REMARKS ON THE DETERMINANT IN NONLINEAR ELASTICITY AND FRACTURE MECHANICS

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Abstract The role of the determinant in ensuring local invertibility of Sobolev functions in $W^{1,N}(\Omega;\mathbb{R}^N)$ is studied. Weak continuity of minors of gradients of functions in $W^{1,p}(\Omega;\mathbb{R}^N)$ for $p < N$ is fully characterized. Properties of the determinant are addressed within the framework of functions of bounded variation, and a change of variables formula is obtained. These results are relevant in the study of equilibria, cavitation, and fracture of nonlinear elastic materials.

Keywords: topological degree, weak lower semicontinuity, functions of bounded variation

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1. INTRODUCTION

Remarkable advances in industry and technology have motivated the study of instabilities in certain advanced materials. The need to understand and predict macroscopic behavior from microscopic and mesoscopic data, as well and the analysis of questions related to phase transformations, defectiveness, the onset of microstructures in smart materials, and other issues related to optimal design and homogenization of composite materials and very thin films, have opened new areas of mathematics virtually unexplored until recently. As it turns out, mathematical models for equilibria and dynamical evolution of phase boundaries for these materials fall outside the scope of classical theories, mostly due to the facts that the underlying energies are nonconvex, or that the admissible fields may exhibit discontinuities. Indeed, spaces of discontinuous mappings have proven to be useful in the modeling of deformations of continua which may undergo fracture or develop defects (see e.g. [13], [14], [15], [27]). A natural space for the underlying deformations is that of functions of bounded variation, BV, or its subclass introduced by De Giorgi and Ambrosio [24] of functions of special bounded variation, SBV.

It is well known that in nonlinear elasticity interpenetration of matter is prevented by assuming that energy densities blow up as the jacobian of the deformation gradient approaches zero. Precisely, if $\Omega \subset \mathbb{R}^N$ represents the reference configuration of an elastic body, and if the bulk energy of an admissible deformation $u : \Omega \to \mathbb{R}^N$ is given by

$$E(u) := \int_\Omega W(\nabla u)\, dx,$$

where $W : \mathbb{R}^N \times \mathbb{R}^{N\times N} \to [0, +\infty)$ is the energy density, then it is commonly assumed that $\det \nabla u > 0$ a.e. in Ω, and $W(\xi) \to +\infty$ as $\det \xi \to 0^+$. Under this degeneracy hypothesis, a question that has challenged mathematicians for several years concerns the search for physically reasonable hypotheses on W and on the class of admissible fields u guaranteeing weak lower semicontinuity of the energy E.

Ciarlet and Nečas [18] introduced a "local invertibility condition"

$$\int_\Omega \det \nabla u\, dx \leq \mathcal{L}^N(u(\Omega))$$

where $\mathcal{L}^N$ stands for the Lebesgue measure in $\mathbb{R}^N$, which, together with the condition $\det \nabla u > 0$ a.e. in Ω, ensures local invertibility of u in appropriate Sobolev spaces.

In this paper, we start by recalling a local inverse function theorem obtained by Fonseca and Gango [28] for mappings u in $W^{1,N}(\Omega; \mathbb{R}^N)$ such that $\det \nabla u(x) > 0$ for a.e. $x \in \Omega$.

If we require less regularity of the admissible fields, as it happens in the study of cavitation in rubber-like materials, then weak continuity properties of minors become very challenging. To this end, we fully characterize weak convergence of det in Sobolev spaces $W^{1,p}(\Omega; \mathbb{R}^N)$ for $p < N$. This analysis was carried out in Fonseca and Malý [32].

Finally, and drawing on the work of Choksi and Fonseca [17], we show how properties of $\det \nabla u$ may ensure a change of variables formula for functions of bounded variation, where ∇u denotes the part of the distributional derivative Du which is absolutely continuous with respect to $\mathcal{L}^N$.

In what follows, if ξ is a $d \times N$ matrix then $\mathbb{M}(\xi)$ stands for the list of all minors of ξ, and $\mathbb{M}_k(\xi)$ denotes all minors of order k, with $k \leq \min\{d, N\}$. If $d = N$ then $\operatorname{adj} \xi$ is the adjugate of ξ, i.e. the matrix of minors of order $N - 1$ so that

$$\det \xi \, \mathbb{I} = (\operatorname{adj} \xi)^T \xi.$$

If μ is a Radon measure in Ω and if $B \subset \Omega$ is a Borel set, then $\mu \lfloor B$ stands for the restriction of μ to B, i.e. $\mu \lfloor B(X) := \mu(X \cap B)$ for all Borel set $X \subset \Omega$.

2. LOCAL INVERTIBILITY IN SOBOLEV SPACES

The study of equilibria for defective crystals has motivated the introduction of variational problems where the domain of integration is varying as well as its deformations. In particular, Fonseca and Parry [35] proposed a model where the underlying energy is given by

$$E(u, v) := \int_{v(\Omega)} W(\nabla(u \circ v^{-1})) \, dx,$$

where u denotes a deformation of Ω and v represents the slip or plastic deformation, with $\det \nabla v = 1$. Invertibility of v will guarantee that the energy may be reformulated as

$$E(u, v) := \int_{\Omega} W(\nabla u (\nabla v)^{-1}) \, dx$$

where now the domain of integration is held fixed.

The theorem below provides sufficient conditions under which we may ensure local invertibility (see [28]).

