SOME REMARKS ON LOWER SEMICONTINUITY

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Dedicated to James Serrin

Abstract. Classical lower semicontinuity results obtained by Serrin in the scalar case are extended to the vectorial setting for convex and polyconvex integrands.

§1. Introduction.

The following theorems on lower semicontinuity were proved by Serrin in [S]:

Theorem A. (Serrin [S, Th. 12]) *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a continuous function, with $f(x,u,\cdot)$ convex in $\mathbb{R}^N$, where Ω is an open subset of $\mathbb{R}^N$. Assume that f satisfies any one of the following conditions:*

(i) *$f(x,u,\xi) \to \infty$ as $|\xi| \to \infty$ for each $(x,u) \in \Omega \times \mathbb{R}$.*
(ii) *$f(x,u,\cdot)$ is strictly convex in $\mathbb{R}^N$ for each $(x,u) \in \Omega \times \mathbb{R}$.*
(iii) *The derivatives f_x, f_ξ and $f_{\xi\, x}$ exist and are continuous.*

Then the functional

$$F(u,\Omega) := \int_\Omega f(x,u(x),\nabla u(x))\,dx$$

is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ with respect to local convergence in L^1.

Theorem B. (Serrin [S, Th. 13]) *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a continuous function, with $f(x,u,\cdot)$ convex in $\mathbb{R}^N$, where Ω is an open subset of $\mathbb{R}^N$. Then the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ with respect to weak convergence in $BV_{\mathrm{loc}}(\Omega;\mathbb{R})$.*

We recall that a sequence of functions $\{u_n\} \subset W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R})$ (or $BV_{\mathrm{loc}}(\Omega;\mathbb{R})$) converges weakly in $BV_{\mathrm{loc}}(\Omega;\mathbb{R})$ to some function u if u_n converges to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R})$ and

$$\sup_n \int_\Omega |\nabla u_n(x)|\,dx < \infty. \tag{1.1}$$

Theorem B was later improved by Ambrosio [A] who established the following result:

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Theorem C. (Ambrosio [A, Th. 3.2]) *Let $f : \Omega\times\mathbb{R}\times\mathbb{R}^N \to [0,\infty]$ be a lower semicontinuous function, with $f(x,u,\cdot)$ convex in $\mathbb{R}^N$, where Ω is an open bounded subset of $\mathbb{R}^N$. Assume that there exists a continuous function $g : \Omega\times\mathbb{R} \to \mathbb{R}^N$ such that the function*

$$(x,u) \to f(x,u,g(x,u)) \qquad \text{is continuous and real valued.} \tag{1.2}$$

Then the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}(\Omega;\mathbb{R})$ with respect to weak convergence in $BV(\Omega;\mathbb{R})$.

Theorems A and B were also extended to the vectorial case by Morrey in his book on Calculus of Variations [M, Thms. 4.1.1, 4.1.2]. However, several years later Eisen [E] found a gap in Morrey's proof, thus placing in doubt the validity of Theorem A and B when $d > 1$, and constructed counterexamples for Theorem A(iii) when $d > 1$ (see also [St] for more details and an extensive bibliography). To our knowledge the validity of Theorems A(i)–(ii) and B when $d > 1$ remains open.

Note also that the techniques used by Ambrosio in Theorem C are rather different from those of Serrin's paper, and are based on a paper of De Giorgi, Buttazzo and Dal Maso [DBDM], where the authors proved lower semicontinuity with respect to local convergence in L^1 for integrands of the form $f = f(u,\xi)$, which are only measurable in the u variable and not necessarily coercive. One of the main ideas in [DBDM] was to approximate from below $f = f(u,\xi)$ by functions of the form $(a(u) + b(u)\cdot\xi)^+$ and then to study lower semicontinuity for these special integrands. A crucial observation is that $b(u(x))\cdot\nabla u(x)$ is the gradient of $B(u(x)) := \int_0^{u(x)} b(s)\,ds$, and thus integration by parts allows one to eliminate the troublesome term $b(u(x))\cdot\nabla u(x)$. Clearly this approach cannot be extended to the vectorial case.

In this paper we show that Theorems A(i) and B continue to hold in the vectorial case. More precisely, we can prove the following:

Theorem 1.1. *Let $f : \Omega\times\mathbb{R}^d\times\mathbb{M}^{d\times N} \to [0,\infty]$ be a lower semicontinuous function, with $f(x,u,\cdot)$ convex in $\mathbb{M}^{d\times N}$. Suppose that for all $(x_0,u_0)\in\Omega\times\mathbb{R}^d$ either $f(x_0,u_0,\xi)\equiv 0$ for all $\xi\in\mathbb{M}^{d\times N}$, or there exist C, $\delta_0 > 0$, and a continuous function $g : B(x_0,\delta_0)\times B(u_0,\delta_0) \to \mathbb{M}^{d\times N}$ such that*

$$f(x,u,g(x,u)) \in L^\infty\left(B(x_0,\delta_0)\times B(u_0,\delta_0);\mathbb{R}\right), \tag{1.3}$$

$$f(x,u,\xi) \geq C|\xi| - \frac{1}{C} \tag{1.4}$$

for all $(x,u)\in\Omega\times\mathbb{R}^d$ with $|x-x_0|+|u-u_0|\leq\delta_0$ and for all $\xi\in\mathbb{M}^{d\times N}$. Let $u\in BV_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x,u,\nabla u)\,dx \leq \liminf_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)\,dx.$$

Here ∇u is the Radon–Nikodym derivative of the distributional derivative Du of u, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$.

The argument of the proof of in Theorem 1.1 is based on the blow–up and truncation methods introduced by Fonseca and Müller [FM], [FM1], together with an approximation result for convex functions due to Ambrosio [A].

If in Theorem 1.1 we require only measurability in x, rather than lower semicontinuity, then the situation becomes significantly more delicate. This problem has been taken up by Acerbi, Bouchitté and Fonseca [ABF] for integrands of the form $f = f(x,\xi)$, convex in ξ, which satisfy the growth condition

$$C|\xi|^p - \frac{1}{C} \leq f(x,\xi) \leq C_2(|\xi|^q + 1),$$

where $p > \max\{1, q(N-1)N\}$ and $C > 0$.

