R. Kohn - Lectures 2+3 (CMU): The Internal Structure of a Crosstie Wall.

Outline:

(1) Explain phenomenon - what is a crosstie wall? What is the underlying calc of var'ns problem? Plan of attack: matching upper + lower bds

(2) More about the var'l pbm: link to micromagnetics

(3) Warmups toward matching upper + lower bds: Modica - Mortola pbm, Aviles - Giga pbm

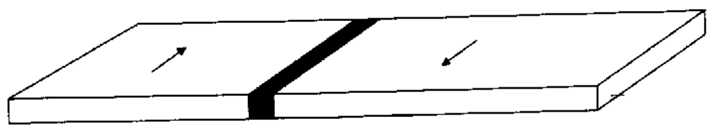
(4) Sketch of successful arg't for crosstie wall.

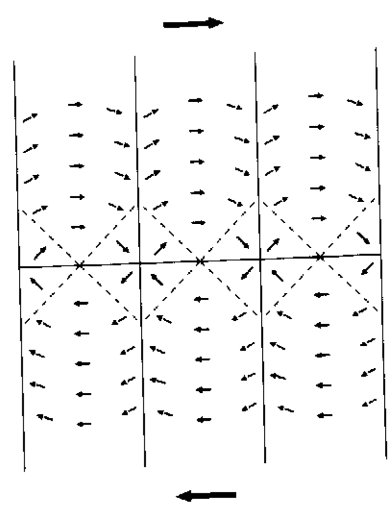
Related reading:

- my article in Proc ICM 2006 (very close to viewpt of this lecture)
- review article in "The Science of Hysteresis II" (pp 269-381, by DeSimone, Kohn, Müller, Otto) has same viewpt, more detail, in Sect 6.5
- original expl'n (same essence, but different viewpt) is due to Alouges, Rivière, Serfaty ESAIM - COCV 8 (2002) 31-68.

What is a cross-tie wall?

Ans: special type of "domain wall" seen in "soft", thin ferromagnets (but not too thin!)





Our goal: explain why it forms (& why this particular pattern) by arguing that it "solves" a certain var'l problem.

Why is this interesting?

a) challenge from physics to calc of vars

b) paradigm for "energy-driven pattern formation"

The variational problem:

$$E(m) = \int_{\text{walls}} \text{"wall energy"}$$

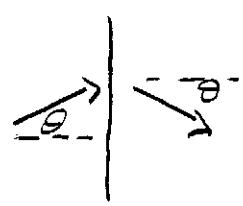
where

$$m = (m_1, m_2)(x_1, x_2) \qquad \text{"magnetization"}$$

is piecewise smooth; divergence-free (even at walls) and unit-length $|m| = 1$, and

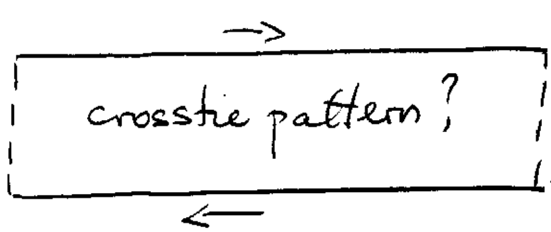
$$\text{wall energy density} = \sin\theta - \theta\cos\theta,$$

where θ = half-angle of discont'y



($m \cdot \nu$ must be cont's at a wall, since $\operatorname{div} m = 0$)

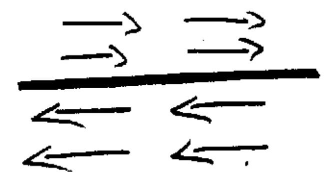
What domain + bc? A strip, imposing per bc on sides:



Claim: crosstie pattern achieves min for this var'l pbm

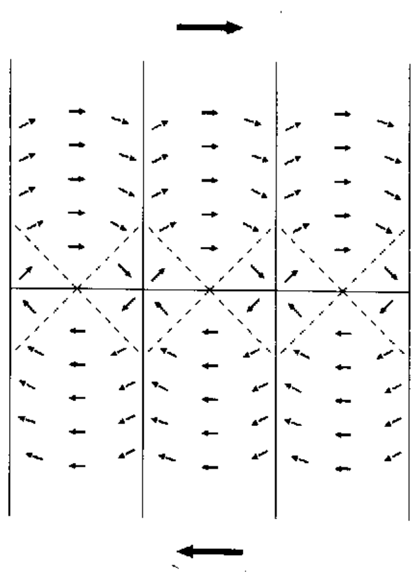
Getting used to the pbm:

