On a Volume Constrained Variational Problem

Luigi Ambrosio [*], Irene Fonseca [†], Paolo Marcellini [‡], Luc Tartar [§]

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Abstract Existence of minimizers for a volume constrained energy

$$E(u) := \int_\Omega W(\nabla u)\, dx$$

where $\mathcal{L}^N(\{u = z_i\}) = \alpha_i, i = 1, \ldots, P$, is proved for the case in which z_i are extremal points of a compact, convex set in $\mathbb{R}^d$ and under suitable assumptions on a class of quasiconvex energy densities W. Optimality properties are studied in the scalar-valued problem where $d = 1$, $P = 2$, $W(\xi) = |\xi|^2$, and the Γ-limit as the sum of the measures of the 2 phases tends to $\mathcal{L}^N(\Omega)$ is identified. Minimizers are fully characterized when $N = 1$, and candidates for solutions are studied for the circle and the square in the plane.

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1 Introduction

In recent years there has been a remarkable development of techniques in applied analysis motivated in part by questions arising in the study of materials. Some of the underlying mathematical problems lie at the boundary of classical analytical methods, requiring new ideas. In this paper we treat a seemingly simple constrained variational problem which falls outside the usual techniques of the Calculus of Variations for proving existence of minimizers.

In 1992 Morton Gurtin, motivated by a problem related to the interface between immiscible fluids [10], suggested that we study existence of minimizers and possible optimal designs for the energy

$$I(u) := \int_\Omega |\nabla u|^2\, dx$$

where $\Omega \subset \mathbb{R}^N$ is an open, bounded, connected Lipschitz domain, and $u : \Omega \to \mathbb{R}$ is subjected to the volume constraints

$$\mathcal{L}^N(\{u = 0\}) = \alpha \quad \text{and} \quad \mathcal{L}^N(\{u = 1\}) = \beta. \tag{1.1}$$

Here $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$ and $\alpha, \beta > 0$ satisfy $\alpha + \beta < \mathcal{L}^N(\Omega)$.

Previous works by Alt and Caffarelli [3] and Aguilera, Alt and Caffarelli [2] address a similar problem in which only one volume constraint is present and for which Dirichlet boundary conditions are imposed on u. They obtain the existence of minimizers for I and regularity properties for solutions and for their free boundaries. In our context, and in the presence of two or more constraints, a priori continuity of minimizers would imply separability of the phases $\{u = 0\}$ and $\{u = 1\}$, thus

[*] Dipartimento di Matematica "F.Casorati", Università di Pavia, Via Abbiategrasso 205, 27100 Pavia, Italy
[†] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA
[‡] Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, 50131 Firenze, Italy
[§] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA

enabling us to use their arguments to obtain additional regularity for both u and the free boundaries. Unfortunately, we were unable to establish continuity, and this seriously limited the choice of variations and required the introduction of analytical methods specific to the multi-phase framework. However, since this work has been completed Tilli, (see [17]) established locally Lipschitz continuity of minimizers of I.

In this paper we obtain the existence of minimizers for I subjected to (1.1). More generally, in Theorem 2.3 we prove existence for solutions of

$$\min\left\{\int_\Omega W(\nabla u)\,dx : u \in W^{1,p}(\Omega;\mathbb{R}^d),\ \mathcal{L}^N(\{u=z_i\})=\alpha_i,\ i=1,\dots,P\right\},$$

where $\{z_1,\dots,z_P\}$ are extremal points of a compact, convex set $K\subset\mathbb{R}^d$, with $P\geq 1$, $\alpha_i>0$ and $\sum_i \alpha_i < \mathcal{L}^N(\Omega)$, provided $W:\mathbb{R}^{d\times N}\to[0,+\infty)$ is a C^1 quasiconvex function with p-growth, $p>1$, satisfying the structure condition

$$\sum_{i,j=1}^{d}\sum_{k=1}^{N}\frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i\nu^j>0 \quad \text{whenever } \xi^T\nu\neq 0, \xi\in\mathbb{R}^{d\times N}, \nu\in S^{d-1}, \tag{1.2}$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$. A characterization of (1.2) in terms of the behavior of W along rank-one lines can be found in Remark 2.4(ii). Certain isotropic energy densities, such as functions of the type $W(\xi)=g(|\xi|,|\mathrm{adj}\,\xi|,|\det\xi|)$, verify (1.2) (see Proposition 2.5).

In Section 4, using optimality properties of minimizers obtained in Section 3 for $W(\xi):=|\xi|^2$, we characterize the asymptotic behavior of minimizers of I subjected to (1.1) as $\alpha\to\mathcal{L}^N(\Omega)-\gamma$ and $\beta\to\gamma$, with $\gamma\in(0,\mathcal{L}^N(\Omega))$. Precisely, we show that the limiting configurations satisfy the constrained least-area problem

$$p_\gamma := \min\left\{P_\Omega(E)\ :\ E\subset\Omega,\ \mathcal{L}^N(E)=\gamma\right\},$$

where $P_\Omega(E)$ denotes the perimeter of E in Ω. Similar results have been obtained for phase-transitions problems in which the formation of phases is driven by a double-well potential (see [13], [5], [8]); here the creation of interfaces is due to the volume constraints.

In Section 5 we characterize fully the solutions of (M) when $W(\xi)=|\xi|^2$, Ω is an interval and $d>1$ (see Subsection 5.1). Explicit solutions are unknown when $\Omega\subset\mathbb{R}^N$ and $N>1$. We study the cases in which Ω is a circle or a square on the plane. If Ω is a circle, then, in Subsection 5.2, we determine the minimum energy among radial configurations, and we construct a family of competing configurations u_{ab} with energy strictly lower than that for radial functions if $\alpha+\beta<<1$. However, u_{ab} are not solutions, as they violate some of the optimality conditions obtained in Section 3. We remark that due to Theorem 4.1, if $\alpha+\beta\to\mathcal{L}^2(\Omega)$ then radial configurations will still not be minimizers for (M). Finally, in Subsection 5.3 we address briefly the case $\Omega=(0,1)^2$; we show that, although the piecewise affine configurations of the form

$$u(x)=\begin{cases} 0 & \text{if } x_1\leq\alpha,\\ \frac{1}{1-\alpha-\beta}x_1-\frac{\alpha}{1-\alpha-\beta} & \text{if } \alpha<x_1<1-\beta,\\ 1 & \text{if } x_1\geq 1-\beta\end{cases}$$

satisfy the optimality conditions, they are not minimizers for (M) if $\alpha+\beta<<1$, and, once again by virtue of Theorem 4.1, they will not have least energy when $\alpha+\beta$ approaches the measure of the unit square.

2 Existence

We first fix notation. $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure in $\mathbb{R}^N$; H^{N-1} is the $(N-1)$-dimensional Hausdorff measure; $\mathcal{M}$ is the space of Lebesgue measurable functions $u:\Omega\to\mathbb{R}^d$; χ_A

is the characteristic function of a set A; $\mathbb{R}^{d\times N}$ is the vector space of $d\times N$ matrices ξ (d rows, N columns) with components ξ_{ij}, $1\le i\le d$, $1\le j\le N$; Ω is an open, bounded, connected Lipschitz domain; $C_c^k(\Omega)$ is the space of k-differentiable functions with compact support in Ω, $k\in\mathbb{N}\cup\{+\infty\}$.

Proposition 2.1. *For any sequence $(u_h)\subset\mathcal{M}$ converging a.e. to $u\in\mathcal{M}$ and for any closed set $C\subset\mathbb{R}^d$ we have*

$$\mathcal{L}^N(\{x\in\Omega: u(x)\in C\})\ge\limsup_{n\to+\infty}\mathcal{L}^N(\{x\in\Omega: u_n(x)\in C\}).$$

PROOF. Since $A:=\mathbb{R}^d\setminus C$ is open,

$$\chi_A(u)\le\liminf_{n\to+\infty}\chi_A(u_n)\qquad\text{a.e. in }\Omega.$$

By Fatou's Lemma,

$$\begin{aligned}\mathcal{L}^N(\{x\in\Omega\,:\,u(x)\in A\}) &= \int_\Omega\chi_A(u)\,dx\le\int_\Omega\liminf_{n\to+\infty}\chi_A(u_n)\,dx\\ &\le \liminf_{n\to+\infty}\int_\Omega\chi_A(u_n)\,dx=\liminf_{n\to+\infty}\mathcal{L}^N(\{x\in\Omega\,:\,u_n(x)\in A\}),\end{aligned}$$

and the proof follows upon passing to the complementary sets. □

Note that since any $L^p(\Omega)$-converging sequence has subsequences which converge almost everywhere, the upper semicontinuity property asserted in Proposition 2.1 is still valid with respect to $L^p(\Omega)$-convergence.

Consider a finite collection of points $\{z_1,\dots,z_P\}$ in $\mathbb{R}^d$, with $P\ge 1$. In this section we obtain the existence of solutions for the minimization problem

$$\text{(M)}\qquad \min\left\{\int_\Omega W(\nabla u)\,dx: u\in W^{1,p}(\Omega;\mathbb{R}^d),\ \mathcal{L}^N(\{u=z_i\})=\alpha_i,\ i=1,\dots,P\right\}$$

where $p>1$, $\alpha_i>0$ and $\sum_i\alpha_i<\mathcal{L}^N(\Omega)$, under certain technical assumptions on W.

