi_h - \xi_0| \to 0$. We also have

$$\operatorname{dist}(\xi; \partial E_h) \leq |\xi_h - \xi| \leq |\xi_h - \xi_0| + \operatorname{dist}(\xi; \partial E)\ .$$

Since $\xi \in E_h$, we conclude that

$$\liminf_{h\to+\infty} d_h(\xi) = \liminf_{h\to+\infty} [-\operatorname{dist}(\xi; \partial E_h)] \geq -\operatorname{dist}(\xi; \partial E) = d(\xi).$$

■

Lemma 4.3 *Let E be a compact, convex set of $\mathbb{R}^n$. There exists a sequence E_h of compact, convex sets of $\mathbb{R}^n$ such that $E_h \supset E$, $\partial E_h \in C^\infty$ for every $h \in \mathbb{N}$, and $E_h \to E$ in the sense of Kuratowski.*

Proof. Set $\delta(\xi) := \operatorname{dist}(\xi; E)$, and $\delta_h(\xi) := (\rho_{\varepsilon_h} * \delta)(\xi)$, where $\varepsilon_h \to 0$ and ρ_{ε_h} is a sequence of smooth *mollifiers*. The functions δ_h are of class C^∞, convex, and converge to δ uniformly in $\mathbb{R}^n$. Therefore, with no loss of generality, we may assume that

$$|\delta_h(\xi) - \delta(\xi)| < \frac{1}{h} \qquad \forall\, \xi \in \mathbb{R}^n.$$

By Sard's theorem, for every $h \in \mathbb{N}$ there exists $t_h \in (1/h, 2/h)$ such that the level set $\{\xi \in \mathbb{R}^n : \delta_h(\xi) = t_h\}$ is a C^∞-hypersurface. Hence

$$E_h := \{\xi \in \mathbb{R}^n : \ \delta_h(\xi) \leq t_h\}$$

is a compact convex set with C^∞ boundary ∂E_h. If $\xi \in E$ then $\delta_h(\xi) = \delta_h(\xi) - \delta(\xi) < 1/h < t_h$ and so $\xi \in E_h$. This proves that $E \subset E_h$.

On the other hand, if $\xi \in E_h$ then

$$\delta(\xi) = \delta(\xi) - \delta_h(\xi) + \delta_h(\xi) < \frac{1}{h} + t_h < \frac{3}{h},$$

ans so $E_h \subset I_{3/h}(E)$; hence, given $\varepsilon > 0$, $E_h \subset I_\varepsilon(E)$ for h sufficiently large. ■

We conclude this appendix by proving the convexity of the signed distance function to a convex set.

Lemma 4.4 *If E is a compact, convex subset of $\mathbb{R}^n$ then the signed distance d_E, defined in (4.1), is a convex function.*

Proof. We may assume that ∂E is of class C^2, the general case being a consequence of Lemma 4.3 and Lemma 4.2. We recall that when the boundary ∂E is of class C^2 there exists a tubular neighborhood U of ∂E where the projection on ∂E is unique, $d_E \in C^2(U)$, $Dd_E(\xi) = \nu(\xi)$, the unit exterior normal vector to ∂E at ξ. To prove the convexity of d_E it is enough to show that

$$d_E\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{2}d_E(\xi_1) + \frac{1}{2}d_E(\xi_2) \quad \forall\, \xi_1, \xi_2 \in \mathbb{R}^n. \tag{4.2}$$

We consider three cases.

Case 1: $\xi_1, \xi_2 \in \mathbb{R}^n \setminus E$. If the midpoint $(\xi_1+\xi_2)/2$ belongs to E, then (4.2) is trivial, since the left hand side is less than or equal to zero and the right hand side is nonnegative. If the midpoint $(\xi_1+\xi_2)/2$ does not belongs to E, then let $\eta_1, \eta_2 \in \partial E$ be such that $d_E(\xi_i) = |\xi_i - \eta_i|$ for $i = 1, 2$. Since $(\eta_1+\eta_2)/2$ belongs to E, then

$$d_E\left(\frac{\xi_1+\xi_2}{2}\right) \leq \left|\frac{\xi_1+\xi_2}{2} - \frac{\eta_1+\eta_2}{2}\right|$$

$$\leq \frac{1}{2}|\xi_1-\eta_1| + \frac{1}{2}|\xi_2-\eta_2| = \frac{1}{2}d_E(\xi_1) + \frac{1}{2}d_E(\xi_2).$$

