The derivation of a hierarchy of plate models from nonlinear elasticity by Γ-convergence CNA Summer School 2004

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1 Abstract

A fundamental problem in elasticity is to derive theories for lower dimensional objects such as plates, shells or rods from the fully nonlinear three dimensional theory. The usual approach is to make certain assumptions on the three dimensional solutions and then to deduce a lower dimensional theory by formal or rigorous asymptotical analysis. These has lead to large variety of theories, which are sometimes not mutually compatible.

Since the early 90's a new, mathematically rigorous, approach has emerged, which is based on the variational principle and the associated notion of Γ-convergence. Le Dret and Raoult have used Γ-convergence to derive a theory for elastic membranes (these have only stretching stiffness, but no bending stiffness and cannot resist compression). In these lectures I will report on ongoing work with G. Friesecke (Munich/ Warwick) and R.D. James (Minnesota) to derive a full hierarchy of limiting theories, which are distinguished by the scaling of the elastic energy as a function of thickness (see Table 1 below). In particular I will discuss the derivation of Kirchhoff's plate theory (which captures bending) and the much debated von Kármán theory.

A key mathematical ingredient is a quantitative rigidity estimate which generalizes results of F. John for deformations with small nonlinear strain. A classical result says that any Lipschitz map from a (bounded) connected set in $\mathbf{R}^n$ to $\mathbf{R}^n$ (we are interested in $n \geq 2$) whose derivative is an element of $SO(n)$ a.e. has in fact constant derivative. The quantative rigidity estimate says that this can be extended to a linear estimate in L^2. More precisely for every $u \in W^{1,2}$ and every Lipschitz domain Ω we have

$$\min_{Q \in SO(n)} \int_\Omega |\nabla u - Q|^2 \leq C(\Omega) \int_\Omega \operatorname{dist}^2(\nabla, SO(n)). \tag{1}$$

The proof of this result is surprisingly simple and will be presented in the first lecture.

The results discussed in the lectures can be summarized in the table below. Here we consider a cylindrical domain $\Omega_h = S \times (-h/2, h/2) \subset \mathbb{R}^3$ of thickness h and elastic deformations $\phi : \Omega_h \to \mathbb{R}^3$. Using the rescaling $y(x_1, x_2, x_3) = \phi(x_1, x_2, hx_3)$ the elastic energy per unit height is

$$I^h(y) = \int_\Omega W(\nabla_h y)\, dx, \quad \text{where } \Omega = S \times (-1/2, 1/2),\ \nabla_h y = (y_{,1}, y_{,2}, \frac{1}{h} y_{,3}). \tag{2}$$

We consider applied body forces $f^{(h)} : S \to \mathbb{R}^3$ leading to the total energy

$$J^h(y) = I^h(y) - \int_\Omega f^{(h)} \cdot y\, dx. \tag{3}$$

We assume that the total applied force vanishes and that the force is of order h^α. More precisely we suppose that

$$\int_S f^{(h)}\, dx_1 dx_2 = 0, \quad h^{-\alpha} f^{(h)} \rightharpoonup f \quad \text{in } L^2(S; \mathbb{R}^3) \tag{4}$$

Given α we ask which rescaling of the energy J^h leads to a nontrivial limit functional (in the sense of Γ-convergence). Moreover we look at minimizers (or almost minimizers) $y^{(h)}$ of J^h and ask which rescaling of the (averaged) in-plane and out-of plane deformation

$$U_i^{(h)}(x_1, x_2) = \int_{-1/2}^{1/2} y_i^{(h)}(x) - x_i\, dx_3, \quad i \in \{1, 2\}, \qquad V^{(h)} = \int_{-1/2}^{1/2} y_3^{(h)}(x)\, dx_3 \tag{5}$$

leads to a nontrivial limit. Given α we thus seek exponents such that (at least for a subsequence)

$$h^{-\beta} J^h \xrightarrow{\Gamma} J, \quad h^{-\gamma} U^{(h)} \rightharpoonup U, \quad h^{-\delta} V^{(h)} \rightharpoonup V. \tag{6}$$

Strictly speaking we allow to apply a (h-dependent) rigid motion to $y^{(h)}$ before computing the deviation $U^{(h)}$ and $V^{(h)}$.

2 References, outlook, open questions

2.1 References for material covered in the lectures

The first three lectures follow [25] where many additional references can be found, for a short summary which emphasizes the main ideas see [24]. The Kirchhoff limit was indepedently established by O. Pantz [49, 50]. His setting is more restrictive since it relies on the celebrated classical estimates by F. John [32] rather than the new rigidity result.