We first find conditions ensuring that the relaxed problem

$$\text{(M)}^*\qquad \min\left\{\int_\Omega W(\nabla u)\,dx: u\in W^{1,p}(\Omega;\mathbb{R}^d),\ \mathcal{L}^N(\{u=z_i\})\ge\alpha_i,\ i=1,\dots,P\right\}$$

has a solution.

Proposition 2.2. *Assume that $W:\mathbb{R}^{d\times N}\to[0,\infty)$ is a quasiconvex function satisfying*

$$c|\xi|^p\le W(\xi)\le C(|\xi|^p+1)\qquad\forall\xi\in\mathbb{R}^{d\times N}\tag{2.1}$$

for some constants $c,C>0$ and some $p\in(1,+\infty)$. Then problem (M)* has at least one solution.

PROOF. It is easy to check that the class of competing functions in (M)* is not empty. Let (u_h) be a minimizing sequence for the problem and denote by $\bar u_h$ the average of u_h on Ω. By Poincaré's inequality and Rellich's Theorem, we may assume, without loss of generality, that $v_h:=u_h-\bar u_h$ converges in $L^p(\Omega)$ to some function v. Since

$$\mathcal{L}^N(\{u_h=P_1\})\bar u_h=\int_{\{u_h=P_1\}}\bar u_h\,dx=\int_{\{u_h=P_1\}}(P_1-v_h)\,dx,$$

$(\bar u_h)$ is bounded; hence, extracting a subsequence, if necessary, the functions (u_h) converge in $L^p(\Omega)$ to some function $u\in W^{1,p}(\Omega;\mathbb{R}^d)$. By Proposition 2.1, the function u satisfies the constraints of

$(\mathrm{M})^*$, and because of the growth condition (2.1), we may use the lower semicontinuity theorem of Acerbi and Fusco (see [1]) to verify that

$$\int_\Omega W(\nabla u)\,dx \leq \liminf_{h\to\infty} \int_\Omega W(\nabla u_h)\,dx.$$

Thus u is a solution of $(\mathrm{M})^*$. □

Note that the previous argument remains valid when the upper bound on W in (2.1) is replaced by the weaker assumption that $u \mapsto \int_\Omega W(\nabla u)$ is lower semicontinuous in the weak topology of $W^{1,p}(\Omega;\mathbb{R}^d)$.

Next we find conditions on W which ensure that any solution of $(\mathrm{M})^*$ actually solves (M).

Theorem 2.3. *Let u be a solution of $(M)^*$ and assume that*

(i) W is differentiable and satisfies

$$\sum_{i=1}^{d}\sum_{k=1}^{N}\left|\frac{\partial W}{\partial \xi_{ik}}\right| \leq C(1+|\xi|^{p-1}) \tag{2.2}$$

for some $C>0$ and all $\xi \in \mathbb{R}^{d\times N}$, and

$$\sum_{i,j=1}^{d}\sum_{k=1}^{N}\frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i\nu^j > 0 \quad \textit{whenever } \xi^T\nu \neq 0, \xi\in\mathbb{R}^{d\times N}, \nu\in S^{d-1}. \tag{2.3}$$

(ii) $z_1,\dots,z_P$ are extremal points of a compact convex set K.

Then u is a solution of (M).

Proof. We must prove that

$$\mathcal{L}^N(\{u=z_i\}) = \gamma_i, \quad \text{for all } i=1,\dots,P.$$

Suppose that $\mathcal{L}^N(\{u=z_1\}) > \alpha_1$. Let $r>0$ be such that $\mathcal{L}^N(\{u=z_1\}) - r > \alpha_1$, and consider a smooth cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N;[0,1])$ such that $\mathcal{L}^N(\operatorname{supp}\varphi) < r$. Without loss of generality, we may assume that the extremal point z_1 is the origin, and let $\nu\in S^{d-1}$ be such that $K\setminus z_1 \subset \{y\in\mathbb{R}^d : y\cdot\nu > 0\}$. Let $0<\delta<\min\{z_i\cdot\nu : i=2,\dots,P\}$, and define $f:\mathbb{R}\to[0,+\infty)$ as

$$f(t) := \begin{cases} -t+\delta & \text{if} \quad t\leq\delta \\ 0 & \text{otherwise.}\end{cases}$$

Set $w := u\cdot\nu$, and consider the perturbations $u_\varepsilon := u + \varepsilon\varphi f(w)\nu$. If $i=2,\dots,P$, and if $u(x)=z_i$, then $w(x)>\delta$ and $f(w(x))=0$, so that $u_\varepsilon(x)=u(x)$. Therefore

$$\{u_\varepsilon = z_i\} \supset \{u=z_i\}.$$

On the other hand,

$$\mathcal{L}^N(\{u_\varepsilon=z_1\}) \geq \mathcal{L}^N(\{u=z_1\}) - \mathcal{L}^N(\operatorname{supp}\varphi) > \mathcal{L}^N(\{u=z_1\}) - r > \alpha_1,$$

and we conclude that u_ε is admissible for $(\mathrm{M})^*$. Thus, taking into account the growth assumption (2.2) and by virtue of Lebesgue's Dominated Convergence Theorem, we can differentiate under the integral to find

$$
\begin{aligned}
0 &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega W\left(\nabla u + \varepsilon \nabla\left[\varphi f(w)\nu\right]|\right)\, dx \\
&= \sum_{i=1}^{d}\sum_{k=1}^{N} \int_\Omega \frac{\partial W}{\partial \xi_{ik}}(\nabla u)\left[\frac{\partial \varphi}{\partial x_k} f(w)\nu^i + \varphi f'(w)\nu^i \sum_{j=1}^{d} \nu^j \frac{\partial u^j}{\partial x_k}\right] dx. \qquad (2.4)
\end{aligned}
$$

Using a partition of unity, any smooth function with compact support may be written as a finite sum of cut-off functions φ, each of small support; we therefore take $\varphi = 1$ in Ω, and (2.4) reduces to

$$
\int_{\{w\in(0,\delta)\}} \sum_{i,j=1}^{d}\sum_{k=1}^{N} \frac{\partial W}{\partial \xi_{ik}}(\nabla u)\frac{\partial u^j}{\partial x_k}\nu^i\nu^j\, dx = 0.
$$

By (2.3) we deduce that $\nabla w = \nabla u^T \nu = 0$ a.e. on $\{0 < w < \delta\}$, hence the function $\max\{0, \min\{w, \delta\}\}$ is constant in Ω. On the other hand, $\mathcal{L}^N(\{w = 0\}) \geq \mathcal{L}^N(\{u = z_1\}) > 0$ and $\mathcal{L}^N(\{w > \delta\}) \geq \mathcal{L}^N(\{u = z_2\}) > 0$. We have reached a contradiction; therefore $\mathcal{L}^N(\{u = z_1\}) = \alpha_1$. □

Remark 2.4. (i) Any differentiable quasiconvex function satisfying the growth condition (2.1) also satisfies (2.2) (see [12]).
(ii) In the scalar-valued case where ($d = 1$) quasiconvexity reduces to convexity, and (2.3) may be rewritten as

$$
\sum_{i=1}^{N} \frac{\partial W}{\partial \xi_i}(\xi)\xi_i > 0 \qquad \text{for all } \xi \in \mathbb{R}^N \setminus \{0\}.
$$

Since W is convex, this is equivalent to saying that W has a strict minimum at the origin. More generally, if $W : \mathbb{R}^{d\times N} \to \mathbb{R}$ is a C^1 rank-one convex function, then (2.3) holds if and only if

$$
t \mapsto W(A + t\nu \otimes \mu) \quad \text{has a strict minimum at } t = 0 \qquad (2.5)
$$

whenever $\mu \in \mathbb{R}\backslash\{0\}$, $A \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$ and $A^T\nu = 0$. Note that in Proposition 2.2, W is assumed to be quasiconvex, and, consequently, it is rank-one convex. In order to prove the equivalence between (2.3) and (2.5), assume first that $\mu \in \mathbb{R} \setminus \{0\}$, $A \in \mathbb{R}^{d\times N}$, $\nu \in S^{N-1}$, $A^T\nu = 0$, and set

$$
\psi(t) := W(A + t\nu \otimes \mu).
$$

Since ψ is convex and C^1, ψ has a strict minimum at the origin if and only if $\operatorname{sign}\psi'(t) = \operatorname{sign} t$ for $t \neq 0$. As

$$
\sum_{j=1}^{d} (A + t\nu \otimes \mu)_{jk}\nu^j = t\mu_k, \qquad (2.6)
$$

we have, for $t \neq 0$,

$$
\begin{aligned}
\psi'(t) &= \sum_{k=1}^{N}\sum_{i=1}^{d} \frac{\partial W}{\partial \xi_{ik}}(A + t\nu \otimes \mu)\nu^i\mu_k \\
&= \frac{1}{t}\sum_{k=1}^{N}\sum_{i,j=1}^{d} \frac{\partial W}{\partial \xi_{ik}}(A + t\nu \otimes \mu)(A + t\nu \otimes \mu)_{jk}\nu^i\nu^j. \qquad (2.7)
\end{aligned}
$$