Theorem 2.1 *Let Ω be a bounded, open subset of $\mathbb{R}^N$, and let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$ be such that $\det \nabla u(x) > 0$ for a.e. $x \in \Omega$. Then for $\mathcal{L}^N$ almost every $x_0 \in \Omega$ the function u is locally almost invertible in a neighborhood of x_0, i.e., there exist $r = r(x_0) > 0$, an open set $D = D(x_0) \subset\subset \Omega$, and a function $w : B(y_0; r) \to D$, with $y_0 := u(x_0)$, such that*

$$\begin{gathered} w \in W^{1,1}(B(y_0, r); \mathbb{R}^N), \quad w \circ u(x) = x \text{ a.e. } x \in D, \\ u \circ w(y) = y \text{ a.e. } y \in B(y_0, r), \end{gathered} \tag{2.1}$$

and

$$\nabla w(y) = (\nabla u)^{-1}(w(y)) \text{ a.e. } y \in B(y_0, r).$$

If, in addition, $\left|\frac{\operatorname{adj}\nabla u}{\det \nabla u}\right|^s \det \nabla u \in L^1(\Omega)$ *for some* $1 \le s < +\infty$*, then* $w \in W^{1,s}(B(y_0, r); D)$*.*

The latter part of this result was independently obtained by Šverák [59].

If u is more regular, precisely if $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ with $p \ge N(N-1)$, then $u : D \to u(D)$ and $w : u(D) \to D$ are homeomorphisms, and there exists a set $\mathcal{N} \subset \Omega$ with $\mathcal{L}^N(\mathcal{N}) = 0$ such that $u : \Omega \setminus \mathcal{N} \to \mathbb{R}^N$ is an open mapping (see also [44], where a stronger version of this result was established). Also, Heinonen and Koskela [43] showed that if $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ with $p > N(N-1)$, $\det \nabla u > 0$ a.e. in Ω, and $N \ge 3$, then u is open and discrete.

We note that there are Lipschitz homeomorphisms which do not satisfy $\det \nabla u \neq 0$ a.e. in Ω. Indeed, Martio and Ziemer [50] proved that for every bounded, open set $\Omega \subset \mathbb{R}^N$, there exist a measurable set $A \subset \Omega$ with $\mathcal{L}^N(A) > 0$, and a homeomorphism $u \in W^{1,\infty}(\Omega; \mathbb{R}^N)$ such that $\det \nabla u = 0$ for every $x \in A$.

The proof of Theorem 2.1 uses properties of degree theory, and departs considerably from earlier work on this subject (see e.g. [8], [18], [59], [60]). We remark that we cannot expect to prove this result by approximating u by a sequence of smooth functions with positive determinant, since Ball [9] has provided an example of a mapping $v \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ with $\det \nabla v = 1$ a.e. in Ω, and for which there is no sequence $\{v_n\} \in C^1(\bar{\Omega}; \mathbb{R}^2)$ such that $v_n \to u$ uniformly and $\det \nabla u_n > 0$ in Ω.

Theorem 2.1 illustrates how in $W^{1,N}(\Omega; \mathbb{R}^N)$ the determinant still behaves, essentially, as the jacobian of a smooth deformation. There are situations, however, where we are led to the study of properties of minors for deformations in $W^{1,p}(\Omega; \mathbb{R}^N)$ for $p < N$. This question is addressed in the next section.

3. WEAK CONTINUITY OF THE JACOBIAN INTEGRAL

In [32] we search for minimal conditions ensuring weak convergence of minors in Sobolev spaces.

Certain energies for nonlinear elastic materials may be represented as

$$E(u) = \int_\Omega W(\nabla u)\, dx$$

where W is a *polyconvex integrand*, i.e. W is a convex function of $\mathbb{M}(\nabla u)$. It follows that E is $W^{1,p}$-sequentially weakly lower semicontinuous, i.e.

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^d) \Rightarrow E(u) \leq \liminf_{n\to\infty} E(u_n), \tag{3.1}$$

if

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^d) \Rightarrow \mathbb{M}(\nabla u_n) \overset{*}{\rightharpoonup} \mathbb{M}(\nabla u) \text{ in the sense of measures.}$$

We recall that if $g_n, g \in L^1(\Omega)$, then we say that $g_n \overset{*}{\rightharpoonup} g$ *in the sense of measures* if

$$\int_\Omega g_n \varphi\, dx \to \int_\Omega g\varphi\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$.

Weak continuity of minors has been studied in depth by Murat and Tartar (see [56], [61]) within the framework of compensated compactness, and lower semicontinuity for polyconvex functions was undertaken by several authors (e.g. see [2], [7], [8], [10], [16], [20], [21], [23], [30], [36], [37], [38], [45], [51], [52], [55], [61]).

Here, and for simplicity, we are going to restrict the study to the higher order minor, det, and when $d = N$, although most of the results may be extended to lower order minors. As it is usual, the admissible fields are assumed to be in $W^{1,N}(\Omega;\mathbb{R}^N)$ because in this space we may integrate by parts and write the determinant as a divergence operator. Precisely, if $u \in W^{1,N}(\Omega;\mathbb{R}^N)$ then

$$\det \nabla u = \mathrm{Det}\nabla u, \tag{3.2}$$

where

$$\mathrm{Det}\nabla u := \frac{1}{N}\mathrm{div}((\mathrm{adj}\nabla u)^T u).$$

The relation (3.2) may fail if u is not sufficiently regular. As an example, consider

$$u(x) := \sqrt[N]{a^N + |x|^N}\,\frac{x}{|x|}, \quad \Omega := B(0,1).$$

Then $u \in W^{1,p}(B(0,1);\mathbb{R}^N)$ for all $p < N$, $\det \nabla u = 1$ a.e. in $B(0,1)$, but

$$\mathrm{Det}\nabla u = \mathcal{L}^N \lfloor B(0,1) + \omega_N\, a^N \delta_0, \qquad \omega_N := \mathcal{L}^N(B(0,1))$$

Moreover, it was proven by Müller [54] (see also [53]) that if S is a closed set with Hausdorff dimension $\alpha \in (0,N)$, then there exists $u \in W^{1,p}(B(0,1);\mathbb{R}^N) \cap C^0(\overline{\Omega})$ for all $p < N$, such that

$$\mathrm{Det}\nabla u = \det \nabla u\, \mathcal{L}^N \lfloor B(0,1) + \mu_s, \tag{3.3}$$

where μ_s is a positive Radon measure, singular with respect to the Lebesgue measure $\mathcal{L}^N$, and such that $\operatorname{supp} \mu_s = S$.