Corollary 1.2. *Let $f:\Omega\times\mathbb{R}^d\times\mathbb{M}^{d\times N}\to[0,\infty]$ be a lower semicontinuous function, with $f(x,u,\cdot)$ convex in $\mathbb{M}^{d\times N}$. Suppose that*

$$f(x,u,\xi)\to\infty \quad \text{as } |\xi|\to\infty$$

and that $f(x,u,0)\in L^\infty_{\mathrm{loc}}\left(\Omega\times\mathbb{R}^d;\mathbb{R}\right)$. Let $u\in BV_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ converging to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$. Then

$$\int_\Omega f(x,u,\nabla u)\,dx\le\liminf_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)\,dx.$$

Corollary 1.2 extends Theorem A(i) of Serrin to the vectorial case.

Corollary 1.3. *Assume that f satisfies all the hypotheses of Theorem 1.1, maybe with exception of (1.4). Then the functional $F(u,\Omega)$ is lower semicontinuous in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ with respect to weak convergence in $BV_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$.*

The method used in Theorem 1.1 can also be applied to polyconvex integrands. For each matrix $\xi\in\mathbb{M}^{d\times N}$ let $\mathcal{M}(\xi)\in\mathbb{R}^\tau$ be the vector whose components are all the minors of ξ, where

$$\tau=\tau(d,N):=\sum_{k=1}^{\min\{d,N\}}\binom{d}{k}\binom{N}{k}.$$

Theorem 1.4. *Let $h:\Omega\times\mathbb{R}^d\times\mathbb{R}^\tau\to[0,\infty]$ be a lower semicontinuous function, with $h(x,u,\cdot)$ convex in $\mathbb{R}^\tau$. Suppose that for all $(x_0,u_0)\in\Omega\times\mathbb{R}^d$ either $h(x_0,u_0,v)\equiv 0$ for all $v\in\mathbb{R}^\tau$, or there exist C, $\delta_0>0$, and a continuous function $g:B(x_0,\delta_0)\times B(u_0,\delta_0)\to\mathbb{R}^\tau$ such that*

$$h(x,u,g(x,u))\in L^\infty\left(B(x_0,\delta_0)\times B(u_0,\delta_0);\mathbb{R}\right), \tag{1.5}$$

$$h(x,u,v)\ge C|v|-\frac{1}{C} \tag{1.6}$$

for all $(x,u)\in\Omega\times\mathbb{R}^d$ with $|x-x_0|+|u-u_0|\le\delta_0$ and for all $v\in\mathbb{R}^\tau$. Let $u\in BV_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ which converges to u in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, where $p=\min\{d,N\}$. Then

$$\int_\Omega h(x,u,\mathcal{M}(\nabla u))\,dx\le\liminf_{n\to\infty}\int_\Omega h(x,u_n,\mathcal{M}(\nabla u_n))\,dx.$$

The conclusion of Theorem 1.4 continues to hold for integrands $h:\Omega\times\mathbb{R}^d\times\mathbb{R}^\tau\to[0,\infty]$ which can approximated from below by a monotone sequence of functions $h_j:\Omega\times\mathbb{R}^d\times\mathbb{R}^\tau\to[0,\infty]$ which satisfy the hypotheses of Theorem 1.4. As

$$h(x,u,v)=\sup_j h_j(x,u,v)\qquad\text{for all } (x,u,v)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^\tau, \tag{1.7}$$

h is clearly lower semicontinuous, convex, and coercive in v (in the weak sense (1.6) above), but it may not verify condition (1.5). As an example, consider

$$h(u,\xi):=A(u)\,h_0(v),\qquad\text{where}\qquad A(u):=\begin{cases}1+1/|u| & \text{if } u\neq 0\\ 1 & \text{if } u=0,\end{cases}$$

$h_0 \geq 0$ is convex and $h_0(v) \to \infty$ as $|v| \to \infty$. Clearly (1.5) is not satisfied for $u_0 = 0$, but it is easy to approximate h, by taking $A_j(u) = \min\{A(u), j\}$ and $h_j := A_j(u)\, h_0(v)$.

More generally (1.7) always holds if for every $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$h(x, u, v) \geq (1 - \varepsilon) h(x_0, u_0, v) \tag{1.8}$$

for all $(x, u) \in \Omega \times \mathbb{R}$ with $|x - x_0| \leq \delta$, $|u - u_0| \leq \delta$ and for all $v \in \mathbb{R}^\tau$ (see [DMS] for a proof). Thus we recover the lower semicontinuity results of Dal Maso and Sbordone [DMS], and of Fusco and Hutchinson [FH], where condition (1.5) is replaced by (1.8) (see also [FL] for related results).

It would be interesting to understand whether for *any* lower semicontinuous function h, convex and coercive in v, it is possible to find an approximating sequence h_j as above. This would entail that Theorem 1.4 (resp. 1.1) continues to hold without condition (1.5) (resp. (1.3)). Although this seems likely, we have not been able to prove it.[1]

Corollary 1.5. *Let $\varphi : \Omega \times \mathbb{R}^N \times \mathbb{R} \to [0, \infty)$ be a continuous function, with $\varphi(x, u, \cdot)$ convex in $\mathbb{R}$. Suppose that for all $(x, u) \in \Omega \times \mathbb{R}^N$*

$$\varphi(x, u, s) \to \infty \qquad \textit{as } |s| \to \infty. \tag{1.9}$$

Let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,N}(\Omega; \mathbb{R}^N)$ bounded in $W^{1,N-1}(\Omega; \mathbb{R}^N)$ and converging to u in $L^1(\Omega; \mathbb{R}^N)$. Then

$$\int_\Omega \varphi(x, u, \det \nabla u)\, dx \leq \liminf_{n \to \infty} \int_\Omega \varphi(x, u_n, \det \nabla u_n)\, dx.$$

To the authors' knowledge Corollary 1.5 is new in this generality. Lower semicontinuity for polyconvex and quasiconvex integrands of this type has been studied by several authors in the past years, see in particular the papers of [ADM], [B], [BM], [CDA], [CDM], [DMar], [DMS], [FH], [FMý1], [Fmar], [G], [Mý1], [Mý2], [Mý3] for the polyconvex case and of [BFMý], [FMý], [Mar1], [Mar2] for the quasiconvex case. It is rather interesting to observe that when φ depends only on the gradient variable s, rather than on the full set of variables (x, u, s), then condition (1.9) can be dropped. This was shown by Celada and Dal Maso in [CDM]. No analogous results without a coercivity condition of the type (1.6) are known for the case when $d \neq N$ (see Proposition 3 in the Appendix for the coercive case).