It follows from (2.6) that

$$
(A + t\nu \otimes \mu)^T\nu = t\mu \neq 0
$$

which, together with (2.3) and (2.7), yields

$$\operatorname{sign}\psi'(t) = \operatorname{sign} t.$$

Conversely, if for some $\xi \in \mathbb{R}^{d\times N}$, $\nu \in S^{d-1}$ such that $\xi^T\nu \neq 0$, (2.3) were violated, then setting

$$\psi(t) := W(A + t\nu \otimes \xi^T\nu), \quad A := \xi - \nu \otimes \xi^T\nu,$$

then $A^T\nu = 0$,

$$\psi'(1) = \sum_{i,j=1}^{d}\sum_{k=1}^{N} \frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i\nu^j \leq 0,$$

which contradictis (2.5).
(iii) Note that the function $W(\xi) = |\xi|^2$, corresponding to the Dirichlet integral, satisfies the hypotheses of Theorem 2.3. In this case a simple truncation argument proves that any solution u of (M) satisfies

$$\min\{z_1, z_2\} \leq u \leq \max\{z_1, z_2\}.$$

More generally, in the isotropic (scalar or vectorial) case where $W(\xi) = \Phi(|\xi|)$, the assumption (2.3) reduces to $\Phi'(t) > 0$ for $t > 0$, and it can be shown that any solution u of (M)* takes its values in the closed convex hull K of $\{z_1, \ldots, z_P\}$. This follows by comparing u with $\Pi(u)$, where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal projection onto a half-space containing all points z_i. Precisely, set

$$\gamma := \inf\left\{\int_\Omega \Phi(|\nabla u|)\,dx : u \in W^{1,p}(\Omega; K),\ \mathcal{L}^N(\{u = z_i\}) \geq \alpha_i,\ i = 1, \ldots, P\right\}.$$

We claim that if an admissible u for (M)* takes values outside K then we may modify u and decrease its energy. In fact, if $\mathcal{L}^N(\{x \in \Omega : u(x) \notin K\}) > 0$, then there exists a hyperplane H with normal $\nu \in S^{d-1}$ such that K is contained in one of the half-spaces determined by H, and the other half-space contains a subset E of the range of u with $\mathcal{L}^N(u^{-1}(E)) > 0$. Without loss of generality, we may assume that

$$H := \{y \in \mathbb{R}^d : y \cdot \nu = 0\}, \quad K \subset \{y \in \mathbb{R}^d : y \cdot \nu \leq 0\},$$

and that there exists $\delta > 0$ such that

$$\mathcal{L}^N(\{x \in \Omega : u(x) \cdot \nu > \delta\}) > 0.$$

Let $\{\eta_1, \ldots, \eta_{d-1}, \nu\}$ be an orthonormal basis of $\mathbb{R}^d$, and define

$$\Pi(u)(x) := \sum_{i=1}^{d-1}(u(x) \cdot \eta_i)\,\eta_i + f(x)\,\nu,$$

where

$$f(x) := \begin{cases} u(x) \cdot \nu & \text{if } u(x) \cdot \nu \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

If $u(x) \in K$ then $\Pi(u)(x) = u(x)$, so $\Pi(u)$ remains admissible. Also $|\nabla \Pi(u)(x)| \leq |\nabla u(x)|$ for a.e. $x \in \Omega$, and

$$|\nabla \Pi(u)(x)| < |\nabla u(x)| \text{ on a set of positive measure.} \tag{2.8}$$

In fact, if $|\nabla \Pi(u)(x)| = |\nabla u(x)|$ a.e. in Ω, then $\nabla u \cdot \nu = 0$ a.e. on $\{u \cdot \nu > 0\}$, and the Sobolev function $v := \max\{(u \cdot \nu), 0\}$ would be constant, in contrast to the conditions

$$\mathcal{L}^N(\{v = 0\}) \geq \mathcal{L}^N(\{u = z_1\}) \geq \alpha_1 > 0,\ \mathcal{L}^N(\{v \geq \delta\}) > 0.$$

Since Φ is strictly increasing, by (2.8) we have

$$\int_\Omega \Phi(|\nabla \Pi(u)(x)|)\, dx < \int_\Omega \Phi(|\nabla u(x)|)\, dx.$$

As K is the intersection of a countable family of half-spaces, an iteration of this argument proves that for any u admissible for $(\mathrm{M})^*$ there is a function $\bar{u}$ still admissible for $(\mathrm{M})^*$, with values in K and with smaller energy. Thus every solution of $(\mathrm{M})^*$ takes its values on K.
(iv) We do not know whether solutions of $(\mathrm{M})^*$ are solutions of (M) when the points z_i are not extremal, even if $d = 1$, $W(\xi) = |\xi|^2$, and there are three or more phases. However, in this particular case it can shown that any *continuous* solution of $(\mathrm{M})^*$ is actually a solution of (M). Indeed, if for instance $\mathcal{L}^N(\{u = z_1\}) > \alpha_1$, then we can make local additive variations to show that each component of u is harmonic in the open set

$$\left\{x \in \Omega \,:\, u(x) \in \mathbb{R}^d \setminus \{z_2, z_3, \ldots, z_P\}\right\}.$$

This contradicts the fact that $\mathcal{L}^N(\{u = z_1\}) > 0$.

Next we exhibit a class of isotropic energy densities W which satisfy (2.3). We recall that W is *isotropic* if it can be written as

$$W(\xi) = \varphi(\lambda_1(\xi), \ldots, \lambda_N(\xi))$$

for some function φ of the the list of *principal stretches* $(\lambda_1(\xi), \ldots, \lambda_N(\xi))$, where $0 \le \lambda_1(\xi) \le \lambda_2(\xi) \le \ldots \le \lambda_N(\xi)$, and $\{\lambda_i^2(\xi) : i = 1, \ldots, N\}$ are the eigenvalues of $\xi^T \xi$.

Proposition 2.5. *Let $W : \mathbb{R}^{N\times N} \to \mathbb{R}$ be given by*

$$W(\xi) = \varphi(\lambda_1(\xi), \ldots, \lambda_N(\xi)), \quad \xi \in \mathbb{R}^{N\times N},$$

where $\varphi : \mathbb{R}^N \to \mathbb{R}$ is a symmetric C^1 function such that for every $i = 1, \ldots, N$

$$\frac{\partial \varphi}{\partial \lambda_i}(\lambda) > 0 \quad \textit{whenever } \lambda_i > 0 \textit{ and } \lambda_j \ge 0 \textit{ for all } j = 1, \ldots, N,\ j \ne i.$$

Then W satisfies (2.3).

Proof. Consider first a matrix $\xi \in \mathbb{R}^{N\times N}$ such that

$$\xi^T \xi\, e_i = \lambda_i^2(\xi)\, e_i,\ i = 1, \ldots, N,\ 0 < \lambda_1(\xi) < \ldots < \lambda_N(\xi),$$

and $\{e_1, \ldots, e_N\}$ is an orthonormal basis of $\mathbb{R}^N$. Fix $B \in \mathbb{R}^{N\times N}$. If t is small enough, then

$$0 < \lambda_1(\xi + tB) < \ldots < \lambda_N(\xi + tB),$$

$$\lambda_i(\xi + tB) \to \lambda_i(\xi) \text{ as } t \to 0,$$

and

$$(\xi + tB)^T(\xi + tB)\, e_i(t) = \lambda_i^2(\xi + tB)\, e_i(t), \tag{2.9}$$

where $e_i(t) \to e_i$ as $t \to 0$, and $|e_i(t)| = 1$. Differentiating (2.9) with respect to t, taking the inner product of the resulting equation with $e_i(t)$, and using the fact that $e_i(t) \cdot \frac{d}{dt} e_i(t) = 0$, we find that, at $t = 0$

$$Be_i \cdot \xi e_i = \lambda_i(\xi)\, \frac{d}{dt} \lambda_i(\xi + tB).$$

For the case in which $B := \nu \otimes \xi^T\nu$, since $Be_i \cdot \xi e_i = (\xi^T\nu \cdot e_i)^2$, we conclude that

$$\frac{d}{dt}\lambda_i(\xi + tB) = \frac{1}{\lambda_i(\xi)}(\xi^T\nu \cdot e_i)^2. \tag{2.10}$$

We are now in a position to prove (2.3). Let $\xi \in \mathbb{R}^{N\times N}, \nu \in S^{N-1}$, be such that $\xi^T\nu \neq 0$. Writing

$$\xi^T\xi e_i = \lambda_i^2(\xi)e_i$$

for a suitable orthonormal basis of $\mathbb{R}^N$, $\{e_1, \ldots, e_N\}$, then there is $j \in \{1, \ldots, N\}$ such that $\xi^T\nu \cdot e_j \neq 0$, and so $\lambda_j > 0$. Construct a sequence of matrices ξ^n such that $\xi^n \to \xi$ as $n \to +\infty$,

$$(\xi^n)^T\xi^n e_i = \lambda_i^2(\xi^n)e_i, \; 0 < \lambda_1(\xi^n) < \ldots < \lambda_N(\xi^n).$$