Case 2: $\xi_1, \xi_2 \in E$. Set $d_i := \operatorname{dist}(\xi_i; \partial E)$ for $i = 1, 2$. Let us assume that $d_1, d_2 > 0$, i.e., $\xi_1, \xi_2 \in \operatorname{int} E$, and let us fix $\rho_i \in (0, d_i)$ for $i = 1, 2$, and a unit vector η. Since $\xi_i + \rho_i\eta \in \operatorname{int} E$, then

$$\frac{\xi_1+\xi_2}{2} + \frac{\rho_1+\rho_2}{2}\eta \in \operatorname{int} E,$$

and the arbitrariness of η yields

$$\operatorname{dist}\left(\frac{\xi_1+\xi_2}{2}; \partial E\right) \geq \frac{\rho_1+\rho_2}{2}.$$

Letting $\rho_i \to d_i$ we obtain (4.2) when $d_1, d_2 > 0$. If $d_1 > 0$ and $d_2 = 0$ it is enough to approximate ξ_2 with interior points (which have positive distance from ∂E) and apply (4.2) with ξ_2 replaced by the approximating points. If $d_1 = d_2 = 0$ then (4.2) is trivial.

Case 3: $\xi_1 \in E$, $\xi_2 \in \mathbb{R}^n \setminus E$ (in fact, to avoid a trivial case, we assume that $\xi_1 \in \operatorname{int} E$). In this case the result follows from the fact that the function $t \in [0,1] \mapsto \varphi(t) = d_E((1-t)\xi_1 + t\xi_2)$ is convex on the whole interval $[0,1]$. In fact, from Case 1 and Case 2, we deduce that this function $\varphi(t)$ is separately convex on the two intervals $[0, t_0]$ and $[t_0, 1]$, where t_0 is the value of t such that the point $(1-t_0)\xi_1 + t_0\xi_2 \in \partial E$. Moreover, the differentiability of d_E at the boundary ∂E implies that φ is differentiable at t_0. Therefore $\varphi(t)$ is convex on the whole interval. ■

Acknowledgment. The research of Irene Fonseca was partially supported by the National Science Foundation under Grant No. DMS-9731957 and through Center for Nonlinear Analysis (CNA). The research of Nicola Fusco and Paolo Marcellini was partially supported by the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica (MURST).

References

[1] E. Acerbi, N. Fusco, *Regularity for minimizers of nonquadratic functionals: the case* $1 < p < 2$, J. Math. Anal. Appl., **140** (1989), 115–135.

[2] L. Ambrosio, N. Fusco, D. Pallara, *Special functions of bounded variation and free discontinuity problems*, Oxford University Press, to appear.

[3] J. M. Ball, R. D. James, *Fine phase mixtures as minimizers of energy*, Arch. Rat. Mech. Anal., **100** (1987), 15–52.

[4] J. M. Ball, R. D. James, *Proposed experimental tests of a theory of fine microstructure and the two wells problem*, Phil. Trans. Royal Soc. London, *A* **338** (1991), 389–450.

[5] P. Celada, S. Perrotta, *Minimizing non convex, multiple integrals: a density result*, Preprint SISSA, 1998.

[6] A. Cellina, *On minima of a functional of the gradient: necessary conditions*; Nonlinear Anal. Theory Methods Appl., **20** (1993), 337-341.

[7] A. Cellina, *On minima of a functional of the gradient: sufficient conditions*, Nonlinear Anal. Theory Methods Appl., **20** (1993), 343-347.

[8] B. Dacorogna, P. Marcellini, *Existence of minimizers for non quasiconvex integrals*, Arch. Rat. Mech. Anal., **131** (1995), 359-399.

[9] B. Dacorogna, P. Marcellini, *Théorème d'existence dans le cas scalaire et vectoriel pour les équations de Hamilton-Jacobi*, C.R. Acad. Sci. Paris, **322**, Série I, (1996), 237-240.