a) our var'l pbm <u>permits</u> a 1-D (unpatterned) wall



Its energy per unit length is 1 since $\sin \pi/2 - \pi/2 \cos \pi/2 = 1$

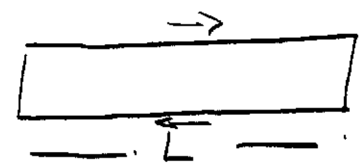
b) proposed pattern does better (energy/unit length turns out to be $\sqrt{2} - 1 \approx .414$); roughly: it trades <u>expensive</u> large-angle wall for many <u>cheaper</u> smaller-angle walls.



(Note: $\sin\theta - \theta\cos\theta \sim \theta^3$ for θ near 0.)

(Rmk: sole free parameter in this pattern is the horizontal periodicity.)

c) Our main assertion is that

min energy for 

is $(\sqrt{2}-1)L$. Pattern shows it is $\leq$ this value.

$\boxed{\text{Key task is to prove corresp lower bd}}$

Our var'l pblm may look a bit strange. Obvious questions:

1) why is the wall energy $\sin\theta - \theta\cos\theta$?

2) why is it defined on piecewise-smooth divergence-free vector fields?

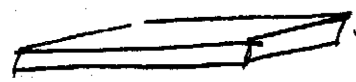
Motivation comes from <u>micromagnetics</u>. There, magnetization $\vec{m} = (m_1, m_2, m_3)$ fn of (x_1, x_2, x_3) satisfies $|m| = 1$ + achieves (local) min of

$$E = \underbrace{d^2 \int_\Omega |\nabla m|^2}_{\text{exchange}} + \underbrace{\cancel{Q \int_\Omega \varphi(m)}}_{0 \;\; \text{anisotropy}} - \underbrace{\cancel{2\int_\Omega H_{ext} \cdot \vec{m}}}_{\text{applied field}} + \underbrace{\int_{\mathbb{R}^3} |\nabla U|^2}_{\text{magnetostatic}}$$

where $\Delta U = \operatorname{div}(m\chi_\Omega)$, i.e.:

$$m\chi_\Omega = \nabla U + \text{div-free} \quad \text{(Helmholtz decompn)}$$

For us: d is very small (abt 10 nm)

Ω is a film  ,
thickness ~ 100 nm

magnetostatic term favors $\operatorname{div} m = 0$, $m \cdot \nu = 0$.

Then film geometry suggests m indep of x_3;
then $m \cdot \nu = 0$ at top/bottom faces $m_3 = 0$.

(Expln of wall energy $\sin\theta - \theta\cos\theta$
postponed till later.)

<u>Key issue</u>: how can we ever hope to prove ansatz-independent <u>lower bds</u>? Warm up by comparing with <u>Modica-Mortola</u> + <u>Aviles-Giga</u>.

<u>Aviles-Giga</u> <u>var'l pbm</u>: $\displaystyle\int_\Omega \varepsilon|\nabla\nabla u|^2 + \frac{1}{\varepsilon}(|\nabla u|^2-1)^2$, $\Omega \subset \mathbb{R}^2$.

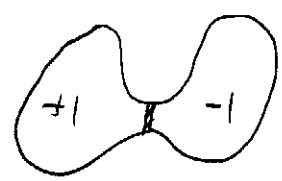
Note strong analogy:

<u>our problem</u>	<u>Aviles-Giga</u>
$m = (m_1, m_2)(x_1, x_2)$	∇u
$\operatorname{div} m = 0$	$\operatorname{curl}(\nabla u) = 0$
$\lvert m \rvert = 1$ exactly	$\lvert \nabla u \rvert = 1$ preferred
wall energy	wall energy as $\varepsilon \to 0$

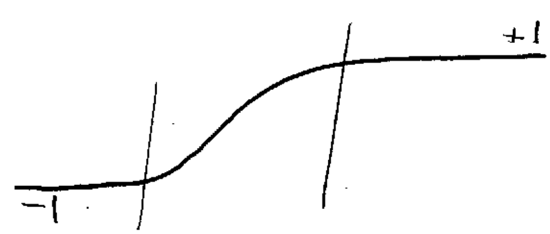
But Aviles-Giga is still complicated. Start with more basic example:

<u>Modica-Mortola</u> $\quad \displaystyle\int_\Omega \varepsilon \lvert \nabla v \rvert^2 + \frac{1}{\varepsilon} \lvert v^2 - 1 \rvert^2 \qquad v : \Omega \to \mathbb{R}$

<u>Claim</u>: as $\varepsilon \to 0$, asymptotic var'l pblm (Γ-limit) involves <u>wall energy</u>

One appl'n: if $\Omega =$ 

then $\exists$ local min v_ε with wall in neck (Kohn-Sternberg, late 80's).