Using (2.10), we conclude that

$$\begin{aligned}
\sum_{i,j=1}^{d}\sum_{k=1}^{N}\frac{\partial W}{\partial \xi_{ik}}(\xi)\xi_{jk}\nu^i\nu^j &= \lim_{n\to\infty}\sum_{i,j=1}^{d}\sum_{k=1}^{N}\frac{\partial W}{\partial \xi_{ik}}(\xi^n)\xi^n_{jk}\nu^i\nu^j \\
&= \lim_{n\to\infty}\left.\frac{d}{dt}\right|_{t=0} W(\xi^n + t\nu\otimes(\xi^n)^T\nu) \\
&= \lim_{n\to\infty}\sum_{i=1}^{N}\frac{\partial\varphi}{\partial\lambda_i}(\lambda_1(\xi^n), \ldots, \lambda_N(\xi^n))\frac{1}{\lambda_i(\xi^n)}((\xi^n)^T\nu \cdot e_i)^2 \\
&\geq \limsup_{n\to\infty}\frac{\partial\varphi}{\partial\lambda_j}(\lambda_1(\xi^n), \ldots, \lambda_N(\xi^n))\frac{1}{\lambda_j(\xi^n)}((\xi^n)^T\nu \cdot e_j)^2 \\
&= \frac{\partial\varphi}{\partial\lambda_j}(\lambda_1(\xi), \ldots, \lambda_N(\xi))\frac{1}{\lambda_j}(\xi^T\nu \cdot e_i)^2 > 0.
\end{aligned}$$

□

A simple class of polyconvex functions satisfying the hypotheses of Proposition 2.5 is formed by energy densities of the type

$$W(\xi) = g(|\xi|, |\mathrm{adj}\,\xi|, |\det\xi|),$$

where $g(\eta, \mu, \lambda)$ is a C^1 convex function on $[0, +\infty)^3$ such that

$$\frac{\partial g}{\partial\mu}(\eta,\mu,\lambda) \geq 0, \; \frac{\partial g}{\partial\lambda}(\eta,\mu,\lambda) \geq 0, \text{ and } \frac{\partial g}{\partial\eta}(\eta,\mu,\lambda) > 0 \qquad \text{for all } (\eta,\mu,\lambda) \text{ with } \eta > 0.$$

Here $\det\xi$ denotes the determinant of the $N \times N$ matrix ξ, and $\mathrm{adj}\,\xi$ is the adjugate of the matrix ξ, i.e. the matrix of the minors of order $N-1$ with the property

$$(\mathrm{adj}\,\xi)^T\xi = \xi^T\mathrm{adj}\,\xi = \det\xi\,\mathbb{I}. \tag{2.11}$$

Let

$$W(\xi) = \varphi(\lambda_1(\xi), \ldots, \lambda_N(\xi)),$$

where the λs again denote the principal streches,

$$\varphi(\lambda_1, \ldots, \lambda_N) := g\left(\sqrt{\sum_{i=1}^{N}\lambda_i^2}, \;\; \sqrt{\sum_{i=1}^{N}\lambda_1^2\ldots\lambda_{i-1}^2\lambda_{i+1}^2, \ldots\lambda_N^2}, \;\; \lambda_1\ldots\lambda_N\right)$$

with $\lambda_{-1}, \lambda_{N+1} := 1$.

3 Optimality Properties of the Solutions

As was shown in the previous section (see Theorem 2.3, Remark 2.4), the problem

$$\text{(M)} \qquad \min\left\{\int_\Omega |\nabla u|^2\,dx : u\in W^{1,1}(\Omega),\, \mathcal{L}^N(\{u=0\})=\alpha,\, \mathcal{L}^N(\{u=1\})=\beta\right\}$$

admits solutions, and any solution belongs to $u \in W^{1,2}(\Omega;[0,1])$. We now we study optimality properties of these solutions.

Theorem 3.1. *Let $u\in W^{1,2}(\Omega;[0,1])$ be a solution of* (M). *Then*
(i)

$$\int_\Omega \varphi f'(u)|\nabla u|^2 + f(u)\nabla\varphi\cdot\nabla u\,dx = 0$$

for all $\varphi\in C^1(\overline{\Omega})$ and all $f\in W^{1,\infty}(\Omega)$ with $f(0)=f(1)=0$;
(ii)

$$\int_\Omega |\nabla u|^2\,g(u)\,dx = \left(\int_\Omega |\nabla u|^2\,dx\right)\left(\int_0^1 g(s)\,ds\right)$$

for all $g\in L^\infty(\mathbb{R})$;
(iii) *Δu is a signed Radon measure in Ω with support contained in $\overline{\{u=0\}}\cup\overline{\{u=1\}}$, and*

$$|\Delta u|(\Omega)\le 2\int_\Omega |\nabla u|^2\,dx.$$

Moreover,

$$\langle \Delta u,\phi\rangle = \lim_{n\to+\infty} n\int_{\{u<1/n\}}\phi\,|\nabla u|^2\,dx - n\int_{\{u>1-1/n\}}\phi\,|\nabla u|^2\,dx$$

for every $\phi\in C_c(\Omega)$;
(iv) *if $F\in W_0^{1,\infty}(\Omega;\mathbb{R}^N)$ satisfies* $\operatorname{div} F=0$ *then*

$$\sum_{i,j=1}^N \int_\Omega \frac{\partial F_i}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx = 0. \tag{3.1}$$

Proof. By Theorem 2.3 and Remark 2.4 (ii) we know that any solution u of (M) belongs to $W^{1,2}(\Omega;[0,1])$. Taking φ and f under the assumptions of part (i), it is clear that

$$\{u=1\}\subset\{u+\varepsilon\varphi f(u)=1\}\quad\text{and}\quad \{u=0\}\subset\{u+\varepsilon\varphi f(u)=0\}.$$

Therefore $u+\varepsilon\varphi f(u)$ is admissible for (M)*, and in light of Remark 2.4 ii),

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_\Omega |\nabla(u+\epsilon\varphi f(u))|^2\,dx \\ &= 2\int_\Omega \varphi f'(u)|\nabla u|^2 + f(u)\nabla\varphi\cdot\nabla u\,dx, \end{aligned}$$

proving (i). Part (ii) follows immediately from (i), setting

$$\varphi\equiv 1,\quad f(t):=\int_0^t g(s)\,ds - t\int_0^1 g(s)\,ds.$$

To obtain (iii), consider the piecewise affine functions

$$f_n(t):=\begin{cases} nt & \text{if } 0\le t\le \frac1n\\ 1 & \text{if } \frac1n\le t\le 1-\frac1n\\ -nt+n & \text{if } 1-\frac1n\le t\le 1.\end{cases}$$

By (i), for all $\varphi \in C^1_c(\Omega)$,

$$\begin{aligned}\langle \Delta u, \varphi\rangle &= -\int_\Omega \nabla u\cdot\nabla\varphi\, dx = -\lim_{n\to+\infty}\int_\Omega \nabla u\cdot\nabla\varphi\, f_n(u)\, dx\\ &= -\lim_{n\to+\infty}\int_\Omega |\nabla u|^2\varphi\, f_n'(u)\, dx = -\lim_{n\to+\infty}\langle \mu_n,\varphi\rangle,\end{aligned}$$

where the finite Radon measures μ_n are defined as

$$\mu_n := |\nabla u|^2\, f_n'(u)\,\mathcal{L}^N\lfloor\Omega.$$

By (ii),

$$|\mu_n|(\Omega) = \int_\Omega |\nabla u|^2\,|f_n'(u)|\, dx = \left(\int_\Omega |\nabla u|^2\, dx\right)\left(\int_0^1 |f_n'(s)|\, ds\right) \le 2\int_\Omega |\nabla u|^2\, dx;$$

thus there is a Radon measure μ such that, up to a subsequence,

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{and} \quad |\mu(\Omega)| \le 2\int_\Omega |\nabla u|^2\, dx.$$

We therefore conclude that

$$\Delta u = -\mu = \text{weak-*}\lim_{n\to+\infty} -n|\nabla u|^2\mathcal{L}^N\lfloor\{u<1/n\} + n|\nabla u|^2\mathcal{L}^N\lfloor\{u>1-1/n\}.$$

Finally, let F be a Lipschitz mapping on Ω, with $F=0$ on $\partial\Omega$, and such that $\operatorname{div} F = 0$. Consider the flow

$$\begin{cases} \dfrac{dw}{dt}(t,x) & = F\,(w(t,x)) \\ w(0,x) & = x \end{cases} \qquad (t,x)\in\mathbb{R}\times\overline{\Omega}.$$

It is well known that

$$\det \nabla_x w(t,x) = 1. \tag{3.2}$$

Indeed, using (2.11) we have

$$\begin{aligned} N\frac{d}{dt}\det\nabla_x w(t,x) &= \operatorname{adj}\nabla_x w(t,x)\cdot\frac{d}{dt}\nabla_x w(t,x) = \operatorname{adj}\nabla_x w(t,x)\cdot\nabla_x(F(w(t,x)))\\ &= \operatorname{adj}\nabla_x w(t,x)\nabla_x w^T(t,x)\cdot\nabla F(w(t,x)) = \det\nabla_x w(t,x)\mathbb{I}\cdot\nabla F(w(t,x))\\ &= \det\nabla_x w(t,x)\operatorname{div}F(w(t,x)) = 0.\end{aligned}$$

