Brittle Thin Films

Andrea Braides
SISSA
Via Beirut 4
34014 Trieste, Italy

Irene Fonseca
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213, USA

Abstract A 2-dimensional model for brittle thin films is obtained from a 3-dimensional fracture model for elastic material. Γ-convergence techniques are used to identify the limiting effective energy as the thickness of the sample approaches zero.
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Introduction

Elastic thin films may be modeled through a limit procedure by considering three-dimensional structures with vanishing thickness ε. In the framework of nonlinear hyperelasticity, the elastic energy associated to these structures is of the form

$$\int_{\Omega_\varepsilon} W(\nabla v)\, dx,$$

where $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$ represents the reference configuration, ω is an open subset of $\mathbb{R}^2$, $\varepsilon > 0$, and $v \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3)$ is a deformation of the body. In order to describe the limiting behavior of these functionals, and upon a change of variables, it is more convenient to study the limit of the family of scaled energies

$$\int_\Omega W\Big(\nabla_\alpha u \Big| \frac{1}{\varepsilon}\nabla_3 u\Big)\, dx,$$

where $\Omega := \Omega_1 = \omega \times (-1/2, 1/2)$, ∇_α denotes differentiation with respect to the variable $x_\alpha = (x_1, x_2)$, thus reducing the analysis to the fixed reference configuration Ω and to $u \in W^{1,p}(\Omega; \mathbb{R}^3)$. It was shown in [30] that, under appropriate hypotheses on the bulk energy density W, the limit energy is of the form

$$\int_\omega Q_2\overline{W}(\nabla_\alpha u)\, dx_\alpha,$$

where $u \in W^{1,p}(\omega; \mathbb{R}^3)$, $\overline{W}(\overline{F}) := \min\{W(\overline{F}|F_3) : F_3 \in \mathbb{R}^3\}$, $\overline{F}$ is a 3×2 matrix, and Q_2 denotes the operation of quasiconvexification on 3×2 matrices. It is worth noting that, unless W is convex, the quasiconvexification operation is essential in the description of the limiting energy, and it underlines the formation of microstructure.

Here we take into account the possibility of fracture in a thin film, and, accordingly, we add to the formulation above a crack initiation energy. Precisely, we consider

$$\int_{\Omega_\varepsilon \setminus K} W(\nabla v)\ dx + \int_{K \cap \Omega_\varepsilon} \vartheta(v^+ - v^-, \nu(K))\ d\mathcal{H}^2,$$

where K denotes an unknown crack surface, $\mathcal{H}^2(K)$ is the 2-dimensional Hausdorff measure of K, $\nu(K)$ is the normal to the crack surface in the reference configuration, $v \in W^{1,p}(\Omega_\varepsilon \setminus K; \mathbb{R}^3)$ is the deformation, defined in the complement of the crack, and $v^\pm$ are the traces of u on both sides of K. The surface energy density ϑ is a fracture initiation term defined on $\mathbb{R}^3 \times S^2$. In Griffith's theory [29] ϑ is constant; we have added a possible dependence on $\nu(K)$ to account for anisotropy, and a dependence on the jump $u^+ - u^-$ to include Barenblatt's approach [11]. Again, by performing a change of variables, and extending ϑ to $\mathbb{R}^3 \times \mathbb{R}^3$ as a homogeneous function of degree 1 in the variable ν, we arrive at the re-scaled energy

$$\int_{\Omega \setminus K_\varepsilon} W\Big(\nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u\Big)\ dx + \int_{K_\varepsilon \cap \Omega} \vartheta\Big(u^+ - u^-, \nu_\alpha(K_\varepsilon), \frac{1}{\varepsilon}\nu_3(K_\varepsilon)\Big)\ d\mathcal{H}^2. \tag{0.1}$$

It is well known that free discontinuity problems formulated in terms of variable domains $\Omega \setminus K_\varepsilon$ and unknown admissible field u are very difficult to handle, and often it is more convenient to set up the problem within the De Giorgi and Ambrosio framework of *special functions of bounded variation* $SBV(\Omega; \mathbb{R}^3)$ (see [7]), by replacing K_ε by the set $S(u)$ of essential discontinuity points for u and interpreting ∇u as an approximate gradient, thus reducing the simultaneous variation in the crack site and in the deformation to a variation on the discontinuous function u only.

In this paper we show that it is possible to characterize the limit of the family of energies (0.1) as

$$\int_\Omega Q_2\overline{W}(\nabla_\alpha u)\, dx + \int_{S(u)\cap\Omega} R_2\overline{\vartheta}(u^+ - u^-, \nu_\alpha(u))\, d\mathcal{H}^2(x), \tag{0.2}$$

where the admissible deformations u belong to $SBV(\Omega; \mathbb{R}^3)$, satisfy $\nabla_3 u = 0$ and $\nu_3(u) = 0$, and where W has superlinear growth (see Theorem 2.1). The effective surface energy density $R_2\overline{\vartheta}$ is obtained first by minimizing ϑ in ν_3, and then considering the *BV-elliptic envelope* on S^1 of the resulting function. We recall that BV-ellipticity is the counterpart of quasiconvexity for surface energies, and for whose properties we refer to [5, 6, 2, 19]. The domain of the limit energy (0.2) is isomorphically contained in $SBV(\omega; \mathbb{R}^3)$ and (0.2) may be written equivalently as

$$\int_\omega Q_2\overline{W}(\nabla_\alpha u)\, dx_\alpha + \int_{S(u)\cap\omega} R_2\overline{\vartheta}(u^+ - u^-, \nu_\alpha(u))\, d\mathcal{H}^1(x_\alpha),$$

for $u \in SBV(\omega; \mathbb{R}^3)$. Again, and as in [30], we recover a model where the admissible fields are restricted to the middle plate ω, although here they may have discontinuities due to the presence fracture.

In Theorem 3.2 we provide an integral representation for thin film relaxed energy in the case where the bulk energy density W grows linearly at infinity. As it is usual, here there is an interaction between bulk and interfacial energies.

1 Notation and Preliminaries

In the sequel $\mathbb{M}^{m\times N}$ stands for the space of $m \times N$ matrices. Given a matrix $F \in \mathbb{M}^{3\times 3}$, and following the notation introduced in [30], we write $F =: (\overline{F}|F_3)$, where F_i denotes the i-th column of F, $1 \leq i \leq 3$, and $\overline{F} := (F_1 F_2) \in \mathbb{M}^{3\times 2}$ is the matrix of the first two columns of F. If $x \in \mathbb{R}^3$ then $x_\alpha := (x_1, x_2) \in \mathbb{R}^2$ is the vector of the first two components of x. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is designated as $\mathcal{L}^N$, and when there is no possibility of confusion we may abbreviate $|E| := \mathcal{L}^N(E)$. We designate by C a positive constant whose value within one formula may vary from on line to the next.

1.1 Γ-convergence

We recall the definition of De Giorgi's Γ-*convergence* in L^p spaces, $1 \leq p < +\infty$: given a family of functionals $J_j : L^p(\Omega; \mathbb{R}^m) \to [0, +\infty)$, $j \in \mathbb{N}$, for $u \in L^p(\Omega; \mathbb{R}^m)$ we define

$$\Gamma(L^p)\text{-}\liminf_{j\to+\infty} J_j(u) := \inf\left\{\liminf_{j\to+\infty} J_j(u_j) \;:\; u_j \to u \text{ in } L^p(\Omega; \mathbb{R}^m)\right\},$$

and

$$\Gamma(L^p)\text{-}\limsup_{j\to+\infty} J_j(u) := \inf\left\{\limsup_{j\to+\infty} J_j(u_j) \;:\; u_j \to u \text{ in } L^p(\Omega; \mathbb{R}^m)\right\}.$$

If these two quantities coincide then their common value is called the Γ-*limit* of the sequence (J_j) at u, and is denoted by $\Gamma(L^p)\text{-}\lim_{j\to+\infty} J_j(u)$.

We say that (J_ε) $\Gamma(L^p)$-*converges to* l *at* u *as* $\varepsilon \to 0$ if for every sequence of positive numbers (ε_j) converging to 0^+ there exists a subsequence (ε_{j_k}) for which

$$l = \Gamma(L^p)\text{-}\lim_{k\to+\infty} J_{\varepsilon_{j_k}}(u).$$

We recall that the Γ-upper and lower limits defined above are L^p-lower semicontinuous functions.

For a comprehensive study of Γ-convergence we refer to the book of Dal Maso [23] (see also [18] Part II).

1.2 Special functions of bounded variation

Let Ω be an open subset of $\mathbb{R}^N$, and let $\mathcal{M}(\Omega)$ be the set of all signed Radon measures on Ω with bounded total variation. We say that $u \in L^1(\Omega;\mathbb{R}^m)$ is a *function of bounded variation*, and we write $u \in BV(\Omega;\mathbb{R}^m)$, if every its distributional first derivatives $D_i u_j$ belong to $\mathcal{M}(\Omega)$. We denote by Du the $\mathbb{M}^{m\times N}$-valued measure whose entries are $D_i u_j$. $S(u)$ is the complement of the Lebesgue set of u; i.e., $x \notin S(u)$ if and only if

$$\lim_{\rho\to 0^+} \rho^{-N} \int_{B_\rho(x)} |u(y) - z|\, dy = 0$$

for some $z \in \mathbb{R}^m$. If $u \in L^1(\Omega;\mathbb{R}^m)$ then the set $S(u)$ is Lebesgue-negligible, and if, in addition, $u \in BV(\Omega;\mathbb{R}^m)$ then the Hausdorff dimension of $S(u)$ is at most $N-1$; precisely, $S(u)$ is *rectifiable* in that there is a countable sequence of C^1 hypersurfaces Γ_i which covers $\mathcal{H}^{N-1}$-almost all of $S(u)$; i.e., $\mathcal{H}^{N-1}\big(S(u) \setminus \bigcup_{i=1}^\infty \Gamma_i\big) = 0$. Moreover, for $\mathcal{H}^{N-1}$-almost every $x \in S(u)$ it is possible to find $a, b \in \mathbb{R}^m$ and $\nu \in S^{N-1}$ such that

$$\lim_{\rho\to 0^+} \rho^{-N} \int_{B_\rho^\nu(x)} |u(y) - a|\, dy = 0, \qquad \lim_{\rho\to 0^+} \rho^{-N} \int_{B_\rho^{-\nu}(x)} |u(y) - b|\, dy = 0,$$

where $B_\rho^\nu(x) := \{y \in B_\rho(x) : \langle y - x, \nu\rangle > 0\}$. The triplet (a, b, ν) is uniquely determined up to a change of sign of ν and an interchange between a and b, and in the sequel it will be denoted by $(u^+, u^-, \nu(u))$.