Calculate the wall energy: 

1D $$\int \varepsilon v_x^2 + \frac{1}{\varepsilon}(v^2-1)^2\,dx \geq 2\int |v^2-1||v_x|\,dx$$

$$\geq \int_{-1}^{+1} 2(1-v^2)\,dv = 8/3$$

2D $$\int \varepsilon |\nabla v|^2 + \frac{1}{\varepsilon}(v^2-1)^2 \geq 2\int |v^2-1||\nabla v|\,dx$$

$$= \int |\nabla \varphi(v)|\,dx$$

$$= \int_{\text{wall}} [\varphi(1)-\varphi(-1)]\,ds$$

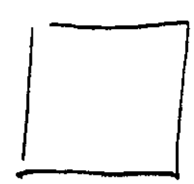
where $\varphi(v) = \int_{-1}^{v} 2|v^2-1|\,dv$, so $\varphi(1)-\varphi(-1) = 8/3$

<u>Argt also reveals opt'l wall profile:</u>

equality $\Leftrightarrow$ $\varepsilon|v_x| = |v^2-1| \underset{v_x>0}{\Longrightarrow} v = \tanh\left(\frac{x-x_0}{\varepsilon}\right)$

OK, now let's rtn to Aviles-Giga, eg on Ω = square $\subset \mathbb{R}^2$

$$\min_{\substack{u=0 \\ \text{at } \partial\Omega}} \int_\Omega \varepsilon|\nabla\nabla u|^2 + \frac{1}{\varepsilon}(|\nabla u|^2-1)^2$$



Ω

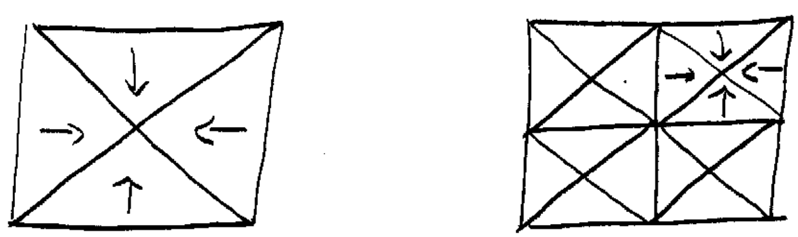
Expect, as $\varepsilon \to 0$

(1) ∇u_ε has no incentive to oscillate, so $|\nabla u| = 1$ in limit

(2) soln of eikonal eqn $|\nabla u| = 1$ with $u = 0$ at $\partial\Omega$ must have "walls" (unless Ω = circle); asymptotic energy should min "wall energy"

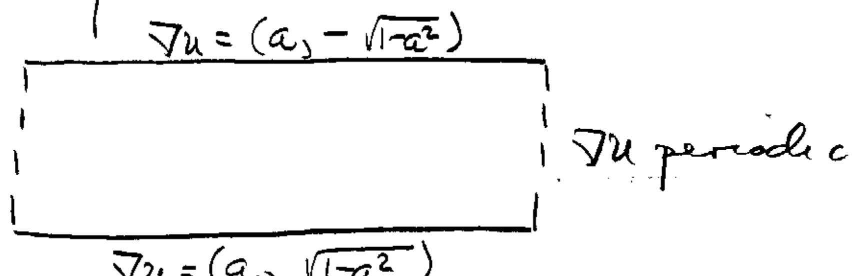
(3) expect internal str of wall to be 1D (like Modica - Mortola)

possible candidates for wall patterns



(Does "viscosity soln" minimize wall energy? open.)

