STABILITY OF VORTEX SOLUTIONS TO AN EXTENDED NAVIER-STOKES SYSTEM

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Abstract

We study the long-time behavior an extended Navier-Stokes system in $\mathbb{R}^2$ where the incompressibility constraint is relaxed. This is one of several "reduced models" of Grubb and Solonnikov '89 and was revisited recently (Liu, Liu, Pego '07) in bounded domains in order to explain the fast convergence of certain numerical schemes (Johnston, Liu '04). Our first result shows that if the initial divergence of the fluid velocity is mean zero, then the Oseen vortex is globally asymptotically stable. This is the same as the Gallay Wayne '05 result for the standard Navier-Stokes equations. When the initial divergence is not mean zero, we show that the analogue of the Oseen vortex exists and is stable under small perturbations. For completeness, we also prove global well-posedness of the system we study.

1. Introduction

The dynamics of vortices of the incompressible Navier-Stokes equations play a central role in the study of many problems. Mathematically, control of the vorticity production [1, 8] will settle a longstanding open problem regarding global existence of smooth solutions [7, 10]. Physically, regions of intense vorticity manifest themselves as cyclones in the atmosphere [9, 30], and at a slightly decreased intensity as eddies in the oceans [6, 32]. In all cases, regions of intense vorticity are of vital geophysical (and astrophysical) interest.

After many years of intense study (see for instance [2, 3, 5, 11, 13, 15, 16, 24, 29, 31, 35–38]), the seminal work of Gallay and Wayne [14] proved the existence of a globally stable (infinite energy) vortex in $\mathbb{R}^2$, known as the Oseen vortex. Physically, this means that any L^1 configuration of vortex patches will eventually combine into a "giant" vortex and then dissipate like the linear heat equation. The main result of this paper is the analogue of this result

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for an extended Navier-Stokes system where the incompressibility constraint is relaxed.

The equations we study are one of several "reduced models" of Grubb and Solonnikov [18, 19]. This model resurfaced recently in [26] to analyze a stable and efficient numerical scheme proposed in [22]. The numerical scheme is a time discrete, pressure Poisson scheme which improves both stability and efficiency of computation by replacing the incompressibility constraint with an auxiliary equation to determine the pressure. The formal time continuous limit of this scheme is the system

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u, \\ \partial_t d = \Delta d, \\ d = \nabla\cdot u, \end{cases} \tag{1.1}$$

where u represents the fluid velocity and p the pressure. We draw attention to the fact that the usual incompressibility constraint, $d=0$, in the Navier-Stokes equations has been replaced with an evolution equation for d. Of course, if $d=0$ at time 0, then it will remain 0 for all time and the system (1.1) reduces to the standard incompressible Navier-Stokes equations.

In domains with boundary the system (1.1) has been studied by numerous authors [20, 21, 23, 26–28] both from an analytical and a numerical perspective. Boundaries, however, cause production of vorticity in a nontrivial manner and make the long time behavior of the vorticity intractable by current methods. Thus, we study the system (1.1) in $\mathbb{R}^2$ where at least the long time behavior of vorticity when $d=0$ is now reasonably understood [14].

Since d approaches 0 asymptotically as $t\to\infty$, we expect that the long time behavior of solutions to (1.1) should be the same as that of the standard incompressible Navier-Stokes equations. Indeed, our first result (theorem 2.1) shows that this is the case, *provided* the initial divergence d_0 has mean 0. In this case, the entropy constructed in [14] can still be used to show global stability of the Oseen vortex. Surprisingly, if d_0 does not have mean 0, the nonlinearity contributes to the entropy non-trivially and we are unable to show global stability of a steady solution using this method. Instead when d_0 has non-zero mean, we use methods similar to [34] and show existence (but not uniqueness) of a solution that is stable under small perturbations globally in time, provided d_0 has a small enough mean. We are unable to show that this solution is stable under large perturbations. Further, if d_0 has large mean, we are unable to show that this solution is stable even under small perturbations.

Plan of this paper. In section 2 we introduce our notation and state our main results. Next, in section 3 we show that if $\beta=0$ the Oseen vortex is the global asymptotically stable steady state. Then, in section 4, we study the analogue of this result when $\beta\neq 0$. We find the analogue of the Oseen vortex in this context, but are unable to show a global stability result like in the case when $\beta=0$. We instead show that the solution is globally stable under perturbations that are small in Gaussian weighted spaces. The proofs in section 3 relied on certain heat kernel like bounds for the vorticity and on relative compactness of complete trajectories. We prove these in sections 5 and 6 respectively. Finally, to ensure our results long time results are not vacuously true, we conclude this paper with section 7, where briefly discuss global well-posedness for the extended Navier-Stokes system in this context.

2. Statement of results.

For our purposes it is more convenient to formulate (1.1) in terms of the vorticity

$$\omega \stackrel{\text{def}}{=} \nabla\times u = \partial_1 u_2 - \partial_2 u_1.$$

Taking the curl of (1.1) gives the system

$$\partial_t\omega + \nabla\cdot(u\omega) = \Delta\omega, \tag{2.1}$$

$$\partial_t d = \Delta d, \tag{2.2}$$

$$u = K_{BS} * \omega + \nabla^{-1} d, \tag{2.3}$$

where, K_{BS} and ∇^{-1} are defined by

$$K_{BS}(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\frac{x^\perp}{|x|^2}, \qquad \text{and} \qquad \nabla^{-1} f \stackrel{\text{def}}{=} \frac{1}{2\pi}\frac{x}{|x|^2} * f.$$

Equation (2.3) simply recovers u as the unique vector field with divergence d and curl ω. When $d=0$, this is simply the Biot-Savart law, hence our notation K_{BS}.

Formally integrating equations (2.1) and (2.2), one immediately sees that the quantities

$$\alpha \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \omega(x,t)\,dx, \qquad \text{and} \qquad \beta \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} d(x,t)dx \tag{2.4}$$

are constant in time. The value of α in the long term vortex dynamics is mainly that of a scaling factor and not too important. The value of β, however, affects the dynamics (or at least our proofs) dramatically. We begin by studying the long term vortex dynamics when $\beta=0$. In this case we show that the Oseen vortex defined by

$$\tilde\omega(x,t) = \frac{1}{t} G\Big(\frac{x}{\sqrt{t}}\Big)$$

is the globally stable solution, where

$$G(x) \stackrel{\text{def}}{=} \frac{1}{4\pi}\exp\Big(\frac{-|x|^2}{4}\Big)$$

is the Gaussian. We state this as our first result.

Theorem 2.1. *Suppose ω_0, $d_0 \in L^1(\mathbb{R}^2)$ are such that $|x| d_0 \in L^1(\mathbb{R}^2)$ and $\beta = 0$. If the pair (ω, d) solves the system* (2.1)–(2.3) *with initial data (ω_0, d_0) then for any $p \in [1,\infty]$ we have*

$$\lim_{t\to\infty} t^{1-1/p}\|\omega(t,\cdot) - \alpha\tilde\omega(t,\cdot)\|_{L^p} = 0 \quad \textit{and} \quad \sup_{t\geqslant 0} t^{\frac{3}{2}-\frac{1}{p}}\|d(t,\cdot)\|_{L^p} < \infty. \tag{2.5}$$

When $\beta \neq 0$, we are unable to prove a result as strong as theorem 2.1, because a key entropy estimate is destroyed by the nonlinearity. To formulate our result in this situation, we first identify the analogue of the Oseen vortex. We show (in section 4.1) that the radial self-similar solutions to the system (2.1)–(2.3) are obtained by rescaling $W_s = W_s(\beta)$, where W_s is the unique, radially symmetric, solution of the ODE

$$\frac{\partial_r W_s}{W_s} = \frac{-r}{2} + \frac{\beta}{2\pi r}\left(1 - e^{-r^2/4}\right), \quad \text{with normalization } \int_{\mathbb{R}^2} W_s\,dx = 1. \tag{2.6}$$

A direct calculation shows that the pair $(\alpha\tilde{\omega}_\beta, \tilde{d}_\beta)$ defined by

$$\tilde{\omega}_\beta(x,t) = \frac{1}{t} W_s\Big(\frac{x}{\sqrt{t}}\Big), \qquad \tilde{d}_\beta(x,t) = \frac{\beta}{t} G\Big(\frac{x}{\sqrt{t}}\Big), \tag{2.7}$$

is a radially symmetric self-similar solution to the system (2.1)–(2.3), making $\tilde{\omega}_\beta$ the analogue of the Oseen vortex. When $\beta = 0$, we see W_s is exactly the Gaussian G, but this is no longer true when $\beta \neq 0$. When $\beta < 4\pi$ the shape of W_s is similar to that of the Gaussian in that W_s attains it's maximum at 0 and is strictly decreasing for $r > 0$. When $\beta > 4\pi$, however, W_s attains its maximum at some $r_0 > 0$ and the profile looks like that of a "vortex ring" (see figure 1). For any $\beta \neq 0$, the interaction between W_s and the nonlinearity is largely responsible for the failure in our proof of theorem 2.1.

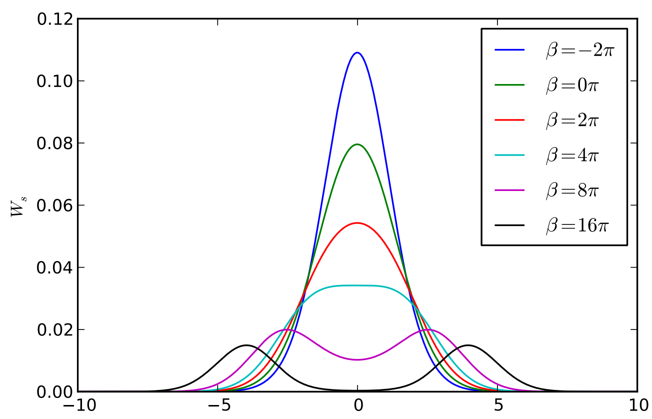

Fig. 1. Plots of W_s vs r for $\beta \in \{-2\pi, 0, 2\pi, 4\pi, 8\pi, 16\pi\}$.

Our main result when $\beta \neq 0$ uses the Gaussian weighted spaces appearing in [13, 14, 34] and shows that the solution $(\alpha\tilde{\omega}_\beta, \tilde{d}_\beta)$ is stable under small perturbations. Explicitly, define the weighted spaces L^2_w by

$$L^2_w \stackrel{\text{def}}{=} \{f \in L^2(\mathbb{R}^2) : \|f\|_w < \infty\}, \quad \text{where} \quad \|f\|_w^2 \stackrel{\text{def}}{=} \int G(x)^{-1} |f(x)|^2 dx. \tag{2.8}$$

Now our stability result when $\beta \neq 0$ is as follows:

Theorem 2.2. *Let $t_0 > 0$ and (ω, d) solve the system (2.1)–(2.3) on the time interval $[t_0, \infty)$. For any $\gamma \in (0, 1/2)$, there exists $\varepsilon_0 = \varepsilon_0(\gamma) > 0$ such that if*

$$|\beta|(1 + |\alpha|) + \big\|\omega(t_0) - \alpha\tilde{\omega}_\beta(t_0)\big\|_w + \big\|d(t_0) - \tilde{d}_\beta(t_0)\big\|_w \leqslant \varepsilon_0,$$

then

$$\lim_{t\to\infty} t^\gamma \|\tilde{G}(t)^{-1/2}(\omega(t) - \alpha\tilde{\omega}_\beta(t))\|_w = 0$$

and

$$\sup_{t\geqslant 0} t^{1/2}\|\tilde{G}(t)^{-1/2}(d(t)-\tilde{d}_\beta(t))\|_w<\infty.$$

Here $\tilde{G}(x,t)=G(x/\sqrt{t})$ *is the rescaled Gaussian.*

When $\beta=0$, the function $\tilde{\omega}_\beta=\tilde{\omega}$, and theorem 2.1 proves stability of $\tilde{\omega}$ (albeit under a different norm) without any smallness assumption on the perturbation.