Therefore

$$\det\nabla_x w(t,x) = \det\nabla_x w(0,x) = 1.$$

The functions

$$u_\varepsilon(x) := u(w_\varepsilon(x)),\ \text{where } w_\varepsilon(x) := w(\varepsilon,x)$$

satisfy the volume constraints of (M) because, by (3.2),

$$\mathcal{L}^N(\{u_\varepsilon=0\}) = \int_{\{u=0\}}\det\nabla w_\varepsilon(x)\, dx = \mathcal{L}^N(\{u=0\}),$$

and, similarly, $\mathcal{L}^N(\{u_\varepsilon = 1\}) = \mathcal{L}^N(\{u = 1\})$. If $u \in C^2$, then

$$\begin{aligned} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |\nabla(u \circ w_\varepsilon)|^2\,dx &= 2\int_\Omega \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{dw_{\varepsilon\, j}}{d\varepsilon} \frac{\partial u}{\partial x_i}\,dx = 2\int_\Omega \sum_{i,j=1}^N F_j \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i}\,dx \\ &= -2\int_\Omega \sum_{i,j=1}^N \frac{\partial F_j}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i}\,dx. \end{aligned}$$

By a simple approximation argument, this formula remains valid if $u \in W^{1,2}(\Omega;\mathbb{R})$ is a solution of (M), and we conclude that

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int_\Omega |\nabla(u \circ w_\varepsilon)|^2\,dx = -2\int_\Omega \nabla F \nabla u \cdot \nabla u\,dx.$$

□

Remark 3.2. If u is locally a Lipschitz function in Ω, statement (ii) can be reformulated as

$$\int_{\{u=t\}} |\nabla u|\,dH^{N-1} = \int_\Omega |\nabla u|^2\,dx$$

for $\mathcal{L}^1$-a.e. $t \in (0,1)$. To prove this assertion we will use the *co-area* formula (see [7], Chapter 3)

$$\int_\Omega h(x)|\nabla v(x)|\,dx = \int_{-\infty}^{+\infty} \left(\int_{\{v=t\}} h(x)\,dH^{N-1}(x) \right) dt, \tag{3.3}$$

which is valid for any Borel function $h : \Omega \to [0,+\infty]$ and $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R})$. By part (i), with $\varphi \equiv 1$ and $f \in C^1_c(\mathbb{R})$, and setting $h(x) := f'(u(x))|\nabla u|$, we obtain

$$0 = \int_\Omega |\nabla u|^2 f'(u)\,dx = \int_0^1 f'(t) \left(\int_{\{u=t\}} |\nabla u|\,dH^{N-1} \right) dt = 0.$$

This proves there is a constant C such that $\int_{\{u=t\}} |\nabla u|\,dH^{N-1} = C$ for $\mathcal{L}^1$-a.e. $t \in (0,1)$. Using the co-area formula once again, we conclude that $C = \int_0^1 |\nabla u|^2\,dx$. As mentioned in the introduction, since the completion of this work, Tilli ([17]) has obtained the locally Lipschitz property of u.

In Proposition 3.4 we will need divergence-free fields with a given trace on a Lipschitz domain; the next result ensures their existence.

Proposition 3.3. *If $\theta \in L^2(\Omega)$ satisfies*

$$\int_\Omega \theta\,dx = 0,$$

then there exists $f \in W^{1,2}_0(\Omega;\mathbb{R}^N)$ such that $\operatorname{div} f = \theta$.

Proof. We first recall a consequence of Tartar's equivalence lemma ([16]), which, in turn, generalizes a result of Peetre (see [15]): let E_1 be a Banach space, and let E_2, E_3, be normed spaces. If $A : E_1 \to E_2$ is a linear bounded operator and $B : E_1 \to E_3$ is a compact linear operator, then Range(A) is closed provided

$$\|u\|_{E_1} \le C\,[\|Au\|_{E_2} + \|Bu\|_{E_3}] \tag{3.4}$$

for some constant $C > 0$. We apply the equivalence lemma with $E_1 := L^2(\Omega)$, $E_2 := [H^{-1}(\Omega)]^N$, $E_3 := H^{-1}(\Omega)$, $Au := \nabla u$, and $Bu := u$. With these choices, the estimate (3.4) reduces to

$$\|u\|_{L^2} \leq C\,[\|\nabla u\|_{H^{-1}} + \|u\|_{H^{-1}}]$$

and the latter inequality has been established by Nečas in [14].

Since Range(A) is closed, so is Range(A^T), where $A^T : [H_0^1(\Omega)]^N \to L^2(\Omega)$ is the divergence operator. Therefore

$$\left\{\theta \in L^2(\Omega) : \int_\Omega \theta\, dx = 0\right\} = [\mathrm{Ker}(A)]^\perp = \overline{\mathrm{Range}(A^T)} = \mathrm{Range}(A^T)$$

and Property 3.3 follows. □

An immediate consequence of Proposition 3.3 is that if $\tau \in H^{1/2}(\partial\Omega, \mathbb{R}^N)$ satisfies

$$\int_{\partial\Omega} \tau \cdot n_\Omega \, dH^{N-1} = 0,$$

where n_Ω is the unit outer normal to $\partial\Omega$, then the problem

$$\begin{cases} \operatorname{div} g = 0 & \text{in } \Omega \\ g = \tau & \text{on } \partial\Omega \end{cases}$$

admits a solution $g \in W^{1,2}(\Omega;\mathbb{R}^N)$. Indeed, it suffices to apply Proposition 3.3 to the function $\theta := -\operatorname{div} h$, where $h \in W^{1,2}(\Omega;\mathbb{R}^N)$ is such that $h = \tau$ on $\partial\Omega$, to obtain a function $f \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ satisfying $\operatorname{div} f = \theta$; $g := f + h$ then has the desired properties.

In the following proposition we use (3.1) to show that the normal derivative on the boundary of the level sets $\{u = 0\}$, $\{u = 1\}$ is locally constant. In fact, the normal derivative for minimizers is globally constant ([17], and also see [2] for the case of one volume constraint).

Proposition 3.4. *If $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ satisfies (3.1), if $\Delta u = 0$ in $\{0 < u < 1\}$, and if the free boundary $S := \{u = 0\} \cup \{u = 1\}$ is C^1, then $\partial u/\partial n$ is locally constant on S.*

Proof. Let $g \in C_c^\infty(B_r)$, where B_r is an open ball of radius r in Ω such that $B_r \cap \{u = 1\} = \emptyset$. Suppose, in addition, that

$$\int_{\partial\{u=0\}} g \cdot \nu \, dH^{N-1} = 0,$$

where ν is the outer normal to $\{u > 0\}$. In view of Proposition 3.3 and the remark thereafter, we consider the fields F^+ and F^- such that

$$\begin{cases} \operatorname{div} F^+ = 0 & \text{in } B_r^+ := B_r \setminus \{u = 0\} \\ F^+ = g & \text{on } S^+ := \partial B_r^+ \cap \{u = 0\} \\ F^+ = 0 & \text{on } \partial B_r^+ \setminus \{u = 0\}, \end{cases}$$

$$\begin{cases} \operatorname{div} F^- = 0 & \text{in } B_r^- := B_r \cap (\mathrm{int}\{\mathrm{u} = 0\}) \\ F^- = g & \text{on } \partial B_r^- \cap \{u = 0\} \\ F^- = 0 & \text{on } \partial B_r^- \setminus \{u = 0\}, \end{cases}$$

and define

$$F := \begin{cases} F^+ & \text{in } B_r^+ \\ F^- & \text{in } B_r^- \\ 0 & \text{otherwise.} \end{cases}$$

A smoothing argument establishes (3.1) (even though, a priori, F is only in $W_0^{1,2}(B_r,\mathbb{R}^N)$ and not necessarily Lipschitz), because $|\nabla u|$ is bounded in B_r by assumption. Hence, we have

$$\begin{aligned}0 &= \sum_{i,j=1}^{N}\int_{B_r}\frac{\partial F_i}{\partial x_j}\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dx=\sum_{i,j=1}^{N}\int_{B_r^+}\frac{\partial F_i^+}{\partial x_j}\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dx\\ &= \sum_{i,j=1}^{N}\int_{S^+}g_i\nu_j\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dH^{N-1}-\int_{B_r^+}F_i^+\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\right)dx.\end{aligned}$$

