R. Kohn - Lectures 2+3 (CMU) : The Internal Structure of a Crottie Wall.

Outline:

1. Explain phenomenon - what is a crottie wall? What is the underlying cause of varns problem? Plan of attack: watching upper + lower buds

2. More about the varns plan: link to microweathering

3. Warming toward watching upper + lower buds: Modica - Mortola plan, Arias - Giga plan

4. Sketch of successful part in crottie wall.

Related reading:

- my article in Proc ICM 2006 (very close to viewpoint of this lecture)
- review article in "The Science of Hysteresis II" (pp 269-381, by DeSimone, Kohn, Müller, Otto) has same viewpoint, more detailed. in Sect 6.5
- original expln (same essence, but different viewpoint) is due to Alouges, Rivièra, Sanfèry ESAHM - COCV 8 (2002) 31-68.
What is a cross-tie well?

Ans: special type of "domain wall" seen in "soft" thin ferromagnets (but not too thin!)

Our goal: explain why it forms (and why this particular pattern) by arguing that it "solves" a certain variational problem.

Why is this interesting?

a) challenge from physics to calculus of variations
b) paradigm for "energy-driven pattern formation"
The variational problem:

\[ E(m) = \int_{\text{walls}} \text{"wall energy"} \]

where

\[ m = (m_1, m_2) (x_1, x_2) \]  \text{"magnetization"}

is piecewise smooth, divergence-free (even at walls) and unit-length \( |m| = 1 \),

and

\[
\text{wall energy density} = \sin \theta - \theta \cos \theta ,
\]

where \( \theta = \text{half-angle of descent} \)

\[ \theta \rightarrow \frac{\theta}{2} \]

\( (m \cdot n) \text{ must be constant at } )

\( (a \text{ well, since } d\omega m = 0) \)

What domain + bc? A strip, imposing \( \text{per bc on sides} \):

\[ \text{crosstie pattern?} \]
Claim: cross tie pattern achieves win for this

proposed pattern does better than the plan.

The energy per unit length is 

\[
T = \frac{1}{2} \cos \frac{T_2}{2} - \frac{T_2}{2}
\]

or our vertical plan permits a

unpatterned wall.

Note 1: \( \theta \) is the inner angle.

\( \theta = \theta_1 + \theta_2 + \theta_3 \)

(Continue: free parameters)

\( \theta_1 - \theta_2 - \theta_3 = 0 \)

This pattern is not circular.
Our main assertion is that the min energy for

is $\sqrt{2}$L. Pattern shows it is $\leq$ this value.

Key task is to prove correspondence.

Our usual plan may look a bit strange. Obvious questions:

1) Why is the wall energy surface-defined?

2) Why is it defined on piecewise-smooth divergence-free vector fields?

Motivation comes from micromagnetics. There, magnetization $\mathbf{m} = (m_1, m_2, m_3)$ in $\mathcal{D}(x_1, x_2, x_3)$ satisfies $|\mathbf{m}| = 1$ and achieves (local) min of

$$E = \frac{\alpha}{2} \int |\mathbf{m}|^2 + \frac{Q}{2} \int \phi(m) - \frac{1}{2} \int \mathbf{H} \cdot \mathbf{m} + \frac{1}{2} \int \mu_0 \mathbf{H}^2$$

exchange anisotropy applied field

magnetostatic
where $AU = \text{div}(m\hat{z})$, i.e.

$m\hat{z} = \nabla U + \text{div-free (Helmholtz decay)}$

For us: $d$ is very small (about 10 nm)

$S_l$ is a film thickness $\sim 100$ nm

magnetostatic term favors $\text{clim m}=0$, $m\cdot\hat{z}=0$

Thin film geometry suggests $m$ undefined $X_3$;

Then $m\cdot\hat{z}=0$ at top/bottom faces $m_3=0$.

(Expln of well energy - & - postpned till later)

Key issue: how can we ever hope to prove
ansatz independent lower Bds? Warm-up
by comparing with Modica-Hart:oba + Aniba-Gipa

Aniba-Gipa var'd plan: $\int E_{1792}u^2 + \frac{1}{2} (1741^2-1)^2, \text{d}cR^2$
Note strong analogy:

\[ m = (m_1, m_2) (x_1, x_2) \]

\[ \text{curl} m = \nabla \times m \]

\[ \text{curl} (\nabla \times m) = 0 \]

\[ \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

\[ \varepsilon \text{curl} \frac{1}{\varepsilon} \text{curl} m = 0 \text{ preferred} \]

\[ \frac{1}{\varepsilon} \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

\[ \frac{1}{\varepsilon^2} \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

But Ariles - Gipa is still complicated. Start with more basic example:

\[ \text{Modica - Mortola} \]

\[ \frac{1}{\varepsilon^2} \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

\[ \frac{1}{\varepsilon} \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

\[ \varepsilon \frac{1}{\varepsilon} \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

\[ \frac{1}{\varepsilon^2} \text{curl} \frac{1}{\varepsilon} \text{curl} m \]

Claim: as \( \varepsilon \to 0 \), asymptotic will plan \((\nabla \cdot \varepsilon)\) involves wall energy

One app: \( \Omega \) if \( \Omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Then I local min \( V \) with wall in wall (Kohn - Sternberg, late 80's).