Finally, to ensure that theorems 2.1 and 2.2 are not vacuously true, we establish global existence of solutions to the system (2.1)–(2.3). While a little work has been done on this system in $\mathbb{R}^2$, the existence and uniqueness theory is not altogether far from the classical theory, and we address this next.

Proposition 2.3. *Define the space* X *to be either* L^1 *or* L^2_w. *If* $\omega_0,d_0\in X$, *then there exists a unique time global solution* (ω,d) *to the system* (2.1)–(2.3) *in* X *with initial data* (ω_0,d_0).

The proof of this proposition follows a similar structure to results in [2,3,13, 14,25,34], and we do not provide a complete proof. However, for convenience of the reader, we sketch a brief outline in section 7.

3. Global stability for mean zero initial divergence.

We devote this section to proving theorem 2.1. The main idea in the case where $\beta=0$ is the same as that used by Gallay and Wayne in [14]. However, to use this method, certain compactness criteria and vorticity bounds need to be established. In order to present a self contained treatment, we begin with the heart of the matter (following [14]), and only state the compactness criteria where required. We postpone the proofs of the vorticity bounds and these criteria to sections 5 and 6 respectively.

3.1. Reformulation using self-similar coordinates. We begin by reformulating theorem 2.1 in the natural self-similar coordinates associated to (1.1).

Proof of theorem 2.1. Define the coordinates ξ and τ by

$$\xi\stackrel{\text{def}}{=}\frac{x}{\sqrt{t}},\quad \tau\stackrel{\text{def}}{=}\log(t), \tag{3.1}$$

and the rescaled velocity, vorticity, and divergence by

$$U(\xi,\tau)\stackrel{\text{def}}{=}\sqrt{t}u(x,t),\quad W(\xi,\tau)\stackrel{\text{def}}{=}t\omega(x,t),\quad \text{and}\quad D(\xi,\tau)\stackrel{\text{def}}{=}td(x,t). \tag{3.2}$$

With this transformation the system (2.1)–(2.3) becomes

$$\partial_\tau W+\nabla\cdot(UW)=\mathcal{L}W, \tag{3.3}$$

$$\partial_\tau D=\mathcal{L}D, \tag{3.4}$$

$$U=K_{BS}*W+\nabla^{-1}D. \tag{3.5}$$

where $\mathcal{L}$ is the operator defined by

$$\mathcal{L}f\stackrel{\text{def}}{=}\Delta f+\frac{1}{2}\xi\cdot\nabla f+f. \tag{3.6}$$

In the rescaled variables we will prove the following result:

Proposition 3.1. *Let (W,D) solve the system (3.3)–(3.5) with initial data (W_0,D_0) such that $W_0,(1+|\xi|)D_0\in L^1(\mathbb{R}^2)$. If $\alpha=\int W_0\,d\xi$ and $\beta=\int D_0\,d\xi=0$, then*

$$\lim_{\tau\to\infty}\|W-\alpha G\|_{L^p}=0 \quad \text{and} \quad \sup_{\tau\geqslant 0} e^{\frac{\tau}{2}}\|D\|_{L^p}<\infty \tag{3.7}$$

for any $p\in[1,\infty]$.

Undoing the change of variables immediately yields theorem 2.1. □

Before proving proposition 3.1 we pause momentarily to explain why the proof in this case is similar to the proof in [14] for the standard Navier-Stokes equations. The only mean zero function D that decays sufficiently at infinity and is an equilibrium solution to (3.4) is the 0 function, in which case the system (3.3)–(3.5) reduces to the standard Navier-Stokes equations in self-similar coordinates. Thus, when $\beta=0$, the long time dynamics of the system (3.3)–(3.5) should be similar to that of the standard Navier-Stokes equations (in self-similar coordinates). Indeed, as we show below, the key step of the proof in [14] goes through almost unchanged. Of course, the required bounds and compactness estimates leading up to this still require work to prove and, for clarity of presentation, we postpone their proofs to sections 5 and 6.

The proof of proposition 3.1 consists of two main steps. The first step is to establish relative compactness of trajectories to the system (3.3)–(3.5) in the space L^1 and is our next lemma.

Lemma 3.2. *Suppose that W and D solve the system (3.3)–(3.5) in $C^0([0,\infty),L^1(\mathbb{R}^2)\times L^1(\mathbb{R}^2))$. Then the trajectory $\{(W(\tau),D(\tau))\}_{\tau\in[0,\infty)}$ is relatively compact in $L^1(\mathbb{R}^2)$. Further,*

$$|W(\xi,\tau)|\leqslant C\int_{\mathbb{R}^2}\exp\Big(\frac{-|\xi-\eta e^{-\tau/2}|^2}{C}\Big)|W_0(\eta)|\,d\eta. \tag{3.8}$$

for some constant C which depends only on $\|W_0\|_{L^1}$ and $\|(1+|\xi|)D_0(\xi)\|_{L^1}$.

The second step in the proof of proposition 3.1 is to characterize complete trajectories of the system (3.3)–(3.5). To do this we need to introduce a weighted L^p space. For any $m\geqslant 0$, $p\in[1,\infty)$ we define the space $L^p(m)$ by

$$L^p(m)=\Big\{f\in L^p:\|f\|_{L^p(m)}<\infty,\ \text{where } \|f\|^p_{L^p(m)}=\int(1+|\xi|^2)^{\frac{pm}{2}}|f(\xi)|^p d\xi\Big\}.$$

It turns out that the only complete trajectories of the system (3.3)–(3.5) that are bounded in $L^2(m)$ are scalar multiples of the Gaussian. This is our next lemma.

Lemma 3.3. *Let $m>3$ and suppose that $\{(W(\tau),D(\tau))\}$ is a complete trajectory of the system (3.3)–(3.5) which is bounded in $L^2(m)$. Then, if $\int W_0=\alpha$ and $\int D_0=0$ we must have $W(\tau)=\alpha G$ and $D=0$ for all τ.*

Momentarily postponing the proofs of lemmas 3.2 and 3.3, we prove proposition 3.1.

Proof of proposition 3.1. Let Ω be the ω-limit set of the trajectory (W,D). Since lemma 3.2 guarantees $\{W(\tau)\}$ and $\{D(\tau)\}$ are relatively compact in L^1, Ω must be non-empty, compact, and fully invariant under the evolution of the system (3.3)–(3.5). Consequently, the trajectory of any $(\overline{W},\overline{D})\in\Omega$ must be complete.

Further, the upper bound (3.8) implies $\overline{W}$ is bounded above by a Gaussian. To see this, choose a sequence of times $\tau_n\to\infty$ such that $(W(\tau_n))\to\overline{W}$ in L^1 and almost everywhere. Now dominated convergence and (3.8) imply

$$\begin{aligned}|\overline{W}(\xi)| &= \lim_{n\to\infty}|W(\xi,\tau_n)| \leqslant \lim_{n\to\infty} C\int \exp\Big(\frac{-|\xi-\eta e^{\frac{-\tau_n}{2}}|^2}{C}\Big)|W_0(\eta)|\,d\eta\\ &\leqslant C\|W_0\|_{L^1}\exp\Big(\frac{-|\xi|^2}{C}\Big).\end{aligned}$$

Consequently $\Omega\subset L^2(m)^2$ for every m.

This implies that for any $(\overline{W},\overline{D})$, the associated complete trajectory is bounded in $L^2(m)^2$ for every m. Thus lemma 3.3 shows $\Omega\subset\{(\theta G,0):\theta\in\mathbb{R}\}$. Since total mass is invariant under the flow (and $\Omega\neq\emptyset$), it follows that $\Omega=\{(\alpha G,0)\}$, where α is defined in (2.4). Since Ω contains exactly one element and $(W(\tau),D(\tau))$ is relatively compact in L^1, this immediately implies the first equality in (3.7) for $p=1$. Combined with the Gaussian upper bound implied by (3.8), we obtain the first inequality in (3.7) for any $p<\infty$.

The proof for $p=\infty$ uses bounds on the semigroup generated by the operator $\mathcal{L}$ and an integral representation for W. Since we develop these bounds in section 6, we prove L^∞ convergence as lemma 6.5 at the end of section 6.

The second inequality in (3.7) follows directly from the explicit solution formula for the heat equation. Since this will also be used later, we extract it as a lemma.

Lemma 3.4. *Let D be a solution to* (3.4) *with initial data D_0. Suppose*

$$\int_{\mathbb{R}^2}(1+|\xi|)|D_0(\xi)|\,d\xi<\infty \quad \textit{and} \quad \int_{\mathbb{R}^2}D_0\,d\xi=0.$$

There there exists a universal constant $C>0$ such that

$$\|D(\tau)\|_{L^p}\leqslant Ce^{-\tau/2}\int_{\mathbb{R}^2}(1+|\xi|)|D_0(\xi)|\,d\xi \tag{3.9}$$

for all $p\in[1,\infty]$.

We remark that the decay rate of D to 0 being faster than that of the rescaled heat kernel is because the initial data has mean-zero. This concludes the proof of proposition 3.1. □

It remains to prove lemmas 3.2–3.4. The proof of lemma 3.4 is short, and we present it here.

Proof of lemma 3.4. Since the heat kernel in x-t coordinates is common knowledge, we return to the x-t coordinates and prove d satisfies the second inequality in (2.5). Let $\bar{G}(x,t)=G(x/\sqrt{t})/t$ be the heat-kernel. Observe

$$\begin{aligned}\|d(t)\|_{L^p} &= \|d_0*\bar{G}(t)\|_{L^p} = \Big\|\int_{\mathbb{R}^2}d_0(y)\bar{G}(x-y,t)\,dy\Big\|_{L^p(x)}\\ &= \Big\|\int_{\mathbb{R}^2}d_0(y)\big(\bar{G}(x-y,t)-\bar{G}(x,t)\big)\,dy\Big\|_{L^p(x)}\end{aligned}$$

$$\begin{aligned}&\leqslant \frac{1}{t^{1-1/p}} \int_{\mathbb{R}^2} |d_0(y)| \big\|G(x-t^{-1/2}y)-G(x)\big\|_{L^p(x)}\,dy\\&\leqslant \frac{C}{t^{\frac{3}{2}-\frac{1}{p}}} \int_{\mathbb{R}^2} |y d_0(y)|\,dy,\end{aligned}$$

which implies the second inequality in (2.5) and concludes the proof. □

The proof of lemma 3.2 is technical; we postpone the proof of (3.8) to section 5 and the proof of compactness to section 6. We prove lemma 3.3 in section 3.2.

