Cloaking via Change of Variables in Electric Impedance Tomography

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Abstract

A recent paper by Pendry, Schurig, and Smith [Science 312, 2006, 1780-1782] used the coordinate-invariance of Maxwell's equations to show how a region of space can be "cloaked" – in other words, made inaccessible to electromagnetic sensing – by surrounding it with a suitable (anisotropic and heterogenous) dielectric shield. Essentially the same observation was made several years earlier by Greenleaf, Lassas, and Uhlmann [Mathematical Research Letters 10, 2003, 685-693 and Physiological Measurement 24, 2003, 413-419] in the closely related setting of electric impedance tomography. These papers, though brilliant, have two shortcomings: (a) the cloaks they consider are rather singular; and (b) the analysis by Greenleaf, Lassas, and Uhlmann does not apply in space dimension $n = 2$. The present paper provides a fresh treatment that remedies these shortcomings in the context of electric impedance tomography. In particular, we show how a regular near-cloak can be obtained using a nonsingular change of variables, and we prove that the change-of-variable-based scheme achieves perfect cloaking in any dimension $n \geq 2$.

Contents

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1 Introduction

We say a region of space is "cloaked" with respect to electromagnetic sensing if its contents – and even the existence of the cloak – are inaccessible to such measurements.

Is cloaking possible? The answer is yes, at least in principle. A cloaking scheme based on change-of-variables was discussed for electric impedance tomography by Greenleaf, Lassas, and Uhlmann in 2003 [19, 20], and for the time-harmonic Maxwell's equation by Pendry, Schurig, and Smith in 2006 [35, 38]. Other schemes have also been discussed, including one based on optical conformal mapping [28, 29], another based on anomalous localized resonances [32], and a third based on use of sensors and active sources [30, 36]. Recent developments include numerical [6, 12, 45] and experimental [39] implementations of change-of-variable-based cloaking; adaptations of the change-of-variable-based scheme to acoustic or elastic sensing [13, 31]; and the introduction of related schemes for cloaking active objects such as light sources [17].

Is cloaking interesting? The answer is clearly yes. One reason is theoretical: the existence of cloaks reveals intrinsic limitations of electromagnetic-based schemes for remote sensing, such as inverse scattering and impedance tomography. A second reason is practical: cloaking provides an easy method for making any object invisible – by simply surrounding it with a cloak. The appeal of this idea has attracted a lot of attention, e.g. [8, 44].

Is cloaking practical? The answer is not yet clear. All approaches to cloaking require the design of materials with exotic dielectric properties. One hopes that the desired properties can be achieved (or at least approximated) by means of "metamaterials" [40]; for the schemes based on change-of-variables this seems to be the case [39]. For a cloaking scheme to be practical it must be reasonably insensitive to imperfection; the robustness of the change-of-variable-based scheme has just begun to be addressed [10, 18, 37] (see Section 2.3 for comments on this work.)

The present paper is related to the first and last of the preceding questions. We ask:

(i) Does the change-of-variables-based scheme really achieve a perfect cloak?

(ii) What about a regularized version of the this scheme? How close does it come to achieving cloaking?

Our analysis is restricted to electric impedance tomography. This amounts to considering electromagnetic sensing in the low-frequency limit [26]; it is simpler than the finite-frequency setting, due

to the ready availability of variational principles. But we do discuss the finite-frequency setting, in Section 2.5.

Concerning (i): there is cause for concern, because the underlying change of variables is highly singular (see Section 2.3). Singularities are sometimes significant; for example, the fundamental solution of Laplace's equation is harmonic except at a point. The physics literature recognizes this issue; for example, Cummer et al. write in [12] that "whether perfect cloaking is achievable, even in theory, is . . . an open question." They also suggest, using an argument based on geometrical optics, that the presence of a singularity "may degrade cloaking performance to an unknown degree."

Actually, (i) was settled for electric impedance tomography by [19] in space dimension $n \geq 3$, using a method that does not work in space dimension two. One goal of the present paper is to show that the situation is not significantly different when $n = 2$: perfect cloaking is also possible in space dimension two. Our discussion of perfect cloaking, presented in Section 4, is not fundamentally different from that in [19]; in particular, our main tool (like [19]) is a result about the removability of singularities for harmonic functions. However our discussion differs from [19] by treating all dimensions $n \geq 2$ simultaneously, and by working directly with the divergence-form PDE of electrostatics rather than rewriting it as the Laplace-Beltrami equation of an associated Riemannian metric. In addition, our exposition is perhaps more elementary (thus more accessible to non-expert readers).

Concerning (ii): the question is as important as the answer. We suggest that the "perfect cloak," obtained using a singular change of variables, not be taken literally. Instead, it should be used to design a more regular "near-cloak," based on a less singular change of variable. The near-cloak is physically more plausible (for example, its dielectric tensor is strictly positive and finite). Moreover, the mathematical analysis of the near-cloak is actually easier, since nothing is singular. Basically, the problem reduces to understanding how boundary measurements are influenced by dielectric inclusions (see Section 2.3 for further explanation).

The paper is organized as follows. We begin, in Section 2, by introducing electric impedance tomography and giving a brief, nontechnical explanation of the change-of-variable-based cloaking scheme. That section also puts our work in context, discussing its relation to known uniqueness results and explaining why the finite-frequency case is similar to but different from the one considered here. Then, in Sections 3 and 4, we give a rigorous analysis of the change-of-variable-based cloaking scheme. In Section 3 we use a regular change of variables and prove that the inclusion is almost cloaked. In Section 4 we use a singular change of variables and prove that the inclusion is perfectly cloaked.

2 The main ideas

2.1 Electric impedance tomography

In electric impedance tomography, one uses static voltage and current measurements at the boundary of an object to gain information about its internal structure.

Mathematically, we suppose the object occupies a (known) bounded region $\Omega \subset \mathrm{I\!R}^n$, $n \geq 2$. Its (unknown) electrical conductivity $\sigma(x)$ is a non-negative symmetric-matrix-valued function on Ω.

The PDE of electrostatics is

$$\nabla \cdot (\sigma \nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sigma_{ij}(x) \frac{\partial u}{\partial x_j} \right) = 0 \quad \text{in } \Omega; \tag{1}$$

it relates the voltage u and the associated electric field ∇u to the resulting current $\sigma \nabla u$ (see Section 2.5). The PDE (1) determines a "Dirichlet to Neumann map" Λ_σ; by definition, it takes an arbitrary boundary voltage to the associated current flux:

$$\Lambda_\sigma : u|_{\partial \Omega} \rightarrow (\sigma \nabla u) \cdot \nu|_{\partial \Omega} \tag{2}$$

where ν is the outward unit normal to $\partial \Omega$. Electric impedance tomography seeks information on σ, given knowledge of the mapping Λ_σ. In the mathematics literature this problem was first proposed and partially addressed by Calderón [7].

Does Λ_σ determine σ? In general, the answer is no: the PDE is invariant under change of variables, so σ can at best be determined "up to change of variables." We shall explain this statement in Section 2.2. If, however, σ is scalar-valued, positive, and finite[1], then the answer is basically yes: under some modest (apparently technical) conditions on the regularity of σ, knowledge of the Dirichlet-to-Neumann map Λ_σ determines an internal isotropic conductivity $\sigma(x)$ uniquely. We shall review these results in Section 2.4.

σ = 1

$\sigma_A = A(x)$

$\sigma_A = \sigma_c(x)$

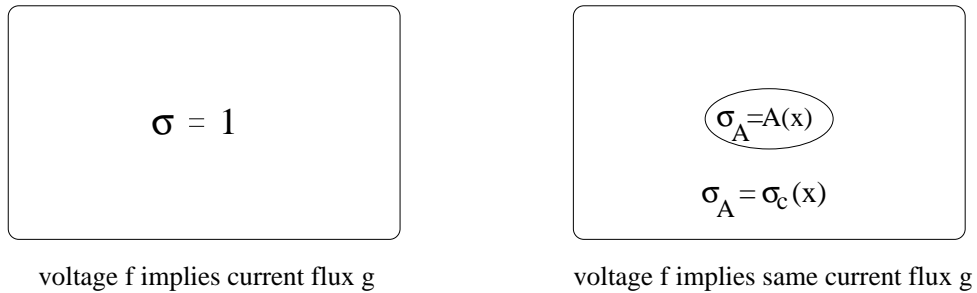voltage f implies current flux g   voltage f implies same current flux g

Figure 1: *The region D is cloaked by σ_c if, regardless of the conductivity distribution $A(x)$ in D, the boundary measurements at $\partial \Omega$ are identical to those of a uniform region with conductivity 1.*

What does it mean in this context for a subset D of Ω to be cloaked? In principle, it means that the contents of D – and even the existence of the cloak – are invisible to electrostatic boundary measurements. To keep things simple, however, we shall use a slightly more restrictive definition: we say $D \subset \Omega$ is cloaked by a conductivity distribution $\sigma_c(x)$ defined outside D if the associated boundary measurements at $\partial \Omega$ are identical to those of a homogeneous, isotropic region with conductivity 1 – regardless of the conductivity in D (see Figure 1). More precisely:

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[1]When we say σ is "positive and finite" we mean it is a bounded, measurable function with $\sigma(x) \geq c_0$ a.e. in Ω for some $c_0 > 0$.