The distributional derivative of a function $u \in BV(\Omega;\mathbb{R}^m)$ admits the decomposition

$$Du = \nabla u\, \mathcal{L}^N \llcorner \Omega + J_u + C_u,$$

where $\nabla u\, \mathcal{L}^N$ is the *Lebesgue part* of Du,

$$J_u := (u^+ - u^-) \otimes \nu(u)\, \mathcal{H}^{N-1} \llcorner S(u)$$

is the *Hausdorff part* or *jump* part of Du, and C_u is the *Cantor part* of Du. We recall that the measure C_u is singular with respect to the Lebesgue measure and it is 'diffuse'; i.e., $C_u(S) = 0$ for every set S with $\mathcal{H}^{N-1}(S) < +\infty$. For the general exposition of the theory of functions of bounded variation we refer the reader to Federer [27], Evans and Gariepy [26], Ambrosio, Fusco and Pallara [8] and Ziemer [32]. A quick introduction can be found in [15] Section 1.

We say that a function $u \in BV(\Omega;\mathbb{R}^m)$ is a *special function of bounded variation* if $C_u \equiv 0$, or, equivalently, if

$$Du = \nabla u\, \mathcal{L}_N \llcorner \Omega + (u^+ - u^-) \otimes \nu(u)\, \mathcal{H}^{N-1} \llcorner S(u).$$

We denote the space of the special functions of bounded variation by $SBV(\Omega;\mathbb{R}^m)$. The introduction of this space is due to De Giorgi and Ambrosio [24]. $GSBV(\Omega)$ is defined as the space of scalar functions in $L^1(\Omega)$ such that for all $T > 0$ the truncations $u_T := (-T)\wedge(u\vee T)$ belong to $SBV(\Omega)$. We say that $u \in GSBV(\Omega;\mathbb{R}^m)$ if every component of u is in $GSBV(\Omega)$. For the properties of functions in $SBV(\Omega;\mathbb{R}^m)$ and $GSBV(\Omega;\mathbb{R}^m)$ we refer to the book by Ambrosio, Fusco and Pallara [8].

Let $p > 1$. $SBV_p(\Omega;\mathbb{R}^m)$ and $GSBV_p(\Omega;\mathbb{R}^m)$ are defined as the subspaces of functions u of $SBV(\Omega;\mathbb{R}^m)$ and $GSBV(\Omega;\mathbb{R}^m)$, respectively, such that

$$\mathcal{H}^{N-1}(S(u)\cap\Omega) < +\infty \qquad \text{and} \qquad \nabla u \in L^p(\Omega;\mathbb{M}^{m\times N}).$$

These spaces are natural domains for the treatment of energies with bulk and surface contributions in the case where the bulk energy density grows superlinearly at infinity.

We recall that a continuous function $W : \mathbb{M}^{m\times N} \to \mathbb{R}$ is said to be *quasiconvex* if affine functions are local minimizers for the corresponding volume integral functional on Sobolev functions with respect to periodic perturbations; precisely, if $W^{1,\infty}_{\#}((0,1)^N;\mathbb{R}^m)$ denotes the space of 1-periodic $\varphi \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^m)$ then

$$W(F) \le \int_{(0,1)^N} W(F + \nabla\varphi(x))\,dx$$

for all $F \in \mathbb{M}^{m\times N}$ and $\varphi \in W^{1,\infty}_{\#}((0,1)^N;\mathbb{R}^m)$. The *quasiconvex envelope* QW of $W : \mathbb{M}^{m\times N} \to \mathbb{R}$ is the greatest quasiconvex function not larger than W. We have

$$QW(F) = \inf\left\{\int_{(0,1)^N} W(F + D\varphi(x))\,dx : \ \varphi \in W^{1,\infty}_{\#}((0,1)^n;\mathbb{R}^m)\right\}$$

for all $F \in \mathbb{M}^{m\times N}$ (see e.g. [10], [22]).

Similarly, a continuous function $\vartheta : \mathbb{R}^m \times S^{N-1} \to \mathbb{R}$ is said to be *BV-elliptic* if functions with an hyperplane as jump set are local minimizers for the corresponding surface integral functional among piecewise constant functions with respect to periodic perturbations; precisely, if Q_ν is a unit cube in $\mathbb{R}^N$ with center 0 and one face orthogonal to ν and $GSBV_{\#}(Q_\nu;\mathbb{R}^m,z)$ denotes the space of functions $u \in GSBV_\infty(Q_\nu;\mathbb{R}^m)$ which are 1-periodic in the direction orthogonal to ν, with $\nabla u = 0$ a.e., and such that $u(x) = \pm z/2$ if $\langle x,\nu\rangle = \pm 1/2$, then

$$\vartheta(z,\nu) \le \int_{S(u)\cap Q_\nu} \vartheta(u^+ - u^-,\nu(u))\,d\mathcal{H}^{N-1}$$

for all $z \in \mathbb{R}^m$, $\nu \in S^{N-1}$ and for all $u \in GSBV_{\#}(Q_\nu;\mathbb{R}^m,z)$. The *$BV$-elliptic envelope* $R\vartheta$ of a continuous function $\vartheta : \mathbb{R}^m \times S^{N-1} \to \mathbb{R}$ is the largest BV-elliptic function not larger than ϑ. We have

$$R\vartheta(z,\nu) := \inf\left\{\int_{S(u)\cap Q_\nu} \vartheta(u^+ - u^-,\nu(u))\,d\mathcal{H}^{N-1} : \ u \in GSBV_{\#}(Q_\nu;\mathbb{R}^m,z)\right\}$$

for all $z \in \mathbb{R}^m$ and $\nu \in S^{N-1}$ (see [6], [19]).

The following closure and lower semicontinuity theorem holds on $GSBV_p(\Omega)$ (see [1], [8]).

Theorem 1.1 *Let W be a quasiconvex function, let ϑ be a BV-elliptic function, with*

$$0 \le W(F) \le C(1 + |F|^p), \qquad 0 \le \vartheta(z, \nu) \le C(1 + |z|)$$

for some $C > 0$, and for all $F \in \mathbb{M}^{m \times N}$, $z \in \mathbb{R}^m$, and $\nu \in S^{N-1}$. If (u_j) is a sequence in $GSBV_p(\Omega; \mathbb{R}^m)$ with $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$, and if

$$\sup_j \left\{ \int_\Omega |\nabla u_j|^p \, dx + \mathcal{H}^{N-1}(S(u_j) \cap \Omega) \right\} < +\infty, \tag{1.1}$$

then $u \in GSBV_p(\Omega; \mathbb{R}^m)$,

$$\int_{S(u)} \vartheta(u^+ - u^-, \nu(u)) \, d\mathcal{H}^{N-1} \le \liminf_j \int_{S(u_j)} \vartheta(u_j^+ - u_j^-, \nu(u_j)) \, d\mathcal{H}^{N-1},$$

and

$$\int_\Omega W(\nabla u) \, dx \le \liminf_j \int_\Omega W(\nabla u_j) \, dx.$$

Definition 1.2 *We say that a sequence (u_j) in $GSBV_p(\Omega; \mathbb{R}^m)$* converges weakly *in $GSBV_p(\Omega; \mathbb{R}^m)$ to a function $u \in GSBV_p(\Omega; \mathbb{R}^m)$ if $u_j \to u$ in $L^1(\Omega; \mathbb{R}^m)$, and* (1.1) *holds.*

In view of this definition, we may say that Theorem 1.1 provides a closure result for weak convergence in $GSBV_p(\Omega; \mathbb{R}^m)$.

The following relaxation theorem may be found in [19].

Theorem 1.3 [Relaxation in $GSBV_p(\Omega; \mathbb{R}^m)$] *If W and ϑ are Borel functions satisfying the growth conditions*

$$|F|^p \le W(F) \le C(1 + |F|^p) \qquad 1 + |z| \le \vartheta(z, \nu) \le C(1 + |z|)$$

for all $F \in \mathbb{M}^{m \times N}$, $z \in \mathbb{R}^m$, $\nu \in S^{n-1}$, and for some $C > 0$, then the lower semicontinuous envelope *of the functional*

$$u \mapsto \int_\Omega W(\nabla u) \, dx + \int_{S(u)} \vartheta(u^+ - u^-, \nu(u)) \, d\mathcal{H}^{N-1}$$

on $SBV_p(\Omega; \mathbb{R}^m)$ with respect to the $L^1(\Omega; \mathbb{R}^m)$ convergence is given by

$$u \mapsto \int_\Omega QW(\nabla u) \, dx + \int_{S(u)} R\vartheta(u^+ - u^-, \nu(u)) \, d\mathcal{H}^{N-1}.$$

2 Brittle thin films

Let ω be a bounded open subset of $\mathbb{R}^2$, let $\Omega_\varepsilon := \omega \times (-\varepsilon/2, \varepsilon/2)$, $\Omega := \Omega_1 = \omega \times (-1/2, 1/2)$, and fix $p > 1$. Let $W : \mathbb{M}^{3\times 3} \to [0, +\infty)$ and $\vartheta : (\mathbb{R}^3 \setminus \{0\}) \times S^2 \to [0, +\infty)$ denote bulk and surface energy density, respectively. The '$3D$-energy' of a deformation $v \in GSBV_p(\Omega_\varepsilon; \mathbb{R}^3)$ of the thin film occupying Ω_ε as a reference configuration is given by

$$\int_{\Omega_\varepsilon} W(\nabla v)\, dx + \int_{S(v)\cap\Omega_\varepsilon} \vartheta(v^+ - v^-, \nu(v))\, d\mathcal{H}^2$$

for functions $v \in GSBV_p(\Omega_\varepsilon; \mathbb{R}^3)$. Note that there is a 1-1 correspondence between rank-1-matrices in $\mathbb{M}^{3\times 3}$ and $(\mathbb{R}^3 \setminus \{0\}) \times S^2$, so that we could as well consider the surface energy as defined on matrices by $\theta(z \otimes \nu) = \vartheta(z, \nu)$, so as to have a symmetric notation with that of the bulk part. However, we prefer to adopt the above notation, as it better describes the decoupled dependence of surface energy densities with respect to the amplitude of the jump, z, and the normal to the jump set, ν.