Further results by Müller, Tang and Yan [55] established that if $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ with $q \geq N^2/(N+1)$ then (3.3) holds, and $\mu_s = 0$ if

$$u \in W^{1,N-1}(\Omega; \mathbb{R}^N), \quad \operatorname{adj} \nabla u \in L^{N/(N-1)}.$$

The exploitation of spaces such as BMO and Hardy spaces allows one to refine these results along the lines of the work of Müller [52], [53], [54], and Coifman, Lions, Meyers and Semmes [19], and, in particular, it can be shown that if $u \in W^{1,N-1}(\Omega; \mathbb{R}^N)$ is such that $\det \nabla u \geq 0$ and $\operatorname{adj} \nabla u \in L^{N/(N-1)}$ then

$$\det \nabla u \, \ln(2 + \det \nabla u) \in L^1_{\text{loc}}.$$

In earlier works, Ball [8], Dal Maso and Celada [16], Dacorogna and Murat [22], Dal Maso and Sbordone [23], Giaquinta, Modica and Souček [38], [39], [40], (see also [41]), and Reshetnyak [57] established that

$$u_n \rightharpoonup u \text{ in } W^{1,N}(\Omega; \mathbb{R}^N) \Rightarrow \det \nabla u_n \overset{*}{\rightharpoonup} \det \nabla u,$$

where the convergence $\overset{*}{\rightharpoonup}$ is in the sense of measures. Moreover, if $u_n, u \in W^{1,N}(\Omega; \mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^N)$ with $p > \frac{N^2}{N+1}$, then we still have

$$\det \nabla u_n \overset{*}{\rightharpoonup} \det \nabla u \quad \text{in the sense of distributions.} \tag{3.4}$$

Also, if $p > N - 1$ and if $\sup_n ||u_n||_\infty < +\infty$, with $\{\det \nabla u_n\}$ equi-integrable, then (3.4) is still valid (see [21]).

A complete characterization of weak convergence of the determinant has been obtained by Fonseca and Malý in [32], where the results below may be found.

Theorem 3.1 *If the sequence $\{u_n\} \subset W^{1,N}(\Omega; \mathbb{R}^N)$ converges to u in $L^1(\Omega; \mathbb{R}^N)$, if $\{\mathbb{M}(\nabla u_n)\}$ is bounded in L^1, and if $\{\det \nabla u_n\}$ is equi-integrable, then*

$$\det \nabla u_n \rightharpoonup \det \nabla u \textit{ in } L^1.$$

Remarks 3.2 (i) This result was proven earlier by Giaquinta, Modica and Souček, see [41], Theorem II.3.2.1, under the additional assumption that all minors of all orders are equi-integrable. Their proof relies of tools from Geometric Measure Theory. The proof presented in [32] is entirely analytical.
(ii) Note that from the hypotheses of Theorem 3.1, one can only guarantee apriori that $u \in BV$, so ∇u must be understood as the Radon

Nikodym derivative of the distributional derivative Du with respect to the Lebesgue measure $\mathcal{L}^N$.
(iii) In Theorem 3.1 equi-integrability is necessary. Indeed, it is possible to construct a sequence $u_n \in W^{1,N}(\Omega;\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,N-1}$, u is affine, $\{\det \nabla u_n\}$ is bounded in L^1, and still

$$\det \nabla u_n \overset{*}{\rightharpoonup} \det \nabla u + \mu_s,$$

where μ_s is a nonzero Radon measure singular with respect to $\mathcal{L}^N$. Let us consider a nonincreasing smooth function $\psi : [0,\infty) \to \mathbb{R}$ such that

$$\psi(r) = \begin{cases} \frac{\pi}{2} & \text{for } 0 \le r \le 1, \\ -\frac{\pi}{2} & \text{for } r \ge 2, \end{cases}$$

and set

$$\begin{aligned} (u_n)_1(x) &:= \frac{x_1 x_3}{\rho} \cos\psi(k\rho), \\ (u_n)_2(x) &:= \frac{x_2 x_3}{\rho} \cos\psi(k\rho), \\ (u_n)_3(x) &:= x_3 \sin\psi(k\rho), \end{aligned}$$

where $\rho := \sqrt{x_1^2 + x_2^2}$. Then $u_n \rightharpoonup u$, where $u(x) = (0,0,x_3)$, and the weak* limit of $\{\det \nabla u_n\}$ is equal to $\det \nabla u \mathcal{L}^N \lfloor B(0,1) + \mu_s$, where the density of the singular measure μ_s is supported on the x_3-axis.

Theorem 3.3 *If the sequence* $\{u_n\} \subset W^{1,N}(\Omega;\mathbb{R}^N)$ *converges to* u *in* $L^1(\Omega;\mathbb{R}^N)$, *if* $\{u_n\}$ *is bounded in* $W^{1,N-1}(\Omega;\mathbb{R}^N)$, *and if* $\det \nabla u_n \overset{*}{\rightharpoonup} \mu$ *for some Radon measure* μ, *then*

$$\frac{d\mu}{d\mathcal{L}^N} = \det \nabla u \quad a.e.\ x \in \Omega.$$

The two theorems above are sharp, in that they are complemented by the following result.