Definition 1 *Let $D \subset \Omega$ be fixed, and let $\sigma_c : \Omega \setminus D$ be a non-negative, matrix-valued conductivity defined on the complement of D. We say σ_c cloaks the region D if its extensions across D,*

$$\sigma_A(x) = \begin{cases} A(x) & for \ x \in D \\ \sigma_c(x) & for \ x \in \Omega \setminus D \end{cases} \tag{3}$$

produce the same boundary measurements as a uniform region with conductivity $\sigma \equiv 1$, regardless of the choice of the conductivity $A(x)$ in D.

The name is appropriate: a cloak makes the associated region D invisible with respect to electric impedance tomography. Indeed, suppose σ_c cloaks $D \subset \Omega$ in the sense of of Definition 1, and let Ω' be any domain containing Ω. Then the Dirichlet-to-Neumann map of

$$\sigma(x) = \begin{cases} A(x) & \text{for } x \in D \\ \sigma_c(x) & \text{for } x \in \Omega \setminus D \\ 1 & \text{for } x \in \Omega' \setminus \Omega \end{cases} \tag{4}$$

is independent of A, and identical to that of the domain Ω' with constant conductivity 1. This holds because Ω communicates with its exterior only through its Dirichlet-to-Neumann map.

Notice that from a single example of cloaking, this extension argument produces many other examples. Indeed, according to (4), if σ_c cloaks $D \subset \Omega$ in the sense of Definition 1, then the extension of σ_c by 1 cloaks D in any larger domain Ω'.

We shall explain in Section 2.3, following [20, 35], how the invariance of electrostatics under change of variables leads to examples of cloaks.

2.2 Invariance by change of variables

The invariance of the PDE (1) by change-of-variables is well known. So is the fact that Λ_σ can determine σ at best "up to change of variables." This observation is explicit e.g. in [22, 25], with an attribution to Luc Tartar.

It is convenient to think variationally. Recall that if $\sigma(x)$ is bounded and positive definite, then the solution of (1) with Dirichlet data f solves the variational problem

$$\min_{u=f \text{ at } \partial\Omega} \int_\Omega \langle \sigma \nabla u, \nabla u \rangle \, dx. \tag{5}$$

Moreover the minimum "energy" is determined by Λ_σ, since when u solves (1) we have

$$\int_\Omega \langle \sigma \nabla u, \nabla u \rangle \, dx = \int_{\partial\Omega} f \Lambda_\sigma(f). \tag{6}$$

Thus, knowledge of Λ_σ determines the minimum energy, viewed as a quadratic form on Dirichlet data. The converse is also true: knowledge of the minimum energy for all Dirichlet data determines the boundary map Λ_σ. This follows from the well known polarization identity: for any f and g,

$$4 \int_{\partial\Omega} f \Lambda_\sigma g = \int_{\partial\Omega} (f+g) \Lambda_\sigma (f+g) \ - \ \int_{\partial\Omega} (f-g) \Lambda_\sigma (f-g). \tag{7}$$

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The right hand side is the minimum energy for $f + g$ minus that for $f - g$, while the left hand side is the boundary map, viewed as a bilinear form on Dirichlet data.

We turn now to change of variables. Suppose $y = F(x)$ is an invertible, orientation-preserving change of variables on Ω. Then we can change variables in the variational principle (5):

$$\int_\Omega \sum \sigma_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx = \int_\Omega \sum \sigma_{ij} \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_i} \frac{\partial u}{\partial y_l} \frac{\partial y_l}{\partial x_j} \det\left(\frac{\partial x}{\partial y}\right) dy.$$

We can write this more compactly as

$$\int_\Omega \langle \sigma(x) \nabla_x u, \nabla_x u \rangle \, dx = \int_\Omega \langle F_* \sigma(y) \nabla_y u, \nabla_y u \rangle \, dy$$

where

$$F_* \sigma(y) = \frac{1}{\det(DF)(x)} DF(x) \sigma(x) (DF(x))^T \tag{8}$$

in which DF is the matrix with i, j element $\partial y_i / \partial x_j$ and the right hand side is evaluated at $x = F^{-1}(y)$. We call $F_* \sigma$ the *push-forward* of σ by the change of variables F.

We come finally to the main point: *if $F(x) = x$ at $\partial\Omega$, then the boundary measurements associated with σ and $F_* \sigma$ are identical*, in other words

$$\Lambda_\sigma(f) = \Lambda_{F_* \sigma}(f) \quad \text{for all } f. \tag{9}$$

Indeed, if $F(x) = x$ at $\partial\Omega$ then the change of variables does not affect the Dirichlet data. So for any f,

$$
\begin{aligned}
\int_{\partial\Omega} f \Lambda_\sigma f &= \min_{u = f \text{ at } \partial\Omega} \int_\Omega \langle \sigma(x) \nabla_x u, \nabla_x u \rangle \, dx \\
&= \min_{u = f \text{ at } \partial\Omega} \int_\Omega \langle F_* \sigma(y) \nabla_y u, \nabla_y u \rangle \, dy \\
&= \int_{\partial\Omega} f \Lambda_{F_* \sigma} f.
\end{aligned}
$$

Thus Λ_σ and $\Lambda_{F_* \sigma}$ determine identical quadratic forms, from which it follows by (7) that $\Lambda_\sigma = \Lambda_{F_* \sigma}$.

2.3 Cloaking via change of variables

We now explain how change-of-variables-based cloaking works. For simplicity we focus on the radial case: $\Omega = B_2$ is a ball of radius 2, and the region D to be cloaked is B_1, the concentric ball of radius 1 (see Figure 2). It will be clear, however, that the method is much more general.

We start by explaining how B_1 can be nearly cloaked using a regular change of variables. Fixing a small parameter $\rho > 0$, consider the piecewise-smooth change of variables

$$F(x) = \begin{cases} \dfrac{x}{\rho} & \text{if } |x| \le \rho \\ \left(\dfrac{2 - 2\rho}{2 - \rho} + \dfrac{1}{2 - \rho}|x|\right) \dfrac{x}{|x|} & \text{if } \rho \le |x| \le 2. \end{cases} \tag{10}$$

Its key properties are that

- F is continuous and piecewise smooth,
- F expands B_ρ to B_1, while mapping the full domain B_2 to itself,
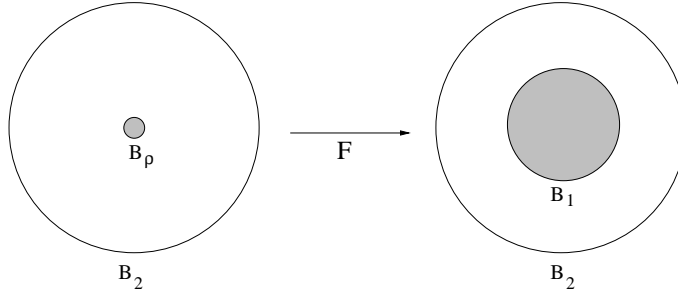- $F(x) = x$ at the outer boundary $|x| = 2$.