Wall energy is easy to guess, by solving "wall var'l pbm"



Guess $u(x,y) = ax + v(y)$. Then v_y solves a 1D "Modica - Mortola - type pblm"

$$\min \int \frac{1}{\varepsilon}(v_y^2 + a^2 - 1)^2 + \varepsilon v_{yy}^2 \, dy$$

best v has $\varepsilon v_{yy} = |v_y^2 + a^2 - 1|$

$$\text{energy} = 2 \int_{-\sqrt{1-a^2}}^{+\sqrt{1-a^2}} |t^2 + a^2 - 1| \, dt = \frac{8}{3}(1-a^2)^{3/2}$$

<u>But is this optimal</u>? — Is the wall really so simple?
— Can we prove a lower bd without any ansatz abt structure of wall?

Answer is yes! Proof uses clever integr by parts. Look for $\Sigma : \mathbb{R}^2 \to \mathbb{R}^2$ ("entropy") s.t.

① $|\operatorname{div} \Sigma(\nabla u)| \le \frac{1}{\varepsilon}(|\nabla u|^2 - 1)^2 + \varepsilon |\nabla\nabla u|^2$

② $\Sigma(a, -\sqrt{1-a^2}) \cdot (0,1) + \Sigma(a, \sqrt{1-a^2}) \cdot (0,-1) = \frac{8}{3}(1-a^2)^{3/2}$.

Then for Aviles - Giga, energy of "wall var'l pblm" is

$$\geq \int |\operatorname{div} \Sigma'(\nabla u)| \qquad \text{by } ①$$

$$\geq \left| \int \operatorname{div} \Sigma'(\nabla u) \right|$$

$$= \frac{8}{3}(1-a^2)^{3/2} \cdot \text{length} \qquad \text{by } ②$$

Successful Σ' is explicit:

$$\Sigma'(\nabla u) = 2\left(-\tfrac{1}{3}u_x^2 - u_x u_y^2 + u_x,\ \tfrac{1}{3}u_y^2 + u_y u_x^2 - u_y\right)$$

for which, by elementary algebra,

$$\operatorname{div} \Sigma'(\nabla u) = 2(1-|\nabla u|^2)(u_{xx} - u_{yy})$$

$$\leq \frac{1}{\varepsilon}(1-|\nabla u|^2)^2 + \varepsilon(u_{xx}-u_{yy})^2$$

$$= \text{Aviles-Giga energy} + 2\varepsilon \int_\Omega u_{xy}^2 - u_{xx}u_{yy} - 4\varepsilon \int_\Omega u_{xy}^2$$

$$\leq \text{Aviles-Giga energy} + O(\varepsilon)$$

since $u_{xy}^2 - u_{xx}u_{yy}$ is a null-Lagrangian (so its integral is fully determined by bc)

Our 1D ansatz is opt'l (as $\varepsilon \to 0$) since all the inequalities above are asymptotically sharp for it.

How to think of Σ'? As $\varepsilon \to 0$ in condn ① we see that

$$u \text{ smooth}, \ |\nabla u|^2 = 1 \implies \operatorname{div} \Sigma'(\nabla u) = 0$$

Thus $\left(\Sigma'_1(\nabla u), \Sigma'_2(\nabla u)\right) = \Sigma'(\nabla u)$ are like an "entropy - entropy flux" pair for a scalar cons law.

Rtn now to question postponed earlier: why is the "wall energy" of our "crosstie var'l pblm" equal to $\sin\theta - \theta\cos\theta$, when θ = half-angle of wall?

<u>Ans</u>: this comes from minimizing the micromagnetic energy in "thick film regime" assuming a 1D ansatz

Micromagnetic energy, reduced to m indep of x_3

$$m = (m_1, m_2, m_3)(x_1, x_2) = (m', m_3)$$

on thin domain $\Omega = \Omega' \times (0,t)$, without anisotropy or applied field, is

$$d^2 t \int_{\Omega'} |\nabla m|^2 + \int_{\mathbb{R}^3} |\nabla U|^2$$

exchange magnetostatic

Can solve for U in Fourier space ("sep of vars")

$$\int_{\mathbb{R}^3} |\nabla U|^2 = t \int_{\mathbb{R}^2} f\left(\tfrac{t}{2}|\xi'|\right) \left| \frac{\xi'}{|\xi'|} \cdot \hat{m}' \right|^2 d\xi' + t \int_{\mathbb{R}^2} g\left(\tfrac{t}{2}|\xi'|\right) |\hat{m}_3|^2 d\xi'$$

$$g(z) = \frac{1 - e^{-2z}}{2z}, \quad f(z) = 1 - g(z)$$

If $t \gg$ length scale on which m' varies ($\approx d$) we get (since $f(z) \sim 1$, $g(z) \sim \frac{1}{2z}$ as $z \to \infty$)

$$\int_{\mathbb{R}^3} |\nabla U|^2 \sim t \, \|\operatorname{div} m'\|^2_{H^{-1}} + \|m_3\|^2_{H^{-1/2}}$$