3.2. Characterization of complete trajectories. The characterization of complete trajectories to the system (3.3)–(3.5) when $\beta = 0$ is identical to the characterization of complete trajectories of the 2D Navier-Stokes equations presented in [14]. Since the proof is short and elegant, we reproduce it here for the reader's convenience.

There are two steps to this proof: First, show that in a complete trajectory both W and D must have constant sign. Of course, since D is mean-zero, this forces $D = 0$ identically, and reduces to the situation already considered by Gallay and Wayne [14]. Second, the most interesting step, is to use the Boltzmann entropy functional to show that W must be a scalar multiple of a Gaussian. This is exactly what fails in the case where D is not mean zero.

We state each of these steps as lemmas, below:

Lemma 3.5. *Suppose $m > 3$ and $(W, D) \in C^0(\mathbb{R}, L^2(m)^2)$ is a solution of the system* (3.3)–(3.5) *which is bounded in $L^2(m)$. Then both W and D must have constant sign.*

Lemma 3.6. *Let (W, D) be a solution to the system* (3.3)–(3.5) *with $W_0 \in L^2(m)$, $D_0 = 0$, $W_0 \geqslant 0$. For the relative entropy H given by*

$$H(W) = \int_{\mathbb{R}^2} W \ln\Big(\frac{W}{G}\Big)\,d\xi, \tag{3.10}$$

we have

$$\partial_\tau H = -\int_{\mathbb{R}^2} W \Big|\nabla \ln\Big(\frac{W}{G}\Big)\Big|^2\,d\xi. \tag{3.11}$$

Lemma 3.3 immediately follows from lemmas 3.5–3.6, and we spell it out here for completeness.

Proof of lemma 3.3. By lemma 3.5, we know that both W and D have constant sign. Since $\int D = 0$, this forces $D = 0$ identically. Further, by symmetry we can assume $W \geqslant 0$.

Note that by the comparison principle the set $L^2(m) \cap \{\tilde{W} \geqslant 0\}$ is invariant under the dynamics of the system (3.3)–(3.5). Restricting our attention to this set, we observe that the entropy H is strictly decreasing except on the set of equilibria $\tilde{W} = \theta G$. By LaSalle's invariance principle this implies that $W = \theta G$ for some θ. Since $\int W = \alpha$ this forces $\theta = \alpha$ concluding the proof. □

It remains to prove lemmas 3.5 and 3.6, which we do in sections 3.2.1 and 3.2.2 respectively.

3.2.1. The sign of complete trajectories. The main idea behind the proof of lemma 3.5 is that the L^1 norm can be used as a Lyapunov functional. However, we first need a relative compactness lemma to guarantee that the α and ω-limit sets are non-empty, and we state this next.

Lemma 3.7. *Let $m>3$ and suppose $(W,D)\in C^0(\mathbb{R},L^2(m)^2)$ is a solution to the system (3.3)–(3.5) which is bounded in $L^2(m)$. The trajectory $\{(W(\tau),D(\tau))\}_{\tau\in\mathbb{R}}$ is relatively compact in $L^2(m)$.*

Lemma 3.7 is also used in the proof of lemma 3.2, and we defer its proof to section 6. We prove lemma 3.5 next.

Proof of lemma 3.5. Define the Lyapunov function Φ by $\Phi(W,D)=\|W\|_{L^1}+\|D\|_{L^1}$. We claim that Φ is always decreasing, and is strictly decreasing in time if and only if one of W and D does not have a constant sign. To see this, define W^+ and W^- to be the solutions to

$$\partial_\tau W^+ + \nabla\cdot(UW^+) = \mathcal{L}W^+ \quad\text{and}\quad \partial_\tau W^- + \nabla\cdot(UW^-) = \mathcal{L}W^-,$$

with initial data $W_0^+=\max\{W,0\}$ and $W_0^-=\max\{-W,0\}$ respectively. We clarify that $U=K_{BS}*W$ here, and does not depend on W^+ or W^-. Clearly $W^\pm\geqslant 0$ and $W=W^+-W^-$ for all time. Further, if both W^+ and W^- are non-zero initially, the strong maximum principle implies that for any $\tau>0$ the supports of $W^\pm(\tau)$ will necessarily intersect. Consequently, for any $\tau>0$,

$$\begin{aligned}(3.12)\qquad \int_{\mathbb{R}^2}|W(\xi,\tau)|\,d\xi &< \int_{\mathbb{R}^2}(W^+(\xi,\tau)+W^-(\xi,\tau))\,d\xi\\ &= \int_{\mathbb{R}^2}(W_0^+(\xi)+W_0^-(\xi))\,d\xi = \int_{\mathbb{R}^2}|W_0|\,d\xi.\end{aligned}$$

A similar argument can be applied to D and replacing $\tau=0$ with any arbitrary time τ_0 will show that Φ is strictly decreasing in time if and only if either W or D do not have a constant sign.

To see that complete trajectories have constant sign, we appeal to lemma 3.7 to guarantee that the trajectory $\{(W(\tau),D(\tau)\}_{\tau\in\mathbb{R}}$ has both an α and an ω-limit. Now choose two sequences of times $(\overline{\tau}_n)\to\infty$ and $(\underline{\tau}_n)\to-\infty$ such that

$$\underline{W}=\lim W(\underline{\tau}_n) \quad\text{and}\quad \overline{W}=\lim W(\overline{\tau}_n) \quad \text{in } L^2(m).$$

Since $\int W$ is conserved we must have $\int\overline{W}=\int\underline{W}$. Further, by LaSalle's invariance principle both $\overline{W}$ and $\underline{W}$ have constant sign. Consequently, for any $\tau\in\mathbb{R}$,

$$\Big|\int_{\mathbb{R}^2}\underline{W}\,d\xi\Big| = \int_{\mathbb{R}^2}|\underline{W}|\,d\xi \geqslant \int_{\mathbb{R}^2}|W(\tau)|\,d\xi \geqslant \int_{\mathbb{R}^2}|\overline{W}|\,d\xi = \Big|\int_{\mathbb{R}^2}\overline{W}\,d\xi\Big| = \Big|\int_{\mathbb{R}^2}\underline{W}\,d\xi\Big|.$$

Hence, $\int|W(\tau)|d\xi$ is constant in τ. This, along with (3.12), shows that W has a constant sign. A similar argument can be applied to D. This shows that Φ is constant in time and hence both W and D must have constant sign. □

3.2.2. Decay of the Boltzmann entropy. The use of the relative entropy H in this context was suggested by C. Villani, and the decay (when $D=0$) is a direct calculation that was carried out in [14, lemma 3.2]. We briefly sketch a few details here for the readers convenience.

Proof of lemma 3.6. Differentiating (3.10) with respect to τ gives

$$\partial_\tau H = \int_{\mathbb{R}^2} \Big(1 + \ln\Big(\frac{W}{G}\Big)\Big) \partial_\tau W = \int_{\mathbb{R}^2} \Big(1 + \ln\Big(\frac{W}{G}\Big)\Big) \big(\mathcal{L}W - \nabla \cdot (UW)\big).$$

Using the identity $\nabla G/G = -\xi/2$ and the term involving $\mathcal{L}$ simplifies to

$$\begin{aligned}\int_{\mathbb{R}^2} \Big(1 + \ln\Big(\frac{W}{G}\Big)\Big) \mathcal{L}W \, d\xi &= -\int_{\mathbb{R}^2} \Big(\nabla W + \frac{\xi}{2} W\Big) \cdot \Big(\frac{\nabla W}{W} - \frac{\nabla G}{G}\Big) \, d\xi \\ &= -\int_{\mathbb{R}^2} W \Big| \frac{\nabla W}{W} - \frac{\nabla G}{G} \Big|^2 d\xi = -\int_{\mathbb{R}^2} W \Big| \nabla \ln\Big(\frac{W}{G}\Big)\Big|^2 d\xi.\end{aligned}$$

We claim the convection terms integrate to 0. Indeed,

$$-\int_{\mathbb{R}^2} \Big(1 + \ln\Big(\frac{W}{G}\Big)\Big) \nabla \cdot (UW) \, d\xi = \int_{\mathbb{R}^2} U \cdot \nabla W \, d\xi + \frac{1}{2} \int_{\mathbb{R}^2} WU \cdot \xi \, d\xi.$$

The first term on the right clearly integrates to 0. If U decayed sufficiently at infinity, we can write $W = \nabla \times U$, integrate the second term by parts, and obtain

$$\frac{1}{2} \int_{\mathbb{R}^2} WU \cdot \xi \, d\xi = \frac{1}{4} \int_{\mathbb{R}^2} \xi \cdot \nabla^\perp |U|^2 \, d\xi = 0. \tag{3.13}$$

Without the decay assumption one can use the Biot-Savart law and Fubini's theorem (see for instance [14, lemma 3.2]) and still show this term integrates to 0. This immediately yields (3.11) as desired. □

4. Stability when the initial divergence has non-zero mean

In this section, we study the long time behaviour of the system (2.1)–(2.3) when $\beta \neq 0$ (i.e. when the mean of the initial divergence is non-zero) and prove theorem 2.2. Unlike the behaviour in section 3, the divergence D of the equilibrium solution to the system (3.3)–(3.5) is non-zero. Consequently, the steady state of the system (3.3)–(3.5) is no longer a Gaussian (like the Oseen vortex), but the radial function W_s defined by (2.6). We remark, however, that different, non-radial, steady solutions to the system (3.3)–(3.5) may exist and we can neither prove nor disprove their existence.

Further it turns out that the radial state W_s doesn't "play nice" with the non-linearity. We are unable to show decay of the analogue of the Boltzmann entropy (3.10), which is a key step in both [14] and the proof of theorem 2.1. We can, however, show that W_s is stable under small perturbations globally in time (theorem 2.2) using techniques that are similar to those in [12, 34]. This is the main goal of this section.

In section 4.1, we derive an explicit equation for the radial steady state W_s. In section 4.2, we compute the evolution of the Boltzmann entropy functional mainly to point out the breaking point of the argument of Gallay and Wayne [14]. In section 4.3, we use a different method (similar to that in [34]) to prove stability under small perturbations (theorem 2.2) modulo the proofs of a few estimates which are presented in section 4.4.

4.1. The radial steady state. Since the equation for D is linear, we find that $D \to \beta G$ as $\tau \to \infty$. This can be seen, for instance, by noticing that $D - \beta G$ satisfies the heat equation in Euclidean coordinates with initial mean zero. An argument analogous to the proof of lemma 3.4 gives the precise decay. Turning to W, we denote the steady state by W_s. For convenience, we normalize W_s so that $\int W_s d\xi = 1$. We claim that a unique radial steady state exists, and is exactly given by (2.6). (We can not, however, rule out the possibility that other non-radial steady states exist.)