We extend ϑ by positive 1-homogeneity as follows:

$$\vartheta(z, \eta) = \begin{cases} |\eta|\vartheta\Big(z, \frac{\eta}{|\eta|}\Big) & \text{if } \eta \neq 0, \\ 0 & \text{if } \eta = 0. \end{cases}$$

Changing variables and setting

$$u(x_\alpha, x_3) := v(x_\alpha, \varepsilon x_3), \quad (\nu_\alpha, \nu_3) := \nu,$$

clearly $u \in GSBV_p(\Omega; \mathbb{R}^3)$ and the integral above becomes

$$\begin{aligned}
&\int_{\Omega_\varepsilon} W\Big(\nabla_\alpha u\left(x_\alpha, \frac{x_3}{\varepsilon}\right) \Big| \frac{1}{\varepsilon}\nabla_3 u\left(x_\alpha, \frac{x_3}{\varepsilon}\right)\Big)\, dx \\
&\quad + \int_{S(v)\cap\Omega_\varepsilon} \vartheta\Big(u^+\left(x_\alpha, \frac{x_3}{\varepsilon}\right) - u^-\left(x_\alpha, \frac{x_3}{\varepsilon}\right), \\
&\qquad \nu_\alpha(u)\left(x_\alpha, \frac{x_3}{\varepsilon}\right), \frac{1}{\varepsilon}\nu_3(u)\left(x_\alpha, \frac{x_3}{\varepsilon}\right)\Big) \frac{\varepsilon}{\sqrt{\varepsilon^2|\nu_\alpha|^2(v) + \nu_3^2(v)}}\, d\mathcal{H}^2 \\
&= \varepsilon\Bigg[\int_\Omega W\Big(\nabla_\alpha u(x)\Big|\frac{1}{\varepsilon}\nabla_3 u(x)\Big)\, dx \\
&\quad + \int_{S(u)\cap\Omega} \vartheta\Big(u^+(x) - u^-(x), \nu_\alpha(u)(x), \frac{1}{\varepsilon}\nu_3(u)(x)\Big)\, d\mathcal{H}^2\Bigg].
\end{aligned}$$

We will consider the ε-*scaled 3D-energies* $J_\varepsilon : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by

$$J_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega W\Big(\nabla_\alpha u\Big|\frac{1}{\varepsilon}\nabla_3 u\Big)\,dx & \\ \displaystyle\quad + \int_{S(u)} \vartheta\Big(u^+ - u^-, \nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u)\Big)\,d\mathcal{H}^2 & \text{if } u \in GSBV_p(\Omega;\mathbb{R}^3), \\ +\infty. & \text{otherwise.} \end{cases} \tag{2.2}$$

We introduce the space

$$\mathcal{V} := \{u \in SBV_p(\Omega;\mathbb{R}^3) : \ \nabla_3 u = 0 \text{ a.e. and } \nu_3(u) = 0 \ \mathcal{H}^2\text{-a.e.}\}.$$

Note that if $u \in \mathcal{V}$ then $D_3 u = 0$ in the sense of distributions, so that $\mathcal{V}$ can be identified with $SBV_p(\omega;\mathbb{R}^3)$.

An energy density $\widehat{W} : \mathbb{M}^{3\times 2} \to \mathbb{R}$ determines naturally two quasiconvex envelopes; precisely, given $\overline{F} \in \mathbb{M}^{2\times 3}$ we may define

$$Q_2\widehat{W}(\overline{F}) := \inf\left\{\int_{(0,1)^2} \widehat{W}(\overline{F} + D_\alpha\varphi(x))\,dx_\alpha : \ \varphi \in W^{1,\infty}_{\#}((0,1)^2;\mathbb{R}^3)\right\},$$

and also

$$Q_3\widehat{W}(\overline{F}) := \inf\left\{\int_{(0,1)^3} \widehat{W}(\overline{F} + D_\alpha\varphi(x))\,dx : \ \varphi \in W^{1,\infty}_{\#}((0,1)^3;\mathbb{R}^3)\right\}.$$

Similarly, associated to a function $\widehat{\vartheta} : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^2 \to \mathbb{R}$ we may introduce two BV-elliptic envelopes, namely

$$R_2\widehat{\vartheta}(z,\eta) := \inf\left\{\int_{S(u)\cap Q'_\eta} \widehat{\vartheta}(u^+ - u^-, \nu(u))\,d\mathcal{H}^1 : \ u \in GSBV_{\#}(Q'_\eta;\mathbb{R}^3, z)\right\}$$

for all $z \in \mathbb{R}^3$ and $\eta \in S^1$, where Q'_η is a unit square on the plane, centered at zero and with two faces orthogonal to η, and

$$R_3\widehat{\vartheta}(z,\eta) := \inf\left\{\int_{S(u)\cap Q_{(\eta,0)}} \widehat{\vartheta}(u^+ - u^-, \nu_\alpha(u))\,d\mathcal{H}^2 : \ u \in GSBV_{\#}(Q_{(\eta,0)};\mathbb{R}^3, z)\right\}.$$

In the latter envelope $Q_{(\eta,0)}$ stands for a unit cube in $\mathbb{R}^3$ centered at zero and with two faces orthogonal to $(\eta, 0)$.

By Fubini's Theorem it can be easily seen that

$$Q_3\widehat{W}(\overline{F}) \ge Q_2\widehat{W}(\overline{F}). \tag{2.3}$$

The main result of the paper is the following.

Theorem 2.1 *Let* $W : \mathbb{M}^{3\times 3} \to [0,+\infty)$ *and* $\vartheta : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3 \to [0,+\infty)$ *be continuous functions,* ϑ *is symmetric and positively homogeneous of degree* 1 *in the second variable; i.e.,*

$$\vartheta(z,\nu) = \vartheta(-z,-\nu), \qquad \vartheta(z,t\nu) = t\vartheta(z,\nu) \text{ for all } t>0,\ z,\nu \in \mathbb{R}^3.$$

Let W *and* ϑ *satisfy the growth and coercivity conditions*

$$|F|^p \le W(F) \le C(1+|F|^p), \qquad 1+|z| \le \vartheta(z,\nu) \le C(1+|z|) \tag{2.4}$$

for all $F \in \mathbb{M}^{3\times 3}$*,* $z \in \mathbb{R}^3$*,* $\nu \in S^2$*, and for some* $C > 0$*. Suppose, in addition, that* ϑ *satisfies the Lipschitz condition*

$$|\vartheta(z,\nu) - \vartheta(z',\nu)| \le L\,|z-z'| \tag{2.5}$$

for all $F, F' \in \mathbb{M}^{3\times 3}$*,* $z, z' \in \mathbb{R}^3$*,* $\nu \in S^2$*, and for some* $C > 0$*. Then the functionals* J_ε *defined in* (2.2) Γ*-converge with respect to the* $L^1(\Omega;\mathbb{R}^3)$ *convergence as* $\varepsilon \to 0^+$ *to the functional* $J_0 : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ *defined by*

$$J_0(u) := \begin{cases} \displaystyle\int_\omega Q_2\overline{W}(\nabla_\alpha u)\,dx_\alpha + \int_{S(u)\cap\omega} R_2\overline{\vartheta}(u^+ - u^-, \nu_\alpha(u))\,d\mathcal{H}^1 & \text{if } u \in \mathcal{V}, \\ +\infty & \textit{otherwise}, \end{cases}$$

where

$$\overline{W}(\overline{F}) := \inf\{W(F) :\ F = (\overline{F}|F_3),\ F_3 \in \mathbb{R}^3\}$$

for $F \in \mathbb{M}^{3\times 2}$*, and*

$$\overline{\vartheta}(z,\eta) := \inf\{\vartheta(z,\eta,\xi) :\ \xi \in \mathbb{R}\}$$

for $z \in \mathbb{R}^3$ *and* $\eta \in \mathbb{R}^2 \setminus \{0\}$*.*

Remark 2.2 (i) The Lipschitz condition (2.5) imposed on the surface energy density ϑ is of technical nature, and will be used to comply with the hypotheses of the integral representation theorem in [16]. It can be easily weakened to a uniform-continuity hypothesis. As the representation result in [16] is designed to deal with Carathéodory integrands of a more general form, it may be possible to completely remove it by modifying the arguments of other results which do not use continuity hypotheses such as the one in [14]. Note, however, that in general, contrary to quasiconvexity, BV-ellipticity does not imply continuity with respect to the jump argument.
(ii) The result of Theorem 2.1 still holds in the case where the surface energy density satisfies a growth condition of the type $0 < c_1 \le \vartheta \le c_2$ (see Example 2.10; for details we refer to the final chapter of [19]).
(iii) Theorem 2.1 may be reformulated for mappings $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m$ for any pair of positive integers N and m. The changes are straightforward, and the proofs are similar to the ones presented in this work. For simplicity, and having in mind the motivation for modeling thin films, we will restrict ourselves to the case where $N = m = 3$.

The proof of the theorem will follow from the propositions and lemmas in the rest of this section. The 'localized' versions of J_ε on cylinders $U \times (-1/2, 1/2)$ for all U open subsets of ω are defined as

$$J_\varepsilon(u, U) := \begin{cases} \displaystyle\int_{U \times (-1/2,1/2)} W\Big(\nabla_\alpha u \Big| \frac{1}{\varepsilon} \nabla_3 u\Big)\, dx \\ \displaystyle + \int_{S(u) \cap (U \times (-1/2,1/2))} \vartheta\Big(u^+ - u^-, \nu_\alpha(u), \frac{1}{\varepsilon}\nu_3(u)\Big)\, d\mathcal{H}^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in GSBV_p(\Omega; \mathbb{R}^3), \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{2.6}$$

2.1 Proof of the lower bound

We begin by giving a lower bound for the Γ-limit.