Calculate the wall energy:
1D \( \int 3v_x^2 + \frac{1}{E} (v^2-1)^2 \, dx \geq 2 \int 1v^2 - 11v_x \, dx \)
  \[ \geq \int_{-1}^{+1} 2(1-v^2) \, dv = \frac{8}{3} \]

2D \( \int 31v_x^2 + \frac{1}{E} (v^2-1)^2 \geq 2 \int 1v^2 - 11v_x \, dx \)
  \[ = \int 17g(v) \, dx \]
  \[ = \int \left[ g(1) - g(-1) \right] \, dx \]
  \[ \text{well} \]
  where \( g(v) = \int_{-1}^{v} 21v^2 - 11 \, dv \), so \( g(1) - g(-1) = \frac{8}{3} \)

Arg also reveals opt'w wall profle:

Equality \( \iff E1v_x = 1v^2 - 11 \Rightarrow v = \tan \theta \left( \frac{x-x_0}{E} \right) \)

\( v_x > 0 \)

OK, now let's run to Aviles-Giga, eq on

\( \text{square } cR^2 \)

\[ \min_{u=0} \int_{SL} \int 31v_x^2 + \frac{1}{E} (1v_x^2 - 1)^2 \, dx \]

\[ \text{at } SL \]
Expect, as $\varepsilon \to 0$

1. $\nabla u$ has no incentive to oscillate, so $\|\nabla u\|_1 = 1$ in limit

2. Solve $\nabla_q \varepsilon \nabla u = 1$ with $u=0$ at $\partial Q$ must have "wells" (unless $\Omega$ = circle); asymptotic energy should min "wall energy"

3. Expect interval size of wall to be $1D$ (like Modica - Mortola)

Possible candidates for wall patterns

(Does "viscous" $\omega$ minimize wall energy? proves.)

Wall energy is easy to guess, by solving "wall var" problem:

$\nabla u = (a, -\sqrt{\varepsilon a^2})$

$\nabla u = (a, \sqrt{\varepsilon a^2})$

$\nabla u$ periodic
Guess \( u(x, y) = ax + \phi(y) \). Then \( \phi \) solves a 1D "Modica - Mortola - type" pbm

\[
\min \int \frac{1}{\varepsilon} \left( \phi_y^2 + a^2 \right) + \varepsilon \phi_y^2 \, dy
\]

best \( \phi \) has \( \varepsilon \phi_y = |\phi_y + a^2 - 1| \)

energy \[
= 2 \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} \left( t^2 + a^2 - 1 \right) dt = \frac{\delta}{3} (1-a^2)^{3/2}
\]

But is this optimal? — Is the wall really so simple?

\[\text{Can we prove a lower bd without any ansatz about structure of wall?}\]

Answer is yes! Proof uses clever interplay by parts. Look for \( \Sigma : \mathbb{R}^2 \to \mathbb{R}^2 \) ("entropy") \( \Sigma \)

1. \( |\text{div} \Sigma(\nabla u)| \leq \frac{1}{\varepsilon} (|\nabla u|^2 - 1)^2 + \varepsilon |\nabla u|^2 \)

2. \( \Sigma \left( a - \sqrt{1-a^2} \right) \cdot (0, 1) + \Sigma \left( a, \sqrt{1-a^2} \right) \cdot (0, 1) = \frac{\delta}{3} (1-a^2)^{3/2} \)

Then, for Aviles - Giga, energy of "wall" would pbm is
2.11

\[ \geq \int \left| \text{div} \, \Sigma' (7u) \right| \quad \text{by } 0 \]

\[ \geq 1 \int \text{div} \, \Sigma (7u) \]

\[ = \frac{8}{3} (1-a^2)^{3/2} \cdot \text{length} \quad \text{by } 0 \]

Successful $\Sigma$ is explicit:

\[ \Sigma (7u) = 2 \left( \frac{1}{2} u_x^2 - u_x u_y^2 + u_x, \frac{1}{2} u_y^2 + u_y u_x^2 - u_y \right) \]

for which, by elementary algebra,

\[ \text{div} \, \Sigma' (7u) = 2 \left( 1 - 17u^2 \right) \left( 2u_{xx} - 2u_{yy} \right) \]

\[ \leq \frac{1}{3} \left( 1 - 17u^2 \right)^2 + \frac{1}{3} \left( u_{xx} - u_{yy} \right)^2 \]

\[ = \text{Avere - Giga energy} + 2 \int_{\Sigma} u_{xy}^2 - u_{xx} u_{yy} - 4 \varepsilon \int_{\Sigma} u_{xy}^2 \]

\[ \leq \text{Averes - Giga energy} + O(\varepsilon) \]

since $u_{xy}^2 - u_{xx} u_{yy}$ is a null-Lagrangian (so its integral is fully determined by bd).