To see that the unique radial steady state satisfies (2.6), we use equation (3.3) to obtain

$$0 = -(K_{BS} * W_s) \cdot \nabla W_s - \beta \nabla \cdot \big((\nabla^{-1} G) W_s \big) + \mathcal{L} W_s,$$

in L^2_w. Under the assumption that W_s is radial, $K_{BS} * W_s \cdot \nabla W_s = 0$ and hence

$$\beta \nabla \cdot \big((\nabla^{-1} G) W_s \big) = \mathcal{L} W_s = \nabla \cdot \Big(G \nabla \frac{W_s}{G} \Big).$$

Consequently,

$$\nabla^{\perp} \varphi = -\beta \nabla^{-1} G W_s + G \nabla \frac{W_s}{G}.$$

for some function φ. Since the right hand side is radial and smooth, we must have $\nabla^{\perp} \varphi = 0$ identically.

Switching to polar coordinates immediately shows that W_s satisfies (2.6), and reverting back to the x and t coordinates shows that $(\tilde{\omega}_\beta, \tilde{d}_\beta)$, defined in (2.7), is the unique radially symmetric, self-similar solution to the system (2.1)–(2.3).

4.2. The Boltzmann entropy. Before embarking on the proof of theorem 2.2, we briefly study the analogue of the Boltzmann entropy in this situation. Naturally, the Gaussian in this context needs to be replaced with W_s, the solution to (2.6), and so (3.10) now becomes

$$H(W) = \int_{\mathbb{R}^2} W \ln\Big(\frac{W}{W_s} \Big) d\xi.$$

Computing $\partial_\tau H$ and performing a calculation similar to that in section 3.2.2 we obtain

$$\begin{aligned} \partial_\tau H &= \int_{\mathbb{R}^2} W (K_{BS} * W) \cdot \Big(\frac{\nabla W}{W} - \frac{\nabla W_s}{W_s} \Big) d\xi - \int_{\mathbb{R}^2} W \Big| \frac{\nabla W}{W} - \frac{\nabla W_s}{W_s} \Big|^2 d\xi \\ &= -\int_{\mathbb{R}^2} W (K_{BS} * W) \cdot \frac{\nabla W_s}{W_s} d\xi - \int_{\mathbb{R}^2} W \Big| \frac{\nabla W}{W} - \frac{\nabla W_s}{W_s} \Big|^2 d\xi. \end{aligned}$$

The second term is of course always negative. The first term can be simplified using (2.6) to

$$\begin{aligned} &-\int_{\mathbb{R}^2} W (K_{BS} * W) \cdot \frac{\nabla W_s}{W_s} d\xi \\ &\qquad = \int_{\mathbb{R}^2} W (K_{BS} * W) \cdot \frac{\xi}{2} d\xi + \beta \int_{\mathbb{R}^2} W (K_{BS} * W) \cdot \frac{\xi}{2\pi |\xi|^2} \Big(1 - 4\pi G \Big) d\xi. \end{aligned}$$

The first term on the right integrates to 0 (by equation (3.13)). Further for any radial function (hence certainly for $W = W_s$) the second term vanishes. Consequently,

$$\partial_\tau H = -\int_{\mathbb{R}^2} W\Big|\frac{\nabla W}{W} - \frac{\nabla W_s}{W_s}\Big|^2 d\xi. \\ + \beta\int_{\mathbb{R}^2} (W - W_s) K_{BS} * (W - W_s) \cdot \frac{\xi}{2\pi|\xi|^2}\Big(1 - 4\pi G\Big)\, d\xi.$$

While the second term on the right should, in principle, be small (at least for small values of β and when W is close to W_s), we are (presently) unable to dominate this by the first term and show that $\partial_\tau H \leqslant 0$. Thus we do not know whether the steady state W_s is stable under large perturbations.

4.3. Stability under small perturbations. We now turn to proving stability of $(\tilde\omega_\beta, \tilde d_\beta)$ as stated in theorem 2.2.

Proof of theorem 2.2. Using the ξ-τ coordinates, let (W, D) be solutions to the system (3.3)–(3.5) with initial data $W_0, D_0 \in L^2_w$. Define the perturbations from the steady state D_p, U_p and W_p by

$$W_p \stackrel{\text{def}}{=} W - \alpha W_s, \quad D_p \stackrel{\text{def}}{=} D - \beta G, \quad \text{and} \quad U_p \stackrel{\text{def}}{=} K_{BS} * W_p + \nabla^{-1} W_p. \tag{4.1}$$

In this setting, theorem 2.2 will follow if we establish

$$\|W_p(\tau)\|_w \leqslant C\big(\|W_p(\tau_0)\|_w e^{-\gamma\tau} + \|D_p(\tau_0)\|_{L^1(1)} e^{-\tau/2}\big) \tag{4.2}$$

for some constant C, where $\tau_0 = \log(t_0)$. As before, the estimate for D in theorem 2.2 is analogous to lemma 3.4.

To begin we state one basic result without proof. First, a straightforward adaptation of the work in [34, theorem 1] yields the following existence result.

Lemma 4.1. *For $\varepsilon_0 > 0$, there exists $\delta_0 > 0$, depending only on α, such that if $W(0), D(0) \in L^2_w$ and*

$$|\beta| + \|W_p(\tau_0)\|_w + \|D_p(\tau_0)\|_w \leqslant \delta_0,$$

then there is a unique solution to the system (3.3)–(3.5) *such that, for all τ,*

$$\|D_p(\tau)\|_w + \|W_p(\tau)\|_w \leqslant \varepsilon_0. \tag{4.3}$$

In order to show convergence to the steady state, we work with the equation for the perturbation,

$$\partial_\tau W_p + \nabla\cdot\big(UW_p + \alpha K_{BS} * W_p W_s + \alpha\nabla^{-1} D_p W_s\big) = \mathcal{L} W_p. \tag{4.4}$$

We multiply (4.4) by $G^{-1} W_p$ and integrate to obtain

$$\frac{1}{2}\partial_\tau \|W_p\|_w^2 + \int_{\mathbb{R}^2} G^{-1} W_p \nabla\cdot\big(UW_p + \alpha K_{BS} * W_p W_s + \alpha\nabla^{-1} D_p W_s\big) \\ = \int_{\mathbb{R}^2} G^{-1} W_p \mathcal{L} W_p. \tag{4.5}$$

We estimate each term individually. First, for the right hand side, we use a coercivity estimate proven in [34]. Namely, since $\int W_p\, d\xi = 0$, for any $\gamma \in (0, 1/2)$ and $\varepsilon > 0$ such that $\gamma + 1000\varepsilon < 1/2$, we have

$$-\int G^{-1} W_p \mathcal{L} W_p \\ \geqslant (\gamma + \varepsilon)\|W_p\|_w^2 + \frac{1 - 2(\gamma + \varepsilon)}{2}\left[\frac{1}{3}\|\nabla W_p\|_w^2 + \frac{1}{32}\|\xi W_p\|_w^2\right]. \tag{4.6}$$

This is proved by first observing operator $L \stackrel{\text{def}}{=} -G^{-1/2}\mathcal{L}G^{1/2}$ is a harmonic oscillator with spectrum $\{0,1/2,1,3/2,\dots\}$ where 0 is a simple eigenvalue. Combining this with a standard energy estimate shows (4.6), and we refer the reader to [13, Appendix A] or [34, §3.1] for the details. We assume, without loss of generality, that $\gamma > 1/4$.

For the first term in the integral on the left of (4.5), observe

$$\begin{aligned}\int_{\mathbb{R}^2} G^{-1}W_p\nabla\cdot(UW_p)\,d\xi &= \int_{\mathbb{R}^2}\big(G^{-1}W_p^2D + \frac{1}{2}G^{-1}U\cdot\nabla\big(W_p^2\big)\big)\,d\xi\\ &= \frac{1}{2}\int_{\mathbb{R}^2} G^{-1}W_p^2\big(D - \frac{1}{2}\xi\cdot U\big)\,d\xi\\ &= \frac{1}{2}\int_{\mathbb{R}^2} G^{-1}W_p^2\big(D - \frac{1}{2}\xi\cdot\nabla^{-1}D\big)\,d\xi + \int_{\mathbb{R}^2} G^{-1}W_p(K_{BS}*W_p)\cdot\nabla W_p\,d\xi,\end{aligned} \tag{4.7}$$

since $K_{BS}*W_s\cdot\xi = 0$.

To estimate this we claim

$$\|D\|_w + \|D\|_{L^\infty} + \|\nabla^{-1}D\|_{L^\infty} \leqslant C\,[|\beta| + \|D_p(0)\|_w], \tag{4.8}$$

$$\text{and}\qquad \|K_{BS}*W_p\|_{L^\infty} \leqslant C\big(\|W_p\|_w + \|\nabla W_p\|_w\big), \tag{4.9}$$

for some constant C that is independent of D_0, W_p and β. To avoid breaking continuity we defer the proof of these estimates to section 4.4 and continue with our proof of theorem 2.2 here.

Let ε_0 to be a small constant to be determined later. Using lemma 4.1, choose δ_0 to guarantee (4.3) holds. Then, returning to (4.7) we see

$$\Big|\int_{\mathbb{R}^2} G^{-1}W_p\nabla\cdot(UW_p)\,d\xi\Big| \leqslant C\big(|\beta| + \|D_p(\tau_0)\|_w + \varepsilon_0\big)\big(\|W_p\|_w^2 + \|\nabla W_p\|_w^2\big).$$

For the second term in the integral on the left of (4.5) we obtain smallness by using the fact that this term vanishes when $W_s = G$. Indeed,

$$\alpha\int_{\mathbb{R}^2} G^{-1}W_pK_{BS}*W_p\cdot\nabla W_s\,d\xi = -\alpha\int_{\mathbb{R}^2} G^{-1}W_sK_{BS}*W_p\big(\nabla W_p + \frac{\xi}{2}W_p\big)\,d\xi,$$

which vanishes when $W_s = G$ due to the identity (3.13). Consequently,

$$\begin{aligned}&\alpha\int_{\mathbb{R}^2} G^{-1}W_pK_{BS}*W_p\cdot\nabla W_s\,d\xi\\ &\qquad = -\alpha\int_{\mathbb{R}^2} G^{-1}(W_s - G)K_{BS}*W_p\big(\nabla W_p + \frac{\xi}{2}W_p\big)\,d\xi.\end{aligned} \tag{4.10}$$

We claim that for all β sufficiently small,

$$\|W_s - G\|_w \leqslant C|\beta|, \tag{4.11}$$

for some universal constant C. Again, to avoid breaking continuity, we defer the proof of (4.11) to section 4.4, and continue with the proof theorem 2.2.

Equations (4.10) and (4.11) immediately show

$$\begin{aligned}&\Big|\alpha\int_{\mathbb{R}^2} G^{-1}W_pK_{BS}*W_p\cdot\nabla W_s\,d\xi\Big|\\ &\qquad\leqslant C|\alpha\beta|\|K_{BS}*W_p\|_{L^\infty}\|W_p\|_w\|\nabla W_s\|_w\\ &\qquad\leqslant C|\alpha\beta|(\|W_p\|_w^2 + \|\nabla W_p\|_w^2).\end{aligned} \tag{4.12}$$

For the last inequality above we absorbed $\|\nabla W_s\|_w$ into the constant C, and used (4.9) and interpolation.