Proposition 2.3 *Let J_ε be defined as in* (2.2) *with W and ϑ Borel functions satisfying the growth conditions*

$$|F|^p \le W(F), \qquad 1 \le \vartheta(z, \nu)$$

for all $F \in \mathbb{M}^{3\times 3}$, $z \in \mathbb{R}^m$, $\nu \in S^{N-1}$, and where $\vartheta(z, \cdot)$ is positively homogeneous of degree 1. *Let (u_ε) be a family of functions such that $\sup_\varepsilon J_\varepsilon(u_\varepsilon) < +\infty$ and $u_\varepsilon \to u$ in $L^1(\Omega; \mathbb{R}^3)$. Then $u \in \mathcal{V}$ and (u_ε) converges weakly to u in $GSBV_p(\Omega; \mathbb{R}^3)$.*

Proof. By the growth conditions on W and ϑ we have

$$\int_\Omega |\nabla u_\varepsilon|^p\, dx + \mathcal{H}^2(S(u_\varepsilon)) \le J_\varepsilon(u_\varepsilon)$$

if $\varepsilon \le 1$, from which we obtain that (u_ε) is pre-compact in $GSBV_p(\Omega; \mathbb{R}^3)$; hence, by Theorem 1.1 the function u belongs to $GSBV_p(\Omega; \mathbb{R}^3)$. Moreover, by the lower semicontinuity of

$$v \mapsto \int_\Omega |\nabla_3 v|^p\, dx, \qquad v \mapsto \int_{S(v) \cap \Omega} |\nu_3(v)|\, d\mathcal{H}^2$$

with respect to the weak $GSBV_p$-convergence, we obtain

$$\int_\Omega |\nabla_3 u|^p\, dx \le \liminf_{\varepsilon \to 0} \int_\Omega (1 + |u_\varepsilon^+ - u_\varepsilon^-|)\, |\nabla_3 u_\varepsilon|^p\, dx \le \liminf_{\varepsilon \to 0} \varepsilon^p J_\varepsilon(u_\varepsilon) = 0, \tag{2.7}$$

and so $u \in \mathcal{V}$. $\square$

Proposition 2.4 *We have $\Gamma\text{-}\liminf_{\varepsilon \to 0} J_\varepsilon \ge J_0$.*

Proof. Given sequences (ε_j) converging to 0^+ and (u_j) converging to u in $L^1(\Omega;\mathbb{R}^3)$ with $\liminf_{j\to+\infty} J_{\varepsilon_j}(u_j) < +\infty$, by the previous proposition we have that $u \in \mathcal{V}$, and we may consider weak $GSBV_p(\Omega;\mathbb{R}^3)$-convergence in place of $L^1(\Omega;\mathbb{R}^3)$-convergence.

Since

$$W(F) \geq \overline{W}(\overline{F}) \geq Q_2\overline{W}(\overline{F}),$$

and

$$\vartheta(z,\nu) \geq \overline{\vartheta}(z,\nu_\alpha) \geq R_2\overline{\vartheta}(z,\nu_\alpha),$$

we have

$$\begin{aligned}
&\liminf_{j\to+\infty} J_{\varepsilon_j}(u_j)\\
&\geq \liminf_{j\to+\infty} \int_{-1/2}^{1/2} \bigg(\int_\omega Q_2\overline{W}(\nabla_\alpha u_j)dx_\alpha\\
&\quad + \int_{S(u_j(\cdot,t))\cap\omega} R_2\overline{\vartheta}(u_j^+ - u_j^-, \nu(u_j(\cdot,t))(x_\alpha))\,|\nu_\alpha(u_j)(x_\alpha,t)| d\mathcal{H}^1(x_\alpha)\bigg)\,dt\\
&\geq \int_{-1/2}^{1/2} \left(\liminf_{j\to+\infty} \int_\omega Q_2\overline{W}(\nabla_\alpha u_j)dx_\alpha \right) dt\\
&\quad + \int_{-1/2}^{1/2} \left(\liminf_{j\to+\infty} \int_{S(u_j(\cdot,t))\cap\omega} R_2\overline{\vartheta}(u_j^+ - u_j^-, \nu(u_j(\cdot,t))(x_\alpha)) d\mathcal{H}^1(x_\alpha) \right) dt\\
&\quad - \limsup_{j\to+\infty} \int_{S(u_j)\cap\Omega} (1 + |u_j^+ - u_j^-|)|\nu_3(u_j)| d\mathcal{H}^2\\
&\geq J_0(u),
\end{aligned}$$

where we have used Fatou's Lemma, the homogeneity of ϑ, the facts that

$$\nu(u_j(\cdot,t))(x_\alpha) = \frac{\nu_\alpha(u_j)(x_\alpha,t)}{|\nu_\alpha(u_j)(x_\alpha,t)|} \text{ for } \mathcal{H}^1 \text{ a.e. } x_\alpha \in S(u(\cdot,t)),$$

property (2.7) together with the inequality $1 - |\nu_\alpha| \leq |\nu_3|$, and finally Theorem 1.1 with $u \in \mathcal{V}$. □

2.2 Proof of the upper bound

We start by establishing the existence of the Γ-limit and its integral representation.

Theorem 2.5 *For all $\varepsilon > 0$ let $W_\varepsilon : \Omega \times \mathbb{M}^{3\times 3} \to [0,+\infty)$ and $\vartheta_\varepsilon : \Omega \times \mathbb{R}^3 \times \mathbb{R}^3 \to [0,+\infty)$ be Borel functions satisfying*

$$|F|^p \leq W_\varepsilon(x,F) \leq C(1+|F|^p),$$

and

$$1+|z| \le \vartheta_\varepsilon(x,z,\nu) \le C(1+|z|).$$

Let $\mathcal{J}_\varepsilon(u,U)$ be defined by

$$\mathcal{J}_\varepsilon(u,U) := \begin{cases} \displaystyle\int_{U\times(-1/2,1/2)} W_\varepsilon\Big(x,\nabla_\alpha u\Big|\frac{1}{\varepsilon}\nabla_3 u\Big)\,dx \\ \displaystyle+\int_{S(u)\cap(U\times(-1/2,1/2))} \vartheta_\varepsilon\Big(x,u^+-u^-,\nu_\alpha(u),\frac{1}{\varepsilon}\nu_3(u)\Big)\,d\mathcal{H}^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u\in GSBV_p(\Omega;\mathbb{R}^3), \\ +\infty \qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases} \tag{2.8}$$

where U is an open subset of ω and $u \in L^p(\Omega;\mathbb{R}^3)$. Then for all (ε_j) converging to 0 there exists a subsequence, not relabeled, such that the Γ-limit

$$\Gamma\text{-}\lim_j \mathcal{J}_{\varepsilon_j}(u,U) =: \mathcal{J}_0(u,U)$$

exists for all $u\in\mathcal{V}$ and U open subset of ω. Moreover, $\mathcal{J}_0$ satisfies:

(i) *for all $u\in\mathcal{V}$ the set function $\mathcal{J}_0(u,\cdot)$ is the restriction of a Borel measure to the family of open subsets of ω;*

(ii) *if $u=v$ a.e. on an open set $U\subset\omega$ then $\mathcal{J}_0(u,U)=\mathcal{J}_0(v,U)$;*

(iii) *for all open sets $U\subset\omega$ the functional $\mathcal{J}_0(\cdot,U)$ is lower semicontinuous on $\mathcal{V}$ with respect to the $L^1(\omega;\mathbb{R}^3)$-convergence;*

(iv) *there exists a constant $C>0$ such that for all $u\in\mathcal{V}$ and every open set $U\subset\omega$*

$$0\le \mathcal{J}_0(u,U)\le C\Big(\int_U (1+|\nabla_\alpha u|^p)\,dx_\alpha + \int_{S(u)\cap U}(1+|u^+-u^-|)d\mathcal{H}^1\Big).$$

The result above follows from standard Γ-convergence arguments and subadditivite properties of $\mathcal{J}_\varepsilon$, which will be established in the proposition below.

Proposition 2.6 [Fundamental estimate] *Let $(\mathcal{J}_\varepsilon)$ be the family of functionals defined in (2.8). For every $\eta>0$ and for every U, U', V open subsets of ω with $U\subset\subset U'$, there exists a constant $M>0$ with the following property: for every $\varepsilon>0$ and for every $u\in SBV_p(U'\times(-1/2,1/2);\mathbb{R}^3)$, $v\in SBV_p(V\times(-1/2,1/2);\mathbb{R}^3)$ there exists a function $\varphi\in C_0^\infty(U')$ with $\varphi=1$ in a neighborhood of $\overline{U}$ and $0\le\varphi\le 1$ such that*

$$\mathcal{J}_\varepsilon(\varphi u+(1-\varphi)v,U\cup V)\le(1+\eta)\big(\mathcal{J}_\varepsilon(u,U')+\mathcal{J}_\varepsilon(v,V)\big)+M\|u-v\|^p_{L^p(S;\mathbb{R}^3)}+\eta\,,$$

where $S:=((U'\setminus U)\cap V)\times(-1/2,1/2)$.