§2. Proof of Theorem 1.1.

Proof of Theorem 1.1. Without loss of generality we may assume that

$$\liminf_{n \to \infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx = \lim_{n \to \infty} \int_\Omega f(x, u_n(x), \nabla u_n(x))\, dx < \infty.$$

Passing to a subsequence, if necessary, there exists a nonnegative Radon measure μ such that

$$f(x, u_n(x), \nabla u_n(x))\, \mathcal{L}^N \lfloor \Omega \overset{\star}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly $\star$ in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(Q(x_0, \varepsilon))}{\varepsilon^N} \geq f(x_0, u(x_0), \nabla u(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega, \tag{2.1}$$

[1] Note added to proof: in a recent paper [CMý] Cerný and Malý have constructed an integrand f which satisfies all the hypotheses of Theorem 1.3 except (1.5) and such that the corresponding functional F fails to be lower semicontinuous with respect to L^1 convergence.

where $Q_\nu(x_0,\varepsilon) := x_0 + \varepsilon\, Q_\nu$. If (2.1) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega;\mathbb{R})$, $0 \le \varphi \le 1$. We have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n)\,dx &\ge \liminf_{n\to\infty}\int_\Omega \varphi\, f(x,u_n,\nabla u_n)\,dx = \int_\Omega \varphi\, d\mu\\ &\ge \int_\Omega \varphi\,\frac{d\mu}{d\mathcal{L}^N}\,dx \ge \int_\Omega \varphi\, f(x,u,\nabla u)\,dx.\end{aligned}$$

By letting $\varphi \to 1$, and using Lebesgue Dominated Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem, it suffices to show (2.1).

Take $x_0 \in \Omega$ such that

$$\text{(2.2)}\quad \frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+}\frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N} < \infty, \qquad \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N+1}}\int_{Q(x_0,\varepsilon)} |u(x)-u(x_0)-\nabla u(x_0)(x-x_0)|dx = 0.$$

Choosing $\varepsilon_k \searrow 0$ such that $\mu(\partial Q(x_0,\varepsilon_k)) = 0$, then

$$\begin{aligned}\lim_{k\to\infty}\frac{\mu(Q(x_0,\varepsilon_k))}{\varepsilon_k^N} &= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\varepsilon_k^N}\int_{Q(x_0,\varepsilon_k)} f(x,u_n,\nabla u_n)\,dx\\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int_Q f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_{n,k}(y), \nabla w_{n,k}(y))\,dy,\end{aligned}$$

where

$$w_{n,k}(y) := \frac{u_n(x_0+\varepsilon_k y) - u(x_0)}{\varepsilon_k}.$$

Clearly $w_{n,k} \in W^{1,1}(Q;\mathbb{R}^d)$, and by (2.2), $\lim\limits_{k\to\infty}\lim\limits_{n\to\infty} ||w_{n,k} - w_0||_{L^1(Q;\mathbb{R}^d)} = 0$, where $w_0(y) := \nabla u(x_0)y$. By a standard diagonalization argument, we may extract a subsequence $w_k := w_{n_k,k}$ which converges to w_0 in $L^1(Q;\mathbb{R}^d)$ and such that

$$\text{(2.3)}\qquad \frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to\infty}\int_Q f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k(y), \nabla w_k(y))\,dy.$$

If $f(x_0,u(x_0),\xi) \equiv 0$ for all ξ then there is nothing to prove. Otherwise, let $\delta_0 > 0$ be given by (1.3), (1.4). From (2.3) and (1.4), and for k large

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) + 1 &\ge \int_{\{|w_k|\le\delta_0/\varepsilon_k\}} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k(y), \nabla w_k(y))\,dy\\ &\ge C\int_{\{|w_k|\le\delta_0/\varepsilon_k\}} |\nabla w_k(y)|\,dy - 1/C,\end{aligned}$$

and so there exists a constant $K > 0$ such that

$$\text{(2.4)}\qquad \int_{\{|w_k|\le\delta_0/\varepsilon_k\}} |\nabla w_k(y)|\,dy \le K.$$

By Proposition 1 and Remark 2(ii) in the Appendix, with $M = (x_0+\varepsilon_1\overline{Q}) \times \overline{B(u(x_0),\delta_0/2)}$ and $V = \mathbb{M}^{d\times N}$, there exist two sequences of continuous functions

$$a_j : M \to \mathbb{R}, \qquad b_j : M \to \mathbb{M}^{d\times N},$$

such that

$$f(x,u,\xi) = \sup_j \left(a_j(x,u) + b_j(x,u)\cdot\xi\right)^+ \tag{2.5}$$

for all $(x,u) \in M$ and $\xi \in \mathbb{M}^{d\times N}$. Define

$$f_j(x,u,\xi) := \left(a_j(x,u) + b_j(x,u)\cdot\xi\right)^+ .$$

Then f_j is continuous, convex in ξ and

$$0 \le f_j(x,u,\xi) \le C_j(|\xi|+1) \tag{2.6}$$

for all $(x,u) \in M$ and $\xi \in \mathbb{M}^{d\times N}$, where

$$C_j := \max\{|a_j(x,u)| + |b_j(x,u)| : \, (x,u) \in M\}.$$

For any fixed $\varepsilon > 0$ we may find $0 < \delta_j \le \delta_0/2$ such that