OK, now consider 1D wall with $m_3 = 0$:

$$m = (\cos\theta, \sin\theta, 0), \quad \theta = \theta(x)$$

Profile $\theta(x)$ should minimize

$$t d^2 \int_a^b \theta_x^2 \, dx + t \, \| \partial_x m_1 \|_{H^{-1}}^2$$

$$= t d^2 \int_a^b \theta_x^2 \, dx + t \, \| m_1 - \cos\theta_\infty \|_{L^2}^2$$

$$= \min_{\substack{\theta(a) = \theta_\infty \\ \theta(b) = -\theta_\infty}} t \int_a^b d^2 \theta_x^2 + |\cos\theta - \cos\theta_\infty|^2 \, dx$$

$$= 2 d t \int_{-\theta_\infty}^{+\theta_\infty} |\cos\theta - \cos\theta_\infty| \, d\theta$$

$$= 4 d t \, (\sin\theta_\infty - \theta_\infty \cos\theta_\infty)$$

(Is the 1D ansatz opt'l? Well, no! In 3D micromagnetics there are lower-energy walls, achieved via nontrivial x_3-dependence. But 1D wall is apparently a local min of energy, at least more or less.)

Rtn to crosstie wall. Recall our assertion, that our "cartoon" of crosstie pattern achieves min energy. Have upper bd, need matching lower bd. Method (via "entropy") is to come up with a map $\Sigma' : S^1 \to \mathbb{R}^2$ s.t.

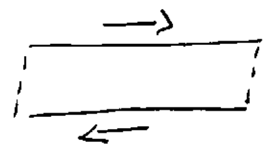
① if m is smooth, div-free, + $|m|=1$ then $\operatorname{div} \Sigma'(m) = 0$.

② if m is piecewise smooth, weakly div-free, + $|m|=1$ then along any wall

$$\left| [\Sigma'(m) \cdot \nu] \right| \leq \text{magnetic wall energy}$$

with equality for angles $\leq 90°$

③ $\int_{\text{bdry}} \Sigma'(m) \cdot \nu = (\sqrt{2} - 1) \cdot \text{length of strip}$

Recall why this is suff't: for any m 

$$\left| \int_{\text{bdry}} \Sigma'(m) \cdot n \right| \leq \int_{\text{interior}} |\operatorname{div} \Sigma'(m)| \leq \int \text{wall energy of } m$$

$(\sqrt{2} - 1) \cdot \text{length}$ by ③

by ② + ①, with equality for our cartoon since all its walls have angle $\leq 90°$

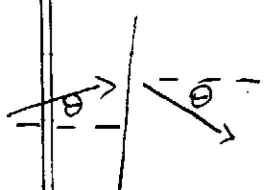
How to find Σ'? Condns that ② hold with equality for small angles leaves little freedom.

First pass (wrong, but informative): try

$$\Sigma'(m) = \tfrac{1}{2}\left(\theta m + m^{\perp}\right) \quad \text{when } m = e^{i\theta}$$

$$= \tfrac{1}{2}\left(\theta\cos\theta - \sin\theta,\ \theta\sin\theta + \cos\theta\right)$$

Then $\operatorname{div} \Sigma(m) = \theta \operatorname{div} m$ (elementary calcn) so ① holds. Also

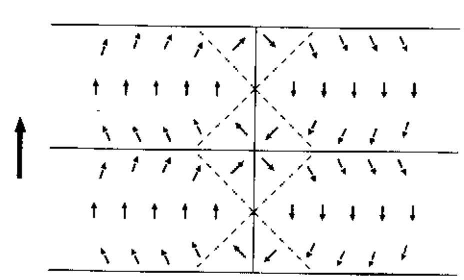


$\nu = (1,0)$, $m = (\cos\theta, \pm\sin\theta)$

$$\Rightarrow \left|[\Sigma'(m)\cdot\nu]\right| = |\Sigma_1^R - \Sigma_1^L| = \sin\theta - \theta\cos\theta$$

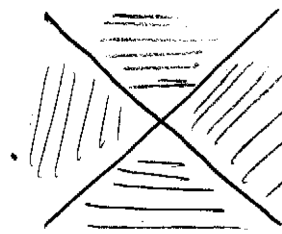
so ② holds (for any θ!).