Figure 2: *The change of variables leading to a regular near-cloak: F expands a small ball B_ρ to a ball of radius 1.*

The associated near-cloak is the push-forward via F of the constant conductivity $\sigma = 1$, restricted to the annulus $B_2 \setminus B_1$. (Abusing notation a bit, we write this as F_*1.) To explain why, consider any conductivity of the form

$$\sigma_A(y) = \begin{cases} A(y) & \text{for } y \in B_1 \\ F_*1(y) & \text{for } y \in B_2 \setminus B_1. \end{cases} \tag{11}$$

By the change-of-variables principle (9) its boundary measurements are identical to those of

$$F_*^{-1}\sigma_A(x) = \begin{cases} F_*^{-1}A(x) & \text{for } x \in B_\rho \\ 1 & \text{for } x \in B_2 \setminus B_\rho \end{cases}$$

where

$$F_*^{-1}\sigma_A = (F^{-1})_*\sigma_A$$

denotes the push-forward of the conductivity distribution σ_A by the map F^{-1}. Thus, the boundary measurements associated with σ_A are the same as those of a uniform ball perturbed by a small inclusion at the center. The contents of the inclusion are uncontrolled, since A is arbitrary. But the radius of the inclusion is small, namely ρ. As we explain in Section 3, this is enough to assure that the boundary measurements are close to those of a completely uniform ball. Thus: when ρ is sufficiently small, this scheme comes close to cloaking the unit ball (see Theorem 1 in Section 3.3).

Now we show how B_1 can be perfectly cloaked using a singular change of variables. The idea is obvious: just take $\rho = 0$ in (10). The resulting change of variables

$$F(x) = \left(1 + \frac{1}{2}|x|\right)\frac{x}{|x|} \tag{12}$$

is the same one used in [19, 20] for electrostatics and in [35] for electromagnetics. Its key properties are that:

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- F is smooth except at 0;

- F blows up the point 0 to the ball B_1, while mapping the full domain B_2 to itself; and

- $F(x) = x$ at the outer boundary $|x| = 2$.

A heuristic "proof" that F_*1 gives a perfect cloak uses the same argument as before. This time $F_*^{-1}A$ occupies a point rather than a ball. Changing the conductivity at a point should have no effect on the boundary measurements. Therefore we expect that when σ_A is given by (11) with F given by (12), the boundary measurements should be identical to those obtained for a uniform ball with $\sigma \equiv 1$.

This heuristic proof needs some clarification. The validity of the change of variables formula is open to question when F is so singular. Worse: our cloak F_*1 is quite singular near its inner boundary $|x| = 1$; some care is therefore needed concerning what we mean by a solution of the PDE (1). These topics will be addressed in Section 4.

We have focused on the radial case because the simple, explicit form of the diffeomorphism F leads to an equally simple, explicit formula for the associated cloak (see Section 4.1). However the method is clearly not limited to the radial case (see Theorems 2 and 4).

Our "regular near-cloak" is quite different from the approximate cloaking scheme considered in [18, 37]. Those papers start with a perfect cylindrical cloak, obtained using the 2D version of the familiar construction (12). This cloak fills the annulus $1 < |y| < 2$ with an anisotropic, heterogeneous medium, whose behavior is rather singular near the inner boundary $|y| = 1$ (see Section 4.1). The approximate cloak considered in [18, 37] is obtained by restricting the perfect cloak to a slightly smaller annulus $1+\delta < |y| < 2$. Perfect cloaking (at any frequency) is obtained as $\delta \to 0$, however the convergence is extremely slow. The convergence can be greatly improved by introducing a layer at the edge of the cloak that permits surface currents [18]. In summary: our "regular near-cloak" avoids singular behavior by using a regularized change of variables, whereas [18, 37] avoid singular behavior by truncation. We also note the interesting article [10], which explores the sensitivity of the ideal cloak to various types of material or manufacturing imperfections.

The focus of this paper is on cloaking. But we note in passing that it might be possible to design other interesting devices using similar change-of-variable-based techniques. A recent example of this type is the scheme of [9] for rotating electromagnetic fields.

2.4   Relation to known uniqueness results

The uniqueness problem for electric impedance tomography asks whether it is possible, in principle, to determine $\sigma(x)$ using boundary measurements. In other words, does Λ_σ determine σ?

If it is known in advance that the conductivity is scalar-valued, positive, and finite, then the answer is basically yes. The earliest uniqueness results – in the class of analytic or piecewise analytic conductivities – date from the early 80's [14, 23, 24]. A few years later, using entirely different methods, uniqueness was proved for conductivities that are several times differentiable in dimension $n \geq 3$ [42] and in dimension $n = 2$ [33]. Recently, using yet another method, uniqueness has been shown in two space dimensions with no regularity hypothesis at all, assuming only that $\sigma(x)$ is

scalar-valued, strictly positive, and finite [4]. We have given just a few of the most important references; for more complete surveys see [11, 21, 43].

We observed in Section 2.2 that when $\sigma(x)$ is symmetric-matrix-valued, boundary measurements can at best determine it "up to change of variables." Is this the only invariance? In other words, if two conductivities give the same boundary measurements, must they be related by change of variables? If cloaking is possible then the answer should be no, since the conductivities σ_A in (3) are not related, as A varies, by change of variables.

Paradoxically, Sylvester proved that in two space dimensions, boundary measurements <u>do</u> determine σ up to change of variables [41]! [2] The heart of his proof was the introduction of isothermal coordinates – i.e. construction of a (unique) map $G : \Omega \to \Omega$ such that $G_*\sigma$ is isotropic and $G(x) = x$ at $\partial\Omega$. By uniqueness in the isotropic setting, Λ_σ determines $G_*\sigma$; thus boundary measurements determine σ up to change of variables.

Does cloaking contradict Sylvester's result? Not at all. The resolution of the paradox is that the introduction of isothermal coordinates *depends crucially on having upper and lower bounds for $\sigma(x)$.* Indeed, if Ω is a ball and $\sigma = F_*1$ with F given by (10), then the associated isothermal coordinate transformation is $G = F^{-1}$. As $\rho \to 0$ in (10) the isothermal coordinates become singular. When ρ is positive we do not get perfect cloaking (consistent with Sylvester's theorem). When $\rho = 0$ we do get cloaking – but the eigenvalues of σ are unbounded both above and below near $|x| = 1$ (see Section 4.1), Sylvester's argument no longer applies, and indeed there is no isothermal coordinate system.

Do boundary measurements determine σ up to change of variables in three or more space dimensions? If we assume only that σ is nonnegative then the answer is no, since cloaking is possible. If, however, we assume that σ is strictly positive and finite, then such a result could still be true. A proof for real-analytic conductivities is given in [27].

2.5   Comments on cloaking at nonzero frequency

This paper focuses on electric impedance tomography, because we can explain the essence of change-of-variable-based cloaking in this electrostatic setting with a minimum of mathematical complexity. The practical applications of cloaking are, however, mainly at nonzero frequencies – for example, making objects invisible at optical wavelengths, or undetectable by electromagnetic scattering measurements. We therefore discuss briefly how the positive-frequency problem is similar to, yet different from, the static case.

For time-harmonic fields in a linear medium, Maxwell's equations become

$$\nabla \times H = (\sigma - i\omega\epsilon)E, \quad \nabla \times E = i\omega\mu H. \tag{13}$$

Here E and H are complex vector fields representing the electric and magnetic fields; σ, ϵ, and μ are real-valued, positive-definite symmetric tensors representing the electrical conductivity, dielectric permittivity, and magnetic permeability of the medium; and $\omega > 0$ is the frequency. The physical electric and magnetic fields are $\mathrm{Re}\left\{Ee^{-i\omega t}\right\}$ and $\mathrm{Re}\left\{He^{-i\omega t}\right\}$.

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[2]Sylvester's paper proved only a local result, and required σ to be C^3. When combined with [33], however, his analysis gives a global C^3 result. The recent improvement in [5] assumes only that σ is bounded and positive-definite.

When $\omega = 0$, (13) reduces formally to (1). Indeed, Maxwell's equations become $\nabla \times H = \sigma E$ and $\nabla \times E = 0$. The latter implies $E = \nabla u$ and the former implies that $\sigma \nabla u$ is divergence-free.

The analogue of the Dirichlet-to-Neumann map Λ_σ at finite frequency is the correspondence between the tangential component of E and the tangential component of H at $\partial\Omega$. When ω is not an eigenfrequency this can be expressed as a map from $E|_{\partial\Omega} \times \nu$ to $H|_{\partial\Omega} \times \nu$, sometimes known as the admittance. (When ω is an eigenfrequency the map is not well-defined and one should consider instead all pairs $(E|_{\partial\Omega} \times \nu, H|_{\partial\Omega} \times \nu)$.) Mathematically, the admittance specifies the set of possible Cauchy data for (13) at frequency ω. Physically, a body interacts with its exterior only through its admittance; therefore two objects with the same admittance are indistinguishable by electromagnetic measurements at frequency ω – for example, by scattering measurements.