$$|a_j(x,u) - a_j(x_0,u(x_0))| + |b_j(x,u) - b_j(x_0,u(x_0))| \le \varepsilon$$

for all $(x,u) \in (x_0+\delta_j\overline{Q})\times\overline{B(u(x_0),\delta_j)}$. Since the function $g(s) := s^+$ is Lipschitz continuous with Lipschitz constant 1, we have

$$|f_j(x,u,\xi) - f_j(x_0,u(x_0),\xi)| \le |a_j(x,u) - a_j(x_0,u(x_0))| + |b_j(x,u) - b_j(x_0,u(x_0))|\,|\xi| \le \varepsilon(1+|\xi|)$$

for all $(x,u) \in (x_0+\delta_j\overline{Q})\times\overline{B(u(x_0),\delta_j)}$ and all $\xi \in \mathbb{M}^{d\times N}$. Hence, by (2.3), (2.4) and (2.5), for any fixed j

$$\begin{aligned}
\frac{d\mu}{d\mathcal{L}^N}(x_0) &\ge \liminf_{k\to\infty}\int_{\{|w_k|\le\delta_j/\varepsilon_k\}} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k(y), \nabla w_k(y))\,dy \\
&\ge \liminf_{k\to\infty}\int_{\{|w_k|\le\delta_j/\varepsilon_k\}} f_j(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k(y), \nabla w_k(y))\,dy \\
&\ge \liminf_{k\to\infty}\left(\int_{\{|w_k|\le\delta_j/\varepsilon_k\}} f_j(x_0, u(x_0), \nabla w_k(y))\,dy - \varepsilon - \varepsilon\int_{\{|w_k|\le\delta_j/\varepsilon_k\}}|\nabla w_k(y)|\,dy\right) \\
&\ge \liminf_{k\to\infty}\int_{\{|w_k|\le\delta_j/\varepsilon_k\}} f_j(x_0, u(x_0), \nabla w_k(y))\,dy - \varepsilon - \varepsilon K.
\end{aligned} \tag{2.7}$$

In order to truncate w_k, let $s := ||w_0||_{L^\infty(Q;\mathbb{R}^d)} + 1$ and let $\{m_k\} \subset \mathbb{N}$ be an increasing sequence of positive integers such that $2^{m_k} \le \delta_j/\varepsilon_k$, with $m_k \to \infty$. Choose an interval

$$(s_k, L_k) \in \left\{(s,2s),\dots,(2^{m_k-1}s, 2^{m_k}s)\right\}$$

such that

$$\int_{\{s_k\le|w_k|\le L_k\}} |\nabla w_k(y)|\,dy \le \frac{K}{m_k}. \tag{2.8}$$

Construct a smooth cut–off function $g_k : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$g_k(u) = \begin{cases} u & \text{if } |u| \le s_k, \\ 0 & \text{if } |u| \ge L_k, \end{cases} \tag{2.9}$$

with $|g_k(u)| \leq |u|$ and $|\nabla g_k(u)| \leq C\, L_k/(L_k - s_k)$ for all $u \in \mathbb{R}^d$. Note that by construction $L_k/(L_k - s_k) = 2$. Define $v_k(y) := g_k(w_k(y))$, and

$$E_k := \{y \in Q : \, |w_k(y)| < s_k\}, \quad E_k^+ := \{y \in Q : \, |w_k(y)| > L_k\}, \quad E_k^- := \{y \in Q : \, s_k \leq |w_k(y)| \leq L_k\}.$$

Then

$$\begin{aligned}
\int_Q |v_k(y) - w_0(y)| dy &= \int_{E_k} |w_k(y) - w_0(y)| dy + \int_{E_k^+} |w_0(y)| dy + \int_{E_k^-} |g_k(w_k(y)) - w_0(y)| dy \\
&\leq ||w_k - w_0||_{L^1(Q;\mathbb{R}^d)} + ||w_0||_{L^\infty(Q;\mathbb{R}^d)} \mathcal{L}^N(E_k^+ \cup E_k^-) + \int_{E_k^-} |w_k(y)| dy \\
&\leq 2||w_k - w_0||_{L^1(Q;\mathbb{R}^d)} + 2||w_0||_{L^\infty(Q;\mathbb{R}^d)} \mathcal{L}^N(E_k^+ \cup E_k^-) \to 0 \qquad \text{as } k \to \infty,
\end{aligned}$$

because

$$\begin{aligned}
0 \leq \mathcal{L}^N(E_k^+ \cup E_k^-) &= \mathcal{L}^N(\{y \in Q : \, |w_k(y)| \geq s_k\}) \\
&\leq \mathcal{L}^N(\{y \in Q : \, |w_k(y) - w_0(y)| \geq 1\}) \leq ||w_k - w_0||_{L^1(Q;\mathbb{R}^d)}.
\end{aligned} \tag{2.10}$$

Moreover

$$\begin{aligned}
\int_Q f_j(x_0, u(x_0), \nabla v_k(y))\, dy &= \int_{E_k} f_j(x_0, u(x_0), \nabla w_k(y))\, dy \\
&\quad + \int_{E_k^+} f_j(x_0, u(x_0), 0)\, dy + \int_{E_k^-} f_j(x_0, u(x_0), \nabla v_k(y))\, dy.
\end{aligned} \tag{2.11}$$