Proof. Let $\eta>0$, and fix open sets U, U' and V as in the statement. Fix $k\in\mathbb{N}$ and let $U_1,\dots,U_{k+1}$ be open subsets of $\mathbb{R}^3$ such that $U\subset\subset U_1\subset\subset$

$U_2 \subset\subset \cdots \subset\subset U_{k+1} \subset\subset U'$. For every $i = 1, \ldots, k$, let φ_i be a cut-off function in $C_0^\infty(U_{i+1})$ with $\varphi_i = 1$ on a neighborhood V_i of $\overline{U_i}$. Fixed $\varepsilon > 0$, $u \in SBV_p(U' \times (-1/2, 1/2); \mathbb{R}^3)$, and $v \in SBV_p(V \times (-1/2, 1/2); \mathbb{R}^3)$, define on $(U' \cup V) \times (-1/2, 1/2)$ the function

$$w_i(x) := \varphi_i(x_\alpha)u(x) + (1 - \varphi_i(x_\alpha))v(x),$$

where u and v are extended arbitrarily outside the sets $U' \times (-1/2, 1/2)$ and $V \times (-1/2, 1/2)$, respectively. Then for $i = 1, \ldots, k$

$$\begin{aligned}
\mathcal{J}_\varepsilon(w_i, U \cup V) &\leq \mathcal{J}_\varepsilon(u, (U \cup V) \cap V_i) + \mathcal{J}_\varepsilon(v, V \setminus \mathrm{spt}\varphi_i) \\
&\quad + \mathcal{J}_\varepsilon(w_i, V \cap (U_{i+1} \setminus \overline{U_i})) \\
&\leq \mathcal{J}_\varepsilon(u, U') + \mathcal{J}_\varepsilon(v, V) + \mathcal{J}_\varepsilon(w_i, V \cap (U_{i+1} \setminus \overline{U_i}))\,. \qquad (2.9)
\end{aligned}$$

Set $T_i := V \cap (U_{i+1} \setminus \overline{U_i})$. We estimate the last term in (2.9) as

$$\begin{aligned}
\mathcal{J}_\varepsilon(w_i, T_i) &\leq C \Bigg[\int_{T_i \times (-1/2,1/2)} (1 + |\nabla w_i|^p)\, dx \\
&\quad + \int_{S(w_i) \cap (T_i \times (-1/2,1/2))} (1 + |w_i^+ - w_i^-|)\, d\mathcal{H}^2 \Bigg] \\
&\leq C \Bigg[|T_i| + \int_{T_i \times (-1/2,1/2)} |\varphi_i \nabla u + (1 - \varphi_i)\nabla v + (\nabla_\alpha \varphi_i \otimes (u - v)|0)|^p dx \\
&\quad + \int_{S(u) \cap (T_i \times (-1/2,1/2))} (1 + |u^+ - u^-|)\, d\mathcal{H}^2 \\
&\quad + \int_{S(v) \cap (T_i \times (-1/2,1/2))} (1 + |v^+ - v^-|)\, d\mathcal{H}^2 \Bigg] \\
&\leq C \Bigg[|T_i| + \int_{T_i \times (-1/2,1/2)} (|\nabla u|^p + |\nabla v|^p + |\nabla_\alpha \varphi_i|^p |u - v|^p)\, dx \Big) \\
&\quad + \int_{S(u) \cap (T_i \times (-1/2,1/2))} (1 + |u^+ - u^-|)\, d\mathcal{H}^2 \\
&\quad + \int_{S(v) \cap (T_i \times (-1/2,1/2))} (1 + |v^+ - v^-|)\, d\mathcal{H}^2 \Bigg] \\
&\leq C \Bigg[\int_{T_i \times (-1/2,1/2)} W_\varepsilon(x, \nabla u)\, dx + \int_{T_i \times (-1/2,1/2)} W_\varepsilon(x, \nabla v)\, dx \\
&\quad + \int_{S(u) \cap (T_i \times (-1/2,1/2))} \vartheta_\varepsilon(x, u^+ - u^-, \nu(u))\, d\mathcal{H}^2 \\
&\quad + \int_{S(v) \cap (T_i \times (-1/2,1/2))} \vartheta_\varepsilon(x, v^+ - v^-, \nu(v))\, d\mathcal{H}^2 \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&\quad +C\left(|T_i| + \|\nabla_\alpha \varphi_i\|^p_{L^\infty}\int_{T_i\times(-1/2,1/2)}|u-v|^p\,dx\right)\\
&\leq \quad C(\mathcal{J}_\varepsilon(u,T_i)+\mathcal{J}_\varepsilon(v,T_i))\\
&\quad +C\left(|T_i| + \max_{1\leq i\leq k}\|\nabla_\alpha \varphi_i\|^p_{L^\infty}\int_{T_i\times(-1/2,1/2)}|u-v|^p\,dx\right).
\end{aligned}
$$

Hence there exists $i_0 \in \{1,\ldots,k\}$ such that

$$
\begin{aligned}
\mathcal{J}_\varepsilon(w_{i_0},T_{i_0}) &\leq \frac{1}{k}\sum_{i=0}^{k}\mathcal{J}_\varepsilon(w_i,T_i)\\
&\leq \frac{C}{k}\big(\mathcal{J}_\varepsilon(u,U')+\mathcal{J}_\varepsilon(v,V)\big)+\frac{C}{k}|S|+M\int_S|u-v|^p\,dx
\end{aligned}
$$

where $M = O(k^p)$ and $S := ((U'\setminus U)\cap V)\times(-1/2,1/2)$. From (2.9) it follows that

$$
\mathcal{J}_\varepsilon(w_{i_0},U\cup V)\leq\left(1+\frac{C}{k}\right)\big(\mathcal{J}_\varepsilon(u,U')+\mathcal{J}_\varepsilon(v,V)\big)+\frac{C}{k}|S|+M\int_S|u-v|^p\,dx\,.
$$

It suffices to choose k large enough. □

PROOF OF THEOREM 2.5. The proof follows by using the standard 'direct methods' of Γ-convergence (described in their general features in [23] (Chapters 16-19) and [18] (Chapters 10-11)), with obvious adaptations to the SBV_p setting as in [19], and to the thin films framework (see [18] Section 24.3 and [20]). We just outline the necessary steps. First, using the previous proposition the thesis of Theorem 2.5 is established on $\mathcal{V}\cap L^p(\Omega;\mathbb{R}^3)$ considering the $L^p(\Omega;\mathbb{R}^3)$ convergence in place of the $L^1(\Omega;\mathbb{R}^3)$ convergence (see [19] Proposition 3.3 and [18] Theorem 24.20). Then, a cut-off argument (see [19] Lemma 3.5) allows to obtain the thesis using the $L^1(\Omega;\mathbb{R}^3)$ convergence (see [19] Proposition 3.4 and the last steps in the proof of Theorem 2.3). Note that property (ii) follows directly from the definition of Γ-limit, property (iii) is a general property of Γ-limits, and (iv) is obtained from the growth condition taking $u_\varepsilon := u$ in the definition of Γ-limsup. □

We can apply the compactness result above and the integral representation theorems by Braides and Chiadò Piat [16] to our functionals, obtaining the following proposition.

Proposition 2.7 [Integral representation] *Let W and ϑ satisfy the hypotheses of Theorem 2.1, and let J_ε be defined by (2.2). Suppose, in addition, that W satisfies the Lipschitz continuity condition*

$$
|W(F)-W(F')|\leq L(1+|F|^{p-1}+|F'|^{p-1})|F-F'|, \tag{2.10}
$$

for all $F, F' \in \mathbb{M}^{3\times 3}$, and for some $L > 0$. Then every sequence (ε_j) converging to 0 admits a a subsequence, not relabeled, such that the Γ-limit

$$\Gamma\text{-}\lim_j J_{\varepsilon_j}(u, U) =: \mathcal{J}_0(u, U)$$

exists for all $u \in \mathcal{V}$ and U open subset of ω. Moreover, there exist W_0 and ϑ_0 such that

$$\mathcal{J}_0(u, U) = \int_{U\times(-1/2,1/2)} W_0(\nabla_\alpha u)\, dx + \int_{S(u)\cap(U\times(-1/2,1/2))} \vartheta_0(u^+ - u^-, \nu(u)) d\mathcal{H}^2$$

for all $u \in \mathcal{V}$ and for all U open subset of ω.

Proof. We apply Theorem 2.5 with

$$W_\varepsilon(x, F) := W(F), \quad \vartheta_\varepsilon(x, z, \nu) := W(z, \nu),$$

to find a corresponding Γ-limit functional $\mathcal{J}_0(u, U)$, for all $u \in SBV_p(\omega; \mathbb{R}^3)$. We denote by $\mathcal{J}_0(u, \cdot)$ its extension as a Borel measure on ω. Then $\mathcal{J}_0$ satisfies properties (i)–(iv) of Theorem 2.5, and, in addition, we have

(v) $\mathcal{J}_0(u + s, U) = \mathcal{J}_0(u, U)$ for all $s \in \mathbb{R}^3$;

(vi) for all $u \in SBV_p(\omega; \mathbb{R}^3)$ and Φ affine transformation in $\mathbb{R}^3$

$$|\mathcal{J}_0(u, S(u)\cap U) - \mathcal{J}_0(\Phi(u), S(u)\cap U)| \le CL\|\Phi - Id\|_{1,\infty} \int_{S(u)\cap U} (1+|u^+ - u^-|)\, d\mathcal{H}^1.$$