Digressing a bit, we remark that many of the uniqueness results sketched in Section 2.4 have been extended to finite frequency. In particular, the admittance of a 3D body at a single frequency determines σ, μ, and ϵ provided they are known in advance to be scalar-valued, sufficiently smooth, and constant near the boundary [34]. A different connection between the positive-frequency and electrostatic cases is provided by [26], which shows that the admittance determines the electrostatic Dirichlet-to-Neumann map in the limit $\omega \to 0$.

Let us focus now on cloaking. The positive-frequency analogue of our definition of cloaking is clear: three nonnegative matrix-valued functions σ, ϵ, and μ defined on $\Omega \setminus D$ cloak a region D if the associated admittance at $\partial\Omega$ does not depend on how σ, ϵ, and μ are extended across D. The positive-frequency analogue of our change-of-variables scheme is also clear: if $\Omega = B_2$, $D = B_1$, and $F(x) = \left(1 + \frac{1}{2}|x|\right)\frac{x}{|x|}$ as in (12) and [19, 20, 35], we should be able to cloak D by taking $\sigma|_{\Omega \setminus D}$, $\epsilon|_{\Omega \setminus D}$, and $\mu|_{\Omega \setminus D}$ each to be the "push-forward" of the constant 1. The correctness of this scheme is demonstrated in [17], though it is not the main focus of that paper. Their argument is, roughly speaking, a finite-frequency (and more general) analogue of the one in presented here in Section 4.

What about our regular near-cloak? The discussion in Section 3 has an obvious extension to the time-harmonic Maxwell setting. To analyze the performance of this near-cloak, we would need an estimate for the effect of a small inclusion (with uncontrolled dielectric properties) upon the boundary measurements (admittance). Unfortunately, this question is to the best of our knowledge open, though the effect of a uniform inclusion is very well understood [3]. We anticipate a result similar to the electrostatic setting – the effect of an inclusion should tend to zero as its radius tends to zero. Such a result would, as an immediate consequence, extend the analysis of Section 3 to the time-harmonic Maxwell setting.

We refer to [17] for further discussion of the time-harmonic problem. That paper includes, among other things, a new change-of-variable-based scheme for cloaking an active device (such as a light source).

3   Analysis of the regular near-cloak

This section reviews some well known facts about the Dirichlet-to-Neumann map, then analyzes the near-cloak obtained using the change of variable (10).

3.1 The Dirichlet-to-Neumann map

In discussing the PDE (1), we assume throughout this section that the conductivity is strictly positive and bounded in the sense that for some constants $0 < m, M < \infty$,

$$m|\xi|^2 \leq \langle \sigma(x)\xi, \xi \rangle \leq M|\xi|^2 \tag{14}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Our discussion of cloaking focused on the case when Ω is a ball, but in this section Ω can be any bounded domain in $\mathbb{R}^n$ with sufficiently regular boundary.

We will make essential use of the variational principle (5). Therefore we must restrict our attention to Dirichlet data f for which there exists a "finite energy" solution. When σ satisfies (14) it is well known that this occurs precisely when

$$f \in H^{1/2}(\partial\Omega) = \left\{ f \, : \, f = v|_{\partial\Omega} \text{ for some } v \text{ such that } \int_\Omega |\nabla v|^2 \, dx < \infty \right\}.$$

When f is constant the solution is also constant – a trivial case – so it is natural to restrict attention to the subspace $H^{1/2}_*(\partial\Omega) = H^{1/2}(\partial\Omega) \cap \{\int_{\partial\Omega} f = 0\}$, with the natural norm

$$\|f\|^2_{H^{1/2}_*(\partial\Omega)} = \min_{v=f \text{ at } \partial\Omega} \int_\Omega |\nabla v|^2 \, dx. \tag{15}$$

This is a fractional Sobolev space, consisting of functions with "one-half derivative in $L^2(\partial\Omega)$" (see e.g. [1]). We shall not try to explain what this means in general, but we note that when Ω is a ball B_R in $\mathbb{R}^2$ the interpretation is quite simple. In fact, if $f = \sum_{k=1}^\infty a_k \sin(k\theta) + b_k \cos(k\theta)$ at the boundary then the optimal v for (15) is the harmonic function $v = \sum_{k=1}^\infty (r/R)^k (a_k \sin(k\theta) + b_k \cos(k\theta))$, and direct calculation gives

$$\|f\|^2_{H^{1/2}_*(\partial B_R)} = \pi \sum_{k=1}^\infty k(a_k^2 + b_k^2).$$

Sometimes it is convenient to specify Neumann rather than Dirichlet data. Note that when σ is anisotropic, the phrase "Neumann data" refers to $g = (\sigma\nabla u) \cdot \nu$. It is well known that the space of finite energy Neumann data is $H^{-1/2}_*(\partial\Omega) = H^{-1/2}(\partial\Omega) \cap \{\int_{\partial\Omega} f = 0\}$. It consists of mean-value-zero functions with "minus one-half derivative in $L^2(\partial\Omega)$". In general

$$\|g\|_{H^{-1/2}_*(\partial\Omega)} = \sup \left\{ \int_{\partial\Omega} fg \, : \, \|f\|_{H^{1/2}_*(\partial\Omega)} \leq 1 \right\};$$

when Ω is a ball of radius R in $\mathbb{R}^2$ and $g = \sum_{k=1}^\infty a_k \sin(k\theta) + b_k \cos(k\theta)$ this reduces to

$$\|g\|^2_{H^{-1/2}_*(\partial B_R)} = \pi R^2 \sum_{k=1}^\infty k^{-1}(a_k^2 + b_k^2).$$

We defined the Dirichlet-to-Neumann map Λ_σ in (2) as the operator that takes Dirichlet to Neumann data. It is a bounded linear map from $H^{1/2}_*(\partial\Omega)$ to $H^{-1/2}_*(\partial\Omega)$. Moreover it is positive

and symmetric (in the L^2 inner product) and invertible, so it defines a positive definite quadratic form on $H_*^{1/2}(\partial\Omega)$. This form can be written "explicitly" as

$$\langle \Lambda_\sigma f_1, f_2 \rangle = \int_{\partial\Omega} \Lambda_\sigma(f_1) f_2 = \int_\Omega \langle \sigma \nabla u_1, \nabla u_2 \rangle \, dx$$

where u_1 and u_2 solve the PDE (1) with Dirichlet data f_1 and f_2 respectively. The natural norm on symmetric linear maps of this type is

$$\|\Lambda\| = \sup \left\{ |\langle \Lambda f, f \rangle| \, : \, \|f\|_{H_*^{1/2}(\partial\Omega)} \leq 1 \right\}. \tag{16}$$

This is equivalent to the operator norm of Λ viewed as a map from $H_*^{1/2}$ to $H_*^{-1/2}$, as a consequence of the polarization identity (7).

When two conductivities are ordered, the associated Dirichlet-to-Neumann maps are also ordered. More precisely: if σ and η satisfy

$$\langle \sigma(x)\xi, \xi \rangle \leq \langle \eta(x)\xi, \xi \rangle$$

for all $x \in \Omega$ and all $\xi \in {I\!\!R}^n$ then $\Lambda_\sigma \leq \Lambda_\eta$ in the sense that

$$\langle \Lambda_\sigma(f), f \rangle \leq \langle \Lambda_\eta(f), f \rangle \tag{17}$$

for all $f \in H_*^{1/2}(\partial\Omega)$. This follows easily from the variational principle (5), since if $\nabla \cdot (\sigma \nabla u) = 0$ and $\nabla \cdot (\eta \nabla v) = 0$ in Ω with $u = v = f$ at $\partial\Omega$, then

$$
\begin{aligned}
\langle \Lambda_\sigma f, f \rangle &= \int_\Omega \langle \sigma \nabla u, \nabla u \rangle \\
&\leq \int_\Omega \langle \sigma \nabla v, \nabla v \rangle \\
&\leq \int_\Omega \langle \eta \nabla v, \nabla v \rangle = \langle \Lambda_\eta f, f \rangle.
\end{aligned}
$$

3.2 Dielectric inclusions

The simplest special case of our PDE (1) is when $\sigma \equiv 1$. Then the solution u is harmonic. We understand almost everything about harmonic functions and the associated Dirichlet-to-Neumann map.