Theorem 3.4 *Let* $u \in W^{1,N}(\Omega;\mathbb{R}^N)$.

(i) Let μ *be a Radon measure on* Ω, *and let* $1 \le p < N-1$. *Then there exists a sequence* $\{u_n\} \subset C^1(\overline{\Omega};\mathbb{R}^N)$ *such that*

$$u_n \to u \ in\ W^{1,p}(\Omega;\mathbb{R}^N) \quad \text{and } \det \nabla u_n \overset{*}{\rightharpoonup} \mu.$$

(ii) Let $f \in L^q(\Omega;\mathbb{R})$, $1 \le q < +\infty$. *Then there exists a sequence* $\{u_n\} \subset C^1(\overline{\Omega};\mathbb{R}^N)$ *such that*

$$u_n \rightharpoonup u \ in\ W^{1,p}(\Omega;\mathbb{R}^N) \quad \text{and } \det \nabla u_n \to f \text{ in } L^q.$$

Finally, we notice that the argument of the proof of Theorem 3.1 may be used to provide an alternative proof for the following lower semicontinuity result of Dal Maso and Sbordone [23] and Celada and Dal Maso [16] (see [32], and also the paper by Fusco and Hutchinson [36] for analytical proofs which do not need Geometric Measure Theory tools such as currents).

Theorem 3.5 *Let $u_n \in W^{1,N}(\Omega;\mathbb{R}^N)$, $u_n \to u$ in $L^1(\Omega;\mathbb{R}^N)$, and let $\{\mathbb{M}(\nabla u_n)\}$ be bounded in L^1. If g is a nonnegative, convex function, then*

$$\int_\Omega g(\det \nabla u)\, dx \le \liminf_{n\to\infty} \int_\Omega g(\det \nabla u_n)\, dx.$$

We end this section with a brief overview of some recently obtained relaxation results for quasiconvex and polyconvex energies (see [1], [12], [30], [33], [31], [46], [49]). If $f : \Omega\times\mathbb{R}^{d\times N} \to \mathbb{R}$ is a Carathéodory function, then the effective (or relaxed) energy is defined as

$$\mathcal{F}_{p,q}(u;\Omega) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} \int_\Omega f(x,\nabla u_n)\, dx : u_n \in W^{1,q}_{\mathrm{loc}}, \right.$$
$$\left. u_n \rightharpoonup u \text{ in } W^{1,p} \right\}.$$

First we consider the case where

$$f(x,\xi) := g(\det \xi),$$

$g : \mathbb{R} \to [0,+\infty)$ is a convex function. We have shown that (see [32])

$$\mathcal{F}_{p,N}(u;\Omega) \ge \int_\Omega g(\det \nabla u)\, dx \quad \text{if } p \ge N-1,$$

and if $p > N-1$ then (see [12], [31])

$$\mathcal{F}_{p,N}(u;\Omega) = \int_\Omega g(\det \nabla u)\, dx + \mu_s(\Omega)$$

for some Radon measure μ, singular with respect to the Lebesgue measure $\mathcal{L}^N$. For a general f, and under the growth condition

$$0 \le f(x,\xi) \le C(1+|\xi|^q)$$

and

$$p > \frac{N-1}{N} q,$$

we have

$$\mathcal{F}_{p,q}(u;\Omega) = h_u \mathcal{L}^N \lfloor \Omega + \mu_s,$$

where (see [1])

$$h_u \le Qf(x, \nabla u)$$

and μ_s is a singular measure. In the case where $f = f(\xi)$, we have

$$h_u = Qf(\nabla u), \tag{3.5}$$

where Qf stands for the *quasiconvexification* of f, precisely,

$$Qf(\xi) := \inf \left\{ \int_{(0,1)^N} f(\xi + \nabla\varphi(x))\, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^N). \right\}$$

This may no longer be valid when f depends also on x and $p < q$. Indeed, Gangbo [37] has constructed an example where

$$f(x, \xi) = \chi_K(x)\, |\det \xi|,$$

and $h_u = f$ if and only if $\mathcal{L}^N(\partial K) = 0$. Hence, in general, (3.5) fails and

$$f^{**}(x, \nabla u) \le h_u$$

is the only known lower estimate.

4. A CHANGE OF VARIABLES FORMULA IN *BV*

In the previous two sections we dealt with properties of $\det \nabla u$ for $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ and for certain ranges of p. Here we go outside the Sobolev spaces framework to handle situations relevant to the study of phase transitions of incoherent phase deformations, to the analysis of fracture, or to tackle problems where the growth of bulk energy densities is at most linear. In the latter case minimizing sequences may convergence to macroscopic states which are only in BV (see [6], [34]).

Given $u \in BV(\Omega; \mathbb{R}^N)$, its distributional derivative may be written as

$$D\varphi = \nabla u\, \mathcal{L}^N \lfloor \Omega + (u^+ - u^-) \otimes \nu H^{N-1} \lfloor S(u) + C(u),$$

where H^{N-1} denotes the $(N-1)$-dimensional Hausdorff measure, $S(u)$ is the jump set of u, which differs from the complement of the set of Lebesgue points, $\Gamma(u)$, by a set of H^{N-1} measure zero, $C(u)$ is the Cantor part, u^+ and u^- are the traces of u on $S(u)$ at x, $\nu(x)$ is the unit normal vector to $S(u)$ pointing towards the side of u^+ (see [25], [62]). Here, and in the sequel, we assume that u is appropriately represented, namely that

$$u(x) = \lim_{r \to 0+} \mathcal{L}^N(B(x,r))^{-1} \int_{B(x,r)} u\, dy$$

whenever the limit on the right exists.