Property (v) follows immediately from the translation invariance of J_ε. To prove (iv) we will exploit the Lipschitz continuity of ϑ (see (2.5)). In fact, let V be an open subset of ω, let $u \in SBV_p(\omega; \mathbb{R}^3)$, and let $u_j \to u$ in $L^1(V \times (-1/2, 1/2); \mathbb{R}^3)$ be such that

$$\lim_j J_{\varepsilon_j}(u_j, V) = \mathcal{J}_0(u, V).$$

By definition of Γ-liminf, we have

$$\begin{aligned}
&\mathcal{J}_0(\Phi(u), V) \le \liminf_j J_{\varepsilon_j}(\Phi(u_j), V)\\
&= \liminf_j \Bigg(\int_{V\times(-1/2,1/2)} W\Big(\nabla\Phi\Big(\nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j\Big)\Big) dx\\
&\quad + \int_{S(u_j)\cap(V\times(-1/2,1/2))} \vartheta\Big(\Phi(u_j^+) - \Phi(u_j^-), \nu_\alpha(u_j), \frac{1}{\varepsilon_j}\nu_3(u_j)\Big)\, d\mathcal{H}^2 \Bigg]\\
&\le \liminf_j \Bigg[\int_{V\times(-1/2,1/2)} W\Big(\nabla_\alpha u_j \Big| \frac{1}{\varepsilon_j}\nabla_3 u_j\Big)\, dx
\end{aligned}$$

$$
\begin{aligned}
&+L\|\nabla\Phi - I\|_\infty \int_{V\times(-1/2,1/2)} \Big(1 + |\nabla_\alpha u_j|^p + \Big|\frac{1}{\varepsilon_j}\nabla_3 u_j\Big|^p\Big)\,dx \\
&+ \int_{S(u_j)\cap(V\times(-1/2,1/2))} \vartheta\Big(u_j^+ - u_j^-, \nu_\alpha(u_j), \frac{1}{\varepsilon_j}\nu_3(u_j)\Big)\,d\mathcal{H}^2\Big) \\
&+ L\|\Phi - Id\|_\infty \int_{S(u_j)\cap(V\times(-1/2,1/2))} |u_j^+ - u_j^-|\sqrt{|\nu_\alpha(u_j)|^2 + \frac{1}{\varepsilon_j^2}|\nu_3(u_j)|^2}\,d\mathcal{H}^2\Big] \\
&\leq \lim_j J_{\varepsilon_j}(u_j, V) + L\|\Phi - Id\|_{1,\infty}(J_{\varepsilon_j}(u_j, V) + |V|) \\
&\leq \mathcal{J}_0(u, V) + CL\|\Phi - Id\|_{1,\infty}\Big(\int_V (1 + |\nabla_\alpha u|^p)\,dx_\alpha \\
&+ \int_{S(u)\cap V} (1 + |u^+ - u^-|)\,d\mathcal{H}^1\Big),
\end{aligned}
$$

and where we have used (2.5) and (2.10). Letting $V \to S(u)\cap U$, and using the fact that $\mathcal{J}_0(u,\cdot)$ is the trace of a Radon measure on the open sets of ω, we obtain

$$
\mathcal{J}_0(\Phi(u), S(u)\cap U) - \mathcal{J}_0(u, S(u)\cap U) \leq L\|\Phi - Id\|_{1,\infty} \int_{S(u)\cap U} (1 + |u^+ - u^-|)\,d\mathcal{H}^1,
$$

and (vi) follows by a symmetric argument.

Conditions (i)–(vi) above assert hypotheses (i)–(v) and (vi)' of Corollary 2.8, Remark 2.5 and Lemma 6.2 in [16], whose thesis is the existence of Carathéodory functions W_0 and ϑ_0 such that

$$
\mathcal{J}_0(u, B) = \int_B W_0(x_\alpha, u, \nabla_\alpha u)\,dx_\alpha + \int_{B\cap S(u)} \vartheta_0(x_\alpha, u^+, u^-, \nu(u))\,d\mathcal{H}^1
$$

for all $u \in SBV_p(\omega;\mathbb{R}^3)$ and for all Borel subsets B of ω. Condition (v) further implies that

$$
W_0(x, z, F) = W_0(x, F), \qquad \vartheta_0(x, u^+, u^-, \nu) = \vartheta_0(x, u^+ - u^-, \nu).
$$

Finally, as W and ϑ are independent of x, we have

$$
W_0(x, z, F) = W_0(F), \qquad \vartheta_0(x, u^+, u^-, \nu) = \vartheta_0(u^+ - u^-, \nu),
$$

and the proof of the proposition is concluded. □

We now provide some upper bound estimates for the value of the Γ-limit on affine functions and on piecewise-constant functions jumping across a hyperplane.

Proposition 2.8 [Estimate for the bulk energy density] *We have*

$$
\Gamma\text{-}\limsup_{\varepsilon\to 0} J_\varepsilon(\overline{F}x_\alpha, U) \leq |U|\,\overline{W}(\overline{F})
$$

for all U bounded open subsets of ω and $\overline{F} \in \mathbb{M}^{3\times 2}$.

PROOF. Let $\eta > 0$ and let $F_3 \in \mathbb{R}^3$ be such that $W(\overline{F}|F_3) \leq \overline{W}(\overline{F}) + \eta$. The claim follows trivially by taking $u_\varepsilon(x) = \overline{F}x_\alpha + \varepsilon F_3 x_3$ as a test function. □

To estimate the surface energy density, we test the Γ-limit on piecewise constant functions which jump across a line. Define the function $u_y^{z,\nu}$ by

$$u_y^{z,\nu}(x) = \begin{cases} z/2 & \text{if } \langle x - y, \nu \rangle > 0, \\ \\ -z/2 & \text{if } \langle x - y, \nu \rangle \leq 0, \end{cases}$$

and let $\Pi_y^\nu = \{x \in \mathbb{R}^2 : \ \langle x - y, \nu \rangle = 0\}$.

Proposition 2.9 [Estimate for the surface energy density] *We have*

$$\Gamma\text{-}\limsup_{\varepsilon \to 0} J_\varepsilon(u_y^{z,\nu}, U) \leq \mathcal{H}^1(U \cap \Pi_y^\nu)\, \overline{\vartheta}(z, \nu) + C|U|$$

for all bounded open subsets U of ω, $y \in \omega$, $z \in \mathbb{R}^3$, and $\nu \in S^1$.

PROOF. Up to a translation, we can suppose that $y = 0$. Let $\Pi^\nu := \Pi_0^\nu$. Let $\eta > 0$, and let $\overline{\nu}_3 \in \mathbb{R}$ be such that

$$\vartheta(z, \nu, \overline{\nu}_3) \leq \overline{\vartheta}(z, \nu) + \eta.$$

The estimate with $C = W(0)$ then follows by taking

$$u_\varepsilon(x) := \begin{cases} z/2 & \text{if } \langle x, (\nu, \varepsilon\overline{\nu}_3) \rangle > 0, \\ \\ -z/2 & \text{if } \langle x, (\nu, \varepsilon\overline{\nu}_3) \rangle \leq 0, \end{cases}$$

and by the arbitrariness of $\eta > 0$. □

2.3 Proof of Theorem 2.1

In view of Propositions 2.7, 2.8 and 2.9, from every sequence (ε_j) of positive numbers converging to 0 we can extract a further subsequence (not relabelled), and find a functional $\mathcal{J}_0$ with densities W_0 and ϑ_0 such that for all $u \in \mathcal{V}$

$$\begin{aligned} \Gamma\text{-}\limsup_{j \to +\infty} J_{\varepsilon_j}(u) &= \int_\Omega W_0(\nabla_\alpha u)dx + \int_{S(u)\cap\Omega} \vartheta_0(u^+ - u^-, \nu(u))d\mathcal{H}^2 \\ &\leq \int_\omega \overline{W}(\nabla_\alpha u)dx_\alpha + \int_{S(u)\cap\omega} \overline{\vartheta}(u^+ - u^-, \nu_\alpha(u))d\mathcal{H}^1. \end{aligned}$$

This, together with Proposition 2.4, yields

$$Q_2\overline{W} \leq W_0 \leq \overline{W}, \quad R_2\overline{\vartheta} \leq \vartheta_0 \leq \overline{\vartheta}.$$

By virtue of Theorem 1.3, applied to the functional $\mathcal{J}_0$, we have

$$Q_2 W_0 = W_0, \quad R_2 \vartheta_0 = \vartheta_0,$$

so we may deduce that

$$W_0 \leq Q_2 \overline{W}, \ \textit{and similarly} \ \vartheta_0 = R_2 \overline{\vartheta}.$$

The result of the theorem follows by definition of Γ-limit and by virtue of the arbitrariness of the sequence (ε_j). □

Example 2.10 We may consider thin films of a hyperelastic material obeying to Griffith theory, with underlying elastic energy given by

$$\int_{\Omega_\varepsilon} W(\nabla u)\, dx + \lambda \mathcal{H}^2(S(u) \cap \Omega_\varepsilon),$$

where λ is a positive constant. In this case we have

$$\vartheta(z, \nu) = \lambda |\nu| = \lambda \sqrt{|\nu_\alpha|^2 + \nu_3^2},$$

so that $\overline{\vartheta}(\nu) = \vartheta(\nu) = \lambda|\nu|$, and the thin-film limit is simply given by

$$\int_\omega Q\overline{W}(\nabla_\alpha u)\, dx_\alpha + \lambda \mathcal{H}^1(S(u) \cap \omega).$$

This result does not follow immediately from Theorem 2.1, but can be easily deduced from it. In fact, a lower estimate for the Γ-limit is still valid by Proposition 2.4, while the upper estimate can be deduced by considering the auxiliary energies

$$\int_{\Omega_\varepsilon} W(\nabla u)\, dx + \lambda \int_{S(u)\cap\Omega_\varepsilon} \vartheta_M(u^+ - u^-)\, d\mathcal{H}^2,$$

where $\vartheta_M(z) = (|z| - M) \vee 1$, with arbitrary $M > 0$, and applying Theorem 2.1 to the scaled versions of these energies (a similar approach is considered in detail in [19]).

3 The case $p = 1$

In the case where $p = 1$ it is well known that the analogues of Theorem 1.1 and 1.3 do not hold. Γ-limit and relaxation problems must be described in the whole space of functions of bounded variation, and in general there is an interaction phenomenon between the bulk and interfacial energy densities. In this section we show that we can still deal with the thin-film limit in this case by combining the description given for $p = 1$ and a relaxation result for functionals in BV, established by Bouchitté, Fonseca and Mascarenhas [14] Section 4.2 (see also [12], [17]).