Our ansatz is optimal (as $\varepsilon \to 0$) since all the inequalities above are asymptotically sharp in it.
How to think of $\Sigma'$? As $\epsilon \to 0$ in condition 1, we see that

$$u \text{ smooth, } \int u^2 = 1 \implies \text{div } \Sigma'(7\omega) = 0$$

Thus $\langle \Sigma'_1(7\omega), \Sigma'_2(7\omega) \rangle = \Sigma'(7\omega)$ are like an "entropy - entropy flux" pair for a scalar conservation law.

Let's now to question postponed earlier: why is the "wall energy" of our "crescent wall point" equal to $\sin \theta - \theta \cos \theta$, when $\theta = \text{half-angle of wall}$?

Ans: this comes from minimizing the microwavnetic energy in "thick film regime", assuming a 1D ansatz.

Microwave energy, reduced to width of $x_3$

$$m = (m_1, m_2, m_3), (x_1, x_2) = (m', m_3)$$

on the domain $\mathcal{L} = \mathcal{L} \times (0,t)$, without anisotropy or applied field, is
\[ \frac{\partial^2 t}{\partial \xi^2} + \int_{\mathbb{R}^2} \nabla^2 \psi \, d\xi + \int_{\mathbb{R}^3} \psi \, d\xi \]

**Exchange equation**

Can solve for \( U \) in Fourier space ("sep of vars")

\[ \int_{\mathbb{R}^3} \psi \, d\xi = \int_{\mathbb{R}^2} f(\frac{\xi}{\xi_0}) \frac{\xi_0}{\xi} \, d\xi_0 \cdot m' \xi_0^2 \, d\xi_0 \]

\[ + \int_{\mathbb{R}^2} g(\frac{\xi}{\xi_0}) \frac{\xi_0^2}{4} \, d\xi_0 \]

\[ g(z) = \frac{1 - e^{-2z}}{2z}, \quad f(z) = 1 - g(z) \]

If \( t \gg \text{length scale on which } m' \text{ varies (w.d)} \)
we get (since \( f(z) \approx 1 \), \( g(z) \approx \frac{1}{2z} \) as \( z \to \infty \))

\[ \int_{\mathbb{R}^3} \psi \, d\xi \approx \frac{1}{2a} \frac{\partial m'}{\partial \xi} \left|_{\xi_0} \right|^2 + \frac{1}{2a} \frac{\partial m_3}{\partial \xi} \left|_{\xi_0} \right|^2 \]

OK, now consider 1D wall with \( m_3 = 0 \):

\[ m = (\cos \theta, \sin \theta, 0) \quad \Rightarrow \quad \theta = \theta(x) \]

\[ \theta = \theta_\infty \quad \frac{x = a}{x = b} \]

Profile \( \theta(x) \) should minimize
\[ t \int_a^b \frac{d}{dx} \Theta_x \, dx + t \| x_m \|_{H^{-1}}^2 \]

\[ = t \int_a^b \Theta_x^2 \, dx + t \| \Theta_1 - \cos \Theta_{\infty} \|_{L^2}^2 \]

\[ = \min \left[ \int_a^b \frac{d^2 \Theta_x}{dx^2} + 1 \cos \Theta_1 - \cos \Theta_{\infty} \right] \, dx \]

\[ \Theta(a) = \Theta_{\infty} \]

\[ \Theta(b) = -\Theta_{\infty} \]

\[ = 2 dt \int_{-\Theta_{\infty}}^{+\Theta_{\infty}} | \cos \Theta_1 - \cos \Theta_{\infty} | \, d\Theta \]

\[ = 4 dt \left( \sin \Theta_{\infty} - \Theta_{\infty} \cos \Theta_{\infty} \right) \]

(Is the 1D ansatz optimal? Well, no! In 3D micromagnetics there are lower-energy wells, achieved via nontrivial \( x \)-dependence. But 1D well is apparently a local min of energy, at least more or less.)