α applied force	β energy	γ in-plane	δ out-of-plane	limit model
$\alpha = 0$	0	0	0	Membrane [36, 37]
$0 < \alpha < 1$	α	0	0	Constrained membrane [17]
$\alpha = 2$	α	0	0	Bending, isometric midplane[24, 25, 49, 50]
$2 < \alpha < 3$	$2\alpha - 2$	$2(\alpha - 2)$	$\alpha - 2$	Linearized isometry constraint [26, 27]
$\alpha = 3$	$2\alpha - 2$	$2(\alpha - 2)$	$\alpha - 2$	Föppl-von Kármán [26, 27]
$\alpha > 3$	$2\alpha - 2$	$\alpha - 1$	$\alpha - 2$	Linearized FvK [26, 27]

Table 1: Relation between the scaling exponents α of the applied forces, β of the energy, γ of the in-plane deformation and δ of the out-of-plane deformation. For $\alpha > 2$ we assume that the limit force is parallel to a fixed direction (which we may choose perpendicular to the plate) and that the total moment exerted by $f^{(h)}$ is zero.

The derivation of membrane theory from nonlinear elasticity by Γ-convergence is due to LeDret and Raoult [36, 37, 38] (partially inspired by earlier work of Acerbi, Buttazzo and Percivale on rods [1]) and was the first rigorous derivation of a two-dimensional theory from nonlinear three dimensional elasticity. Their result shows in particular that the traditional ansatz-based approach gives the wrong answer for membranes, it misses the fact that in the appropriate limit of zero thickness the membrane has no resistance to compression (while it can withstand tension). The study of two-dimensional objects which can only withstand tension has a long history in the engineering literature ('tension-field theory') and Pipkin [51, 52] discusses this in the context of relaxation of two-dimensional elastic energy functionals.

A brief description of the von Kármán limit is given in [26], a detailed version should appear on the MPI MIS website (www.mis.mpg.de) soon. The original formulation of the theory can be found in the works of Föppl (who only considered the stretching energy) [23][pp. 132–139] and von Kármán [35][pp. 348–351]. Truesdell's critique was taken from [55]. In the regime in between Kirchhoff and von Kármán ($2 < \beta < 4$) one is lead to a number of interesting questions about isometric immersions (which satisfy Gauss curvature $K = 0$) and their infinitesimal cousins for maps with low regularity ([53, 33, 48, 47]). A completely different approach to von Kármán theory was pursued by Monneau [43]. Using a subtle application of the implicit function theorem he shows that for every sufficiently small and sufficiently regular solution of the vK equations there exists a family $y^{(h)}$ of nearby solutions of the three dimensional equations of nonlinear elasticity.

Another interesting result which makes use of the rigidity estimate is the work by DalMaso, Negri and Percivale [19] which shows that the geometrically linear theory of elasticity (which replaces the $SO(3)$ invariance by an invariance under the tangent space of skew symmetric matrices) arises as a rigorous $\Gamma-$limit of nonlinear elasticity. This is particular interesting in view of the fact that geometrically linear elasticity is often used to analyse singular behaviour of the solutions, e.g. at corners. A common criticism is that the usual derivation of geometrically linear theory by Taylor expansion exactly fails to be valid in this setting. The DalMaso-Negri-Percivale result shows that at least singularities which are in the Sobolev space $W^{1,2}$ are compatible with the passage from nonlinear to geometrically linear elasticity, in the limit of small overall energy.

2.2 Outlook, selected further results and open problems

While the energy scalings h^β with $\beta > 2$ are by now relatively well understood very little is known in the range $0 < \beta < 2$ (see the comments on the exponents $\beta = 1$ and $\beta = 5/3$ below and Conti's work [17] for $\beta < 1$). Another major open problem is to go beyond limiting theories and study more precisely the behaviour at small, but finite h. Again this is particularly interesting in the range $0 < \beta < 2$. Here the membrane theory ($\beta = 0$) leads to a limit which is too soft (no resistance to compression) while the Kirchhoff bending theory ($\beta = 2$) is too rigid (only isometric immersions are allowed as competitors, and this poses a very strong restriction on the boundary conditions which can be prescribed). Engineers and physicists frequently uses two-dimensional theories where the total energy is written as a membrane energy term (which still penalizes compressions) plus h^2 times a bending term. The usual choices are consistent with the table above (in the sense that they lead the same Γ-limits as three dimensional nonlinear elasticity, when these limits are known). The real question is what additional information one can get from these theories, which choices of the membrane and the bending energy are 'optimal' (in a sense which needs to be defined) and which choices are asymptocially equivalent. The question whether Γ-convergence can be used to identify higher order corrections ('Γ-development') has been studied before (see e.g. [4]), but so far results have been mostly obtained for linear problems. Braides and Truskinovsky [11] have addressed the issue of higher order corrections in Γ-convergence in a number of other contexts.