We remark that ∇u does not have the structure of a gradient, i.e. it is not necessarily curl-free. Indeed, according to a result by Alberti [3] ∇u may be any L^1 function. In spite of this degeneracy, the usual change of variables formula still holds (see [29]).

It is well known that the change of variables formula

$$\int_A v(x)|Ju(x)| = \int_{\mathbb{R}^N} \Big(\sum_{x \in A \cap G \cap u^{-1}(y)} v(x) \Big)\, dy \tag{4.1}$$

holds for a set G of full measure in Ω, provided u is an almost everywhere approximately differentiable function (hence, if $u \in BV$), u has the N-property on G, and v is a measurable function on Ω such that $v|Ju| \in L^1(\Omega)$. Here Ju is the Jacobian computed from the approximate derivative, which, in turn, coincides a.e. with the absolutely continuous part ∇u of Du. We recall that u is said to have the N-*property* on G if $\mathcal{L}^N(u(E)) = 0$ for each set $E \subset G$ with $\mathcal{L}^N(E) = 0$.

In the case where $v = 1$, equation (4.1) is often called the area formula.

Federer (see [26], 3.2.1) showed that u has the N-property on G if

$$G \subset \left\{ x : \mathrm{ap}\ \limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|} < \infty \right\}.$$

In [17] another class of admissible domains was proposed. We denote by $M(Du)(x)$, the Hardy-Littlewood maximal function of the total variation measure $|Du|$ computed using the balls contained in Ω. As it is usual, it μ is a finite, positive Radon measure on Ω, we define the *maximal function of* μ as (see [58])

$$M(\mu)(x) := \sup \left\{ \frac{\mu(B(x,r))}{\mathcal{L}^N(B(x,r))} : 0 < r < \mathrm{dist}(x, \partial\Omega) \right\},$$

and it can be shown that (see [4])

$$\mathcal{L}^N\big(\{x \in \Omega : M(\mu)(x) = +\infty\}\big) = 0.$$

Theorem 4.1 *Let* $u \in BV(\Omega; \mathbb{R}^N)$ *and let*

$$G \subset \{x : M(|Du|)(x) < \infty\}$$

be a measurable set with $\mathcal{L}^N(G) = \mathcal{L}^N(\Omega)$. *Then the change of variables* (4.1) *holds on* G *for any measurable function* v *such that* $v|Ju| \in L^1(\Omega)$.

Proof. Set

$$(u)_{x,r} := \mathcal{L}^N(B(x,r))^{-1} \int_{B(x,r)} u\, dy.$$

By Poincaré's inequality,

$$
\begin{aligned}
\left|(u)_{x,r/2} - (u)_{x,r}\right| &\le C\mathcal{L}^N(B(x,r))^{-1} \int_{B(x,r)} |u(y) - (u)_{x,r}|\, dy \\
&\le Cr\mathcal{L}^N(B(x,r))^{-1} \int_{B(x,r)} |Du| \\
&\le CrM(|Du|)(x).
\end{aligned}
\tag{4.2}
$$

Iterating this inequality for r, $r/2$, $r/4, \dots$, we obtain

$$
|u(x) - (u)_{x,r}| \le CrM(|Du|)(x)
$$

for all $x \in G$. Using Poincaré's inequality once again on a ball $B(x,R)$, where $R = 2\operatorname{dist}(x,y)$ and $B(x,R)$ is contained in Ω, we obtain the Bojarski-Hajłasz inequality (see [11])

$$
|u(y) - u(x)| \le C|y-x|(M(|Du|)(x) + M(|Du|)(y)).
$$

Hence the graph of $u\lfloor G$ can be covered by a countable union of graphs of Lipschitz functions and thus from the N-property of Lipschitz functions we conclude the N-property of u on G. □

A consequence of the above theorem results in conservation of volume.

Corollary 4.2 *Let $u \in BV(\Omega;\mathbb{R}^N)$ be such that $|\det \nabla u(x)| = 1$ for a.e. $x \in \Omega$, and let*

$$
G \subset \{x : M(|Du|)(x) < \infty\}
$$

be a measurable set such that

$$
\mathcal{L}^N(u(G)) = \mathcal{L}^N(G) = \mathcal{L}^N(\Omega).
$$

Then for any measurable set $A \subset G$ we have

$$
\mathcal{L}^N(u(A)) = \mathcal{L}^N(A).
$$

Next we present another another version of change of variable formula for BV functions. Namely, we show that if the absolutely continuous part of ∇u is in L^N, we may essentially enlarge the set G for which (4.1) is valid. This is a generalization of change of variable formula for $W^{1,N}$ functions by Malý and Martio [48] and Malý [46], following some ideas from the above mentioned works (see also [43], [44], [47], [50]).