For the last term in the integral on the left of (4.5) observe

$$\begin{aligned}\Big|\alpha\int_{\mathbb{R}^2} G^{-1}W_p\nabla\cdot(\nabla^{-1}D_pW_s)\,d\xi\Big| &= \Big|\alpha\int_{\mathbb{R}^2} G^{-1}W_p(D_pW_s+\nabla^{-1}D_p\cdot\nabla W_s)\,d\xi\Big|\\ &\leqslant |\alpha|\|W_p\|_w\big(\|D_p\|_{L^\infty}+\|\nabla^{-1}D_p\|_{L^\infty}\big)\|W_s\|_w\\ &\leqslant C|\alpha|\|W_p\|_w\big(\|D_p\|_{L^\infty}+\|D_p\|_{L^1}^{1/2}\|D_p\|_{L^\infty}^{1/2}\big).\end{aligned}$$

The last estimate followed from the interpolation inequality

$$\|\nabla^{-1}D_p\|_{L^\infty}\leqslant C\|D_p\|_{L^1}^{1/2}\|D_p\|_{L^\infty}^{1/2},\tag{4.13}$$

the proof of which can be found in [34] or [13] (see also proposition 6.2 in section 6, below).

Since D_p satisfies (3.4) with mean-zero initial data $D_p(\tau_0)\in L^1(1)$, it must satisfy the decay estimate (3.9). Thus

$$\begin{aligned}\Big|\alpha\int_{\mathbb{R}^2} G^{-1}W_p\nabla\cdot(\nabla^{-1}D_pW_s)\,d\xi\Big| &\leqslant C\|D_p(\tau_0)\|_{L^1(1)}e^{-\tau/2}\|W_p\|_w\\ &\leqslant \frac{\varepsilon}{8}\|W_p\|_w^2+C\|D_p(\tau_0)\|_{L^1(1)}^2e^{-\tau}.\end{aligned}$$

Making $(1+|\alpha|)|\beta|$, δ_0 and ε_0 small enough, our estimates so far give

$$\begin{aligned}\frac12\partial_\tau\|W_p\|_w^2+(\gamma+\varepsilon)\|W_p\|_w^2+\frac{1-2(\gamma+\varepsilon)}{2}\left[\frac13\|\nabla W_p\|_w^2+\frac{1}{32}\|\xi W_p\|_w^2\right]&\\ \leqslant \varepsilon\left[\|W_p\|_w^2+\|\xi W_{p,k+1}\|_w^2+\|\nabla W_p\|_w^2\right]+Ce^{-\tau}\|D_p(\tau_0)\|_{L^1(1)}^2&.\end{aligned}$$

Because we chose ε small enough, the first three terms on the right can be absorbed in the left. Consequently,

$$\partial_\tau\|W_p(\tau)\|_w^2+2\gamma\|W_p\|_w^2\leqslant Ce^{-\tau}\|D_p(0)\|_{L^1(1)}^2,$$

which immediately implies (4.2). □

4.4. Proofs of estimates. In this section, we prove the bounds (4.8), (4.9) and (4.11), which were used in the proof of theorem 2.2. We begin with the bounds on the divergence.

Lemma 4.2. *Let D satisfy* (3.4) *with initial data $D_0\in L^2(w)$, and let $\beta=\int D_0d\xi$. Then if $D_p=D-\beta G$, there exists a uniform constant $C>0$ such that* (4.8) *holds.*

Proof. Multiplying (3.4) by $G^{-1}D$, integrating and using the coercivity estimate (4.6) gives

$$\frac12\partial_\tau\|D\|_w^2+\frac14\left[\|D\|_w^2+\frac13\|\nabla D\|_w^2+\frac{1}{32}\|\xi D\|_w^2\right]\leqslant 0.$$

Integrating this inequality in τ gives us the desired inequality for $\|D\|_w$.

Further, in the standard x-t coordinates, D solves the heat equation. The classical estimates for solutions to the heat equation give us

$$\|D(\tau)\|_{L^\infty}+\|D(\tau)\|_{L^1}\leqslant C\|D(\tau_0)\|_{L^1}\leqslant C\|D(\tau_0)\|_w.$$

Combined with the interpolation inequality (4.13) this yields the same bound for $\|\nabla^{-1}D\|_{L^\infty}$, completing the proof. □

Now we turn to (4.9), which follows using the Sobolev embedding theorem and interpolation.

Proof of inequality (4.9). We know that the Biot-Savart operator satisfies the interpolation inequality

$$\|K_{BS} * W_p\|_{L^\infty} \leqslant C\|W_p\|_{L^{4/3}}^{1/2} \|W_p\|_{L^4}^{1/2}.$$

The proof is the same as that of (4.13), and can be found in [13, 34] (see also proposition 6.2 in section 6, below). Combining this with Sobolev inequality we obtain

$$\begin{aligned}\|K_{BS} * W_p\|_{L^\infty} &\leqslant C\|W_p\|_{L^{4/3}}^{1/2} \|W_p\|_{L^4}^{1/2} \leqslant C\|W_p\|_{L^{4/3}}^{1/2} \|\nabla W_p\|_{L^{4/3}}^{1/2} \\ &\leqslant C\|W_p\|_{L^2(w)}^{1/2} \|\nabla W_p\|_{L^2(w)}^{1/2} \leqslant C\big(\|W_p\|_{L^2(w)} + \|\nabla W_p\|_{L^2(w)}\big),\end{aligned}$$

as desired. ☐

Finally, we prove (4.11) showing W_s is close to G when β is small.

Lemma 4.3. *Let $W_s \in L^2_w$ be a solution to equation* (2.6)*. Then there is a universal constant $C > 0$ such that such that the inequality* (4.11) *holds for all β sufficiently small.*

Proof. Define $P_s = W_s - G$. Notice that this solves

$$\mathcal{L}P_s = \beta G P_s + \beta \nabla^{-1} G \cdot \nabla P_s + \beta G^2 + \beta \nabla^{-1} G \cdot \nabla G.$$

Multiply this equation by $G^{-1} P_s$ and using (4.6), with $\gamma = 1/4$, to obtain that

$$\begin{aligned}\frac{1}{4}\|P_s\|_w^2 + \frac{1}{4}\left[\frac{1}{3}\|\nabla P_s\|_w^2 + \frac{1}{32}\|\xi P_s\|_w^2\right] &\leqslant -\int G^{-1} P_s \mathcal{L} P_s \\ &= -\beta \int P_s^2 - \beta \int G^{-1} P_s \nabla^{-1} G \cdot \nabla P_s \\ &\quad -\beta \int G P_s - \beta \int G^{-1} P_s \nabla^{-1} G \cdot \nabla G \\ &\leqslant (2|\beta| + \varepsilon)\left[\|P_s\|_w^2 + \|\nabla P_s\|_w^2\right] + |\beta|^2 C_\varepsilon.\end{aligned}$$

Here $\varepsilon < 1/20$ is a positive constant. Then when β is sufficiently small, we may absorb the terms on the last line into the left hand side, giving (4.11) as desired. ☐

5. Bounds for the vorticity

Bounds on the vorticity to the standard 2D incompressible Navier-Stokes equations are well known. In this section we prove the analogues of these bounds for the extended Navier-Stokes equations (1.1).

We begin with the vorticity decay in L^p. The strategy for this proof is not entirely different from the classical case, however, the appearance of a divergence term complicates matters and yields a slightly different final estimate. We will use this estimate in the proof of (3.8) and in our discussion of well-posedness in section 7.

Lemma 5.1. *Let p be an element of $[1,\infty]$, and suppose that (ω,d) solve the system (2.1)–(2.3) with $\omega_0, d_0 \in L^1$. Then there exists $C>0$, depending only on p, $\|\omega_0\|_{L^1}$, and $\|d_0\|_{L^1}$ such that*

$$\|\omega\|_{L^p} + t^{1/2}\|\nabla\omega\|_{L^p} \leqslant \frac{C}{t^{1-1/p}} \tag{5.1}$$

and

$$\|\nabla\omega\|_{L^p} \leqslant \frac{C}{t^{3/2-1/p}}. \tag{5.2}$$

Proof. We omit the proof of the bound on the gradient. Indeed, by following the work in [25, proposition 4.1], we note that the estimate relies only on (5.1) and Duhamel's principle. In view of this, obtaining this result is a straightforward adaptation.

Now, we obtain the L^p bound by obtaining a bound in L^1 and L^∞ and interpolating. The L^1 bound follows by splitting ω_0 into its positive and negative parts, using the maximum principle, and using that the mass is preserved.

The classical technique for obtaining the L^∞ bound has three steps: (i) get a bound on the L^2 norm in terms of the L^1 norm divided by $t^{1/2}$, (ii) show that this gives a bound on the L^∞ norm in terms of the L^2 norm divided by $t^{1/2}$ for the adjoint problem, and (iii) apply these inequalities over $[0,t/2]$ and $[t/2,t]$ to finish. Since the work in (ii) is the same as the work in (i) and since (iii) is unchanged from the classical setting, we simply show the first step (i). To this end, multiplying our equation by ω and integrating by parts gives us

$$\frac{d}{dt}\|\omega\|_{L^2}^2 \leqslant -2\|\nabla\omega\|_{L^2}^2 + 2\|d\|_{L^\infty}\|\omega\|_{L^2}^2.$$

Using the Fourier transform, we see that there is a constant $C>0$ such that for any R,

$$\begin{aligned}
\|\hat{\omega}\|_{L^2}^2 &\leqslant \int_{B_R^c} \frac{|\xi|^2}{R^2}|\hat{\omega}|^2 d\xi + \int_{B_R} |\hat{\omega}|^2 d\xi \\
&\leqslant \frac{1}{R^2}\int |\xi|^2|\omega|^2 d\xi + \int_{B_R} \|\hat{\omega}\|_{L^\infty}^2 d\xi \\
&\leqslant \frac{1}{R^2}\|\nabla\omega\|_{L^2}^2 + CR^2\|\omega\|_{L^1}^2.
\end{aligned}$$

Using $R = \|\omega\|_{L^2}^2/(2C\|\omega_0\|_{L^1}^2)$ along with these inequalities yields

$$\frac{d}{dt}\|\omega\|_{L^2}^2 \leqslant \left[\frac{C\|d_0\|_{L^1}}{t} - \frac{\|\omega\|_{L^2}^2}{2C\|\omega_0\|_{L^1}^2}\right]\|\omega\|_{L^2}^2. \tag{5.3}$$

Here we used the standard estimates for the heat equation, and then we used Young's inequality. Define $\phi(t) = t\|\omega\|_{L^2}^2$ to obtain

$$\phi'(t) \leqslant \frac{\phi}{t}\left[\|d_0\|_{L^1} - \frac{\phi}{2C\|\omega_0\|_{L^1}^2} + 1\right].$$

This implies that $\phi \leqslant 2C\|\omega_0\|_{L^1}^2[\|d_0\|_{L^1}^2 + 1]$, which proves our claim. □

Now, we prove the pointwise, heat kernel type bound on the vorticity when $\beta = 0$ stated in lemma 3.2. We use the increased decay of the heat equation when the initial data is mean-zero here. The key point here is that the L^∞

norm of the divergence is integrable in time, so we may reproduce the classical arguments in this case. We follow the work of Carlen and Loss in [4] in order to do this.