Another relatively simple case arises when σ is uniform except for a constant-conductivity spherical inclusion of radius ρ centered at some $x_0 \in \Omega$:

$$\sigma_{\alpha,\rho}(x) = \begin{cases} \alpha & \text{for } x \in B_\rho(x_0) \\ 1 & \text{for } x \in \Omega \setminus B_\rho(x_0). \end{cases} \tag{18}$$

In view of (17), the effect of the inclusion depends monotonically on its conductivity α. It is therefore natural to consider the extreme limits as $\alpha \to 0$ and $\alpha \to \infty$.

We now discuss these limits in detail, since they are important to our analysis. Given any $f \in H^{1/2}(\partial\Omega)$ let u_0^ρ denote the solution to

$$\Delta u_0^\rho = 0 \quad \text{in } \Omega \setminus \overline{B_\rho(x_0)} \ , \quad \text{with} \quad \frac{\partial u_0^\rho}{\partial \nu} = 0 \quad \text{on } \partial B_\rho(x_0) \ , \quad \text{and} \quad u_0^\rho = f \quad \text{on } \partial\Omega \ . \tag{19}$$

Similarly let let u_∞^ρ denote the solution to

$$\Delta u_\infty^\rho = 0 \quad \text{in } \Omega \setminus \overline{B_\rho(x_0)} \ , \quad \text{with} \quad u_\infty^\rho = c_\infty \quad \text{on } \partial B_\rho(x_0) \ , \quad \text{and} \quad u_\infty^\rho = f \quad \text{on } \partial\Omega \ , \tag{20}$$

where the constant c_∞ is (uniquely) determined by

$$\int_{\partial B_\rho(x_0)} \frac{\partial u_\infty^\rho}{\partial \nu} = 0 \ . \tag{21}$$

Using very standard energy arguments it is easy to see that

$$u_\alpha^\rho \to u_0^\rho \quad \text{as } \alpha \to 0 \ , \quad \text{and} \quad u_\alpha^\rho \to u_\infty^\rho \quad \text{as } \alpha \to \infty \ ,$$

weakly in $H^1(\Omega \setminus \overline{B_\rho(x_0)})$. Indeed, energy considerations immediately yield that $\|\nabla u_\alpha^\rho\|_{L^2(\Omega \setminus \overline{B_\rho(x_0)})}$ is bounded uniformly in α, that $\|\nabla u_\alpha^\rho\|_{L^2(B_\rho(x_0))} \to 0$ as $\alpha \to \infty$, and that $\|\alpha \nabla u_\alpha^\rho\|_{L^2(B_\rho(x_0))} \to 0$ as $\alpha \to 0$. By extraction of subsequences we now get weak $H^1(\Omega \setminus \overline{B_\rho(x_0)})$ limits, u_0^ρ and u_∞^ρ, that satisfy (19) and (20), respectively. The boundary conditions on $\partial B_\rho(x_0)$ follow from the continuity of $(\sigma \nabla u) \cdot \nu$ across this "interface." The condition (21), determining c_∞, follows since

$$\int_{\partial B_\rho(x_0)} \frac{\partial u_\alpha^\rho}{\partial \nu}^+ = \int_{\partial B_\rho(x_0)} \alpha \frac{\partial u_\alpha^\rho}{\partial \nu}^- = 0 \ ,$$

and therefore

$$\int_{\partial B_\rho(x_0)} \frac{\partial u_\infty^\rho}{\partial \nu} = \lim_{\alpha \to \infty} \int_{\partial B_\rho(x_0)} \frac{\partial u_\alpha^\rho}{\partial \nu}^+ = 0 \ .$$

It is not hard to see that this same c_∞ may also be characterized as the constant that gives rise to the smallest energy (of u_∞^ρ). The fact that we get single limits as $\alpha \to 0$ and $\alpha \to \infty$, respectively, is a consequence of the uniqueness of the solution to (19), and the solution to (20). We now define

$$\Lambda_0^\rho f = \nabla u_0^\rho \cdot \nu|_{\partial\Omega} \ ,$$

and

$$\Lambda_\infty^\rho f = \nabla u_\infty^\rho \cdot \nu|_{\partial\Omega} \ .$$

Integration by parts, together with the weak H^1 convergence, gives that $\Lambda_{\sigma_\alpha,\rho} f \to \Lambda_0^\rho f$ and $\Lambda_{\sigma_\alpha,\rho} f \to \Lambda_\infty^\rho f$ as $\alpha \to 0$ and $\alpha \to \infty$, respectively. In particular

$$\langle \Lambda_{\sigma_\alpha,\rho} f, f \rangle \to \langle \Lambda_0^\rho f, f \rangle \quad \text{as } \alpha \to 0 \ ,$$

and

$$\langle \Lambda_{\sigma_\alpha,\rho} f, f \rangle \to \langle \Lambda_\infty^\rho f, f \rangle \quad \text{as } \alpha \to \infty \ .$$

Finally we note that if Ω is a ball of radius R in $I\!\!R^2$ and the inclusion lies at its center, then the above convergence of the Dirichlet-to-Neumann maps can easily be derived by explicit solution of (19) and (20), using separation of variables.

In the small-particle limit $\rho \to 0$, the perturbation introduced by the presence of a small inclusion (extreme or not) is well understood. We shall not use its exact form; rather what matters to us is its magnitude, which is proportional to the *volume* of the inclusion:

Proposition 1 *Let Λ_1 be the Dirichlet-to-Neumann map when $\sigma \equiv 1$, and let Λ_0^ρ and Λ_∞^ρ be the Dirichlet-to-Neumann maps associated with the problems (19) and (20) respectively. Then*

$$\|\Lambda_1 - \Lambda_0^\rho\| \leq C\rho^n \quad and \quad \|\Lambda_1 - \Lambda_\infty^\rho\| \leq C\rho^n$$

when ρ is sufficiently small. Here n is the spatial dimension and we mean the operator norm (16) on the left hand side of each inequality.

A proof of the estimate for $\Lambda_1 - \Lambda_\infty^\rho$ is given in Section 2 of [16] and the same argument can be used for $\Lambda_1 - \Lambda_0^\rho$. The constant C depends of course on the location of x_0 and the shape of Ω. Much more detailed results are known, including a full asymptotic expansion for the dependence of the Dirichlet-to-Neumann map on ρ; see e.g. [2] for a recent review.

We have focused on spherical inclusions only for the sake of simplicity. The preceding discussion extends straightforwardly to inclusions of any fixed shape, i.e. to the situation when $B_\rho(x_0)$ is replaced by $x_0 + \rho D$ where D is any "inclusion shape" (a bounded domain in $I\!\!R^n$, containing the origin, with sufficiently regular boundary).

3.3 The regular near-cloak is almost invisible

Now consider the "regular near-cloak" discussed in Section 2.3: $\Omega = B_2$ is a ball about the origin of radius 2, and $\sigma = \sigma_A$ has the form

$$\sigma_A(y) = \begin{cases} A(y) & \text{for } y \in B_1 \\ F_*1(y) & \text{for } y \in B_2 \setminus B_1. \end{cases}$$

where F is given by (10). The symbol A stands for "arbitrary:" $A(x)$ is the (scalar or matrix-valued) conductivity in the region being cloaked. We assume it is positive definite and finite,

$$m|\xi|^2 \leq \langle A(y)\xi, \xi \rangle \leq M|\xi|^2 \quad \text{for } y \in B_1, \tag{22}$$

so the solution of the PDE (1) is well-defined and unique. However our estimates will not depend on the lower and upper bounds m and M.