If a function $u \in BV(\Omega;\mathbb{R}^N)$ then we write

$$
\begin{aligned}
G_1(u) &:= \left\{x \in \Omega : M(|(Du)_s|)(x) < \infty\right\}, \\
G_2(u) &:= \Big\{x \in \Omega :\ \text{there exists } \alpha \in (0,1) \text{ such that} \\
&\qquad \limsup_{r\to 0^+} r^{-N-\alpha} \int_{B(x,r)} |u(y) - u(x)|\, dy < \infty\Big\}.
\end{aligned}
$$

We recall that for $\mathcal{L}^N$ a.e. $x \in \Omega$

$$\lim_{r\to 0+} r^{-N-1} \int_{B(x,r)} |u(y) - u(x) - \nabla u(x) \cdot (y-x)| \, dy = 0,$$

therefore for all $\alpha \in (0,1)$

$$\limsup_{r\to 0+} r^{-N-\alpha} \int_{B(x,r)} |u(y) - u(x)| \, dy = 0;$$

hence

$$\mathcal{L}^N(G_1(u) \cap G_2(u)) = \mathcal{L}^N(\Omega).$$

Theorem 4.3 *Let $u \in BV(\Omega; \mathbb{R}^N)$ be such that $\nabla u \in L^N(\Omega)$. Let*

$$G \subset G_1(u) \cap G_2(u)$$

be a measurable set with $\mathcal{L}^N(G) = \mathcal{L}^N(\Omega)$. Let v be a measurable function on Ω such that $v\,|Ju| \in L^1(\Omega)$. Then the change of variables formula (4.1) holds on G.

Proof. We will verify the N-property of u on G. Let $E \subset G$ be a set of measure zero, and choose an open set $U \subset \Omega$ containing E. For $x \in E$ write

$$r_k := 2^{-k} r_0, \quad B_k := B(x, r_k), \text{ with } B(x, r_0) \subset U.$$

Fix a point $x \in E$, and find an $\alpha \in (0,1)$ according to the definition of G_2. With each $k = 1, 2, \ldots$ we associate a value $\rho_k = \rho_k(x) \geq 0$ such that

$$\begin{aligned} |B_k \cap \{y : |u(y) - u(x)| \leq \rho_k\}| &\geq \mathcal{L}^N(B_k)/2, \\ |B_k \cap \{y : |u(y) - u(x)| \geq \rho_k\}| &\geq \mathcal{L}^N(B_k)/2. \end{aligned} \tag{4.3}$$

Notice that, by Chebychev's inequality and the definition of G_2,

$$\rho_k \leq C(x) r_k^{\alpha} \tag{4.4}$$

for some $C(x) > 0$. We write

$$\begin{aligned} V_k &:= \{y \in B_k : |u(y) - u(x)| \leq \rho_k\}, \\ b_k &:= \int_{V_k} (1 + |\nabla u|^N) \, dy, \end{aligned}$$

and set

$$\begin{aligned} I_1 &:= \{k : b_k \geq r_k^{N\alpha}\}, \\ I_2 &:= \{k : 2 r_k^{-N\alpha} b_k \geq \sup_{j \geq k} r_j^{-N\alpha} b_j\}, \\ I = I(x) &:= I_1 \cup I_2. \end{aligned}$$

If I_1 is finite, then the sequence $\{r_k^{-N\alpha} b_k\}$ is bounded and thus I_2 is infinite. It follows that $I_1 \cup I_2$ is infinite. We claim that

$$k \in I \implies \rho_k^N \le K(x) \int_{V_k} (1 + |\nabla u|^N)\, dx, \tag{4.5}$$

where $K = K(x)$ is a positive constant which may depend on x and u but not on k. Let η_k be an auxiliary smooth function with values

$$\eta_k := \begin{cases} (\rho_k - \rho_{k+1})^+ & \text{on } (-\infty, \rho_{k+1}), \\ 0 & \text{on } (\rho_k, +\infty), \end{cases} \tag{4.6}$$

and satisfying

$$0 \le \eta_k \le (\rho_k - \rho_{k+1})^+, \quad 0 \le -\eta_k' \le 2.$$

Set

$$u_k(y) := \eta_k(|u(y) - u(x)|).$$

By the chain rule (see Ambrosio and Dal Maso [5]), $u \in BV(B_k; \mathbb{R}^N)$ and

$$|Du_k| \le 2|(Du)_s| + 2|\nabla u| \chi_{V_k}. \tag{4.7}$$

In order to estimate $(\rho_k - \rho_{k+1})^+$, without loss of generality we may assume that $\rho_k \ge \rho_{k+1}$. We have

$$|\{y : u_k(y) = \rho_k - \rho_{k+1}\}| \ge 2^{-N-1} \mathcal{L}^N(B_k),$$
$$|\{y : u_k(y) = 0\}| \ge 2^{-1} \mathcal{L}^N(B_k),$$

and thus by a Poincaré type inequality and (4.7) we obtain

$$\begin{aligned} \rho_k - \rho_{k+1} &\le C r_k^{-N+1} \int_{B_k} |Du_k| \le C r_k M(|(Du)_s|)(x) + C r_k^{-N+1} \int_{V_k} |\nabla u|\, dy \\ &\le K b_k^{1/N}, \end{aligned} \tag{4.8}$$

where we have used the fact that $M(|(Du)_s|)(x) < +\infty$ and that $b_k \ge \mathcal{L}^N(B_k)/2$. Now, if $k \in I_1$ then by (4.4) we have

$$\rho_k^N \le K r_k^{N\alpha} \le K b_k,$$

and this asserts (4.5). If $k \in I_2$, then by (4.8)

$$\rho_k \le \sum_{j \ge k} (\rho_j - \rho_{j+1})^+ \le K \sum_{j \ge k} b_j^{1/N} \le K \sum_{j \ge k} (r_j / r_k)^\alpha b_k^{1/N} \le K b_k^{1/N},$$

which yields (4.5) as well. We conclude the proof using Vitali's covering argument. We write the set E as