Proof of (3.8). Our first step is to obtain bounds for the equation

$$\phi_t = \Delta\phi + \nabla\cdot(b\phi) + c\phi. \tag{5.4}$$

which depend only on certain norms of b and c. To this end, fix $T>0$ and we let $r(t)$ be a monotone increasing, smooth function defined on $[0,T]$ to be determined later. In addition, we may assume without loss of generality that ϕ is non-negative. Then we calculate

$$\begin{aligned}
r(t)^2\|\phi\|_{L^r}^{r-1}\frac{d}{dt}\|\phi\|_{L^r} &= \dot r\int\phi^r\log\left(\frac{\phi^r(x)}{\|\phi\|_{L^r}}\right)dx + r(t)^2\int\phi^{r-1}\phi_t dx\\
&= \dot r\int\phi^r\log\left(\frac{\phi^r(x)}{\|\phi\|_{L^r}}\right)dx\\
&\quad + r(t)^2\int\phi^{r-1}(\Delta\phi+\nabla\cdot(b\phi)+c\phi)\,dx\\
&= \dot r\int\phi^r\log\left(\frac{\phi^r(x)}{\|\phi\|_{L^r}}\right)dx + 4(r-1)\int\left|\nabla\left(\phi^{r/2}\right)\right|^2 dx\\
&\quad + \int r(r-1)\phi^r(\nabla\cdot b) + r^2\int c\phi^r dx.
\end{aligned}$$

The log-Sobolev inequality [4, Equation (1.17)], which the authors derive from the work in [17], is

$$\int|f|^2\log\left(\frac{f^2}{\|f\|_{L^2}^2}\right)dx + (2+\log(a))\int|f|^2dx \leqslant \frac{a}{\pi}\int|\nabla f|^2dx, \tag{5.5}$$

for any $f\in H^1$ and $a\in(0,\infty)$. Applying this with $a=4\pi(r-1)/\dot r$, gives us

$$\begin{aligned}
r(t)^2\|\phi\|_{L^r}^{r-1}\frac{d}{dt}\|\phi\|_{L^r} &\leqslant -\dot r\left(2+\log\left(\frac{4\pi(r-1)}{\dot r}\right)\right)\|\phi\|_{L^r}^r\\
&\quad + \left(r(r-1)B(t)+r^2C(t)\right)\|\phi\|_{L^r}^r,
\end{aligned}$$

where $B(t)=\|\nabla\cdot b(t,\cdot)\|_{L^\infty}$ and $C(t)=\|c(t,\cdot)\|_{L^\infty}$. Now we set $G(t)=\log\|\phi\|_{L^r}$ and $s=1/r$ to obtain

$$\frac{dG}{dt}\leqslant \dot s(2+\log(4\pi s(1-s))) - \dot s\log(-\dot s) + (1-s)B(t) + C(t).$$

Letting $s(t)$ be a linear interpolation of 1 and 0 over $[0,T]$, we see that $\dot s = -T^{-1}$. Then we may integrate this to obtain

$$G(T)-G(0)\leqslant 4-\log(4\pi)-\log(T)+\int_0^T[B(t)+C(t)]\,dt.$$

Exponentiating gives us

$$\|\phi(T)\|_{L^\infty}\leqslant\frac{K}{T}\exp\left(\int_0^T[B(t)+C(t)]\,dt\right). \tag{5.6}$$

In order to get pointwise decay from (5.6), we look at the operator

$$P^{(\alpha)}(T,x,y) := e^{-\alpha\cdot x}P(T,x,y)e^{\alpha\cdot y},$$

where P is the solution kernel for our linear problem (5.4) with $c \equiv 0$ and $\alpha(x,y)$ is a function to be identified later. We assume that b can be written as $b = b_1 + b_2$ where $\nabla \cdot b_1 = 0$

$$\|b_1(t)\|_{L^\infty} \leqslant \frac{K_1}{\sqrt{t+1}}, \quad \|\nabla \cdot b_2\|_{L^\infty} \leqslant \frac{K_2}{(t+1)^{3/2}}, \quad \text{and} \quad \|b_2\|_{L^\infty} \leqslant \frac{K_2}{(t+1)}. \tag{5.7}$$

In the application we have in mind, b_1 comes from the Biot-Savart kernel of the vorticity, while b_2 comes from ∇^{-1} of the divergence.

We wish to obtain bounds for P through our integral bounds on $P^{(\alpha)}$. To this end, we notice that $P^{(\alpha)}$ is the solution kernel for the problem

$$\phi_t = \Delta\phi + \nabla \cdot ((b + 2\alpha)\phi) + (\alpha \cdot b + |\alpha|^2)\phi.$$

Applying (5.6), and noticing that $\nabla \cdot (b + 2\alpha) = \nabla \cdot b$, we obtain, for any α,

$$P^{(\alpha)}(T,x,y) \leqslant \frac{K}{T} \exp\left(2\int_0^T \left[K_2(t+1)^{-3/2} + K_2^2(t+1)^{-2} + |\alpha| K_1 (t+1)^{-1/2} + |\alpha|^2\right] dt\right).$$

Choosing

$$\alpha = -\frac{1}{4T}\frac{(x-y)}{|x-y|}\left[|x-y| - 2K_1\sqrt{T+1}\right]_+,$$

using the definition of $P^{(\alpha)}$, and integrating in time, we obtain

$$P(T,x,y) \leqslant \frac{K}{T}\exp\left((4K_2 + 2K_2^2) - \frac{1}{8T}\left[|x-y| - 2K_1\sqrt{T+1}\right]_+^2\right).$$

By possibly changing the constants, we may obtain

$$P(T,x,y) \leqslant \frac{C}{T}\exp\left(-\frac{|x-y|^2}{CT}\right).$$

To conclude, we apply the above to equation (2.1), by choosing $b_1 = K_{BS} * \omega$ and $b_2 = \nabla^{-1} d$. Lemmas 3.4 and 5.1 and interpolation inequalities of the form (4.13) show that (5.7) is satisfied, concluding the proof. □

6. Relative compactness of complete trajectories

In this section we prove lemmas 3.2 and 3.7, showing that complete trajectories in L^1 are relatively compact. The development is similar to [14], and the main difference here is the additional divergence term which requires us to alter many of the proofs. We first work up towards proving lemma 3.7, and then use this to prove lemma 3.2.

6.1. The semi-group of $\mathcal{L}$ and apriori bounds. In order to obtain the desired compactness results, we will need estimates on various quantities. We will state these estimates here, but we will omit the proofs and provide references.

Let $S(\tau) \stackrel{\text{def}}{=} \exp(\tau\mathcal{L})$ be the semigroup generated by the operator $\mathcal{L}$. First we recall some estimates on the operator $S(\tau)$. In order to state these, we define the function

$$a(\tau) \stackrel{\text{def}}{=} 1 - e^{-\tau}.$$

This function appears naturally with the change of variables. We recall a lemma on the operator S from [13].

Lemma 6.1. [13, Appendix A]

1) *For $m>1$, $S(\tau)$ is a bounded operator on $L^2(m)$. In addition, $\nabla S(\tau)$ is bounded away from $\tau=0$. More precisely, there is a universal constant C such that*

$$\|S(\tau)\|_{L^2(m)\to L^2(m)}\leqslant C,\quad \|\nabla S(\tau)\|_{L^2(m)\to L^2(m)}\leqslant \frac{C}{\sqrt{a(\tau)}}.$$

2) *Let $L^2_0(m)$ be the space of $L^2(m)$ functions with integral zero. For $\mu\in(0,1/2]$ and $m>1+2\mu$ and $\tau>0$, there is a universal constant C such that*

$$\|S(\tau)\|_{L^2_0(m)\to L^2_0(m)}\leqslant Ce^{-\mu\tau},\quad \|\nabla S(\tau)\|_{L^2_0(m)\to L^2_0(m)}\leqslant C\frac{e^{-\mu\tau}}{\sqrt{a(\tau)}}.$$

3) *For $1\leqslant q\leqslant p\leqslant\infty$, $T>0$, $m\in[0,\infty)$ and $\alpha\in\mathbb{N}^2$, there is a constant C_T, depending on T, such that*

$$\|\partial^\alpha S(\tau)f\|_{L^p(m)}\leqslant \frac{C_T}{a(\tau)^{(q^{-1}-p^{-1})+|\alpha|/2}}\|f\|_{L^q(m)},$$

for any $f\in L^q(m)$ and any $0<\tau\leqslant T$.

We note that the commutator of ∇ and $S(\tau)$ is computed as

$$\partial_i S(\tau)=e^{\tau/2}S(\tau)\partial_i.$$

In addition, we need the well-known bounds on Biot-Savart kernel and ∇^{-1}. The proof of this proposition may be found in [34, proposition 1] and [13, Appendix B].

Proposition 6.2. *Denote by K either the operator $K_{BS}*$ or the operator ∇^{-1}. Then the following inequalities hold for any f such that the right hand side of each inequality is finite.*

1) *If $1<p<2<q$ and $1+q^{-1}-p^{-1}=1/2$ then there is a constant C such that*

$$\|Kf\|_{L^q}\leqslant C\|f\|_{L^p}.$$

2) *If $1\leqslant p<2<q\leqslant\infty$ and $0<\theta<1$ satisfy*

$$\frac{\theta}{p}+\frac{1-\theta}{q}=\frac{1}{2},$$

then there is a constant C such that

$$\|Kf\|_{L^\infty}\leqslant C\|f\|^{\theta}_{L^p}\|f\|^{1-\theta}_{L^q}.$$

3) *There exists a constant $C_p>0$ depending only on p such that if $p>1$ then*

$$\|\nabla Kf\|_{L^p}\leqslant C_p\|f\|_{L^p}.$$

4) *If $0<m<1$ and $q>2$ then there is a constant C_q, depending only on q, such that*

$$\|Kf\|_{L^q(m-2/q)}\leqslant C_q\|f\|_{L^2(m)}.$$

Finally, we state an *a priori* bound on solutions to the system (3.3)–(3.5). The proof of this lemma is a straightforward adaptation of [14, lemma 2.1].

Lemma 6.3. *Suppose that* (W,D) *solves the system* (3.3)–(3.5) *in the space*

$$C^0([0,T],L^2(m))\cap C^0((0,T],H^1(m))$$

with $W_0\in L^2(m)$ *and* $D_0\in L^2(m)$ *as the initial conditions for* W *and* D *respectively. Then there is a constant* C *such that*

$$\|W(\tau)\|_{L^2(m)}+a(\tau)^{1/2}\|\nabla W(\tau)\|_{L^2(m)}\leqslant C.$$

6.2. Compactness in $L^2(m)$. First we show relative compactness of complete trajectories on $\mathbb{R}_+$ in $L^2(m)$. This is accomplished by decomposing the remainder term into convenient functions, two of which decay to zero and one whose trajectory is relatively compact.