As we explained in Section 2.3, the Dirichlet-to-Neumann map of σ_A is identical to that of

$$F_*^{-1}\sigma_A(x) = \begin{cases} F_*^{-1}A(x) & \text{for } x \in B_\rho \\ 1 & \text{for } x \in B_2 \setminus B_\rho. \end{cases}$$

By the ordering relation (17), and the convergence results described in the previous section, we conclude that

$$\lim_{\alpha \to 0} \Lambda_{\sigma_\alpha,\rho} = \Lambda_0^\rho \leq \Lambda_{\sigma_A} = \Lambda_{F_*^{-1}\sigma_A} \leq \Lambda_\infty^\rho = \lim_{\alpha \to \infty} \Lambda_{\sigma_\alpha,\rho} \quad ,$$

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whence
$$\Lambda_0^\rho - \Lambda_1 \le \Lambda_{\sigma_A} - \Lambda_1 \le \Lambda_\infty^\rho - \Lambda_1 \ .$$

It follows using Proposition 1 that the boundary measurements obtained using this near-cloak are almost identical to those of a uniform ball with conductivity 1:

$$\|\Lambda_{\sigma_A} - \Lambda_1\| \le C\rho^n \ , \tag{23}$$

where the left hand side is the operator norm (16). The constant C is independent of A; in fact it does not even depend on the values of m and M in (22). We have proved:

Theorem 1 *Suppose the shell $B_2 \setminus B_1$ has conductivity F_*1, where F is given by (10). If ρ is sufficiently small then B_1 is nearly cloaked, in the sense made precise by (23).*

We have focused on the spherically symmetric setting due to its simple, explicit character. However our argument did not use this symmetry in any essential way. Indeed, the same argument proves (see Figure 3):

Theorem 2 *Let $G : B_2 \to \Omega$ be a Lipschitz continuous map with a Lipschitz continuous inverse, and let $D = G(B_1)$. Then $H = G \circ F \circ G^{-1} : \Omega \to \Omega$ is piecewise Lipschitz; moreover*

- *H expands $G(B_\rho)$ to D, and*

- *$H(x) = x$ at $\partial\Omega$.*

If the shell $\Omega \setminus D$ has conductivity H_*1 then D is nearly cloaked when ρ is small. More precisely: when the conductivity of Ω has the form

$$\sigma_A(y) = \begin{cases} A(y) & \text{for } y \in D \\ H_*1(y) & \text{for } y \in \Omega \setminus D, \end{cases}$$

the Dirichlet-to-Neumann map is nearly independent of A in the sense that

$$\|\Lambda_{\sigma_A} - \Lambda_1\| \le C\rho^n$$

4 Analysis of the singular cloak

This section discusses the perfect cloak obtained using the singular change of variables (12). We focus on the radial case for simplicity, but our argument extends straightforwardly to a broad class of non-radial examples (see Theorem 4).

As we explained in Section 2.3, the basic assertion of cloaking is that for conductivities of the form (11) with F given by (12), the Dirichlet-to-Neumann map is identical to that of the uniform ball with conductivity 1. Thus, if the shell $B_2 \setminus B_1$ has conductivity F_*1 then the ball B_1 is cloaked.

This assertion follows from Theorem 1 by passing to the limit $\rho \to 0$ (see Remark 1 in Section 4.2). But it can also be proved directly, and the direct argument – being very different – gives
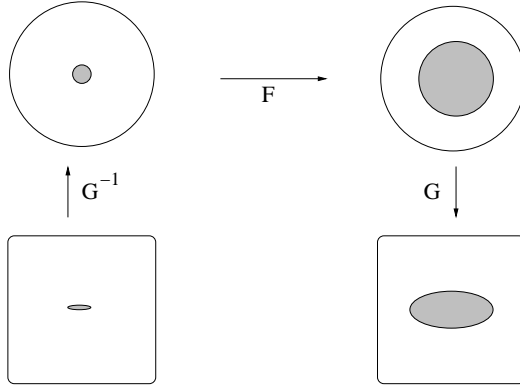
Figure 3: *The map $H = G \circ F \circ G^{-1}$ blows up $G(B_\rho)$ to $D = G(B_1)$ while acting as the identity on $\partial\Omega = \partial G(B_2)$.*

additional insight. In particular, it reveals the mechanism of cloaking: the potential in B_1 is constant, rendering the conductivity in this region irrelevant.

The essence of the argument presented in this section is similar to that of [19]. In particular, our main tool is a well-known result on the removability of isolated singularities for solutions of Laplace's equation (see the proof of Proposition 2).

4.1 Explicit form of the cloak

Recall that F_*1 is defined by (8). When $F : B_2 \to B_2$ is given by (12) it is easy to make F_*1 explicit. Indeed, the Jacobian matrix $DF = (\partial F_i/\partial x_j)$ is

$$DF = \left(\frac{1}{2} + \frac{1}{|x|}\right) I - \frac{1}{|x|}\hat{x}\,\hat{x}^T, \tag{24}$$

for $x \neq 0$, where I is the identity matrix and $\hat{x} = x/|x|$. Thus DF is symmetric; $\hat{x}$ is an eigenvector with eigenvalue $1/2$, and (in space dimension n) $\hat{x}^\perp$ is an $n-1$-dimensional eigenspace with eigenvalue $\frac{1}{2} + \frac{1}{|x|}$. The determinant is evidently

$$\det(DF) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{|x|}\right)^{n-1} = \frac{(|x| + 2)^{n-1}}{2^n|x|^{n-1}}. \tag{25}$$

It follows by a brief calculation that in the shell $1 < |y| < 2$,

$$F_*1(y) = \frac{2^n}{(2 + |x|)^{n-1}}\Big[\left(\tfrac{1}{4}|x|^{n-1} + |x|^{n-2} + |x|^{n-3}\right)\left(I - \hat{x}\,\hat{x}^T\right) + \tfrac{1}{4}|x|^{n-1}\hat{x}\,\hat{x}^T\Big], \tag{26}$$

where the right hand side is evaluated at

$$x = F^{-1}(y) = 2(|y| - 1)\frac{y}{|y|}. \tag{27}$$

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Since F is singular at $x = 0$ we expect F_*1 to be a bit strange near the inner boundary of the shell. The details depend on the spatial dimension n:

$$\text{when } n = 2, \text{ one eigenvalue of } F_*1 \text{ tends to 0 and the other to } \infty; \qquad (28)$$

$$\text{when } n = 3, \text{ one eigenvalue tends to 0 while the others remain finite;} \qquad (29)$$

$$\text{when } n \geq 4, \text{ all eigenvalues tend to 0.} \qquad (30)$$

In fact: writing $r = |x| = 2(|y| - 1)$, when $n = 2$ the eigenvalues behave like r and r^{-1} as $r \to 0$; when $n = 3$ one eigenvalue behaves like r^2 and two like r^0; when $n \geq 4$ one eigenvalue behaves like r^{n-1} and the remaining $n - 1$ like r^{n-3}. Notice that for $n \geq 3$, the conductivity F_*1 depends smoothly on y near the inner boundary of the shell. The "strangeness" we mentioned above is not a lack of smoothness but rather a degeneracy (lack of a uniform lower bound). In space dimension $n = 2$ the situation is a little different: F_*1 becomes degenerate but also lacks smoothness since the circumferential eigenvalue becomes infinite. This difference between $n = 2$ and $n \geq 3$ will play no essential role in our analysis.

4.2 The potential outside the cloaked region

Let v be the potential associated with Dirichlet data f:

$$\nabla \cdot (\sigma_A \nabla v) = 0 \quad \text{in } B_2, \text{ with } v = f \text{ at } \partial B_2, \qquad (31)$$

where σ_A is given by (11) using the singular change of variable (12). We assume, as in Section 3, that A is bounded above and below in the sense that (22) holds.

Does this PDE have a unique solution? The answer is not immediately obvious, due to the degeneracy of F_*1 near $|y| = 1$. We shall show, here and in Section 4.3, that the only reasonable solution of (31) is

$$v(y) = \begin{cases} u(x) & \text{for } y \in B_2 \setminus B_1 \\ u(0) & \text{for } y \in B_1, \end{cases} \qquad (32)$$

where u is the harmonic function with the same Dirichlet data

$$\Delta u = 0 \quad \text{in } B_2, \text{ with } u = f \text{ at } \partial B_2 \qquad (33)$$

and $x = F^{-1}(y)$.

What can we assume about the solution of (31)? Later, in Section 4.3, we will ask that ∇v and $\sigma_A \nabla v$ both be square-integrable. For the moment, however, we ask only that v be bounded near $|y| = 1$. More precisely, we ask that

$$|v(y)| \leq C \quad \text{for } |y| \leq r \qquad (34)$$

for some constants $C < \infty$ and $r > 1$. (We do not assume v is bounded in the entire ball B_2 because the Dirichlet data can be unbounded – an $H^{1/2}$ function need not be L^∞.) This is a very modest hypothesis. Indeed, since F_*1 is smooth for $|y| > 1$, elliptic regularity assures us that v is uniformly bounded in any compact subset of $B_2 \setminus B_1$. The essential content of (34) is thus that

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v does not diverge as $|y| \to 1$. If the conductivity were positive and finite such growth would be ruled out by the variational principle (5) and an easy truncation argument.