$$E = \bigcup_{m=1}^{\infty} E_m, \quad E_m := \{x \in E : K(x) \le m\}.$$

It is enough to show that $\mathcal{L}^N(u(E_m)) = 0$ for a fixed $m \in \{1, 2, \dots\}$. We cover $u(E_m)$ with balls $B(u(x), \rho_k + r_k)$ where $x \in E$ and $k \in I(x)$. Notice that we add r_k to ρ_k only to avoid degeneracy when $\rho_k = 0$. We have

$$r_k^N \le C\mathcal{L}^N(B_k) \le 2C\mathcal{L}^N(V_k). \tag{4.9}$$

Since $I(x)$ is infinite, by virtue of (4.4) we obtain a fine covering of E; hence; hence we may extract a disjoint subcover $\{B(n)\}$ of $u(E)$ up to a set of measure zero. If $B(n) = B(u(x), \rho_k + r_k)$ is one of selected balls, then the corresponding radius $\rho_k + r_k$ is denoted by $\rho(n)$, and the corresponding set V_k by $V(n)$. Notice that the sets $V(n)$ are disjoint. By (4.5) and (4.9) we obtain

$$\begin{aligned}\mathcal{L}^N(u(E_m)) &\le C\sum_n \rho(n)^N \le C\, m \sum_n \int_{V(n)} (1 + |\nabla u|^N)\, dy \\ &\le C\, m \int_U (1 + |\nabla u|^N)\, dy.\end{aligned}$$

A suitable choice of the set U allows us to deduce that the right hand side on the above set of inequalities may be rendered as small as we want. Hence the N-property is verified, which, in turn, implies that the (4.1) holds as well. $\square$

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References

[1] E. G. Acerbi, G. Bouchitté and I. Fonseca. Relaxation of convex functionals and the Lavrentiev phenomenon. In preparation.

[2] E. Acerbi and G. Dal Maso. New lower semicontinuity results for polyconvex integrals case. *Calc. Var.*, 2:329–372, 1994.

[3] G. Alberti. A Lusin type theorem for gradients. *J. Funct. Anal.*, 100:110–118, 1991.

[4] L. Ambrosio. On the lower semicontinuity of quasiconvex integrals in $SBV(\Omega;\mathbb{R}^k)$. *Nonlinear Analysis*, 23:691–702, 1994.

[5] L. Ambrosio and G. Dal Maso. A general chain rule for distributional derivatives. *Proc. Amer. Math. Soc.*, 108:691–702, 1990.

[6] L. Ambrosio and G. Dal Maso. On the representation in $BV(\Omega;\mathbb{R}^m)$ of quasiconvex integrals. *J. Funct. Anal.*, 109:76–97, 1992.

[7] E. Acerbi and N. Fusco. New lower semicontinuity results for polyconvex integrals case. *Arch. Rat. Mech. Anal.*, 86:125–145, 1984.

[8] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.*, 63:337–403, 1977.

[9] J. M. Ball. Global invertibility of Sobolev functions and the interpenetration of the matter. *Proc. Roy. Soc. Edin.*, 88A:315–328, 1988.

[10] J. M. Ball and F. Murat. $W^{1,p}$ quasiconvexity and variational problems for multiple integrals. *J. Funct. Anal.*, 58:225–253, 1984.

[11] B. Bojarski and P. Hajłasz. Pointwise inequalities for Sobolev functions and some applications. *Studia Math.*, 106:77–92, 1993.

[12] G. Bouchitté, I. Fonseca and J. Malý. The effective bulk energy of the relaxed energy of multiple integrals below the growth exponent. *Proc. Roy. Soc. Edin.*, 128A:463–479, 1998.

[13] A. Braides. *Approximation of Free-Discontinuity Problems.* Lecture Notes in Mathematics, Springer Verlag, Berlin 1998.

[14] A. Braides and A. Coscia. The interaction between bulk energy and surface energy in multiple integrals. *Proc. Royal Soc. Edin.*, 124A:737–756, 1994.

[15] A. Braides and V. Chiadò Piat. Integral representation results for functionals defined on $SBV(\Omega;\mathbb{R}^m)$. *J. Math. Pures Appl.*, 75:595–626, 1996.

[16] P. Celada and G. Dal Maso. Further remarks on the lower semicontinuity of polyconvex integrals. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 11:661–691, 1994.

[17] R. Choksi and I. Fonseca. A change of variables formula for mappings in BV. *Proc. A.M.S.*, 125:2065–2072, 1997.

[18] P. G. Ciarlet and J. Nečas. Injectivity and self contact in non linear elasticity. *Arch. Rat. Mech. Anal.*, 97:171–188, 1987.

[19] R. Coifman P.-L. Lions, Y. Meyer and S. Semmes. Compacité par compensation et espaces de Hardy. *CRAS Paris*, 309:945–949, 1989.

[20] B. Dacorogna. *Direct Methods in Calculus of Variations.* Appl. Math. Sciences 78, Springer–Verlag, Berlin 1989.

[21] B. Dacorogna and P. Marcellini. Semicontinuité pour des intégrandes polyconvexes sans continuité des determinants. *C. R. Acad. Sci. Paris Sér. I Math.*, 311, 6:393–396, 1990.

[22] B. Dacorogna and F. Murat. On the optimality of certain Sobolev exponents for the weak continuity of determinants. *J. Funct. Anal.*, 105:42–62, 1992.

[23] G. Dal Maso and C. Sbordone. Weak lower semicontinuity of polyconvex integrals: a borderline case. *Math. Z.*, 218:603–609, 1995.

[24] E. De Giorgi and L. Ambrosio. Un nuovo funzionale del calcolo delle variazioni. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, 82:199–210, 1988.

[25] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, 1992.

[26] H. Federer. *Geometric Measure Theory.* Springer, 1969 (Second edition 1996).

