REMARKS ON A 25 YEAR OLD THEOREM ON TWO-DIMENSIONAL CELLULAR AUTOMATA

J. M. Greenberg, C. Greene, and S. P. Hastings
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The purpose of this brief note is to call attention to a theorem published almost twenty-five years ago [GGH] on two dimensional cellular automata which we believe is still of interest, but which seems to be very little known. One reason for this obscurity may be that when we wrote this paper, we did not know the term "cellular automata", and so it does not appear in the title or elsewhere in the paper. A second reason may be that the journal itself was discontinued more than 15 years ago. Despite the long intervening period, we are not aware of another theorem like it in this field.

The model in question is quite well known, as a discrete model of "excitable media". See for example [W], where the model is described (with acknowledgement) and some of the patterns are shown. What seems to be largely unknown is that we proved a theorem which allows one to predict from the initial condition an important aspect of the long time behavior of this model. The theorem is based on defining a "topological invariant" for the model. We now state a special case of the main result in the paper, in order for the reader to decide quickly if the result indeed merits reading further, and perhaps even looking up the original reference.

We consider a specific three-state cellular automata on a two-dimensional square lattice. This means that to each index pair (i,j), and for each integer valued "time" $n \geq 0$. we associate a number from the set $\{0,1,2\}$, denoted by $s_n(i,j)$, and this mapping obeys a set of rules which enable us to determine $s_{n+1}(i,j)$ if we know $s_n(i,j)$ and also the values of $\{s_n(i',j')\}$ for (i',j') lying in some "neighborhood set" of (i,j). The nature of this neighborhood does not matter much in the statement of the theorem. Indeed, the result is not restricted to two dimensions. For the purpose of this basic example, consider that the neighborhood of the "cell" (i,j) consists of the set $N_{ij} = \{(i+1,j),(i-1,j),(i,j+1),(i,j-1)\}$. This is often called the "von Neumann" neighborhood of the cell (i,j).

Our rules are as follows:

$$s_{n+1}(i,j) = \begin{cases} 2 & \text{if} \quad s_n(i,j)=1 \\ 0 & \text{if} \quad s_n(i,j)=2 \\ 1 & \text{if} \quad s_n(i,j)=0 \text{ and } s_n(i',j')=1 \text{ for at least one } (i',j') \in N_{ij} \\ 0 & \text{otherwise} \end{cases}$$

The reader who has not seen this model before may wish to consider the following example. In this example we pick out for examination a four square block whose cells at time $n = 0$ are in the states shown

$$\begin{matrix} 1 & 2 \\ 0 & 0 \end{matrix} .$$

Use the rule to determine the states at future times $n = 1, 2, 3$. Actually, we have not given quite enough information to do this. For example, the states of three of these cells at time $n = 1$ are completely determined by what we can see above, but the state of the lower right cell depends also on neighbors which we have not shown. Nevertheless, if the we follow all possible alternatives for the arrangement above, we find only two possibilities:

After one time step we have the same pattern except that it is rotated by 90 degrees or, after three time steps, we have returned to the original pattern in these four squares.

We quickly conclude that if one has this pattern to begin with, then for all $n \geq 0$, at least two of these four cells must have non-zero state. We call the pattern "persistent", because within a fixed region it never goes to all zeros.

Our theorem was more interesting than this. As restricted to this model it says:

Theorem: *If the initial state space* $\{s_0(i,j) \,|\, -\infty < i, j < \infty\}$ *contains any copies or reflections of the pattern above, or of one of the patterns*

$$\begin{matrix} 1 & 2 \\ 0 & 2 \end{matrix} , \quad \begin{matrix} 1 & 1 \\ 2 & 0 \end{matrix}$$

then the pattern is persistent. **If none of these patterns exist at $t = 0$, and the number of non-zero cells initially is finite, then the pattern is not persistent.**

We have emphasized the interesting part of the theorem.

Above we referred to a "topological invariant" which we discovered for this model. This is used in the proof of the theorem. To define this invariant we consider any cycle of cells, which we denote by

$$C = \{C_1, C_2, ..., C_n\} ,$$

where C_{j+1} is a neighbor of C_j and where C_n is a neighbor of C_1. We also consider that the states $\{0, 1, 2\}$ are on a circle. At a certain time n we proceed from C_1 along the cycle of cells, and as we do so, we move from point to adjacent point on the circle. That is, at step k we are at state $\sigma \in \{0, 1, 2\}$ if the state $s_k(n)$ of cell C_k at time n is equal to σ. When we return to C_1 we will have made some net integer number of clockwise rotations around the circle. We denote this number by $W_n(C)$. Our basic lemma is that $W_n(C)$ is independent of time n.

The theorem as stated requires more work, to reduce consideration to the shortest possible cycles of four cells in a square. We refer the interested reader to the original paper. If such a reader is unable to locate the paper, please email sph@pitt.edu for a hard copy. Electronic copies are not presently available, but should be obtainable from SIAM starting in 2005.

Remarks: The result above was stated in [GH], but without proof. The winding number concept introduced in [GGH] greatly facilitates the proof. The theorem in [GGH] is actually that the pattern is persistent if and only if, for some "time" n_0, there is a cycle with non-zero winding number. For the three state case considered here, all cycles immediately have a well-defined, and constant, winding number. It is easily shown that in this case if there is any cycle with non-zero winding number, then this can be "contracted" to one of the patterns described in the theorem. In the general case of many states, we obtain an upper bound for the smallest n_0 at which we can examine the winding numbers of all cycles and deduce persistence or non-persistence.

References

[GH] Greenberg, J and Hastings, S. P, Spatial patterns for discrete models of diffusion in excitable media, Siam J. Applied Math vol 34 (1978), 515-523.

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[W] Wolfram, S, A New Kind of Science (2003), pg. 1013.