With this modest hypothesis on v, we can identify its values in $B_2 \setminus B_1$ by changing variables then using a standard theorem about the removability of point singularities for harmonic functions.

Proposition 2 *If v solves (31) and satisfies (34) then*

$$v(y) = u(x) \quad for \ 1 < |y| < 2 \tag{35}$$

where $x = F^{-1}(y)$ and u is the harmonic function on B_2 with the same Dirichlet data as v.

Proof. Since $\sigma_A(y) = F_* 1(y)$ is smooth and bounded away from zero for $|y|$ strictly larger than 1, elliptic regularity applies and v is a classical solution of the PDE in $B_2 \setminus \overline{B_1}$. When ϕ is supported in $B_2 \setminus \overline{B_1}$, the PDE combines with the definition of F_* and the change of variables formula to give

$$0 = \int \langle \sigma_A \nabla_y v(y), \nabla_y \phi(y) \rangle \, dy = \int \langle \nabla_x v(F(x)), \nabla_x \phi(F(x)) \rangle \, dx. \tag{36}$$

Since $\phi(y)$ is supported on $B_2 \setminus \overline{B_1}$, the test function $\phi(F(x))$ vanishes at 0 and ∂B_2 but is otherwise arbitrary. So (36) tells us that $w(x) = v(F(x))$ is a weak solution of $\Delta w = 0$ in the punctured ball $B_2 \setminus \{0\}$. By elliptic regularity, it is also a classical solution.

We now use the following well known result about removable singularities for harmonic functions: if $\Delta w = 0$ in a punctured ball about 0 and if

$$\begin{aligned} |w(x)| &= o\left(|x|^{2-n}\right) &&\text{in dimension } n \geq 3, \text{ or} \\ |w(x)| &= o\left(\log |x|^{-1}\right) &&\text{in dimension } n = 2 \end{aligned} \tag{37}$$

as $x \to 0$, then w has a removable singularity at 0 (see e.g. [15]). In other words, $w(0)$ is determined by continuity and (so extended) w is harmonic in the entire ball.

Our $w(x) = v(F(x))$ satisfies (37) – indeed, it is uniformly bounded near 0 as a consequence of (34). So w is harmonic on B_2. Moreover w has the same Dirichlet data as v, since $F(x) = x$ at ∂B_2. Thus w is precisely the function u that appears in (35), and the proof is complete. $\square$

Remark 1 *We have shown using elliptic theory that for the cloak constructed using the singular change of variable (12), the potential outside the cloaked region is given by (35). An alternative, more physical justification of (35) is this: it gives the limiting value of the potential associated with our regular near-cloak (10) in the singular limit $\rho \to 0$.*

To justify the Remark, let F_ρ be the regularized change of variable (10), and let v_ρ be the potential in the near-cloak for a given choice of the Dirichlet data. Then $u_\rho(x) = v_\rho(F_\rho(x))$ is harmonic outside B_ρ. It is also uniformly bounded (away from the outer boundary $|y| = 2$), with a bound independent of ρ. So by a standard compactness argument, the limit as $\rho \to 0$ exists and is harmonic in $B_2 \setminus \{0\}$. Since the limit is bounded, 0 is a removable singularity and $u_0(x) = \lim_{\rho \to 0} u_\rho(x)$ is the unique harmonic function in B_2 with the given Dirichlet data. Now for any fixed $1 < |y| < 2$ we can pass to the limit $\rho \to 0$ in the relation $v_\rho(y) = u_\rho(F_\rho^{-1}(y))$ to get $v_0(y) = u_0(F_0^{-1}(y))$, confirming (35).

4.3 The potential inside the cloaked region

We have asserted that the solution of (31) is given by (32). Proposition 2 justifies this assertion outside B_1; this section completes the justification by showing that (i) the proposed v is indeed a solution, and (ii) it is the only reasonable solution.

To show that v is a solution, we must demonstrate that $\sigma_A \nabla v$ is divergence-free. This is the main goal of the following Proposition.

Proposition 3 *Fixing $f \in H_*^{1/2}(\partial B_2)$, let v be defined by (32). Then*

(a) *v is Lipschitz continuous away from ∂B_2, i.e. $|\nabla v|$ is uniformly bounded in B_r for every $r < 2$.*

(b) *$\sigma_A \nabla v$ is also uniformly bounded away from ∂B_2,*

(c) *$(\sigma_A \nabla v) \cdot \nu \to 0$ uniformly as $|y| \downarrow 1$, where $\nu = y/|y|$ is the normal to ∂B_1, and*

(d) *$\sigma_A \nabla v$ is weakly divergence-free in the entire domain B_2.*

Proof. We observe first that (d) follows immediately from (b), (c), and (36). Indeed, a bounded vector-field ξ is weakly divergence-free on B_2 if and only if it is weakly divergence-free on the subdomains B_1 and $B_2 \setminus \overline{B_1}$ and its normal flux $\xi \cdot \nu$ is continuous across the interface ∂B_1. (The normal flux is well-defined from either side, as a consequence of ξ being divergence free in B_1 and its complement.) We apply this to $\xi = \sigma_A \nabla v$, which is clearly clearly divergence-free in B_1 (where it vanishes) and in $B_2 \setminus \overline{B_1}$ (by equation (36)). If (c) holds then the normal flux $\xi \cdot \nu = 0$ vanishes on both sides of ∂B_1. In particular it is continuous, so (d) holds.

The proofs of (a)-(c) are straightforward calculations based on the change of variable formula and the smoothness of $u(x) = v(F(x))$, together with our explicit formulas for DF (24) and F_*1 (26). To see that ∇v is bounded away from ∂B_2 we observe that, by chain rule and the symmetry of DF, we have
$$\nabla_y v = (DF^{-1})^T \nabla_x u = (DF)^{-1} \nabla_x u$$
for $1 < |y| < 2$. The matrix $(DF)^{-1}$ is uniformly bounded, by (24); and $\nabla_x u$ is bounded (except perhaps near ∂B_2) since u is harmonic in x. Thus $|\nabla v|$ is bounded and v is Lipschitz continuous on $1 \le |y| < r$ for any $r < 2$. It is moreover constant on B_1, and continuous across ∂B_1. Therefore v is Lipschitz continuous on the entire ball B_r for every $r < 2$.

In dimensions $n \ge 3$ (b) follows immediately from (a), since F_*1 is uniformly bounded. In dimension $n = 2$ however we must be more careful, since F_*1 becomes unbounded as $|y| \downarrow 1$. Using the definition of σ_A, chain rule, and the symmetry of DF we have

$$\sigma_A \nabla_y v = F_*1 (DF)^{-1} \nabla_x u \tag{38}$$

for $1 < |y| < 2$. The symmetric matrices F_*1 and $(DF)^{-1}$ have the same eigenvectors, namely $\hat{x}$ and $\hat{x}^\perp$. Taking $n = 2$ in (24) and (26) we see that the eigenvalue of F_*1 in direction $\hat{x}^\perp$ behaves like $|x|^{-1}$, while that of $(DF)^{-1}$ behaves like $|x|$. The eigenvalues of both matrices in direction $\hat{x}$

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are bounded. Thus the product $F_*1(DF)^{-1}$ is bounded. This yields (b), since $\nabla_x u$ is bounded away from ∂B_2 and $\sigma_A \nabla v = 0$ for $y \in B_1$.

The proof of (c) is similar to that of (b). Since $|y| \downarrow 1$ corresponds to $|x| \to 0$ and $y/|y| = x/|x| = \hat{x}$, we must show that the $\hat{x}$ component of (38) tends to zero as $|x| \to 0$. Since $F_*1(DF)^{-1}$ is symmetric and $\hat{x}$ is an eigenvector, it suffices to show that the corresponding eigenvalue tends to 0. In fact, its value according to (24) and (26) is

$$\frac{2^{n-1}}{(2+|x|)^{n-1}}|x|^{n-1} \le |x|^{n-1}$$

which tends to zero linearly (if $n = 2$) or better (if $n \ge 3$). The proof is now complete. $\qquad \square$

We have shown that the function defined by (32) solves the PDE (31). Is it the only solution? If σ_A were strictly positive and finite, uniqueness would be standard. When σ_A is degenerate, however, uniqueness can sometimes fail. For example, if σ_A were identically 0 in B_1 then the solution would not be unique: v would be arbitrary in B_1. Our situation, however, is much more controlled: the degeneracy occurs only at ∂B_1, and it has a very specific form.