We claim that the last two integral are infinitesimal as $k \to \infty$. Indeed, by (2.6) and (2.10)

$$0 \leq \int_{E_k^+} f_j(x_0, u(x_0), 0)\, dy \leq C_j \mathcal{L}^N(E_k^+) \to 0, \tag{2.12}$$

while, from (2.6) and (2.8)

$$\int_{E_k^-} f_j(x_0, u(x_0), \nabla v_k(y))\, dy \leq C_j \int_{E_k^-} (1 + |\nabla g_k(w_k) \nabla w_k|\, dy \tag{2.13}$$

$$\leq C_j \left(\mathcal{L}^N(E_k^-) + ||\nabla g_k||_\infty \int_{E_k^-} |\nabla w_k|\, dy \right) \leq C_j \left(\mathcal{L}^N(E_k^-) + 2C\, \frac{K}{m_k} \right) \to 0 \tag{2.14}$$

as $k \to \infty$. Therefore from (2.7), (2.11), (2.12), (2.13) and (2.14)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{k \to \infty} \int_Q f_j(x_0, u(x_0), \nabla v_k(y))\, dy - \varepsilon - \varepsilon\, K,$$

where we used the fact that $E_k \subset \{y \in Q : \, |w_k| \leq \delta_j/\varepsilon_k\}$ by choice of m_k. By a result of Serrin [S,Theorem 2] we obtain

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f_j(x_0, u(x_0), \nabla u(x_0)) - \varepsilon - \varepsilon\, K.$$

By taking the sup in j and using (2.5) we have

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \nabla u(x_0)) - \varepsilon - \varepsilon\, K.$$

Now we let $\varepsilon \to 0^+$.

$\square$

Proof of Corollary 1.2. We only need to show that condition (1.4) in Theorem 1.1 is satisfied. The argument follows closely that of Morrey [M] in Lemma 4.1.3.

Suppose that (1.4) is false. Then we may find $(x_0, u_0) \in \Omega \times \mathbb{R}^d$ and a sequence $\{(x_n, u_n, \xi_n)\} \subset \Omega \times \mathbb{R}^d \times \mathbb{M}^{d\times N}$, with $(x_n, u_n) \to (x_0, u_0)$ and $|\xi_n| \to \infty$ as $n \to \infty$, such that

$$f(x_n, u_n, \xi_n) \le \frac{1}{n}|\xi_n| - 1. \tag{2.15}$$

By hypothesis $f(x, u, 0) \in L^\infty_{\mathrm{loc}}\left(\Omega \times \mathbb{R}^d; \mathbb{R}\right)$, hence there exists $M > 0$ such that

$$f(x_n, u_n, 0) \le M \quad \text{for all } n. \tag{2.16}$$

Define $v_n := \xi_n/|\xi_n|$. By extracting a subsequence, if necessary, we may assume that $v_n \to v_0 \in \mathbb{M}^{d\times N}\setminus\{0\}$ as $n \to \infty$. Since $f(x, u, \cdot)$ is convex, for any $R > 0$ we have

$$\begin{aligned} f\left(x_n, u_n, R\,v_n\right) &= f\left(x_n, u_n, \frac{R}{|\xi_n|}\,\xi_n\right) \\ &\le \left(1 - \frac{R}{|\xi_n|}\right) f(x_n, u_n, 0) + \frac{R}{|\xi_n|} f(x_n, u_n, \xi_n) \le \left(1 - \frac{R}{|\xi_n|}\right) M + \frac{R}{|\xi_n|}\left(\frac{1}{n}|\xi_n| - 1\right), \end{aligned}$$

where we have used (2.15) and (2.16). If we now let $n \to \infty$ in the previous inequality, and use the facts that $|\xi_n| \to \infty$, $(x_n, u_n, R\,v_n) \to (x_0, u_0, Rv_0)$, together with the lower semicontinuity of the function f, we get

$$f\left(x_0, u_0, R\,v_0\right) \le M,$$

which is a contradiction since $f\left(x_0, u_0, R\,v_0\right) \to \infty$ as $R \to \infty$.

$\square$

Proof of Corollary 1.3. Let $\{u_n\} \subset W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R})$ be a sequence of functions converging weakly in $BV_{\mathrm{loc}}(\Omega; \mathbb{R})$ to u. Then by (1.1)

$$\sup_n \int_\Omega |\nabla u_n(x)|\, dx \le K < \infty,$$

for some constant $K > 0$ independent of n. Fix $\varepsilon > 0$. The integrand

$$f_\varepsilon(x, u, \xi) := f(x, u, \xi) + \varepsilon\,|\xi|$$

satisfies all the hypotheses of Theorem 1. Therefore

$$\int_\Omega f(x, u, \nabla u)\, dx \le \int_\Omega f_\varepsilon(x, u, \nabla u)\, dx \le \liminf_{n\to\infty} \int_\Omega f_\varepsilon(x, u_n, \nabla u_n)\, dx \le \liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n)\, dx + \varepsilon\, K.$$

It suffices to let $\varepsilon \to 0^+$. $\square$

§3. Proof of Theorem 1.4.

Proof of Theorem 1.4. The proof of Theorem 1.4 is similar to the previous one, and we will only indicate the main differences. Let $f(x, u, \xi) := h(x, u, \mathcal{M}(\xi))$. We proceed as in the proof of Theorem 1.1 up to (2.3), where now $w_k \in W^{1,p}(Q; \mathbb{R}^d)$. Inequality (2.4) should be replaced by

$$\int_{\{|w_k| \le \delta_0/\varepsilon_k\}} |\mathcal{M}(\nabla w_k(y))|\, dy \le K. \tag{3.1}$$

By Proposition 1 and Remark 2(ii) in the Appendix, with $M = (x_0 + \varepsilon_1 \overline{Q}) \times \overline{B(u(x_0), \delta_0/2)}$ and $V = \mathbb{R}^\tau$, there exist two sequences of continuous functions

$$a_j : M \to \mathbb{R}, \qquad b_j : M \to \mathbb{R}^\tau,$$

such that

$$h(x,u,v) = \sup_j \left(a_j(x,u) + b_j(x,u) \cdot v\right)^+$$

for all $(x,u) \in M$ and $v \in \mathbb{R}^\tau$. Define

$$h_j(x,u,v) := \left(a_j(x,u) + b_j(x,u) \cdot v\right)^+, \quad \text{and} \quad f_j(x,u,\xi) := h_j(x,u,\mathcal{M}(\xi)).$$