3.1 Relaxation in BV

Let $W : \mathbb{M}^{m\times N} \to [0,+\infty)$ and $\vartheta : \mathbb{R}^m \times S^{N-1} \to [0,+\infty)$ be continuous functions. We recall that the *recession function of* f is defined by

$$W^\infty(F) := \limsup_{t\to+\infty} \frac{W(tF)}{t}.$$

If W is Lipschitz (respectively, quasiconvex) and has linear growth (see (H1) below), then W^∞ is a Lipschitz (respectively, quasiconvex), positively 1-homogeneous function (see [28]). Moreover, we define the positively 1-homogeneous function

$$\vartheta_0(z,\nu) := \limsup_{t\to 0^+} \frac{\vartheta(tz,\nu)}{t}.$$

We suppose that W and ϑ satisfy the following hypotheses:

(H1) there exists $C > 0$ such that

$$\frac{1}{C}|F| \le W(x,F) \le C(1+|F|)$$

for all $F \in \mathbb{M}^{m\times N}$;

(H2) there exist constants $C, L > 0,\ 0 < r < 1$, such that

$$\left| W^\infty(F) - \frac{W(tF)}{t} \right| \le C\frac{1}{t^r}$$

for every $F \in \mathbb{M}^{m\times N}$ with $|F| = 1$ and for all $t > 0$ such that $t > L$;

(H3) there exists $C > 0$ such that

$$\frac{1}{C}|z| \le \vartheta(z,\nu) \le C|z|$$

for all $(z,\nu) \in \mathbb{R}^m S^{N-1}$;

(H4) there exist constants $C, l, s > 0$ such that

$$\left| \vartheta_0(z,\nu) - \frac{\vartheta(tz,\nu)}{t} \right| \le Ct^s$$

for every $(z,\nu) \in \mathbb{R}^m \times S^{N-1}$ with $|z| = 1$ and for all t such that $0 < t < l$.

Let $(z,\nu) \in \mathbb{R}^m \times S^{N-1}$, let $\{\nu_1,\dots,\nu_{N-1},\nu\}$ form an orthonormal basis of $\mathbb{R}^N$, and define the class of admissible functions

$$\mathcal{A}(z,\nu) := \Big\{ v \in SBV_{\rm loc}(S_\nu;\mathbb{R}^m) : v(y) = 0 \text{ if } \langle y,\nu\rangle = -\frac{1}{2},\ v(y) = z \text{ if } \langle y,\nu\rangle = \frac{1}{2},$$
$$v \text{ is 1-periodic in the directions of } \nu_1,\dots,\nu_{N-1} \Big\},$$

where S_ν is the strip

$$S_\nu := \left\{ y \in \mathbb{R}^N \, : \, |\langle y, \nu \rangle| < \frac{1}{2} \right\},$$

and where the boundary values of v are understood in the sense of traces.

We use the notation

$$\int_\Omega W(dC_u) := \int_\Omega W\Big(\frac{dC_u}{d|C_u|}\Big) d|C_u|,$$

where $|C_u|$ is the total variation measure of C_u.

Theorem 3.1 *Under hypotheses* (H1)–(H4)*, with W Lipschitz continuous, if $u \in BV(\Omega;\mathbb{R}^m)$ then the lower semicontinuous envelope $\mathcal{F}$ with respect to the L^1 convergence of the functional*

$$u \in SBV(\Omega;\mathbb{R}^m) \mapsto \int_\Omega W(\nabla u)\ dx + \int_{S(u)\cap\Omega} \vartheta(u^+ - u^-, \nu) d\mathcal{H}^{N-1}, \tag{3.11}$$

is given by

$$\mathcal{F}(u) = \int_\Omega g(\nabla u) dx + \int_{S(u)\cap\Omega} h(u^+ - u^-, \nu) d\mathcal{H}^{N-1} + \int_\Omega g^\infty(dC_u),$$

where

$$\begin{aligned} g(A) \quad := \quad \inf \Bigg\{ & \int_{(0,1)^N} W(\nabla u) dx + \int_{S(u)\cap(0,1)^N} \vartheta_0(u^+ - u^-, \nu) d\mathcal{H}^{N-1} : \\ & u \in SBV((0,1)^N;\mathbb{R}^m), u\,|_{\partial(0,1)^N}\,(x) = Ax \Bigg\}, \end{aligned} \tag{3.12}$$

and

$$h(z,\nu) := \inf \left\{ \int_{Q_\nu} W^\infty(\nabla u(x)) dx + \int_{S(u)\cap Q_\nu} \vartheta(u^+ - u^-, \nu) d\mathcal{H}^{N-1} : u \in \mathcal{A}(z,\nu) \right\}. \tag{3.13}$$

3.2 Thin-film limit

Now we state the thin-film approximation result, where $\overline{W}$ and $\overline{\vartheta}$ are as defined in Theorem 2.1,

$$\begin{aligned} g_2(\overline{F}) \quad := \quad \inf \Bigg\{ & \int_{(0,1)^2} \overline{W}(\nabla_\alpha u)\ dx_\alpha + \int_{S(u)\cap(0,1)^2} (\overline{\vartheta})_0(u^+ - u^-, \nu_\alpha) d\mathcal{H}^1 : \\ & u \in SBV((0,1)^2;\mathbb{R}^3), u\,|_{\partial(0,1)^2}\,(x) = \overline{F}x \Bigg\}, \end{aligned}$$

$$
\begin{aligned}
g_3(\overline{F}) \quad := \quad \inf \Bigg\{ \int_{(0,1)^3} \overline{W}(\nabla_\alpha u)\ dx + \int_{S(u)\cap(0,1)^3} (\overline{\vartheta})_0(u^+ - u^-, \nu_\alpha) d\mathcal{H}^2 : \\
u \in SBV((0,1)^3;\mathbb{R}^3), u\,|_{\partial(0,1)^3}\,(x) = \overline{F}x \Bigg\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
h_2(z,\eta) \quad &:= \quad \inf \Bigg\{ \int_{Q'_\eta} \overline{W}^\infty(\nabla_\alpha u(x)) dx_\alpha \\
&\qquad + \int_{S(u)\cap Q'_\eta} \overline{\vartheta}(u^+ - u^-, \eta) d\mathcal{H}^1 : u \in \mathcal{A}'(z,\eta) \Bigg\}, \\
h_3(z,\eta) \quad &:= \quad \inf \Bigg\{ \int_{Q_{(\eta,0)}} \overline{W}^\infty(\nabla_\alpha u(x)) dx \\
&\qquad + \int_{S(u)\cap Q_{(\eta,0)}} \overline{\vartheta}(u^+ - u^-, \eta) d\mathcal{H}^2 : u \in \mathcal{A}(z,(\eta,0)) \Bigg\},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}'(z,\eta) := \Bigg\{ v \in SBV_{\mathrm{loc}}(S_\nu \cap \omega;\mathbb{R}^m) : v(y) = 0 \text{ if } \langle y,\eta\rangle = -\frac{1}{2}, \\
v(y) = z \text{ if } \langle y,\eta\rangle = \frac{1}{2}, v \text{ is 1-periodic in the direction of } \rho \Bigg\},
\end{aligned}
$$

with ρ orthogonal to η on the plane.

Theorem 3.2 *Let $W : \mathbb{M}^{3\times 3} \to [0,+\infty)$ and $\vartheta : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3 \to [0,+\infty)$ be continuous functions, and let ϑ be symmetric and positively homogeneous of degree 1 in the second variable; i.e.,*

$$
\vartheta(z,\nu) = \vartheta(-z,-\nu), \qquad \vartheta(z,t\nu) = t\vartheta(z,\nu) \text{ for all } t > 0,\ z,\nu \in \mathbb{R}^3.
$$

Suppose that W and ϑ satisfy the growth and coercivity conditions

$$
|F| \le W(F) \le C(1+|F|), \qquad |z| \le \vartheta(z,\nu) \le C|z|
$$

for all $F \in \mathbb{M}^{3\times 3}$, $z \in \mathbb{R}^3$, $\nu \in S^2$, and for some $C > 0$. Let W and ϑ satisfy (H2) *and* (H4)*, respectively. The functionals J_ε defined in* (2.2)*, with $GSBV_p(\Omega;\mathbb{R}^m)$ replaced by $SBV(\Omega;\mathbb{R}^m)$, Γ-converge with respect to the $L^1(\Omega;\mathbb{R}^3)$ convergence as $\varepsilon \to 0^+$ to the functional $J_0 : L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by*

$$
J_0(u) = \begin{cases} \displaystyle\int_\omega g_2(\nabla_\alpha u)\, dx_\alpha + \int_{S(u)\cap\omega} h_2(u^+ - u^-, \nu_\alpha(u))\, d\mathcal{H}^1 + \int_\Omega g_2^\infty(dC_u) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u \in BV(\Omega;\mathbb{R}^m),\ D_3 u = 0 \\ +\infty \qquad\qquad\qquad\qquad\qquad \textit{otherwise.} \end{cases} \tag{3.14}
$$

Remark 3.3 Note that, contrary to the case $p > 1$, we do not assume any Lipschitz continuity on ϑ. This discrepancy is not surprising, since BV-ellipticity combined with the growth conditions at 0 does imply the Lipschitz continuity of ϑ, and hence this condition can be overcome by a relaxation argument, which is implicit in [12]. In some special cases it is possible to apply other relaxation results, which do not assume hypotheses (H2) and (H4) (see [13] for the scalar isotropic case and [17] when ϑ is positively homogeneous of degree 1).

PROOF. The reasoning of the proof is close to the one used for $p > 1$. We briefly indicate some of the necessary changes.

Note that, since $J_\varepsilon(u) \geq |D_\alpha u|(\Omega) + \frac{1}{\varepsilon}|D_3 u|(\Omega)$, an argument similar to that of Proposition 2.3 yields trivially that $J_0(u)$ is finite only on functions $u \in BV(\Omega;\mathbb{R}^m)$ with $D_3 u = 0$.