[10] B. Dacorogna, P. Marcellini, *Sur le problème de Cauchy-Dirichlet pour les systèmes d'équations non linéaires du premier ordre*, C. R. Acad. Sci. Paris, **323**, Série I, (1996), 599–602.

[11] B. Dacorogna, P. Marcellini, *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial case*, Acta Mathematica, **178** (1997), 1-37.

[12] B. Dacorogna, P. Marcellini, *Implicit partial differential equations*, Birkhäuser, Boston, 1999.

[13] B. Dacorogna, P. Marcellini, *Attainment of minima and implicit partial differential equations*, Ricerche di Matematica, **48** (1999), 311-346.

[14] F. S. De Blasi, G. Pianigiani, *On the Dirichlet problem for first order partial differential equations. A Baire category approach*, NoDEA Nonlinear Diff. Eq. Appl., **6** (1999), 13-34.

[15] G. Dolzmann, B. Kirchheim, S. Müller, V. Sverak, *The two-well problem in three dimensions*, Max-Planck Inst., Preprint 21, 1999.

[16] L. C. Evans, *Quasiconvexity and partial regularity in the calculus of variations*, Arch. Rat. Mech. Anal., **95** (1986), 227–252.

[17] L. C. Evans, R. F. Gariepy, *Blowup, compactness and partial regularity in the calculus of variations*, Ind. Univ. Math. J., **36** (1987), 361-371.

[18] I. Fonseca, G. Francfort, *$3D-2D$ asymptotic analysis of an optimal design problem for thin films*, J. Reine Angew. Math., **505** (1998), 173–202.

[19] I. Fonseca, N. Fusco, *Regularity results for anisotropic image segmentation models,* Ann. Sc. Norm. Sup. Pisa, **24** (1997), 463-499.

[20] I. Fonseca, G. Leoni, *Bulk and contact energies: nucleation and relaxation*, SIAM J. Math. Anal., **30** (1998), 190–219.

[21] G. Friesecke, *A necessary and sufficient condition for non attainment and formation of microstructure almost everywhere in scalar variational problems*, Proc. Royal Soc. Edinburgh, **124A** (1994), 437-471.

[22] P. Marcellini, *A relation between existence of minima for nonconvex integrals and uniqueness for not strictly convex integrals of the calculus of variations*, Math. Theories of Optimization, Cecconi J.P. and Zolezzi T. Ed., Lecture Notes in Math., Springer-Verlag, 979 (1983), 216-231.

[23] E. Mascolo, R. Schianchi, *Existence theorems for nonconvex problems*, J. Math. Pures Appl., **62** (1983), 349–359.

[24] E. Mascolo, R. Schianchi, *Nonconvex problems in the calculus of variations*, Nonlinear Analysis, Theory Meth. Appl., **9** (1985), 371–379.

[25] E. Mascolo, R. Schianchi, *Existence theorems in the calculus of variations*, J. Diff. Eq., **67** (1987), 185–198.

[26] S. Müller, V. Sverak, *Attainment results for the two-well problem by convex integration*, ed. Jost J., International Press, 1996, 239–251.

[27] J. P. Raymond, *Existence of minimizers for vector problems without quasiconvexity conditions*, Nonlinear Analysis, Theory Meth. Appl., **18** (1992), 815–828.

[28] M. A. Sychev, *Characterization of homogeneous scalar variational problems solvable for all boundary data*, Preprint CMU Pittsburgh, 97-203, October 1997.

[29] S. Zagatti, *Minimization of functionals of the gradient by Baire's theorem*, Preprint SISSA, 71/97, May 1997.

[30] Ziemer W.P., *Weakly differentiable functions*, Graduate Texts in Math., Springer-Verlag, New York, 1989.

I. Fonseca: Carnegie-Mellon University, Pittsburgh, Pennsylvania 15213, USA (fonseca+@andrew.cmu.edu)

N. Fusco: Dipartimento di Matematica "R. Caccioppoli", Università di Napoli, Via Cintia, 80126 Napoli, Italy (fusco@matna1.dma.unina.it)

P. Marcellini: Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67 A, 50134 Firenze, Italy (marcell@udini.math.unifi.it)