Then h_j is continuous, convex in v and

$$0 \le h_j(x,u,v) \le C_j(|v|+1), \tag{3.2}$$

for all $(x,u) \in M$ and $v \in \mathbb{R}^\tau$, where

$$C_j := \max\{|a_j(x,u)| + |b_j(x,u)| : (x,u) \in M\}.$$

We may now proceed as in Theorem 1.1, replacing (2.8) with

$$\int_{\{s_k \le |w_k| \le L_k\}} |\mathcal{M}(\nabla w_k(y))| \, dy \le \frac{K}{m_k}, \tag{3.3}$$

up to (2.13) which becomes

$$\int_{E_k^-} f_j(x_0, u(x_0), \nabla v_k(y)) \, dy \le C_j \int_{E_k^-} (1 + |\mathcal{M}(\nabla g_k(w_k) \nabla w_k)|) \, dy. \tag{3.4}$$

Since $||\nabla g_k||_\infty \le C$ for some constant C, and in view of the fact that for $x \in E_k^-$

$$|\mathcal{M}(\nabla g_k(w_k) \nabla w_k)| \le |\mathcal{M}(\nabla g_k(w_k)| \, |\mathcal{M}(\nabla w_k)| \, ,$$

it follows that for $x \in E_k^-$

$$|\mathcal{M}(\nabla g_k(w_k) \nabla w_k)| \le C |\mathcal{M}(\nabla w_k)|. \tag{3.5}$$

Then, by (3.3) and (3.5),

$$C_j \int_{E_k^-} (1 + |\mathcal{M}(\nabla g_k(w_k) \nabla w_k)|) \, dy \le C_j \left(\mathcal{L}^N(E_k^-) + \frac{CK}{m_k} \right) \to 0. \tag{3.6}$$

Consequently, in view of (2.7), (2.11), (2.12), (3.4) and (3.6)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \ge \liminf_{k \to \infty} \int_Q f_j(x_0, u(x_0), \nabla v_k(y)) \, dy - \varepsilon - \varepsilon K. \tag{3.7}$$

At this stage we are not yet in position to apply Proposition 2 in the Appendix due to the lack of coercivity of h_j. By (3.1), (3.6) and since $v_k = 0$ on E_k^+, we have

$$\sup_k \int_Q |\mathcal{M}(\nabla v_k(y))| \, dy \le K_1 < \infty, \tag{3.8}$$

where K_1 is independent of δ and j. Define

$$g_{j,\varepsilon}(v) := h_j(x_0, u(x_0), v) + \varepsilon\,|v|.$$

By (3.7) and (3.8)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{k\to\infty} \int_Q g_{j,\varepsilon}(\mathcal{M}(\nabla v_k(y)))\,dy - \varepsilon - \varepsilon(K + K_1).$$

By Proposition 2 in the Appendix we obtain

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq g_{j,\varepsilon}(\mathcal{M}(\nabla u(x_0))) - \varepsilon - \varepsilon(K+K_1) = f_j(x_0, u(x_0), \nabla u(x_0)) + \varepsilon\,|\mathcal{M}(\nabla u(x_0))| - \varepsilon - \varepsilon(K+K_1).$$

By taking the sup in j we have

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \nabla u(x_0)) + \varepsilon\,|\mathcal{M}(\nabla u(x_0))| - \varepsilon - \varepsilon(K+K_1).$$

Now we let $\varepsilon \to 0^+$. This concludes the proof of the theorem □

Remark 3.1. The truncation arguments in the proofs of Theorem 1.1 and 1.4 simplify respectively the ones of Fonseca and Müller [FM] and of Fusco and Hutchinson [FH]. Moreover they can be used when the target function w_0 is *any* L^∞ function, not necessarily smooth. See [FL] for more details.

Proof of Corollary 1.5. The proof is similar to that of Corollary 1.3. In this case we consider the integrand

$$f_\varepsilon(x, u, \xi) := h(x, u, \mathcal{M}(\xi)) + \varepsilon\,|\xi|^{N-1}, \qquad \varepsilon > 0.$$

We omit the details. □

§4. Appendix.

Proposition 1. *Let M be a closed set of $\mathbb{R}^p$, let V be a reflexive and separable Banach space. Let $f : M \times V \to (-\infty, +\infty]$ be a $M \times w$-V sequentially lower semicontinuous function, convex in the last variable and such that there exists a continuous function $v_0 : M \to V$ with*

$$(f(\cdot, v_0(\cdot)))^+ \in L^\infty(M;\mathbb{R}). \tag{4.1}$$

Then there exist two sequences of continuous functions

$$a_j : M \to \mathbb{R}, \qquad b_j : M \to V^*,$$

where V^* is the dual space of V, such that

$$f(t, v) = \sup_j\,(a_j(t) + < b_j(t), v >)$$

for all $t \in M$ and $v \in V$.

Moreover if f is bounded from below, then (4.1) can be weakened to

$$(f(\cdot, v_0(\cdot)))^+ \in L^\infty_{\mathrm{loc}}(M;\mathbb{R}). \tag{4.2}$$

Remark 2. (i) Proposition 1 was proved by Ambrosio in [A, Lemma 2.5], in the case where (4.1) is replaced by the assumption that $f(\cdot, v_0(\cdot))$ is continuous.