Note that

$$\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\right) = \frac{\partial u}{\partial x_i}\Delta u+\frac{\partial}{\partial x_i}\left[\frac{1}{2}|\nabla u|^2\right]=\frac{\partial}{\partial x_i}\left[\frac{1}{2}|\nabla u|^2\right],$$

hence

$$\begin{aligned}0 &= \sum_{i=1}^{N}\int_{S^+}g_i\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial n}\,dH^{N-1}-\int_{B_r^+}F_i^+\frac{\partial}{\partial x_i}\left[\frac{1}{2}|\nabla u|^2\right]dy\\ &= \int_{S^+}\sum_{i=1}^{N}g_i\nu_i\left[\left|\frac{\partial u}{\partial n}\right|^2-\frac{1}{2}|\nabla u|^2\right]dH^{N-1}=\frac{1}{2}\int_{S^+}g\cdot\nu\left|\frac{\partial u}{\partial n}\right|^2\,dH^{N-1}\end{aligned}$$

by the identity $\partial u/\partial x_i=\nu_i\partial u/\partial n$ on the boundary. We have proved the implication

$$\int_{S^+}g\cdot\nu\,dH^{N-1}=0\qquad\Longrightarrow\qquad\int_{S^+}g\cdot\nu\left|\frac{\partial u}{\partial n}\right|^2\,dH^{N-1}=0\tag{3.5}$$

for $g\in C_c^\infty(B_r,\mathbb{R}^N)$, and this ensures there is a constant λ such that

$$\int_{S^+}g\cdot\nu\left|\frac{\partial u}{\partial n}\right|^2\,dH^{N-1}=\lambda\int_{S^+}g\cdot\nu\,dH^{N-1}\qquad\text{for all } g\in C_0^\infty(B_r,\mathbb{R}^N).$$

Thus the normal derivative of u is locally constant on S^+. □

4 Asymptotic Behavior of the Solutions

In this section we investigate the asymptotic behavior of solutions $u_{\alpha\beta}$ of

$$(\mathrm{M})_{\alpha\beta}\qquad \min\left\{\int_\Omega|\nabla u|^2\,dx:\ u\in W^{1,2}(\Omega),\mathcal{L}^N(\{u=0\})=\alpha,\,\mathcal{L}^N(\{u=1\})=\beta\right\}$$

as $(\alpha+\beta)\nearrow\mathcal{L}^N(\Omega)$. We denote by $m_{\alpha\beta}$ the Dirichlet integral of $u_{\alpha\beta}$ and, for any constant $\gamma\in(0,|\Omega|)$, we set

$$p_\gamma:=\min\left\{P_\Omega(E)\ :\ E\subset\Omega,\ \mathcal{L}^N(E)=\gamma\right\},\tag{4.1}$$

where $P_\Omega(E)$ denotes, as usual, the perimeter of E in Ω. The main result of this section is the following:

Theorem 4.1. *For any $\gamma \in (0, \mathcal{L}^N(\Omega))$,*

$$\lim_{\substack{\alpha \to \mathcal{L}^N(\Omega)-\gamma \\ \beta \to \gamma \\ \alpha+\beta<\mathcal{L}^N(\Omega)}} \big(\mathcal{L}^N(\Omega) - (\alpha+\beta)\big) m_{\alpha\beta} = p_\gamma^2. \tag{4.2}$$

Moreover, any limit point in the $L^2(\Omega)$ topology of $u_{\alpha\beta}$ is the characteristic function of a minimizing set for (4.1).

Theorem 4.1 will be deduced from Theorem 4.2 below, recalling that Γ-convergence ensures that minimizers of $(\mathrm{M})_{\alpha\beta}$ converge to minimizers for (4.1), and that minima for $(\mathrm{M})_{\alpha\beta}$ tend to the minimum for the limiting problem, so that (4.2) holds.

Theorem 4.2. *For any $u \in L^2(\Omega)$ and $\alpha, \beta > 0$ with $\alpha + \beta < \mathcal{L}^N(\Omega)$, we define*

$$F_{\alpha\beta}(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|^2 & \text{if } u \in H^1(\Omega), \mathcal{L}^N(\{u \le 0\}) \ge \alpha, \mathcal{L}^N(\{u \ge 1\}) \ge \beta \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$G_\gamma(u) := \begin{cases} [P_\Omega(E)]^2 & \text{if } u = \chi_E \text{ and } \mathcal{L}^N(E) = \gamma \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\Gamma\big(L^2(\Omega)\big) - \lim_{\substack{\alpha \to \mathcal{L}^N(\Omega)-\gamma \\ \beta \to \gamma \\ \alpha+\beta<\mathcal{L}^N(\Omega)}} \big(\mathcal{L}^N(\Omega) - (\alpha+\beta)\big) F_{\alpha\beta}(u) = G_\gamma(u) \qquad \text{for all } u \in L^2(\Omega).$$

Proof. Without loss of generality, we can assume that $\mathcal{L}^N(\Omega) = 1$. We fix sequences $\{\alpha_n\}$ and $\{\beta_n\}$, converging to $(1-\gamma)$ and γ, respectively, and we denote by $F^+(u)$, $F^-(u)$, the upper and lower Γ-limits:

$$F^+(u) := \inf_{\{u_n\}} \left\{ \limsup_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) F_{\alpha_n\beta_n}(u_n) : u_n \to u \text{ in } L^2(\Omega) \right\}$$

and

$$F^-(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) F_{\alpha_n\beta_n}(u_n) : u_n \to u \text{ in } L^2(\Omega) \right\}.$$

We must prove that $F^- \ge G_\gamma \ge F^+$.

Step 1. We first establish the inequality $F^- \ge G_\gamma$ by showin that

$$\liminf_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) F_{\alpha_n\beta_n}(u) \ge G_\gamma(u) \tag{4.3}$$

for any sequence $\{u_n\}$ converging to u in $L^2(\Omega)$. It is not restrictive to assume that the lim inf in (4.3) is a finite limit, and to assume, by a truncation argument, that $0 \le u_n \le 1$.

We first prove that $u = \chi_E$ is a characteristic function and that $\mathcal{L}^N(E) = \gamma$. Indeed, by Proposition 2.1 with $C = \{0\}$ and $C = \{1\}$, we infer that

$$\mathcal{L}^N(\{u = 0\}) \ge \limsup_{n\to+\infty} \mathcal{L}^N(\{u_n = 0\}) = (1-\gamma), \qquad \mathcal{L}^N(\{u = 1\}) \ge \limsup_{n\to+\infty} \mathcal{L}^N(\{u_n = 1\}) = \gamma.$$

In particular, there exists a Borel set $E \subset \Omega$ such that $u = \chi_E$. Since

$$\int_\Omega u_n \, dx \ge \beta_n + \int_{L_n} u_n \, dx,$$

with $L_n = \{0 < u_n < 1\}$, passing to the limit as $n \to +\infty$ we obtain

$$\mathcal{L}^N(E) = \int_\Omega u\, dx \geq \gamma,$$

as claimed.

Denoting by $\int_\Omega |Du|$ the total variation of a $L^1_{\rm loc}(\Omega)$ function u (see for instance [9]), we notice that

$$\begin{aligned}\int_\Omega |Du_n| &= \int_{L_n} |\nabla u_n|\, dx \leq \big(\mathcal{L}^N(L_n)\big)^{1/2} \left(\int_\Omega |\nabla u_n|^2\, dx\right)^{1/2} \\ &\leq \left[\big(1-(\alpha_n+\beta_n)\big)\int_\Omega |\nabla u_n|^2\, dx\right]^{1/2}.\end{aligned}$$

Therefore, as $P_\Omega(E) = \int_\Omega |D\chi_E|$ and $u \mapsto \int_\Omega |Du|$ is $L^1_{\rm loc}(\Omega)$ lower semicontinuous,

$$\begin{aligned}G_\gamma(u) &= [P_\Omega(E)]^2 = \left(\int_\Omega |Du|\right)^2 \leq \liminf_{n\to+\infty}\left(\int_\Omega |Du_n|\right)^2 \\ &\leq \liminf_{n\to+\infty}\big(1-(\alpha_n+\beta_n)\big)\int_\Omega |\nabla u_n|^2\, dx,\end{aligned} \tag{4.4}$$

and this proves (4.3).

Step 2. Next we establish the inequality $F^+(u) \leq G_\gamma(u)$. It is not restrictive to assume that $u = \chi_E$ is a characteristic function, $\mathcal{L}^N(E) = \gamma$ and $P_\Omega(E) < +\infty$.