In the following I discuss very briefly some other possible extensions and open questions.

Shells. Instead of plates (whose undeformed states is flat) one can consider shells (which are curved in the undeformed state). The analogue of the Kirchhoff theory is derived in [28] (for membrane theory see [38]). It should be possible to extend also the results for the other scaling regimes. In this context the distinction between shells which are flexible (i.e. those where the boundary conditions allow for nontrivial infinitesimal isometric deformations) and those which are not, which has been emphazised by Ciarlet and his coworkers, should be important.

Rods. The analogues of the Kirchhoff and von Kármán theories can be found in [44, 45], respectively. For rods there exists also a very interesting alternative approach by Mielke [40, 41] which avoids passing to the limit $h \to 0$ and thus contains more information. Instead of letting the diameter h of the rod go to zero Mielke considers longer and longer rods of fixed cross-section. For the infinitely long rod he shows that all solutions whose nonlinear elastic strain $(\nabla y)^T \nabla y - Id$ is sufficiently small (in some Hölder space C^α) lie on a twelve dimensional manifold and that on that manifold the three dimensional elasticity equations (without body forces) reduce exactly to the Timoshenko beam equations. It would be very interesting to extend that approach to plates - so far there are only results starting from linear elasticity [42, 5]

Buckling of plates and von Kármán theory. Von Kármán theory is frequently used to analyse the onset of buckling in thin plates. While the theory discussed in the lectures does assert that vK theory arises at a limit in a certain scaling regime it is not a priori clear that the loading conditions leading to buckling are compatible with that regime. In ongoing work we have established a stability alternative which specifies when vK is appropriate and we are currently exploring this condition in specific examples. A corresponding preprint should appear soon on the MPI-MIS website.

Euler-Lagrange equations and dynamics. The analysis presented in the lectures relies very strongly on the fact that we look at (global) minimizers of the energy. It would be very interesting to see whether this analysis can also help to understand the evolution equations of elasticity (the rigidity estimate does not use any minimizing property, so certain estimates are available from energy estimates alone). Marsden and coworkers [29] have studied the limiting dynamics for shells by viewing the evolution equations as an infinite dimensional Hamiltonian system and passing to the limit in the Hamiltonian. It is not obvious, however, that the limit of the three dimensional time-dependent solutions will indeed be governed by the evolution according the the limiting Hamiltonian. As a first step in the direction of dynamics M. Schultz is currently studying the behaviour of sequences of deformations which satisfy the stationary Euler-Lagrange equations (exactly or approximately).

Multiwell problems. Models for solid-solid phase transitions [6, 15] (see [9, 21, 46] for recent sets of lecture notes) lead to elastic energies for which the minimum is attained not on $SO(3)$ but on a set $K = \cup_i SO(3)U_i$ consisting of multiple copies of $SO(3)$ (so called energy wells). The rigidity theorem can be extended to the situation of two wells, provided the wells are incompatible (in a sufficiently strong sense), i.e. in particular there is no non-affine deformation whose gradient takes values in K almost everywhere [13]. This result can be used to deduce optimal scaling laws for thin films of two-well materials [14], improving the results in [10]. Another natural generalisation is to replace $SO(n)$ by the set of conformal matrices. For a (partial) quantative version of Reshetnyak's rigidity result for almost conformal maps see [22].

Interesting scaling exponents in the range $0 < \beta < 2$. The exponent $\beta = 1$ arises in the study of thin films subject to compressive Dirichlet boundary conditions [8, 31]. Such films develop a complex blistering pattern, see [30] for a discussion of the background and the engineering interest in this problem.

The compression of thin films under more general boundary conditions has recently attracted a lot of interest in the physics literature [7, 12, 20, 39]. The exponent $\beta = 5/3$ is conjectured to be important here, for first rigorous results in this direction see [56, 18]. A related but different problem arises in the study of complex folding patterns a free boundaries after rupture and it has been suggested that similar patterns might be important in certain growth models in biology [54, 3].

Numerical analysis. It seems natural to use the rigidity estimate to study a priori and a posteriori estimates for numerical schemes for thin elastic bodies. In my view that would be an interesting direction to explore.

Disclaimer: I would like to emphasize that the above account represents my personal bias and is by no means an attempt to review the enormous literature on plates and shells. In particular I have not at all discussed the large body of work based on linear elasticity and results based on (formal) asymptotic expansion (see e.g. the book [16] by Ciarlet for further information on this).

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