(ii) If f is non negative then, without loss of generality, we may assume that

$$f(t,v) = \sup_j \left(a_j(t) + < b_j(t), v >\right)^+ .$$

Proof. Define

$$g(t,v) := f(t, v + v_0(t)) - C_0, \qquad \text{where} \quad C_0 := ||\left(f(\cdot, v_0(\cdot))\right)^+||_{L^\infty(M;\mathbb{R})}. \tag{4.3}$$

Clearly g satisfies the same hypotheses of f and

$$-\infty < g(t,0) = f(t, v_0(t)) - C_0 \leq 0. \tag{4.4}$$

Following Ambrosio [A], we consider the multifunction $\Gamma : M \to \{C \subset \mathbb{R} \times V^* : \ C \neq \emptyset, \text{ convex, closed}\}$ defined by

$$\Gamma(t) := \{(\alpha, \beta) \in \mathbb{R} \times V^* : \ g(t,v) \geq \alpha + < \beta, v > \quad \text{for all } v \in V\} .$$

It is easy to see by the hypotheses on g that $\Gamma(t)$ is well defined. We only note that $(4.4)_1$ and the convexity of $g(t, \cdot)$ imply that $\Gamma(t) \neq \emptyset$. We claim that Γ is lower semicontinuous. Fix an open set $W \subset \mathbb{R} \times V^*$. We need to show that

$$\Gamma^-(W) := \{t \in M : \ \Gamma(t) \cap W \neq \emptyset\}$$

is open in M. Assume that $\Gamma^-(W) \neq \emptyset$ and fix $t_0 \in \Gamma^-(W)$, $(\alpha_0, \beta_0) \in W$ and $\varepsilon > 0$ such that

$$\begin{aligned} & g(t_0, v) \geq \alpha_0 + < \beta_0, v > \quad \text{for all } v \in V, \\ & \{(\alpha, \beta) \in \mathbb{R} \times V^* : \ |\alpha - \alpha_0| + ||\beta - \beta_0|| < 4\varepsilon\} \subset W. \end{aligned} \tag{4.5}$$

We claim that there exists an open ball B in M, containing t_0, such that

$$g(t,v) \geq \alpha_0 + < \beta_0, v > - \varepsilon(1 + ||v||) \quad \text{for all } t \in B \text{ and } v \in V. \tag{4.6}$$

Assume that (4.6) is false. Then there exist two sequences $\{t_n\} \subset \mathbb{R}$ converging to t_0 in M and $\{v_n\} \subset V$ such that

$$g(t_n, v_n) < \alpha_0 + < \beta_0, v_n > - \varepsilon(1 + ||v_n||) \quad \text{for all } n \in \mathbb{N}. \tag{4.7}$$

If $\sup_n ||v_n|| < \infty$ then we can assume that v_n converges weakly to some $v_0 \in V$ and, due to the lower semicontinuity of g, letting $n \to \infty$ in (4.7) we get

$$g(t_0, v_0) \leq \alpha_0 + < \beta_0, v_0 > - \varepsilon(1 + ||v_0||)$$

which contradicts (4.5). Therefore $\sup_n ||v_n|| = \infty$, and without loss of generality we can assume that $||v_n|| \to \infty$, with $p_n := v_n / ||v_n||$ converging weakly to some $p_0 \in V$ as $n \to \infty$. Fix $\gamma > 0$ and let n be so large that $||v_n|| \, \gamma > 1$; by the convexity of $g(t, \cdot)$ and (4.7)

$$\begin{aligned} g(t_n, p_n/\gamma) - \alpha_0 - < \beta_0, p_n/\gamma > &\leq \frac{1}{\gamma \, ||v_n||} \left[g(t_n, v_n) - \alpha_0 - < \beta_0, v_n > \right] + \left(1 - \frac{1}{\gamma \, ||v_n||}\right) \left(g(t_n, 0) - \alpha_0\right) \\ &\leq - \frac{\varepsilon(1 + ||v_n||)}{\gamma \, ||v_n||} - \alpha_0 \left(1 - \frac{1}{\gamma \, ||v_n||}\right), \end{aligned}$$

where we have used the fact that $g(t_n,0) \le 0$ by $(4.4)_2$. Letting $n \to \infty$ and using the lower semicontinuity of g and (4.5) gives

$$0 \le g(t_0, p_0/\gamma) - \alpha_0 - < \beta_0, p_0/\gamma > \le -\frac{\varepsilon}{\gamma} - \alpha_0 < 0,$$

provided γ is taken sufficiently small. We arrive at a contradiction, and this proves (4.6). We can now proceed as in [A].

We claim that $B \subset \Gamma^-(W)$. Fix $b \in B$. By the Hahn–Banach Separation Theorem there exist $\alpha \in \mathbb{R}$ and $\beta \in V^*$ such that

$$g(b,v) - \alpha_0 - < \beta_0, v > \ge \alpha + < \beta, v > \ge -2\varepsilon - \varepsilon ||v|| \quad \text{for all } v \in V.$$

Consequently $\alpha \ge -2\varepsilon$ and $||\beta|| \le \varepsilon$. Let $\omega := \inf\{\alpha, 2\varepsilon\}$. Then

$$g(b,v) \ge \alpha_0 + \omega + < \beta_0 + \beta, v > \quad \text{for all } v \in V$$

and thus, by (4.5) it follows that $b \in \Gamma^-(W)$ and the claim is proved. Since Γ is lower semicontinuous by the Continuous Selection Theorem (see [A, Lemma 1.1]) for every $(t,v) \in M \times V$ we can write

$$g(t,v) = \sup \{\sigma_1(t) + < \sigma_2(t), v >: (\sigma_1, \sigma_2) \text{ is a continuous selection for } \Gamma\}$$

and by Lindelöf Theorem (see e.g. Lemma 9.2 in [FL]) we get the desired result. This completes the proof of the proposition in the case where (4.1) holds.

When f is bounded from below and (4.2) is satisfied, without loss of generality we can assume that f is nonnegative. Let $\psi_n \in C_0^\infty(\mathbb{R}^p;\mathbb{R})$, $0 \le \psi_n(t) \le 1$ be a cut–off function such that $\psi_n(t) \equiv 1$ for $|t| \le n$ and $\psi_n(t) \equiv 0$ for $|t| \ge n+1$. Define

$$f_n(t,v) := \psi_n(t)\, f(t,v).$$

Then f_n satisfies the same hypotheses of f and also (4.1). Thus we can apply the first part of the Proposition to f_n. To conclude the proof it suffices to observe that, since $f \ge 0$,

$$f(t,v) = \sup_n f_n(t,v).$$

□

Proposition 2. *Let $h : \mathbb{R}^\tau \to [0,\infty)$ be a convex function such that*

$$h(v) \to \infty \qquad \text{as } |v| \to \infty.$$

Let $u \in W^{1,p}(\Omega;\mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,p}(\Omega;\mathbb{R}^d)$ which converges to u in $L^1(\Omega;\mathbb{R}^d)$, where $p = \min\{d,N\}$. Then

$$\int_\Omega h(\mathcal{M}(\nabla u))\,dx \le \liminf_{n\to\infty} \int_\Omega h(\mathcal{M}(\nabla u_n))\,dx.$$

Proposition 2 has been proved by Dal Maso and Sbordone (cf. Theorem 2.2 in [DMS]) using cartesian currents and by Fusco and Hutchinson (cf. Theorem 2.6 in [FH]).