Also, the proof of Proposition 2.4 can be reproduced in this context as follows: by Theorem 3.1 and for every $\delta > 0$ the functional in $BV(\Omega;\mathbb{R}^m)$

$$u \mapsto \int_\Omega g_\delta(\nabla_\alpha u)\ dx + \int_{S(u)\cap\Omega} h_\delta(u^+ - u^-, \nu_\alpha) d\mathcal{H}^2 + \int_\Omega g_\delta^\infty(dC_u)$$

is lower semicontinuous in L^1, where g_δ and h_δ are, respectively, the bulk and interfacial densities of the relaxed energy according to Theorem 3.1 and corresponding to

$$F \mapsto \overline{W}(\overline{F}) + \delta|F_3|, \quad (z,\nu) \mapsto \overline{\vartheta}(z,\nu_\alpha) + \delta|\nu_3|.$$

Therefore, if (ε_j) is a sequence converging to zero and (u_j) converges to u in L^1 with

$$\liminf_{\varepsilon_j} J_{\varepsilon_j}(u_j) < +\infty,$$

without loss of generality we may assume that

$$\lim_{\varepsilon_j} J_{\varepsilon_j}(u_j) = \liminf_{\varepsilon_j} J_{\varepsilon_j}(u_j), \quad \sup_j |D_3 u_j|(\Omega) \leq C < +\infty,$$

and we have

$$\begin{aligned}
J_0(u) &\leq \int_\Omega g_3(\nabla_\alpha u)\ dx + \int_{S(u)\cap\Omega} h_3(u^+ - u^-, \nu_\alpha) d\mathcal{H}^2 + \int_\Omega g_3^\infty(dC_u) \\
&\leq \liminf_{\delta\to 0} \left[\int_\Omega g_\delta(\nabla u)\ dx + \int_{S(u)\cap\Omega} h_\delta(u^+ - u^-, \nu) d\mathcal{H}^2 + \int_\Omega g_\delta^\infty(dC_u)\right] \\
&\leq \liminf_{\delta\to 0} \liminf_{\varepsilon_j\to 0} \left[\int_\Omega g_\delta(\nabla u_j)\ dx + \int_{S(u_j)\cap\Omega} h_\delta(u_j^+ - u_j^-, (\nu_j)) d\mathcal{H}^2\right] \\
&\leq \liminf_{\delta\to 0} \liminf_{\varepsilon_j\to 0} \big(J_{\varepsilon_j}(u_j) + \delta\,|D_3 u_j|(\Omega)\big) \\
&= \liminf_{\varepsilon_j\to 0} J_{\varepsilon_j}(u_j).
\end{aligned} \tag{3.15}$$

The next step is aimed at proving the existence of Γ-limits $\mathcal{J}_0$ for subsequences of arbitrarily chosen subsequences. As before, this relies on the fundamental estimate which may be obtained exactly as in Proposition 2.6. The argument of the proof of Theorem 2.5 is now somewhat simplified as there is no need to reduce from L^p to L^1 convergence. We obtain that J_0 satisfies, upon identifying $\{u \in BV(\Omega;\mathbb{R}^m),\ D_3 u = 0\}$ with $BV(\omega;\mathbb{R}^3)$,

(i) for all $v \in BV(\omega;\mathbb{R}^3)$ the set function $\mathcal{J}_0(u,\cdot)$ is the restriction of a Borel measure to the family of open subsets of ω;

(ii) if $u = v$ a.e. on an open set $U \subset \omega$ then $\mathcal{J}_0(u,U) = \mathcal{J}_0(v,U)$;

(iii) for all open sets $U \subset \omega$ the functional $\mathcal{J}_0(\cdot,U)$ is lower semicontinuous on $BV(\omega;\mathbb{R}^3)$ with respect to the $L^1(\omega;\mathbb{R}^3)$-convergence;

(iv) for all $v \in BV(\omega;\mathbb{R}^3)$ and open sets $U \subset \omega$

$$|Dv|(U) \le \mathcal{J}_0(u,U) \le C(|U| + |Dv|(U)).$$

Furthermore, after assuming a Lipschitz condition on ϑ (which is not restrictive up to the relaxation argument in the proof of Theorem 2.1), the same argument as in the proof of Theorem 2.7 shows that $\mathcal{J}_0$ depends continuously on vertical translations, and is independent of horizontal translations; more precisely,

(v) $\mathcal{J}_0(u+s,U) = \mathcal{J}_0(u,U)$ for all $s \in R^3$;

(vi) for all $u \in BV(\omega;\mathbb{R}^3)$ and Φ affine transformation in $\mathbb{R}^3$

$$|\mathcal{J}_0(u,U) - \mathcal{J}_0(\Phi(u),\cap U)| \le CL\|\Phi - Id\|_{1,\infty}\left(|U| + |Du|(U)\right).$$

Conditions (i)–(vi) imply that $\mathcal{J}_0$ satisfies the hypotheses of the integral representation Theorem 3.12 in [14], so that $\mathcal{J}_0$ can be represented on $BV(\omega;\mathbb{R}^m)$ as

$$\mathcal{J}_0(u,U) = \int_U W_0(\nabla_\alpha u)dx_\alpha + \int_{S(u)\cap U} \vartheta_0(u^+ - u^-,\nu_\alpha)d\mathcal{H}^2 + \int_U W_0^\infty(dC_u),$$

for some W_0 and ϑ_0.

Clearly Propositions 2.8 and 2.9 still hold, so that we get $W_0 \le \overline{W}$ and $\vartheta_0 \le \overline{\vartheta}$. As $\overline{W}$ and $\overline{\vartheta}$ satisfy the hypotheses of Theorem 3.1, and $\mathcal{J}_0$ is equal to its own relaxation on $BV(\omega;\mathbb{R}^3)$, we conclude that

$$W_0 \le g_2, \quad \vartheta_0 \le h_2,$$

which, together with (3.15), concludes the proof. □

Remark 3.4 Since $g_2 \le Q_2\overline{W} \le \overline{W}$ and $h_2 \le R_2\overline{\vartheta} \le \overline{\vartheta}$, then J_0 is the lower semicontinuous envelope of the functional defined by

$$\begin{cases} \displaystyle\int_\omega Q_2\overline{W}(\nabla_\alpha u)\,dx_\alpha + \int_{S(u)\cap\omega} R_2\overline{\vartheta}(u^+ - u^-,\nu_\alpha(u))\,d\mathcal{H}^1 & \text{if } u \in SBV(\omega;\mathbb{R}^3) \\ +\infty & \text{otherwise;} \end{cases} \tag{3.16}$$

that is, the one obtained in Theorem 2.1. Hence, the case $p = 1$ may be seen as giving the same result, up to a final relaxation.

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References

[1] L. Ambrosio, A compactness theorem for a new class of functions of bounded variation, *Boll. Un. Mat. Ital.* **3-B** (1989), 857–881.

[2] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291–322.

[3] L. Ambrosio, Variational problems in SBV and image segmentation, *Acta Appl. Math.* **17** (1989), 1–40.

[4] L. Ambrosio, On the lower semicontinuity of quasi-convex integrals defined in $SBV(\Omega;\mathbb{R}^k)$, *Nonlinear Anal.* **23** (1994), 405–425.

[5] L. Ambrosio and A. Braides, Functionals defined on partitions of sets of finite perimeter, I: integral representation and Γ-convergence, *J. Math. Pures. Appl.* **69** (1990), 285–305.

[6] L. Ambrosio and A. Braides, Functionals defined on partitions of sets of finite perimeter, II: semicontinuity, relaxation and homogenization, *J. Math. Pures. Appl.* **69** (1990), 307–333.

[7] L. Ambrosio and A. Braides, Energies in SBV and variational models in fracture mechanics, *Homogenization and Applications to Material Sciences*, (D. Cioranescu, A. Damlamian, P. Donato eds.), GAKUTO, Gakkōtosho, Tokio, Japan, 1997, p. 1-22.

[8] L. Ambrosio, N. Fusco and D. Pallara, *Special Functions of Bounded Variation and Free Discontinuity Problems*, Mathematical Monographs, Oxford University Press, 2000.

[9] G. Anzellotti, S. Baldo and D. Percivale, Dimension-reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity, *Asymptotic Anal.* **9** (1994), 61–100.

[10] J. M. Ball and F. Murat, $W^{1,p}$ quasiconvexity and variational problems for multiple integrals, *J. Funct. Anal.* **58** (1984), 225–253.

[11] G. I. Barenblatt, The mathematical theory of equilibrium cracks in brittle fracture, *Adv. Appl. Mech.* **7** (1962), 55–129.

[12] A. C. Barroso, G. Bouchitté, G. Buttazzo and I. Fonseca, Relaxation in $BV(\Omega;\mathbb{R}^p)$ of energies involving bulk and surface energy contributions, *Arch. Rat. Mech. Anal.* **135** (1996), 107–173.

[13] G. Bouchitté, A. Braides and G. Buttazzo, Relaxation results for some free discontinuity problems, *J. Reine Angew. Math.* **458** (1995), 1–18.

[14] G. Bouchitté, I. Fonseca and L. Mascarenhas, A global method for relaxation *Arch Rational Mech Anal* **145** (1998) 1, 51–98

[15] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1998.

[16] A. Braides and V. Chiadò Piat, Integral representation results for functionals defined on $SBV(\Omega;\mathbb{R}^m)$, *J. Math. Pures Appl.* **75** (1996), 595–626.

[17] A. Braides and A. Coscia, The interaction between bulk energy and surface energy in multiple integrals, *Proc. Royal Soc. Edin.* **124A** (1994), 737–756.

[18] A. Braides and A. Defranceschi, *Homogenization of Multiple Integral*, Oxford University Press, Oxford, 1998.

[19] A. Braides, A. Defranceschi and E. Vitali, Homogenization of free-discontinuity problems, *Arch. Rational Mech. Anal.* **135** (1996), 297–356.

[20] A. Braides, I. Fonseca and G. Francfort, 3D-2D Asymptotic analysis for inhomogeneous thin films. Submitted to *Indiana U. Math. J.*

[21] D. Caillerie, Thin elastic and periodic plates, *Math. Meth Appl. Sci.* **6** (1984), 159–191.

[22] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.

[23] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[24] E. De Giorgi and L. Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199–210.

[25] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Mat.* **58** (1975), 842–850.

[26] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton 1992.

[27] H. Federer, *Geometric Measure Theory.* Springer Verlag, New York, 1969.

[28] I. Fonseca and S. Müller. Relaxation of quasiconvex functionals in $BV(\Omega;\mathbb{R}^p)$ for integrands $f(x,u,\nabla u)$, *Arch. Rat. Mech. Anal.* **123** (1993), 1–49.

[29] A. A. Griffith, The phenomenon of rupture and flow in solids, *Phil Trans. Royal Soc. London A* **221** (1920), 163–198.

[30] H. Le Dret and A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures Appl.* **74** (1995), 549–578.

[31] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989), 577–685.

[32] W. Ziemer, *Weakly Differentiable Functions*, Springer, Berlin, 1989.