Lemma 6.4. *Assume that* $m>3$ *and that* $(W,D)\in C^0([0,\infty),L^2(m)^2)$ *is a solution to the system* (3.3)–(3.5)*, and is bounded in* $L^2(m)$*. The trajectory* $\{(W,D)\}_{\tau\in\mathbb{R}_{\geqslant 0}}$ *is relatively compact in* $L^2(m)$*.*

Proof. We work here with W only, but the proof for D is similar and simpler. We define the remainder, R, to be such that $W=\alpha G+R$. One can check that

$$\partial_\tau R=\mathcal{L}R-\alpha\Lambda R-N(R)-\nabla\cdot(W\nabla^{-1}D).$$

where

$$\alpha\Lambda R\stackrel{\text{def}}{=}(\alpha K_{BS}*G\cdot\nabla R+\alpha K_{BS}*R\cdot\nabla G)\quad\text{and}\quad N(R)\stackrel{\text{def}}{=}K_{BS}*R\cdot\nabla R.$$

Hence we may write

$$R(\tau,\xi)=S(\tau)R_0-R_1-R_2 \tag{6.1}$$

where

$$R_1\stackrel{\text{def}}{=}\int_0^\tau S(\tau-s)(\alpha\Lambda R(s)+N(R)(s))\,ds$$

$$\text{and}\quad R_2\stackrel{\text{def}}{=}\int_0^\tau S(\tau-s)\nabla\cdot(W(s)\nabla^{-1}D(s))\,ds.$$

The first term tends to zero by part two of lemma 6.1 and the fact that $\int R_0 d\xi=0$. It follows from the work in lemma 2.2 in [14] that R_1 is bounded in $L^2(m+1)$, and, hence, is a relatively compact trajectory. Thus, we need only show that R_2 tends to zero.

To this end, we use the first inequality in lemma 6.1 to obtain

$$\begin{aligned}\|R_2\|_{L^2(m)}&\leqslant\int_0^\tau e^{-\frac{1}{2}(\tau-s)}\|\nabla S(\tau-s)(W\nabla^{-1}D)(s)\|_{L^2(m)}ds\\&\leqslant C\int_0^\tau\frac{e^{-\frac{1}{2}(\tau-s)}}{\sqrt{a(\tau-s)}}\|(W\nabla^{-1}D)(s)\|_{L^2(m)}ds\\&\leqslant C\int_0^\tau\frac{e^{-\frac{1}{2}(\tau-s)}}{\sqrt{a(\tau-s)}}\|\nabla^{-1}D(s)\|_{L^\infty}\|W(s)\|_{L^2(m)}ds.\end{aligned}$$

The results of lemma 3.4 and proposition 6.2 imply that $\|\nabla^{-1}D(s)\|_{L^\infty}$ tends to zero as s tends to infinity. Hence, we see that $\|R_2\|_{L^2(m)}$ tends to zero as τ tends to infinity. □

Now we will show relative compactness of complete trajectories in $L^2(m)$, i.e. we will prove lemma 3.7. Our method of proof will be similar to above.

Proof of lemma 3.7. Again we will look at R as above and only work with W. This time we will decompose R as

$$R(\tau)=S(\tau-\tau_0)R(\tau_0)-\int_{\tau_0}^{\tau}S(\tau-s)\,(\alpha\Lambda R(s)+N(R)(s))\,ds$$
$$-\int_{\tau_0}^{\tau}S(\tau-s)\nabla\cdot(W(s)\nabla^{-1}D(s))ds,$$

where $\tau_0<\tau$. Since $R\in L^2_0(m)$, by construction, it follows from lemma 6.1 that $S(\tau-\tau_0)R(\tau_0)$ tends to zero as τ_0 tends to negative infinity. Hence we may write

$$R(\tau)=-R_1-R_2,$$

where

$$R_1\stackrel{\text{def}}{=}\int_{-\infty}^{\tau}S(\tau-s)\,(\alpha\Lambda R(s)+N(R)(s))\,ds$$
$$\text{and}\quad R_2\stackrel{\text{def}}{=}\int_{-\infty}^{\tau}S(\tau-s)\nabla\cdot(W(s)\nabla^{-1}D(s))\,ds.$$

As before, showing that R_1 is relatively compact is exactly as in [14]. Thus, we need only investigate R_2, which we handle similarly to the previous lemma.

We will show that R_2 is bounded in $L^2(m+r)$ for some $r>0$. For any $q\in(1,2)$, lemma 6.1 gives us

$$\|R_2\|_{L^2(m+r)}\leqslant C\int_{-\infty}^{\tau}\frac{e^{-\frac{1}{2}(\tau-s)}}{a(\tau-s)^{1/q}}\|W\nabla^{-1}D\|_{L^q(m+r)}ds.$$

Hölder's inequality implies that

$$\|W\nabla^{-1}D\|_{L^q(m)}\leqslant\|W\|_{L^2(m)}\|\nabla^{-1}D\|_{L^{2q/(2-q)}(r)}.$$

The first term is bounded due to the assumptions in the statement of the current lemma. For the remaining term concerning the divergence D, we apply proposition 6.2 to see that, letting $\tilde{m}=r+(2-q)/q$, and choosing r and q such that $\tilde{m}\leqslant m$,

$$\|\nabla^{-1}D\|_{L^{2q/(2-q)}(r)}\leqslant C\|D\|_{L^2(\tilde{m})}\leqslant C\|D\|_{L^2(m)}.$$

Hence R_2 is bounded in $L^2(m+r)$. Lemma 6.1 and lemma 6.3 imply that R_2 is also bounded in $H^1(m)$, so that Rellich's theorem, see e.g. [33, theorem XIII.65] implies that R_2 is relatively compact in $L^2(m)$, finishing the proof. □

In order to conclude, we need that bounded trajectories in L^1 are relatively compact. In order to show this, one may reproduce the proof of [14, lemma 2.5] as it relies only on a pointwise estimate on W, which we recreate in (3.8). This yields the final lemma we need to prove the necessary compactness.

6.3. Convergence in L^∞.

In this section we prove convergence of W to αG in L^∞, as stated in theorem 2.1.

Lemma 6.5. *Let* (W,D) *solve the system* (3.3)–(3.5) *with initial data* (W_0,D_0) *such that* $W_0,D_0\in L^1(1)$. *If* $\alpha=\int W_0\,d\xi$ *and* $\beta=\int D_0\,d\xi=0$, *then*

$$\lim_{\tau\to\infty}\|W-\alpha G\|_{L^\infty}=0.$$

Proof. Recall that we have shown that W converges to αG in L^p for all $p\in[1,\infty)$. As in (6.1), letting $R=W-\alpha G$, we may write an integral equation for R using the semigroup S. We will use this to show that $\|R\|_{L^\infty}$ tends to zero. As above, R satisfies

$$R(\tau)=S(1)R(\tau-1)-\int_{\tau-1}^{\tau} S(\tau-s)\,(\alpha\Lambda R(s)+N(R)(s))\,ds \\ -\int_{\tau-1}^{\tau} S(\tau-s)\nabla\cdot(W(s)\nabla^{-1}D(s))ds.$$

First, we use the third conclusion of lemma 6.1 with $p=\infty$, $q=1$, $\alpha=0$ and $m=0$ on the first term. Hence, we have that

$$\|S(1)R(\tau-1)\|_{L^\infty}\leqslant C\|R(\tau-1)\|_{L^1}.$$

Since $\|R(\tau-1)\|_{L^1}$ tends to zero, then $\|S(1)R(\tau-1)\|_{L^\infty}$ tends to zero. We may use this same strategy to deal with the rest of the terms.

First we look at

$$\Lambda R=(K_{BS}*G)\cdot R+(K_{BS}*R)\cdot\nabla G=\nabla\cdot((K_{BS}*G)R+(K_{BS}*R)G).$$

Then lemma 6.1, implies that,

$$\Big\|\int_{\tau-1}^{\tau} S(\tau-s)\Lambda R(s)ds\Big\|_{L^\infty}\leqslant\int_{\tau-1}^{\tau}(\|(K_{BS}*G)R\|_{L^1}+\|(K_{BS}*R)G\|_{L^1})ds \\ \leqslant C\int_{\tau-1}^{\tau}(\|R\|_{L^1}+\|K_{BS}*R\|_{L^\infty})ds.$$

Since R tends to zero in L^p for all p, then lemma 6.2 implies that $K_{BS}*R$ tends to zero in L^∞.

Next, we deal with the term involving $N(R)$. Notice that $N(R)=\nabla\cdot((K_{BS}*R)R)$. Hence, as above, we obtain

$$\Big\|\int_{\tau-1}^{\tau} S(\tau-s)N(R)(s)ds\Big\|_{L^\infty}\leqslant\int_{\tau-1}^{\tau}\|(K_{BS}*R)R\|_{L^1}ds \\ \leqslant C\int_{\tau-1}^{\tau}\|K_{BS}*R\|_{L^\infty}\|R\|_{L^1}ds.$$

Hence, this term tends to zero as well.

Finally, for the last term, we obtain

$$\Big\|\int_{\tau-1}^{\tau} S(\tau-s)\nabla\cdot(W\nabla^{-1}D)(s)ds\Big\|_{L^\infty}\leqslant\int_{\tau-1}^{\tau}\|W\nabla^{-1}D\|_{L^1}ds \\ \leqslant C\int_{\tau-1}^{\tau}\|\nabla^{-1}D\|_{L^\infty}\|W\|_{L^1}ds.$$

Using lemma 3.4 and lemma 6.2, we see that $\|\nabla^{-1}D\|_{L^\infty}$ tends to zero. This finishes the proof that $\|R\|_{L^\infty}$ tends to zero. □

7. Brief Remarks on Well-posedness

The well-posedness of the system (2.1)–(2.3) in classical or Lebesgue spaces is very similar to the development in [2, 3, 25]. For the weighted spaces, one may look to the strategies of [13, 34]. Since the adaptations required in our setting are minimal, we only briefly comment on the manner of proof. First, we discuss the primary a priori estimates in each of these spaces. Then, we discuss the iterative scheme used to prove local existence.