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References

[ABF] Acerbi E., Bouchitté G. and I. Fonseca, *Relaxation of convex functionals and the Lavrentiev phenomenon*, in preparation.

[ADM] Acerbi E. and G. Dal Maso, *New lower semicontinuity results for polyconvex integrals case*, Cal. Var. **2** (1994), 329-372.

[AF] Acerbi E. and F. Fusco, *Semicontinuity problems in the Calculus of Variations*, Arch. Rat. Mech. Anal. **86** (1984), 125-145.

[A] Ambrosio L., *New lower semicontinuity results for integral functionals*, Rend. Accad. Naz. Sci. XL **11** (1987), 1-42.

[B] Ball J. M., *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rat. Mech. Anal. **63** (1977), 337-403.

[BM] Ball J. M. and F. Murat, *$W^{1,p}$ quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225-253.

[BFMý] Bouchitté G., Fonseca, I. and J. Malý, *Relaxation of multiple integrals below the growth exponent*, Proc. Royal Soc. Edinburgh **128A** (1998), 463-479.

[CDA] Carbone L. and R. De Arcangelis, *Further results on Γ-convergence and lower semicontinuity of integral functionals depending on vector-valued functions*, Richerche Mat. **39** (1990), 99-129.

[CDM] Celada P. and G. Dal Maso, *Further remarks on the lower semicontinuity of polyconvex integrals*, Ann. Inst. Henri Poincaré, Ann. Non Lin. **11** (1994), 661-691.

[CMý] Cerný R. and J. Malý, *Counterexample to lower semicontinuity in Calculus of Variations* (to appear).

[D] Dacorogna B., *Direct Methods in the Calculus of Variations*, Applied Math. Sciences 78, Springer-Verlag, 1989.

[DMar] Dacorogna B. and P. Marcellini, *Semicontinuité pour des intégrandes polyconvexes sans continuité des determinants*, C. R. Acad. Sci. Paris Sér. I Math. **311, 6** (1990), 393-396.

[DMS] Dal Maso G. and C. Sbordone, *Weak lower semicontinuity of polyconvex integrals: a borderline case*, Math. Z. **218** (1995), 603-609.

[DBDM] De Giorgi E., Buttazzo G. and G. Dal Maso, *On the lower semicontinuity of certain integral functions*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Natur., Rend. **74** (1983), 274-282.

[E] Eisen G., *A counterexample for some lower semicontinuity results*, Math. Z. **162** (1978), 241-243.

[FL] Fonseca I. and G. Leoni, *On lower semicontinuity and relaxation*, Proc. Royal Soc. Edinburgh (to appear).

[FMý] Fonseca I. and J. Malý, *Relaxation of multiple integrals below the growth exponent*, Ann. Inst. Henri Poincaré **14** (1997), 308-338.

[FMý1] Fonseca I. and J. Malý, *Weak convergence of minors*, in preparation.

[FMar] Fonseca I. and P. Marcellini, *Relaxation of multiple integrals in subcritical Sobolev spaces*, J. Geom. Anal. **7** (1997), 57-81.

[FM] Fonseca I. and S. Müller, *Quasi–convex integrands and lower semicontinuity in L^1*, SIAM J. Math. Anal. **23** (1992), 1081-1098.

[FM1] Fonseca I. and S. Müller, *Relaxation of quasiconvex functionals in $BV(\Omega,\mathbb{R}^p)$ for integrands $f(x,u,\nabla u)$*, Arch. Rat. Mech. Anal. **123** (1993), 1-49.

[FH] Fusco N. and J. E. Hutchinson, *A direct proof for lower semicontinuity of polyconvex functionals*, Man. Math. **85** (1995), 35-50.

[G] Gangbo W., *On the weak lower semicontinuity of energies with polyconvex integrands*, J. Math. Pures et Appl. **73, 5** (1994).

[GMS] Giaquinta M., Modica G. and J. Souček, *Cartesian currents in the Calculus of Variations. I.*, Springer-Verlag, 1998.

[H] Hajłasz P., *A note on weak approximation of minors*, Ann. Inst. H. Poincaré **12, 4** (1995), 415-424.

[Mý1] Malý J., *Weak lower semicontinuity of polyconvex integrals*, Proc. Royal Soc. Edinburgh **123A** (1993), 681-691.

[Mý2] Malý J., *Weak lower semicontinuity of polyconvex and quasiconvex integrals*, Preprint, Vortragsreihe 1993, Bonn.

[Mý3] Malý J., *Lower semicontinuity of quasiconvex integrals*, Manuscripta Math. **85** (1994), 419-428.

[Mar1] Marcellini P., *Approximation of quasiconvex functions and lower semicontinuity of multiple integrals quasiconvex integrals*, Manus. Math. **51** (1985), 1-28.

[Mar2] Marcellini P., *On the definition and the lower semicontinuity of certain quasiconvex integrals*, Ann. Inst. H. Poincaré, Analyse non Linéaire **3** (1986), 391-409.

[M] Morrey C. B., *Multiple integrals in the Calculus of Variations*, Springer, Berlin, 1966.

[R] Rudin W., *Real and Complex Analysis*, McGraw–Hill, 1987.

[S] Serrin J., *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc. **161** (1961), 139-167.

[St] Struwe M., *Variational methods*, Springer, Berlin, 1996.