We first assume that $E = D \cap \Omega$ for some bounded open set D with smooth boundary in $\mathbb{R}^N$, and we prove that

$$F^+(u) \leq \big[H^{N-1}(\partial D \cap \overline{\Omega})\big]^2. \tag{4.5}$$

Let

$$d(x) := \begin{cases} \operatorname{dist}(x, \partial D) & \text{if } x \notin D \\ -\operatorname{dist}(x, \partial D) & \text{if } x \in D \end{cases}$$

be the signed-distance function from D. Due to the smoothness of D, for $\sigma > 0$ sufficiently small we have that

$$\big\{x \in \mathbb{R}^N \,:\, d(x) = t\big\} = \big\{\Phi_t(x) \,:\, x \in \partial D\big\} \tag{4.6}$$

for $t \in (-\sigma, \sigma)$, where $\Phi_t(x) := x + t\nu(x)$ and ν is the unit outer normal to D. For n large enough,

$$\mathcal{L}^N(\{x \in \Omega \,:\, |d(x)| < \sigma\}) > 1 - (\alpha_n + \beta_n),$$

and hence we can find λ_n, $\mu_n \in (-\sigma, \sigma)$ such that $\lambda_n < \mu_n$ and

$$\mathcal{L}^N(\{x \in \Omega \,:\, d(x) \leq \lambda_n\}) = \alpha_n, \qquad \mathcal{L}^N(\{x \in \Omega \,:\, d(x) \geq \mu_n\}) = \beta_n.$$

By construction, the functions

$$u_n(x) = \frac{[\min\{d(x), \mu_n\} - \lambda_n]^+}{\mu_n - \lambda_n}$$

satisfy the constraint $\mathcal{L}^N(\{u_n = 0\}) = \alpha_n$, $\mathcal{L}^N(\{u_n = 1\}) = \beta_n$, and $u_n \to u$ in $L^2(\Omega)$. Using the identity $|\nabla d| = 1$ and the co-area formula (3.3) with $h \equiv 1$ we can estimate

$$\begin{aligned}
\big(1-(\alpha_n+\beta_n)\big)\int_\Omega |\nabla u_n|^2\,dx &= \big(1-(\alpha_n+\beta_n)\big)\int_{\{x\in\Omega\,:\,\lambda_n<d(x)<\mu_n\}} |\nabla u_n|^2\,dx \\
&= \frac{1-(\alpha_n+\beta_n)}{(\mu_n-\lambda_n)^2}\mathcal{L}^N(\{x\in\Omega\,:\,\lambda_n<d(x)<\mu_n\}) \\
&= \left[\frac{\mathcal{L}^N(\{x\in\Omega\,:\,\lambda_n<d(x)<\mu_n\})}{\mu_n-\lambda_n}\right]^2 \\
&= \left[\frac{1}{\mu_n-\lambda_n}\int_{\lambda_n}^{\mu_n} H^{N-1}(\{x\in\Omega\,:\,d(x)=t\})\,dt\right]^2.
\end{aligned}$$

Hence, to get (4.5) we need only to prove the inequality

$$\limsup_{t\to 0} H^{N-1}(\{x\in\Omega\,:\,d(x)=t\}) \le H^{N-1}(\partial D\cap\overline{\Omega}). \tag{4.7}$$

Indeed, let us fix an open set $A \supset \overline{\Omega}$. By (4.6), for $|t| < \min\{\sigma, \operatorname{dist}(\overline{\Omega}, \partial A)\}$,

$$\{x\in\Omega\,:\,d(x)=t\} \subset \Phi_t(A\cap\partial D);$$

hence

$$\limsup_{t\to 0} H^{N-1}(\{x\in\Omega\,:\,d(x)=t\}) \le \limsup_{t\to 0} H^{N-1}(\Phi_t(A\cap\partial D)) = H^{N-1}(A\cap\partial D),$$

and (4.7) follows by letting $A \downarrow \overline{\Omega}$.

Finally, by Lemma 4.3 below we can find a sequence of bounded open sets D_n with smooth boundary in $\mathbb{R}^N$ such that $u_n := \chi_{D_n\cap\Omega}$ converge to $u = \chi_E$ in $L^2(\Omega)$, $\mathcal{L}^N(D_n\cap\Omega) = \gamma$, and

$$\lim_{n\to+\infty} H^{n-1}(\partial D_n\cap\overline{\Omega}) = P_\Omega(E).$$

Applying (4.4) to u_n and using the lower semicontinuity of $u \mapsto F^+(u)$ (see [6]), we obtain

$$F^+(u) \le \liminf_{n\to+\infty} F^+(u_n) \le \liminf_{n\to+\infty} H^{N-1}(\partial D_n\cap\overline{\Omega}) = P_\Omega(E).$$

□

Lemma 4.3. *Let $E \subset \Omega$ be a set with finite perimeter such that $0 < \mathcal{L}^N(E) < \mathcal{L}^N(\Omega)$. There exists a sequence of bounded open sets $D_n \subset \mathbb{R}^N$ with smooth boundary in $\mathbb{R}^N$ such that $\mathcal{L}^N(E) = \mathcal{L}^N(D_n\cap\Omega)$, χ_{D_n} converges to χ_E in $L^2(\Omega)$, and*

$$\lim_{n\to+\infty} H^{N-1}(\partial D_n\cap\overline{\Omega}) = P_\Omega(E).$$

Proof. Let us first assume the existence of nonempty balls B, B' such that $B \subset E$ and $B' \subset \Omega\setminus E$. By a local reflection argument (see for instance [4]) we can extend E to a bounded set with finite perimeter E' in $\mathbb{R}^N$ such that $|D\chi_{E'}|(\partial\Omega) = 0$. It is possible to find bounded open sets E_n with smooth boundary that converge to E' and are such that (see [9])

$$\lim_{n\to+\infty} P_{\mathbb{R}^N}(E_n) = P_{\mathbb{R}^N}(E').$$

By the lower semicontinuity of the perimeter function on open sets,

$$
\begin{aligned}
P_\Omega(E) = P_\Omega(E') &\leq \liminf_{n\to+\infty} P_\Omega(E_n) \leq \limsup_{n\to+\infty} H^{N-1}(\partial E_n \cap \overline{\Omega}) \\
&= \limsup_{n\to+\infty} P_{\mathbb{R}^N}(E_n) - P_\Omega(E_n) \\
&\leq \limsup_{n\to+\infty} P_{\mathbb{R}^N}(E_n) - \liminf_{n\to+\infty} P_{\mathbb{R}^N\setminus\overline{\Omega}}(E_n) \\
&\leq P_{\mathbb{R}^N}(E') - P_{\mathbb{R}^N\setminus\overline{\Omega}}(E') = P_\Omega(E') = P_\Omega(E);
\end{aligned}
$$

whence $H^{N-1}(\partial E_n \cap \overline{\Omega})$ converges tp $P_\Omega(E)$.

Since $\mathcal{L}^N(E_n \cap \Omega)$ converges to $\mathcal{L}^N(E)$, possibly adding to E_n small balls contained in B' and possibly removing from E_n small balls contained in B, we obtain sets D_n with the same properties and with $\mathcal{L}^N(D_n \cap \Omega) = \gamma$.

To prove the general case, we notice that any set $E \subset \Omega$ with $0 < P_\Omega(E) < +\infty$ can be approximated, in area and perimeter, by sets E_h such that $\mathcal{L}^N(E_h) = \mathcal{L}^N(E)$ and such that both E_h and $\Omega \setminus E_h$ have nonempty interior: the approximation can, for instance, be achieved choosing a point $x \in \Omega$ where the density of E is $1/2$ and defining

$$E_h := E \cup B_{1/h}(x) \setminus B_{\rho_h}(x) \qquad h \geq 1,$$

with $\rho_h = \big(\mathcal{L}^N(B_{1/h}(x) \setminus E)/\omega_N\big)^{1/N}$ chosen to satisfy the volume constraint. Hence, since the approximation property is true for E_h, a diagonal argument leads to the existence of D_n also in the general case. □

Proof of Theorem 4.1. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences converging to $(1-\gamma)$, γ, respectively, and let $u_n \in W^{1,2}(\Omega;[0,1])$ be the corresponding solutions to $(\mathrm{M})_{\alpha_n,\beta_n}$ (see Remark 2.4(ii)) . By the general properties of Γ-convergence (see [6]), we need only to know that the sequence $\{u_n\}$ is relatively compact in $L^2(\Omega)$.

Let $E \subset \Omega$ be a set of finite perimeter with $\mathcal{L}^N(E) = \gamma$, and, in view of Theorem 4.2, let $\{v_n\}$ be a sequence converging to χ_E in $L^2(\Omega)$ and such that

$$\lim_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) \int_\Omega |\nabla v_n|^2 \, dx = [P_\Omega(E)]^2.$$

Since u_n are minimizing, we have

$$\limsup_{n\to+\infty} \big(1 - (\alpha_n + \beta_n)\big) \int_\Omega |\nabla u_n|^2 \, dx \leq [P_\Omega(E)]^2,$$

and (4.4) yields

$$\limsup_{n\to+\infty} \int_\Omega |\nabla u_n| \, dx \leq P_\Omega(E) < +\infty.$$

Since the embedding $BV(\Omega) \subset L^1(\Omega)$ is compact, and as the functions are equibounded, we conclude that $\{u_n\}$ is relatively compact in the $L^2(\Omega)$ topology. □

5 Competing Configurations

5.1 One-dimensional solutions

In the scalar case, where Ω is an interval, $d = 1$ and $\{z_1, z_2\} = \{0, 1\}$, the solutions of (M) are easily characterized. Assuming with no loss of generality that $\Omega = (0, 1)$, we claim that any minimizer u in Ω is affine in $\{0 < u < 1\}$. In fact, denoting by $\{A_i\}_{i \in I}$ the connected components of $\{0 < u < 1\}$, we have

$$\int_0^1 |u'|^2 \, dx = \sum_i \int_{A_i} |u'|^2 \, dx = \sum_i \frac{1}{\mathcal{L}^1(A_i)}.$$

The inequality between arithmetic mean and harmonic mean gives

$$\int_0^1 |u'|^2 \, dx \geq \frac{[\mathrm{card}(I)]^2}{\sum_i \mathcal{L}^1(A_i)} = \frac{[\mathrm{card}(I)]^2}{1 - \alpha - \beta}.$$

This proves that $\mathrm{card}(I) = 1$, i.e. $\{0 < u < 1\}$ has only one connected component, and that the least energy is $1/(1 - \alpha - \beta)$.