Problem: θ isn't well-defined, since walls can (and do!) contain vertices.



Successful choice is similar in each quadrant:

$$\Sigma'(m) = \begin{cases} \theta m + m^\perp + (0, -\sqrt{2}) & -\pi/4 < \theta < \pi/4 \\ (\pi/2 - \theta) m - m^\perp + (-\sqrt{2}, 0) & \pi/4 \le \theta \le 3\pi/4 \\ (\theta - \pi) m + m^\perp + (0, \sqrt{2}) & 3\pi/4 \le \theta \le 5\pi/4 \\ (\frac{3\pi}{2} - \theta) m - m^\perp + (\sqrt{2}, 0) & 5\pi/4 \le \theta \le 7\pi/4 \end{cases}$$



slightly different formula in each quadrant

Fact: This defines a cont's $\Sigma : S^1 \to \mathbb{R}^2$ satisfying all our conditions. In particular,

$$|[\Sigma(m) \cdot \nu]| \le \text{wallenergy density}$$

with equality when total angle < 90°.

No need for arithmetic: our pattern achieves the bd because it uses only walls with angles < 90°.

Some rmks:

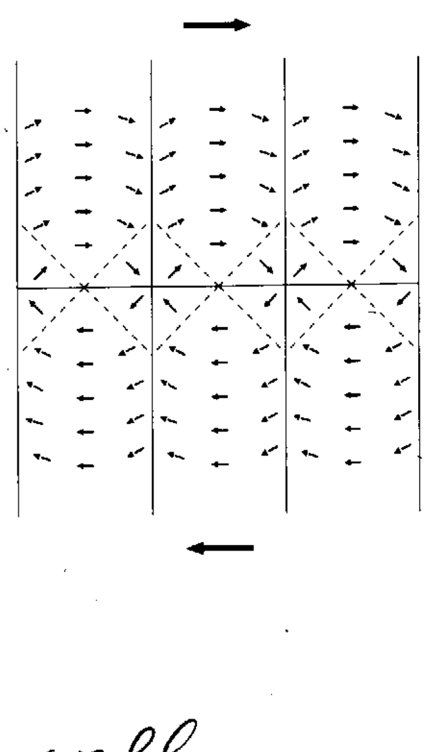
1. We proved matching upper + lower bds, but we have not shown crosstie pattern

is unique. (Defining feature: achieves net 180° wall, by piecewise smooth, div-free pattern using only walls with angle $\leq 90°$.)

(2) There's a cross tie pattern for every angle $>90°$. (A simple discontuity is opt'l for angle $<90°$.)

(3) Internal length scale of crosstie wall is not determined by preceding discn; actually it's set by competition btwn

- anisotropy - if energy prefers $m_z = 0$ then this favors smaller periodicity



- finite ratio $\frac{\text{thickness}}{\text{exchange par}} = \frac{t}{d}$.
  Then our calcn of energy is only approximate; actually walls have "tails" that repel each other. This favors larger periodicity

(4) We did not prove cross-tie wall achieves global min of micromagnetic

energy. (It doesn't!). Actually it's a local min. Our modeling hypotheses recognize it as global min of a more restricted fnl

(5) Method (via well-chosen "entropy") resembles
- use of "calibrations" to study minimal surfaces
- use of "null-Lagrangians" to estimate relaxed energies

but it seems rather mysterious nevertheless. Were we lucky? Or did there have to be an integr-by-pts proof of this lower bd?

(6) Can other patterns (assoc "Landau-type" theories from physics) be explained this way?

Certainly not all energy-min patterns. For example, analysis of vortex patterns in type-II superconductivity is entirely different (see recent book of Sandier + Serfaty).

(7) Can similar methods be used for other math'l pblms?

One appln: cptness of low-energy states for Aviles-Giga as $\varepsilon \to 0$ (eg pf that as $\varepsilon \to 0$, limiting u has $|\nabla u| = 1$). Pf by DeSimone/Kohn/Müller/Otto uses existence of many entropies, then argues as Tartar did for scalar cons laws.