[27] I. Fonseca and G. Francfort. Relaxation in BV versus quasiconvexification in $W^{1,p}$: a model for the interaction between damage and fracture. *Calc. Var.*, 3:407–446, 1995.

[28] I. Fonseca and W. Gangbo. Local invertibility of Sobolev functions. *SIAM J. Math. Anal.*, 26,2:280–304, 1995.

[29] I. Fonseca and W. Gangbo. *Degree Theory in Analysis and Applications.* Clarendon Press, Oxford, 1995.

[30] I. Fonseca and G. Leoni. On lower semicontinuity and relaxation. To appear.

[31] I. Fonseca and J. Malý. Relaxation of Multiple Integrals below the growth exponent. *Anal. Inst. H. Poincaré. Anal. Non Linéaire*, 14,3:309–338, 1997.

[32] I. Fonseca and J. Malý. Weak continuity of jacobian integrals. In preparation.

[33] I. Fonseca and P. Marcellini. Relaxation of multiple integrals in subcritical Sobolev spaces. *J. Geom. Anal.*, 7,1:57–81, 1997.

[34] I. Fonseca and S. Müller. Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Rational Mech. Anal.*, 123:1–49, 1993.

[35] I. Fonseca and G. Parry. Equilibrium configurations of defective crystals. *Arch. Rat. Mech. and Anal.*, 120:245–283, 1992.

[36] N. Fusco and J. E. Hutchinson. A direct proof for lower semicontinuity of polyconvex functionals. *Manuscripta Math.*, 85:35–50, 1995.

[37] W. Gangbo. On the weak lower semicontinuity of energies with polyconvex integrands. *J. Math. Pures et Appl.*, 73, 5:455–469, 1994.

[38] M. Giaquinta, G. Modica and J. Souček. Cartesian currents, weak dipheomorphisms and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.* **106** *(1989), 97-159. Erratum and addendum. Arch. Rat. Mech. Anal.*, 109:385–592, 1990.

[39] M. Giaquinta, G. Modica and J. Souček. Cartesian Currents and Variational Problems for Mappings into Spheres. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 16:393–485, 1989.

[40] M. Giaquinta, G. Modica and J. Souček. Liquid crystals: relaxed energies, dipoles, singular lines and singular points. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 17,3:415–437, 1990.

[41] M. Giaquinta, G. Modica and J. Souček. *Cartesian Currents in the Calculus of Variations I, II.* Ergebnisse der Mathematik und Ihrer Grenzgebiete Vol. 38, Springer, Berlin–Heidelberg, 1998.

[42] P. Hajłasz. Note on weak approximation of minors. Ann. Inst. H. Poincaré, to appear.

[43] J. Heinonen and P. Koskela. Sobolev mappings with integral dilatations. *Arch. Rat. Mech. Anal.*, 125:81–97, 1993.

[44] T. Iwaniec and V. Šverák. On mappings with integrable dilatation. *Proc. Amer. Math. Soc.*, 118,1:181–188, 1993.

[45] J. Malý. Weak lower semicontinuity of polyconvex integrals. *Proc. Roy. Soc. Edinburgh*, 123A:681–691, 1993.

[46] J. Malý. Weak lower semicontinuity of quasiconvex integrals. *Manuscripta Math.*, 85:419–428, 1994.

[47] J. Malý. The area formula for $W^{1,n}$-mappings. *Comment. Math. Univ. Carolinae*, 35,2:291–298, 1994.

[48] J. Malý and O. Martio. Lusin's condition (N) and mappings of the class $W^{1,n}$. *J. Reine Angew. Math.*, 458:19–36, 1995.

[49] P. Marcellini. Approximation of quasiconvex functions and lower semicontinuity of multiple quasiconvex integrals. *Manuscripta Math.*, 51:1–28, 1985.

[50] O. Martio and W. P. Ziemer. Lusin's condition (N) and mappings with non-negative jacobians. *Michigan Math. J.*, 39:495–508, 1992.

[51] C. B. Morrey. *Multiple Integrals in the Calculus of Variations.* Springer, 1996.

[52] S. Müller. Weak continuity of determinants and nonlinear elasticity. *C. R. A. S. Paris*, 307:501–506, 1988.

[53] S. Müller. Det=det. A Remark on the distributional determinant. *C. R. A. S. Paris*, 311:13–17 , 1990.

[54] S. Müller. On the singular support of the distributional determinant. *Analyse Nonlinéaire*, 10:657–696, 1993.

[55] S. Müller, Q. Tang and S. B. Yan. On a new class of elastic deformations not allowing for cavitation. *Ann. IHP*, 11:217–243, 1994.

[56] F. Murat. Compacité par compensation: condition necessaire et suffisante de continuité faible sous une hypothése de rang constant. *Ann. Sc. Norm. Sup. Pisa*, 4,8:68–102, 1981.

[57] Y. Reshetnyak. Weak convergence and completely additive vector functions on a set. *Sibir. Math.*, 9:1039–1045, 1968.

[58] E. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press, 1970.

[59] V. Šverák. Regularity properties of deformations with finite energy. *Arch. Rat. Mech. Anal.*, 100:105–127, 1988.

[60] Q. Tang. Almost-everywhere injectivity in nonlinear elasticity. *Proc. Roy. Soc. Edinburgh*, 109:79–95, 1988.

[61] L. Tartar. Compensated compactness and applications to partial differential equations. *Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, vol. IV (ed. R. Knops), Pitman Res. Notes Math.*, 39:136–212, 1979.

[62] W. P. Ziemer. *Weakly Differentiable Functions.* Springer-Verlag, Berlin 1989.