This argument can be repeated in the vector-valued case with $\Omega = (0, 1)$ and $d > 1$. Recalling that in this case we have existence for any finite set of constrained points $K = \{z_1, \ldots, z_P\}$ (not necessarily extremal points of a convex set, see Remark 2.4(iv)), it can be shown that the problem is equivalent to finding the shortest connection between these points. In fact, setting $\gamma := 1 - \sum_i \alpha_i$, we denote by *path* any finite sequence $w := \{w_1, \ldots, w_r\}$ such that

$$\{w_1, \ldots, w_r\} = K,$$

and we claim that the infimum of (M) is given by

$$\gamma^{-1} \inf \left\{ \sum_{j=1}^{r-1} |w_{j+1} - w_j| \, : \, w \text{ is a path} \right\}^2.$$

In fact, using the Lagrange multiplier rule, this infimum can be represented by

$$\text{(P)} \qquad \inf \left\{ \sum_{j=1}^{r-1} \frac{|w_{j+1} - w_j|^2}{a_j} \, : \, w \text{ is a path}, \ \sum_{j=1}^{r} a_j = \gamma \right\}.$$

If $u \in H^1(0, 1)$ is any admissible function for (M) and I is any connected component of $A_u := \{u \notin K\}$, then the condition $u(\partial I) \subset K$ implies that

$$\int_I |u'|^2 \, dt \geq \frac{[\mathrm{osc}(u, I)]^2}{\mathcal{L}^1(I)} \geq \delta^2,$$

where $\delta > 0$ is the least distance between two points in K. Hence, A_u has only finitely many connected components. It is now easy to establish a one to one correspondence, with equivalence of the energies, between admissible functions $u \in H^1(0, 1)$ for (M) and admissible pairs (w_i, a_i) for (P). In fact, given $u \in H^1(0, 1)$ admissible for (M), if $I = (s, t)$ is a connected component of A_u with length a_j, then we set

$$w_j := \lim_{x \downarrow s} u(x), \qquad w_{j+1} := \lim_{x \uparrow t} u(x).$$

If u is a solution for (M) then

$$\int_I |u'|^2 \, dx = \frac{|w_{j+1} - w_j|^2}{a_j},$$

and clearly (w_j, a_j) is admissible for (P). Conversely, any $\{(w_j, a_j)\}$ admissible for (P) corresponds to a function u admissible for (M), where u is piecewise affine, has slope $|w_{j+1} - w_j|/a_j$ in intervals with length a_j, and whose level sets $\{u = z_i\}$ are formed by n_i intervals (possibly reducing to a single point) with total length α_i, where

$$n_i := \mathrm{card}\left(\{j \,:\, w_j = z_i\}\right).$$

5.2 The circle : radial and comparison configurations

Here we study candidates for solutions of the problem

$$\text{(Mr)} \qquad \min\left\{\int_B |\nabla u|^2\,dx : u \in W_r^{1,2}(B),\, \mathcal{L}^N(\{u=0\}) = \alpha, \mathcal{L}^N(\{u=1\}) = \beta\right\},$$

where B is the unit ball of $\mathbb{R}^2$ and $W_r^{1,2}(B)$ denotes the space of radial functions in $W^{1,2}(B)$.

Let $u(x) := g(|x|) \in W_r^{1,2}(B)$, where g is continuous in $(0,1]$. We define

$$\bar{r} := \sup\left\{r \in (0,1] \,:\, g(r) \in \{0,1\}\right\}.$$

If $g(\bar{r}) = 0$ we can make a nonincreasing rearrangement of g that preserves the measure of level sets of u (see for instance [11], Lemma 7.17) to obtain a new function $\tilde{u}(x) = \tilde{g}(|x|)$ whose Dirichlet integral does not exceed that of u, and which is still admissible for (M). If $g(\bar{r}) = 1$ the same argument can be applied to $1-u$. To determine the minimum energy we may therefore restrict ourselves to nonincreasing or nondecreasing functions g.

A computation shows that $g(r) = a + b\ln r$ in $\{0 < g < 1\}$ for suitable constants a, b. In the nonincreasing case these constants can be computed using the volume constraints to find $b = 1/\ln(r_0/r_1)$ and $a = -b\ln r_1$, where

$$r_0 := \sqrt{\frac{\beta}{\pi}}, \qquad r_1 := \sqrt{\frac{\pi-\alpha}{\pi}}.$$

With these choices of a, b, the Dirichlet integral reduces to $4\pi/\ln((\pi-\alpha)/\beta)$. Taking into account also the nondecreasing case, we find that the minimal energy of (M) is

$$\min\left\{\frac{4\pi}{\ln\frac{\pi-\alpha}{\beta}}, \frac{4\pi}{\ln\frac{\pi-\beta}{\alpha}}\right\}.$$

We claim that the solutions of (M) are generally not radial. Consider the family of functions

$$u_{ab}(z) := a + b\ln\left|\frac{z+1}{z-1}\right|^2$$

in the complex variable $z = x + iy$, defined in the unit disk $\Omega = \{|z| < 1\}$. These functions are harmonic, and their level sets are circles orthogonal to $\partial\Omega$, i.e., the solutions of the constrained least area problem (4.1). This might suggest that the functions $\min\{\max\{0, u_{ab}\}, 1\}$ are solutions of (M), for suitable a, b depending on α, β, at least when $\alpha + \beta$ is close to π. However, this is not true because the normal derivative is not constant on level sets thereby violating the necessary condition for minimality stated in Proposition 3.4. This can be seen either by direct computation or by the conformal change of variables $w = \log[(z+1)/(z-1)]$, mapping the circles on vertical segments in the w plane and Ω onto a strip; in the new configuration the functions have constant normal derivative, hence in the original configuration this property cannot be true.

However, the functions u_{ab} can be used to show that for α, $\beta << 1$, the solutions of (M) are not radial; in fact, using the equations $\Delta u_{ab} = 0$, $\partial u_{ab}/\partial n = 0$, it can be shown that

$$\int_\Omega |\nabla u_{ab}|^2\, dz = \int_{\{x=0\}} \frac{\partial u_{ab}}{\partial x}\, dH^1 = 4b \int_{-1}^{1} \frac{1}{y^2+1}\, dy = 2b\pi.$$

Denoting by r_0 and r_1 the radii of the circles $\{u_{ab} = 0\}$, $\{u_{ab} = 1\}$, respectively, for α, $\beta << 1$,

$$1 \sim a + b \ln \frac{4}{r_1^2}, \qquad 0 \sim a + b \ln \frac{r_0^2}{4}, \qquad \alpha \sim \frac{\pi r_0^2}{2}, \qquad \beta \sim \frac{\pi r_1^2}{2},$$

and thus the least energy of (M) cannot exceed

$$\frac{2\pi}{\ln \frac{4\pi^2}{\alpha\beta}}.$$

For $\alpha = \beta$, this quantity is asymptotically 4 times smaller than the least energy of radial solutions.

5.3 The square: piecewise affine and comparison configurations

Let $\Omega = (0,1)^2$, $d = 1$, fix $\alpha, \beta \in (0,1)$, with $\alpha + \beta < 1$, and consider the piecewise linear function $u : (0,1)^2 \to \mathbb{R}^2$ defined by

$$u(x) = \begin{cases} 0 & \text{if } x_1 \le \alpha, \\ \frac{1}{1-\alpha-\beta} x_1 - \frac{\alpha}{1-\alpha-\beta} & \text{if } \alpha < x_1 < 1-\beta, \\ 1 & \text{if } x_1 \ge 1-\beta \end{cases}$$

We claim that, even though u satisfies the optimality conditions of Section 3, u will not solve (M) when $\alpha + \beta << 1$, nor when $\alpha + \beta$ is close to 1. Indeed,

$$\int_\Omega |\nabla u|^2\, dx = \frac{1}{1-\alpha-\beta},$$

and if we consider a competing configuration v such that $v = 0$ on a right triangle with right angle at $(0,0)$, $v = 1$ on a right triangle with right angle at the vertex $(1,1)$, and v is linear in the region between these two triangles, then it can be shown that

$$\int_\Omega |\nabla v|^2\, dx = \frac{1-\alpha-\beta}{(\sqrt{2} - \sqrt{\alpha} - \sqrt{\beta})^2}.$$

In particular,

$$\int_\Omega |\nabla v|^2\, dx < \int_\Omega |\nabla u|^2\, dx$$

for $\alpha + \beta$ sufficiently small, for instance, if $\alpha + \beta < 3 - 2\sqrt{2}$. Finally, considering the limiting configuration, which is equal to one on a quarter of a circle centered at $(0,0)$ with radius r, and it is constantly equal to zero elsewhere on the square, then $r = 2\sqrt{\beta/\pi}$, and the perimeter of the interface is $\sqrt{\pi\beta}$. By Theorem 4.1 we conclude that if $\sqrt{\pi\beta} < 1$, then u cannot be a solution for (M).

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