Uniqueness should be proved in a specific class. We assumed in Section 4.2 that v was uniformly bounded near ∂B_1. Here we assume further that

$$\nabla v \in L^2(B_2) \quad \text{and} \quad \sigma_A \nabla v \in L^2(B_2). \tag{39}$$

Proposition 4 *If v is a weak solution of the PDE (31) which also satisfies (34) and (39) then v must be given by the formula (32).*

Proof. We know from Proposition 2 that $v(y) = u(x)$ outside B_1. What remains to be proved is that $v \equiv u(0)$ in B_1.

Recall that u has a removable singularity at 0. In particular it is continuous there. Since F^{-1} maps ∂B_1 to 0, it follows that $v(y) \to u(0)$ as y approaches ∂B_1 from outside.

Since $\nabla v \in L^2(B_2)$ by hypothesis, the restriction of v to ∂B_1 makes sense, and it is the same from outside or inside. Evidently this restriction is constant, identically equal to $u(0)$. It follows, by uniqueness for the PDE $\nabla \cdot (A \nabla v) = 0$ in B_1, that $v \equiv u(0)$ throughout B_1, as asserted. $\qquad \square$

The preceding argument actually uses somewhat less than (39). Any condition that makes v continuous across ∂B_1 would be sufficient. However we also need a hypothesis on $\sigma_A \nabla v$ (for example that it be integrable) for the PDE (31) to make sense.

4.4   The singular cloak is invisible

Our main point is that if the shell $B_2 \setminus B_1$ has conductivity F_*1 then the ball B_1 is cloaked. This is an easy consequence of the preceding results:

Theorem 3 *Suppose σ_A is given by (11), where F is given by (12) and A is uniformly positive and finite (22). Then the associated Dirichlet-to-Neumann map Λ_{σ_A} is the same as that of a uniform ball B_2 with conductivity 1.*

Proof. It suffices to prove that Λ_{σ_A} and Λ_1 determine the same quadratic form on Dirichlet data, where Λ_1 is the Dirichlet-to-Neumann map of the uniform ball. But by (32) we have

$$\int_{B_2} \langle \sigma_A \nabla v, \nabla v \rangle \, dy = \int_{B_2 \setminus B_1} \langle \sigma_A \nabla v, \nabla v \rangle \, dy,$$

and the definition of σ_A combined with the change of variables formula gives

$$\int_{B_2 \setminus B_1} \langle \sigma_A \nabla v \nabla v \rangle \, dy = \int_{B_2} |\nabla_x u|^2 \, dx$$

where u is harmonic with the same Dirichlet data as v. Thus

$$\langle \Lambda_{\sigma_A} f, f \rangle = \langle \Lambda_1 f, f \rangle$$

for all $f \in H_*^{1/2}$, whence $\Lambda_{\sigma_A} = \Lambda_1$ as asserted. $\qquad\square$

We have focused on the radial setting for the sake of simplicity. However the analysis in this section extends straightforwardly to the nonradial cloaks discussed at the end of Section 3.

Theorem 4 *Let $G : B_2 \to \Omega$ be a Lipschitz continuous map with Lipschitz continuous inverse, and let $D = G(B_1)$. Then $H = G \circ F \circ G^{-1} : \Omega \to \Omega$ acts as the identity on $\partial \Omega$, while "blowing up" the point $z_0 = G(0)$ to D. (This is the $\rho = 0$ limit of Figure 3). Consider a conductivity σ_A defined on Ω of the form*

$$\sigma_A(y) = \begin{cases} A(w) & \text{for } w \in D \\ H_* 1(w) & \text{for } w \in \Omega \setminus D, \end{cases}$$

where A is symmetric, positive, and finite but otherwise arbitrary. The associated Dirichlet-to-Neumann map Λ_{σ_A} is independent of A; in fact, $\Lambda_{\sigma_A} = \Lambda_1$ is the Dirichlet-to-Neumann map associated with conductivity 1.

Proof. We claim that

$$v(w) = \begin{cases} u(z) & \text{for } w \in \Omega \setminus D \\ u(z_0) & \text{for } w \in D, \end{cases} \tag{40}$$

where $w = H(z)$ and u solves $\Delta u = 0$ in Ω with the same Dirichlet data as v. The proof is parallel to our argument in the radial case, so we shall be relatively brief.

The proof of Proposition 2 made no use of radial symmetry; it applies equally in the present setting. We must assume of course that v is bounded away from $\partial \Omega$, and we conclude that (40) is correct outside D.

The analogue of Proposition 3(a) is the statement that v is uniformly Lipschitz in $\Omega \setminus \overline{D}$ except perhaps near $\partial \Omega$. With the conventions $x = G^{-1}(z)$, $y = F(x)$, and $w = G(y)$, we have

$$DH(z) = DG(y) \, DF(x) \, DG^{-1}(z)$$

by chain rule. By hypothesis, DG and DG^{-1} are uniformly bounded. Therefore $(DH)^{-1}$ is uniformly bounded too. Since $\Delta_z u = 0$, u is a smooth function of z except perhaps near $\partial \Omega$. It follows that $v(w) = u(H^{-1}(w))$ is uniformly Lipschitz continuous away from $\partial \Omega$.

The analogue of Proposition 3(b) is the statement that $H_*1 \nabla_w v$ is uniformly bounded away from $\partial\Omega$. Recalling the definition

$$H_*1 = \frac{1}{\det DH} DH DH^T$$

and using that $\nabla_w v = (DH^T)^{-1} \nabla_z u$, we see that

$$H_*1 \nabla_w v = \frac{1}{\det DH} DH \nabla_z u.$$

Since u is harmonic, it is smooth away from $\partial\Omega$. As for $DH/\det(DH)$: it has the same behavior as $DF/\det(DF)$, since DG and DG^{-1} are bounded. One verifies using the explicit formula (24) that $DF/\det(DF)$ stays bounded as $x \to 0$.

The analogue of Proposition 3(b) is the statement that the normal flux $(H_*1 \nabla_w v) \cdot n_w \to 0$ as w approaches ∂D from outside, where n_w is the unit normal at ∂D. We use the fact that n_w is parallel to $(DG^{-1})^T(\nu_y)$, if ν_y is the unit normal to ∂B_1 at the corresponding point $y = G^{-1}(w)$. (To see this, note that if τ is tangent to ∂B_1 then $DG\,\tau$ is tangent to ∂D, and $\langle DG\,\tau, (DG^{-1})^T \nu \rangle = \langle \tau, \nu \rangle = 0$.) It follows that

$$|(H_*1 \nabla_w v) \cdot n_w| \leq C |\langle H_*1 \nabla_w v, (DG^{-1})^T \nu_y \rangle|. \tag{41}$$

Now,

$$H_*1 \nabla_w v = (\det DH)^{-1} DH \nabla_z u = (\det DH)^{-1} DG\, DF\, DG^{-1} \nabla_z u.$$

So the inner product on the right side of (41) is equal to

$$(\det DH)^{-1} \left| \langle DG\, DF\, DG^{-1} \nabla_z u, (DG^{-1})^T \nu_y \rangle \right| = (\det DH)^{-1} \left| \langle DF\, DG^{-1} \nabla_z u, \nu_y \rangle \right|.$$

Since DG and DG^{-1} are bounded, this is bounded by a constant times

$$(\det DF)^{-1} \left| \langle DF\, DG^{-1} \nabla_z u, \nu_y \rangle \right|.$$

But recall that $\nu_y = y/|y| = x/|x|$ is an eigenvector of the symmetric matrix $(\det DF)^{-1} DF$, with an eigenvalue that tends to 0 as $x \to 0$. Therefore

$$(H_*1 \nabla_w v) \cdot n_w \to 0 \quad \text{as } w \to D,$$

as asserted.

The arguments used for Proposition 3(d), Proposition 4 and Theorem 3 did not use radial symmetry or the explicit form of the cloak, so they extend immediately to the present setting. $\square$

We note that for $n \geq 3$ the results in Theorem 3 and Theorem 4 coincide with those already established in [19].

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