A Priori Estimates. The main a priori estimates in L^p and in L^2_w follow as in the work of lemma 5.1 and section 4, respectively. The a priori estimate in $L^2(m)^2$ is a slight modification of the argument of [13]. To this end, multiply (3.3) by $|\xi|^{2m}W$ to obtain

$$\frac{1}{2}\frac{d}{d\tau}\int|\xi|^{2m}W^2d\xi+\int|\xi|^{2m}W\nabla\cdot(UW)d\xi$$
$$=\int|\xi|^{2m}\left\{W\Delta W+\frac{W}{2}(\xi\cdot\nabla)W+W^2\right\}d\xi.$$

Integrating by parts, we see that these terms can be rewritten as

$$\int|\xi|^{2m}W(\Delta W)d\xi=-\int|\xi|^{2m}|\nabla W|^2d\xi+2m(m-1)\int|\xi|^{2m-2}W^2d\xi,$$
$$\int|\xi|^{2m}\frac{W}{2}(\xi\cdot\nabla)Wd\xi=-\frac{m+1}{2}\int|\xi|^{2m}W^2d\xi,$$
$$\int|\xi|^{2m}W\nabla\cdot(UW)d\xi=\frac{1}{2}\int|\xi|^{2m}DW^2d\xi+\frac{1}{2}\int|\xi|^{2m}\nabla\cdot(UW^2)d\xi$$
$$=\frac{1}{2}\int|\xi|^{2m}DW^2d\xi-m\int|\xi|^{2m-2}(\xi\cdot U)W^2d\xi.$$

By noting that for any $\varepsilon>0$ there is a $C_\varepsilon>0$ so that $|\xi|^{2m-2}\leqslant\varepsilon|\xi|^{2m}+C_\varepsilon$, we see that

$$\frac{1}{2}\frac{d}{d\tau}\int|\xi|^{2m}W^2d\xi+\int|\xi|^{2m}|\nabla W|^2d\xi+\frac{m-1-4\varepsilon}{2}\int|\xi|^{2m}W^2d\xi$$
$$\leqslant C_\varepsilon\int W^2d\xi+C_\varepsilon\|U\|_\infty^{2m}\int W^2d\xi+\frac{\|D\|_\infty}{2}\int|\xi|^{2m}W^2d\xi.$$

We know that $\|D\|_{L^\infty}$ decays to zero, and there is sufficient control over $\|W\|_{L^2}$ and $\|U\|_{L^\infty}$ by lemma 5.1 and proposition 6.2. Hence choosing $\varepsilon>0$ sufficiently small and integrating the above inequality yields the apriori estimate required in $L^2(m)^2$. These a priori estimates are summarized in the following proposition.

Proposition 7.1. *Fix* $(W_0,D_0)\in X$ *where* X *is either* $L^2(m)^2$, *with* $m>1$ *and* $\int D_0d\xi=0$, *or* $(L^2_w)^2$. *Then there exists a unique solution to the system* (3.3)–(3.5) *which satisfies*

$$\|W(\tau)\|_X\leqslant C.$$

Here C *is a constant depending only on the initial data and which tends to zero as* $\|W_0\|_X$ *tends to zero.*

An Iterative Scheme. To prove existence and uniqueness of classical solutions with initial data in L^1 we follow [2]. For existence, we begin with smooth initial data, and use an iterative argument to obtain the existence of solutions which are bounded in L^p for every p. The key contribution here is that we iterate only in the vorticity, leaving the divergence fixed as solutions to the heat equation follow from the classical theory. We define $\omega_0 = 0$ and then let ω_k be the solution to the linear system

$$\partial_t \omega_k + \nabla \cdot (u_{k-1}\omega_k) = \Delta \omega_k$$
$$u_k = \nabla^{-1} d + K_{BS} * \omega_k.$$

Bounds similar to lemma 5.1 can be obtained for this system, establishing the existence of a solution. Uniqueness follows by directly estimating the difference of two solutions. Afterwards, a continuity argument is used to extend this to any initial data in L^1.

In general, this argument differs from that in [2] only in the appearance of an extra term involving d in several of the estimates. However, this extra term behaves much better than the non-linear term as the classical theory on the heat equation for d yields appropriate bounds on the divergence in any of the required spaces. In particular, this gives us the following result which we state without proof.

Proposition 7.2. *Suppose that ω_0 and d_0 are elements of $L^1(\mathbb{R}^2)$. Then there exist $\omega, d \in C(\mathbb{R}_+, L^1) \cap C(\mathbb{R}_+, W^{1,1} \cap W^{1,\infty})$, where $\mathbb{R}_+ := (0,\infty)$, which are the unique solutions to the system* (2.1)–(2.3).

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References

[1] J. T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys. **94** (1984), no. 1, 61–66. MR763762 (85j:35154)

[2] M. Ben-Artzi, *Global solutions of two-dimensional Navier-Stokes and Euler equations*, Arch. Rational Mech. Anal. **128** (1994), no. 4, 329–358.

[3] H. Brezis, *Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations" [Arch. Rational Mech. Anal.* **128** *(1994), no. 4, 329–358; MR1308857 (96h:35148)]*, Arch. Rational Mech. Anal. **128** (1994), no. 4, 359–360.

[4] E. A. Carlen and M. Loss, *Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier-Stokes equation*, Duke Math. J. **81** (1995), no. 1, 135–157 (1996). A celebration of John F. Nash, Jr.

[5] A. Carpio, *Asymptotic behavior for the vorticity equations in dimensions two and three*, Comm. Partial Differential Equations **19** (1994), no. 5-6, 827–872.

[6] F. Colas, J. C McWilliams, X. Capet, and J. Kurian, *Heat balance and eddies in the peru-chile current system*, Climate dynamics **39** (2012), no. 1-2, 509–529.

[7] P. Constantin, *Some open problems and research directions in the mathematical study of fluid dynamics*, Mathematics unlimited—2001 and beyond, 2001, pp. 353–360. MR1852164

[8] P. Constantin and C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations*, Indiana Univ. Math. J. **42** (1993), no. 3, 775–789. MR1254117 (95j:35169)

[9] Q. Deng, L. Smith, and A. Majda, *Tropical cyclogenesis and vertical shear in a moist Boussinesq model*, J. Fluid Mech. **706** (2012), 384–412. MR2971553

[10] C. L. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, The millennium prize problems, 2006, pp. 57–67. MR2238274

[11] Y. Fujigaki and T. Miyakawa, *Asymptotic profiles of nonstationary incompressible navier–stokes flows in the whole space*, SIAM Journal on Mathematical Analysis **33** (2001), no. 3, 523–544.

[12] T. Gallay and Y. Maekawa, *Long-time asymptotics for two-dimensional exterior flows with small circulation at infinity*, Anal. PDE **6** (2013), no. 4, 973–991.

[13] T. Gallay and C. E. Wayne, *Invariant manifolds and the long-time asymptotics of the Navier-Stokes and vorticity equations on* $\mathbf{R}^2$, Arch. Ration. Mech. Anal. **163** (2002), no. 3, 209–258.

[14] T. Gallay and C. E. Wayne, *Global stability of vortex solutions of the two-dimensional navier-stokes equations*, Commun. Math. Phys. **255** (2005), no. 10, 97–129.

[15] Y. Giga and T. Kambe, *Large time behavior of the vorticity of two-dimensional viscous flow and its application to vortex formation*, Comm. Math. Phys. **117** (1988), no. 4, 549–568.

[16] Y. Giga, T. Miyakawa, and H. Osada, *Two-dimensional Navier-Stokes flow with measures as initial vorticity*, Arch. Rational Mech. Anal. **104** (1988), no. 3, 223–250.

[17] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), no. 4, 1061–1083.

[18] G. Grubb and V. A. Solonnikov, *Reduction of basic initial-boundary value problems for Navier-Stokes equations to initial-boundary value problems for nonlinear parabolic systems of pseudo-differential equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **171** (1989), no. Kraev. Zadachi Mat. Fiz. i Smezh. Voprosy Teor. Funktsii. 20, 36–52, 183–184.

[19] G. Grubb and V. A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand. **69** (1991), no. 2, 217–290 (1992).

[20] M. Ignatova, G. Iyer, J. P. Kelliher, R. L. Pego, and A. D. Zarnescu, *Global existence for two extended Navier-Stokes systems*, Commun. Math. Sci. **13** (2015), no. 1, 249–267.

[21] G. Iyer, R. L. Pego, and A. Zarnescu, *Coercivity and stability results for an extended Navier-Stokes system*, J. Math. Phys. **53** (2012), no. 11, 115605, 26. MR3026550

[22] H. Johnston and J.-G. Liu, *Accurate, stable and efficient Navier-Stokes solvers based on explicit treatment of the pressure term*, J. Comput. Phys. **199** (2004), no. 1, 221–259.

[23] H. Johnston, C. Wang, and J.-G. Liu, *A local pressure boundary condition spectral collocation scheme for the three-dimensional Navier-Stokes equations*, J. Sci. Comput. **60** (2014), no. 3, 612–626.

[24] R. Kajikiya and T. Miyakawa, *On* L^2 *decay of weak solutions of the Navier-Stokes equations in* $\mathbf{R}^n$, Math. Z. **192** (1986), no. 1, 135–148.

[25] T. Kato, *The Navier-Stokes equation for an incompressible fluid in* $\mathbf{R}^2$ *with a measure as the initial vorticity*, Differential Integral Equations **7** (1994), no. 3-4, 949–966.

[26] J.-G. Liu, J. Liu, and R. L. Pego, *Stability and convergence of efficient Navier-Stokes solvers via a commutator estimate*, Comm. Pure Appl. Math. **60** (2007), no. 10, 1443–1487. MR2342954 (2008k:76039)

[27] J.-G. Liu, J. Liu, and R. L. Pego, *Error estimates for finite-element Navier-Stokes solvers without standard inf-sup conditions*, Chin. Ann. Math. Ser. B **30** (2009), no. 6, 743–768.

[28] J.-G. Liu, J. Liu, and R. L. Pego, *Stable and accurate pressure approximation for unsteady incompressible viscous flow*, J. Comput. Phys. **229** (2010), no. 9, 3428–3453.

[29] K. Masuda, *Weak solutions of Navier-Stokes equations*, Tohoku Math. J. (2) **36** (1984), no. 4, 623–646.

[30] M. T Montgomery and R. K Smith, *Paradigms for tropical-cyclone intensification*, Mon. Wea. Rev **140** (2011), 3278–3299.

[31] M. Oliver and E. S. Titi, *Remark on the rate of decay of higher order derivatives for solutions to the Navier-Stokes equations in* $\mathbf{R}^n$, J. Funct. Anal. **172** (2000), no. 1, 1–18.

[32] M. R Petersen, S. J Williams, M. E Maltrud, M. W Hecht, and B. Hamann, *A three-dimensional eddy census of a high-resolution global ocean simulation*, Journal of Geophysical Research: Oceans **118** (2013), no. 4, 1759–1774.

[33] M. Reed and B. Simon, *Methods of modern mathematical physics. IV. Analysis of operators*, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978.

[34] L. M. Rodrigues, *Asymptotic stability of Oseen vortices for a density-dependent incompressible viscous fluid*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 2, 625–648. MR2504046 (2010i:35286)

[35] M. E. Schonbek, *L^2 decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **88** (1985), no. 3, 209–222.

[36] M. E. Schonbek, *Lower bounds of rates of decay for solutions to the Navier-Stokes equations*, J. Amer. Math. Soc. **4** (1991), no. 3, 423–449.

[37] M. E. Schonbek, *On decay of solutions to the Navier-Stokes equations*, Applied nonlinear analysis, 1999, pp. 505–512.

[38] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on $\mathbf{R}^n$*, J. London Math. Soc. (2) **35** (1987), no. 2, 303–313.

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