[Underline]Rm to crosstie wall. Recall our assertion, that our 'cartoon' of crosstie pattern achieves min energy. Have upper bd, need matching lower bd. Method (via 'entropy') is to
Come up with a map \( \Sigma': S^1 \to \mathbb{R}^2 \) s.t.

1. If \( m \) is smooth, div-free, \( \| m \|_1 = 1 \) then \( \text{div} \Sigma'(m) = 0 \).

2. If \( m \) is piecewise smooth, weakly div-free, \( \| m \|_1 = 1 \) then along any wall
   \[ | [\Sigma'(m) \cdot n] | \leq \text{magnetc wall energy} \]
   with equality for angles \( \leq 90^\circ \).

3. \[ \int \Sigma'(m) \cdot n = (\sqrt{2} - 1) \cdot \text{length of strip} \]

Recall why this is so: for any \( m \)

\[ \left| \int \Sigma'(m) \cdot n \right| \leq \int \| \text{div} \Sigma'(m) \|_1 \leq \int \text{well energy of } m \]

By (2) + (1),

with equality for our cartoon

since all its walls have angle \( \leq 90^\circ \)
How to find $\Sigma$? Condr. that $\alpha$ hold with equality for small angles leaves little freedom.

First pass (wrong, but informative): try

$$\Sigma'(m) = \frac{1}{2} (\theta m + m^2) \quad \text{when } m = e^{i\theta}$$

$$= \frac{1}{2} (\theta \cos \theta - \sin \theta, \theta \sin \theta + \cos \theta)$$

Then $\text{div } \Sigma(m) = \theta \text{div } m$ (elementary calc)
so $\alpha$ holds. Also

$$\Rightarrow \quad v = (1,0), \quad m = (\cos \theta, \pm \sin \theta)$$
$$\Rightarrow |\Sigma'(m) \cdot v| = |\Sigma^R - \Sigma^L| = \sin \theta - \theta \cos \theta$$

so $\alpha$ holds (for any $\theta$!),

Problem: $\theta$ isn't well-defined, since wells can (and do!) contain vertices.
Successful choice is similar in each quadrant:

\[
\Sigma'(m) = \begin{cases} 
(\pi/2 - \theta) m - m^\perp + (0, -\sqrt{2}) & -\pi/4 < \theta < \pi/4 \\
(\theta - \pi) m + m^\perp + (0, \sqrt{2}) & \pi/4 < \theta < 3\pi/4 \\
(3\pi/2 - \theta) m - m^\perp + (\sqrt{2}, 0) & 3\pi/4 < \theta < 5\pi/4 \\
\end{cases}
\]

slightly different formula in each quadrant

Fact: This defines a count's \( \Sigma : S^1 \to \mathbb{R}^2 \) satisfying all our conditions. In particular,

\[ |[\Sigma'(m), v]| \leq \text{well energy density} \]

with equality when total angle \( < 90^\circ \).

No need for arithmetic: our pattern achieves the old because it uses only walls with angles \( < 90^\circ \).

Some remarks:

1. We proved watching upper + lower beds, but we have not shown crosstie pattern
is unique. (Defining feature: achieves net 180° well, by presence smooth, dew-free pattern using only walls with angle ≤ 90°.)

2. There's a cross-tie pattern for every angle > 90°. (A simple discontinuity is optimal for angle < 90°.)

3. Internal length scale of cross-tie wall is not determined by preceding discontinuity. Actually, it's set by competition &

   • anisotropy - if energy prefers $m_z = 0$ then this favors smaller periodicity

   • finite ratio $\frac{\text{thickness}}{\text{exchange barrier}} = \frac{1}{d}$

Then our calculation of energy is only approximate, actually walls have "tails" that overlap each other, thus favors larger periodicity.

4. We did not prove cross-tie wall achieves global minimum of micromagnetic...
energy. (It doesn't!) Actually it's a local min. Our modeling hypotheses recognize it as global min of a more restricted forbidden method (via well-chosen "entropy") resembles:
- use of "calibrations" to study
  minimal surfaces
- use of "null-Lagrangians" to
  estimate relaxed energies

but it seems rather mysterious nevertheless, were we lucky? Or did there have to be an integr-by-parts proof of this lower bd?

Can other patterns (assoc Landau-type theories from physics) be explained this way?

Certainly not all energy-min patterns. For example, analysis of vortex patterns in type-II superconductivity is entirely different (see recent book of Landau & Serfaty).
Can similar methods be used for other nuclear plans?

One applies the notion of low-energy states for large potentials, e.g., that as $E \to 0$, limiting $V$ has $\frac{1}{1+4N^2} = 1$. Pf by DeSimone/Kohn/Müller/Otto uses existence of many eigenvalues, then argues as Tartar did, for scalar cases.